2D Quantum Gravity and continuum random surfaces models from a physicist's point of view

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François David,
IPhT and CNRS

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Quantum field theories and path integrals

- Reminder: matrix elements of (local field) operators $\mathbf{\Phi}(x)$ in a QFT can be calculated by path integrals (Feynman-Kac)

$$\langle OUT|\mathbf{\Phi}(x_1)\cdots\mathbf{\Phi}(x_N)|IN\rangle \propto \int_{\phi(in),\phi(out)\ \text{fixed}} D[\phi(x)]\ e^{\frac{i}{\hbar}S[\phi]}\ \phi(x_1)\cdots\phi(x_N) \tag{1}$$

- $S[\phi]$ is the classical action (here for a scalar field)

$$S[\phi] = \int dt\, d^3\vec{x}\ \mathcal{L}(\partial_\mu\phi,\phi) \tag{2}$$

and $\mathcal{L}(\partial_\mu\phi,\phi)$ the Lagrangian density (keep only renormalizable terms in $V(\phi)$)

$$\mathcal{L}(\partial_\mu\phi,\phi) = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial\vec{x}}\right)^2 - V(\phi) \tag{3}$$

- The evolution operator $U(t) = e^{\frac{t}{i\hbar}H}$ is given by the path (functional) integral, and

$$\text{quantum states } |\Psi\rangle = \text{boundary conditions on } \ \phi(x) \tag{4}$$

$$\text{local operators } \ \mathbf{A}(x) \ = \ \text{local field functionals } A(x) = A(\phi,\partial\phi,\partial^2\pi\ldots) \tag{5}$$

Euclidean path integral

- One can go to the Euclidean theory by a "Wick rotation"

$$\text{time } t \quad \to \quad \text{Euclidean time } \; x_0 = it \tag{6}$$

- So Euclidean functional integral become (static) expectation on a probability space

$$\langle \boldsymbol{\Phi}(x_1)\cdots\boldsymbol{\Phi}(x_N)\rangle = E[\phi(x_1)\cdots\phi(x_N)] = \frac{1}{Z}\int D[\phi(x)]\; e^{-\frac{1}{\hbar}S_E[\phi]}\;\phi(x_1)\cdots\phi(x_N) \tag{7}$$

- in Euclidean space $x = (x_0, \vec{x})$ for the Euclidean action

$$S_E[\phi] = \int d^d x\; \frac{1}{2}\left(\frac{\partial\phi}{\partial x_\mu}\right)^2 + V(\phi) \tag{8}$$

- **NB:** Wick rotation and Euclidean QFT are not (completely...) black magic. This is justified by the mathematical quantum formalism: Operator algebras and Tomita-Takesaki theory, algebraic QFT, axiomatic and constructive field theory, reconstruction theorem, etc.
- **NB:** Algebraic formulation of QFT's is especially important for Conformal Field Theories (CFT), where the Hilbert space and the algebra of observables (the operators) are structured by the action of **Vir** and of the associated current algebra and short distance OPE.
- **NB:** In particular, strong analyticity properties of correlation functions. Euclidean and Minkovski theories treated on the same footing (space-time is a complex manifold).

2d gravity and string theories

- Question: How to define a functional integral over 2d Riemannian geometries?
- For simplicity: fixed topology: $M=$ sphere S_2 or disk D_2

$$Z = \int_{\mathcal{M}} \mathcal{D}[m]\, e^{-\mathcal{S}_{gr}[g]} \int \mathcal{D}[\phi_{matter}]\, e^{-S_{matter}[\phi_{matter}]}$$

  - $\mathcal{M} = \{$Riemannian structures m on the manifold M$\}$
  - $\mathcal{G} = \{$Riemannian metric g on the manifold M$\}$
  - $\mathcal{M} = \mathcal{G}/\mathrm{Diff}(M)$

- The measure $\mathcal{D}[m]$ has to be local, so it should comes from the measure $\mathcal{D}[g]$ induced from the local diffeomorphism invariant metric (*DeWitt metric*) on the space of all smooth metrics $\mathcal{G}$

$$||\delta \mathrm{g}||^2 = \int_M d^2z\, \sqrt{|\mathrm{g}|}\, \delta \mathrm{g}_{\mu\nu} \delta \mathrm{g}_{\rho\sigma}\, \mathrm{g}^{\mu\rho}\mathrm{g}^{\nu\sigma} = \int_M \mathrm{tr}[g^{-1}\ \delta g\, g^{-1}\, \delta g]$$

  I leave aside the (important) question of topology changes!

- Motivations and applications:
  - A toy model for quantum gravity (4d, Lorentz signature)
  - A model for bosonic string $X = X^A(\tau,\sigma)$ propagating in D-dimensional space with background metric G_{AB}, $A \in \{1,\cdots,D\}$.
  - The coordinates of the string are the matter fields "living" on the "world sheet" of the string. These are D (Gaussian) free fields (i.e. GFF)

$$S_{matter} = \frac{1}{4\pi} \int_M d^2z\ \sqrt{|g|}\ D^\mu X^A D_\mu X^B G_{AB}(X)$$

- In 2 dimension, for fixed topology, there is only one possible term in the classical gravitational action
$$S_{gr}[g] = \mu_0 \int_M d^2z \sqrt{|g|}$$
since the Einstein-Hilbert term $-\int_M R$ is a topological invariant (R is a local derivative).
- This makes the classical theory trivial $g = 0$ unless $\mu = 0$, then Einstein equations are (since $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$)
$$0 = 0 \quad !$$
- But this is not the end of the story... There is another possible non trivial classical local equation
$$R + \mu = 0 \tag{9}$$
constant Gaussian curvature = Liouville equation
- But is does not come seemingly) from a local action principle (at least Diff invariant).

Conformal gauge & conformal anomaly

- By an adequate change of coordinate $z = \xi(w)$ any metric $g(w)$ over M can always be written (locally) in the coordinate system z as

$$\mathrm{g}_{\mu\nu} = e^{\phi}\hat{\mathrm{g}}_{\mu\nu}\ ,\quad \phi(z)\ \text{conformal factor}$$

  where $\hat{\mathrm{g}}_{\mu\nu}$ is an a priori chosen fixed metric (this is gauge fixing of Diff).
- $\hat{g}$ is fixed up to possible moduli (in finite numbers) , and the diffeomorphism ξ_g ($g \to g' = e^{\phi}\hat{g}$) is unique up to possible conformal zero modes.
- In general the coordinate system z is chosen to be conformal, such that the reference metric $\hat{g}$ is

$$\hat{g}_{\mu\nu}dz^{\mu}dz^{\nu} = e^{\hat{\varphi}}\,dz\,d\bar{z}\ ,\quad z = z^1 + iz^2$$

- The functional integral over g reduces to a integral over ϕ (Feynman & de Witt, Faddev & Popov)

$$\int \mathcal{D}[\mathrm{g}] = \int \mathcal{D}[\mathrm{g}] \int_{Diff} \mathcal{D}[\xi]\ \delta[\xi - \xi_g] = \int \mathcal{D}_g[\phi]\ \det[\nabla_g]\det[\bar{\nabla}_g] \tag{10}$$

- $\mathcal{D}_g[\phi]$ is the functional measure for a scalar field ϕ in metric g induced by the metric

$$||\delta\phi||^2 = \int_M d^2z\,\sqrt{|\mathrm{g}|}\,(\delta\phi)^2$$

  but $g = \hat{g}e^{\phi}$ depend on ϕ , so the measure $\mathcal{D}_g[\phi]$ is non-linear.
- WARNING! These functional integrals require UV regularization and a proper continuum limit prescription (renormalization) to be defined properly.

