Liouville Quantum Gravity on the Riemann sphere

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Abstract

In this paper, we rigorously construct $2d$ Liouville Quantum Field Theory on the Riemann sphere introduced in the 1981 seminal work by Polyakov **Quantum Geometry of bosonic strings**. We also establish some of its fundamental properties like conformal covariance under $\mathrm{PSL}_2(\mathbb{C})$-action, Seiberg bounds, KPZ scaling laws, KPZ formula and the Weyl anomaly (Polyakov-Ray-Singer) formula for Liouville Quantum Gravity.

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1 Introduction

1.1 Liouville quantum gravity

The two dimensional Liouville Quantum Field Theory (Liouville QFT for short, or also LQG [1] as a short-cut for Liouville Quantum Gravity as is now usual in the mathematics literature) was introduced by A. Polyakov in 1981 [47] as a model for quantizing the bosonic string in the conformal gauge and gravity in two space-time dimensions. Liouville QFT is one of the most important $2d$-Conformal Field Theories (CFT).

In its simplest formulation, the Liouville QFT is a quantum field theory of the conformal mode X of a two dimensional "Riemannian metric" g conformally equivalent (in a heuristic sense) to the flat metric dx^2, namely $g = e^{\gamma X} dx^2$, with the random field X governed by the action

$$S_L(X) = \frac{1}{4\pi} \int \left((\partial X)^2 + 4\pi\mu \, e^{\gamma X} \right) d^2x. \tag{1.1}$$

Heuristically, the field $X : \mathbb{R}^2 \to \mathbb{R}$ (called Liouville field) has to be understood as a random function with probability distribution of the form (up to a renormalizing constant)

$$e^{-S_L(X)} DX$$

where DX stands for a uniform measure on some space of functions X. The parameter μ is the analog of a"cosmological constant" and the measure $e^{\gamma X} d^2x$ is an interaction term. We stress that for $\mu > 0$ this field is non Gaussian whereas for $\mu = 0$ it is Gaussian, in which case it is known under the name Gaussian Free Field (GFF), free referring to the fact that the field is "free" of interactions (here the term $\mu \, e^{\gamma X} dx^2$).

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[1] not to be confused with Loop Quantum Gravity, another approach to quantize gravity in 3 and 4 dimensions...

The *classical Liouville theory* is the study of the extrema of the Liouville action: these are found by solving the classical *Liouville equation* (with unknown function u)

$$\triangle u = 2\pi\mu e^u.$$

This equation describes the conformal metrics $e^u dx^2$ with constant Ricci scalar curvature $R = -2\pi\mu$. The Liouville QFT can be seen as a randomization of the Liouville equation, hence its name.

In the quantum conformal theory, the structure of the action (1.1) entails that the field X must obey some reparametrization rules: the Liouville QFT can be formulated with respect to any background metric ((1.1) is just the expression in the flat background metric) and all these formulations must satisfy some consistency conditions. In particular, the random field X must transform under a conformal change of coordinates $z \to w = w(z)$ as (see section 3.2)

$$X(z) \to \widetilde{X}(w) = X(z(w)) + Q\log\left|\frac{\partial z}{\partial w}\right|. \tag{1.2}$$

In the jargon of CFT this is equivalent to stating that the analytic part of the energy-momentum tensor (or Schwarzian connection) is $T_{zz} = -(\partial_z X)^2 + Q\,\partial_z^2 X$. The interaction term $\mathrm{e}^{\gamma X} d^2x$ (or Liouville measure) must be invariant under this rule. The requirement of such an invariance fixes the value of the constant $Q = 2/\gamma + \gamma/2$. In the standard branches of LQG, the parameter γ belongs to $]0,2]$. Furthermore, the central charge of a CFT is a one-dimensional parameter that reflects the way the CFT reacts to changes of the background metric (see section 3.4). For the Liouville quantum theory, the central charge of LQG is $c_L = 1 + 6Q^2$: thus it can range continuously in the interval $[25, +\infty[$ and this is one of the interesting features of this theory.

Since its introduction, Liouville QFT has been and is still much studied in theoretical physics, in the context of integrable systems and conformal field theories, of string theories, of quantum gravity, for its relation with random matrix models and topological gravity (see [46] for a review), and more recently in the context of its relations with 4 dimensional supersymmetric gauge theories and the AGT conjecture [1].

Liouville theory has also raised recently much interest in mathematics and theoretical physics in the (slightly different) context of probability theory and random geometry where the conjectured link between large planar maps and LQG is intensively studied (see the next subsection for a discussion on this point). Up to now such studies have exclusively focused on the incarnation of the Liouville theory as a free field theory where the parameter μ is set to zero[2]: see for instance [23, 50] and the review [27]. Within this framework, the Liouville measure $\mathrm{e}^{\gamma X} d^2x$ is formally the exponential of the GFF and is mathematically defined via Kahane's theory of Gaussian multiplicative chaos [35] for $\gamma \in]0,2[$. It is then possible to study in depth the properties of the measure in relation with SLE curves or geometrical objects in the plane that can be constructed out of the GFF [4, 20, 59]. In particular, in this geometrical and probabilistic context, a precise mathematical formulation of the KPZ scaling relations can be given [5, 23, 50]. Let us also mention that defining a random metric is an important open problem in the field and steps towards this problem have been achieved in [12] and [45] in the special case $\gamma = \sqrt{8/3}$: recall that for this value of γ (and in the context $\mu = 0$) the work [45] constructs a random growth process which is conjectured to be the growth of balls of a metric space formally corresponding to the "tangent plane" of LQG. This metric space is supposed to correspond to the conformal embedding in the plane of the so-called Brownian plane, recently constructed in [13]. Also, as originally suggested in [15], one can define rigorously the associated diffusion process called Liouville Brownian motion [28] (see also [7] for a construction starting from one point). This has led to further understanding of the geometry of ($\mu = 0$) LQG via heat kernel techniques, see [3, 8, 29, 43, 51] for recent progresses.

Treating the Liouville theory as a GFF (thus setting $\mu = 0$) is justified in some cases. It provides an approximation sufficient to derive the conformal weights of the operators (see the seminal work [36] and also [14, 17] for the framework considered here) and is the basis of many calculations of the correlation functions of the Liouville theory, i.e. expectations of product of vertex operators of the form $V_\alpha(x) = \mathrm{e}^{\alpha X(x)}$

[2]In the mathematics literature, one speaks of critical LQG when $\mu = 0$ though the terminology is misleading because non critical LQG, which is the object of this work, is also a CFT.

[16, 32, 33]. Indeed for some specific choices of the parameters, one may reduce the calculation of Liouville theory correlations to the calculation of correlations with respect to the GFF and then compute the latter quantity using Coulomb gas and CFT techniques [19]. This leads in particular to the famous DOZZ formula for the 3-point correlation functions of Liouville theory on the sphere [18, 61] (see section 5 for further explanations). Thanks to such calculations, many checks have been done between the results of Liouville theory for the correlation functions and the corresponding calculation using random matrix models and integrable hierarchies [46].

Nevertheless for many questions the "interaction" exponential term has to be taken into account. This is for instance the case for the open string (Liouville theory in the disk) where the negative curvature metric and the boundary conditions play an essential role. The purpose of this paper is precisely to define the full Liouville theory for all $\mu > 0$ in the simple case of the theory defined on the Riemann sphere $\mathbb{S}_2$ (the theory in the disk can be defined along the same lines but details of the construction will appear elsewhere). More generally, the purpose of this paper is to rigorously define the general k-point correlation function of vertex operators on the sphere for $k \geq 3$. We will also study the conformal invariance properties of these correlation functions and study the associated Liouville measure. Our results should not appear as a surprise for theoretical physicists as we recover (in a rigorous setting) many known properties of LQG but they are the first rigorous probabilistic results about the full Liouville theory (on the sphere), as it was introduced by Polyakov in his 1981 seminal paper [47].

1.2 Relation with discretized $2d$ quantum gravity

In this Section we will present some precise conjectures on the connection of our results to the work on discrete models of 2d gravity, randoms surfaces and random maps.

The standard way to discretize 2d quantum gravity coupled to matter fields is to consider a statistical mechanics model (corresponding to a conformal field theory with central charge $c_{\rm m}$) defined on a random lattice (or random map), corresponding to the random metric, for instance a random triangulation of the sphere. We formulate below precise mathematical conjectures on the relationship of LQG to that setup..

Let $\mathcal{T}_N$ be the set of triangulations of $\mathbb{S}_2$ with N faces and $\mathcal{T}_{N,3}$ be the set of triangulations with N faces and 3 marked faces or points (called roots).

Next consider a model of statistical physics (matter field) that can be defined on every $T \in \mathcal{T}_N$. The list of such models contains pure gravity (no matter field), Ising model (a spin ± 1 on each triangle or vertex), the multicritical discrete spin models (which correspond to the discrete series of the minimal CFT with $1/2 \leq c_{\rm m} < 1$), the $\mathrm{O}(N)$ dilute and dense loop models with $0 \leq N < 2$, the $q = 3$ or $q = 4$ Potts models and discrete models associated to minimal or rational conformal field theories with central charge $c_{\rm m}$ such that $-2 < c_{\rm m} \leq 1$). We refer to [39] for a review and references.

For $T \in \mathcal{T}_N$, define the partition function of the matter field on T

$$Z_{\rm m}(T,\beta) = \sum_{C_T} W(C_T,\beta)$$

as a sum of configurations C_T (defined as ensemble of some local or geometric discrete degrees of freedom) over T with positive local Boltzmann weights $W(C_T,\beta)$. These Boltzmann weights depend on some parameters denoted β and these parameters are tuned to their critical point β_c such that the statistical model coupled to gravity is critical. At this point, the triangulation T has no marked points. Call Z_N the partition function at criticality for triangulations of size N

$$Z_N = \sum_{T \in \mathcal{T}_{N,3}} Z_{\rm m}(T,\beta_c), \tag{1.3}$$

where we extend straightforwardly the above definition of $Z_{\rm m}(T,\beta_c)$ to triangulations T with marked points (the marked points play no role in the definition of $Z_{\rm m}(T,\beta_c)$). It is known (see [2]) that Z_N, diverges as N goes to infinity as

$$Z_N \sim \; N^{3-(2-\gamma_s)-1} e^{\mu_c^{\rm m} N}(1+o(1)) \tag{1.4}$$

with μ_c^{m} some critical "cosmological constant" or "fugacity" that depends on the critical model considered, and the string exponent γ_s can be explicitly expressed in terms of the central charge c_{m} of the CFT for the matter field through the relations

$$2-\gamma_s=\frac{2Q}{\gamma}\qquad\text{for}\quad Q=2/\gamma+\gamma/2=\sqrt{(25-c_{\mathrm{m}})/6}. \tag{1.5}$$

Therefore, for $\bar{\mu}>\mu_c^{\mathrm{m}}$, the full partition of the system triangulations+matter field

$$Z_{\bar{\mu}}=\sum_N e^{-\bar{\mu}N}Z_N \tag{1.6}$$

converges and we can sample a random triangulation according to this partition function. We are interested in the regime where the system samples preferably the triangulations with a large number of faces. Notice that for $-2<c_{\mathrm{m}}\leqslant 1$, we have $\sqrt{2}<\gamma\leqslant 2$ and therefore $-1<\gamma_s\leqslant 0$. From (1.4), we see that the closer $\bar{\mu}$ is to μ_c^{m}, the larger the typical area of the random triangulation (with 3 marked points) is and for $\mu\sim\mu_c^{\mathrm{m}}$, the size of the typical area diverges. Therefore, we are interested in the limit $\bar{\mu}\to\mu_c^{\mathrm{m}}$ in the following regime: we assume that $\bar{\mu}$ depends on a parameter $a>0$ such that

$$\overline{\mu}=\mu_c^{\mathrm{m}}+\mu a^2 \tag{1.7}$$

where μ is a fixed positive constant.

Let us now explain how to embed a triangulation $T\in\mathcal{T}_{N,3}$ onto the sphere $\mathbb{S}_2$ and define a random measure on $\mathbb{S}_2$ out of it. Following [31] (see also [12, section 2.2]), we can equip such a triangulation with a conformal structure (where each face has the geometry of an equilateral triangle). The uniformization theorem tells us that we can then conformally map the triangulation onto the sphere $\mathbb{S}_2$ and the conformal map is unique if we pick three distinct points x_1,x_2,x_3 on the sphere $\mathbb{S}_2$ and demand the map to send the three marked points to x_1,x_2,x_3. We denote by $\nu_{T,a}$ the corresponding deterministic measure where each triangle of the sphere is given a volume a^2. Concretely, the uniformization provides for each face $t\in T$ a conformal map $\psi_t:\Delta\to\mathbb{S}_2$ where Δ is an equilateral triangle of volume 1. Then $\nu_{T,a}(dz)=a^2|(\psi_t^{-1})'|^2dz$ on the image triangle $\psi_t(\Delta)$. In particular, the volume of the total space $\mathbb{S}_2$ is Na^2. Now, we consider the random measure $\nu_{a,\overline{\mu}}$ defined by

$$\mathbb{E}^{a,\overline{\mu}}[F(\nu_{a,\overline{\mu}})]=\frac{1}{Z_a}\sum_N e^{-(\overline{\mu}-\mu_c^{\mathrm{m}})N}\sum_{T\in\mathcal{T}_{N,3}}F(\nu_{T,a}),$$

for positive bounded functions F where Z_a is a normalization constant. We denote by $\mathbb{P}^{a,\overline{\mu}}$ the probability law associated to $\mathbb{E}^{a,\overline{\mu}}$.

We can now state a precise mathematical conjecture:

Conjecture 1. *Under $\mathbb{P}^{a,\overline{\mu}}$ and under the relation (1.7), the family of random measures $(\nu_{a,\overline{\mu}})_{a>0}$ converges in law as $a\to 0$ in the space of Radon measures equipped with the topology of weak convergence towards the law of the Liouville measure of LQG with parameter γ given by (1.5), cosmological constant μ and vertex operators at the points x_1,x_2,x_3 with weights $\alpha_i=\gamma$ for all i.*

Note that $\nu_{a,\overline{\mu}}(\mathbb{S}_2)$ converges in law under $\mathbb{P}^{a,\overline{\mu}}$ as $a\to 0$ towards a $\Gamma(\frac{\sum_i\alpha_i-2Q}{\gamma},\mu)$ distribution with parameter γ, μ and $\alpha_i=\gamma$ for all i, which corresponds precisely to the law of the volume of the space for LQG with these parameters (see Subsection 3.3).

Exemple 1: Pure gravity $c_{\mathbf{m}}=0,\gamma=\sqrt{\frac{8}{3}}$

Pure gravity corresponds to the case when no matter field is put on the triangulation, in which case $Z_{\mathrm{m}}(T,\beta)=\sum_{C_T}W(C_T,\beta)=1$ for all T. Z_N thus stands for the cardinal of $\mathcal{T}_{N,3}$ and it is known (see [2]) that

$$Z_N\sim N^{3-\frac{5}{2}-1}e^{\mu_c^{\mathrm{m}}N}(1+o(1))$$

as N goes to infinity. Notice that $3 = \frac{\sum_{i=1}^{3}\alpha_i}{\gamma}$ where $\alpha_i = \gamma$ for all i and $\frac{5}{2} = \frac{2Q}{\gamma}$ for $\gamma = \sqrt{\frac{8}{3}}$.

One can check that $\nu_{a,\overline{\mu}}(\mathbb{S}_2)$ converges in law under $\mathbb{P}^{a,\overline{\mu}}$ as $a \to 0$ towards a $\Gamma(\frac{1}{2}, \mu)$ distribution with parameter $\gamma = \sqrt{\frac{8}{3}}$, μ and $\alpha_i = \gamma$ for all i.

