Higgs in the SM and Beyond : 4° lecture

Today : - Higgs sum rule
- Higgs potential at 1-loop, and stability

- We have seen that SM Higgs is a UV completion of the theory of massive gauge boson

$$A(W_L W_L \to W_L W_L) \sim \frac{s}{v^2}$$

it grows w/ energy^2 because $\varepsilon_L^\mu \sim \frac{k^\mu}{m_W}$

In fact, $W_L \sim \frac{\partial_\mu \pi}{m}$ they are the eaten GB's at $E \gg m_W$

$$\mathcal{L} \xrightarrow{E \gg m_W} \simeq \mathcal{L}_\pi = \frac{v^2}{2}\left(\frac{\partial \pi^{\hat{a}}}{v} + \dots\right)^2$$

$\sim \frac{s}{v^2}$

$(\partial\pi)^2\pi^2$

CCWZ

$$\mathcal{L}_\pi = \frac{v^2}{2} d_\mu^{\hat{a}} d_\mu^{\hat{a}}$$

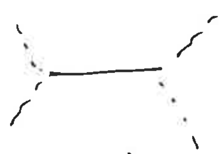
$$d_\mu^{\hat{a}} = \mathrm{Tr}\left[T^{\hat{a}}\left(\frac{\partial \pi}{v} + \frac{1}{2v^2}[\pi, \partial\pi] - \frac{i}{6v^3}[\pi,[\pi,\partial\pi]] + \dots\right)\right]$$

<u>With a Higgs</u> : $\frac{(\partial h)^2}{2} + \frac{h}{v} a\left(2m_W^2 (W)^2 + m_Z^2 Z_\mu^2\right)$

$$\xrightarrow{E \gg m_W} \frac{(\partial h)^2}{2} + \frac{h}{v} a\left(\partial\pi + \dots\right)^2$$

$\frac{s}{v^2}$

$+$



$s \cdot \frac{a^2}{s-m_h^2} \cdot s$

$$\approx \frac{s}{v^2}\left(1 - \frac{a^2 s}{s-m_h^2}\right) = \begin{cases} s \gg m_h^2 & (1-a^2)\frac{s}{v^2} \\ s \ll m_h^2 & \frac{s}{v^2} \end{cases}$$

$\Rightarrow$ well behaved for $\underline{a = 1 = \text{SM-value}}$

- We have seen that other UV completions are possible

$$3 \otimes 3 = 1 \oplus 3 \oplus 5$$

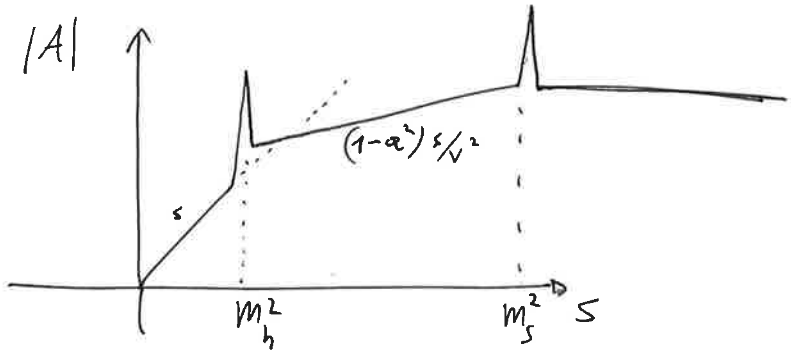
exchange of other scalar singlets or quintuplet

$$\frac{1}{2}\left(1 + 2\frac{a h}{v} + 2\frac{b S}{v}\right)(\partial\pi + \dots)^2 + (\partial h)^2 - \frac{m_h^2 h^2}{2} + (\partial S)^2 - \frac{M_S^2 S^2}{2}$$

$$\Rightarrow A(\pi\pi \to \pi\pi) \simeq \frac{s}{v^2}\left(1 - \frac{a^2 s}{s - m_h^2} - \frac{b^2 s}{s - m_S^2}\right)$$

$\Rightarrow$ well behaved if $\underline{a^2 + b^2 = 1}$

like in weakly coupled theory



$(1 - a^2 - b^2) s/v^2 \to 0 \qquad \underline{a^2 + b^2 = 1}$

can we overunitarize? $\quad a > 1$

More generally, what can we say on "a" from first principle?

We'll see the unitarity + analyticity + crossing symmetry tell something about $(1 - a^2)$.

