Random matrix ensembles for quantum spins and decoherence

François David
IPhT Saclay & CNRS

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+ work in progress

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In honor of Tom Spencer

Plan

1. The model: quantum spin + random matrices
2. The evolution functional: exact solution
3. Evolution of coherent and incoherent states
4. Quantum diffusion regime & initial conditions: to be or not to be Markovian
5. Extensions: spin clusters

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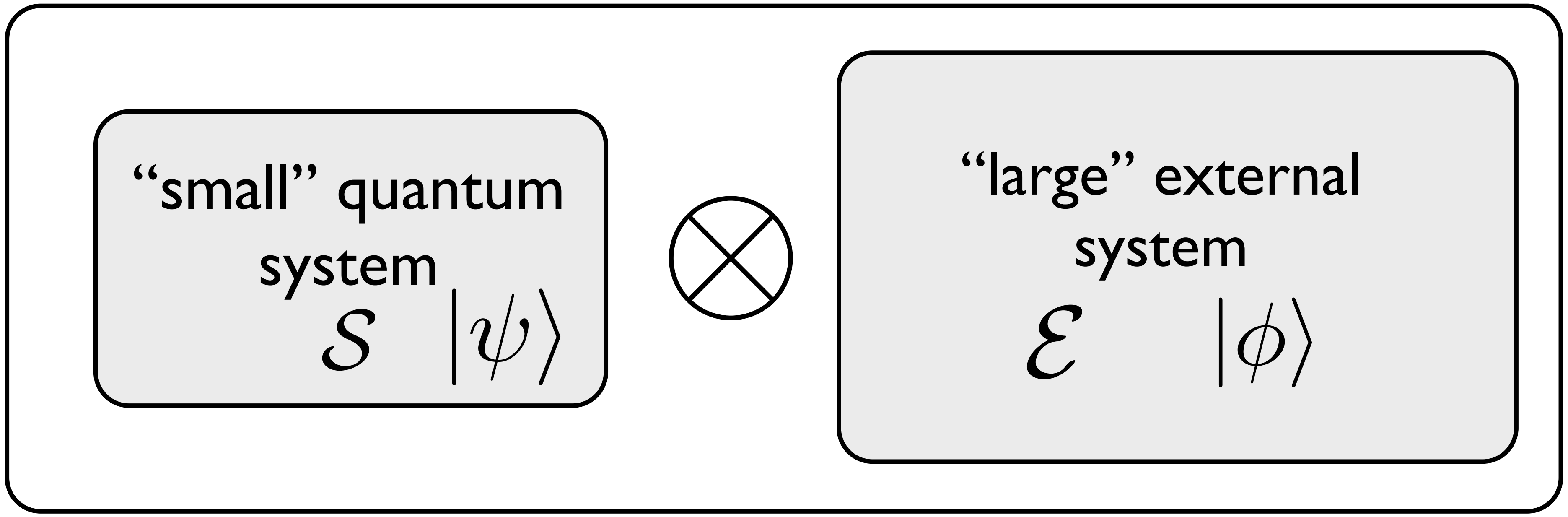
5. Extensions: spin clusters

Apologies

No disorder .... No SUSY ...

Decoherence



- Decoherence = disappearance - or rather inobservability - of the quantum correlations between
  - some states of a system ***s,*** through its (weak) coupling with an external system ***E*** (heat bath, environment, etc.)
  - or more generally a few “individualized” degrees of freedom (pointer states, semi-classical variables, collective coordinates, etc.) of a large isolated macroscopic system

$$(a_1|\psi_1\rangle + a_2|\psi_2\rangle) \otimes |\phi\rangle \rightarrow a_1|\psi_1'\rangle \otimes |\phi_1'\rangle + a_2|\psi_2'\rangle \otimes |\phi_2'\rangle$$

- I shall present a simple toy model
- based on very standard ideas:
  - spin and coherent states *(Takahashi & Shibata, 1975)*
  - random matrix hamiltonians *(Mello, Pereyra & Kumar, 1988)*
  - which have been much applied for the spin 1/2 case ($j = 1/2$, Q-bit, 2 level system)
- but some (relatively) novel aspects
  - general spin j (from quantum to classical spin)
  - generic interaction (novel random matrix ensembles)
- It allows to study analytically several aspects decoherence
- in particular the crossover between unitary quantum dynamics and stochastic diffusion in classical phase space

I - The model

A quantum SU(2) spin $\mathcal{S}$ + an external system $\mathcal{E}$

spin $= j \qquad \dim(\mathcal{H}_\mathcal{S}) = 2j+1 \qquad \dim(\mathcal{H}_\mathcal{E}) = N \gg j$

Single spin:

For large spin $j \to \infty$ the spin becomes a classical object

Classical phase space is the 2-sphere

The coherent states behave as quasi classical states

$$|\vec{n}\rangle \quad , \quad (\vec{n}\cdot\vec{\mathbf{S}})|\vec{n}\rangle = j|\vec{n}\rangle$$

Dynamics of the coupled spin:

$$H = H_\mathcal{S} \otimes \mathbf{1}_\mathcal{E} + H_{\mathcal{S}\mathcal{E}} + \mathbf{1}_\mathcal{S} \otimes H_\mathcal{E}$$

The Hamiltonians:

- Slow spin dynamics $\qquad H_\mathcal{S} = 0$
  (no dissipative & thermalisation effects)
- Dynamic of the external system generic $\qquad H_\mathcal{E} \to H_{\mathcal{S}\mathcal{E}}$

The interaction Hamiltonian

The interaction hamiltonian is given by a Gaussian random matrix ensemble, with the only constraint that the ensemble in invariant under

$$SU(2) \times U(N)$$

spin → $SU(2)$; external system → $U(N)$

For this, go to Wigner representation of spin operators

$$\langle r\alpha|H|s\beta\rangle = H^{rs}_{\alpha\beta} \to W^{(lm)}_{\alpha\beta} \qquad \mathbf{j} \otimes \mathbf{j} = \mathbf{0} \oplus \mathbf{1} \oplus \cdots \oplus \mathbf{2j}$$

$$A_{rs} = \langle r|A|s\rangle \qquad W^{(l,m)}_{A} = \sum_{r,s=-j}^{j} \sqrt{\frac{2l+1}{2j+1}} \left\langle \begin{matrix} j & l \\ r & m \end{matrix} \middle| \begin{matrix} j \\ s \end{matrix} \right\rangle A_{rs}$$

It is enough to take for the $W^{(lm)}_{\alpha\beta}$ independent gaussian random variables with zero mean and variance depending only on l and with the Hermiticity constraint.

$$\mathrm{Var}\left(W^{(lm)}_{\alpha\beta}\right) = \Delta(l) \qquad W^{(l,m)}_{\alpha\beta} = (-1)^m \overline{W}^{(l,-m)}_{\beta\alpha}$$

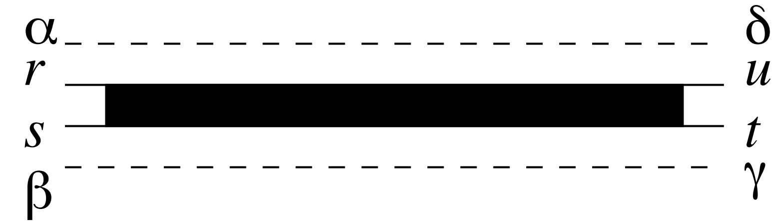
We thus get a matrix ensemble characterized by the variances

$$\Delta = \{\Delta(l)\,,\; l = 0, 1, \cdots 2j\}$$

NB: The *l=m=0* term represents the $H_{\mathcal{E}}$ Hamiltonian

With this GU(2)xU(N) ensemble, the 2-points correlator is

$$\overline{H^{rs}_{\alpha\beta} H^{tu}_{\gamma\delta}} = \delta_{\alpha\delta}\delta_{\beta\gamma}\mathcal{D}_{rs,tu}$$



$$\mathcal{D}_{rs,tu} = \quad \delta_{s-r,t-u} \sum_{l=0}^{2j} \Delta(l) \frac{2l+1}{2j+1} \left\langle \begin{matrix} j & l \\ s & r-s \end{matrix} \middle| \begin{matrix} j \\ r \end{matrix} \right\rangle \left\langle \begin{matrix} j & l \\ t & u-t \end{matrix} \middle| \begin{matrix} j \\ u \end{matrix} \right\rangle$$

This representation allows to use diagrammatic rules to resum perturbative expansions in the interaction.
Standard ribbon propagator for the N indices, more complicated structure for the spin indices, but still planar.