The Faddev-Popov determinant

- The Fadeev-Popov determinant

$$J[g] = \det[\nabla_g]\det[\bar{\nabla}_g] = \exp(-\Gamma_{ghost}[g])$$

is the Jacobian for the change of variables $\text{metric} \to \phi$.

- $\nabla = \nabla^z$ is the holomorphic derivative $c^z \to \nabla^z c^z$, 1-form$\to$ (-2)-form.
- The FP determinant can be written as a functional integral of a ghost-antighost system, b and c are anticommuting (Berezin) -2 and 1 forms. This integral defines a (non unitary) Conformal Field Theory.

$$\det[\nabla^z] = e^{-\Gamma_{ghost}} = \int \mathcal{D}[b,c]\, e^{-\int b_{zz}\nabla^z_g c^z} \tag{11}$$

- To compute $J[g]$, Polyakov uses the trace anomaly method: under any local conformal variation of the metric $g = \hat{g}\, e^{\phi}$,

$$\phi \to \phi + \delta\phi$$

the variation of the bc system effective action Γ_{ghost} must be a local operator (a local functional of the metric g and its derivatives).

- This is a general theorem of local QFT, and of Analysis. The only operator with the right dimension is the curvature R. The calculation, using for instance a heat kernel regularisation of the functional determinants, gives

$$\frac{\delta}{\delta\phi(x)}\Gamma_{ghost}[g] = \frac{26}{48\pi}\sqrt{|g|}R(x) = \frac{26}{48\pi}\sqrt{|\hat{g}|}\left(\hat{R} - \Delta_{\hat{g}}\phi\right)$$

The functional integral (continued)

- Integration gives the (free) Liouville action

$$\Gamma_{ghost}[g] = \frac{26}{48\pi} \int_M \sqrt{\hat{g}} \left(\frac{1}{2}(\hat{\nabla}\phi)^2 + \hat{R}\phi \right) + \Gamma_{ghost}[\hat{g}]$$

- Can we reduce the complicated functional measure $\mathcal{D}_g[\phi]$ to the simpler $\mathcal{D}_{\hat{g}}[\phi]$?
- Yes, also by arguments of locality and conformal anomaly (modulo a little rescaling of ϕ, more later)

$$\mathcal{D}_g[\phi] = \mathcal{D}_{\hat{g}}[\phi] \ \exp\left(\frac{1}{48\pi} \int_M \sqrt{\hat{g}} \left(\frac{1}{2}(\hat{\nabla}\phi)^2 + \hat{R}\phi \right) \right)$$

- The contribution of matter fields (if it is a conformal field theory, i.e. a massless theory corresponding to a 2d statistical system at a critical point) is of the same form (given also by conformal anomaly)

$$\Gamma_{matter}[g] = -\frac{c_{matter}}{48\pi} \int_M \sqrt{\hat{g}} \left(\frac{1}{2}(\hat{\nabla}\phi)^2 + \hat{R}\phi \right) + \Gamma_{matter}[\hat{g}]$$

where the effective action for matter is defined as

$$\int \mathcal{D}_g[\phi_m]\, e^{-S_m[\phi_m]} = e^{-\Gamma_m[g]}$$

The Liouville theory

- The initial pure gravity (geometric) action becomes

$$S_{gr}[g] = \mu_0 \int_M \sqrt{g} = \mu_0 \int_M \sqrt{\hat{g}}\, e^{A_0 \phi}$$

- A_0 is a renormalization factor (I have been a bit sketchy in the derivation) but its value is fixed by consistency, i.e. the absence of conformal anomaly (see later).
- The functional integral over metrics reduces to the the functional integral for the quantum Liouville theory for ϕ

$$Z = \int \mathcal{D}_{\hat{g}}[\phi]\, e^{-S_L[\phi]}\, e^{-\hat{\Gamma}[\hat{g}]}$$

- Action of the Liouville theory

$$S_L[\phi] = \frac{25 - c_m}{48\pi} \int_M \sqrt{|\hat{g}|} \left(\frac{1}{2} (\hat{\nabla}\phi)^2 + \hat{R}\phi + \mu\, e^{A_0 \phi} \right)$$

- By consistency of the conformal anomaly, the contribution of ghost+matter in the background $\hat{g}$ must be given by the functional integral of the ghost + matter field in the fixed background metric $\hat{g}$.

$$e^{-\hat{\Gamma}[\hat{g}]} = \int \mathcal{D}_{\hat{g}}[b, c] \int \mathcal{D}_{\hat{g}}[\phi_m]\, e^{-S_{\hat{g}}[b,c] - S_{\hat{g}}[\phi_m]}$$

The Liouville theory (continued)

- The Liouville action depends explicitely on the background metric $\hat{g}$ via the $\hat{R}\phi$ term. It is a conformally invariant theory even with $\mu > 0$ (do not view μ as a mass term).
- The central charge of the Liouville theory is

$$c_L = (25 - c_m) + 1$$

so that

$$c_L + c_{ghost} + c_{matter} = 0$$

- The usual and convenient normalisation is (Zamolodchikov & Zamolodchikov)

$$\phi(x) = \gamma\,\varphi(x)\ , \qquad \frac{2}{\gamma} + \frac{\gamma}{2} = Q \ \text{ background charge}\ , \quad c_L = 1 + 6Q^2$$

$$S_L[\varphi] = \frac{1}{2\pi}\int_M \sqrt{|\hat{g}|}\left(\frac{1}{2}(\hat{\nabla}\varphi)^2 + \frac{Q}{2}\,\hat{R}\,\varphi + \mu\, e^{\gamma\varphi}\right)$$

- Note that this parametrization makes sense if

$$0 \leq \gamma \leq 2\ , \quad \infty \geq Q \geq 4\ , \quad \text{i.e.}\ -\infty \leq c_m \leq 1$$

.

- Indeed there are problems when $c_m > 1$ (the $c = 1$ barrier).
- For a given Q (here c_m), the other branch $\gamma > 1$ is interesting in its own.

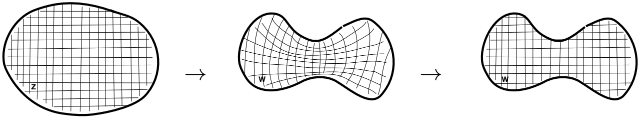
Relation between Q and γ

- The "quantum" volume measure $d^2z\, e^{\gamma\varphi}$ requires an UV regulator ϵ (e.g. lattice) and a renormalization prescription to be properly defined.

$$d^2z\, e^{\gamma\varphi} = \lim_{\epsilon\to 0} d^2z\ \epsilon^{\frac{\gamma^2}{2}}\ e^{\gamma\varphi_\epsilon}$$

- Under a conformal (analytic) mapping

$$z \to w = f(z) \iff w \to z = g(w)$$



$$d^2z\, e^{\gamma\varphi_\epsilon} \quad = \quad d^2w\ |g'|^2\, e^{\gamma\, \varphi\circ g_{\epsilon'}} \quad = \quad d^2w\ |g'|^{2+\frac{\gamma^2}{2}}\, e^{\gamma\, \varphi\circ g_\epsilon} \quad = \quad d^2w\, e^{\gamma\,\tilde\varphi}$$

- Hence the conformal transformation law for the quantum Liouville field φ

$$\varphi(z) \ \to\ \tilde\varphi(w) = \varphi(z) - \left(\frac{2}{\gamma} + \frac{\gamma}{2}\right) \log|f'(z)|$$

- In order to be consistent with the Liouville action (e.g. its saddle point equation), this fixes the relation between Q and γ,

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2}$$

Summary of the physicist's derivations

Polyakov '81: first calculation of the FP determinant and derivation of the Liouville theory.
'81-'87: Study of the theory by CFT an integrability methods (Otto & Dorn, Gervais & Neveu ...)
A simpler presentation (e.g. David 87)

$$\boxed{\text{Assume that the effective theory exists \& determine it by consistency conditions}}$$

The form of the action is fixed by dimensional analysis (keep IR relevant/UV renormalizable terms) compatible with global conformal symmetry.
The couplings Q and γ are fixed by the anomaly consistency condition.
The whole quantum theory

$$\text{Liouville} + \text{ghosts} + \text{matter}$$

must not depend on the background metric $\hat{g}$, since it is just a gauge fixing choice. This fixes Q.
The full theory is in fact a topological theory.