Exemple 2: Ising model: $c_{\mathbf{m}} = \frac{1}{2}, \gamma = \sqrt{3}$

From [2], the partition function of the Ising model on triangulations at criticality Z_N^{Is} (corresponding to (1.3)) diverges as $N^{3-\frac{7}{3}-1} e^{\mu_c^{\mathrm{Is}} N}(1+o(1))$ as N goes to infinity (note that the critical temperature is different on the random lattice models from the regular lattice). Once again, notice that $3 = \frac{\sum_{i=1}^{3}\alpha_i}{\gamma}$ where $\alpha_i = \gamma$ for all i and $\frac{7}{3} = \frac{2Q}{\gamma}$ for $\gamma = \sqrt{3}$. Again, $\nu_{a,\overline{\mu}}(\mathbb{S}_2)$ converges in law under $\mathbb{P}^{a,\overline{\mu}}$ as $a \to 0$ towards a $\Gamma(\frac{2}{3}, \mu)$ distribution.

Conjecture with general vertex operators

Finally one may ask what is the relation between the general vertex operators $V_\alpha(x) = \exp(\alpha X(x))$ (with $\alpha < Q$) that we consider in this paper, the Liouville measure given by (3.33) with more than 3 points x_i and some $\alpha_i \neq \gamma$, and local observables in discrete 2 dimensional gravity. Since the 3 original $V_\gamma(x)$ correspond to fixing through conformal invariance the points on $\mathbb{S}_2$, hence to the local density of vertices of the triangulation T through the conformal mapping onto the sphere, it is natural to consider the local density moment defined as follows. In addition to the points x_1, x_2, x_3 (to which the centers of the marked faces of the triangulation T are sent), we consider additional fixed points x_i with $i > 3$ on the sphere, around which a small disc $\mathcal{D}_{x_i,\epsilon_i}$ centered at x_i with radius ϵ_i is drawn. Then we consider the number of vertices $N_{x_i,\epsilon_i}(T)$ of the triangulation T mapped inside the disk $\mathcal{D}_{x_i,\epsilon_i}$. We consider the random measure defined for all positive bounded functions F as

$$\mathbb{E}^{a,\overline{\mu},(\epsilon_i)_i}[F(\nu_{a,\overline{\mu},(\epsilon_i)_i})] = \frac{1}{Z_{a,(\epsilon_i)}} \sum_N e^{-(\overline{\mu}-\mu_c^m)N} \sum_{T \in \mathcal{T}_{N,3}} \prod_{i>3} \epsilon_i^{2\Delta_i} (a^2 N_{x_i,\epsilon_i}(T))^{\frac{\alpha_i}{\gamma}} F(\nu_{T,a}),$$

where $Z_{a,(\epsilon_i)_i}$ is a normalization constant, $\Delta_i = \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$ the conformal weight (see next sections). We denote by $\mathbb{P}^{a,\overline{\mu},(\epsilon_i)_i}$ the probability law associated to $\mathbb{E}^{a,\overline{\mu},(\epsilon_i)_i}$. Notice that we have included the renormalization terms a^2 and $\epsilon_i^{2\Delta_i}$ although they cancel with the same terms in $Z_{a,(\epsilon_i)_i}$. However, they are needed if one were consider the limit for the partition function $Z_{a,(\epsilon_i)_i}$. We can now state our conjecture:

Conjecture 2. *Under $\mathbb{P}^{a,\overline{\mu},(\epsilon_i)_i}$ and under the relation (1.7), the family of random measures $(\nu_{a,\overline{\mu},(\epsilon_i)_i})_{a>0}$ converges in law as $a \to 0$ and then as $\epsilon_i \to 0$ in the space of Radon measures equipped with the topology of weak convergence towards the law of the Liouville measure of LQG with parameter γ given by (1.5), cosmological constant μ and vertex operators at the points x_1, x_2, x_3 with weights $\alpha_i = \gamma$ for all $i \leqslant 3$ and vertex operators at the points x_i with weights α_i for $i > 3$.*

Relation with the Brownian map

It is natural to ask if, in conjecture 1, one can reinforce the convergence of measures to a convergence in the space of random metric spaces (equipped with a natural volume form). More precisely, in the case of pure gravity $c_{\mathrm{m}} = 0$, consider the Riemannian metric defined on each image triangle $\psi_t(\Delta) \subset \mathbb{S}_2$ of the uniformization by $a|(\psi_t^{-1})'|^2 dz^2$ (hence the lengths of the edges of the image triangles are $\sqrt{a}$). Let $d_{T,a}$ be the corresponding distance function on $\mathbb{S}_2$ and $d_{a,\bar{\mu}}$ the random metric on $\mathbb{S}_2$ defined analogously to the random measure $\nu_{a,\bar{\mu}}$. Then, it is widely believed that the metric space (equipped with a volume measure) $(\mathbb{S}_2, d_{a,\bar{\mu}}, \nu_{a,\bar{\mu}})$ converges in law as $a \to 0$ towards a metric space $(\mathbb{S}_2, d, \nu)$, where ν is the LQG measure of conjecture 1. If this is the case, then the space $(\mathbb{S}_2, d, \nu)$ should be related to the Brownian map equipped with its volume measure (see [41, 44]): more precisely, for all fixed $A > 0$, both metric spaces should be isometric (up to some global constant) once conditioned to have same volume A. The isometry should also send the Brownian map volume measure to the measure ν.

1.3 Summary of our results

The sphere can be seen as the compactified plane $\overline{\mathbb{R}^2}$ equipped with the spherical metric

$$\hat{g} = \frac{4}{(1+|x|^2)^2} dx^2.$$

Our main goal is here to explain how to give sense to the Liouville partition function on the sphere corresponding to the action (for $\gamma \in [0,2[$)

$$S_L(X,\hat{g}) := \frac{1}{4\pi} \int_{\mathbb{R}^2} \left(|\partial^{\hat{g}} X|^2 + Q R_{\hat{g}} X + 4\pi \mu e^{\gamma X} \right) \lambda_{\hat{g}}, \tag{1.8}$$

where $\partial^{\hat{g}}$, $R_{\hat{g}}$ and $\lambda_{\hat{g}}$ respectively stand for the gradient, Ricci scalar curvature and volume form in the metric $\hat{g}$, which is called background metric. Observe here that the curved background of the sphere imposes to consider an additional curvature term $QR_{\hat{g}}X$, which is not seemingly present in the flat metric action (1.1)[3].

Formally, considering an action like (1.8) means that we want to define a random function $X : \overline{\mathbb{R}^2} \to \mathbb{R}$ such that its probability distribution is given for all suitable functionals F by

$$\mathbb{E}[F(X)] = Z^{-1} \int F(X) e^{-S_L(X,\hat{g})} DX \tag{1.9}$$

where Z is a normalization constant and DX stands for some uniform measure on the Sobolev space $H^1(\mathbb{R}^2,\hat{g})$ (see subsection 2.1 for a precise definition). There are several reasons that make the construction of such an action not straightforward.

First recall that such a uniform measure does not exist. Actually, the theory of linear Gaussian spaces may give sense to the squared gradient in the action (1.8) but this gives rise to an infinite measure. This infinite measure can be "localized" by requiring, for instance, the mean value of the field X over the sphere to vanish. This gives rise to the notion of Gaussian Free Field $X_{\hat{g}}$ with vanishing mean in the metric of the sphere $\hat{g}$, which is a random distribution living almost surely in the dual Sobolev space $H^{-1}(\mathbb{R}^2,\hat{g})$ (see [20]). To recover the full meaning of the squared gradient in the action (1.8), one has to tensorize the law $\mathbb{P}$ of the GFF $X_{\hat{g}}$ with vanishing mean with the Lebesgue measure dc on $\mathbb{R}$ and consider the "law" of the field X as the image of the measure $dc \otimes \mathbb{P}$ under the mapping $(c, X_{\hat{g}}) \mapsto c + X_{\hat{g}}$. The "random variable" c stands for the mean value of the field X, i.e. $\int_{\mathbb{R}^2} X \, \lambda_{\hat{g}}$, and is known under the name "zero mode" in physics. Once the meaning of this squared gradient term is understood, the first mathematical issue is the non-continuity of the mapping $X \in H^{-1}(\mathbb{R}^2,\hat{g}) \mapsto \int e^{\gamma X} \, d\lambda_{\hat{g}}$, which can be handled via renormalization (here Gaussian multiplicative chaos [35]).

Second, it turns out (as is well known in physics) that the partition function (1.9) is diverging because of the instability of the Liouville potential on the sphere. More precisely, the divergence comes from the summation over the zero modes (recall that they are distributed as the Lebesgue measure). By using the Gauss-Bonnet theorem, it is plain to check that the contribution of the zero modes basically reduces to the following diverging integral (the divergence occurs at $c \to -\infty$) called **mini-superspace approximation** (see [34])

$$\int_{\mathbb{R}} e^{-2Qc - \mu e^{\gamma c}} \, dc.$$

This divergence has a geometric flavor: if one computes the saddle point of the Liouville action (1.8), we guess that the field X, if it exists, should concentrate on the solution of the classical Liouville equation on the sphere with negative curvature. Such an equation has no solution and it is well known that one must insert conical singularities in the shape of the sphere to make it support a metric with negative curvature (see [60]). From the probabilistic angle, conical singularities can be understood via Girsanov transforms as p-point correlation functions of the vertex operators: more precisely, one can choose p points $z_1, \dots, z_p$ on

[3] In the flat metric (1.1), this curvature term is hidden in the boundary condition on the field X at infinity, $X(x) \sim -Q \ln |x|$, hence the name of "background charge at infinity". See [34] for instance.

the sphere and $\alpha_1, \dots \alpha_p \in \mathbb{R}$ (with additional constraints that we will precise later) and make sense of the following partition function

$$\int e^{\sum_i \alpha_i X(z_i)} e^{-S_L(X,\hat{g}_\varphi)} DX. \tag{1.10}$$

When focusing on the zero mode contribution, we get the following mini-superspace approximation of (1.10)

$$\int_{\mathbb{R}} e^{(\sum_i \alpha_i - 2Q)c - \mu e^{\gamma c}} \, dc. \tag{1.11}$$

We will establish that the whole partition function makes sense if and only if the weights $(\alpha_i)_i$ satisfy the so-called **Seiberg bounds** [54]

$$\sum_i \alpha_i > 2Q, \quad \text{ and } \quad \alpha_i < Q \;\; \forall i. \tag{1.12}$$

One can notice that the first condition comes from the contribution of the zero modes and ensures that the integral (1.11) converges. The second part is related to Gaussian multiplicative theory and ensures that the Liouville measure (i.e. the term $\int e^{\gamma X} \, d\lambda_{\hat{g}}$ in (1.8)) does not blow up under the effect of the insertions. Furthermore, it is straightforward to see from (1.12) that we must at least consider three insertions in order to make sense of (1.10).

Once we have achieved the construction of the Liouville action, we establish the main properties of such a theory. In particular, we establish the well known **KPZ scaling laws** (see [36, 46, 34]) about the μ-scaling properties of the partition function (1.10), the **KPZ formula** which quantifies the way the partition function (1.10) changes under the action of Möbius transforms of the sphere [36, 14, 46, 34] and finally we determine the way the partition function of LQG behaves under conformal changes of metrics, known as the **Weyl anomaly** formula (see [9, 11, 10, 21, 47] for early references on the scale and Weyl anomalies in the physics literature and [48, 53] on related mathematical work) thereby recovering $c_L = 1 + 6Q^2$ as the central charge of the Liouville theory. Finally, we discuss possible approaches of the $\gamma \geqslant 2$ branches of LQG.

2 Background

Throughout the paper, given a metric tensor g on $\overline{\mathbb{R}^2}$, we will denote by ∂^g the gradient, $\triangle_g$ the Laplace-Beltrami operator, $R_g = -\triangle_g \ln g$ the Ricci scalar curvature and λ_g the volume form in the metric g. When no index is given, this means that the object has to be understood in terms of the usual Euclidean metric (i.e. ∂, $\triangle$, R and λ).

$C(\overline{\mathbb{R}^2})$ stands for the space of continuous functions on $\mathbb{R}^2$ admitting a finite limit at infinity. In the same way, $C^k(\overline{\mathbb{R}^2})$ for $k \geqslant 1$ stands for the space of k-times differentiable functions on $\mathbb{R}^2$ such that all the derivatives up to order k belong to $C(\overline{\mathbb{R}^2})$.

2.1 Metrics on the sphere $\mathbb{R}^2$

The Riemann sphere can be mapped onto the whole plane $\mathbb{R}^2$ via stereographic projection. The corresponding spherical metric on $\mathbb{R}^2$ then reads

$$\hat{g} = \frac{4}{(1+|x|^2)^2} dx^2.$$

Its Ricci scalar curvature is 2 (its Gaussian curvature is 1) and its volume 4π.

More generally, we say a metric $g = g(x)dx^2$ is conformally equivalent to $\hat{g}$ if $g(x) = e^{\varphi(x)}\hat{g}(x)$ with $\varphi \in C^2(\overline{\mathbb{R}^2})$ such that $\int_{\mathbb{R}^2} |\partial\varphi|^2 \, d\lambda < \infty$. Its curvature R_g can be obtained from the curvature relation

$$R_g = e^{-\varphi}\big(R_{\hat{g}} - \Delta_{\hat{g}}\varphi\big). \tag{2.1}$$

In what follows, we will denote by $m_g(h)$ the mean value of h in the metric g, that is

$$m_g(h) = \frac{1}{\lambda_g(\mathbb{R}^2)} \int_{\mathbb{R}^2} h \, d\lambda_g. \tag{2.2}$$

Given any metric g conformally equivalent to the spherical metric, one can consider the Sobolev space $H^1(\mathbb{R}^2, g)$, which is the closure of $C^\infty(\overline{\mathbb{R}^2})$ with respect to the Hilbert-norm

$$\int_{\mathbb{R}^2} h^2 \, d\lambda_g + \int_{\mathbb{R}^2} |\partial h|^2 \, d\lambda. \tag{2.3}$$

The topological dual of $H^1(\mathbb{R}^2, g)$ will be denoted by $H^{-1}(\mathbb{R}^2, g)$ and it does not depend on g.

2.2 Log-correlated field and Gaussian free fields

Here we introduce the various free fields that we will use throughout the paper. They are all based on the notion of log-correlated field (LGF) (see [25]) and related Gaussian Free Fields (GFF) (see [20, 30, 56]).

The purpose of this section is to give a precise meaning to the measure on the space of functions corresponding to the "probability density"

$$\exp\Big(- \frac{1}{4\pi} \int_{\mathbb{R}^2} |\partial^g X(x)|^2 \lambda_g(dx) \Big) DX \tag{2.4}$$

where g is any metric conformally equivalent to the spherical one and DX stands for the "uniform measure" on the space of functions $X : \mathbb{R}^2 \to \mathbb{R}$. Though we could give straight away the mathematical definition, we choose to explain first the motivations for the forthcoming definitions.