II - The evolution functional

separable state $\to$ entangled state $\to$ mixed state for $\mathcal{S}$

$$|\psi_0\rangle \otimes |\phi_0\rangle \to |\Phi(t)\rangle \; , \quad \rho_{\mathcal{S}}(t) = \mathrm{tr}_{\mathcal{E}}(|\Phi(t)\rangle\langle\Phi(t)|$$

Evolution functional

$$\rho_{\mathcal{S}}(t) = \mathcal{M}(t) \cdot \rho_{\mathcal{S}}(0) \; , \quad \mathcal{M}(t) = \mathrm{tr}_{\mathcal{E}} \left(e^{-itH} \left(\cdot \otimes \rho_{\mathcal{E}}(0) \right) e^{itH} \right)$$

For simplicity, start from a random state $|\psi_{\mathrm{E}}\rangle$

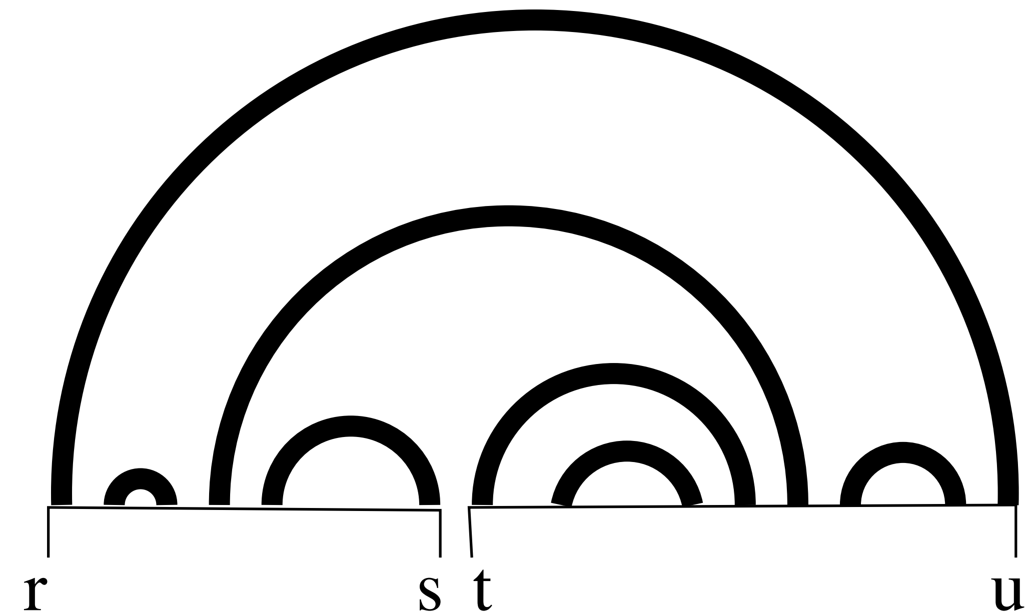
Then the evolution functional is

$$\mathcal{M}(t) = \oint \frac{dx}{2i\pi} \oint \frac{dy}{2i\pi} \; e^{it(x-y)} \, \mathcal{G}(x,y)$$

$$\mathcal{G}(x,y) = \frac{1}{N} \mathrm{tr}_{\mathcal{E}} \left[\frac{1}{x-H} \otimes_{\mathcal{S}} \frac{1}{y-H} \right]$$

We take the large N limit (large external system) and make the average over H, assuming self averaging as usual.

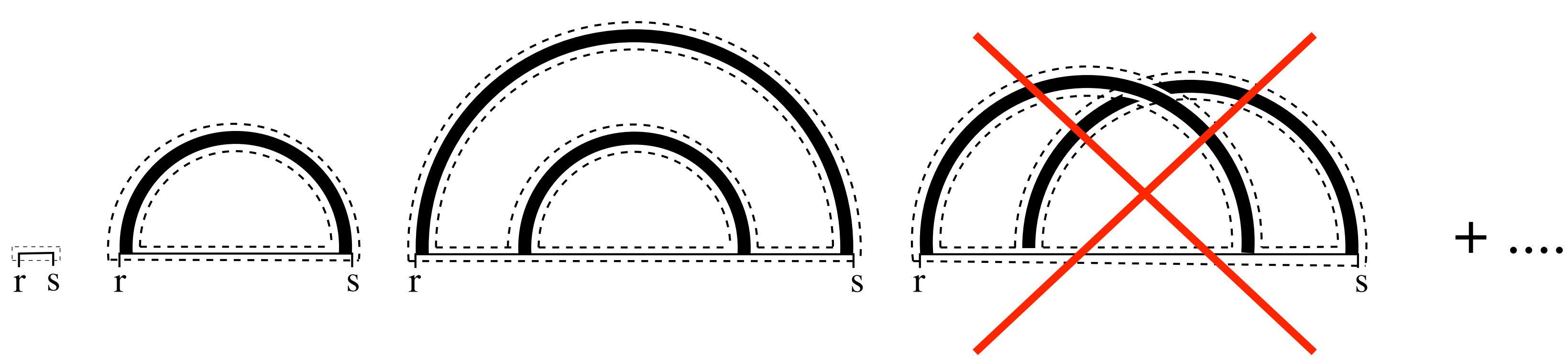
$\overline{\mathcal{G}(x,y)}$ is given by a sum of planar diagrams of the standard form (rainbow diagrams)



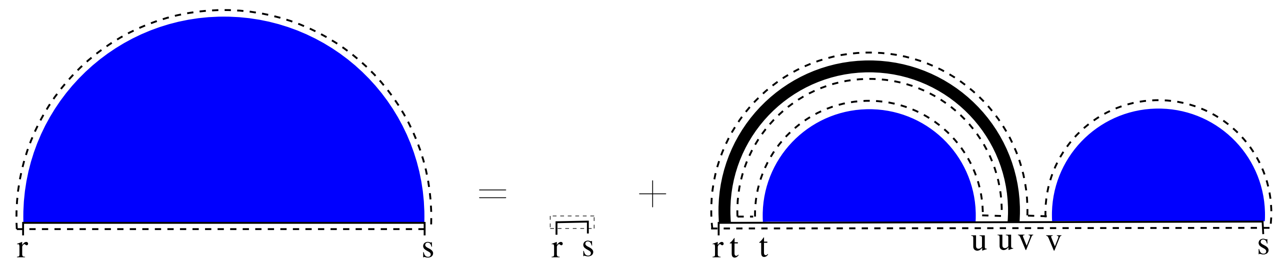
It is useful to start from the single resolvent

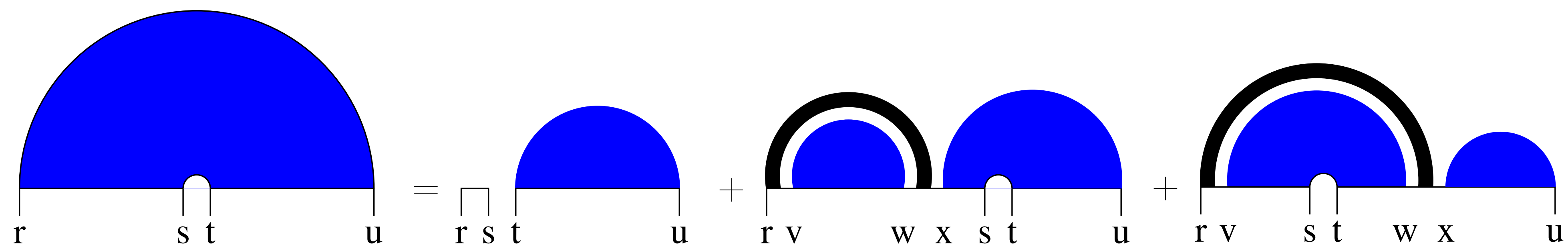
$$\mathcal{H}(x) = \frac{1}{N}\mathrm{tr}_{\mathcal{E}}\left[\frac{1}{x-H}\right]$$

$\overline{\mathcal{H}(x)}$ is given by a sum of planar rainbow diagrams



These resolvents obey recursion relations





Thanks to the SU(2) invariance, the solution of these equations takes a simple diagonal form in the Wigner representation

$$\overline{\mathcal{H}}_{rs}(x) = \delta_{rs}\widehat{\mathcal{H}}(x)$$

$$\overline{\mathcal{G}}_{rs,tu}(x,y) \rightarrow \overline{\mathcal{W}}_{\mathcal{G}}^{(l_1,m_1),(l_2,m_2)}(x,y) = \delta_{l_1 l_2}\delta_{m_1+m_2,0}\,(-1)^{m_1}\widehat{\mathcal{G}}^{(l)}(x,y)$$

with

$$\widehat{\mathcal{H}}(x) = \frac{1}{2\widehat{\Delta}(0)}\left(x - \sqrt{x^2 - 4\widehat{\Delta}(0)}\right)$$

$$\widehat{\Delta}(0) = N\sum_{l=0}^{2j}\frac{2l+1}{2j+1}\,\Delta(l)$$

Resolvent for a single Wigner matrix (semi circle law)

