Chapter 3

The algebraic quantum formalism

3.1 Introduction

3.1.1 Observables as operators

The physical observables of a quantum system are represented by the symmetric (self-adjoint) operators on the Hilbert space of pure states of the system (see 2.3.1 and in 2.3.3). They thus generate (by addition and multiplication) the set of all (not necessary symmetric) operators on the Hilbert space. This set forms an associative but non-commutative complex algebra of operators.

We have also seen that the mixed states ω (density matrices) correspond to the positive normalized linear forms on this algebra of operators, that associate to a self-adjoint operator the expectation value of the corresponding observable on the state ω.

Finally in 2.3.3 we have already seen that it is of interest to consider the set of "bounded operators" $\mathcal{B}(\mathcal{H})$ over some Hilbert space $\mathcal{H}$. We also mentioned that in classical mechanics, the set of smooth functions (i.e. the set of classical observables) over the phase space of a classical system form in general a Poisson algebra, i.e. a commutative algebra equipped with a Poisson bracket.

In this section I shall present and discuss the "algebraic approach" of quantum mechanics. This formulation of the principles of quantum mechanics relies precisely on the mathematical theory of algebras of operators, and is a formalisation of the above considerations. It can be viewed as an extension and as a mathematically rigorous formulation of the "canonical formalism". Of course the idea of "non-commutative observables" goes back to the "matrix mechanics" initiated by Heisenberg and was a crucial element in the elaboration the canonical formalism itself. As we shall see, in the algebraic formulation one focuses on the abstract structure of the set of observables and of the set of states of a system, and on the rules that must satisfy the probabilities associated to measurements of observables over states. The explicit realization of these (observables, states and probabilities) as an algebra of operators acting on an Hilbert space representing the pure states of the system, as well as the explicit form of the probabilities as given by the Born rule, can be deduced from some more abstract, but still mostly physical axioms.

3.1.2 Operator algebras

Let me first recall briefly which kind of operator algebras play a role in quantum mechanics. The Hilbert space $\mathcal{H}$ is a complex vector state, embodied with a scalar product (a sesquiinear form) $(\psi,\phi) \to \langle\psi|\phi\rangle = \psi\cdot\phi$ which is linear in ϕ and antilinear on ψ, symmetric, positive and non degenerate. This defines the standard norm on $\mathcal{H}$, $\|\psi\|^2 = \psi\cdot\psi$. To be a Hilbert space, the vector space $\mathcal{H}$ has to be complete under this norm, namely any Cauchy sequence of ψ_n, $n\in\mathbb{N}$ has a limit. Apart from the finite dimensional Hilbert spaces, the simplest and most useful Hilbert space is the separable Hilbert space, that admits a denumbrable orthonormal basis (i.e. a complete basis of orthonormal vectors labelled by the integers e_n, $n\in\mathbb{N}$, $\langle e_n|e_m\rangle = \delta_{n,m}$).

Among the linear operators acting on the Hilbert space $\mathcal{H}$, an interesting class is the algebra of bounded operators $\mathcal{B}(\mathcal{H})$. An operator $A\in GL(\mathcal{H})$ is bounded if its sup-norm is finite. The norm of an operator is defined as

$$\|A\|^2 = \sup_{\psi\in\mathcal{H}^*} \frac{\langle A\psi|A\psi\rangle}{\langle\psi|\psi\rangle} < \infty \tag{3.1.1}$$

In quantum mechanics many physical observables, starting with the position operator, the momentum operator and the energy operator (the Hamiltonian), as well as the number of particles operator in quantum field theories, are not bounded operators. They may however be reconstructed from the bounded operators, within the theory of rigged Hilbert spaces (see for instance [BLOT90]).

The norm of an operator A corresponds to the diameter of its spectrum. The spectrum of A, spec(A), is the set of $z\in\mathbb{C}$ such that the operator $A-z$ is not invertible, this is the infinite dimensional generalization of the set of eigenvalues of a matrix. So the norm $\|A\|$ is roughly speaking the sup of the modulus of the eigenvalues of A.

This norm has many interesting property. Firstly it is indeed a norm, such that

$$\|\lambda_1 A_1 + \lambda_2 A_2\| \le |\lambda_1|\,\|A_1\| + |\lambda_2|\,\|A_2\| \tag{3.1.2}$$

and $\mathcal{B}(\mathcal{H})$ is complete under this norm. Moreover for products of operators it satisfy the inequality

$$\|AB\| \le \|A\|\,\|B\| \tag{3.1.3}$$

that makes this algebra of operators a Banach algebra. But it satisfies also the non-trivial identity ($A^\dagger$ is the adjoint of A)

$$\|A^\dagger A\| = \|A\|^2 = \|A^\dagger\|^2 \tag{3.1.4}$$

that makes it a C^*-algebra. C^*-algebras have many interesting properties and are at the basis of the mathematical theory of operator algebras. They were introduced by Gelfand (under a different name). Note here that, although $\mathcal{B}(\mathcal{H})$ is a complex algebra, the "C" in the denomination is not for "complex" but rather for "compact" (the unit sphere for the norm is compact) and the "*" is for the adjoint conjugation †, which is rather denoted * in the mathematical literature.

For a finite dimensional Hilbert state $\mathcal{H}=\mathbb{C}^n$, the corresponding complex C*-algebra is the standard algebra of $n\times n$ complex matrices $\mathrm{M}(n,\mathbb{C})$. Reciprocally, any simple finite dimensional complex C*-algebra can be represented as a $\mathrm{M}(n,\mathbb{C})$ matrix algebra, and this representation is unique. The situation is more interesting (both mathematically and also for physics) in the infinite dimensional case. There exists C*-algebras (i.e. subalgebras $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ that satisfy the above conditions) that cannot be represented as the algebra of bounded operators of some "smaller" Hilbert space $\mathcal{H}'$. Moreover such C*-algebras may have several inequivalent irreducible representations over a Hilbert space. A simple example will be given with application to one dimensional quantum mechanics later.

The mathematical theory of operator algebras was initiated by F. J. Murray and J. von Neumann in the end of the 1930's. One of J. von Neumann's motivations was precisely to formulate more precisely the mathematics of quantum mechanics, in cases where the canonical formalism and the concept of "wave function" is not sufficient or well defined. A very interesting and useful subclass of C*-algebras is the class of the so-called W*-algebras or von Neumann algebras. They are very important both in mathematics and for the mathematical proper formulation of quantum field theories. They will be presented and discussed (a little bit) in section 3.7.

Finally a very important property of these operator algebra will be used in this chapter. Although C*-algebras are usually defined and studied as algebras of operators actng on a Hilbert space $\mathcal{H}$, they can be also defined in an abstract way, without reference to an underlying Hilbert space. In this approach, a C*-algebra is an abstract associative complex algebra $\mathcal{A}$, together with a conjugation * (acting as the standard conjugation $A \to A^\dagger$ for operators, and a norm $\|\cdot\|$ satisfying the same properties of the sup-norm that I discussed above. So

$$\text{abstract C}^*\text{-algebra} = (\mathcal{A}, {}^*, \|\cdot\|) \tag{3.1.5}$$

The two approach are equivalent, a famous mathematical result – the GNS construction – shows that one can reconstruct all the representations of the algebra as algebras of operators acting on a Hilbert space from its abstract definition and from its states (in the sense of quantum mixed states, i.e. positive linear normalized forms) that can be constructed on the abstract algebra. Thus the Hilbert space of pure states of a quantum system can be "reconstructed" from the observables of the system.

Standard references on operator algebras in the mathematical litterature are the books by J. Dixmier (1981, 1982) [Dix69], by Sakai (1971) [Sak71], and by P. de la Harpe and V. Jones (1995) [dlHJ95].

3.1.3 The algebraic approach

Let me know present the algebraic formulation and what I am going to do in this chapter. In the algebraic formulation, quantum mechanics is still constructed from the classical concepts of observables and states, but one makes the assumption that the observables are not commuting objects. They will generate an associative but non-commutative algebra. The properties of this algebra of observables, and the dynamics allowed by the theory, turns out to be quite constrained, in particular by enforcing causality, locality and unitarity in order to obtain physical theories consistent with special relativity (quantum field theories). The adequate mathematical object to treat quantum field theories in a mathematically consistent way is indeed the theory of algebras of operators, in particular C* and W*-algebras, and of their representations.

As we already explained, the mathematical theory of operator algebras started in the 1930's and was developed in the 1940's and 1950's, up to now. It is still a major and expanding field of mathematics. Its applications for quantum physics, in particular for quantum field theory and quantum statistical mechanics, were notably initiated by Segal in the end of the 1940's [Seg47b] and further developed by the creation of axiomatic quantum field theory (the Wightman's axioms) [SA00] and shortly after through the development of algebraic quantum field theory [HK64](the Haag-Kastler-Ruelle theory) in the 1960's.

The standard and excellent reference on the algebraic and axiomatic approaches to quantum field theory is the book by R. Haag, Local Quantum Physics, especially the second edition (1996) [Haa96]. Another good, but older, reference is the book by N. N. Bogoliubov; A. A. Logunov, A.I. Oksak and I.T. Todorov (1975, 1990) [BLOT90]. Another useful reference on axiomatic QFT is the famous book by R. F. Streater and A. S. Wightman (1964, 1989) [SA00] . Besides the mathematical references on operator algebras given above, some mathematical references more oriented towards mathematical physics and theoretical physics are the books by Bratteli and Robinson(1979) [BR02] (for statistical physics) and the books by A. Connes (1994) [Con94] and A. Connes and M. Marcolli [CM07] (for high energy physics, quantum gravity and string theory).