Example: shift sym: $\mathcal{L}_{IR} = \frac{(\partial\sigma)^2}{2} + a(\partial\sigma)^4 + \ldots$

$\underline{\sigma \to \sigma + c}$

Arkani-Hamed, Adams, Nicolis, Rattazzi and Dubovsky have shown that $a \geq 0$ hep-th/0602178 using a dispersion relation based on unitarity + crossing + analyticity

$$\underbrace{\underset{\substack{\sigma\sigma\to\sigma\sigma \\ \uparrow \\ \text{forward, } t=0}}{A''(s=0)}}_{} = \frac{4}{\pi}\int_0^\infty \frac{ds}{s^2}\,\sigma^{TOT}_{\sigma\sigma\to\text{anything}}(s) \geq 0 \qquad \leftarrow \text{twice-subtracted sum rule}$$

this is an amplitude at $s \simeq 0$, in the IR $\Rightarrow$ I can use $\mathcal{L}_{IR}$

$\Rightarrow \boxed{a \geq 0}$ $\boxed{\text{UV} - \text{IR connection}}$

no all IR theories admit UV completions.

Let's generalize it to our system with a Higgs:

- We need Analyticity, Crossing Symmetry and Unitarity

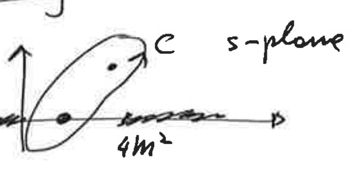
1) Analyticity: $A_{\pi\pi\to\pi\pi}(s,t)$ is analytic is s at fixed t except at cuts and poles

$$\Rightarrow A(s,0) \equiv A(s) = A(0) + s\,A'(0) + \frac{s^2}{2}A''(0) + \ldots$$

it can be Taylor expanded

$$\Rightarrow A^{(n)}(0) = \frac{n!}{2\pi i}\oint_C \frac{A(s)}{s^{n+1}} \qquad \text{cauchy theorem}$$

add other residues if there are IR poles



2) Crossing Symmetry

$$A(\pi^a\pi^b\to\pi^c\pi^d)(s,t=0) \equiv A_{ab\to cd}(s)$$

$$\underset{\text{crossing}}{=} A_{a\bar{d}\to c\bar{b}}(u)$$

$$u = 4m^2 - s$$

We eventually take the limit $m\to 0$ for simplicity

$$\Rightarrow \boxed{A_{ab\to cd}(s) = A_{a\bar{d}\to c\bar{b}}(-s)}$$

3) Unitarity: $S^\dagger S = S S^\dagger = \mathbb{1}$

$\Rightarrow$ optical theorem $\boxed{\operatorname{Im} A_{ab\to ab}(s) = s\,\sigma^{TOT}_{ab}(s)}$

$\boxed{\operatorname{Im}(\text{elastic-forward}) \propto \text{Total cross-section}}$

(for massless particle)

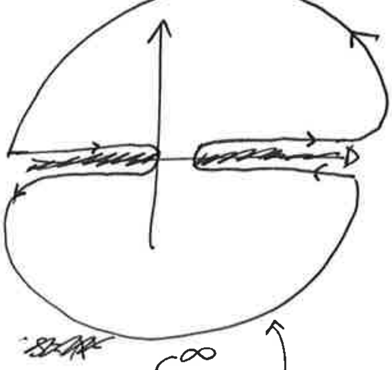
– We are interested in the coefficient of $A'(0)$

$$A_{\pi\pi\to\pi\pi}(s) \simeq (1-e^2)\frac{s}{v^2} + \dots \Rightarrow A'(0) \sim \frac{(1-e^2)}{v^2}$$

We need one subtraction:

$$\underset{t=0}{A'_{\pi\pi\to\pi\pi}(0)} = \frac{1}{2\pi i}\oint ds \frac{A_{\pi\pi\to\pi\pi}(s)}{s^2}$$

← regulate it with a mass and eventually take $m\to 0$



$$A'_{ab\to ab}(0) = C^{\infty} + \text{“integrals along cuts”}$$

$$C^{\infty} \simeq \int_0^{2\pi} \frac{d\theta}{2\pi}\, \frac{A(\Lambda^2 e^{i\theta})}{\Lambda^2 e^{i\theta}} \xrightarrow[\Lambda^2\to\infty]{} 0 \quad \text{if}$$