Factorization

The evolution functional for the density matrix of the spin $\rho_{\mathcal{S}}(t)$ takes a simple diagonal form in the Wigner representation basis

$$\rho_{\mathcal{S}\,rs}(t) \;\to\; W_{\mathcal{S}}^{(l,m)}(t) = \widehat{\mathcal{M}}^{(l)}(t) \cdot W_{\mathcal{S}}^{(l,m)}(0)$$

with the kernel given by a universal decoherence function

$$\widehat{\mathcal{M}}^{(l)}(t) = M(t/\tau_0, Z(l))$$

depending on a rescaled time $t' = t/\tau_0$ and a factor $Z(l)$

$$\tau_0 = 1/\sqrt{\widehat{\Delta}(0)} \qquad\qquad Z(l) = \frac{\widehat{\Delta}(l)}{\widehat{\Delta}(0)}$$

τ_0 is the dynamical time scale of the system (more later)

The parameter $Z(l)$ depends on the spin sector considered.

The $Z(l)$ function

The l dependence of the factor $Z(l)$ depends on the initial variances of the GU(2) ensemble for the Hamiltonian.

$$\widehat{\Delta}(l) = N \sum_{l'=0}^{2j} \Delta(l')(2l'+1)(-1)^{2j+l'+l} \begin{Bmatrix} j & j & l' \\ j & j & l \end{Bmatrix}$$ ← 6-j symbol

$$Z(l) = \widehat{\Delta}(l)/\widehat{\Delta}(0) \qquad\qquad Z(l) \in [-1,1]$$

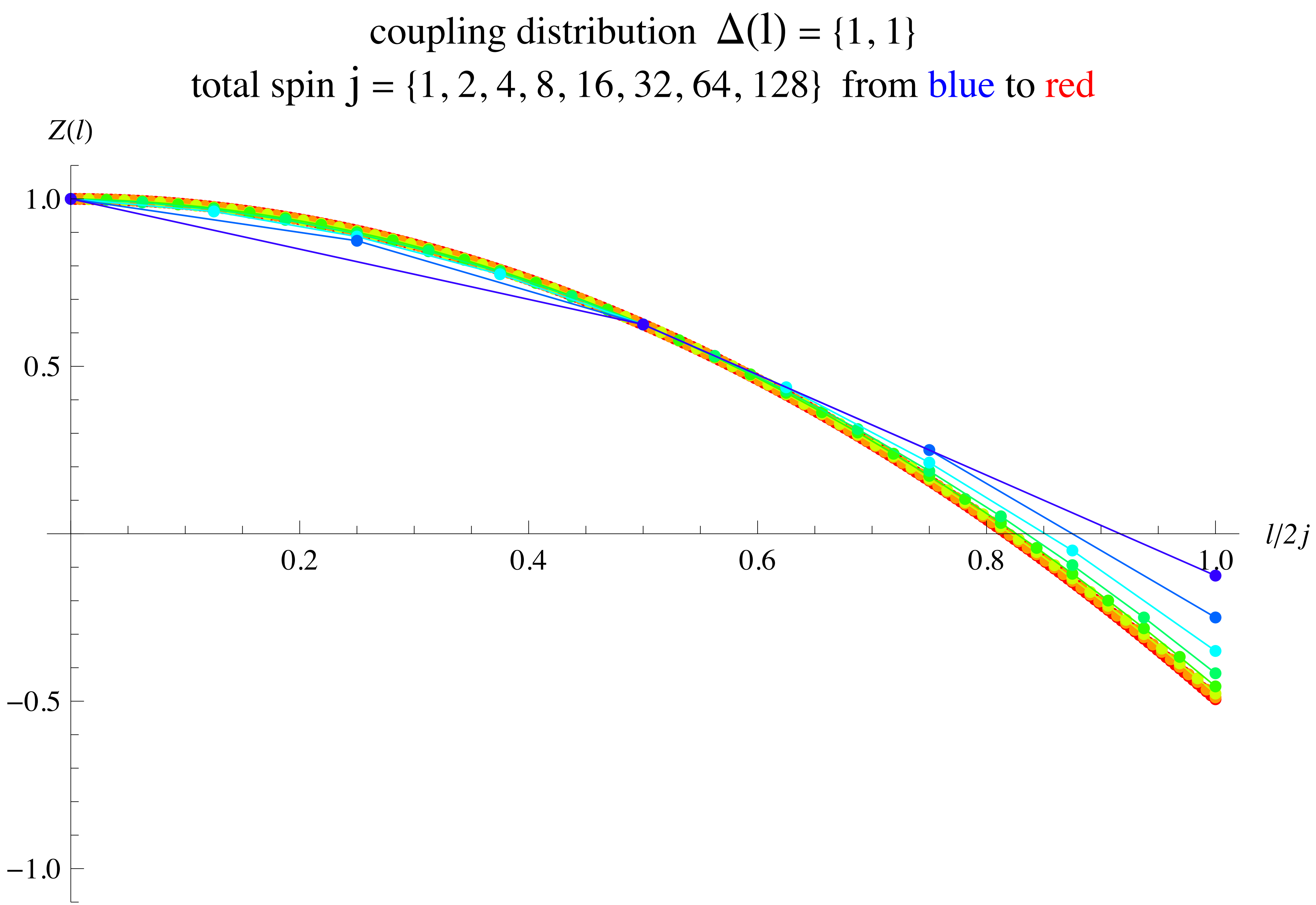
$Z(l)$ is maximal for $l=0$
$Z(l)$ takes a scaling form in the large spin limit

$$Z(l) = \widehat{\Delta}(l)/\widehat{\Delta}(0) \rightarrow Y(x) \text{ with } x = l/2j$$

Its small l behavior is quadratic in l

$$Z(l) = 1 - l(l+1)\,\frac{1}{4}\frac{D_0}{j(j+1)} + \cdots \quad , \quad D_0 = \frac{\sum_{l'=1}^{l_0} \bar{\Delta}(l')\,(2l'+1)\,l'(l'+1)}{\sum_{l'=0}^{l_0} \bar{\Delta}(l')(2l'+1)}$$