We stress that the conformal invariance of the Dirichlet energy entails that the action in (2.4) does not depend on the metric chosen among a fixed conformal class of metrics and that the corresponding random field X must be invariant under all the automorphisms of the sphere, i.e. the Möbius transforms. It is then easy to convince oneself that this action must correspond to the LGF, i.e. a centered Gaussian field with covariance structure

$$\mathbb{E}[X(x)X(y)] = \ln \frac{1}{|x-y|}. \tag{2.5}$$

The point is that this field is defined only up to a constant. One way to define this field is to consider its restriction to the space of test functions f with vanishing mean $\int_{\mathbb{R}^2} f d\lambda = 0$ (see [25]). This is not the approach that we will develop here. Given a metric g conformally equivalent to that of the sphere, we will rather consider this field conditioned on having vanishing mean in the metric g, call it X_g. Formally, X_g can be understood as

$$X_g = X - m_g(X). \tag{2.6}$$

The constant has thus been fixed by imposing the condition

$$\int X_g \, d\lambda_g = 0. \tag{2.7}$$

Though this description is not rigorous as the field X does not exist as a function, each field X_g is perfectly defined on the space of test functions and its covariance structure can be explicitly given

$$\begin{aligned} G_g(x,y) &:= \mathbb{E}[X_g(x)X_g(y)] \\ &= \ln \frac{1}{|x-y|} - m_g(\ln \frac{1}{|x-\cdot|}) - m_g(\ln \frac{1}{|y-\cdot|}) + \theta_g, \end{aligned} \tag{2.8}$$

with

$$\theta_g := \frac{1}{\lambda_g(\mathbb{R}^2)^2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \ln \frac{1}{|z-z'|} \lambda_g(dz) \lambda_g(dz'). \tag{2.9}$$

It is then plain to check that X_g is a Gaussian Free Field with vanishing λ_g-mean on the sphere, that is a Gaussian random distribution with covariance kernel given by the Green function G_g of the problem

$$\triangle_g u = -2\pi f \quad \text{on } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} u \, d\lambda_g = 0$$

i.e.

$$u = \int G_g(\cdot, z) f(z) \lambda_g(dz) := G_g f. \tag{2.10}$$

Furthermore, X_g lives almost surely in the dual space $H^{-1}(\mathbb{R}^2, g)$ of $H^1(\mathbb{R}^2, g)$, see [20, 56], and this space does not depend on the choice of the metric g in the conformal equivalence class of $\hat{g}$. We state the following classical result on the Green function G_g (see the appendix for a short proof)

Proposition 2.1. (Conformal covariance) *Let ψ be a Möbius transform of the sphere and consider the metric $g_\psi = |\psi'(x)|^2 g(\psi(x)) dx^2$ on the sphere. We have*

$$G_{g_\psi}(x, y) = G_g(\psi(x), \psi(y)).$$

Furthermore, a simple check of covariance structure with the help of (2.8) entails

Proposition 2.2. (Rule for changing metrics) *For every metrics g, g' conformally equivalent to the spherical metric, we have the following equality in law*

$$X_g - m_{g'}(X_g) \stackrel{law}{=} X_{g'}.$$

Specializing to the round metric, let us register the explicit formula

$$G_{\hat{g}}(z, z') = \ln \frac{1}{|z - z'|} - \frac{1}{4}(\ln \hat{g}(z) + \ln \hat{g}(z')) - \frac{1}{2} \tag{2.11}$$

and the transformation rule under Möbius maps

$$G_{\hat{g}}(\psi(z), \psi(z')) = G_{e^\phi \hat{g}}(z, z') = G_{\hat{g}}(z, z') - \frac{1}{4}(\phi(z) + \phi(z')) \tag{2.12}$$

where $e^\phi = \hat{g}_\psi / \hat{g}$ (see Appendix).

All these GFFs X_g (g conformally equivalent to $\hat{g}$) may be thought of as centerings in λ_g-mean of the same LGF. They all differ by a constant. To absorb the dependence on the constant, we tensorize the law $\mathbb{P}$ of the field X_g (whatever the metric g) with the Lebesgue measure dc on $\mathbb{R}$ and we consider the image of the measure $\mathbb{P} \otimes dc$ under the mapping $(X_g, c) \mapsto X_g + c$. This measure will be understood as the "law" (it is not finite) of the field X corresponding to the action (2.4). In particular, this measure will be invariant under the shifts $X \to X + a$ for any constant $a \in \mathbb{R}$. This also ensures that the choice of the metric g to fix the constant for the LGF, yielding the GFF X_g, is irrelevant as it will be absorbed by the shift invariance of the Lebesgue measure.

To sum up, in what follows, we will formally understand the measure (2.4) as the image of the product measure $\mathbb{P} \otimes dc$ on $H^{-1}(\mathbb{R}^2, \hat{g}) \times \mathbb{R}$ by the mapping $(X_g, c) \mapsto X_g + c$, where dc is the Lebesgue measure on $\mathbb{R}$ and X_g has the law of a GFF X_g with vanishing λ_g-mean, no matter the choice of the metric g.

2.3 Gaussian multiplicative chaos

In what follows, we need to introduce some cut-off approximation of the GFF X_g for any metric g conformally equivalent to the spherical metric. Natural cut-off approximations can be defined via convolution. We need that these cut-off approximations be defined with respect to a fixed background metric: we consider Euclidean circle averages of the field because they facilitate some computations (especially Proposition 2.4 below) but we could consider ball averages, convolutions with a smooth function or white noise decompositions of the GFF as well.

Definition 2.3. (Circle average regularizations of the free field) *We consider the field $X_{g,\epsilon}$*

$$X_{g,\epsilon}(x) = \frac{1}{2\pi} \int_0^{2\pi} X_g(x + \epsilon e^{i\theta}) \, d\theta.$$

Proposition 2.4. *We claim (recall (2.9))*

1. $\lim_{\epsilon\to 0}\mathbb{E}[X_{\hat{g},\epsilon}(x)^2]+\ln\epsilon+\frac{1}{2}\ln\hat{g}(x)=\theta_{\hat{g}}+\ln 2$ *uniformly on* $\mathbb{R}^2$.
2. *Let* ψ *be a Möbius transform of the sphere. Denote by* $(X_{\hat{g}}\circ\psi)_\epsilon$ *the* ϵ*-circle average of the field* $X_{\hat{g}}\circ\psi$*. Then*
$$\lim_{\epsilon\to 0}\mathbb{E}[(X_{\hat{g}}\circ\psi)_\epsilon(x)^2]+\frac{1}{2}\ln\hat{g}(\psi(x))+\ln|\psi'(x)|+\ln\epsilon=\theta_{\hat{g}}+\ln 2$$
uniformly on $\mathbb{R}^2$.

Proof. To prove the first statement results, apply the ϵ-circle average regularization to the Green function $G_{\hat{g}}$ in (2.8) and use
$$\int_0^{2\pi}\int_0^{2\pi}\ln\frac{1}{|e^{i\theta}-e^{i\theta'}|}\,d\theta d\theta'=0.$$
Defining $f(x):=2m_{\hat{g}}(\ln\frac{1}{|x-\cdot|})$ and letting f_ϵ be the circle average of f we then get that $\mathbb{E}[X_{\hat{g},\epsilon}(x)^2]+f_\epsilon(x)+\ln\epsilon$ converges uniformly to $\theta_{\hat{g}}$. Then use (A.2) i.e. $f(x)=\frac{1}{2}\ln\hat{g}(x)-\ln 2$ to get the claim.

Concerning the second statement, observe that $X_{\hat{g}}\circ\psi$ is a GFF with vanishing mean in the metric $g_\psi=|\psi'|^2\hat{g}\circ\psi$ (see Proposition 2.1). Therefore, the Green function of this GFF is given by (2.8) with $g=g_\psi$. The same argument as the first item shows that

$$\lim_{\epsilon\to 0}(\mathbb{E}[X_{\hat{g}}\circ\psi_\epsilon(x)^2]+f^\psi_\epsilon(x)+\ln\epsilon)=\theta_{g_\psi}$$

uniformly on $\mathbb{R}^2$ where f^ψ_ϵ is the circle average of $f^\psi(x)=2m_{g_\psi}(\ln\frac{1}{|x-\cdot|})$. By (A.3)

$$f^\psi(x)=\frac{1}{2}\ln\hat{g}(\psi(x))+\theta_{g_\psi}+\ln|\psi'(x)|-\theta_{\hat{g}}-\ln 2$$

which yields the claim. ◻

Define now the measure
$$M_{\gamma,\epsilon}:=\epsilon^{\frac{\gamma^2}{2}}e^{\gamma(X_{\hat{g},\epsilon}+Q/2\ln\hat{g})}\,d\lambda. \tag{2.13}$$

Proposition 2.5. *For* $\gamma\in[0,2[$*, the following limit exists in probability*

$$M_\gamma=\lim_{\epsilon\to 0}M_{\gamma,\epsilon}=e^{\frac{\gamma^2}{2}\theta_{\hat{g}}+\ln 2}\lim_{\epsilon\to 0}e^{\gamma X_{\hat{g},\epsilon}-\frac{\gamma^2}{2}\mathbb{E}[X^2_{\hat{g},\epsilon}]}\,d\lambda_{\hat{g}}$$

in the sense of weak convergence of measures. This limiting measure is non trivial and is a (up to a multiplicative constant) Gaussian multiplicative chaos of the field $X_{\hat{g}}$ *with respect to the measure* $\lambda_{\hat{g}}$.

Proof. This results from standard tools of the general theory of Gaussian multiplicative chaos (see [49] and references therein) and Proposition 2.4. We also stress that all these methods were recently unified in a powerful framework in [55]. ◻

The following Proposition summarizes the behavior of this measure under Möbius transformations:

Proposition 2.6. *Let* F *be a bounded continuous function on* $H^{-1}(\mathbb{R}^2,\hat{g})$*,* $f\in C(\overline{\mathbb{R}^2})$ *and* ψ *be a Möbius transformation of the sphere. Then*

$$(F(X_{\hat{g}}),\int_{\mathbb{R}^2}f\,dM_\gamma)\stackrel{law}{=}(F(X_{\hat{g}}\circ\psi^{-1}-m_{\hat{g}_\psi}(X_{\hat{g}})),e^{-\gamma m_{\hat{g}_\psi}(X_{\hat{g}})}\int_{\mathbb{R}^2}f\circ\psi e^{\gamma\frac{Q}{2}\phi}dM_\gamma)$$

where $\hat{g}_\psi=|\psi'|^2g\circ\psi$ *and* $e^\phi=\hat{g}_\psi/\hat{g}$.

Proof. We have

$$\int f\epsilon^{\frac{\gamma^2}{2}} e^{\gamma(X_{\hat g,\epsilon}+Q/2\ln \hat g)}d\lambda = \int f\circ\psi\ \epsilon^{\frac{\gamma^2}{2}} e^{\gamma(X_{\hat g,\epsilon}\circ\psi+Q/2\ln\hat g\circ\psi)}|\psi'|^2\,d\lambda$$
$$= \int f\circ\psi\ (\frac{\epsilon}{|\psi'|})^{\frac{\gamma^2}{2}} e^{\gamma(X_{\hat g,\epsilon}\circ\psi+Q/2\ln\hat g)} e^{\gamma\frac{Q}{2}\phi}\,d\lambda.$$

Let $\psi(z)=\frac{az+b}{cz+d}$ where $ad-bc=1$. Then $\psi'(z)=(cz+d)^{-2}$ and

$$\phi(z)=2(\ln(1+|z|^2)-\ln(|az+b|^2+|cz+d|^2)$$

is in $C(\overline{\mathbb{R}^2})$. Let $\eta>0$. Using Proposition 2.4 we get that on the set $A_\eta:=B(0,\frac{1}{\eta})\setminus B(-\frac{d}{c},\eta)$

$$\lim_{\epsilon\to0}\mathbb{E}[X_{\hat g,\epsilon}(\psi(z))^2]-\mathbb{E}[(X_{\hat g}\circ\psi)_{\frac{\epsilon}{|\psi'(z)|}}(z)^2]=0.$$

We may then use the results of [55] to conclude that the measures

$$(\frac{\epsilon}{|\psi'|})^{\frac{\gamma^2}{2}} e^{\gamma(X_{\hat g,\epsilon}\circ\psi+Q/2\ln\hat g)}\,d\lambda$$

and

$$\epsilon^{\frac{\gamma^2}{2}} e^{\gamma(X_{\hat g}\circ\psi)_\epsilon+Q/2\ln\hat g)}\,d\lambda$$

converge in probability to the same random measure on A_η. By Proposition 2.4

$$\mathbb{E}\int_{A_\eta^c}(\frac{\epsilon}{|\psi'|})^{\frac{\gamma^2}{2}} e^{\gamma(X_{\hat g,\epsilon}\circ\psi+Q/2\ln\hat g)}\lambda \leqslant C\int_{A_\eta^c}(\hat g/\hat g_\psi)^{\frac{\gamma^2}{4}}\lambda_{\hat g}=C\int_{A_\eta^c}e^{-\frac{\gamma^2}{4}\phi}\lambda_{\hat g}\to 0 \tag{2.14}$$

as $\eta\to0$. By Propositions 2.1 and 2.2, $X_{\hat g}\circ\psi$ is equal in law with $X_{\hat g}-m_{\hat g_\psi}(X_{\hat g})$ yielding the claim. □

3 Liouville Quantum Gravity on the sphere

We will now give the formal definition of the functional integral (1.9) in the presence of the vertex operators $e^{\alpha_i X(z_i)}$. Let $g=e^\varphi\hat g$ be a metric conformally equivalent to the spherical metric in the sense of Section 2.1 and let F be a continuous bounded functional on $H^{-1}(\mathbb{R}^2,\hat g)$. We define

$$\begin{aligned}&\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(g,F;\epsilon)\\ &:= e^{\frac{1}{96\pi}\int_{\mathbb{R}^2}|\partial^{\hat g}\varphi|^2+2R_{\hat g}\varphi\,d\lambda_{\hat g}}\int_{\mathbb{R}}\mathbb{E}\Big[F(c+X_g+Q/2\ln g)\prod_i\epsilon^{\frac{\alpha_i^2}{2}}e^{\alpha_i(c+X_{g,\epsilon}+Q/2\ln g)(z_i)}\\ &\exp\Big(-\frac{Q}{4\pi}\int_{\mathbb{R}^2}R_g(c+X_g)\,d\lambda_g-\mu\epsilon^{\frac{\gamma^2}{2}}\int_{\mathbb{R}^2}e^{\gamma(c+X_{g,\epsilon}+Q/2\ln g)}\,d\lambda\Big)\Big]\,dc.\end{aligned}\tag{3.1}$$

and want to inquire when the limit $\lim_{\epsilon\to0}\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(g,F;\epsilon)=:\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(g,F)$ exists.

Remark 3.1. *We include the additional factor $e^{\frac{1}{96\pi}\int_{\mathbb{R}^2}|\partial^{\hat g}\varphi|^2+2R_{\hat g}\varphi\,d\lambda_{\hat g}}$ to conform to the physics conventions. Indeed the formal expression (1.9) differs from (3.1) in that in the latter we use a normalized expectation for the Free Field. Thus to get (1.9) we would need to multiply by the Free Field partition function $z(g)$. The latter is not uniquely defined but its variation with metric is:*

$$z(e^\varphi\hat g)=e^{\frac{1}{96\pi}\int_{\mathbb{R}^2}|\partial^{\hat g}\varphi|^2+2R_{\hat g}\varphi\,d\lambda_{\hat g}}z(\hat g)$$

see [20, 30]. This additional factor makes the Weyl anomaly formula conform with the standard one in Conformal Field Theory. We note also that the translation by $Q/2\ln g$ in the argument of F is necessary for conformal invariance (Section 3.2).