In this chapter my goal is to give a brief and heuristic presentation of the algebraic formulation of quantum theory. So I shall try to introduce more or less precisely the mathematical concepts, but with no attempts at mathematical rigor and precision in the derivations. In contrast with the usual (and useful) mathematical presentations, where the axioms and the principles are first stated and then discussed and applied, I shall try to motivate these axioms by logical and physical considerations, and "reconstruct" the algebraic formalism step by step, trying to precise at which steps the different physical principles that we expect/assume for a sensible physical theory enter in the game. Of peculiar importance are the principles of causality, of reversibility and of locality (both to ensure relativistic invariance and to enforce the fact that one can causally separate domains in space-time and decompose an extended physical system into its subparts).

Such a presentation is somehow original, and I hope that it will be useful. A particular feature of this approach is that I start from the concept of observables and states of a physical system, and try to reconstruct and justify the mathematical structure (why an algebra? why a conjugation? why a sup-norm that makes the set generated by the physical observables a C^*-algebra? etc..) . It then appears at that first stage that the natural structure that emerges is the structure of real C^*-algebras (i.e. algebras build on the field of real numbers $\mathbb{R}$, not necessarily on the field of complex numbers $\mathbb{C}$). Fortunately there is a mathematical theory of real C^*-algebras (less developped than the theory of complex C^*-algebras since the former ones are less interesting, and for many problems equivalent to the latter ones), that can be used as well. The only good reference on real C^*-algebras I am aware of is the monograph by Goodearl (1982) [Goo82], and I shall refer to it.

It is only at the second stage that I shall explain which physical requirements (basically these are the requirements of locality and of causal separability) enforce the use of complex algebras and of the standard complex Hilbert space formalism.

3.2 The algebra of observables

3.2.1 The mathematical principles

A quantum system is described by its observables, its states and a causal involution acting on the observables and enforcing constraints on the states. Rather than discussing the physics concepts behind these terms, let me first give the mathematical axioms and motivate them physically after.

3.2.1.a - Observables

The physical observables of the system generate a *real associative unital algebra* $\mathcal{A}$ (whose elements will still be denoted "observables") . $\mathcal{A}$ is a linear vector space

$$\mathbf{a},\mathbf{b}\in\mathcal{A}\quad \lambda,\mu\in\mathbb{R}\qquad \lambda\mathbf{a}+\mu\mathbf{b}\in A$$

with an associative product (distributive w.r.t the addition)

$$\mathbf{a},\mathbf{b},\mathbf{c}\in\mathcal{A}\qquad \mathbf{ab}\in A\qquad (\mathbf{ab})\mathbf{c}=\mathbf{a}(\mathbf{bc}) \tag{3.2.1}$$

and a multiplicative unity element $\mathbf{1}$ such that

$$\mathbf{1a}=\mathbf{a1}=\mathbf{a}\ ,\ \ \forall\ \mathbf{a}\in\mathcal{A} \tag{3.2.2}$$

I shall precise later what is meant by "physical observables".

3.2.1.b - The *-conjugation

There is an involution * on $\mathcal{A}$ (denoted conjugation). It is an anti-automorphism whose square is the identity. This means that

$$(\lambda\mathbf{a}+\mu\mathbf{b})^\star=\lambda\mathbf{a}^\star+\mu\mathbf{b}^\star \tag{3.2.3}$$

and

$$(\mathbf{a}^*)^*=\mathbf{a}\qquad (\mathbf{ab})^\star=\mathbf{b}^\star\mathbf{a}^\star \tag{3.2.4}$$

3.2.1.c - States

Each state φ associates to an observable $\mathbf{a}$ its expectation value $\varphi(\mathbf{a})\in\mathbb{R}$ in the state φ. The states satisfy

$$\varphi(\lambda\mathbf{a}+\mu\mathbf{b})=\lambda\varphi(\mathbf{a})+\mu\varphi(\mathbf{b}) \tag{3.2.5}$$

and

$$\varphi(\mathbf{a}^\star)=\varphi(\mathbf{a})\qquad \varphi(\mathbf{1})=1\qquad \varphi(\mathbf{a}^\star\mathbf{a})\geq 0 \tag{3.2.6}$$

The set of states is denoted $\mathcal{E}$. It is natural to assume that it allows to discriminate between observables, i.e.

$$\forall\ \mathbf{a}\neq\mathbf{b}\in\mathcal{A}\ (\text{and }\neq\mathbf{0}),\exists\ \varphi\in\mathcal{E}\text{ such that }\varphi(\mathbf{a})\neq\varphi(\mathbf{b}) \tag{3.2.7}$$

For the unfamiliar reader the symbol $\forall$ means "for all" and $\exists$ means "there exists".

I do not discuss the concepts of time and dynamics at that stage. This will be done later. I first discuss the relation between these "axioms" and the physical concepts of causality, reversibility and probabilities.

3.2.2 Physical discussion

3.2.2.a - Observables and causality

In quantum physics, the concept of physical observable corresponds both to an operation on the system (measurement) and to the response on the system (result on the measure), but I shall not elaborate further. I already discussed why in classical physics observables form a real commutative algebra. The removal of the commutativity assumption is the simplest modification imaginable compatible with the uncertainty principle (Heisenberg 1925).

Keeping the mathematical structure of an associative but non commutative algebra reflects the assumption that there is still some concept of "causal ordering" between observables (not necessarily physical), in a formal but loose sense. Indeed the multiplication and its associativity means that we can "combine" successive observables, e.g. $\mathbf{ab} \simeq (\mathbf{b}$ then $\mathbf{a})$, in a linear process such that $((\mathbf{c}$ then $\mathbf{b})$ then $\mathbf{a}) \simeq (\mathbf{c}$ then $(\mathbf{b}$ then $\mathbf{a}))$. This "combination" is different from the concept of "successive measurement".

Without commutativity the existence of an addition law is already a non trivial fact, it means that we can "combine" two non compatible observations into a new one whose mean value is always the sum of the first two mean values.

These operations of addition and mutiplication of observables are in fact more natural in the context of relativistic theories, via the analyticity properties of correlation functions and the short time and short distance expansions.

3.2.2.b - The *-conjugation and reversibility

The existence of the involution (or conjugation) * is the second and very important feature of quantum physics. It implies that although the observables do not commute, there is no favored arrow of time (or causal ordering) in the formulation of a physical theory. To any causal description of a system in term of a set of observables $\{\mathbf{a}, \mathbf{b}, \dots\}$ corresponds an equivalent "anti-causal" description it terms of conjugate observables $\{\mathbf{a}^*, \mathbf{b}^*, \dots\}$. In other word the properties of the * conjugation amount to assuming microscopic reversibility. Again although there is no precise concept of time or dynamics yet, the involution * must not be confused with the time reversal operator $\mathbf{T}$ (which may or may not be a symmetry of the dynamics).

3.2.2.c - States, mesurements and probabilities

The states φ are the simple generalisation of the classical concept of statistic (or probabilistic) states describing our knowledge of a system through the expectation value of the outcome of measurements for each possible observables. At that stage I do not assume anything about whether there are states such that all the values of the observables can be determined or not. Thus a state can be viewed also as the characterization of all the information which can be extracted from a system through a measurement process (this is the point of view often taken in quantum information theory). I do not consider how states are prepared, nor how the measurements are performed (this is the object of the subpart of quantum theory known as the theory of quantum measurement) and just look at the consistency requirements on the outcome of measurements.

The "expectation value" $\varphi(\mathbf{a})$ of an observable $\mathbf{a}$ has not been defined precisely either at that stage. In statistics the expectation of an observable can be considered as well as given by the average of the outcome of measurements $\mathbf{a}$ over many realisations of the system in the same state (frequentist point of view) or as the sum over the possible outcomes a_i times the plausibility for the outcomes in a given state (Bayesian point of view). As already done in 2.5.2, and we shall see in the discussion of the algebraic formalism and of the quantum logic formalism, discussing the role of reversibility amounts to treat simultaneously the statistics for predictions (which can be done using the frequentist point of view) and the statistics for retrodictions (which is better done using the Bayesian point of view). Therefore in my opinion both point of views have to be considered on a same footing, and are somehow unified, in the quantum formalism.

The linearity of the states considered as function over the observables Eq. 3.2.5 follows from (or is equivalent to) the assumptions that the observables form a linear vector space on $\mathbb{R}$. The very important condition in 3.2.6

$$\varphi(\mathbf{a}^*) = \varphi(\mathbf{a})$$

for any $\mathbf{a}$ follows from the assumption of reversibility discussed above. If it was not satisfied, there would be observables that would allow to favor one causal ordering, irrespective of the possible dynamics and of the possible states of the system.

The positivity condition $\varphi(\mathbf{a}^*\mathbf{a}) \geq 0$ ensures that the states have a probabilistic interpretation, so that on any state the expectation value of a positive observable is positive, and that there are no negative probabilities, in other word it will ensure unitarity. It is the simplest consistent positivity condition compatible with reversibility, and in fact the only possible without assuming more structure on the observables. Of course the condition $\varphi(\mathbf{1}) = 1$ is the normalisation condition for probabilities.

3.2.3 Physical observables and pure states

Three important concepts follow from the mathematical principles assumed in 3.2.1, and tentatively justified physically in 3.2.2.