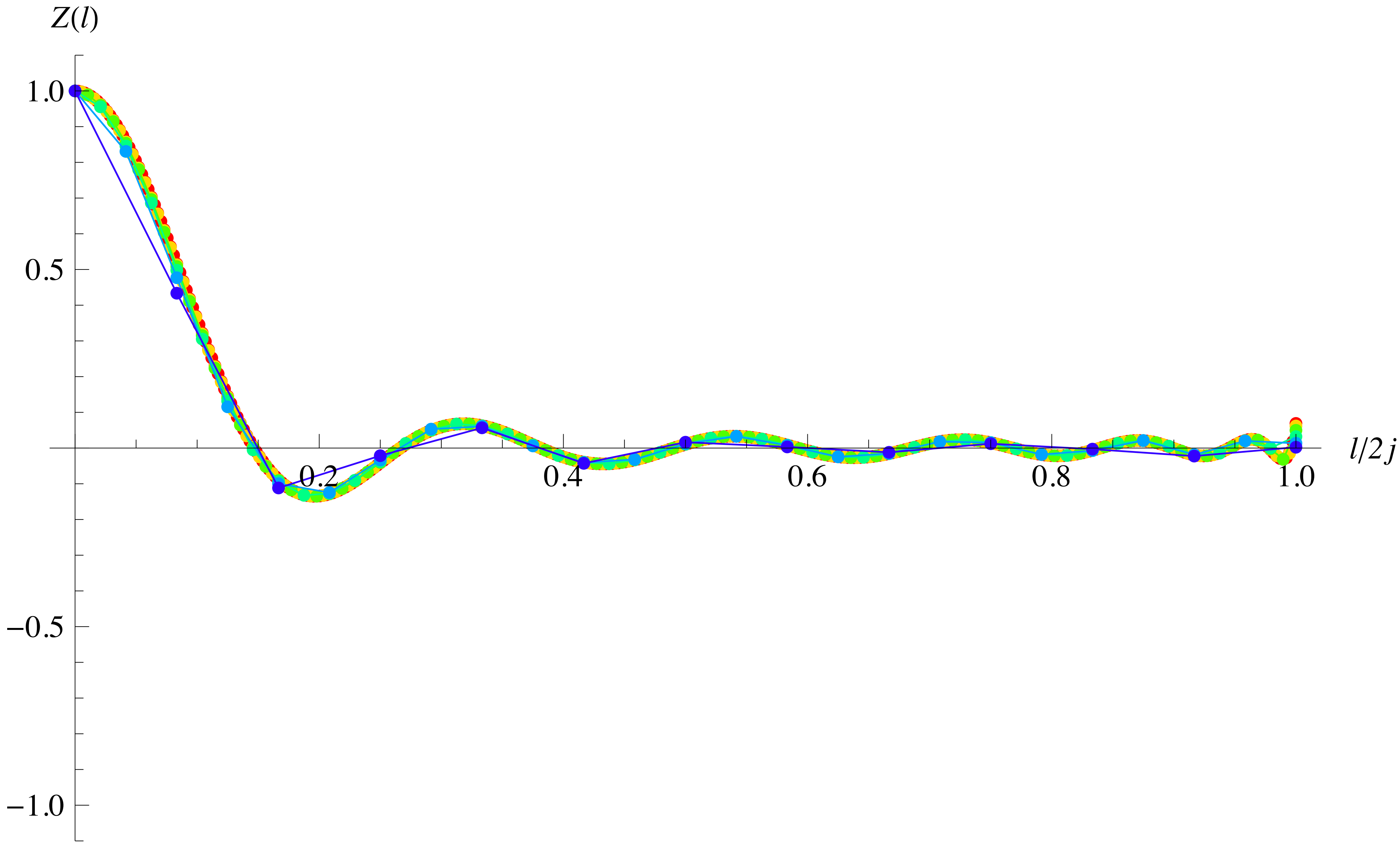
Example 1: l=0 and 1 channels only



Example 2: l=0 to 12 channels

coupling distribution $\Delta(l) = \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$

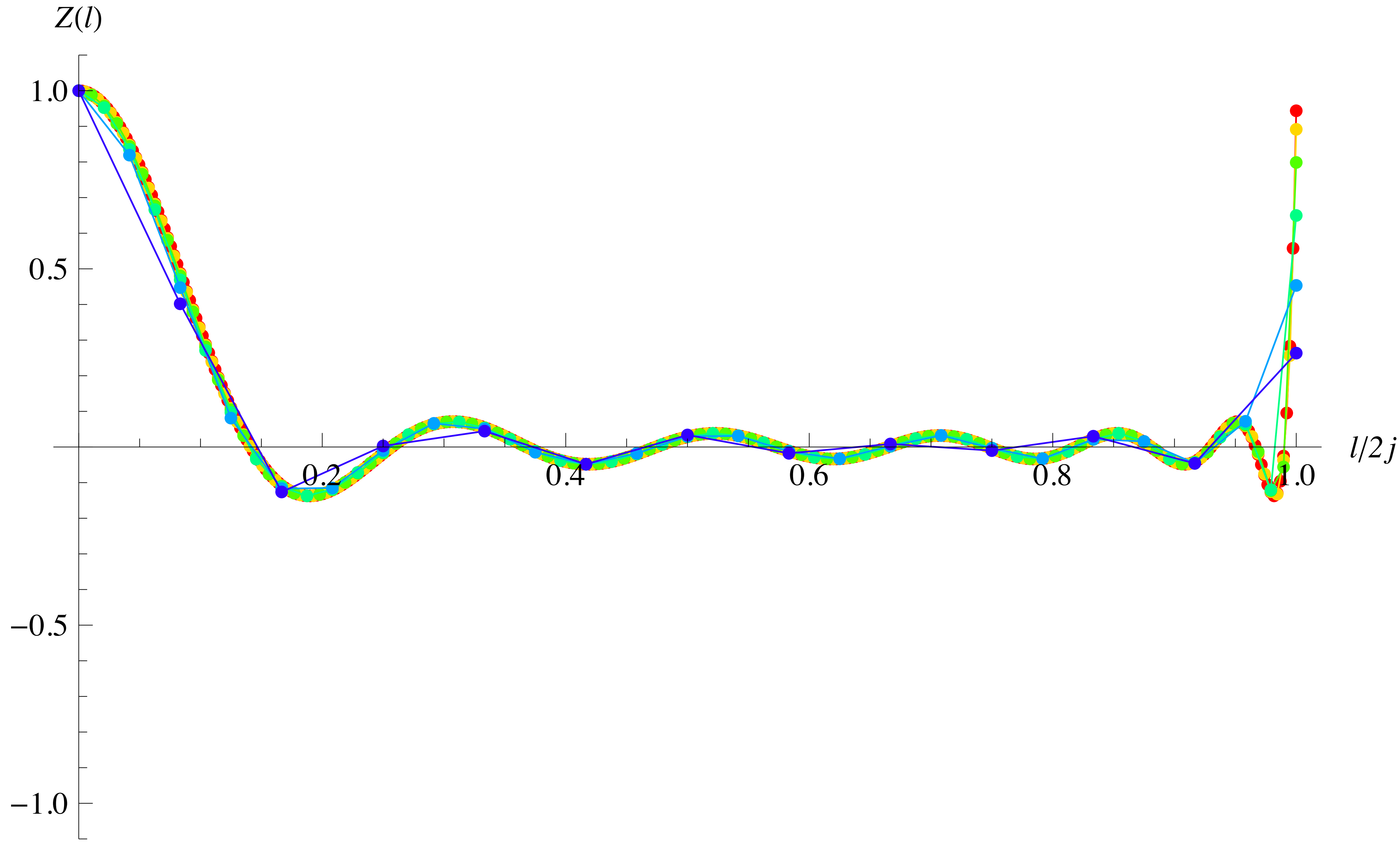
total spin $j = \{24, 48, 96, 192, 384, 768\}$ from blue to red



Example 3: l=0 to 12 but even only channels

coupling distribution $\Delta(l) = \{1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1\}$

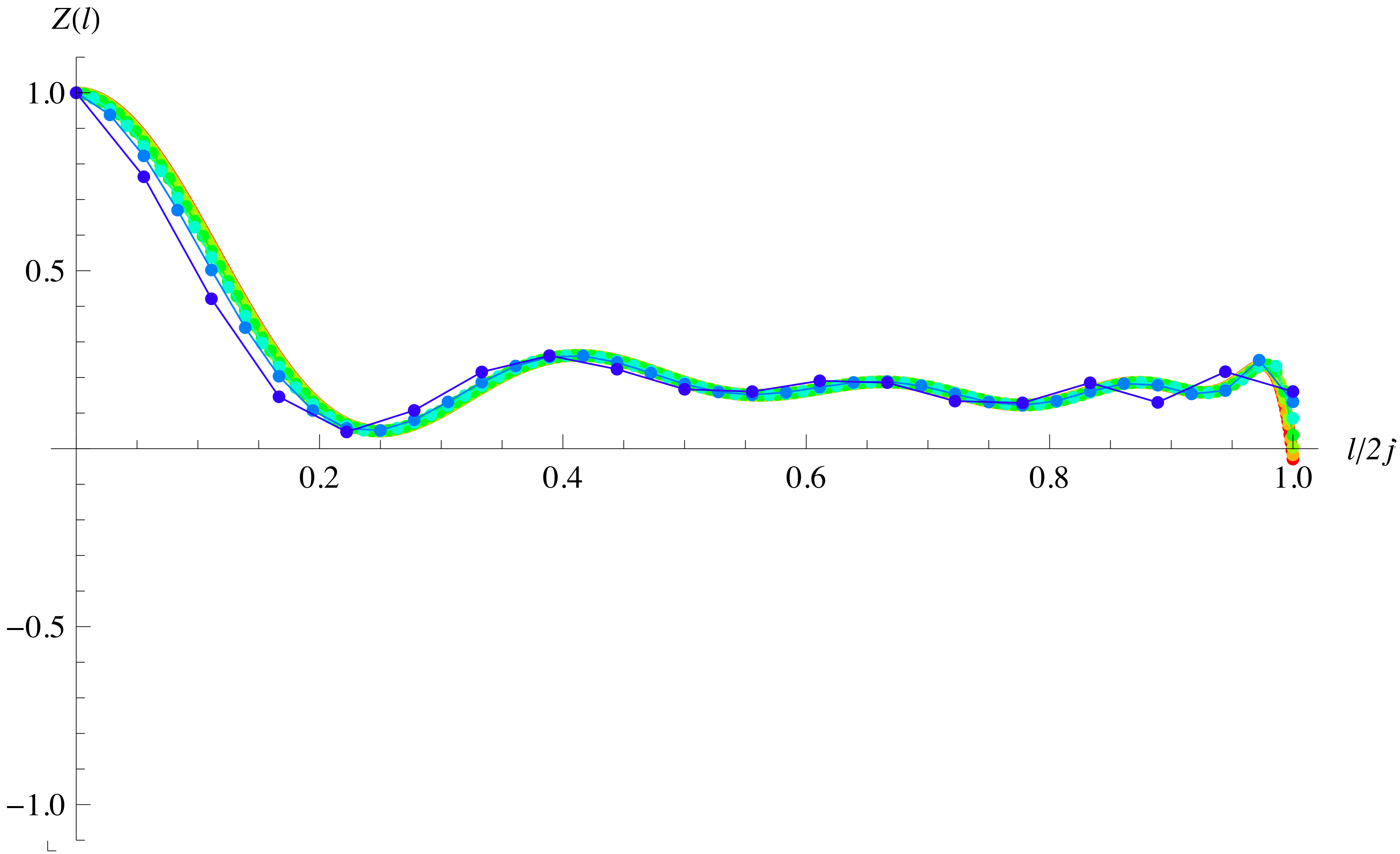
total spin $j = \{24, 48, 96, 192, 384, 768\}$ from blue to red