We start by considering the round metric, $g = \hat{g}$. We first handle the curvature term. Since $R_{\hat{g}} = 2$ and $X_{\hat{g}}$ has vanishing $\lambda_{\hat{g}}$-mean we obtain

$$\begin{aligned}
&\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(\hat{g}, F; \epsilon) \\
&\quad = \int_{\mathbb{R}} e^{-2Qc}\,\mathbb{E}\Big[F(c + X_{\hat{g}} + Q/2\ln\hat{g}) \prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i(c+X_{\hat{g},\epsilon}+Q/2\ln\hat{g})(z_i)} \\
&\qquad \exp\Big(-\mu\epsilon^{\frac{\gamma^2}{2}} \int_{\mathbb{R}^2} e^{\gamma(c+X_{\hat{g},\epsilon}+Q/2\ln\hat{g})}\, d\lambda\Big)\Big]\, dc.
\end{aligned} \tag{3.2}$$

Now we handle the insertions operators $e^{\alpha_i X_{\hat{g},\epsilon}(z_i)}$. In view of Proposition 2.4, we can write (with the Landau notation)

$$\epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i X_{\hat{g},\epsilon}(z_i)} = e^{\frac{\alpha_i^2}{2}(\theta_{\hat{g}}+\ln 2)} \hat{g}(z_i)^{-\frac{\alpha_i^2}{4}} e^{\alpha_i X_{\hat{g},\epsilon}(z_i) - \frac{\alpha_i^2}{2}\mathbb{E}[X_{\hat{g},\epsilon}(z_i)^2]}(1 + o(1)). \tag{3.3}$$

Note that the $o(1)$ term is deterministic as it just comes from the normalization of variances. Then, by applying the Girsanov transform and setting

$$H_{\hat{g},\epsilon}(x) = \sum_i \alpha_i \int_0^{2\pi} G_{\hat{g}}(z_i + \epsilon e^{i\theta}, x)\frac{d\theta}{2\pi}, \tag{3.4}$$

we obtain

$$\begin{aligned}
\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(\hat{g}, F; \epsilon) &= e^{C_\epsilon(\mathbf{z})} \Big(\prod_i \hat{g}(z_i)^{-\frac{\alpha_i^2}{4}+\frac{Q}{2}\alpha_i}\Big) \\
&\int_{\mathbb{R}} e^{\left(\sum_i \alpha_i - 2Q\right)c}\,\mathbb{E}\Big[F(c + X_{\hat{g}} + H_{\hat{g},\epsilon} + Q/2\ln\hat{g})(1 + o(1)) \\
&\times \exp\Big(-\mu e^{\gamma c}\epsilon^{\frac{\gamma^2}{2}} \int_{\mathbb{R}^2} e^{\gamma(X_{\hat{g},\epsilon}+H_{\hat{g},\epsilon}+Q/2\ln\hat{g})}\, d\lambda\Big)\Big]\, dc,
\end{aligned} \tag{3.5}$$

with

$$\lim_{\epsilon\to 0} C_\epsilon(\mathbf{z}) = \frac{1}{2}\sum_{i\neq j}\alpha_i\alpha_j G_{\hat{g}}(z_i, z_j) + \frac{\theta_{\hat{g}} + \ln 2}{2}\sum_i \alpha_i^2 := C(\mathbf{z}). \tag{3.6}$$

In the next subsection we study under what conditions the limit in (3.5) exists.

3.1 Seiberg bounds and KPZ scaling laws

Since $H_{\hat{g},\epsilon}$ converges in $H^{-1}(\mathbb{R}^2, \hat{g})$ to

$$H_{\hat{g}}(x) = \sum_i \alpha_i G_{\hat{g}}(z_i, x) \tag{3.7}$$

it suffices to study the convergence of the partition function $\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(\hat{g}, 1; \epsilon)$. We show that a necessary and sufficient condition for the Liouville partition function to have a non trivial limit is the validity of the so-called **Seiberg bounds** (see [54, 46, 34])

$$\sum_i \alpha_i > 2Q \quad \text{and} \quad \forall i, \quad \alpha_i < Q. \tag{3.8}$$

The first inequality controls the $c \to -\infty$ divergence of the integral over the zero modes $c \in \mathbb{R}$ and is necessary even for the regularized theory to exist. Indeed, let

$$Z_\epsilon := \epsilon^{\frac{\gamma^2}{2}} \int_{\mathbb{R}^2} e^{\gamma(X_{\hat{g},\epsilon}+H_{\hat{g},\epsilon}+Q/2\ln\hat{g})}\, d\lambda. \tag{3.9}$$

Note that $|H_{\hat{g},\epsilon}(z)| \leqslant C_\epsilon$ since $G(z_i, z)$ tends to constant as $|z| \to \infty$. Hence from Proposition 2.5 we infer $\mathbb{E}[Z_\epsilon] < \infty$ and thus $Z_\epsilon < \infty$ $\mathbb{P}$-almost surely. Hence we can find $A > 0$ such that $\mathbb{P}(Z_\epsilon \leqslant A) > 0$ and then

$$\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(\hat{g}, 1, \epsilon) \geqslant \Big(\prod_i \hat{g}(z_i)^{-\frac{\alpha_i^2}{4}+\frac{Q}{2}\alpha_i}\Big) e^{C_\epsilon(\mathbf{z})} \int_{-\infty}^0 e^{\big(\sum_i \alpha_i - 2Q\big)c} e^{-\mu e^{\gamma c} A} \mathbb{P}(Z_\epsilon \leqslant A)\, dc = +\infty$$

if the first condition in (3.8) fails to hold. The condition $\alpha_i < Q$ is needed to ensure that the integral in (3.9) does not blow up in the neighborhood of the places of insertions $(z_i)_i$ as $\epsilon \to 0$.

Finally, we mention that the bounds (3.8) show that the number of insertions must be at least 3 in order to have well defined correlation functions of the Liouville theory on the sphere. This has a strong geometric flavor (see [60]): on the sphere one must at least insert three conical singularities in order to construct a metric with negative curvature (notice that the saddle points of the Liouville action are precisely these metrics). We claim

Theorem 3.2. (Convergence of the partition function) *Let $\sum_i \alpha_i > 2Q$. Then the limit*

$$\lim_{\epsilon \to 0} \Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(\hat{g}, 1; \epsilon) := \Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(\hat{g}, 1)$$

exists. The limit is nonzero if $\alpha_i < Q$ for all i whereas it vanishes identically if $\alpha_i \geqslant Q$ for some i.

Proof. Eq. (3.5) gives for $F = 1$

$$\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(\hat{g}, 1, \epsilon) = \prod_i \hat{g}(z_i)^{-\frac{\alpha_i^2}{4}+\frac{Q}{2}\alpha_i} e^{C(\mathbf{z})}(1 + o(1))\mathbb{E}\Big[\int_{\mathbb{R}} e^{c(\sum_i \alpha_i - 2Q)} \exp\big(-\mu e^{\gamma c} Z_\epsilon\big)\, dc\Big].$$

As remarked above, $Z_\epsilon > 0$ almost surely. By making the change of variables $u = \mu e^{\gamma c} Z_\epsilon$ in (3.5), we compute

$$\mathbb{E}\Big[\int_{\mathbb{R}} e^{c(\sum_i \alpha_i - 2Q)} \exp\big(-\mu e^{\gamma c} Z_\epsilon\big)\, dc\Big] = \frac{\mu^{\frac{\sum_i \alpha_i - 2Q}{\gamma}}}{\gamma} \Gamma\Big(\gamma^{-1}(\sum_i \alpha_i - 2Q)\Big)\mathbb{E}\Big[\frac{1}{Z_\epsilon^{\frac{\sum_i \alpha_i - 2Q}{\gamma}}}\Big] \tag{3.10}$$

where Γ is the standard Γ function. The claim follows from the following Lemma. ∎

Lemma 3.3. *Let $s < 0$. If $\alpha_i < Q$ for all i then*

$$\lim_{\epsilon \to 0} \mathbb{E}[Z_\epsilon^s] = \mathbb{E}[Z_0^s]$$

where

$$Z_0 = \int_{\mathbb{R}^2} e^{\gamma H_{\hat{g}}(x)} M_\gamma(dx) \tag{3.11}$$

and the limit is nontrivial: $0 < \mathbb{E} Z_0^s < \infty$.

If $\alpha_i \geqslant Q$ for some $i \in \{1, \dots, p\}$ then

$$\lim_{\epsilon \to 0} \mathbb{E} Z_\epsilon^s = 0.$$

As a corollary of the relation (3.10), we obtain a rigorous derivation of the KPZ scaling laws (see [36, 46, 34] for physics references)

Theorem 3.4. (KPZ scaling laws) *We have the following exact scaling relation for the Liouville partition function with insertions $(z_i, \alpha_i)_i$*

$$\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(\hat{g}, 1) = \mu^{\frac{2Q - \sum_i \alpha_i}{\gamma}} \Pi_{\gamma,1}^{(z_i\alpha_i)_i}(\hat{g}, 1)$$

where

$$\Pi_{\gamma,1}^{(z_i\alpha_i)_i}(\hat{g}, 1) = e^{C(\mathbf{z})}\Big(\prod_i \hat{g}(z_i)^{\Delta_{\alpha_i}}\Big)\gamma^{-1}\Gamma\Big(\gamma^{-1}(\sum_i \alpha_i - 2Q)\Big)\mathbb{E}\Big[\frac{1}{Z_0^{\frac{\sum_i \alpha_i - 2Q}{\gamma}}}\Big]$$

and we defined

$$\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2}) \tag{3.12}$$

and $C(\mathbf{z})$ is defined by (3.6). Moreover

$$\Pi_{\gamma,\mu}^{(z_i \alpha_i)_i}(\hat{g}, F) = e^{C(\mathbf{z})} \prod_i \hat{g}(z_i)^{\Delta_{\alpha_i}} \tag{3.13}$$

$$\int_{\mathbb{R}} e^{\left(\sum_i \alpha_i - 2Q\right)c} \, \mathbb{E}\Big[F(c + X_{\hat{g}} + H_{\hat{g}} + Q/2 \ln \hat{g}) \exp\Big(-\mu e^{\gamma c} Z_0\Big)\Big] \, dc.$$

Proof of Lemma 3.3. Note first that $\mathbb{E} Z_\epsilon^s < \infty$ for all $\epsilon \geqslant 0$. Indeed, recalling (2.13)

$$Z_\epsilon = \int_{\mathbb{R}^2} e^{\gamma H_{\hat{g},\epsilon}(z)} M_{\gamma,\epsilon}(dx).$$

Take any non empty ball B that contains no z_i. Then

$$\mathbb{E}[Z_\epsilon^s] \leqslant A^s \mathbb{E}[M_{\gamma,\epsilon}(B)^s]$$

where $A = C \min_{z \in B} \frac{4 e^{\gamma H_{\hat{g}}(z)}}{(1+|z|^2)^2}$. It is a standard fact in Gaussian multiplicative chaos theory (see [49, Th 2.12] again) that the random variable $M_{\gamma,\epsilon}(B)$ possesses negative moments of all orders for $\gamma \in [0, 2[$.

Let now $\alpha_i < Q$ for all i. Let us consider the set $A_r = \cup_i B(z_i, r)$ and write

$$Z_\epsilon = \int_{A_r} e^{\gamma H_{\hat{g},\epsilon}(z)} M_{\gamma,\epsilon}(dx) + \int_{A_r^c} e^{\gamma H_{\hat{g},\epsilon}(z)} M_{\gamma,\epsilon}(dx) := Z_{r,\epsilon} + Z_{r,\epsilon}^c.$$

Since $H_{\hat{g},\epsilon}$ converge uniformly on A_r^c to a continuous limit the limit

$$\lim_{\epsilon \to 0} Z_{r,\epsilon}^c = \int_{A_r^c} e^{\gamma H_{\hat{g},\epsilon}(z)} M_\gamma(dx) := Z_{r,0}^c \tag{3.14}$$

exists in probability by Proposition 2.5.

We study next the r-dependence of $Z_{r,\epsilon}$. Without loss of generality, we may take $\epsilon = 2^{-n}$ and $r = 2^{-m}$ with $n > m$ and $A_r = B(0, r)$. Then, dividing $B(0, r)$ to dyadic annuli $2^{-k-1} \leqslant |z| \leqslant 2^{-k}$ and noting that $e^{\gamma H_{\hat{g},\epsilon}(z)} \leqslant C 2^{\gamma \alpha k}$ on such annulus we get

$$Z_{r,\epsilon} = \int_{B(0,r)} e^{\gamma H_{\hat{g},\epsilon}(z)} M_{\gamma,\epsilon}(dx) \leqslant C \sum_{k=m}^{n} 2^{\gamma \alpha k} M_{\gamma,\epsilon}(B_k) \tag{3.15}$$

where $B_k = B(0, 2^{-k})$.

The distribution of $M_{\gamma,\epsilon}(B_k)$ is easiest to study using the white noise cutoff $(\widetilde{X}_\epsilon)_\epsilon$ of $X_{\hat{g}}$. More precisely, the family $(\widetilde{X}_\epsilon)_\epsilon$ is a family of Gaussian processes defined as follows. Consider the heat kernel $(p_t(\cdot,\cdot))_{t \geqslant 0}$ of the Laplacian $\triangle_{\hat{g}}$ on $\mathbb{R}^2$. Let W be a white noise distributed on $\mathbb{R}_+ \times \mathbb{R}^2$ with intensity $dt \otimes \lambda_{\hat{g}}(dy)$. Then

$$\widetilde{X}_\epsilon(x) = \frac{1}{\sqrt{2\pi}} \int_{\epsilon^2}^{\infty} \big(p_{t/2}(x,y) - \frac{1}{\lambda_{\hat{g}}(\mathbb{R}^2)}\big) W(dt, dy).$$

The correlation structure of the family $(\widetilde{X}_\epsilon)_{\epsilon > 0}$ is given by

$$\mathbb{E}[\widetilde{X}_\epsilon(x) \widetilde{X}_{\epsilon'}(x')] = \frac{1}{2\pi} \int_{(\epsilon \wedge \epsilon')^2}^{\infty} \big(p_t(x,x') - \frac{1}{\lambda_{\hat{g}}(\mathbb{R}^2)}\big) \, dt. \tag{3.16}$$

For $\epsilon > 0$, we define the random measure

$$\widetilde{M}_{\gamma,\epsilon} := e^{\gamma \widetilde{X}_\epsilon - \frac{\gamma^2}{2} \mathbb{E}[(\widetilde{X}_\epsilon(x))^2]} \, d\lambda_{\hat{g}}$$

and $\widetilde{M}_\gamma := \lim_{\epsilon\to 0} \widetilde{M}_{\gamma,\epsilon}$, which has the same law as M_γ (see [49, Thm 3.7]). The covariance of the field $X_{\hat{g},\epsilon}$ is comparable to the one of $\widetilde{X}_\epsilon$. Indeed, uniformly in ϵ,

$$\mathbb{E}[\widetilde{X}_\epsilon(x)\widetilde{X}_\epsilon(y)] \leqslant C + \mathbb{E}[X_\epsilon(x)X_\epsilon(y)]$$

and so by Kahane's convexity inequality (see [35]) we get, for $q \in (0,1)$

$$\mathbb{E}[M_{\gamma,\epsilon}(B_k)^q] \leqslant C\mathbb{E}[\widetilde{M}_{\gamma,\epsilon}(B_k)^q].$$

We have the relation

$$\sup_\epsilon \mathbb{E}[\widetilde{M}_{\gamma,\epsilon}(B_k)^q] \leqslant C_q 2^{-k\xi(q)} \tag{3.17}$$

for all $q < \frac{4}{\gamma^2}$ where $\xi(q) = (2+\frac{\gamma^2}{2})q - \frac{\gamma^2}{2}q^2$. Indeed, the family $(\widetilde{M}_{\gamma,\epsilon}(B_k))_\epsilon$ is a martingale so that, by Jensen, it suffices to prove that the limit $\widetilde{M}_\gamma$ satisfies such a bound. This latter fact is standard, see [49, Th 2.14] for instance.