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{Discrete gravity}\\ \text{Random Matrices}\end{array}} \arrow[rr] \arrow[dr] & & \boxed{\text{Liouville theory}} \arrow[dl] \\
& \boxed{\begin{array}{c}\text{Topological and}\\ \text{non-critical strings}\end{array}} &
\end{tikzcd}$$

Boundary Liouville theory

- Bulk term

$$S_L^{bulk}[\varphi] = \frac{1}{2\pi}\int_M \sqrt{|\hat{g}|}\left(\frac{1}{2}(\hat{\nabla}\varphi)^2 + \frac{Q}{2}\,\hat{R}\,\varphi + \mu\, e^{\gamma\varphi}\right)$$

- Boundary term:

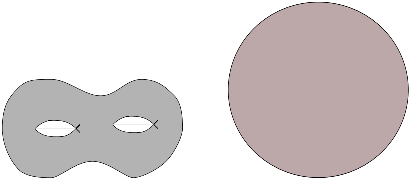
$$S_L^{boundary}[\varphi] = \frac{1}{2\pi}\int_{\partial M} |\hat{g}|^{1/4}\left(Q\,\widehat{K}\,\varphi + \lambda\, e^{\gamma/2\,\varphi}\right)$$

- $\widehat{K}$ extrinsic curvature of the boundary.
- λ boundary cosmological constant (line tension).
- The linear term in $Q\phi$ is topological (not a coincidence...)

$$\int_M \sqrt{|\hat{g}|}\,\hat{R} + \int_{\partial M} \sqrt[4]{|\hat{g}|}\ 2\hat{K} = 4\pi\,\chi$$

- Semiclassical limit $Q \to \infty$ i.e. $\gamma \to 0$
- Classical equations are constant curvature equations (Liouville)
- Bulk: constant scalar curvature $R = -\mu\gamma^2$
- Boundary: constant extrinsic curvature $K = -\lambda\gamma^2/4$

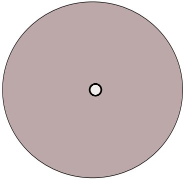
- So no boundary solutions describe genus $g \geq 2$ surfaces or the Poincaré disk



In this first case

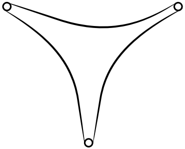
$$\mu \propto \frac{1}{Area}$$

- For $\lambda > \lambda_c = \frac{\sqrt{-8\mu}}{\gamma}$, the solution with one boundary describes a macroscopic hole in the Poincaré disk!



- It is possible to quantize the theory in the whole Poincaré disk with proper boundary conditions at infinity (ZZ and FZZT branes).
- This might corresponds to solutions for $\lambda < -\lambda_c$, the solution is a disk with a finite negative curvature

- Liouville theory must be considered when quantizing a surface with at least 2 (and in general 3) boundaries (holes=macroscopic loops=normalizable states) , or punctures (=local operator= non normalizable states).



- One must check that Liouville+matter has the proper boundary states on ∂M to be consistent (no anomalies).
- A lot is known (heuristics AND rigorous) about the quantum Liouville theory. It is an integrable QFT and a CFT. Hilbert space, spectrum, correlation functions, etc.
- In some specific cases (like 2d gravity coupled to minimal models, 3 points functions)) calculations can be done setting $\mu = 0$ (di Francesco & Kutasov, Seiberg, ...) using CFT methods (Fateev & Dotsenko)
- This gives already very interesting results (e.g. comparison with random matrix models amplitudes)
- Full calculations in the Poincaré disk or for generic cases are much more difficult.

CFT & the KPZ scaling relations

- Initially proposed and proved in a QFT context (using a light cone formulation of 2d gravity) by Knizhnik, Polyakov, and Zamolodchikov '88.
- Simple derivation in a Liouville theory context & conformal gauge by F.D. (+ extension to 2d supergravity by Distler & Kawai). Often denoted the DDK derivation.
- This derivation rely on short distance scaling properties of Liouville QFT, where the full theory $\mu > 0$ can be replaced by a free field $\mu = 0$
- Question: How the scaling dimensions of operators of a 2d CFT are changed by the coupling to 2d gravity?
- We have seen that "operators" in a QFT have a scaling dimension. We have seen already an example: massless free field ϕ

$$V_\alpha = e^{i\alpha\phi} = \lim_{\epsilon\to 0} \epsilon^{\frac{-\alpha^2}{2}} \; e^{i\alpha\phi_\epsilon} \tag{12}$$

ϵ is the UV cutoff (minimal distance)

$$V_\alpha(x)V_\alpha(y) \propto |x-y]^{-4\Delta_\alpha} \tag{13}$$

There are subtleties with IR divergences !

$$\Delta_\alpha = \frac{\alpha^2}{4} = \text{scaling dimension of } V_\alpha \tag{14}$$

NB: dimension in term of $mass^2 = length^{-2}$

- In a CFT operators have two dimensions h and $\bar{h}$ (dimensions w.r.t. z and $\bar{z}$.

$$\Delta = h + \bar{h} \quad , \qquad spin = h - \bar{h}$$

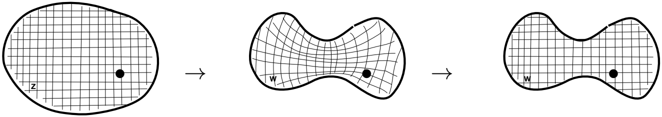
- Geometric statistical 2d model at critical point $\leftrightarrow$ CFT: SAW, Potts models, percolation, etc. are described by CFT (sometimes non-unitary)
- Creation of defect (hence a geometrical interface) $\leftrightarrow$ local operators O_α.
- Fractal dimension of these objects $\leftrightarrow$ conformal dimensions $\Delta(O_\alpha)$
- When coupled to 2d gravity, no simple concept of distance $|x-y|$ between two points x and y, since the metric is quantized (it fluctuates and is integrated out)
- But local operators still have a scaling dimension: characterized by how they transform under global conformal transformations, e.g. how they scale with the length scale of the Liouville theory, i.e. the Area of the quantum surface.

$$\langle Area\rangle = \left\langle \int_M \sqrt{g} \right\rangle \propto \mu^{-1} \ , \ \langle \mathbf{O}\rangle = \left\langle \int_M \sqrt{g}\, O \right\rangle \propto \mu^{-1+\Delta_\mathbf{O}} \tag{15}$$

- This definition allows to make the connexion with scaling dimensions which appear in discretized statistical models on random discrete surfaces (random maps, random matrix models)
- This was the original motivation of KPZ.
- The DDK derivation is in fact a simple extension of the argument which relates γ to Q in the Liouville theory through conformal invariance.
- Remember: $e^{\gamma\varphi}$ is the volume element, Q is the Liouville coupling.