Example 4: l=0 to 10 channels, random variances

coupling distribution $\Delta(l) = \{16., 0.99, 0.94, 0.44, 0.3, 0.94, 0.65, 0.96, 0.64, 0.82\}$

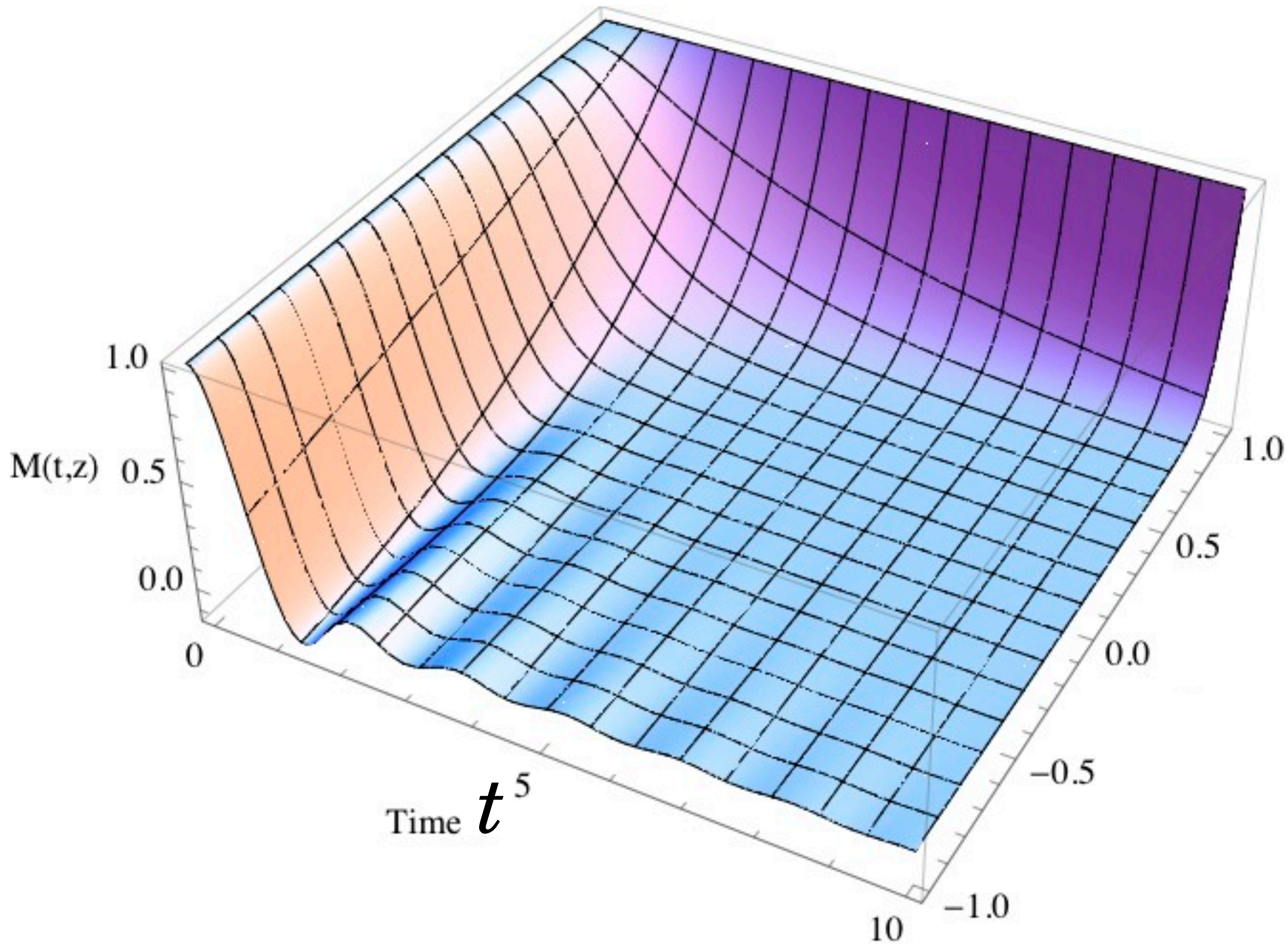
total spin $j = \{9, 18, 36, 72, 144, 288, 576\}$ from blue to red



The decoherence function is a generalized hypergeometric function

$$M(t,Z) = \oint \frac{dx}{2i\pi} \oint \frac{dy}{2i\pi}\, e^{-it(x-y)}\, \frac{H(x)H(y)}{1 - Z\, H(x)H(y)} \quad , \qquad H(x) = \frac{1}{2}(x - \sqrt{x^2-4})$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{m} t^{2m}\, z^n\, (-1)^{m+n}\, \frac{2(2m+1)(n+1)^2(2m)!}{m!(m+1)!(m-n)!(m+n+2)!}$$



