Introduction to 2d CFT in the bootstrap formalism

Sylvain Ribault, November 2014

Abstract: This is a short introduction to the main ideas and techniques of two-dimensional conformal field theory. In paricular, we introduce the Virasoro algebra and its representations, explain the principles of the conformal bootstrap approach, define the conformal blocks, and construct the energy-momentum tensor.

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1 Conformal symmetry and the Virasoro algebra

2d conformal symmetry, unlike conformal symmetry in higher dimensions, implies the existence of an infinite-dimensional symmetry group. This is because, in holomorphic coordinates where the metric is

$$ds^2 = dzd\bar{z} \tag{1}$$

any holomorphic function $f(z)$ defines a conformal transformation – a transformation which preserves the angles:

$$dfd\bar{f} = |f'(z)|^2 dzd\bar{z} \tag{2}$$

Such functions belong to the conformal group, and the corresponding Lie algebra is generated by the differential operators

$$L_n = -z^{n+1}\frac{\partial}{\partial z} \quad , \quad n \in \mathbb{Z} \tag{3}$$

These operators form a Virasoro algebra, with commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{m+n,0} \tag{4}$$

We have added a central term, whose coefficient is the central charge c, which does not follow from the representation of L_ns as differential operators. In a quantum theory the wavefunctions form projective representations of the algebra, which are equivalent to ordinary representations of the centrally extended algebra – so we generally use the centrally extended algebra.

Now which representations will we use? Eigenvalues of L_0 are called conformal dimensions and interpreted as energies, and we want them to be bounded from below. But

$$L_0|v\rangle = \Delta|v\rangle \quad \Rightarrow \quad L_0 L_n|v\rangle = (\Delta - n)L_n|v\rangle \tag{5}$$

so $L_{n>0}$ decrease the energy. A simple way to bound the energy from below is to consider highest-weight representations, generated by a primary state $|\Delta\rangle$ such that

$$L_0|\Delta\rangle = \Delta|\Delta\rangle \tag{6}$$

$$L_{n>0}|\Delta\rangle = 0 \tag{7}$$

Then the eigenvalues of L_0 in this representation R_Δ belong to $\Delta + \mathbb{N}$. The space of states, or spectrum, can be decomposed into such representations, and is of the form

$$S = \sum_\Delta n_\Delta R_\Delta \otimes \bar{R}_\Delta \tag{8}$$

The sum can actually be an integral, depending on the particular CFT. The possibly infinite integer $n_\Delta \in \mathbb{N} \cup \infty$ is the multiplicity of the representation. And the representation $\bar{R}_\Delta$ is a representation of the anti-holomorphic Virasoro algebra – the symmetry algebra must actually contain two copies of the Virasoro algebra, corresponding to holomorphic and anti-holomorphic transformations. For simplicity we have assumed that the spectrum is diagonal so we cannot have terms $R_\Delta \otimes \bar{R}_{\Delta'}$ with $\Delta \neq \Delta'$: this assumption is not necessary, although there are constraints on the violations of diagonality, such as $\Delta - \Delta' \in \frac{1}{2}\mathbb{Z}$.

As in any QFT we have a state-field correspondence: to a state in the spectrum we associate a field – technically a z-dependent operator on the spectrum. We use the notations

$$|\Delta\rangle \quad \mapsto \quad V_\Delta(z) \tag{9}$$

$$\mathcal{L}|\Delta\rangle \quad \mapsto \quad \mathcal{L}V_\Delta(z) \tag{10}$$

where $\mathcal{L} = \prod_i L_{-n_i}$ is a creation operator. The observables which we want to compute are correlation functions, such as the N-point function

$$\left\langle \prod_{i=1}^N \mathcal{L}_i V_{\Delta_i}(z_i) \right\rangle \tag{11}$$