A less fast then s at infinity

$$\Rightarrow A'_{\pi^a\pi^b\to\pi^a\pi^b}(0) = \int_0^{\infty} ds\, \frac{A^{ab\to ab}(s+i\epsilon) - A^{ab\to ab}(s-i\epsilon)}{2\pi i\, s^2} + \int_{-\infty}^{0} ds\, \frac{A^{ab\to ab}(s+i\epsilon) - A^{ab\to ab}(s-i\epsilon)}{2\pi i\, s^2}$$

change variable $s \longrightarrow -s$

$$= \left(\qquad \right) + \int_0^{\infty} ds\, \frac{A^{ab\to ab}(-s+i\epsilon) - A^{ab\to ab}(-s-i\epsilon)}{2\pi i\, s^2}$$

$$\underset{\text{crossing}}{=} \int_0^{\infty} \frac{ds}{\pi}\, \frac{\operatorname{Im} A^{ab\to ab}(s+i\epsilon)}{s^2} + \int_0^{\infty} ds\, \underbrace{\frac{A^{a\bar b\to a\bar b}(s-i\epsilon) - A^{a\bar b\to a\bar b}(s+i\epsilon)}{2\pi i\, s^2}}_{= -\frac{\operatorname{Im} A^{a\bar b\to a\bar b}(s+i\epsilon)}{\pi s^2}}$$

$$\underset{\text{unitarity}}{=} \int_0^{\infty} \frac{ds}{\pi}\, \frac{\left[\sigma^{TOT}_{ab\to \text{anyth.}}(s) - \sigma^{TOT}_{a\bar b\to\text{anyth.}}(s)\right]}{s}$$

$$\Rightarrow \boxed{A'_{ab\to ab}(0) = \int_0^{\infty} \frac{ds}{\pi}\, \frac{\sigma^{TOT}_{ab}(s) - \sigma^{TOT}_{a\bar b}(s)}{s}}$$

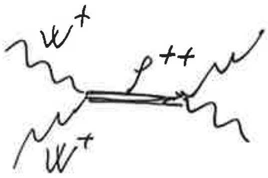
Let's consider now $A(\pi^+\pi^- \longrightarrow \pi^+\pi^-)$ $\qquad \pi^{\pm} = \frac{\pi^1 \pm i\pi^2}{\sqrt{2}}$

$$\Rightarrow \boxed{A'_{+-\to +-}(0) = \int_0^{\infty} \frac{ds}{\pi}\left(\sigma^{TOT}_{+-}(s) - \sigma^{TOT}_{++}(s)\right)}$$

with $\boxed{A'_{+-\to+-}(0) \propto \frac{(1-a^2)}{v^2}}$

$\Rightarrow$ contrary to the case with two subtractions we <u>can't</u> firmly exclude $\underline{1-a^2<0}$ but we can say that if $\boxed{a>1 \Rightarrow \sigma_{++} \text{ must dominate the cross-section}}$

$\Rightarrow$ strong hint that a doubly charged state is coupled to W^+W^- in such a case!



These arguments have been generalized to include masses, arbitrary groups and include C^∞ in <u>1405.2960</u>. (and in fact, much more) (saturating Froissart)

Let's come back to the <u>Higgs Potential</u>

$$V = \lambda\left(|H|^2 - \frac{v^2}{2}\right)^2 \longrightarrow 2 \text{ parameters } \lambda \text{ and } v$$

trade λ for the mass $\underline{m_h^2 = 2\lambda v^2} \Rightarrow \underline{\lambda = \frac{m_h^2}{2v^2} \simeq 0.13 \ll 1}$

As a consequence we have seen that the potential is <u>quite shallow</u>

$\hookrightarrow$ • perhaps it is flat and raised by the ~~...~~ quantum effects? (dilaton type of models) In any case is sensitive to Quantum corrections.

Coleman-Weinberg Potential

$$V = V_{classical} + \hbar V^{(1)} + \hbar^2 V^{(2)} + \dots$$

$V^{(1)} = V_{CW}$ is called Coleman-Weinberg potential.