3.2.3.a - Physical (symmetric) observables:

An observable $\mathbf{a} \in \mathcal{A}$ is denoted symmetric (self adjoint, or self-conjugate) if $\mathbf{a}^* = \mathbf{a}$. Symmetric observables correspond to the physical observables, which are actually measurable. Observable such that $\mathbf{a}^* = -\mathbf{a}$ are denoted skew-symmetric (anti-symmetric or anti-conjugate). They do not correspond to physical observables since for such observables $\varphi(a) = 0$ but they must be included formalism in order to have a consistent algebraic structure.

3.2.3.b - Pure states:

The set of states $\mathcal{E}$ is a convex subset of the set of real linear forms on $\mathcal{A}$ (the dual of $\mathcal{A}$). Indeed if φ_1 et φ_2 are two states and $0 \leq x \leq 1$, $\varphi = x\varphi_1 + (1-x)\varphi_2$ is also a state.

This corresponds to the fact that any statistical mixture of two statistical mixtures is a statistical mixture. The extremal points in $\mathcal{E}$, i.e. the states which cannot be written as a statistical mixture of two differents states in $\mathcal{E}$, are called the pure states. Non pure states are called mixed states. If a system is in a pure state one cannot get more information from this system than the information that what we have already.

3.2.3.c - Bounded observables

I just need to impose two additional technical and natural assumptions: (i) for any observable $\mathbf{a} \neq 0$, there is a state φ such that $\varphi(\mathbf{a}^*\mathbf{a}) > 0$, if this is not the case, the observable $\mathbf{a}$ is indistinguishable from the observable 0 (which is always false); (ii) $\sup_{\varphi\in\mathcal{E}} \varphi(\mathbf{a}^*\mathbf{a}) < \infty$, i.e. I restrict $\mathcal{A}$ to the algebra of bounded observables, this will be enough to characterize the system.

3.3 The C*-algebra of observables

The involution * et the existence of the states $\varphi \in \mathcal{E}$ on $\mathcal{A}$ strongly constrain the structure of the algebra of observables and of its representations. Indeed this allows to associate to $\mathcal{A}$ a unique norm $||\cdot||$ with some specific properties. This norm makes $\mathcal{A}$ a C*-algebra, and more precisely a real abstract C*-algebra. This structure justifies the standard representation of quantum mechanics where pure states are elements of an Hilbert space and physical observables are self-adjoint operators.

3.3.1 The norm on observables, $\mathcal{A}$ is a Banach algebra

Let us consider the function $\mathbf{a} \to ||\mathbf{a}||$ from $\mathcal{A} \to \mathbb{R}^+$ defined by

$$||\mathbf{a}||^2 = \sup_{\text{states}\,\varphi\in\mathcal{E}} \varphi(\mathbf{a}^\star\mathbf{a}) \tag{3.3.1}$$

We have assumed that $||\mathbf{a}|| < \infty$, $\forall a \in \mathcal{A}$ and that $||\mathbf{a}|| = 0 \iff \mathbf{a} = 0$ (this is equivalent to $\mathbf{a} \neq 0 \implies \exists\varphi \in \mathcal{E}$ such that $\varphi(\mathbf{a}^*\mathbf{a}) \neq 0$). It is easy to show that $||\cdot||$ is a norm on $\mathcal{A}$, such that

$$||\lambda\mathbf{a}|| = |\lambda|\,||\mathbf{a}|| \quad , \qquad ||\mathbf{a}+\mathbf{b}|| \leq ||\mathbf{a}|| + ||\mathbf{b}|| \quad , \qquad ||\mathbf{ab}|| \leq ||\mathbf{a}||\,||\mathbf{b}|| \tag{3.3.2}$$

If $\mathcal{A}$ is not closed for this norm, we can take its completion $\overline{\mathcal{A}}$. The algebra of observables is therefore a real Banach algebra.

Derivation:

The first identity in 3.3.2 comes from the definition and the linearity of states. Taking $\mathbf{c} = x\mathbf{a} + (1-x)\mathbf{b}$ and using the positivity of $\varphi(c^*c) \geq 0$ for any $x \in \mathbb{R}$ one obtains Schwartz inequality $\varphi(a^*b)^2 = \varphi(a^*b)\varphi(b^*a) \leq \varphi(a^*a)\varphi(b^*b)$, $\forall\, a, b \in \mathbf{A}$. This implies the second inequality. The third inequality comes from the fact that if $\varphi \in \mathcal{E}$ and $b \in \mathcal{A}$ are such that $\varphi(b^*b) > 0$, then φ_b defined by $\varphi_b(a) = \frac{\varphi(b^*ab)}{\varphi(b^*b)}$ is also a state for $\mathcal{A}$. Then $||ab||^2 = \sup_\varphi \varphi(b^*a^*ab) = \sup_\varphi (\varphi_b(a^*a)\varphi(b^*b)) \leq \sup_\psi \psi(a^*a) \cdot \sup_\varphi \varphi(b^*b) = ||a||^2\,||b||^2$.

3.3.2 The observables form a real C^*-algebra

Moreover the norm satisfies the two non-trivial properties.

$$||\mathbf{a}^*\mathbf{a}|| = ||\mathbf{a}||^2 = ||\mathbf{a}^*||^2 \tag{3.3.3}$$

and

$$\mathbf{1}+\mathbf{a}^*\mathbf{a} \quad \text{is invertible} \quad \forall\, \mathbf{a}\in\mathcal{A} \tag{3.3.4}$$

These two properties are equivalent to state that $\mathcal{A}$ is a real C*-algebra. The first condition 3.3.3 is sometimes called the C* condition. It has already been discussed for complex algebras. The second condition 3.3.4 is specific to real algebras. For a discussion of the definition of real C*-algebras and of their properties, that will be used below, I refer to the by Goodearl [Goo82].

Derivation:

One has $||a^*a|| \leq ||a||\,||a^*||$. Schwartz inequality implies that $\varphi(a^*a)^2 \leq \varphi\big((a^*a)^2\big)\varphi(1)$, hence $||a||^2 \leq ||a^*a||$. This implies (3.3.3).

To obtain (3.3.4), notice that if $1+a^*a$ is not inversible, there is a $b\neq 0$ such that $(1+a^*a)b=0$, hence $b^*b+(ab)^*(ab)=0$. Since there is a state φ such that $\varphi(b^*b)\neq 0$, either $\varphi(b^*b)<0$ or $\varphi((ab)^*(ab)<0$, this contradicts the positivity of states.

The full consequences of the fact that $\mathcal{A}$ is a real C*-algebra will be discussed in next subsection. Before this let me introduce first the concept of spectrum of an observable.

3.3.3 Spectrum of observables and results of measurements

Here I discuss in a slightly more precise way the relationship between the spectrum of observables and results of measurements. The spectrum [1] of an element $\mathbf{a}\in\mathcal{A}$ is defined as

$$\mathrm{Sp}^{\mathbb{C}}(\mathbf{a}) = \{z\in\mathbb{C}:\ (z-\mathbf{a}) \text{ not inversible in the complexified algebra } \mathcal{A}_{\mathbb{C}} \text{ of } \mathcal{A}\}\ .$$

The spectral radius of $\mathbf{a}$ is defined as

$$r^{\mathbb{C}}(\mathbf{a}) = \sup(|z|;\ z\in\mathrm{Sp}^{\mathbb{C}}(\mathbf{a}))$$

For a real C*-algebra it is known that the norm $||\cdot||$ defined by 3.3.1 is

$$||a||^2 = r^{\mathbb{C}}(\mathbf{a}^*\mathbf{a})$$

Moreover the spectrum of any physical observable (symetric) is real

$$\mathbf{a}=\mathbf{a}^* \implies \mathrm{Sp}^{\mathbb{C}}(\mathbf{a})\subset\mathbb{R}$$

and for any $\mathbf{a}$, the product $\mathbf{a}^*\mathbf{a}$ is a symmetric positive element of $\mathcal{A}$, i.e. its spectrum is real and positive

$$\mathrm{Sp}^{\mathbb{C}}(\mathbf{a}^*\mathbf{a})\subset\mathbb{R}^+$$

1. The exact definition of the spectrum is slightly different for a general real Banach algebra.

Finally, for any (continuous) real function $F\ \mathbb{R} \to \mathbb{R}$ and any $\mathbf{a} \in \mathcal{A}$ one can define the observable $F(\mathbf{a})$. Now consider a physical observable $\mathbf{a}$. Physically, measuring $F(\mathbf{a})$ amounts to measure $\mathbf{a}$ and when we get the real number A as the result of the measurement, return $F(A)$ as a result of the measure of $F(\mathbf{a})$ (this is fully consistent with the algebraic definition of $F(\mathbf{a})$ since $F(\mathbf{a})$ commutes with $\mathbf{a}$). Then it can be shown easily that the spectrum of $F(\mathbf{a})$ is the image by F of the spectrum of $\mathbf{a}$, i.e.

$$\mathrm{Sp}^{\mathbb{C}}(F(\mathbf{a})) = F(\mathrm{Sp}^{\mathbb{C}}(\mathbf{a}))$$

In particular, assuming that the spectrum is a discrete set of points, let us choose for F the function

$$F[\mathbf{a}] = 1/(z\mathbf{1} - \mathbf{a})$$

For any state φ, the expectation value of this observable on the state φ is

$$E_\varphi(z) = \varphi(1/(z\mathbf{1} - \mathbf{a})$$

and is an analytic function of z away from the points of the spectrum $\mathrm{Sp}^{\mathbb{C}}(\mathbf{a})$). Assuming that the singularity at each z_p is a single pole, the residue of $E_\varphi(z)$ at z_p is nothing but

$$\begin{aligned} Res_{z_p} E_\varphi &= \varphi(\delta(\mathbf{a} - z_p\mathbf{1})) \\ &= \text{probabiliy to obtain } z_p \text{ when measuring } \mathbf{a} \text{ on the state } \varphi \end{aligned} \tag{3.3.5}$$

with $\delta(z)$ the Dirac distribution.