Scaling dimensions of operators and coupling to classical metrics

- under a local scale transformation $z \to w = g^{-1}(z)$



a local (primary) operator $O(z)$ transforms as

$$O(z) \to O'(w) = \left|\frac{\partial w}{\partial z}\right|^{2\Delta(O)} O(g(w)) \tag{16}$$

- This means that in the definition of a local operators O, the choice of reference classical metric $\hat{g}$ on space is implicitly present.
- For instance, when defining precisely the UV regulator ϵ and the regularized field ϕ_ϵ which give to the vertex operator V_α in metric $\hat{g}_{\mu\nu} = \delta_{\mu\nu}\hat{g}$

$$V_\alpha = e^{i\alpha\phi} = \lim_{\epsilon\to 0} \epsilon^{\frac{-\alpha^2}{2}}\ e^{i\alpha\phi_\epsilon} \tag{17}$$

the lattice steps dx depends on $\hat{g}$ since $\epsilon^2 = dx^2\hat{g}(x)$

- Therefore, (16) is equivalent to state that if in a classical reference metric $\hat{g}$, the operator is $\hat{O}$, if we change the reference metric by a classical scale factor $e^{\gamma\hat{\varphi}}$ (γ is here for normalization, and $\hat{\varphi}$ a classical smooth function), $\hat{O}$ changes as

$$\hat{g} \to \hat{g}' = e^{\gamma\hat{\varphi}}\hat{g} \quad \text{then} \qquad \hat{O} \ \to\ \hat{O}' = e^{-\gamma\Delta_0\hat{\varphi}}\ \hat{O} \tag{18}$$

Coupling to quantum metrics

- In the quantum metric

$$g = \hat{g} e^{\gamma\varphi} \quad \varphi \;=\; \text{Liouville field} \tag{19}$$

  the definition of the operators $O(x)$ still involves the metric, hence φ.
- But the relation is not the classical one.
- This can be shown by explicit calculations in specific models (e.g. for some geometrical/mesure theoretical models using a constructive approach with probabilistic methods, see B. Duplantier lectures).
- Or by using CFT methods: field theoretical models, algebraic formulation of self-consistency conditions, vanishing of conformal anomaly.
- Start from the same form of local ansatz for the "gravitational dressing" of a local operator $A(x)$ as in the classical case
- Classical metric

$$g_c = \hat{g}\, e^{\gamma\varphi_c} \;\rightarrow\; \sqrt{g_c} A_c = \sqrt{\hat{g}} e^{\gamma(1-\Delta^{\mathbf{0}}_A)\varphi_c} A \tag{20}$$

  φ_c a smooth classical field. $\Delta^{\mathbf{0}}_A$ classical dimension of the operator A.
- Quantum metric (with a strong assumption that the matter theory is coupled in a minima way to the metric)

$$g = \hat{g}\, e^{\gamma\varphi} \;\rightarrow\; (\sqrt{g} A)_q = \sqrt{\hat{g}} e^{\gamma(1-\Delta_A)\varphi} A \tag{21}$$

  φ Liouville quantum field. Δ_A quantum dimension of the operator A.

- Quantum dimensions $\neq$ classical dimensions!

$$\Delta_A \neq \Delta_A^0 \tag{22}$$

- This is a general quantum phenomenon: products of quantum local operators have to be renormalized (Operator Product Expansion) and in general

$$(AB)_q \neq A_q \times B_q \tag{23}$$

- Here one has to consider

$$A = e^{-\gamma\Delta_A\varphi}\hat{A} \quad , \quad B = e^{\gamma\varphi} \quad \text{while} \quad AB = e^{\gamma(1-\Delta_A)\varphi}A \tag{24}$$

- The quantum dimensions are fixed by the same consistency condition that fixes γ as a function of Q in Liouville
- The operator $\sqrt{\hat{g}}e^{\gamma(1-\Delta_A)\varphi}A$ must have conformal dimension $h=0$ so that the integral of the operator on the quantum manifold

$$\Delta S_A = \int_M \sqrt{|\hat{g}|}\; e^{\gamma(1-\Delta_\alpha)\varphi(x)}\;\widehat{O}_\alpha(x) \tag{25}$$

does not depend of the choice of reference classical metric $\hat{g}$ (gauge fixing parameter)
- The conformal dimension Δ_α^L of the vertex operator $V_\alpha^L = e^{\alpha\varphi}$ in the Liouville theory is (same calculation than those fixing γ) is

$$V_\alpha^L = e^{\alpha\varphi} \;,\quad \Delta_\alpha^L = \frac{\alpha}{2}Q - \frac{\alpha^2}{4} \tag{26}$$

Note the Q term, different from that for a free Gaussian field ϕ.

Derivation of the KPZ relations

- Take $\alpha = \gamma(1-\Delta_A)$ The total conformal dimension of the operator

$$e^{\gamma(1-\Delta_A)\varphi(x)}\,\widehat{A}(x)$$

  must be 1 for consistency. Hence

$$\Delta^L_\alpha + \Delta^{\mathbf{0}}_A = 1$$

- This gives the general KPZ relation between the "gravitational dimension" $\Delta = \Delta_A$ of an operator A and its "classical" dimension $\Delta^{\mathbf{0}} = \Delta^{\mathbf{0}}_A$ in the original CFT.

$$\Delta^{\mathbf{0}} = \Delta + \frac{\gamma^2}{4}\Delta(\Delta-1)$$

  These are the "algebraic" KPZ relations (derived from consistency conditions for a QFT and its operator content). *Knishnik, Polyakov & Zamoldochikov 87, F. D. 88, Distler & Kawai 88*

- They are valid for the primary operators in any unitary CFT coupled to gravity ...
- ....and some non-unitary ones. This includes the theories describing SAW, some loops models, percolation, in particular many models connected to SLE processes.
- They implies relations between the fractal dimensions in flat space and in 2d gravity for many random geometrical objects described by the critical points of sone statistical models, hence to some 2d scale invariant QFT.

Scaling of observables and couplings

- Indeed, scaling of $\langle\sqrt{g}A\rangle$ (A a matter local operator) with the Area of the surface is obtained by adding a source terms for the Liouville+Matter action

$$S = S_L[\varphi] + S_m[\phi_m] + t_A\,\Delta S_A \tag{27}$$

- t_A is a coupling constant
- In statistical mechanics it_A is called a scaling (external) field.
- ΔS_A is the integral over M of the dressed operator $A(x)$

$$\Delta S_A \;=\; \int_M \sqrt{g}A \quad \underset{\text{conformal gauge}}{=} \quad \int_M \sqrt{\hat{g}}\,e^{\gamma(1-\Delta_A)\varphi}\,\hat{A}[\phi] \tag{28}$$

- The Liouville coupling term $\mu\int e^{\gamma\varphi}$ is nothing but the source term associated to the unity operator $\mathbf{1}$ with scaling dimension 0

$$\mu = t_{\mathbf{1}} \quad , \quad \Delta S_{\mathbf{1}} = \int_M \sqrt{g} = \int_M \sqrt{\hat{g}}\,e^{\gamma\varphi} = \text{quantum area} \tag{29}$$

- The expectation value is simply the derivative with respect to the coupling

$$\langle\int\sqrt{g}A\rangle = \frac{d}{dt_A}\int D[g]D[\phi_m]\;e^{-S}\Big|_{t_A=0} \tag{30}$$

- A simple scaling argument shows that a global rescaling of the $\hat{g}$ (a gauge transformation) is reabsorbed into a translation of the Liouville field φ

$$\hat{g} \to \hat{g}\, e^{\gamma \hat{\varphi}_0} \qquad \Longleftrightarrow \qquad \varphi \to \varphi - \hat{\varphi}_0 \tag{31}$$

  which can be reabsorbed into a change of the t_A, and in particular of $\mu = t_1$.
- So that the scaling dimension of t_A (in dimensions of *length*$^{-2}$) is

$$dim(t_A) = 1 - \Delta_A \tag{32}$$

- This implies that

$$\langle \int \sqrt{g} A \rangle \propto t_1^{1-\Delta_A} = \mu^{-1+\Delta_A} \tag{33}$$