This object is defined by the axioms it obeys. First come the symmetry axioms, called conformal Ward identities, which essentially state that the dependence on $\mathcal{L}_i$ is completely determined by conformal symmetry. More specifically, the action of $\mathcal{L}_i$ is given by a particular differential operator, for example

$$\langle \cdots L_{-1} V_{\Delta_i}(z_i) \cdots \rangle = \frac{\partial}{\partial z_i} \langle \cdots V_{\Delta_i}(z_i) \cdots \rangle \tag{12}$$

2 Conformal bootstrap and conformal blocks

Symmetry axioms are essential, but they cannot be enough for solving a theory, that is computing all the correlation functions. The equations on correlation functions which follow from symmetry are always linear equations, which have many solutions. The bootstrap method consists in systematically exploiting not only symmetry axioms, but also consistency axioms such as the existence and associativity of the Operator Product Expansion (OPE). First, the existence. The OPE of two primary fields can be written as

$$V_{\Delta_1}(z_1) V_{\Delta_2}(z_2) = \sum_{\Delta, \mathcal{L}} C^{\Delta, \mathcal{L}}_{\Delta_1, \Delta_2}(z_1, z_2) \mathcal{L} V_\Delta(z_2) \tag{13}$$

Here the OPE coefficient $C^{\Delta, \mathcal{L}}_{\Delta_1, \Delta_2}(z_1, z_2)$ is a number, not a field. Then conformal symmetry implies

$$C^{\Delta, \mathcal{L}}_{\Delta_1, \Delta_2}(z_1, z_2) = C_{\Delta_1, \Delta_2, \Delta} f^{\Delta, \mathcal{L}}_{\Delta_1, \Delta_2}(z_1, z_2) \tag{14}$$

where $C_{\Delta_1, \Delta_2, \Delta}$ is the permutation-symmetric three-point structure constant, and the structure function $f^{\Delta, \mathcal{L}}_{\Delta_1, \Delta_2}(z_1, z_2)$ is completely determined by conformal symmetry. Symmetry equations being linear, it is actually determined up to an overall normalization, which can be chosen such that

$$f^{\Delta, 1}_{\Delta_1, \Delta_2}(z_1, z_2) = (z_1 - z_2)^{\Delta - \Delta_1 - \Delta_2} \tag{15}$$

Inserting the OPE in a four-point function, we have the decomposition

$$\left\langle \prod_{i=1}^4 V_{\Delta_i}(z_i) \right\rangle = \sum_\Delta C_{\Delta_1, \Delta_2, \Delta} C_{\Delta, \Delta_3, \Delta_4} \left| \mathcal{G}^{(s)}_\Delta(\Delta_i | z_i) \right|^2 \tag{16}$$

where $\mathcal{G}^{(s)}_\Delta(\Delta_i | z_i)$ is an s-channel conformal block, which contains a sum over descendents $\mathcal{L}$. This holomorphic block comes with an antiholomorphic counterpart, which contains the sum over descendents $\bar{\mathcal{L}}$. The idea is now that the OPE is associative, and that we can obtain a t-channel decomposition of the same four-point function using the $V_{\Delta_2} V_{\Delta_3}$ OPE,

$$\left\langle \prod_{i=1}^4 V_{\Delta_i}(z_i) \right\rangle = \sum_\Delta C_{\Delta_2, \Delta_3, \Delta} C_{\Delta, \Delta_1, \Delta_4} \left| \mathcal{G}^{(t)}_\Delta(\Delta_i | z_i) \right|^2 \tag{17}$$

The conformal blocks are universal objects, completely determined by conformal symmetry – just like the characters of representations, which are however much simpler as they depend one only one representation, not five. So the equality between the two decompositions of the four-point function, called crossing symmetry, is a non-trivial quadratic equation for the real unknown here – the three-point structure constant C. Is this really the only unknown? Not quite: the spectrum i.e. the set of allowed values of Δ may not be known at this point, and is also constrained by crossing symmetry.