This implies that for any physical observable $\mathbf{a}$, its spectrum is the set of all the possible real numbers z_p returned by a measurement of $\mathbf{a}$. This is one of the most important axioms of the standard formulation of quantum mechanics, and we see that it is a consequence of the axioms in this formulation. Of course the probability to get a given value z_p (an element of the spectrum) depends on the state φ of the system, and it is given by 3.3.5 which is nothing but the Born rule, as obtained from this abstract definiton of the states.

3.3.4 Complex C*-algebras

The theory of operator algebras (C*-algebras and W*-algebras) and their applications almost exclusively deal with complex algebras, i.e. algebras over $\mathbb{C}$. In the case of quantum physics we shall see a bit later why quantum mechanics and quantum field theories must be represented by complex C*-algebras. I give here some mathematical definitions.

Abstract complex C*-algebras and complex states ϕ are defined as in 3.2.1. A complex C*-algebra $\mathfrak{A}$ is a complex associative involutive algebra. The involution is now anti-linear

$$(\lambda\mathbf{a} + \mu\mathbf{b})^\star = \bar{\lambda}\mathbf{a}^\star + \bar{\mu}\mathbf{b}^\star \qquad \lambda, \mu \in \mathbb{C}$$

$\bar{z}$ denotes the complex conjugate of a complex number z. $\mathfrak{A}$ is equipped with a norm $\mathbf{a} \to ||\mathbf{a}||$ which still satisfy the C* condition 3.3.3,

$$||\mathbf{a}^*\mathbf{a}|| = ||\mathbf{a}||^2 = ||\mathbf{a}^*||^2 \tag{3.3.6}$$

and it is closed under this norm. The condition 3.3.4 is not necessary any more (it follows from 3.3.6 for complex algebras).

The states are defined now as the complex linear forms ϕ on $\mathfrak{A}$ which satisfy

$$\phi(\mathbf{a}^*) = \overline{\phi(\mathbf{a})} \qquad \phi(\mathbf{1}) = 1 \qquad \phi(\mathbf{a}^*\mathbf{a}) \geq 0 \tag{3.3.7}$$

Every complex C*-algebra $\mathfrak{A}$ can be considered as a real C*-algebra $\mathcal{A}_{\mathbb{R}}$ (by considering $\mathrm{i} = \sqrt{-1}$ as an element $\mathbf{i}$ of the center of $\mathcal{A}_{\mathbb{R}}$) but the reverse is not true in general. For instance the algebra of 2×2 real matrices $\mathrm{M}_2(\mathbb{R})$ is not a complex algebra.

However if a real algebra $\mathcal{A}_{\mathbb{R}}$ has an element (denoted $\mathbf{i}$) in its center $\mathcal{C}$ that is isomorphic to $\sqrt{-1}$, i.e. $\mathbf{i}$ is such that

$$\mathbf{i}^* = -\mathbf{i}\ , \quad \mathbf{i}^2 = -\mathbf{1}\ , \quad \mathbf{ia} = \mathbf{ai} \quad \forall\ \mathbf{a} \in \mathcal{A}_{\mathbb{R}} \tag{3.3.8}$$

then the algebra $\mathcal{A}_{\mathbb{R}}$ is isomorphic to a complex algebra $\mathcal{A}_{\mathbb{C}} = \mathfrak{A}$. One identifies $x\mathbf{1} + y\mathbf{i}$ with the complex scalar $z = x + \mathrm{i}y$. The conjugation *, which is linear on $\mathcal{A}_{\mathbb{R}}$, is now anti-linear on $\mathcal{A}_{\mathbb{C}}$. One can associate to each $\mathbf{a} \in \mathcal{A}_{\mathbb{R}}$ its real and imaginary part

$$\mathrm{Re}(\mathbf{a}) = \frac{\mathbf{a} + \mathbf{a}^*}{2}\ , \ \ \mathrm{Im}(\mathbf{a}) = \mathbf{i}\frac{\mathbf{a}^* - \mathbf{a}}{2} \tag{3.3.9}$$

and write it in $\mathcal{A}_{\mathbb{C}}$

$$\mathbf{a} = \mathrm{Re}(\mathbf{a}) + \mathrm{i}\ \mathrm{Im}(\mathbf{a}) \tag{3.3.10}$$

To any real state (and in fact to any real linear form) $\varphi_{\mathbb{R}}$ on $\mathcal{H}_{\mathbb{R}}$ one associates the complex state (the complex linear form) $\phi_{\mathbb{C}}$ on $\mathcal{A}_{\mathbb{C}}$ defined as

$$\phi_{\mathbb{C}}(\mathbf{a}) = \varphi_{\mathbb{R}}(\mathrm{Re}(\mathbf{a})) + \mathrm{i}\,\varphi_{\mathbb{R}}(\mathrm{Im}(\mathbf{a})) \tag{3.3.11}$$

It has the expected properties for a complex state on the complex algebra $\mathfrak{A}$.

3.4 The GNS construction, operators and Hilbert spaces

General theorems show that abstract C*-algebras can always be represented as algebra of operators on some Hilbert space. This is the main reason why pure states are always represented by vectors in a Hilbert space and observables as operators. Let me briefly consider how this works.

3.4.1 Finite dimensional algebra of observables

Let me first consider the case of finite dimensional algebras, which corresponds to quantum system with a finite number of independent quantum states. This is the case considered in general in quantum information theory.

If $\mathcal{A}$ is a finite dimensional real algebra, one can show by purely algebraic methods that $\mathcal{A}$ is a direct sum of matrix algebras over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the quaternions). See [Goo82] for details. The idea is to show that the C*-algebra conditions implies that the real algebra $\mathcal{A}$ is semi-simple (it cannot have a nilpotent two-sided ideal) and to use the Artin-Wedderburn theorem. One can even relax the positivity condition on states that $\varphi(\mathbf{a}^*\mathbf{a}) \geq 0$ for all $\mathbf{a} \in \mathcal{A}$, and replace it by the weaker positivity condition that $\varphi(\mathbf{a}^2) \geq 0$ only for physical symmetric observables such that $\mathbf{a} = \mathbf{a}^*$, which is physically somewhat more satisfactory (F. David unpublished, probably known in the math litterature...). This is physically more satisfactory in my opinion, since at that stage it is not completely obvious that any positive physical observable car be represented as the square of a physical observable). The conclusion is that any finite dimensional real C*-algebra is a direct sum of matrix algebras, of the form

$$\mathcal{A} = \bigoplus_i M_{n_i}(K_i) \qquad K_i = \mathbb{R},\ \mathbb{C},\ \mathbb{H} \tag{3.4.1}$$

The index i label the components of the center of the algebra. Any observable reads

$$\mathbf{a} = \oplus_i \mathbf{a}_i\ ,\ \ \mathbf{a}_i \in \mathcal{A}_i = M_{n_i}(K_i)$$

The multiplication corresponds to the standard matrix multiplication and the involution * to the standard conjugation (transposition, transposition+complex conjugation and transposition+conjugation respectively for real, complex and quaternionic matrices). One thus recovers the familiar matrix ensembles of random matrix theory.

Any state ω can be written as

$$\omega(\mathbf{a}) = \sum_i p_i\,\mathrm{tr}(\boldsymbol{\rho}_i \mathbf{a}_i) \qquad p_i \geq 0, \quad \sum_i p_i = 1$$

and the $\boldsymbol{\rho}_i$'s some symmetric positive normalised matrices in each $\mathcal{A}_i$

$$\boldsymbol{\rho}_i \in \mathcal{A}_i = M_{n_i}(K_i)\,, \quad \boldsymbol{\rho}_i = \boldsymbol{\rho}_i^*\,, \quad \mathrm{tr}(\boldsymbol{\rho}_i) = 1\,, \quad \boldsymbol{\rho}_i \geq 0$$

The algebra of observables is indeed a subalgebra of the algebra of operators on a finite dimensional real Hilbert space $\mathcal{H} = \bigoplus_i K_i^{n_i}$ ($\mathbb{C}$ and $\mathbb{H}$ being considered as 2 dimensional and 4 dimensional real vector spaces respectively). But it is not necessarily the whole algebra $\mathcal{L}(\mathcal{H})$. The system corresponds to a disjoint collection of standard quantum systems described by their Hilbert space $\mathcal{H}_i \simeq K_i^{\oplus n_i}$ and their algebra of observables $\mathcal{A}_i$. This decomposition is (with a bit of abuse of language) a decomposition into "superselection sectors"[2]. The $\boldsymbol{\rho}_i$ are the quantum density matrices corresponding to the state. The p_i's correspond to the classical probability to be in a given sector, i.e. in a state described by $(\mathcal{A}_i, \mathcal{H}_i)$.