Therefore by Tchebychev

$$\mathbb{P}(Z_{r,\epsilon} > R) \leqslant C_{q,\delta} R^{-q} \sum_{k=m}^{n} 2^{-k\xi(q)} 2^{(\gamma\alpha+\delta)qk} \leqslant C_{q,\delta} R^{-q} 2^{-m(\xi(q)-q(\gamma\alpha+\delta))}$$

provided $(\gamma\alpha+\delta)q < \xi(q)$. This holds for q and δ small enough since $\alpha < Q$ i.e. $\gamma\alpha < 2+\frac{\gamma^2}{2}$. Hence, for some $\alpha,\beta > 0$

$$\mathbb{P}(Z_{r,\epsilon} > r^\alpha) \leqslant C r^\beta \quad \forall \epsilon \geqslant 0$$

where we noted that the same argument covers also the $\epsilon = 0$ case.

Let $\chi_r = 1_{Z_{r,\epsilon} > r^\alpha}$. We get by Schwartz

$$|\mathbb{E}[((Z_{r,\epsilon} + Z^c_{r,\epsilon})^s - (Z^c_{r,\epsilon})^s)\chi_r]| \leqslant 2(\mathbb{E}\chi_r \mathbb{E}(Z^c_{r,\epsilon})^{2s})^{1/2} \leqslant C r^{\beta/2}(\mathbb{E}(Z^c_{r,\epsilon})^{2s})^{1/2}$$

and using $|(a+b)^s - b^s| \leqslant Cab^{s-1}$

$$|\mathbb{E}((Z_{r,\epsilon} + Z^c_{r,\epsilon})^s - (Z^c_{r,\epsilon})^s)(1-\chi_r)| \leqslant C r^\alpha \mathbb{E}(Z^c_{r,\epsilon})^{s-1}.$$

Since $\mathbb{E}(Z^c_{r,\epsilon})^s \leqslant \mathbb{E}(Z^c_{1,\epsilon})^s$ and the latter stays bounded as $\epsilon \to 0$ we conclude

$$|\mathbb{E}[(Z_\epsilon)^s - (Z^c_{r,\epsilon})^s)]| \leqslant C(r^\alpha + r^\beta)$$

for all $\epsilon \leqslant r$. In particular, for $\epsilon = 0$ this gives

$$\lim_{r\to 0} \mathbb{E}[(Z^c_{r,0})^s] = \mathbb{E}[Z_0^s]. \tag{3.18}$$

Since $\mathbb{E}[(Z^c_{r,\epsilon})^s] < \infty$ for all $\epsilon \geqslant 0$ and by (3.14) $Z^c_{r,\epsilon}$ converges in probability to $Z^c_{r,0}$ as $\epsilon \to 0$ we have $\lim_{\epsilon\to 0} \mathbb{E}[(Z^c_{r,\epsilon})^s] = \mathbb{E}[(Z^c_{r,0})^s]$. From (3.18) we then conclude our claim $\lim_{\epsilon\to 0} \mathbb{E}[(Z_\epsilon)^s] = \mathbb{E}[(Z_0)^s]$.

For later purpose let us remark that from (3.17) we get

$$M_\gamma(B_k) \leqslant C_\delta(\omega) 2^{-k(2+\frac{\gamma^2}{2}-\delta)}$$

where $C_\delta(\omega) < \infty$ almost surely. This easily leads to

$$\sup_{\epsilon>0} \int_{B_r} e^{\gamma H_{\hat{g},\epsilon}(z)} M_\gamma(dx) \to 0 \tag{3.19}$$

in probability as $r \to 0$.

Let us now prove the second part of the lemma. Without loss of generality, we may assume that $\alpha_1 \geqslant Q$ and $z_1 = 0$. It suffices to prove for the $Z_{1,\epsilon}$ defined in (3.15) that

$$\lim_{\epsilon\to 0} \mathbb{E}[Z_{1,\epsilon}^s] = 0 \tag{3.20}$$

By Kahane convexity [35] (or [49, Thm 2.1]) we get

$$\mathbb{E}[Z_{1,\epsilon}^s] \leqslant C\mathbb{E}[\widetilde{Z}_{1,\epsilon}^s].$$

Next, we bound

$$\widetilde{Z}_{1,\epsilon} \geqslant c\sum_{k=1}^{n} 2^{\alpha\gamma k}\widetilde{M}_{\gamma,\epsilon}(A_k) \geqslant c\max_{k\leqslant n} 2^{(2+\gamma^2/2)k}\widetilde{M}_{\gamma,\epsilon}(A_k) \tag{3.21}$$

where A_k is the annulus with radi 2^{-k} and 2^{-k+1} and we recall that $\epsilon = 2^{-n}$ and $\alpha\gamma \geqslant 2+\gamma^2/2$. We may then decompose, for $r = 2^{-k}$ (and $\epsilon < r$),

$$\widetilde{M}_{\gamma,\epsilon}(dz) = e^{\gamma \widetilde{X}_r(z) - \frac{\gamma^2}{2}\mathbb{E}[\widetilde{X}_r(z)^2]} r^2 \widehat{M}_{\gamma,\epsilon,r}(dz/r) \tag{3.22}$$

where the measure $\widehat{M}_{\gamma,\epsilon,r}$ is independent of the sigma-field $\{\widetilde{X}_u(x); u \geqslant r, x\in\mathbb{R}^2\}$ and has the law

$$\widehat{M}_{\gamma,\epsilon,r}(dz) = e^{\gamma(\widetilde{X}_\epsilon - \widetilde{X}_r)(rz) - \frac{\gamma^2}{2}\mathbb{E}[(\widetilde{X}_\epsilon - \widetilde{X}_r)(rz)^2]}\, dz.$$

We can rewrite (3.22) as

$$\widetilde{M}_{\gamma,\epsilon}(dz) = e^{\gamma\widetilde{X}_r(0) - \frac{\gamma^2}{2}\mathbb{E}[(\widetilde{X}_r(0))^2]} e^{\gamma(\widetilde{X}_r(z) - \widetilde{X}_r(0)) - \frac{\gamma^2}{2}(\mathbb{E}[(\widetilde{X}_r(z))^2] - \mathbb{E}[(\widetilde{X}_r(0))^2])} r^2 \widehat{M}_{\gamma,\epsilon,r}(dz/r) \tag{3.23}$$

to get

$$\widetilde{M}_{\gamma,\epsilon}(A_k) \geqslant r^2 e^{\gamma\widetilde{X}_r(0) - \frac{\gamma^2}{2}\mathbb{E}[(\widetilde{X}_r(0))^2]} e^{\min_{z\in B(0,1)} Y_r(z)} \widehat{M}_{\gamma,\epsilon,r}(A_1) \tag{3.24}$$

with $Y_r(z) = \gamma(\widetilde{X}_r(rz) - \widetilde{X}_r(0)) - \frac{\gamma^2}{2}(\mathbb{E}[(\widetilde{X}_r(rz))^2] - \mathbb{E}[(\widetilde{X}_r(0))^2])$. Now we want to determine the behavior of all the terms involved in the above right-hand side.

By using in turn Doob's inequality and then Kahane convexity [35] (or [49, Thm 2.1]), we get

$$\mathbb{E}[\sup_{\epsilon<r} \widehat{M}_{\gamma,\epsilon,r}(A_1)^{-q}] \leqslant c_q \mathbb{E}[\widehat{M}_{\gamma,0,r}(A_1)^{-q}] \leqslant \mathbb{E}[M_\gamma(A_1)^{-q}] \leqslant C_q. \tag{3.25}$$

uniformly in $r \leqslant 1$. Hence, for all $a > 0$

$$\mathbb{P}(\sup_{\epsilon<r} \widehat{M}_{\gamma,\epsilon,r}(A_1) \leqslant n^{-1}) \leqslant C_a n^{-a}. \tag{3.26}$$

Next, we estimate the min in (3.24). The key point is to observe that the Gaussian process Y_r does not fluctuate too much in such a way that its minimum possesses a Gaussian left tail distribution. To prove this, we write $Y_r(z) = \mathbb{E}[Y_r(z)] + Y_r'(z)$ and we note that using the covariance structure of $(\widetilde{X}_r)_r$ we get for all $z \in B(0,1)$

$$|\mathbb{E}Y_r(z)| = \frac{\gamma^2}{2}|\mathbb{E}[(\widetilde{X}_r(rz))^2] - \mathbb{E}[(\widetilde{X}_r(0))^2]| \leqslant C$$

and for all $z, z' \in B(0,1)$,

$$\mathbb{E}[(Y_r'(z) - Y_r'(z'))^2] \leqslant C|z-z'|,$$

uniformly in $r \leqslant 1$. Using for example [40, Thm. 7.1, Eq. (7.4)], one can then deduce

$$\forall x \geqslant 1, \quad \sup_r \mathbb{P}(\min_{z\in B(0,1)} \gamma Y_r(z) \leqslant -x) \leqslant Ce^{-cx^2}$$

for some constants $C, c > 0$. Hence, for all $a > 0$

$$\mathbb{P}(e^{\min_{z\in B(0,1)} Y_r(z)} \leqslant n^{-1}) \leqslant C_a n^{-a}. \tag{3.27}$$

Combining (3.24), (3.26) and (3.27) with (3.21) we conclude

$$\mathbb{P}(\widetilde{Z}_{1,\epsilon} < n) \leqslant \mathbb{P}(\max_{k\leqslant n} e^{\gamma X_{2^{-k}}(0)} \leqslant n^3) + Cn^{-a}.$$

Since the law of the path $t \mapsto \widetilde{X}_t(0)$ is that of Brownian motion at time $-\ln t$ the first term on the RHS tends to zero as $n \to \infty$ and (3.20) follows. $\square$

3.2 Conformal covariance, KPZ formula and Liouville field

In what follows, we assume that the bounds (3.8) hold and we will study how the n-point correlation functions $\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(\hat{g}, F)$ transform under conformal reparametrization of the sphere. The KPZ formula describes precisely the rule for these transformations. More precisely, let $\psi : \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}$ be a conformal automorphism of the whole sphere, i.e. a Möbius transform. We claim (recall (3.12))

Theorem 3.5. (Field theoretic KPZ formula) *Let ψ be a Möbius transform of the sphere. Then*

$$\Pi_{\gamma,\mu}^{(\psi(z_i),\alpha_i)_i}(\hat{g}, 1) = \prod_i |\psi'(z_i)|^{-2\Delta_{\alpha_i}} \Pi_{\gamma,\mu}^{(z_i,\alpha_i)_i}(\hat{g}, 1).$$

Let us now define the law of the Liouville field on the sphere.

Definition 3.6. (Liouville field) *We define a probability law $\mathbb{P}^{\gamma,\mu}_{(z_i,\alpha_i)_i,\hat{g}}$ on $H^{-1}(\mathbb{R}^2, \hat{g})$(with expectation $\mathbb{E}^{\gamma,\mu}_{(z_i,\alpha_i)_i,\hat{g}}$) by*

$$\mathbb{E}^{\gamma,\mu}_{(z_i,\alpha_i)_i,\hat{g}}[F(\phi)] = \frac{\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(\hat{g}, F)}{\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(\hat{g}, 1)},$$

for all bounded continuous functional on $H^{-1}(\mathbb{R}^2, \hat{g})$.

We have the following result about the behaviour of the Liouville field under the Möbius transforms of the sphere

Theorem 3.7. *Let ψ be a Möbius transform of the sphere. The law of the Liouville field ϕ under $\mathbb{P}^{\gamma,\mu}_{(z_i,\alpha_i)_i,\hat{g}}$ is the same as that of $\phi \circ \psi + Q \ln |\psi'|$ under $\mathbb{P}^{\gamma,\mu}_{(\psi(z_i),\alpha_i)_i,\hat{g}}$.*

Proof of Theorems 3.5 and 3.7. We start from the relation (3.13). Let

$$H_{\hat{g}}^{\psi}(z) = \sum_i \alpha_i G_{\hat{g}}(\psi(z_i), z).$$

We apply Proposition 2.6 to $f = e^{\gamma H^{\psi}_{\hat{g},\epsilon}}$. By (3.19) we can take the limit $\epsilon \to 0$ to get

$$\Pi_{\gamma,\mu}^{(\psi(z_i),\alpha_i)_i}(\hat{g}, F) = e^{C(\psi(\mathbf{z}))} \prod_i \hat{g}(\psi(z_i))^{\Delta_{\alpha_i}} \int_{\mathbb{R}} e^{sc} \mathbb{E}\Big[F(c + X_{\hat{g}} \circ \psi^{-1} - m_{\hat{g}_\psi}(X_{\hat{g}}) + H_{\hat{g}}^{\psi} + Q/2 \ln \hat{g})$$
$$\exp\big(-\mu e^{\gamma(c - m_{\hat{g}_\psi}(X_{\hat{g}}))} \int e^{\gamma(H_{\hat{g}}^{\psi} \circ \psi + \frac{Q}{2}\phi)} dM_\gamma\big)\Big]\, dc.$$

where we denoted $s = \sum_i \alpha_i - 2Q$. Next, use the shift invariance of the Lebesgue measure (we make the change of variables $c = c' + m_{\hat{g}_\psi}(X_{\hat{g}})$) to get

$$\Pi_{\gamma,\mu}^{(\psi(z_i)\alpha_i)_i}(\hat{g}, F) = e^{C(\psi(\mathbf{z}))} \prod_i \hat{g}(\psi(z_i))^{\Delta_{\alpha_i}} \int_{\mathbb{R}} e^{sc} \mathbb{E}\Big[e^{s m_{\hat{g}_\psi}(X_{\hat{g}})} F(c + X_{\hat{g}} \circ \psi^{-1} + H_{\hat{g},\psi} + Q/2 \ln \hat{g}) \tag{3.28}$$
$$\exp\big(-\mu e^{\gamma c} \int e^{\gamma(H_{\hat{g},\psi} \circ \psi + \frac{Q}{2}\phi)} dM_\gamma\big)\Big]\, dc.$$

Now we apply the Girsanov transform to the term $e^{sm_{\hat{g}_\psi}(X_{\hat{g}})}$ where $m_{\hat{g}_\psi}(X_{\hat{g}}) = \frac{1}{4\pi}\int X_{\hat{g}} e^\phi d\lambda_{\hat{g}}$ and $e^\phi = \frac{|\psi'|^2 \hat{g}\circ\psi}{\hat{g}}$. This has the effect of shifting the law of the field $X_{\hat{g}}$, which becomes

$$X_{\hat{g}} + \frac{s}{4\pi} G_{\hat{g}} e^\phi.$$

The variance of this Girsanov transform is $s^2 D_\psi$ where

$$D_\psi = \frac{1}{4\pi} m_{\hat{g}}(e^\phi G_{\hat{g}} e^\phi) = \frac{1}{(4\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_{\hat{g}}(z,z')\lambda_{g_\psi}(dz)\lambda_{g_\psi}(dz'), \tag{3.29}$$

i.e. the whole partition function will be multiplied by $e^{\frac{s^2}{2}D_\psi}$.