large time limit: fast algebraic decay with t except for Z close to unity

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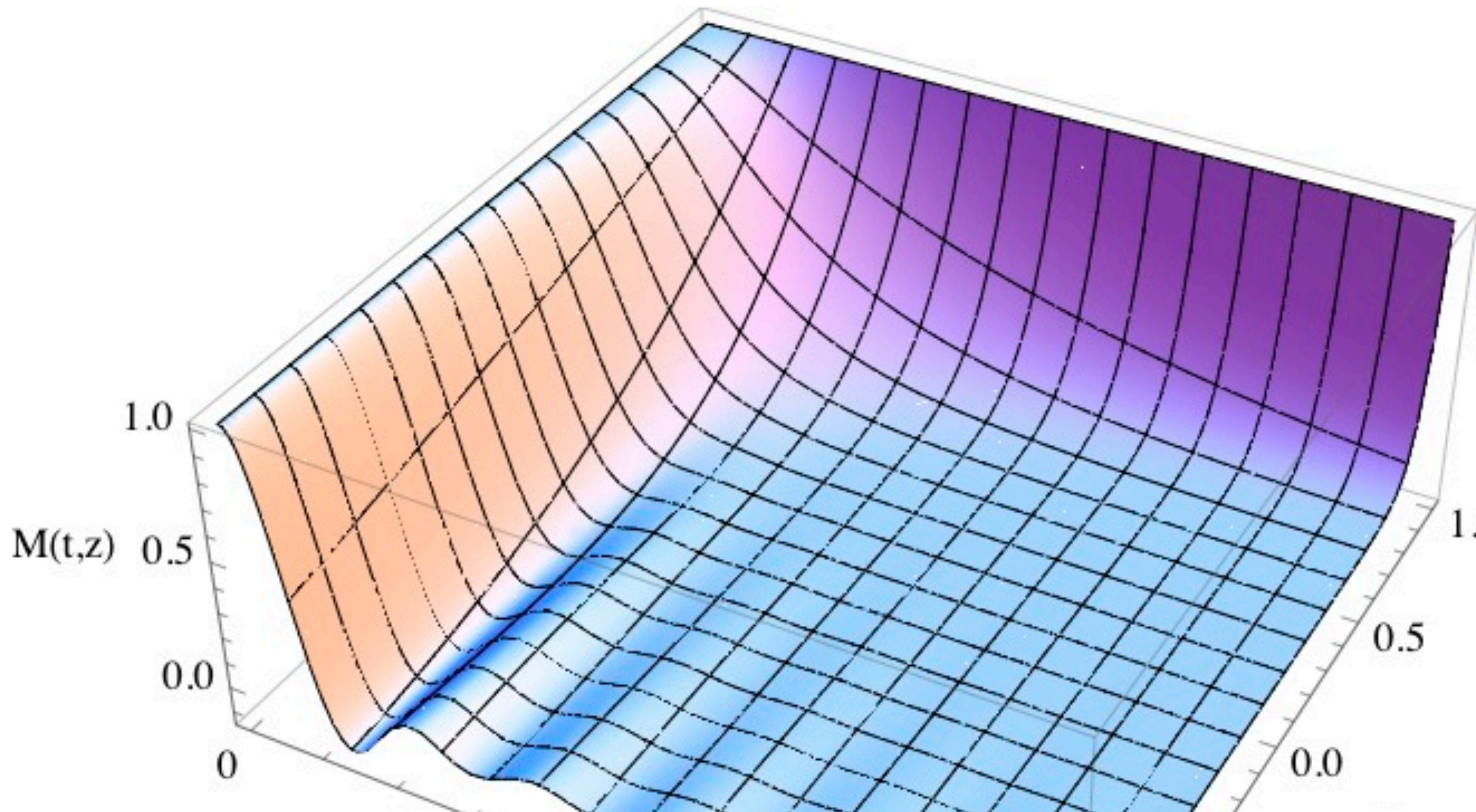
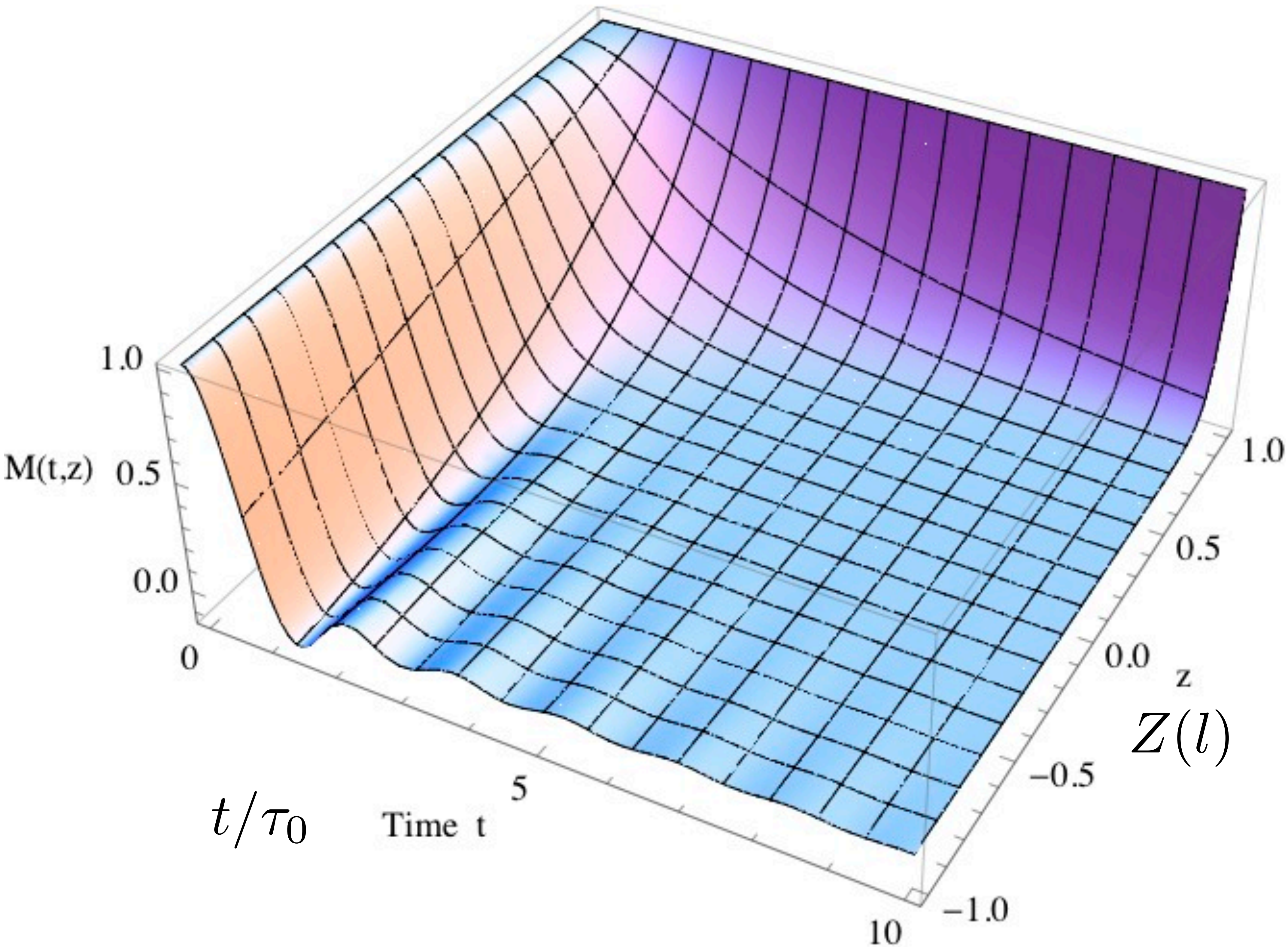


large time limit: fast algebraic decay with t except for Z close to unity

Z

$$M(t,z) = \frac{1}{2\pi}\, t^{-3} \left(\frac{1+z}{(1-z)^3} - \frac{1-z}{(1+z)^3} \sin(4t) \right) \left(1 + \mathcal{O}(t^{-1})\right)$$