A pure state is (the projection onto a) single vector $|\psi_i\rangle$ in a single sector $\mathcal{H}_i$. Linear superpositions of pure states in different sectors $|\psi\rangle = \sum_i c_i|\psi_i\rangle$ do not make sense, since they do not belong to the representation of $\mathcal{A}$. No observable $\mathbf{a}$ in $\mathcal{A}$ allows to discriminate between the seemingly-pure-state $|\psi\rangle\langle\psi|$ and the mixed state $\sum_i |c_i|^2|\psi_i\rangle\langle\psi_i|$. Thus the different sectors can be viewed as describing completely independent systems with no quantum correlations, in other word really parallel universes with no possible interaction or communication between them.

2. See below for a more precise definition and discussion. For many authors the term of superselection sectors is reserved to infinite dimensional algebras which do have inequivalent representations.

3.4.2 Infinite dimensional real algebra of observables

This result generalizes to the case of infinite dimensional real C*-algebras, but it is much more difficult to prove, analysis and topology enter in the game and the fact that the algebra is closed under the norm is crucial (for a physicist this is a natural requirement).

Theorem (Ingelstam NN [Ing64, Goo82]): For any real C*-algebra, there exists a real Hilbert space $\mathcal{H}$ such that $\mathcal{A}$ is isomorphic to a real symmetric closed real sub-algebra of the algebra $B(\mathcal{H})$ of bounded operators on $\mathcal{H}$.

Now any real algebra of symmetric operators on a real Hilbert space $\mathcal{H}$ may be extended (by standard complexification) into a complex algebra of self-adjoint operator on a Hilbert space $\mathcal{H}_{\mathbb{C}}$ on $\mathbb{C}$ and thus one can reduce the study of real algebra to the study of complex algebra. In particular the theory of representations of real C*-algebra is not really richer than that of complex C*-algebra and mathematicians usually. I shall consider only the later case.

I will discuss later why in quantum physics one should restrict oneself also to complex algebras. But note that in physics real (and quaternionic) algebra of observables do appear as the subalgebra of observables of some system described by a complex Hilbert space, subjected to some additional symmetry constraint (time reversal invariance $\mathbf{T}$ for real algebra, time reversal and an additional SU(2) invariance for quaternionic algebras).

3.4.3 The complex case, the GNS construction

Let me discuss more the case of complex C*-algebras, since their representation in term of Hilbert spaces are simpler to deal with. The famous GNS construction (Gelfand-Naimark-Segal [GN43, Seg47a]) allows to construct the representations of the algebra of observables in term of its pure states. It is interesting to see the basic ideas, since they allow to understand how the Hilbert space of physical pure states emerges from the abstract [3] concepts of observables and mixed states.

The idea is quite simple. To every state ϕ we associate a representation of the algebra $\mathcal{A}$ in a Hilbert space $\mathcal{H}_\phi$. This is done as follows. The state ϕ allows to define a bilinear form $\langle\,|\,\rangle_\phi$ on $\mathcal{A}$, considered as a vector space on $\mathbb{C}$, through

$$\langle \mathbf{a}|\mathbf{b}\rangle_\phi = \phi(\mathbf{a}^\star\mathbf{b}) \tag{3.4.2}$$

This form is positive ≥ 0 but is not strictly positive > 0, since there are in general isotropic (or null) vectors such that $\langle \mathbf{a}|\mathbf{a}\rangle_\phi = 0$. Thus $\mathcal{A}$, considered as a vector space equiped with this norm is only a pre-Hilbert space. However, thanks to the C*-condition, the set $\mathcal{I}_\phi$ of these null vectors form a linear subspace $\mathcal{I}_\phi$ of $\mathcal{A}$.

$$\mathcal{I}_\phi = \{\mathbf{a}\in\mathcal{A} : \langle \mathbf{a}|\mathbf{a}\rangle_\phi = 0\} \tag{3.4.3}$$

3. in the mathematic sense: they are not defined with reference to a given representation such as operators in Hilbert space, path integrals, etc.

Taking the completion of the quotient space of $\mathcal{A}$ by $\mathcal{I}_\phi$ one obtains a vector space $\mathcal{H}_\phi$

$$\mathcal{H}_\phi = \overline{\mathcal{A}/\mathcal{I}_\phi} \tag{3.4.4}$$

When there is no ambiguity, if $\mathbf{a}$ is an element of the algebra $\mathcal{A}$ (an observable), we denote by $|a\rangle$ the corresponding vector in the Hilbert space $\mathcal{H}_\phi$, that is the equivalent class of $\mathbf{a}$ in $\mathcal{H}_\phi$

$$|a\rangle = \{\mathbf{b} \in \mathcal{A} :\ \mathbf{b} - \mathbf{a} \in \mathcal{I}_\phi\} \tag{3.4.5}$$

On this space $\mathcal{H}_\phi$ the scalar product $\langle a|b\rangle$ is > 0 (and $\mathcal{H}_\phi$ is closed) hence $\mathcal{H}_\phi$ is a Hilbert space.

The algebra $\mathcal{A}$ acts linearily on $\mathcal{H}_\phi$ through the representation π_ϕ (in the space of bounded linear operators $B(\mathcal{H}_\phi)$ on $\mathcal{H}_\phi$) defined as

$$\pi_\phi(\mathbf{a})|b\rangle = |ab\rangle \tag{3.4.6}$$

If one considers the vector $|\xi_\phi\rangle = |1\rangle \in \mathcal{H}_\phi$ (the equivalence class of the operator identity $\mathbf{1} \in \mathcal{A}$), it is of norm 1 and it is such that

$$\phi(\mathbf{a}) = \langle \xi_\phi|\pi_\phi(\mathbf{a})|\xi_\phi\rangle \tag{3.4.7}$$

(this follows basically from the definition of the representation). Moreover this vector $\xi_\phi\rangle$ is cyclic, this means that the action of the operators on this vector allows to recover the whole Hilbert space $\mathcal{H}_\phi$, more precisely

$$\overline{\pi_\phi(\mathcal{A})|\xi_\phi\rangle} = \mathcal{H}_\phi \tag{3.4.8}$$

However this representation is in general neither faithful (different observables may be represented by the same operator, i.e. the mapping π_ϕ is not injective), nor irreducible ($\mathcal{H}_\phi$ has invariant subspaces). The most important result of the GNS construction is the following theorem

Theorem (Gelfand-Naimark 43): The representation π_ϕ is irreducible if and only if ϕ is a pure state.

The proof is standard and may be found in [dlHJ95] This theorem has far reaching consequences. First it implies that the algebra of observables $\mathcal{A}$ has always a faithful representation in some big Hilbert space $\mathcal{H}$. Any irreducible representation π of $\mathcal{A}$ in some Hilbert space $\mathcal{H}$ is unitarily equivalent to the GNS representation π_ϕ constructed from a unit vector $|\xi\rangle \in \mathcal{H}$ by considering the state

$$\phi(\mathbf{a}) = \langle \xi|\pi(\mathbf{a})|\xi\rangle$$

Equivalent pure states Two pure states ϕ and ψ are equivalent if their GNS representations π_ϕ and π_ψ are equivalent. Then ϕ and ψ are unitarily equivalent, i.e. there is a unitary element $\mathbf{u}$ of $\mathcal{A}$ ($\mathbf{u}^*\mathbf{u} = \mathbf{1}$) such that $\phi(\mathbf{a}) = \psi(\mathbf{u}^*\mathbf{a}\mathbf{u})$ for any $\mathbf{a}$. As a consequence, to this pure state ψ (which is unitarily equivalent to ϕ) is associated a unit vector $|\psi\rangle = \pi_\phi(\mathbf{u})|\xi_\phi\rangle$ in the Hilbert space $\mathcal{H} = \mathcal{H}_\phi$, and we have the representation

$$\psi(\mathbf{a}) = \langle \psi|A|\psi\rangle \quad , \qquad A = \pi_\phi(\mathbf{a}) \tag{3.4.9}$$

In other word, all pures states which are equivalent can be considered as projection operators $|\psi\rangle\langle\psi|$ on some vector $|\psi\rangle$ in the same Hilbert space $\mathcal{H}$. Any observable $\mathbf{a}$ is represented by some bounded operator A and the expectation value of this observable in the state ψ is given by the Born formula 3.4.9. Equivalent classes of equivalent pure states are in one to one correspondence with the irreducible representations of the algebra of observables $\mathcal{A}$.

The standard explicit formulation of quantum mechanics in terms of operators, state vectors and density matrices is thus recovered from the abstract formulation

3.5 Why complex algebras?

In the mathematical presentation of the formalism that I gave here, real algebras play the essential role. However it is known that quantum physics must be described by complex algebras. There are several arguments that point towards the necessity of complex algebras, besides the fact that this actually works. Indeed one must still take into account some essential physical features of the quantum word: time, dynamics and locality.

3.5.1 Dynamics:

Firstly, if one wants the quantum system to have a "classical limit" corresponding to a classical Hamiltonian system, one would like to have conjugate observables P_i, Q_i whose classical limit are conjugate coordinates p_i, q_i with a correspondence between the quantum commutators and the classical Poisson brackets

$$[Q,P] \;\to\; \mathrm{i}\,\{p,q\} \tag{3.5.1}$$

Thus anti-symmetric operators must be in one to one correspondence with symmetric ones. This is possible only if the algebra of operators is a complex one, i.e. if it contains an $\mathbf{i}$ element in its center.