Plugging in the shifted field to (3.28) we need to compute $H_{\hat{g},\psi}\circ\psi + \frac{s}{4\pi}G_{\hat{g}}e^\phi$. First, using (2.12) for $(H^\psi_{\hat{g}}\circ\psi)(z) = \sum_i \alpha_i G_{\hat{g}}(\psi(z),\psi(z_i))$ we get

$$H^\psi_{\hat{g}}\circ\psi = H_{\hat{g}} - \frac{\sum\alpha_i}{4}\phi(z) - \frac{1}{4}\sum_i \alpha_i\phi(z_i).$$

Next, to compute $G_{\hat{g}}e^\phi$ note that both metrics $\hat{g}$ and $\hat{g}_\psi = e^\phi\hat{g}$ have Ricci curvature 2. Hence from (2.1) we infer $e^\phi = 1 - \frac{1}{2}\Delta_{\hat{g}}\phi$ and thus

$$\frac{1}{4\pi}G_{\hat{g}}e^\phi = \frac{1}{4}(\phi - m_{\hat{g}}(\phi)). \tag{3.30}$$

Combining we get

$$H_{\hat{g},\psi}\circ\psi + \frac{s}{4\pi}G_{\hat{g}}e^\phi = H_{\hat{g}} - \frac{Q}{2}\phi(z) - \frac{1}{4}\sum_i\alpha_i\phi(z_i) - \frac{s}{4}m_{\hat{g}}(\phi).$$

Thus (3.28) becomes

$$\Pi^{(\psi(z_i)\alpha_i)_i}_{\gamma,\mu}(\hat{g},F) = e^{C(\psi(\mathbf{z}))}\Big(\prod_i \hat{g}(\psi(z_i))^{\Delta_{\alpha_i}}\Big)\int_{\mathbb{R}} e^{sc}\mathbb{E}\Big[F\big(c' + (X_{\hat{g}} + H_{\hat{g}} + Q/2(\ln\hat{g} - \ln|\psi'|^2))\circ\psi^{-1}\big)$$
$$\exp\big(-\mu e^{\gamma c'}\int e^{\gamma H_{\hat{g}}}dM_\gamma\big)\Big]\, dc\; e^{\frac{s^2}{2}D_\psi}.$$

where

$$c' = c - \frac{s}{4}m_{\hat{g}}(\phi) - \frac{1}{4}\sum_i \alpha_i\phi(z_i).$$

By a shift in the c-integral we get

$$\Pi^{(\psi(z_i)\alpha_i)_i}_{\gamma,\mu}(\hat{g},F) = e^{C(\psi(\mathbf{z}))}\prod_i \hat{g}(\psi(z_i))^{\Delta_{\alpha_i}}\int_{\mathbb{R}} e^{sc}\mathbb{E}\Big[F\big(c + (X_{\hat{g}} + H_{\hat{g}} + Q/2(\ln\hat{g} - \ln|\psi'|^2))\circ\psi^{-1}$$
$$\exp\big(-\mu e^{\gamma c}\int e^{\gamma H_{\hat{g}}}dM_\gamma\big)\Big]\, dc\; e^{\frac{s}{4}\sum_i\alpha_i\phi(z_i)}e^{\frac{s^2}{2}(D_\psi + \frac{1}{2}m_{\hat{g}}(\phi))} \tag{3.31}$$

Combining (3.6) with (2.12) we have

$$C(\psi(\mathbf{z})) = C(\mathbf{z}) - \frac{1}{8}\sum_{i\neq j}\alpha_i\alpha_j(\phi(z_i)+\phi(z_j)) = C(\mathbf{z}) - \frac{\sum_i\alpha_i}{4}\sum_j\alpha_j\phi(z_j) + \frac{1}{4}\sum_i\alpha_i^2\phi(z_i).$$

Since $|\psi'(z_i)|^2\hat{g}(\psi(z_i)) = e^{\phi(z_i)}\hat{g}(z_i)$ and $\Delta_{\alpha_i} = -\frac{1}{4}\alpha_i\alpha_i + \frac{Q}{2}\alpha_i$ we conclude

$$e^{C(\psi(\mathbf{z}))}\prod_i\hat{g}(\psi(z_i))^{\Delta_{\alpha_i}}e^{\frac{s}{4}\sum_i\alpha_i\phi(z_i)} = e^{C(\mathbf{z})}\prod_i(|\psi'(z_i)|^{-2}\hat{g}(z_i))^{\Delta_{\alpha_i}}.$$

The proof is completed by the identity

$$D_\psi = -\frac{1}{2}m_{\hat{g}}(\phi) \tag{3.32}$$

proven in the appendix. □

3.3 The Liouville measure

Here, we study the Liouville measure $Z(\cdot)$, the law of which is defined for all Borel sets $A_1, \cdots, A_k \subset \mathbb{R}^2$ by

$$\begin{aligned}&\mathbb{E}^{\gamma,\mu}_{(z_i,\alpha_i)_i,\hat{g}}[F(Z(A_1),\cdots,Z(A_k))]\\ =&(\Pi_{\gamma,\mu}(z_i,\alpha_i)_i(\hat{g},1))^{-1}\lim_{\epsilon\to 0}\int_{\mathbb{R}}\mathbb{E}\Big[F((e^{\gamma c}\epsilon^{\frac{\gamma^2}{2}}\int_{A_j}e^{\gamma(X_{\hat{g},\epsilon}+Q/2\ln\hat{g})}\,d\lambda)_j)\prod_i\epsilon^{\frac{\alpha_i^2}{2}}e^{\alpha_i(c+X_{\hat{g},\epsilon}+Q/2\ln\hat{g})(z_i)}\\ &\exp\Big(-\frac{Q}{4\pi}\int_{\mathbb{R}^2}R_{\hat{g}}(c+X_{\hat{g}})\,d\lambda_{\hat{g}}-\mu e^{\gamma c}\epsilon^{\frac{\gamma^2}{2}}\int_{\mathbb{R}^2}e^{\gamma(X_{\hat{g},\epsilon}+Q/2\ln\hat{g})}\,d\lambda\Big)\Big]\,dc.\end{aligned}$$

In what follows, we call $Z_0(\cdot)$ the measure defined under $\mathbb{P}$ by

$$Z_0(A):=\int_A e^{\gamma H_{\hat{g}}}\,dM_\gamma$$

so that Z_0 in (3.11) is $Z_0(\mathbb{R}^2)$. We have:

Proposition 3.8. *Under $\mathbb{P}^{\gamma,\mu}_{(z_i,\alpha_i)_i,\hat{g}}$, the Liouville measure is given for all $A_1,\cdots,A_k$ by*

$$\mathbb{E}^{\gamma,\mu}_{(z_i,\alpha_i)_i,\hat{g}}[F(Z(A_1),\cdots,Z(A_k))]=\frac{\int_0^\infty\mathbb{E}\Big[F\big(y\frac{Z_0(A_1)}{Z_0(\mathbb{R}^2)},\cdots,y\frac{Z_0(A_k)}{Z_0(\mathbb{R}^2)}\big)Z_0(\mathbb{R}^2)^{-\frac{\sum_i\alpha_i-2Q}{\gamma}}\Big]e^{-\mu y}y^{\frac{\sum_i\alpha_i-2Q}{\gamma}-1}dy}{\mu^{\frac{2Q-\sum_i\alpha_i}{\gamma}}\Gamma(\frac{\sum_i\alpha_i-2Q}{\gamma})\mathbb{E}\Big[Z_0(\mathbb{R}^2)^{-\frac{\sum_i\alpha_i-2Q}{\gamma}}\Big]}. \tag{3.33}$$

In particular,
1) the volume of the space $Z(\mathbb{R}^2)$ follows the Gamma distribution $\Gamma\Big(\frac{\sum_i\alpha_i-2Q}{\gamma},\mu\Big)$, meaning

$$\forall F\in C_b(\mathbb{R}_+),\quad \mathbb{E}^{\gamma,\mu}_{(z_i,\alpha_i)_i,\hat{g}}[F(Z(\mathbb{R}^2))]=\frac{\mu^{\frac{\sum_i\alpha_i-2Q}{\gamma}}}{\Gamma\Big(\frac{\sum_i\alpha_i-2Q}{\gamma}\Big)}\int_0^\infty F(y)y^{\frac{\sum_i\alpha_i-2Q}{\gamma}-1}e^{-\mu y}\,dy.$$

2) the law of the random measure $Z(\cdot)$ conditionally on $Z(\mathbb{R}^2)=A$ is given by

$$\mathbb{E}^{\gamma,\mu}_{(z_i,\alpha_i)_i,\hat{g}}[F(Z(\cdot))|Z(\mathbb{R}^2)=A)]=\frac{\mathbb{E}\Big[F\big(A\frac{Z_0(\cdot)}{Z_0(\mathbb{R}^2)}\big)Z_0(\mathbb{R}^2)^{-\frac{\sum_i\alpha_i-2Q}{\gamma}}\Big]}{\mathbb{E}\Big[Z_0(\mathbb{R}^2)^{-\frac{\sum_i\alpha_i-2Q}{\gamma}}\Big]}$$

for any continuous bounded functional F on the space of finite measures equipped with the topology of weak convergence.
3) Under $\mathbb{P}^{\gamma,\mu}_{(z_i,\alpha_i)_i,\hat{g}}$, the law of the random measure $Z(\cdot)/A$ conditioned on $Z(\mathbb{R}^2)=A$ does not depend on A and is explicitly given by

$$\mathbb{E}^{\gamma,\mu}_{(z_i,\alpha_i)_i,\hat{g}}[F(Z(\cdot)/A)|Z(\mathbb{R}^2)=A)]=\frac{\mathbb{E}\Big[F\big(\frac{Z_0(\cdot)}{Z_0(\mathbb{R}^2)}\big)Z_0(\mathbb{R}^2)^{-\frac{\sum_i\alpha_i-2Q}{\gamma}}\Big]}{\mathbb{E}\Big[Z_0(\mathbb{R}^2)^{-\frac{\sum_i\alpha_i-2Q}{\gamma}}\Big]}.$$

Proof. Taking the limit $\epsilon\to 0$ in the relation (3.5) gives

$$\begin{aligned}&\mathbb{E}^{\gamma,\mu}_{(z_i,\alpha_i)_i,\hat{g}}[F(Z(A_1),\cdots,Z(A_k))]\\ &=(\Pi^{\gamma,\mu}_{(z_i,\alpha_i)_i}(\hat{g},1))^{-1}\Big(\prod_i\hat{g}(z_i)^{-\frac{\alpha_i^2}{4}+\frac{Q}{2}\alpha_i}\Big)e^{C(\hat{g})}\\ &\int_{\mathbb{R}}e^{\big(\sum_i\alpha_i-2Q\big)c}\,\mathbb{E}\Big[F(e^{\gamma c}Z_0(A_1),\cdots,e^{\gamma c}Z_0(A_k))\exp\big(-\mu e^{\gamma c}Z_0(\mathbb{R}^2)\big)\Big]\,dc.\end{aligned}$$

Finally, let us make the change of variables $e^{\gamma c}Z_0(\mathbb{R}^2)=y$ to complete the proof. ∎

3.4 Changes of conformal metrics, Weyl anomaly and central charge

In this section, we want to study how the Liouville partition function (3.1) depends on the background metric g conformally equivalent to the spherical metric in the sense of Section 2.1, say $g = e^{\varphi}\hat{g}$.

By making the change of variables $y \to y - m_{\hat{g}}(X_g)$ in (3.1) and using Proposition 2.2, we can and will replace X_g by $X_{\hat{g}}$ in the expression (3.1).

Now we apply the Girsanov transform to the curvature term $e^{-\frac{Q}{4\pi}\int_{\mathbb{R}^2} R_g X_{\hat{g}}\, d\lambda_g}$. Since by (2.1) $R_g\lambda_g = (R_{\hat{g}} - \Delta_{\hat{g}}\varphi)\lambda_{\hat{g}}$ this has the effect of shifting the field $X_{\hat{g}}$ by

$$-\frac{Q}{4\pi}G_{\hat{g}}(R_{\hat{g}} - \Delta_{\hat{g}}\varphi) = -\frac{Q}{2}(\varphi - m_{\hat{g}}(\varphi))$$

where we used $G_{\hat{g}}R_{\hat{g}} = 0$ (since $R_{\hat{g}}$ is constant).

This Girsanov transform has also the effect of multiplying the whole partition function by the exponential of

$$\begin{aligned}
&\frac{Q^2}{32\pi^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} R_g(z)G_{\hat{g}}(z,z')R_g(z')\,\lambda_g(dz)\lambda_g(dz')\\
&=\frac{Q^2}{16\pi}\int_{\mathbb{R}^2} R_g(\varphi - m_{\hat{g}}(\varphi))\,d\lambda_g\\
&=\frac{Q^2}{16\pi}\int_{\mathbb{R}^2} (R_{\hat{g}} - \triangle_{\hat{g}}\varphi)(\varphi - m_{\hat{g}}(\varphi))\,d\lambda_{\hat{g}} \qquad \text{(use (2.1))}\\
&=\frac{Q^2}{16\pi}\int_{\mathbb{R}^2} |\partial^{\hat{g}}\varphi|^2\,d\lambda_{\hat{g}}.
\end{aligned}$$

Therefore, by making the change of variables $c \to c + Q/2m_{\hat{g}}(\varphi)$ to get rid of the constant $m_{\hat{g}}(\varphi)$ in the expectation, we get

$$\begin{aligned}
\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(g,F) =& e^{\frac{1}{96\pi}\int_{\mathbb{R}^2}|\partial^{\hat{g}}\varphi|^2 + 2R_{\hat{g}}\varphi\,d\lambda_{\hat{g}} + \frac{Q^2}{16\pi}\int_{\mathbb{R}^2}|\partial^{\hat{g}}\varphi|^2\,d\lambda_{\hat{g}} + Q^2 m_{\hat{g}}(\varphi)} \\
&\lim_{\epsilon\to 0}\int_{\mathbb{R}} \mathbb{E}\Big[F(X_{\hat{g}} + c + Q/2\ln\hat{g})\prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i(c + X_{\hat{g},\epsilon} + Q/2\ln\hat{g})(z_i)} \\
&\exp\Big(-\frac{Q}{4\pi}\int_{\mathbb{R}^2} R_g c\,d\lambda_g - \mu e^{\gamma v}\epsilon^{\frac{\gamma^2}{2}}\int_{\mathbb{R}^2} e^{\gamma X_{\hat{g},\epsilon} + Q/2\ln\hat{g}}\,d\lambda\Big)\Big]\,dc.
\end{aligned} \tag{3.34}$$

Now we observe that the Gauss-Bonnet theorem entails

$$\int_{\mathbb{R}^2} R_g c\,d\lambda_g = \int_{\mathbb{R}^2} R_{\hat{g}} c\,d\lambda_{\hat{g}}$$

because c is a constant. Therefore, using $Q^2 m_{\hat{g}}(\varphi) = \frac{6Q^2}{96\pi}\int_{\mathbb{R}^2} 2R_{\hat{g}}\varphi\,d\lambda_{\hat{g}}$,

$$\begin{aligned}
&\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(g,F) \qquad\qquad (3.35)\\
&= e^{\frac{1+6Q^2}{96\pi}\int_{\mathbb{R}^2}|\partial^{\hat{g}}\varphi|^2 + 2R_{\hat{g}}\varphi\,d\lambda_{\hat{g}}}\lim_{\epsilon\to 0}\int_{\mathbb{R}} \mathbb{E}\Big[F(X_{\hat{g}} + c + Q/2\ln\hat{g})\prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i(c + X_{\hat{g},\epsilon} + Q/2\ln\hat{g})(z_i)} \\
&\quad\exp\Big(-\frac{Q}{4\pi}\int_{\mathbb{R}^2} R_{\hat{g}}(c + X_{\hat{g}})\,d\lambda_{\hat{g}} - \mu e^{\gamma c}\epsilon^{\frac{\gamma^2}{2}}\int_{\mathbb{R}^2} e^{\gamma X_{\hat{g},\epsilon} + Q/2\ln\hat{g}}\,d\lambda\Big)\Big]\,dc \\
&= e^{\frac{1+6Q^2}{96\pi}\int_{\mathbb{R}^2}|\partial^{\hat{g}}\varphi|^2 + 2R_{\hat{g}}\varphi\,d\lambda_{\hat{g}}}\,\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(\hat{g},F). \qquad\qquad (3.36)
\end{aligned}$$

We can rewrite the above relation in a more classical physics language

Theorem 3.9. (Weyl anomaly and central charge)

1. We have the so-called **Weyl anomaly**

$$\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(e^{\varphi}\hat{g},F)=\exp\Big(\frac{c_L}{96\pi}\Big(\int_{\mathbb{R}^2}|\partial\varphi|^2\,d\lambda+\int_{\mathbb{R}^2}2R_{\hat{g}}\varphi\,d\lambda_{\hat{g}}\Big)\Big)\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(\hat{g},F)$$

where

$$c_L=1+6Q^2$$

is the **central charge** *of the Liouville theory.*

2. The law of the Liouville field ϕ under $\mathbb{P}^{\gamma,\mu}_{(z_i,\alpha_i)_i,g}$ is independent of the metric g in the conformal equivalence class of $\hat{g}$.