Small *1-Z* scaling



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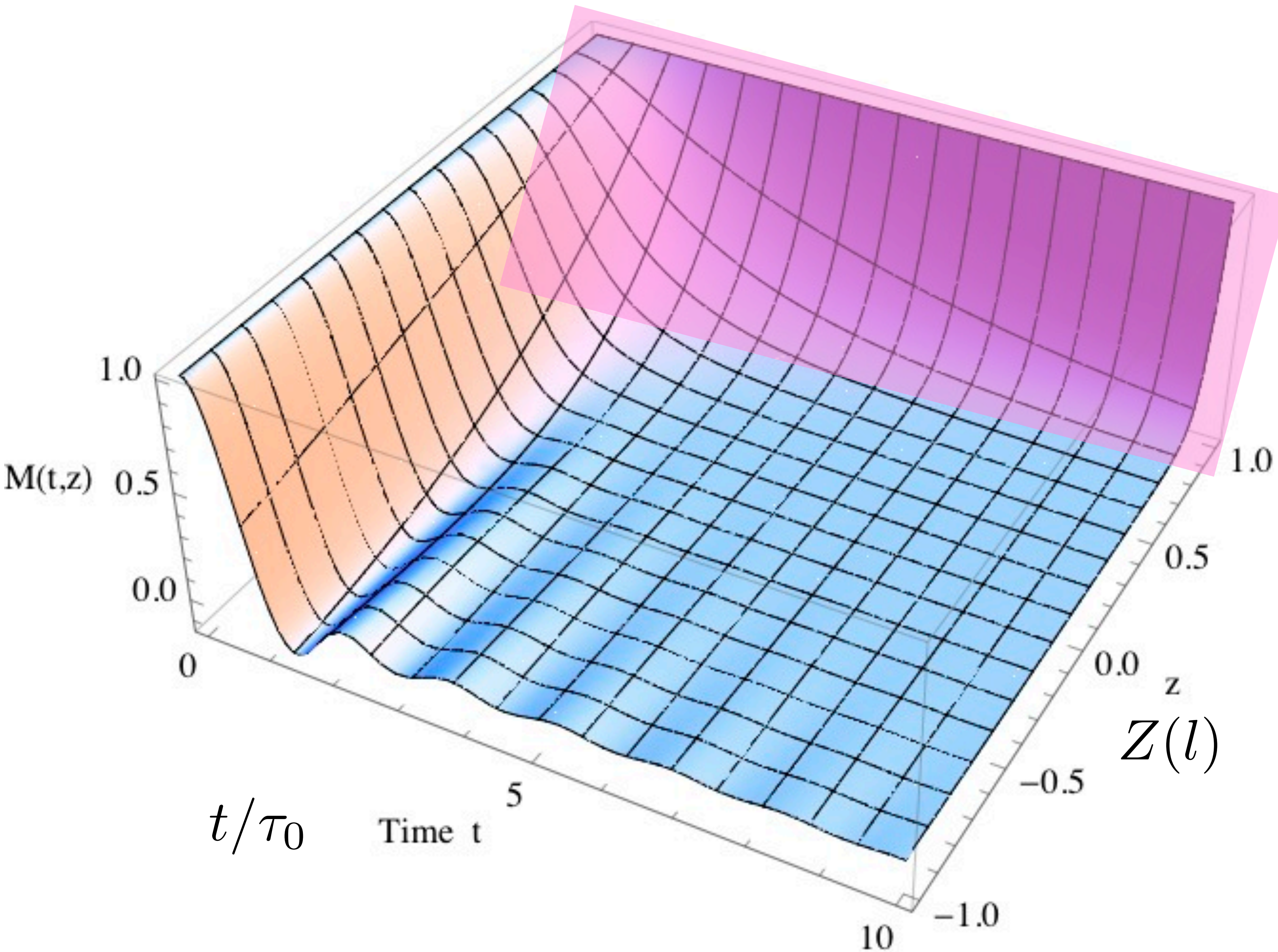
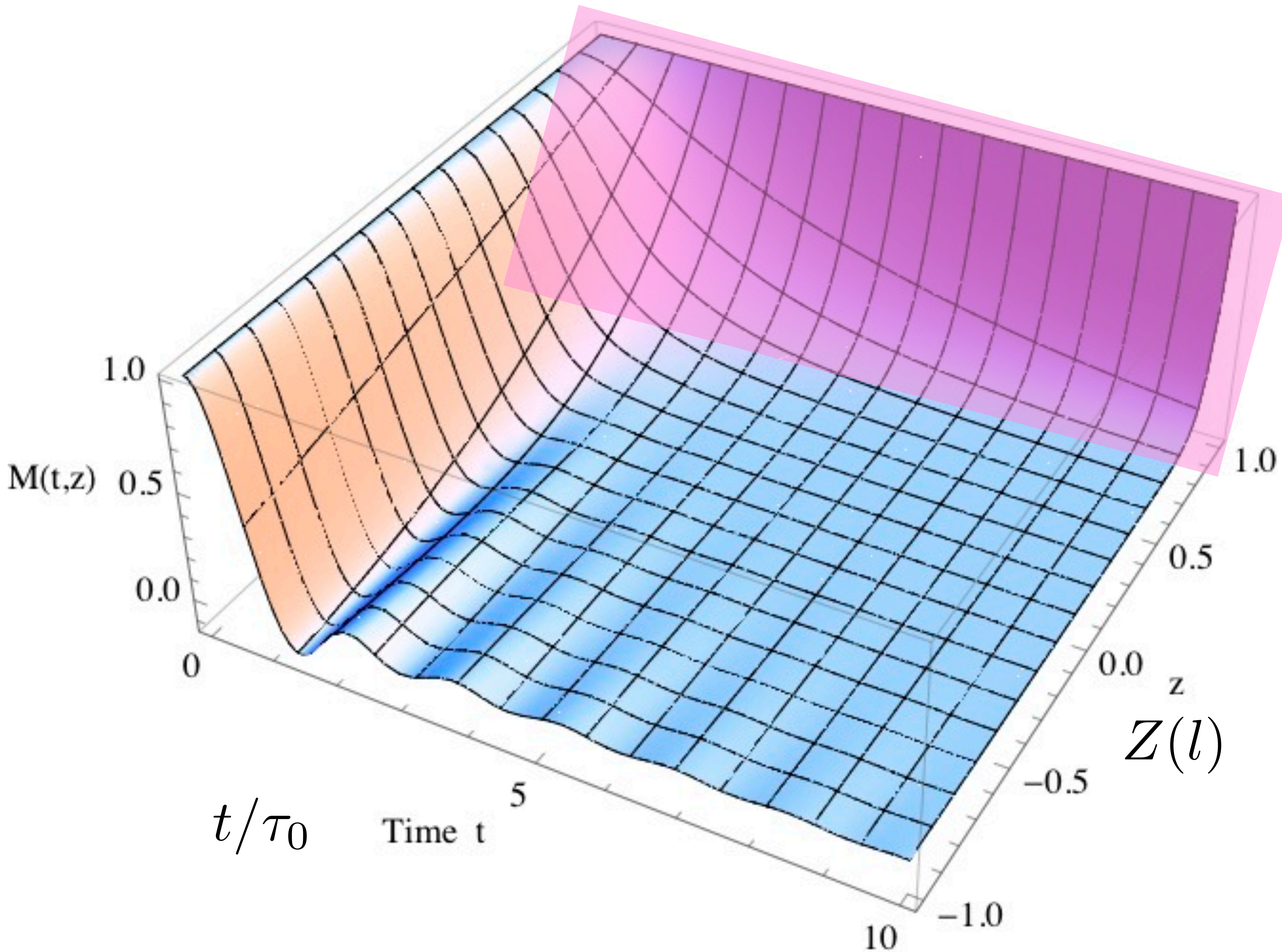


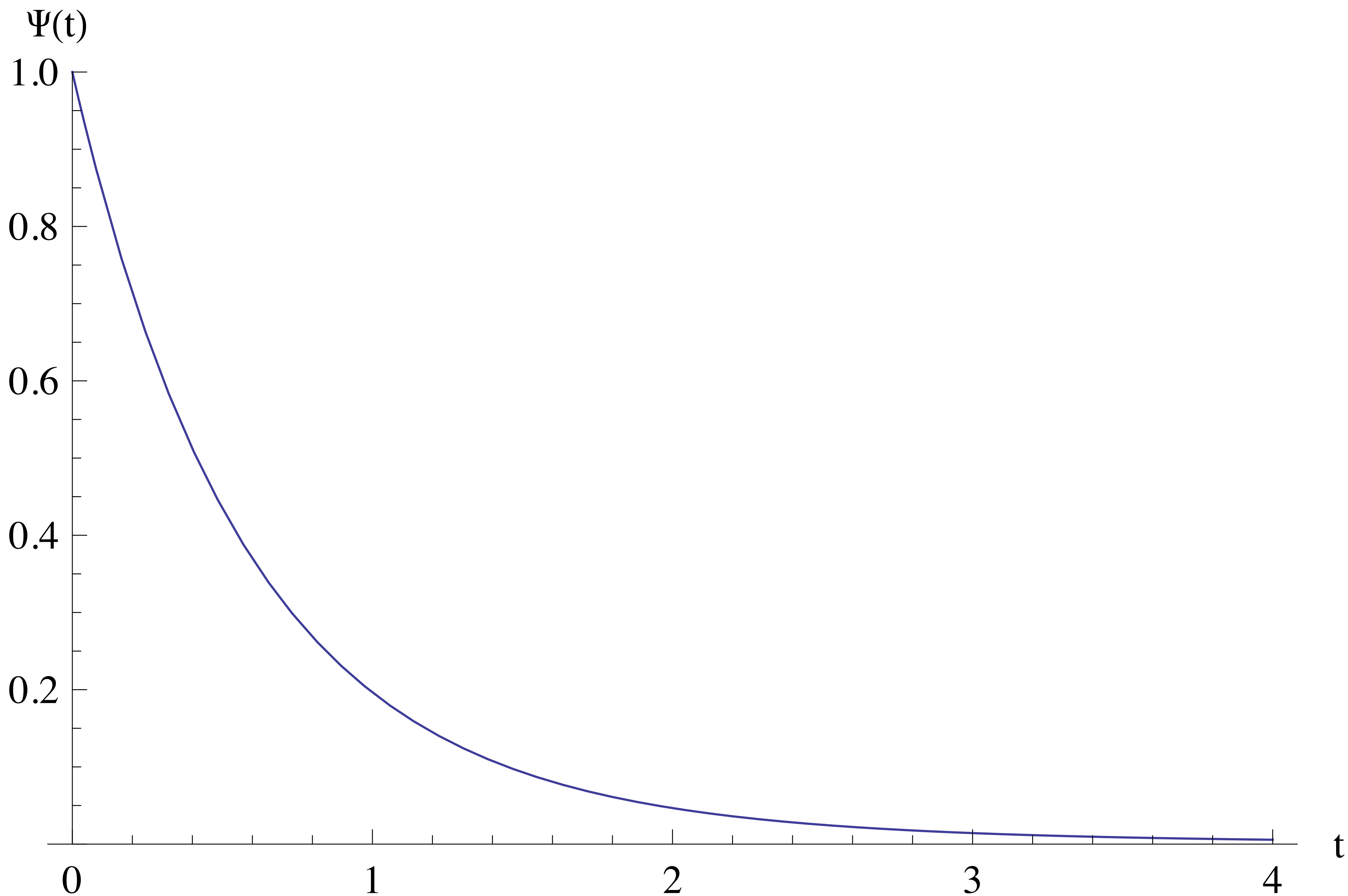
Small *1-Z* scaling

$$M(t', z) = \Psi(t'') \quad \text{with} \quad t'' = t'(1 - z)$$



Small Z scaling function

$$\Psi(t'') \;=\; \frac{1}{2\pi}\int_{-2}^{2} dx\,\sqrt{4-x^2}\; e^{-t''\,\sqrt{4-x^2}}$$