Another, but related, argument goes as follows: if one wants to have a time evolution group of inner automorphism acting on the operators and the states, it is given by unitary evolution operators $U(t)$ of the form

$$U(t)=\exp(tA) \quad , \qquad A=-A^{*} \tag{3.5.2}$$

This corresponds to have an Hamiltonian dynamics with a physical observable corresponding to a conserved energy, and given by a Schrödinger equation. This is possible only if the algebra is complex, so that we can write

$$A=-\mathrm{i}H \tag{3.5.3}$$

There has been various attempts to construct realistic quantum theories of particles or fields based on strictly real Hilbert spaces, most notably by Stueckelberg and his collaborators in the '60. See [Stu60]. None of them is really satisfying.

3.5.2 Locality and separability:

Another problem with real algebras comes from the requirement of locality in quantum field theory, and to the related concept of separability of subsystems. Space-time locality will be discussed a bit more later on. But there is already a problem with real algebras when one wants to characterize the properties of a composite system out of those of its subconstituents. As far as I know, this was first pointed out by Araki, and recovered by various people, for instance by Wooter in [Woo90] (see Auletta [Aul01] page 174 10.1.3).

Let me consider a system $\mathcal{S}$ which consists of two separated subsystem $\mathcal{S}_1$ and $\mathcal{S}_2$. Note that in QFT a subsystem is defined by its subalgebra of observables and of states. These are for instance the "system" generated by the observables in two causally separated regions. Then the algebra of observables $\mathcal{A}$ for the total system $1+2$ is the tensor product of the two algebras $\mathcal{A}_1$ and $\mathcal{A}_2$

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \tag{3.5.4}$$

which means that $\mathcal{A}$ is generated by the linear combinations of the elements $\mathbf{a}$ of the form $\mathbf{a}_1 \otimes \mathbf{a}_2$.

Let me now assume that the algebras of observables $\mathcal{A}_1$ and $\mathcal{A}_2$ are (sub)algebras of the algebra of operators on some *real* Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. The Hilbert space of the whole system is the tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Observables are represented by operators A, and physical (symmetric) operators $\mathbf{a} = \mathbf{a}^*$ correspond to symmetric operators $A = A^T$. Now it is easy to see that the physical (symetric) observables of the whole system are generated by the products of pairs of observables(A_1, A_2) of the two subsystems which are of the form

$$A_1 \otimes A_2 \quad \text{such that} \quad \begin{cases} A_1 \text{ and } A_2 \text{ are both symmetric} \\ \qquad\qquad \text{or} \\ A_1 \text{ and } A_2 \text{ are both skew-symmetric} \end{cases} \tag{3.5.5}$$

In both case the product is symmetric, but these two cases do not generate the same observables. This is different from the case of algebras of operators on *complex* Hilbert spaces, where all symmetric operators on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ are generated by the tensor products of the form

$$A_1 \otimes A_2 \quad \text{such that } A_1 \text{ and } A_2 \text{ are symmetric} \tag{3.5.6}$$

In other word, if a quantum system is composed of two independent subsystems, and the physics is described by a real Hilbert space, there are physical observables of the big system which cannot be constructed out of the physical observables of the two subsystems! This would turn into a problem with locality, since one could not characterize the full quantum state of a composite system by combining the results of separate independent measurements on its subparts. Note that this is also related to the idea of quantum tomography.

3.5.3 Quaternionic Hilbert spaces:

There has been also serious attempts to build quantum theories (in particular of fields) based on quaternionic Hilbert spaces, both in the ′60 and more recently by

S. Adler [Adl95]. One idea was that the SU(2) symmetry associated to quaternions could be related to the symmetries of the quark model and of some gauge interaction models. These models are also problematic. In this case there are less physical observables for a composite system that those one can naively construct out of those of the subsystems, in other word there are many non trivial constraints to be satisfied. A far as I know, no satisfying theory based on $\mathbb{H}$, consistent with locality and special relativity, has been constructed.

3.6 Superselection sectors

3.6.1 Definition

In the general infinite dimensional complex case the decomposition of an algebra of observables $\mathcal{A}$ along its center $Z(\mathcal{A})$ goes in a similar way as in the finite dimensional case. One can write something like

$$\mathcal{A} = \int_{c\in\mathcal{A}'} \mathcal{A}_c \tag{3.6.1}$$

where each $\mathcal{A}_c$ is a simple C*-algebra.

A very important difference with the finite dimensional case is that an infinite dimensional C*-algebra $\mathcal{A}$ has in general many inequivalent irreducible representations in a Hilbert space. Two different irreducible representations π_1 and π_2 of $\mathcal{A}$ in two subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of a Hilbert space $\mathcal{H}$ are generated by two unitarily inequivalent pure states φ_1 and φ_2 of $\mathcal{A}$. Each irreducible representation π_i and the associated Hilbert space $\mathcal{H}_i$ is called a *superselection sector*. The great Hilbert space $\mathcal{H}$ generated by all the unitarily inequivalent pure states on $\mathcal{A}$ is the direct sum of all superselections sectors. The operators in $\mathcal{A}$ do not mix the different superselection sectors. It is however often very important to consider the operators in $\mathcal{B}(\mathcal{H})$ which mixes the different superselection sectors of $\mathcal{A}$ while respecting the structure of the algebra $\mathcal{A}$ (i.e. its symmetries). Such operators are called *intertwinners*.

3.6.2 A simple example: the particle on a circle

One of the simplest examples of superselection sector is the nonrelativistic particle on a one dimensional circle. Let us first consider the particle on a line. The two conjugate operators $\mathbf{Q}$ and $\mathbf{P}$ obey the canonical commutation relations

$$[\mathbf{Q},\mathbf{P}] = \mathrm{i} \tag{3.6.2}$$

They are unbounded, but their exponentials

$$\mathbf{U}(k) = \exp(\mathrm{i}k\mathbf{Q}) \quad , \qquad \mathbf{V}(x) = \exp(\mathrm{i}x\mathbf{Q}) \tag{3.6.3}$$

generates a C*-algebra. Now a famous theorem by Stone and von Neumann states that all representations of their commutation relations are unitary equivalent. In other word, there is only one way to quantize the particle on the line, given by canonical quantization and the standard representation of the operators acting on the Hilbert space of functions on $\mathbb{R}$.

$$\mathbf{Q} = x \quad , \qquad \mathbf{P} = \frac{1}{\mathrm{i}}\frac{\partial}{\partial x} \tag{3.6.4}$$

Now, if the particle is on a circle with radius 1, the position x becomes an angle θ defined mod. 2π. The operator $\mathbf{U}(k)$ is defined only for integer momenta $k = 2\pi n$, $n \in \mathbb{Z}$. The corresponding algebra of operators has now inequivalent irreducible representations, indexed by a number Φ. Each representation π_Φ corresponds to the representation of the $\mathbf{Q}$ and $\mathbf{P}$ operators acting on the Hilbert space $\mathcal{H}$ of functions $\psi(\theta)$ on the circle as

$$\mathbf{Q} = \theta \quad , \qquad \mathbf{P} = \frac{1}{\mathrm{i}}\frac{\partial}{\partial \theta} + A \quad , \qquad A = \frac{\Phi}{2\pi} \tag{3.6.5}$$

So each superselection sector describes the quantum dynamics of a particle with unit charge $e = 1$ on a circle with a magnetic flux Φ. No global unitary transformation (acting on the Hilbert space of periodic functions on the circle) can map one superselection sector onto another one. Indeed this would correspond to the unitary transformation

$$\psi(\theta) \to \psi(\theta)\,\mathrm{e}^{\mathrm{i}\theta\,\Delta A} \tag{3.6.6}$$

and there is a topological obstruction if ΔA is not an integer. Here the different superselection sectors describe different "topological phases" of the same quantum system.

This is of course nothing but the famous Aharonov-Bohm effect [ES49][AB59]. Note that this formulation of the Aharonov-Bohm effect in the algebraic formalism does not contradict the usual formulation in the standard formulation in term of a change of boundary conditions for the wave-functions in real space.

3.6.3 General discussion

The notion of superselection sector was first introduced by Wick, Wightman and Wigner in 1952. They observed (and proved) that is is meaningless in a quantum field theory like QED to speak of the superposition of two states ψ_1 and ψ_2 with integer and half integer total spin respectively, since a rotation by 2π changes by (-1) the relative phase between these two states, but does not change anything physically. This apparent paradox disappear when one realizes that this is a similar situation than above. No physical observable allows to distinguish a linear superposition of two states in different superselection sectors, such as $|1\ \mathrm{fermion}\rangle + |1\ \mathrm{boson}\rangle$ from a statistical mixture of these two states $|1\ \mathrm{fermion}\rangle\langle 1\ \mathrm{fermion}|$ and $|1\ \mathrm{boson}\rangle\langle 1\ \mathrm{boson}|$. Indeed, any operator creating or destroying just one fermion is not a physical operator (bur rather an intertwining operator), but of course an operator creating or destroying a pair of fermions (or rather a pair fermion-antifermion) is physical.

The use of superselection sectors have been sometimes criticized in high energy physics (see a discussion on its use in relation with continuous symmetries in [Wei05]). Superselection sectors are nevertheless a very important feature of the mathematical formulation of quantum field theories, but they do have also physical significance. One encounters superselection sectors in quantum systems with an infinite number of states (non-relativistic or relativistic) as soon as

— the system may be in different phases (for instance in a statistical quantum system with spontatneous symmetry breaking);
— the system has global or local gauge symmetries and sectors with different charges Q_a (abelian or non abelian);
— the system contains fermions;
— the system exhibits different inequivalent topological sectors, this includes the simple case of a particle on a ring discussed above (the Aharonov-Bohm effect), but also gauge theories with θ-vacua;
— more generally, at a mathematical level, a given QFT for different values of its couplings or the masses of particle may corresponds to different superselection sectors of the same operator algebra (Haag's theorem).
— superselection sectors have also been used to discuss measurements in quantum mechanics and the quantum-to-classical transition.