- This argument applies when the couplings depend on the positions

$$t_A \int \sqrt{g} A \;\to\; \int_M t_A(x)\, \sqrt{g(x)}\, A(x) \tag{34}$$

  since the DDK analysis relies on local properties and local scale transformations. Hence this works also for local correlators of local operators

$$A(x) \simeq \frac{\delta}{\delta t_A(x)} \tag{35}$$

KPZ topological scaling relation

- Another important consequence is the scaling of the functional integral over metric as a function of the genus of the surface. For a closed surface with genus h and Euler characteristics
$$\chi = (2-2h)$$
one shows easily by reabsorbing a change of μ into a global constant translation of $\varphi \to \varphi + \varphi_0$ and by taking into account the linear term in $Q\varphi$ in the Liouville action
$$Q\int_M \sqrt{\hat{g}}\frac{1}{2}\hat{R}\varphi_0 + \int_{\partial M} |\hat{g}|^{\frac{1}{4}}\hat{K}\varphi_0 = Q\varphi_0\,\chi \tag{36}$$
that the whole functional integral (the sum over the metrics) of a manifold with fixed genus h
$$Z_h = \int_{\text{genus } h} D[g]D[\phi_m]e^{-\mu\int\limits_M \sqrt{g} - S_m[\phi_m, g]} \tag{37}$$
scales with μ as
$$Z_h \propto \mu^{(1-h)\left(1+\frac{4}{\gamma^2}\right)} \sim Area^{-(1-h)\left(1+\frac{4}{\gamma^2}\right)} \tag{38}$$
- This is a very important result for string theory! It implies that it is possible (at least in perturbation theory) to construct a "double scaling limit" where the surfaces on different topologies are summed up in a consistent way.
- The exponent is often denoted γ_{string}
$$Z_h \propto \mu^{(1-h)\left(2-\gamma_{string}\right)} \quad , \qquad \gamma_{string} = 1 - \frac{4}{\gamma^2} \tag{39}$$
- Up to now, only proof via algebraic KPZ approach (or topological TQFT).

Conformal versus Riemannian random geometries?

- The KPZ relations reflect the scaling properties of random geometries which are related to conformal transformations (local scale transformations).
- Conformal transformations respect the angles and the local complex structure of Riemann surfaces. This is a small subpart of Diff.
- What about metric properties of random geometries? Important question for understanding if the path integral approach may be used for $d > 2$ quantum gravity (especially $d = 4$ and $d > 4$!)
- Simplest question: what is the metric Hausdorff dimension d_H of a random 2d metric?

$$V(r) = Vol(B(x_0, r)) = \text{ number of point at distance } r \text{ from } x_0 \ \propto r^{d_H} \qquad (40)$$

- This question started to be studied in ~ 1989 by F.D.
- This is a surprisingly difficult and subtle problem
- It is known how to formulate the problem for discretized surfaces
- A lot of beautiful and exact results are known in the very special case of pure gravity ($c_m = 0$ or $\gamma = \sqrt{8/3}$).
- First work by Ambjørn J and Watabiki Y 1995. Exact recursion relations and scaling ansatz gives

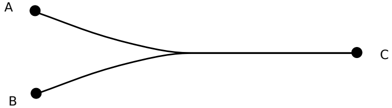
$$d_H = 4 \qquad (41)$$

Discrete random metrics and maps: summary of results

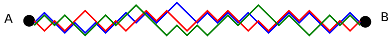
- Problem fully solved using bijections between planar maps and well labeled trees (Cori & Vauquelin '81, Chassaing & Schaeffer 2004, Bouttier, Di Francesco & Guitter 2003)
- Since then, many results and extensions: non-planar maps, correlation functions, generalizations of the models, connexion with continuous random tree (CRT, Aldous).
- Beautiful mixture of combinatorics and probabilitiies.
- Typical results are:
- Hausdorff dimension $= 4$

$$V(r) \propto r^4$$

- Confluence of geodesics



- Macroscopic uniqueness of geodesics (in probability)



- The distance geometry of a planar map is very different from that of a smooth manifold or of a regular lattice.

What is known in the continuum?

- No rigorous or exact results!
- There is a KPZ-like conjecture for d_H by Watabiki 1993

$$d_H = 2\left(\frac{1+\sqrt{\frac{49-c_m}{25-c_m}}}{1+\sqrt{\frac{1-c_m}{25-c_m}}}\right) \tag{42}$$

- This is obtained through a KPZ relation applied to a diffusion process

$$\cancel{d_H = \frac{2}{\Delta} \quad \text{for an ``operator" with dimension} \quad \Delta^0 = -1}$$

$$d_H = \frac{2}{\Delta - 1} \quad \text{for an ``operator" with dimension} \quad \Delta^0 = 2 \tag{43}$$

such as the "operator" $\cancel{O = (-\Delta_{LB})^{-1}}$ $O = (-\Delta_{LB})^2$ (here Δ_{LB} is the scalar Laplace-Beltrami differential operator)
- Advantages:
  - $d_H = 4$ for $c_M = 0$ (exact),
  - $d_H = 2$ for $c_M = -\infty$ (classical limit)
  - $d_H = \frac{3+\sqrt{17}}{2}$ for $c_m = -2$, in good agreement with numerical simulations
- Question: O is not a local primary operator. Not clear at all that KPZ is valid.
- NB: $\frac{3+\sqrt{17}}{2}$ is a "magic" number that appears in many seemingly unrelated problems (fragmentation, growth, index of subfactors in operator algebras, etc.)

An old attempt to compute d_H in the continuum

- This is an old unsuccessful calculation (F.D. 1992)
- It raises problems that I still do not understand
- How to define geodesics and distances in the continuum, i.e. at scales ℓ

$$\text{UV cut-off } a \ll \ell \ll L = \text{size of } \; M \tag{44}$$

- $d(x, y)$ is given by the proper time of a massive quantum particle propagating from x to y
- Indeed the Feynman propagator (Green function) behaves in general (flat or curved space) at large distances as

$$G(x, y) = \langle \Omega | \phi(x) \phi(y) | \Omega \rangle \propto e^{-d(x,y)m} \tag{45}$$

since in the path integral (sum over all paths $x \to y$), the paths close to the shortest geodesic from x to y dominate.

- Thus one can take

$$d(x, y) = \lim_{m \to \infty} -\frac{1}{m} \log \left(G(x, y) \right) \tag{46}$$

- This definition is reparametrization invariant (or rather covariant) and makes sense for any effective theory like the Liouville theory, provided that

$$1/d(x, y) \ll m \ll 1/a \tag{47}$$

The propagator in a random metric

- The scalar Propagator is the Kernel of the inverse of the operator $(-\Delta + m^2)$

$$G(x,y) = \left((-\Delta_g + m^2)^{-1}\right]_{x,y} \tag{48}$$

Δ_g is the Laplace-Beltrami operator in the metric g. In a conformal metric it reads

$$\Delta_g = e^{-\gamma\varphi}\Delta \quad \text{in metric} \quad g = e^{\gamma\varphi} \tag{49}$$

- But also a path integral formulation

$$G(x,y) = \sum_{path\ \mathcal{P}\ x\to y} e^{-m\,\mathrm{length}(\mathcal{P})} \tag{50}$$

- In a metric g both definitions make sense and the propagator depends on the metric. So $G(x,y)$ and $d(x,y)$ seem bilocal observable not that different from the correlation functions of local operators considered in CFT/Gravity and the algebraic KPZ relations.
- But two important differences
- **(i)** The (very) massive field ϕ is not a CFT theory
- **(ii)** We must compute the **logarithm** of a correlation function.
- The first problem is technical. One might hope to have integrability left (not clear), and we still have good old perturbation theory, here in γ.