3 The energy-momentum tensor

So far I have introduced the main ideas of the conformal bootstrap, now I want to introduce useful technical constructions which are useful in particular for deriving the conformal Ward identities. A basic remark is that a 2d CFT really has only one holomorphic Virasoro algebra – the algebra of infinitesimal holomorphic transformations of the plane. But we could have the impression that we have a Virasoro algebra at each point z_0 – the algebra which acts on fields $V_\Delta(z_0)$, via the state-field correspondence. All these Virasoro algebra are of course not independent. This is clear at the level of differential operators, which can be written relative to any base point z_0:

$$L_n^{(z_0)} = -(z-z_0)^{n+1}\frac{\partial}{\partial z} \tag{18}$$

The z_0-dependence of $L_n^{(z_0)}$ is then

$$\frac{\partial}{\partial z_0}L_n^{(z_0)} = -(n+1)L_{n-1}^{(z_0)} \tag{19}$$

This is equivalent to $\frac{\partial}{\partial z_0}T^{(z_0)}(z) = 0$ where

$$T^{(z_0)}(z) = T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n^{(z_0)}}{(z-z_0)^{n+2}} \tag{20}$$

This object should be considered a field, and is called the energy-momentum tensor. It has the property of being locally holomorphic i.e. $\bar{z}$-independent. This is in contrast to the field $V_\Delta(z)$ which has an implicit dependence on $\bar{z}$.

The energy-momentum tensor encodes Virasoro generators. The commutation relations of these generators are equivalent to the OPE

$$T(y)T(z) = \frac{\frac{c}{2}}{(y-z)^4} + \frac{2T(z)}{(y-z)^2} + \frac{\partial T(z)}{y-z} + O(1) \tag{21}$$

Similarly, the definition of the primary field $V_\Delta(z_0)$ is equivalent to the OPE of $T(z)$ with this field,

$$T(z)V_\Delta(z_0) = \sum_{n\in\mathbb{Z}} \frac{L_n^{(z_0)}}{(z-z_0)^{n+2}}V_\Delta(z_0) = \left(\frac{\Delta}{(z-z_0)^2} + \frac{1}{z-z_0}\frac{\partial}{\partial z_0}\right)V_\Delta(z_0) + O(1) \tag{22}$$

All conformal Ward identities can be deduced from these OPEs, and from the fact that $T(z)$ is locally holomorphic – actually holomorphic everywhere except where fields such as $V_\Delta(z_0)$ are inserted, in which case we know its singular behaviour: namely, a second-order pole if the field is primary. For example, Ward identity for translation invariance is obtained by inserting $\oint_\infty dz\, T(z) = 0$ in a correlation function, and we obtain

$$\sum_{i=1}^{N} \frac{\partial}{\partial z_i} \left\langle \prod_{i=1}^{N} V_{\Delta_i}(z_i) \right\rangle = 0 \tag{23}$$

Finally, we can write the spin two field T in terms of a spin one field J such that

$$T = J^2 + Q\partial J \tag{24}$$

The advantage of the field J is its simple self-OPE

$$J(y)J(z) = \frac{\frac{1}{2}}{(y-z)^2} + O(1) \tag{25}$$

from which one can nevertheless recover the TT OPE, with

$$c = 1 + 6Q^2 \tag{26}$$

Equivalently, the modes of $J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}$ obey

$$[J_n, J_m] = \frac{1}{2} n \delta_{n+m,0} \tag{27}$$

The disadvantage of $J(z)$ is that contrary to $T(z)$ it can have branch cuts, not just poles. Assuming that $J(z)$ is single-valued would amount to assuming a $\hat{\mathfrak{u}}_1$ symmetry, so that we would have a free boson theory. (The free boson is then the field ϕ such that $J = \partial\phi$.)

4 Conclusion and further reading

Using the methods which we have sketched, it is possible to

1. compute conformal blocks,
2. in some cases, solve crossing symmetry and find the three-point structure constant,
3. fully solve certain 2d CFTs, that is compute all correlation functions.

For more details, see my recent review article `arXiv:1406.xxxx`.