Notice that the above theorem can be reformulated as a **Polyakov-Ray-Singer formula** for LQG, see [48] and [47, 53] for more on this topic.

4 About the $\gamma\geqslant 2$ branches of Liouville Quantum Gravity

Here we discuss various situations that may arise in the study of the case $\gamma\geqslant 2$. We want this discussion to be very concise, so we just give the results as well as references in order to find the tools required to carry out the computations in full details. Yet, we stress that the computations consist in following verbatim the strategy of this paper. In what follows, we will only give the partition function in the round metric as the Weyl anomaly then gives straightforwardly the partition function for any metric conformally equivalent to the spherical metric.

4.1 The case $\gamma=2$ or string theory

The case $\gamma=2$ corresponds to $Q=2$ and is very important in string theory, see the excellent review [37] as well as the original paper [47]. The partition function of LQG is then the limit

$$\begin{aligned}&\Pi_{2,\mu}^{(z_i\alpha_i)_i}(\hat{g},F) &(4.1)\\&\quad=\lim_{\epsilon\to 0}\int_{\mathbb{R}}\mathbb{E}\Big[F(X_{\hat{g}}+c+\ln\hat{g})\prod_i\epsilon^{\frac{\alpha_i^2}{2}}e^{\alpha_i(c+X_{\hat{g},\epsilon}+\ln\hat{g})(z_i)}\\&\qquad\exp\Big(-\frac{1}{2\pi}\int_{\mathbb{R}^2}R_{\hat{g}}(c+X_{\hat{g},\epsilon})\,d\lambda_{\hat{g}}-\mu\sqrt{2/\pi}e^{2c}(-\ln\epsilon)^{1/2}\epsilon^2\int_{\mathbb{R}^2}e^{2X_{\hat{g},\epsilon}+2\ln\hat{g}}\,d\lambda\Big)\Big]\,dc.\end{aligned}$$

Notice the additional square root $(-\ln\epsilon)^{1/2}$ in order to get a non trivial renormalized interaction term[4]. After carrying the same computations than in (3.5) and taking the limit $\epsilon\to 0$, we get

$$\begin{aligned}\Pi_{2,\mu}^{(z_i\alpha_i)_i}(\hat{g},F)=&\Big(\prod_i\hat{g}(z_i)^{-\frac{\alpha_i^2}{4}+\alpha_i}\Big)e^{C(\mathbf{z})}\int_{\mathbb{R}}e^{\left(\sum_i\alpha_i-4\right)c}\,\mathbb{E}\Big[F(c-\theta_{\hat{g}}+X_{\hat{g}}+H_{\hat{g}}+\ln\hat{g}) &(4.2)\\&\times\exp\Big(-\mu e^{2c}\int_{\mathbb{R}^2}e^{2H_{\hat{g}}(x)}\hat{g}(x)M'(dx)\Big)\Big]\,dc,\end{aligned}$$

where the measure $M'(dx)$ is defined by

$$M'(dx)=(2\mathbb{E}[X_{\hat{g}}^2]-X_{\hat{g}})e^{\gamma X_{\hat{g}}-\frac{\gamma^2}{2}\mathbb{E}[X_{\hat{g}}^2]}\,\lambda_{\hat{g}}(dx)$$

and $C(\mathbf{z})$ defined as in (3.6). One can check as in subsection 3.5 that this partition function is conformally invariant. The convergence of probability of the renormalized measure $(-\ln\epsilon)^{1/2}\epsilon^2\int_{\mathbb{R}^2}e^{2X_{\hat{g},\epsilon}+2\ln\hat{g}}\,d\lambda$ has been investigated in [24, 26] when $X_{\hat{g},\epsilon}$ is a white noise decomposition of the field $X_{\hat{g}}$, which can also be taken as a definition of the regularized field. Convergence in law of of the circle average based regularization measure is carried out via the smooth Gaussian approximations introduced in [49]. Establishing the Seiberg bounds needs some extra care and can be handled via the conditioning techniques used in [52].

[4]The $\sqrt{2/\pi}$ term appears in relation with the results in [26] to make the $\gamma=2$ case appear as a suitable limit of the $\gamma<2$ case, see Conjecture 3 below.

4.2 Freezing in LQG

For $\gamma > 2$ and $Q = 2$, one can define

$$\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(\hat{g},F) = \lim_{\epsilon\to 0}\int_{\mathbb{R}} \mathbb{E}\Big[F(X_{\hat{g}}+c+\ln\hat{g})\prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i(c+X_{\hat{g},\epsilon}+\ln\hat{g})(z_i)} \exp\Big(-\frac{1}{2\pi}\int_{\mathbb{R}^2} R_{\hat{g}}(c+X_{\hat{g}})\,d\lambda_{\hat{g}} - \mu e^{\gamma c}\epsilon^{2\gamma-2}\int_{\mathbb{R}^2} e^{\gamma X_{\hat{g},\epsilon}+\gamma\ln\hat{g}}\,d\lambda\Big)\Big]\,dc. \tag{4.3}$$

Here we choose to use a white noise regularization of the field $X_{\hat{g}}$ to stick to the framework in [42]. Notice the unusual power of ϵ in order to non-trivially renormalize the interaction term, which gets dominated by the near extrema of the field $X_{g,\epsilon}$. Under this framework, the convergence in law of the random measures

$$(-\ln\epsilon)^{\frac{3\gamma}{4}}\epsilon^{2\gamma-2}e^{\gamma X_{\hat{g},\epsilon}}dx \to M'_{\frac{2}{\gamma}}(dx)$$

is established in [42], where $M'_{\frac{2}{\gamma}}(dx)$ is a random measure characterized by

$$\mathbb{E}[e^{M'_{\frac{2}{\gamma}}(f)}] = \mathbb{E}[e^{-c_\gamma\int_{\mathbb{R}^2} f(x)^{\frac{2}{\gamma}}\hat{g}^{-1}(x)M'(dx)}].$$

Hence the convergence in law in the sense of weak convergence of measures

$$(-\ln\epsilon)^{\frac{3\gamma}{4}}\epsilon^{2\gamma-2}e^{\gamma X_{\hat{g},\epsilon}+\gamma\ln\hat{g}}\,d\lambda \to \hat{g}^{\gamma}(x)M'_{\alpha}(dx).$$

We deduce

$$\begin{aligned}
&\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(\hat{g},F) \qquad (4.4)\\
&=\Big(\prod_i \hat{g}(z_i)^{-\frac{\alpha_i^2}{4}+\alpha_i}\Big)e^{C(\mathbf{z})}\int_{\mathbb{R}} e^{\big(\sum_i\alpha_i-4\big)c}\,\mathbb{E}\Big[F(c-\frac{\gamma}{2}\theta_{\hat{g}}+X_{\hat{g}}+H_{\hat{g}}+\ln\hat{g})\\
&\quad\times\exp\Big(-\mu e^{\gamma c}\int_{\mathbb{R}^2} e^{\gamma H_{\hat{g}}(x)}\hat{g}(x)\,M'_{\frac{2}{\gamma}}(dx)\Big)\Big]\,dc,\\
&=\Big(\prod_i \hat{g}(z_i)^{-\frac{\alpha_i^2}{4}+\alpha_i}\Big)e^{C(\mathbf{z})}\int_{\mathbb{R}} e^{\big(\sum_i\alpha_i-4\big)c}\,\mathbb{E}\Big[F(c-\frac{\gamma}{2}\theta_{\hat{g}}+X_{\hat{g}}+H_{\hat{g}}+\ln\hat{g})\\
&\quad\times\exp\Big(-c_\gamma\mu^{\frac{2}{\gamma}}e^{2c}\int_{\mathbb{R}^2} e^{2H_{\hat{g}}(x)}\hat{g}(x)\,M'(dx)\Big)\Big]\,dc,
\end{aligned}$$

with $C(\mathbf{z})$ given by 3.6. Up to the unusual shape of the cosmological constant, this is exactly the same partition function as in the critical case $\gamma = 2$. The difference is here the law of the Liouville measure $M'_{\frac{2}{\gamma}}(dx)$, which can be seen as a $\alpha = \frac{2}{\gamma}$-stable transform of the derivative martingale M' and is now purely atomic (see [42] for further details).

4.3 Duality of LQG

The basic tools in order to carry out the following computations can be found in [6]. Define the dual partition function for $\bar{\gamma} > 2$ and $Q = \frac{2}{\bar{\gamma}} + \frac{\bar{\gamma}}{2}$ as

$$\bar{\Pi}_{\bar{\gamma},\mu}^{(z_i\alpha_i)_i}(\hat{g},F) = \lim_{\epsilon\to 0}\int_{\mathbb{R}} \mathbb{E}\Big[F(X_{\hat{g}}+c+Q/2\ln\hat{g})\prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i(c+X_{\hat{g},\epsilon}+Q/2\ln\hat{g})(z_i)} \exp\Big(-\frac{Q}{4\pi}\int_{\mathbb{R}^2} R_{\hat{g}}(c+X_{\hat{g}})\,d\lambda_{\hat{g}} - \mu e^{\bar{\gamma}c}\epsilon^{2}\int_{\mathbb{R}^2} e^{\bar{\gamma}X_{\hat{g},\epsilon}+\bar{\gamma}Q/2\ln\hat{g}}\,d\lambda_{\alpha}\Big)\Big]\,dc \tag{4.5}$$

where λ_α is a α-stable Poisson measure with spatial intensity λ and $\alpha = 4/\bar{\gamma}^2$. We get

$$\begin{aligned}\bar{\Pi}_{\bar{\gamma},\mu}^{(z_i\alpha_i)_i}(\hat{g},F) \\ =&\Big(\prod_i \hat{g}(z_i)^{-\frac{\alpha_i^2}{4}+\frac{Q}{2}\alpha_i}\Big)e^{C(\mathbf{z})}\int_{\mathbb{R}} e^{\big(\sum_i \alpha_i - 2Q\big)c}\,\mathbb{E}\Big[F(c-\frac{\bar{\gamma}}{2}\theta_{\hat{g}}+X_{\hat{g}}+H_{\hat{g}}+\frac{Q}{2}\ln\hat{g})\\ &\times\exp\Big(-\mu e^{\bar{\gamma}c}\int_{\mathbb{R}^2} e^{\bar{\gamma}H_{\hat{g}}(x)}\hat{g}^{\frac{\bar{\gamma}}{4}}(x)\,S'_\alpha(dx)\Big)\Big]\,dc\end{aligned} \tag{4.6}$$

with $C(\mathbf{z})$ defined as usual and $S'_\alpha(dx)$ is a stable Poisson random measure with spatial intensity $e^{\gamma X_g - \frac{\gamma^2}{2}\mathbb{E}[X_g^2]}\,d\lambda$. By computing the expectation we get

$$\begin{aligned}\bar{\Pi}_{\bar{\gamma},\mu}^{(z_i\alpha_i)_i}(\hat{g},1) =&\Big(\prod_i \hat{g}(z_i)^{-\frac{\alpha_i^2}{4}+\frac{Q}{2}\alpha_i}\Big)e^{C(\mathbf{z})}\int_{\mathbb{R}} e^{\big(\sum_i \alpha_i - 2Q\big)c}\\ &\times\mathbb{E}\Big[\exp\Big(-\mu^{\frac{\gamma^2}{4}}\frac{4\Gamma(1-\gamma^2/4)}{\gamma^2}e^{\gamma c}\int_{\mathbb{R}^2} e^{\gamma H_{\hat{g}}(x)}\hat{g}e^{\gamma X_g - \frac{\gamma^2}{2}\mathbb{E}[X_g^2]}\,d\lambda\Big)\Big]\,dc\\ =&\frac{\mu^{\frac{2Q-\sum_i\alpha_i}{\bar{\gamma}}}}{\mu^{\frac{2Q-\sum_i\alpha_i}{\gamma}}}\Big(\frac{4\Gamma(1-\gamma^2/4)}{\gamma^2}\Big)^{\frac{2Q-\sum_i\alpha_i}{\gamma}}\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(\hat{g},1).\end{aligned} \tag{4.7}$$

Observe that this is an ad-hoc construction of duality (see also [22]). The very problem to fully justifies the duality of LQG is to find a proper analytic continuation of the partition of LQG, i.e. the function

$$\gamma \mapsto \Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(\hat{g},1).$$

First observe that this mapping goes to ∞ as $\gamma \to 2$ and it is necessary to get rid of the pole at $\gamma = 2$. We make the following conjecture

Conjecture 3. *The function*

$$\gamma \mapsto \Big(\frac{4\Gamma(1-\gamma^2/4)}{\gamma^2}\Big)^{\frac{2Q-\sum_i\alpha_i}{\gamma}}\Pi_{\gamma,\mu}^{(z_i\alpha_i)_i}(\hat{g},1)$$

is an analytic function of $\gamma \in]0,2[$, which admits an analytic extension for $\gamma \geqslant 2$ given by $\bar{\Pi}_{\gamma,\mu}^{(z_i\alpha_i)_i}(\hat{g},1)$. Furthermore, this extension at $\gamma = 2$ is the partition function $\Pi_{2,\mu}^{(z_i\alpha_i)_i}(\hat{g},1)$ of the critical case.

We do not know how to establish analyticity but we stress that the above function is continuous on $]0,+\infty[$.

5 Perspectives

In this section, we give a brief overview of perspectives and open problems linked to this work.

The DOZZ formula

One of the interesting features of LQG is that it is a non minimal CFT but nevertheless physicists have conjectured exact formulas for the three point correlation function of the theory. This correlation function is very important because (in theory) one can compute all correlation functions of LQG from the knowledge of the three point function. In LQG, the three point function is quite amazingly supposed to have a completely explicit form, the celebrated DOZZ formula [18, 58, 61].