Replica trick

- To compute a log use the replica trick (makes sense in perturbation theory)
- Consider n "replicas" of ϕ, i.e. n independent identical fields ϕ_α

$$\Phi(x) = \{\phi_\alpha(x);\ \alpha = 1, ...n\} \tag{51}$$

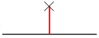
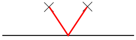
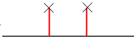
coupled only to the metric g

$$\log\left(\langle\phi(x)\phi(y)\rangle\right) = \frac{d}{dn}\left(\langle\phi(x)\phi(y)\rangle\right)^n\Big|_{n=0} \tag{52}$$

$$= \frac{d}{dn}\left\langle\prod_{\alpha=1}^{n}\left(\phi_\alpha(x)\phi_\alpha(y)\right)\right\rangle\Big|_{n=0} \tag{53}$$

- In Liouville theory, each field ϕ_α describes a replica "α" of a free particle of mass m propagating in the background conformal metric $g = e^{\gamma}\varphi$
- The propagator can be expanded as a weak coupling expansion in the Liouville coupling (each  represents one φ)

$$G = \frac{1}{-\Delta + e^{\gamma\varphi}m^2} = G_0 - \gamma m^2 G_0\varphi G_0 - \gamma^2\frac{m^2}{2}G_0\varphi^2 G_0 + \gamma^2 m^4 G_0\varphi G_0\varphi G_0 + \cdots \tag{54}$$

 $- \gamma m^2$  $- \frac{\gamma^2 m^2}{2}$  $+ \gamma^2 m^4$  $+ \cdots$

Diff. invariant observables

- The geodesic distance between two points $d(x, y; g)$ is not Diff. invariant.
- The area of a disk of radius R centered at some point is better

$$A(x; R) = \int_M d^2y \sqrt{g(y)}\ \theta(R - d(x, y; g)) \tag{55}$$

$$\theta(z) = \begin{cases} 1 & \text{if } z \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{56}$$

- The average area of a disk if radius R is Diff. invariant and is a well defined observable

$$\langle A(R) \rangle = \left\langle \int_M d^2x \sqrt{g(x)}\, A(x; R) \middle/ \int_M d^2x \sqrt{g(x)} \right\rangle \tag{57}$$

- This has to be computed in the full theory

$$gravity + matter + replicas \tag{58}$$

- The effective theory is

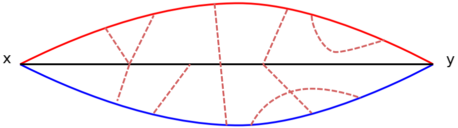
$$Liouville + replicas \tag{59}$$

- No problem of conformal anomaly. The UV central carge of the replicas is

$$c_{UV}(1\, replica) = 1 \quad \text{but} \quad c_{UV}(n = 0\, replica) = 0 \qquad ☺ \tag{60}$$

Perturbative calculation

- One can compute $\langle A(R) \rangle$ in Liouville theory as a perturbative expansion in γ
- Simplest case $\mu = 0$, Liouville field φ = Gaussian Free Field, $M = \mathbb{C}$
- The coupling to the quantum metric induces interactions between replicas.



- Standard phenomena in the physics of disordered systems. We study the propagation of massive quantum particles in a fixed metric, then average over the metric

$$\text{quantum metric} \quad = \quad \text{quenched random environment} \tag{61}$$

- This is very different than in the usual cases

$$\text{KPZ scaling: quantum metric} \quad = \quad \text{annealed random environment} \tag{62}$$

- The Hausdorff dimension d_H of quantum space in 2d gravity may be defined by the scaling

$$\langle A(R) \rangle \propto R^{d_H} \quad \text{as} \qquad R \to \infty \tag{63}$$

Puzzling result: metrics disorder is strongly relevant!

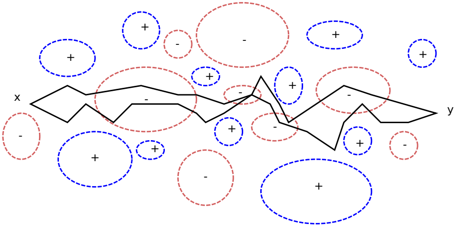
- If a KPZ-like scaling holds, I would expect at first order

$$\langle A(R)\rangle = \pi R^2 + \gamma\, a_1\, R^2 \log R + \cdots \qquad \Longrightarrow \qquad d_H = 2 + \gamma\ a_1/\pi + \cdots \tag{64}$$

- Instead I found at first order

$$\langle A(R)\rangle = \pi R^2 + \gamma\, C_1\, R^{5/2} + \cdots \tag{65}$$

- Where does this come from and what does this means?
- The interaction between the replicas trajectorie mediated by the fluctuations of the metric is attractive!
- The trajectories of the replicas are trapped in domains where $e^{-\gamma\varphi}$ is small , hence $\varphi \ll 0$
- and repelled by domains where $e^{-\gamma\varphi}$ is large , hence $\varphi \gg 0$



Distance geometry as a disordered system problem

- No point in doing higher order calculations if I do not understand what's going on!
- One can argue that geodesics flows are:
  - fast where the scalar curvature is $R < 0$
  - slow where the scalar curvature is $R > 0$
- The results of the one loop calculations (small γ) are qualitatively consistent with the exact results for $c_m = 0$, i.e. $\gamma = \sqrt{8/3}$
- confluence of geodesics $\leftrightarrow$ trapping by $R \ll 0$ domains
- macroscopic uniqueness $\leftrightarrow$ no replica symmetry breaking
- There are analogies between geodesics in random metrics and directed polymers in random environments
- This later system is solvable in 1+1 dim, large distance behavior governed by a non trivial fixed point in the disorder strength
- For geodesics the disorder has non-local correlations. Disorder strength fixed by γ , hence by c_m.

Two puzzling questions

- Question I: Is there a difference between the two regimes for geodesics statistics

  Regime I: $\ell \gg m^{-1} \gg \epsilon$, this is my calculation is valid
  Regime II: $\ell \gg \epsilon \gg m^{-1}$, this is where combinatorics methods are valid

- Question 2: What is the good tool to study the distance geometry?

  KPZ = Knishnik, Polyakov, Zamolodchikov
  OR
  KPZ = Kardar, Parisi, Zang

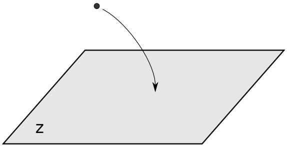
- Any new idea welcome!

Thank you!

The $c_m = 1$ barrier and branching transitions

- Geometrical picture of KPZ scaling: local operator = conical singularity

$$\text{operator } A \text{ at } z_0 \quad \Longleftrightarrow \quad \text{insert } \; e^{\gamma(1-\Delta_A)\varphi(z_0)} A(z_0) \tag{66}$$

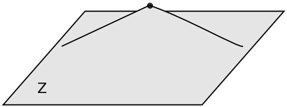


- The Liouville action has now a charge at z_0

$$S_L = \frac{1}{2\pi} \int d^2 z \; \left(\frac{1}{2} (\partial\varphi)^2 - 2\pi \, \gamma(1-\Delta_A)\varphi(z)\delta(z-z_0) \right) \tag{67}$$

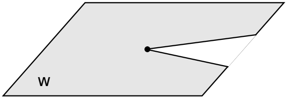
- The classical configuration (extremum of S_L) has a cusp at z_0

$$\varphi_c(z) = -\gamma(1-\Delta_A) \log(|z-z_0|) \tag{68}$$



- Absorbe this in a conical singularity with total angle $\Theta < 2\pi$ by analytic mapping

$$z \to w = z^{\Theta/2\pi} \tag{69}$$



- Transformation of Liouville field φ under an analytic mapping

$$\varphi(z) \ \to \ \tilde{\varphi}(w) = \varphi(z) - Q\log(|dw/dz|) \tag{70}$$

- Conical singularity angle Θ fixed by the condition

$$\tilde{\varphi}_c = 0 \qquad \Longrightarrow \qquad \frac{\Theta}{2\pi} = 1 - \frac{\gamma}{Q}(1 - \Delta_A) \tag{71}$$