Thus one should keep in mind that the abstract algebraic formalism contains as a whole the different possible states, phases and dynamics of a quantum system, while a given representation describes a subclass of states or of possible dynamics.

3.7 von Neumann algebras

A special class of C*-algebras, the so-called von Neumann algebras or W*-algebras, is of special interest in mathematics and for physical applications. As far as I know these were the algebras of operators originally studied by Murray and von Neumann (the ring of operators). Here I just give some definitions and some motivations, without details or applications.

3.7.1 Definitions

There are several equivalent definitions, I give here three classical definitions. The first two refer to an explicit representation of the algebra as an algebra of operators on a Hilbert space, but the definition turns out to be independent of the representation. The third one depends only on the abstract definition of the algebra.

Weak closure: $\mathcal{A}$ a unital *- sub algebra of the algebra of bounded operators $\mathcal{L}(\mathcal{H})$ on a complex Hilbert space $\mathcal{H}$ is a W*-algebra iff $\mathcal{A}$ is closed under the weak topology, namely if for any sequence A_n in $\mathcal{A}$, when all the individual matrix elements $\langle x|A_n|y\rangle$ converge towards some matrix element A_{xy}, this defines an operator in the algebra

$$\forall\ x,y \in \mathcal{H} \quad \langle x|A_n|y\rangle \to A_{xy} \qquad \Longrightarrow \qquad A \in \mathcal{A} \quad \text{such that} \quad \langle x|A|y\rangle = A_{xy} \tag{3.7.1}$$

NB: The weak topology considered here can be replaced in the definition by stronger topologies on $\mathcal{L}(\mathcal{H})$. In the particular case of commutative algebras, one can show that W*-algebras correspond to the set of measurable functions $L^\infty(X)$ on some measurable space X, while C*-algebras corresponds to the set $C_0(Y)$ of continuous functions on some Hausdorff space Y. Thus, as advocated by A. Connes, W*-algebras corresponds to non-commutative measure theory, while C*-algebras to non-commutative topology theory.

The bicommutant theorem: A famous theorem by von Neumann states that $\mathcal{A} \subset L(\mathcal{H})$ is a W*-algebra iff it is a C*-algebra and it is equal to its bicommutant

$$\mathcal{A} = \mathcal{A}'' \tag{3.7.2}$$

(the commutant $\mathcal{A}'$ of $\mathcal{A}$ is the set of operators that commute with all the elements of $\mathcal{A}$, and the bicommutant is the commutant of the commutant).
NB: The equivalence of this "algebraic" definition with the previous "topological" or "analytical" one illustrate the deep relation between algebra and analysis at work in operator algebras and in quantum physics. It is often stated that this property means that a W*-algebra $\mathcal{A}$ is a symmetry algebra (since $\mathcal{A}$ is the algebras of symmetries of $\mathcal{B} = \mathcal{A}'$). But one can also view this as the fact that a W*-algebra is a "causally complete" algebra of observables, in analogy with the notion of causally complete domain (see the next section on algebraic quantum field theory).

The predual property It was shown by Sakai that W*-algebras can also be defined as C*-algebras that have a predual, i.e. when considered as a Banach vector space, $\mathcal{A}$ is the dual of another Banach vector space $\mathcal{B}$ ($\mathcal{A} = \mathcal{B}^\star$).
NB: This definition is unique up to isomorphisms, since $\mathcal{B}$ can be viewed as the set of all (ultra weak) continuous linear functionals on $\mathcal{A}$, which is generated by the positive normal linear functionals on $\mathcal{A}$ (i.e. the states) with adequate topology. So W*-algebras are also algebras with special properties for their states.

3.7.2 Classification of factors

Let me say a few words on the famous classification of factors. Factors are W*-algebras with trivial center $C = \mathbb{C}$ and any W*-algebra can be written as an integral sum over factors. W*-algebra have the property that they are entirely determined by their projectors elements (a projector is such that $\mathbf{a} = \mathbf{a}^* = \mathbf{a}^2$, and corresponds to orthogonal projections onto closed subspaces E of $\mathcal{H}$). The famous classification result of Murray and von Neumann states that there are basically three different classes of factors, depending on the properties of the projectors and on the existence of a trace.

Type I: A factor is of type I if there is a minimal projector E such that there is no other projector F with $0 < F < E$. Type I factors always corresponds to the whole algebra of bounded operators $L(\mathcal{H})$ on some (separable) Hilbert space $\mathcal{H}$. Minimal projector are projectors on pure states (vectors in $\mathcal{H}$). This is the case usually considered by "ordinary physicists". They are denoted I_n if $\dim(\mathcal{H}) = n$ (matrix algebra) and I_∞ if $\dim(\mathcal{H}) = \infty$.

Type II: Type II factors have no minimal projectors, but finite projectors, i.e. any projector E can be decomposed into $E = F + G$ where E, F and G are equivalent projectors. The type II_1 hyperfinite factor has a unique finite trace ω (a state such that $\omega(\mathbf{1}) = 1$ and $\omega(\mathbf{a}\mathbf{a}^*) = \omega(\mathbf{a}^*\mathbf{a})$), while type $\mathrm{II}_\infty = \mathrm{II}_1 \otimes \mathrm{I}_\infty$. They play an important role in non-relativistic statistical mechanics of infinite systems, the mathematics of integrable systems and CFT.

Type III: This is the most general class. Type III factors have no minimal projectors and no trace. They are more complicated. Their classification was achieved by A. Connes. These are the general algebras one must consider in relativistic quantum field theories.

3.7.3 The Tomita-Takesaki theory

Let me say a few words on a important feature of von Neumann algebras, which states that there is a natural "dynamical flow" on these algebras induced by the states. This will be very sketchy and naive. We have seen that in "standard quantum mechanics" (corresponding to a type I factor), the evolution operator $U(t) = \exp(-\mathrm{i}tH)$ is well defined in the lower half plane $\mathrm{Im}(t) \leq 0$.

This correspondence "state $\leftrightarrow$ dynamics" can be generalized to any von Neumann algebra, even when the concept of density matrix and trace is not valid any more. Tomita and Takesaki showed that to any state ϕ on $\mathcal{A}$ (through the GNS construction $\phi(\mathbf{a}) = \langle\Omega|\mathbf{a}\Omega\rangle$ where Ω is a separating cyclic vector of the Hilbert space $\mathcal{H}$) one can associate a one parameter family of modular automorphisms $\sigma_t^\Phi\colon \mathcal{A} \to \mathcal{A}$, such that $\sigma_t^\Phi(\mathbf{a}) = \Delta^{\mathrm{i}t}\mathbf{a}\Delta^{-\mathrm{i}t}$, where Δ is positive selfadjoint modular operator in $\mathcal{A}$. This group depends on the choice of the state ϕ only up to inner automorphisms, i.e. unitary transformations u_t such that $\sigma_t^\Psi(\mathbf{a}) = u_t\sigma_t^\Phi(\mathbf{a})u_t^{-1}$, with the 1-cocycle property $u_{s+t} = u_s\sigma_s(u_t)$.

As advocated by A. Connes, this means that there is a "global dynamical flow" acting on the von Neumann algebra $\mathcal{A}$ (modulo unitaries reflecting the choice of initial state). This Tomita-Takesaki theory is a very important tool in the mathematical theory of operator algebras. It has been speculated by some authors that there is a deep connection between statistics and time (the so called "thermal time hypothesis"), with consequences in quantum gravity. Without defending or discussing more this hypothesis, let me just state that the theory comforts the point of view that operator algebras have a strong link with causality.

3.8 Locality and algebraic quantum field theory

Up to now I have not really discussed the concepts of time and of dynamics, and the role of relativistic invariance and locality in the quantum formalism. One should remember that the concepts of causality and of reversibility are already incorporated within the formalism from the start.

It is not really meaningful to discuss these issues if not in a fully relativistic framework. This is the object of algebraic and axiomatic quantum field theory. Since I am not a specialist I give only a very crude and very succinct account of this formalism and refer to the excellent book by R. Haag [Haa96] for all the details and the mathematical concepts.

3.8.1 Algebraic quantum field theory in a dash

In order to make the quantum formalism compatible with special relativity, one needs three things.

Locality: Firstly the observables must be built out of the local observables, i.e. the observables attached to bounded domains $\mathcal{O}$ of Minkovski space-time $M = \mathbb{R}^{1,d-1}$. They corresponds to measurements made by actions on the system in a finite region of space, during a finite interval of time. Therefore one associate to each domain $\mathcal{O} \subset M$ a subalgebra $\mathcal{A}(\mathcal{O})$ of the algebra of observables.

$$\mathcal{O} \to \mathcal{A}(\mathcal{O}) \subset \mathcal{A} \tag{3.8.1}$$

This algebra is such that is

$$\mathcal{A}(\mathcal{O}_1 \cup \mathcal{O}_2) = \mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2) \tag{3.8.2}$$

where $\vee$ means the union of the two subalgebras (the intersection of all subalgebras containing both $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$.