$$\underbrace{e^{i\Gamma[\Phi]}}_{\text{quantum 1PI action}} = \int D\varphi\, e^{i \overbrace{S[\Phi+\varphi]}^{\text{classical action}}}\Big|_{1PI} = e^{-i\left(\int d^4x\right)\left(V(\Phi) + V^{(1)}\hbar + \dots\right)}\Big|_{\Phi = const}$$

spacetime volume

$$S[\Phi+\varphi] = S[\Phi] + \frac{1}{2}\underbrace{\varphi \cdot S''[\Phi] \cdot \varphi}_{\hookrightarrow \int d^4x \int d^4y\, \varphi(x) \frac{\delta S}{\delta\Phi(x)\delta\Phi(y)} \varphi(y)} + \dots$$

functional Taylor series

at 1-loop only those with 2φ matter

$$\Rightarrow \int D\varphi\, e^{iS[\varphi+\Phi]}\Big|_{1PI} \overset{\text{1-loop}}{=} \underset{\uparrow\ \exp(\text{c.c.})}{const} \cdot e^{iS[\Phi]} \underbrace{\int D\varphi\, e^{\frac{i}{2}\varphi \cdot S'' \cdot \varphi}}_{\text{Gaussian Integral}} = \bigcirc$$

1-loop vacuum-volume in Φ bkg

$$\propto \frac{1}{\sqrt{\det S''[\Phi]}}$$

$$\Rightarrow S_{quantum} = S_{classical} + e^{\frac{1}{2}\ln\det S''[\Phi]}$$

$$= S_{classical} + e^{1/2\, \mathrm{Tr} \ln S''[\Phi]}$$

Φ-dependent mass

$$\Rightarrow -iV_{CW}\int d^4x = \frac{1}{2}\int d^4x \langle x| \ln S''[\phi] |x\rangle = \frac{1}{2}\left(\int d^4x\right)\int \frac{d^4k}{(2\pi)^4} \ln\left[k^2 - m^2(\Phi)\right]$$

$$S''[\Phi] = \frac{\delta S}{\delta\Phi(x)\delta\Phi(y)} = \left(-\partial^2 - m^2(\Phi)\right)\delta(x-y)$$

$$V_{CW} = \frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \ln(k^2 - m^2(\Phi) - i\epsilon) \overset{\text{wick}}{\underset{k^0 = i k^4_{Eucl.}}{=}} \frac{-1}{2}\int \frac{dk^2\, k^2}{16\pi^2} \ln(k^2 + m^2(\phi))$$

badly UV divergent but we have $V_{class.}$ to renormalize it

$$V'''_{CW} = -\frac{1}{32\pi^2}\int dk^2\, k^2 \frac{(-2)}{(k^2 + m^2(\Phi))^3} = \frac{1}{32\pi^2}\frac{1}{m^2(\Phi)}$$

derivative with respect to $m^2(\Phi)$

Integrating back in $m^2(\Phi)$

$$V_{CW} = \frac{m^4(\Phi)}{64\pi^2}\log m^2(\Phi) + a\, m^4(\Phi) + b\, m^2(\Phi) + c$$

only log's are meaningful

~~...~~ divergent but renormalized against Vclassical bare parameters

$$\Rightarrow \boxed{V = V_{class} + \frac{m^4(\Phi)}{64\pi^2}\log m^2(\Phi)}$$

nice result to keep in mind

If I had many fields $m^4(\phi)\log m^2(\phi) \to \sum_i m_i^4(\phi_i) \ln m_i^2(\phi)$

$$= \mathrm{Tr}(M^4(\phi) \ln M^2(\phi))$$

If I had spinors the gaussian integral flip sign, and the trace is also all the spinorial index too

$$\Rightarrow \boxed{\begin{array}{l} V(\Phi) = V_{class} + \dfrac{\mathrm{STr}[M^4(\Phi)\ln M^2(\phi)]}{64\pi^2} \\[2ex] \mathrm{STr} = \underbrace{(+)\mathrm{Tr}}_{\text{bosons}} + \underbrace{(-)\mathrm{Tr}}_{\text{fermions}} = \sum (-1)^{2s_i} n_i \end{array}}$$

spin

number degrees of freedom.

In the SM the largest contribution comes from the top, as usual

$$H = \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} \equiv \begin{pmatrix} 0 \\ \frac{\phi}{\sqrt{2}} \end{pmatrix}$$

$$V_{\text{tree-level}} = \lambda |H|^4 - m^2 |H|^2 = \frac{\lambda \phi^4}{4} - \frac{m^2}{2}\phi^2 \longrightarrow \boxed{m_h^2(\phi) = 3\lambda\phi^2 - m^2}$$

$$\mathcal{L}_{yuk} = y_t \bar{Q}_L H t_R + h.c. = \frac{y_t \phi}{\sqrt{2}} \bar{t}_L t_R + h.c. \longrightarrow \boxed{m_t^2(\phi) = \frac{y_t^2 \phi^2}{2}}$$

$$\Longrightarrow V = \frac{h^4}{4}\left[\lambda + \frac{1}{16\pi^2}\left(9\lambda^2 - \cancel{12}\frac{y_t^4}{\cancel{4}}\right)\log\frac{h^2}{v^2}\right]$$

at large $h \gg v$ only the logs are important; discard h^2

$$\Longrightarrow \boxed{V(h \gg v) = \frac{h^4}{4}\left(\lambda + \frac{3}{16\pi^2}(3\lambda^2 - y_t^4)\log\left(\frac{h}{v}\right)^2\right)}$$