- Change the volume = insert identity operator $\mathbf{1}$, with $\Delta = 0$
- Conical angle is

$$\Theta = 2\pi\frac{4-\gamma^2}{4+\gamma^2} \tag{72}$$

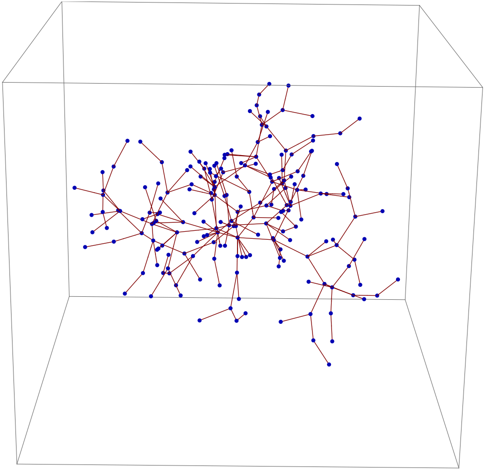
- Problem for $\gamma > 2$

$$\gamma > 2 \qquad \Longrightarrow \qquad \Theta < 0 \tag{73}$$

- In the continuous formulation, inserting a point "tears out the surface"

The $\gamma > 2$ problem

- In discretized models, the surface develops branches and becomes a macroscopic tree made out of microscopic "branches" and "leaves"
- The continuous model is not in the universality class of a random surface model
- The infinite surface is not homeomorphic to the sphere S_2
- But rather to the "generic random tree" (= CRT, continuous random tree of Aldous)



Random tree

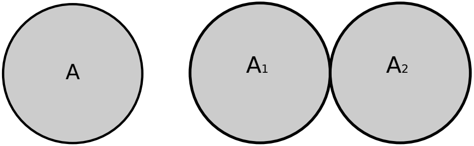
- Other way to see the problem (M. Cates, F.D., Jain, ...): The surface develops "baby universes"
- The partition function for a sphere (genus $h = 0$ surface) of area A scales as

$$Z(A) \propto A^{-\left(1+\frac{4}{\gamma^2}\right)} \tag{74}$$

- The partition function for a pinched sphere made of two pieces of area A_1 and A_2 is

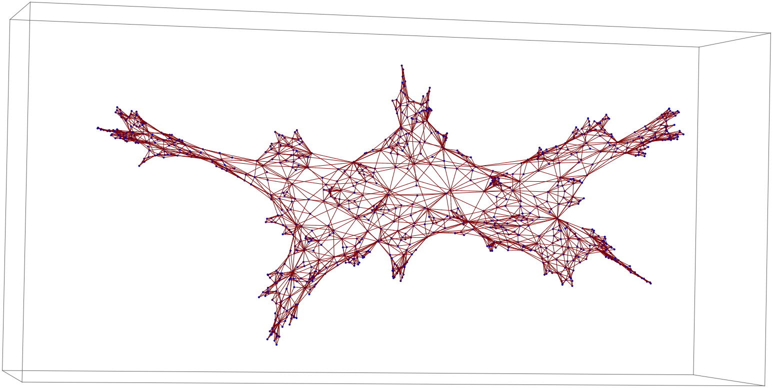
$$Z_(A) \propto A_1^{-\frac{4}{\gamma^2}} A_2^{-\frac{4}{\gamma^2}} \qquad , \qquad A = A_1 + A_2 \tag{75}$$

- For $\gamma < 2$, $A = A_1$ or $A = A_2$ (non big "baby universes"
- For $\gamma > 2$, the surface is pinched with probability 1



- But then splitting within splitting within splitting $\rightarrow$ trees

A random surface is “spiky” but is not a tree



Random triangulation

Diffusions in random metrics and KPZ

François David

Workshop on Manifolds of Metrics and Probabilistic Methods in Geometry and Analysis

CRM, Montréal, July 2-6, 2012

The geometric KPZ relations

- We have seen many times that in a random 2d conformal metric

$$g = e^{\gamma\varphi} \quad ; \qquad \varphi(z) \quad aGFF \tag{76}$$

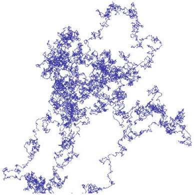
there is the famous KPZ relation

$$\Delta^0 = \Delta + \frac{\gamma^2}{4}\Delta(\Delta - 1) \tag{77}$$

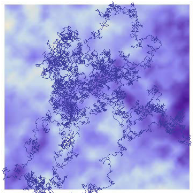
- Geometrical setting: Duplantier & Sheffield '08: Given a fractal $\mathcal{X}$ in the plane

$$\delta_0 = 2(1 - \Delta_0) = \text{ fractal dimension of } \mathcal{X} \text{ at some point } z \tag{78}$$

$$\delta = 2(1 - \Delta) = \text{ quantum fractal dimension of } \mathcal{X} \text{ at } z \tag{79}$$



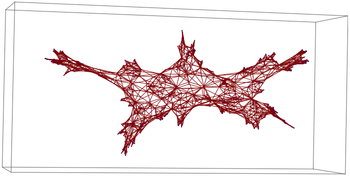
versus



- These KPZ relations hold for generic fractals $\mathcal{X}$ in the "quantum metric" $e^{\gamma\varphi}$.
- The fractal $\mathcal{X}$ does not need to be generated by a CFT, nor to be conformaly invariant! This is a non trivial generalisation!

Geometric versus algebraic KPZ

- $g = e^{\gamma\varphi}$ is treated as a measure $d\mu(z) = e^{\gamma\varphi(z)}d^2z$, not really as a metric.
- Covering $\mathcal{X}$ by Euclidean balls $\mathcal{B}_\epsilon$ is not Diff covariant.
- Is it important?
- Can one rederive this geometric & probabilistic version of the KPZ relations in the original QFT framework?
- Can one find a formulation where Diff(M) invariance is manifest?
- More general question: is it possible to define *local observables* in 2d gravity which are similar (and can be compared)
  (i) in the discretized models (random triangulations and random maps)
  (ii) and in the continuum models (Liouville) as well?



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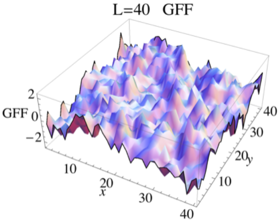
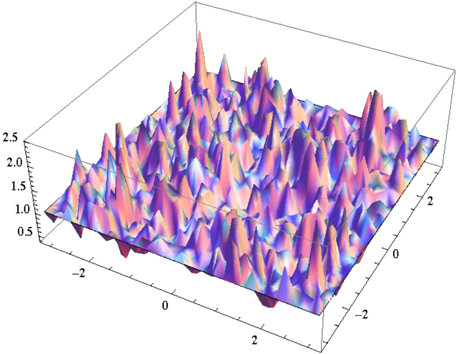


Diffusion and fractal dimensions

- Idea: use a diffusion process and the heat kernel instead of disks to probe the fractal $\mathcal{X}$ (F.D. & M. Bauer '08)
- The "diffusion time" t will be the (Diff. covariant) scale.
- The heat kernel is
$$K(z_1, z_2; t) = \langle z_1 | \mathrm{e}^{t\Delta} | z_2 \rangle \tag{80}$$
- In flat space, let the measure with support on the fractal $\mathcal{X}$ be
$$d\mu^{\mathbf{0}}_{\mathcal{X}}(x)$$
- Let z_0 is a point on the fractal, then the convolution of the measure by the heat kernel
$$B^{\mathbf{0}}_{\mathcal{X}}(z_0; t) = \int d\mu^{\mathbf{0}}_{\mathcal{X}}(z)\, K(z_0, z; t)$$
scales with the time t as
$$B^{\mathbf{0}}_{\mathcal{X}}(z_0; t) \underset{t \to 0}{\sim} t^{-x_0}$$
Where
$$\delta_0 = 2x_0$$
is the fractal dimension of $\mathcal{X}$ at z_0. This is the (weighted by $d\mu$) probability to be on $\mathcal{X}$ at time t, after a random walk starting from z_0.