More precisely, let $z_1, z_2, z_3 \in \mathbb{R}^2$ and $\alpha_1, \alpha_2, \alpha_3$ be three points satisfying the Seiberg bounds (3.8). Applying the Möbius transformation rule (3.4) for the map ψ that takes (z_1, z_2, z_3) to $(0, 1, \infty)$ we get after some calculation

$$\Pi_{\gamma,\mu}^{(z_i,\alpha_i)_i}(\hat{g},1) = |z_1-z_2|^{2\Delta_{12}}|z_2-z_3|^{2\Delta_{23}}|z_1-z_3|^{2\Delta_{13}}C_\gamma(\alpha_1,\alpha_2,\alpha_3)$$

where we denoted $\Delta_{12} = \Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2}$ and similarly for Δ_{13} and Δ_{23}. The coefficient is given by (recall $s = \sum_{i=1}^{3} \alpha_i - 2Q$)

$$C_\gamma(\alpha_1, \alpha_2, \alpha_3) = e^{\frac{1}{4}(s^2+2Qs)+2\ln 2\Delta(\alpha_1)}\gamma^{-1}\mu^{-s/\gamma}\Gamma(s/\gamma)\mathbb{E}\, Z^{-s/\gamma}$$

and

$$Z = \int |z|^{-\alpha_1\gamma}|z-1|^{-\alpha_2\gamma}\hat{g}(z)^{-\frac{\gamma}{4}\sum_{i=1}^{3}\alpha_i} M_\gamma(dz).$$

The DOZZ formula is a conjecture on an exact expression for $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$. It is based on the observation that $\mathbb{E}\, Z^{-s/\gamma}$ can be computed in closed form if $-s/\gamma = n$, a positive integer. We have

$$\mathbb{E}\, Z^n = \int e^{\gamma^2 \sum_{i<j} G_{\hat{g}}(z_i, z_j)} \prod_{i=1}^{n} |z_i|^{-\alpha_1\gamma}|z_i - 1|^{-\alpha_2\gamma}\hat{g}(z_i)^{-\frac{\gamma}{4}\sum_{j=1}^{3}\alpha_j}\lambda_{\hat{g}}(dz_i).$$

Using (2.11) this becomes

$$\mathbb{E}\, Z^n = e^{-\gamma^2\frac{n^2-n}{4}} \int \prod_{i<j} |z_i - z_j|^{-\gamma^2} \prod_{i=1}^{n} |z_i|^{-\alpha_1\gamma}|z_i - 1|^{-\alpha_2\gamma}\lambda(dz_i),$$

an expression that does not depend on the background metric $\hat{g}$. This Coulomb gas integral can be computed in closed form and leads to an expression which can be cast in a form where n enters as a parameter allowing a formal extension of the formula to the negative real axis. This leads to the DOZZ formula for $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$. We will not state it here explicitly as it is quite complicated and involves introducing numerous special functions.

Proving the DOZZ formula seems at this time difficult. Note for instance that for given γ only a finite number of positive moments of Z exist so one can not attempt to solve a moment problem. In the semiclassical $\gamma \to 0$ limit we want to point to an interesting recent approach to the DOZZ formula by performing deformation of the integration contour in function space [34].

The semi-classical limit

The semiclassical limit of LQG is the study of the concentration phenomena of the Liouville field around the extrema of the Liouville action for small γ, see [46, 34]. After a suitable rescaling of the parameters μ and $(\alpha_i)_i$, that is

$$\mu\gamma^2 = \Lambda, \quad \alpha_i = \frac{\chi_i}{\gamma} \tag{5.1}$$

for some fixed constants $\Lambda > 0$ and weights $(\chi_i)_i$ satisfying $\chi_i < 2$ and $\sum_i \chi_i > 4$, the Liouville field $\gamma\phi$ should converge in law towards $U + \ln \hat{g}$, where U is the solution of the classical Liouville equation with sources

$$\triangle_{\hat{g}} U - R_{\hat{g}} = 2\pi\Lambda e^U - 2\pi \sum_i \chi_i \delta_{z_i}, \qquad \text{with } \int_{\mathbb{R}^2} e^U \, d\lambda_{\hat{g}} = \frac{\sum_i \chi_i - 4}{\Lambda}, \tag{5.2}$$

hence the name of the theory "Liouville quantum gravity". The reader may consult [38] for some partial results in the "toy model" situation where the zero modes have been turned off.

A Möbius transform relations

In this section, we gather a few relations concerning Möbius transforms and their behavior with respect to Green functions. Recall that the set of automorphisms of the Riemann sphere can be described in terms of the Möbius transforms

$$\psi(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0.$$

Such a function preserves the cross ratios: for all distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C}$

$$\frac{(z_1-z_3)(z_2-z_4)}{(z_2-z_3)(z_1-z_4)} = \frac{(\psi(z_1)-\psi(z_3))(\psi(z_2)-\psi(z_4))}{(\psi(z_2)-\psi(z_3))(\psi(z_1)-\psi(z_4))}. \tag{A.1}$$

Recall that g_ψ stands for the metric $|\psi'|^2\hat{g}\circ\psi$.

Proof of Proposition 2.1. We can rewrite the expression (2.8) with $g = g_\psi$ in a condensed way

$$G_{g_\psi}(x,y) = \frac{1}{(4\pi)^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \ln\frac{|x-z||y-z'|}{|x-y||z-z'|}\,\lambda_{g_\psi}(dz)\lambda_{g_\psi}(dz').$$

By making a change of variables and use (A.1), we get

$$\begin{aligned}
G_{g_\psi}(x,y) =& \frac{1}{(4\pi)^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \ln\frac{|x-\psi^{-1}(z)||y-\psi^{-1}(z')|}{|x-y||\psi^{-1}(z)-\psi^{-1}(z')|}\,\lambda_{\hat{g}}(dz)\lambda_{\hat{g}}(dz').\\
=& \frac{1}{(4\pi)^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \ln\frac{|\psi(x)-z||\psi(y)-z'|}{|\psi(x)-\psi(y)||z-z'|}\,\lambda_{\hat{g}}(dz)\lambda_{\hat{g}}(dz').
\end{aligned}$$

This is exactly the expression of $G_{\hat{g}}(\psi(x),\psi(y))$. □

Corollary A.1. *We have the following relations for all Möbius transforms ψ*

$$-2m_{\hat{g}}(\ln\frac{1}{|x-\cdot|}) = -\frac{1}{2}\ln\hat{g}(x) + \ln 2 \tag{A.2}$$

$$-2m_{g_\psi}(\ln\frac{1}{|x-\cdot|}) + \theta_{g_\psi} = -\frac{1}{2}\ln\hat{g}(\psi(x)) - \ln|\psi'(x)| + \theta_{\hat{g}} + \ln 2. \tag{A.3}$$

In particular (2.11) holds.

Proof. We use the following relation

$$\int_{\mathbb{R}^2}\ln|x-\cdot|\lambda_{|\psi'|^2\hat{g}(\psi)} = 2\pi(\ln(|ax+b|^2+|cx+d|^2) - \ln(|a|^2+|c|^2)). \tag{A.4}$$

The proof of this identity is based on the fact that both sides have the same Laplacian and the difference of both functions goes to 0 as $|x|$ goes to infinity.

The first relation is a straightforward consequence of (A.4) with $\psi(z) = z$. One could use (A.4) as well to prove the second but another way (which we follow below) is to use (A.1). Write

$$\begin{aligned}
-2m_{g_\psi}(\ln\frac{1}{|x-\cdot|}) + \theta_{g_\psi} =& \frac{1}{(4\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\frac{|x-z||x-z'|}{|z-z'|}\lambda_{g_\psi}(dz)\lambda_{g_\psi}(dz')\\
=& \frac{1}{(4\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\frac{|x-\psi^{-1}(z)||x-\psi^{-1}(z')|}{|\psi^{-1}(z)-\psi^{-1}(z')|}\lambda_{\hat{g}}(dz)\lambda_{\hat{g}}(dz')
\end{aligned}$$

Observe that the mapping $(x,y)\mapsto \frac{1}{(4\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\frac{|x-\psi^{-1}(z)||y-\psi^{-1}(z')|}{|\psi^{-1}(z)-\psi^{-1}(z')|}\lambda_{\hat{g}}(dz)\lambda_{\hat{g}}(dz')$ is a continuous function so that we can write

$$\begin{aligned}
&-2m_{g_\psi}(\ln\frac{1}{|x-\cdot|}) + \theta_{g_\psi}\\
&= \lim_{y\to x}\frac{1}{(4\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\frac{|x-\psi^{-1}(z)||y-\psi^{-1}(z')|}{|\psi^{-1}(z)-\psi^{-1}(z')|}\lambda_{\hat{g}}(dz)\lambda_{\hat{g}}(dz')\\
&= \lim_{y\to x}\big(\frac{1}{(4\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\frac{|x-\psi^{-1}(z)||y-\psi^{-1}(z')|}{|x-y||\psi^{-1}(z)-\psi^{-1}(z')|}\lambda_{\hat{g}}(dz)\lambda_{\hat{g}}(dz') + \ln|x-y|\big).
\end{aligned}$$

Now we can use the invariance of cross-products with respect to Möbius transforms to get

$$
\begin{aligned}
&-2m_{g_\psi}(\ln \frac{1}{|x-\cdot|}) + \theta_{g_\psi} \\
&\quad = \lim_{y\to x} \big(\frac{1}{(4\pi)^2} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln \frac{|\psi(x)-z||\psi(y)-z'|}{|\psi(x)-\psi(y)||z-z'|} \lambda_{\hat g}(dz)\lambda_{\hat g}(dz') + \ln|x-y| \big) \\
&\quad = \lim_{y\to x} \big(\frac{1}{(4\pi)^2} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln \frac{|\psi(x)-z||\psi(y)-z'|}{|z-z'|} \lambda_{\hat g}(dz)\lambda_{\hat g}(dz') - \ln \frac{|\psi(x)-\psi(y)|}{|x-y|} \big) \\
&\quad = -2m_{\hat g}(\ln \frac{1}{|\psi(x)-\cdot|}) + \theta_{\hat g} - \ln|\psi'(x)|.
\end{aligned}
$$

We complete the proof thanks to (A.2). ◻

Lemma A.2. *The relations* (2.12) *and* (3.32) *hold.*

Proof. Using the relation (A.4), we have

$$
\begin{aligned}
&G_{\hat g}(\psi(x),\psi(z)) \\
&= \ln \frac{1}{|x-z|} + \frac{1}{2}(\ln(|ax+b|^2+|cx+d|^2) - \ln(|a|^2+|c|^2)) \\
&\quad + \frac{1}{2}(\ln(|az+b|^2+|cz+d|^2) - \ln(|a|^2+|c|^2)) \\
&\quad - \frac{1}{4\pi}\int_{\mathbb{R}^2} \frac{1}{2}(\ln(|au+b|^2+|cu+d|^2) - \ln(|a|^2+|c|^2))\lambda_{g_\psi}(du) \\
&= \ln \frac{1}{|x-z|} + \frac{1}{2}\ln(|ax+b|^2+|cx+d|^2) + \frac{1}{2}\ln(|az+b|^2+|cz+d|^2) - \frac{1}{2}\ln(|a|^2+|c|^2) \\
&\quad - \frac{1}{4\pi}\int_{\mathbb{R}^2} \frac{1}{2}\ln(|au+b|^2+|cu+d|^2)\lambda_{g_\psi}(du).
\end{aligned}
$$

After integrating, we get that

$$
\begin{aligned}
\int_{\mathbb{R}^2} G_{\hat g}(\psi(x),\psi(z))\hat g(z)dz = &-2\pi\ln(1+|x|^2) + 2\pi\ln(|ax+b|^2+|cx+d|^2) + \\
&\frac{1}{2}\int_{\mathbb{R}^2}\ln(|az+b|^2+|cz+d|^2)\lambda_{\hat g}(dz) - 2\pi\ln(|a|^2+|c|^2) \\
&-\frac{1}{2}\int_{\mathbb{R}^2}\ln(|au+b|^2+|cu+d|^2)\lambda_{g_\psi}(du).
\end{aligned}
$$

At this stage, we will suppose that $ad-bc=1$. Hence, we have

$$
\begin{aligned}
-\frac{1}{2}&\int_{\mathbb{R}^2}\ln(|au+b|^2+|cu+d|^2)\lambda_{g_\psi}(du) \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\ln(|\psi'(u)|^2\hat g(\psi(u)))\lambda_{g_\psi}(du) \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\ln(\hat g(v))\lambda_{\hat g}(dv) + \frac{1}{2}\int_{\mathbb{R}^2}\ln(|\psi'(u)|^2)\lambda_{g_\psi}(du) \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\ln(\hat g(v))\lambda_{\hat g}(dv) - \int_{\mathbb{R}^2}\ln|cu+d|\lambda_{g_\psi}(du).
\end{aligned}
$$

Now, we introduce the function

$$
G(x) = \int_{\mathbb{R}^2}\ln|cx+d-cu-d|\lambda_{g_\psi}(du) = 4\pi\ln|c| + \int_{\mathbb{R}^2}\ln|x-u|\lambda_{g_\psi}(du).
$$

By using equation (A.4), we get that

$$G(x) = 4\pi \ln |c| + 2\pi(\ln(|ax+b|^2 + |cx+d|^2) - \ln(|a|^2 + |c|^2))$$

Hence, we get that

$$\int_{\mathbb{R}^2} \ln |cu+d| \lambda_{g_\psi}(du) = G(-\frac{d}{c}) = 4\pi \ln |c| - 4\pi \ln |c| - 2\pi \ln(|a|^2 + |c|^2) = -2\pi \ln(|a|^2 + |c|^2).$$

At the end, we get

$$\begin{aligned}
\int_{\mathbb{R}^2} & G_{\hat{g}}(\psi(x), \psi(z)) \lambda_{\hat{g}}(dz) \\
&= -2\pi \ln(1+|x|^2) + 2\pi \ln(|ax+b|^2 + |cx+d|^2) \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^2} \ln(|az+b|^2 + |cz+d|^2) \lambda_{\hat{g}}(dz) - 2\pi \ln(|a|^2 + |c|^2) \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^2} \ln(\hat{g}(v)) \lambda_{\hat{g}}(dv) + 2\pi \ln(|a|^2 + |c|^2) \\
&= -\pi \ln \frac{g_\psi(x)}{\hat{g}(x)} - \pi m_{\hat{g}}(\ln \frac{g_\psi(x)}{\hat{g}(x)}) = -\pi\phi(x) - \pi m_{\hat{g}}(\phi)
\end{aligned}$$

which implies that

$$\frac{1}{(4\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_{\hat{g}}(\psi(x), \psi(z)) \lambda_{\hat{g}}(dx) \lambda_{\hat{g}}(dz) = -\frac{1}{2} m_{\hat{g}}(\phi). \tag{A.5}$$

Recall now that $X_{\hat{g}} \circ \psi$ equals in law $X_{\hat{g}} - m_{g_\psi}(X_{\hat{g}})$ so that

$$G_{\hat{g}}(\psi(x), \psi(z)) = G_{\hat{g}}(x, y) - \frac{1}{4\pi}((G_{\hat{g}} e^\phi)(x) + (G_{\hat{g}} e^\phi)(y)) + D_\psi$$

where $D_\psi = \frac{1}{4\pi} m_{\hat{g}}(e^\phi G_{\hat{g}} e^\phi)$. Using (3.30) this becomes

$$D_\psi = \frac{1}{4\pi}(m_{g_\psi}(\phi) - m_{\hat{g}}(\phi)) \tag{A.6}$$

and the applying (3.30) again we get

$$G_{\hat{g}}(\psi(x), \psi(z)) = G_{\hat{g}}(x, y) - \frac{1}{4}(\phi(x) + \phi(y)) + \frac{1}{2}(m_{\hat{g}}(\phi) + m_{g_\psi}(\phi)).$$

(A.5) implies

$$-\frac{1}{2} m_{\hat{g}}(\phi) = \frac{1}{2} m_{g_\psi}(\phi)$$

which yields (2.12) and combining with (A.6) we also get (3.32). □

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