<u>Vacuum stability</u>: V positive up to $h \sim M_{pl}$

for λ and y_t we can use the tree-level relations

$$\begin{cases} m_h^2 = 2\lambda v^2 \longrightarrow \lambda = \dfrac{m_h^2}{2v^2} \\ m_t = \dfrac{y_t v}{\sqrt{2}} \longrightarrow y_t^2 = \dfrac{2m_t^2}{v} \end{cases} \Longrightarrow V(h = M_{pl}) \geq 0$$

gives a <u>constraint on m_h as a function of m_t (or vice versa)</u>

it is about $m_h \gtrsim 200$ GeV for $m_t \simeq 170$ GeV and $h = M_{pl}$.

But, we can live happily with metastability, and most importantly

$$\frac{9\lambda^2}{16\pi^2}\ln\left(\frac{M_{pl}}{v}\right)^2 = \frac{9\, m_h^4}{64\pi^2 v^4}\ln\left(\frac{M_{pl}}{v}\right)^2 \simeq 0.3$$

which is comparable with λ from tree-level with $m_h \simeq 200$ GeV

One should actually resum the large logs

$$V = \cancel{\ } \frac{h^4}{4}\left(\lambda + \lambda^2 \log^2(\ ) - y^4 \log(\ )\ \ \right)$$

$$= \frac{h^4}{4}\,\lambda(\mu = h)$$

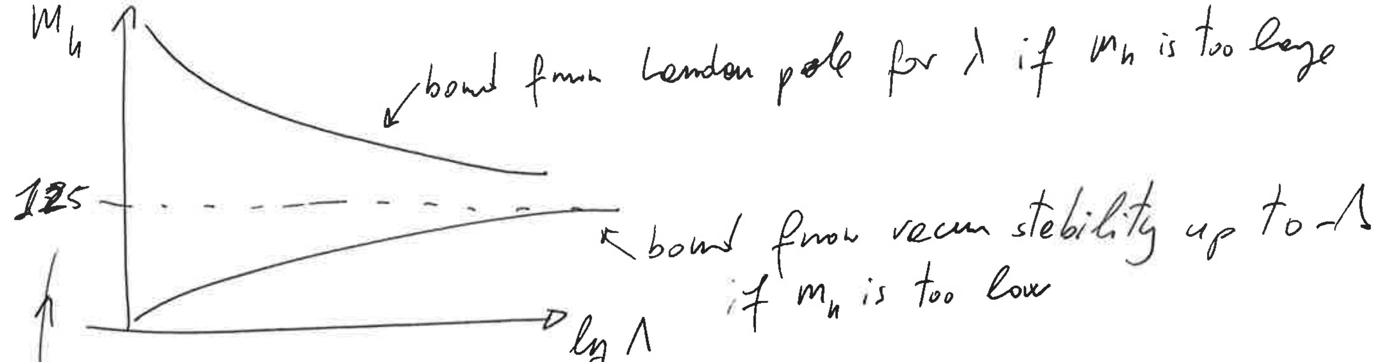
with $\lambda(\mu)$ the correction to the quartic + evaluated at $\mu = h$ to minimize the higher orders

⇒ to resum the log I just need to solve the RG equation for the quartic:

$$\boxed{16\pi^2\beta_\lambda = 16\pi^2\frac{d\ln\lambda}{d\ln\mu} = 24\lambda^2 - 6y_t^4 + \cancel{\ \ \ } + 12y_t^2\lambda}$$

↑ wave function

People have studied this problem up to 3 loops including all EW corrections and calculating the lifetime ($dP = dt\,dV\,\Lambda_B^4 e^{-S(\Lambda_B)}$ BOUNCING SOLUT.)



it is right at the boundary between stability and metastability. Currently, 3σ away from stability and many from actual instability (lifetime less than universe age) ⟶ but lifetime depends on the cosmological history!!

($dt\,d^3x \sim H^{-1}$ Hubble ↑)