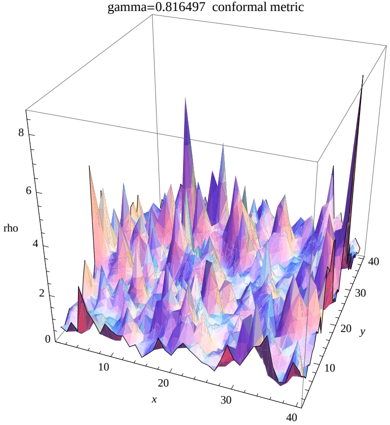
Diffusion in random metrics: an experiment

- Consider a realization of a random metric $g = e^{\gamma\varphi}$ $\quad\varphi$ a GFF



- Can we do the same thing? In principle yes! The Heat Kernel is well defined in any metric.
- Here is a very crude numerical simulation on a quite small lattice.
- A few slides to illustrate the qualitative evolution of the heat Kernel is a random metric.

$$\kappa = \gamma^2 = 2/3$$



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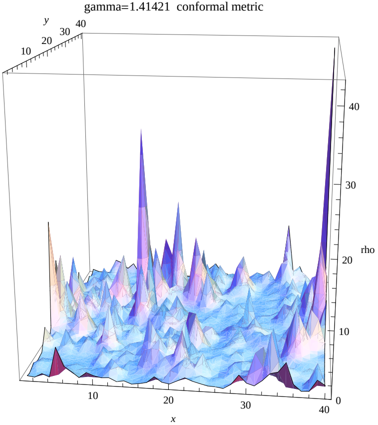
$$\kappa = \gamma^2 = 2$$



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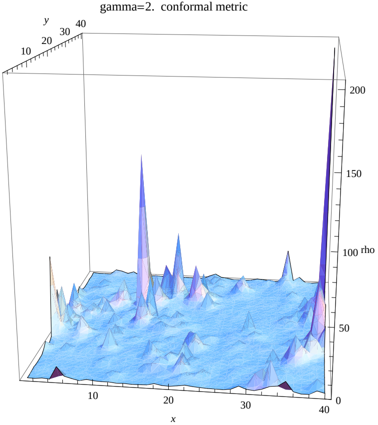
$$\kappa = \gamma^2 = 4$$



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What have we learnt?

- Diffusion (i.e. random walk) in a quite random environnement (the random metric) is not that different from diffusion in flat space, when viewed in the conformal coordinate system.
- Diffusion is fast in regions where $\varphi \gg 0$ (or positive scalar curvature R)
- Diffusion is slow in regions where $\varphi \ll 0$ (or negative scalar curvature R)
- This is the opposite behavior than the behavior of geodesic flows!
- When $\gamma > 2$ the diffusion process is nevertheless trapped in the atoms for the measure $e^{\gamma\varphi}$.
- On the simulations the process gets out of the atoms after a finite time, but this is a finite size effect (very small lattice)

Diffusion in a gravitational background

- In a non-flat "classical" smooth metric $\hat{g}$ this works easily!
- Replace the flat space heat kernel $K(z_1, z_2; t)$ by the heat kernel in the metric $\hat{g} = e^{\gamma\hat{\varphi}}$ (we have chosen a conformal gauge)

$$\hat{K}^{\hat{\varphi}}(z_1, z_2; t) = \langle z_1 | \mathrm{e}^{t\hat{\Delta}} | z_2 \rangle$$

- $\hat{\Delta} = \mathrm{e}^{-\gamma\hat{\varphi}}\,\Delta$ is the Laplacian in the gravitational background.
- The fractal measure on $\mathcal{X}$ is dressed by $\hat{\varphi}$ and becomes

$$d\mu^{\mathbf{0}}_{\mathcal{X}}(z) \rightarrow d\mu^{\hat{\varphi}}(z) = d\mu^{\mathbf{0}}_{\mathcal{X}}(z)\,\mathrm{e}^{-\gamma(1-x_{\mathbf{0}})\hat{\varphi}(z)} \tag{81}$$

- One still has the flat space short time scaling

$$B^{\hat{\varphi}}_{\mathcal{X}}(z_0; t) \underset{t\to 0}{\sim} t^{-x^{\mathbf{0}}}$$

- Now consider a quantum metric $g = e^{\gamma\varphi}$
- As we shall see, and as in KPZ, the quantum fractal dimension of $\mathcal{X}$ is modified $x_0 \rightarrow x$. This means that the fractal measure on $\mathcal{X}$ is dressed by the quantum field φ as

$$d\mu^{\mathbf{0}}_{X}(z) \rightarrow d\mu^{\varphi}(z) = d\mu^{\mathbf{0}}_{X}(z)\,\mathrm{e}^{-\gamma(1-x)\varphi(z)}$$

with

$$x \neq x_0$$

The quantum heat kernel and its Mellin transform

- The unknown dimension x is determined by the consistency condition that the quantum average of the convolution of the quantum measure by the quantum heat kernel

$$B^{\varphi}_{\mathcal{X}}(z_0;t) = \int d\mu^{\varphi}_{\mathcal{X}}(z)\,\tilde{K}^{\varphi}(z_0,z;t)$$

should scale at short times as

$$\langle B^{\varphi}_{\mathcal{X}}(z_0,t)\rangle_{\varphi} \sim t^{-x}$$

- Same scaling dimension x in the measure and in the short time evolution
- The short time behavior of $\langle B^{\varphi}(z_0,t)\rangle_{\varphi}$ can be calculated via the short distance behavior of the Mellin-Barnes transform w.r.t. the time t of the heat kernel,

$$M^{\varphi}(z_1,z_2;s) = \Gamma(s)\,\langle z_1|(-\tilde{\Delta})^{-s}|z_2\rangle$$

- By standard CFT calculation (exponential of free field calculations, as in algebraic KPZ) and a bit of replica trick (analytic continuation in s) one finds the short distange behavior of the e.v. of the dressed Mellin transform

$$\langle \mathrm{e}^{\gamma(1-x)\varphi(z_1)} M^{\varphi}(z_1,z_2;s)\rangle_{\varphi} \sim |z_1-z_2|^{2s-2+\frac{\gamma^2}{2}(s-1)(2x-s)}$$

when

$$|z_1-z_2| \to 0$$

From short distance to short time: back to KPZ

- Now integrate

$$\int \mathrm{e}^{\gamma(1-x)\varphi(z_1)} M^{\varphi}(z_1, z_2; s) \;\times\; d\mu^0_{\mathcal{X}}(z_1)$$

- Look for the smallest singularity s_c, (pole in the complex variable s), of the resulting Mellin transform of $B^{\varphi}(z_0, t)$, coming from short distance divergence in the integral over z.
- One finds for the singularity s_c the quadratic equation

$$x_0 = s_c - 1 + \frac{\gamma^2}{2}(s_c - 1)(2x - s_c)$$

- NB: This involves the classical dimension x_0 of the original measure $d\mu^0_{\mathcal{X}}(z)$ for the fractal $\mathcal{X}$
- This implies the short time scaling for the convolution of the measure with the heat kernel (inverse Mellin transform techniques)

$$\langle B^{\varphi}(z_0, t)\rangle_{\varphi} \sim t^{-s_c}$$

- The self-consistency constraint

$$s_c = x$$

  implies the KPZ relation

$$x_0 = x + \frac{\gamma^2}{4}x(x-1) \quad , \qquad x = 1 - \Delta = \delta/2$$