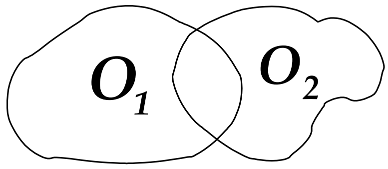

Figure 3.1: The union of two domains

Note that this implies

$$\mathcal{O}_1 \subset \mathcal{O}_2 \implies \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) \tag{3.8.3}$$

The local operators are obtained by taking the limit when a domain reduces to a point (this is not a precise or rigorous definition, in particular in view of the UV divergences of QFT and the renormalization problems).

Caution, the observables of two disjoint domains are not independent if these domains are not causally independent (see below) since they can be related by dynamical/causal evolution.

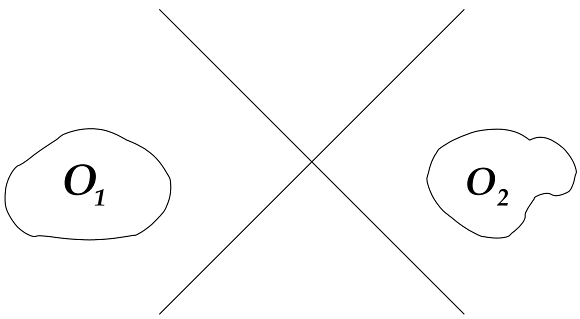

Figure 3.2: For two causally separated domains, the associated observables must commute

Causality: Secondly causality and locality must be respected, this implies that physical local observables which are causally independent must always commute. Indeed the result of measurements of causally independent observables is always independent of the order in which they are performed, independently of the state of the system. Were this not the case, the observables would not be independent and through some measurement process information could be manipulated and transported at a faster than light pace. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are causally separated (i.e. any $x_1 - x_2$, $x_1 \in \mathcal{O}_1$, $x_2 \in \mathcal{O}_2$ is space-like)) then any pair of operators A_1 and A_2 respectively in $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ commutes

$$\mathcal{O}_1 \between \mathcal{O}_2\,, \quad A_1 \in \mathcal{A}(\mathcal{O}_1)\,, \quad A_2 \in \mathcal{A}(\mathcal{O}_2) \quad \Longrightarrow \quad [A_1, A_2] = 0 \tag{3.8.4}$$

This is the crucial requirement to enforce locality in the quantum theory.
NB: As already discussed, in theories with fermion, fermionic field operators like ψ and $\bar{\psi}$ are not physical operators, since they intertwin different sectors (the bosonic and the fermionic one) and hence the anticommutation of fermionic operators does not contradict the above rule.

Causal completion: One needs also to assume causal completion, i.e.

$$\mathcal{A}(\mathcal{O}) = \mathcal{A}(\widehat{\mathcal{O}}) \tag{3.8.5}$$

where the domain $\widehat{\mathcal{O}}$ is the causal completion of the domain $\mathcal{O}$ ($\widehat{\mathcal{O}}$ is defined as the set of points $\mathcal{O}''$ which are causally separated from the points of $\mathcal{O}'$, the set of points causally separated from the points of $\mathcal{O}$, see fig.3.3 for a self explanating illustration).

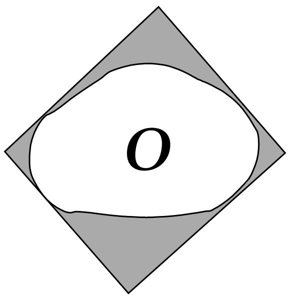

Figure 3.3: A domain $\mathcal{O}$ and its causal completion $\widehat{\mathcal{O}}$ (in gray)

This implies in particular that the whole algebra $\mathcal{A}$ is the (inductive) limit of the subalgebras generated by an increasing sequence of bounded domains whose union is the whole Minkovski space

$$\mathcal{O}_i \subset \mathcal{O}_j \text{ if } i < j \text{ and } \bigcup_i \mathcal{O}_i = \mathbb{M}^4 \quad \Longrightarrow \quad \varinjlim \mathcal{A}(\mathcal{O}_i) = \mathcal{A} \tag{3.8.6}$$

and also that it is equal to the algebra associated to "time slices" with arbitrary small time width.

$$\mathcal{S}_\epsilon = \{\mathsf{x} = (t, \vec{x}) :\ t_0 < t < T_0 + \epsilon\} \tag{3.8.7}$$

Figure 3.4: An arbitrary thin space-like slice of space-time is enough to generate the algebra of observables $\mathcal{A}$

This indicates also why one should concentrate on von Neumann algebras. The set of local subalgebras $\mathcal{L} = \{\mathcal{A}(\mathcal{O}) :\ \mathcal{O}$ subdomains of $M\}$ form an orthocomplemented lattice of algebras (see the next chapter to see what this notion is) with interesting properties.

Poincaré invariance: The Poincaré group $\mathfrak{P}(1,d-1) = \mathbb{R}^{1,d-1} \rtimes \mathrm{O}(1,d-1)$ must act on the space of local observables, so that it corresponds to a symmetry of the theory (the theory must be covariant under translations in space and time and Lorentz transformations). When $\mathcal{A}$ is represented as an algebra of operators on a Hilbert space, the action is usually represented by unitary [4] transformations $U(a,\Lambda)$ (a being a translation and Λ a Lorentz transformation). This implies in particular that the algebra associated to the image of a domain by a Poincaré transformation is the image of the algebra under the action of the Poincaré transformation.

$$U(a,\Lambda)\mathcal{A}(\mathcal{O})U^{-1}(a,\Lambda) = \mathcal{A}(\Lambda\mathcal{O}+a) \tag{3.8.8}$$

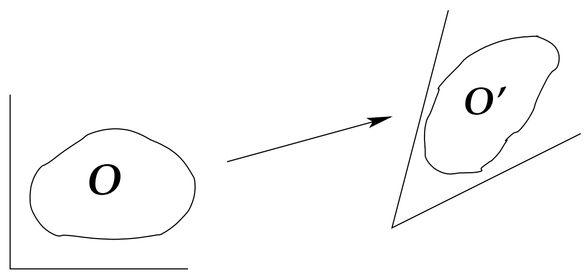

Figure 3.5: The Poincaré group acts on the domains and on the associated algebras

The generator of time translations will be the Hamiltonian $P_0 = H$, and time translations acting on observables corresponds to the dynamical evolution of the system in the Heisenberg picture, in a given Lorentzian reference frame.

The vacuum state: Finally one needs to assume the existence (and the uniqueness, in the absence of spontaneous symmetry breaking) of a special state, the vacuum state

4. Unitary with respect to the real algebra structure, i.e. unitary or antiunitary w.r.t. the complex algebra structure.

$|\Omega\rangle$. The vacuum state must be invariant under the action of the Poincaré transformations, i.e. $U(a,\Lambda)|\Omega\rangle = |\Omega\rangle$. At least in the vacuum sector, the spectrum of $\mathbf{P} = (E, \vec{P})$ (the generators of time and space translations) must lie in the future cone.

$$E^2 - \vec{p}^2 > 0 \quad , \qquad E > 0 \tag{3.8.9}$$

This is required since the dynamics of the quantum states must respect causality. In particular, the condition $E > 0$ (positivity of the energy) implies that dynamical evolution is compatible with the modular automorphisms on the algebra of observables constructed by the Tomita-Takesaki theory.

3.8.2 Axiomatic QFT

3.8.2.a - Wightman axioms

One approach to implement the program of algebraic local quantum field theory is the so-called axiomatic field theory framework (Wightman & Gårding). Actually the axiomatic field theory program was started before the algebraic one. In this formalism, besides the axioms of local, AQFT, the local operators are realized as "local fields". These local fields Φ are represented as distributions (over space-time M) whose values, when applied to some C^∞ test function with compact support f (typically inside some $\mathcal{O}$) are operators $\mathbf{a} = \langle \Phi \cdot f \rangle$. Local fields are thus "operator valued distributions". They must satisfy the Wightman's axioms (see Streater and Wightman's book [SA00] and R. Haag's book, again), which enforce causality, locality, Poincaré covariance, existence (and uniqueness) of the vacuum (and eventually in addition asymptotic completeness, i.e. existence of a scattering S-matrix).

3.8.2.b - CPT and spin-statistics theorems

The axiomatic framework is very important for the definition of quantum theories. It is within this formalism that one can derive the general and fundamental properties of relativistic quantum theories

- Reconstruction theorem: reconstruction of the Hilbert space of states from the vacuum expectation values of product of local fields (the Wightman functions, or correlation functions),
- Derivation of the analyticity properties of the correlation functions with respect to space-time $\mathbf{x} = (t, \vec{x})$ and impulsion $\mathbf{p} = (E, \vec{p})$ variables,
- Analyticity of the S matrix (an essential tool),
- The CPT theorem: locality, Lorentz invariance and unitarity imply CPT invariance,
- The spin statistics theorem,
- Definition of quantum field theories in Euclidean time (Osterwalder-Schrader axioms) and rigorous formulation of the mapping between Euclidean theories and Lorentzian quantum theories.

3.9 Discussion

I gave here a short introduction to the algebraic formulation of quantum mechanics and quantum field theory. I did not aim at mathematical rigor nor completeness. I have not mentioned recent developments and applications in the direction of gauge theories, of two dimensional conformal field theories, of quantum field theory in non trivial (but classical) gravitational background.

However I hope to have conveyed the idea that the "canonical structure of quantum mechanics" – complex Hilbert space of states, algebra of operators, Born rule for probabilities – is quite natural and is a representation of an underlying more abstract structure: a real algebra of observables + states, consistent with the physical concepts of causality, reversibility and locality/separability.