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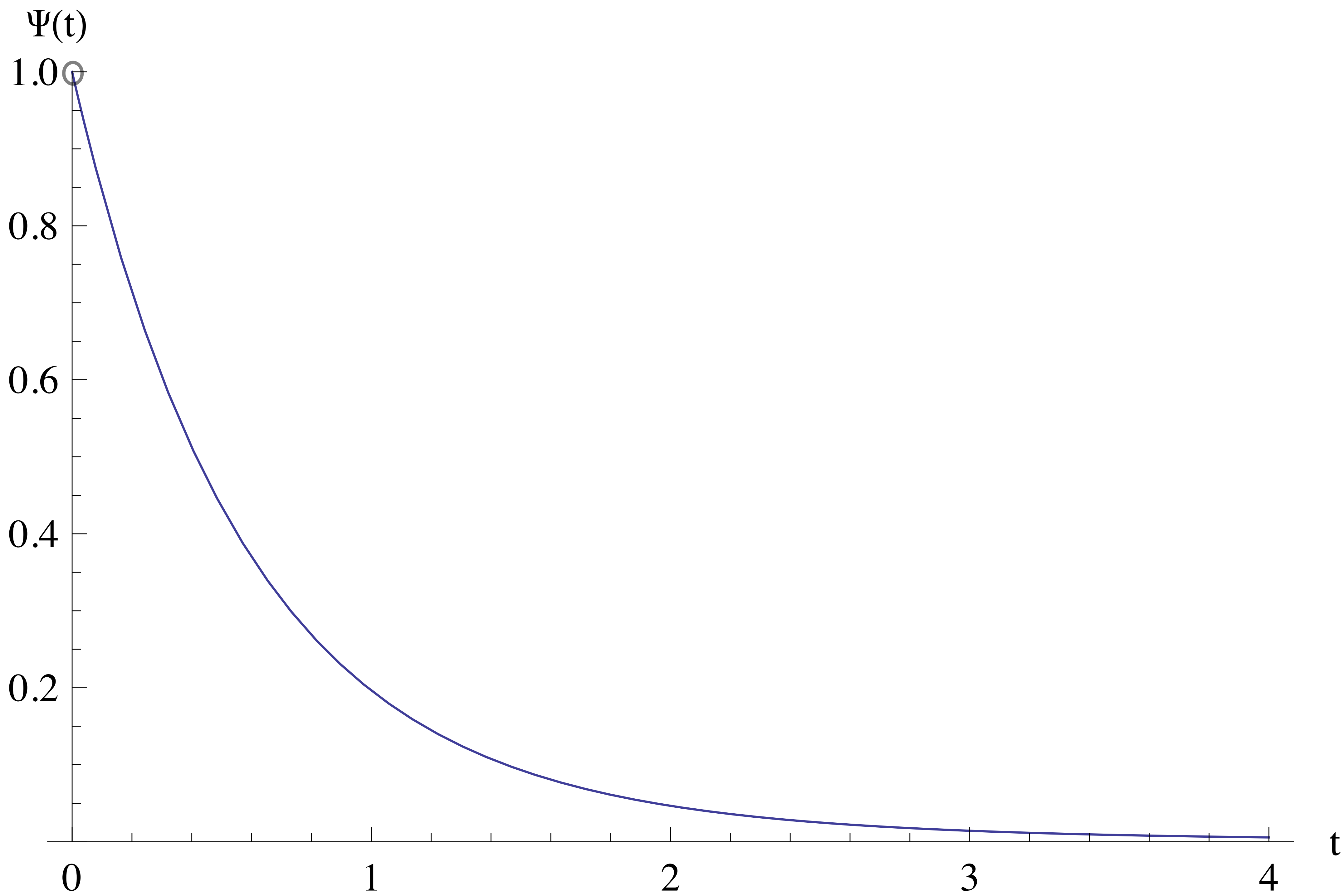

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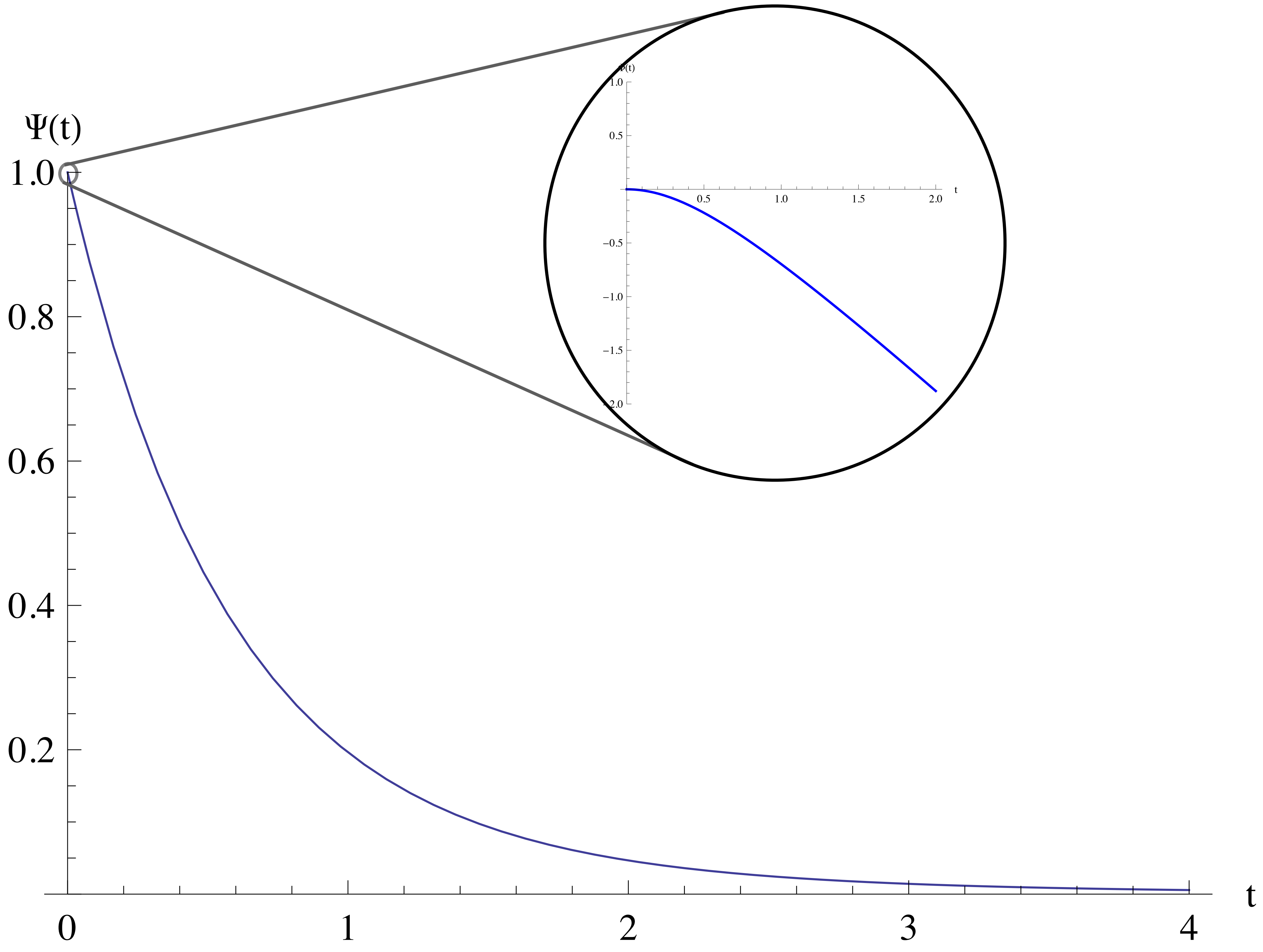
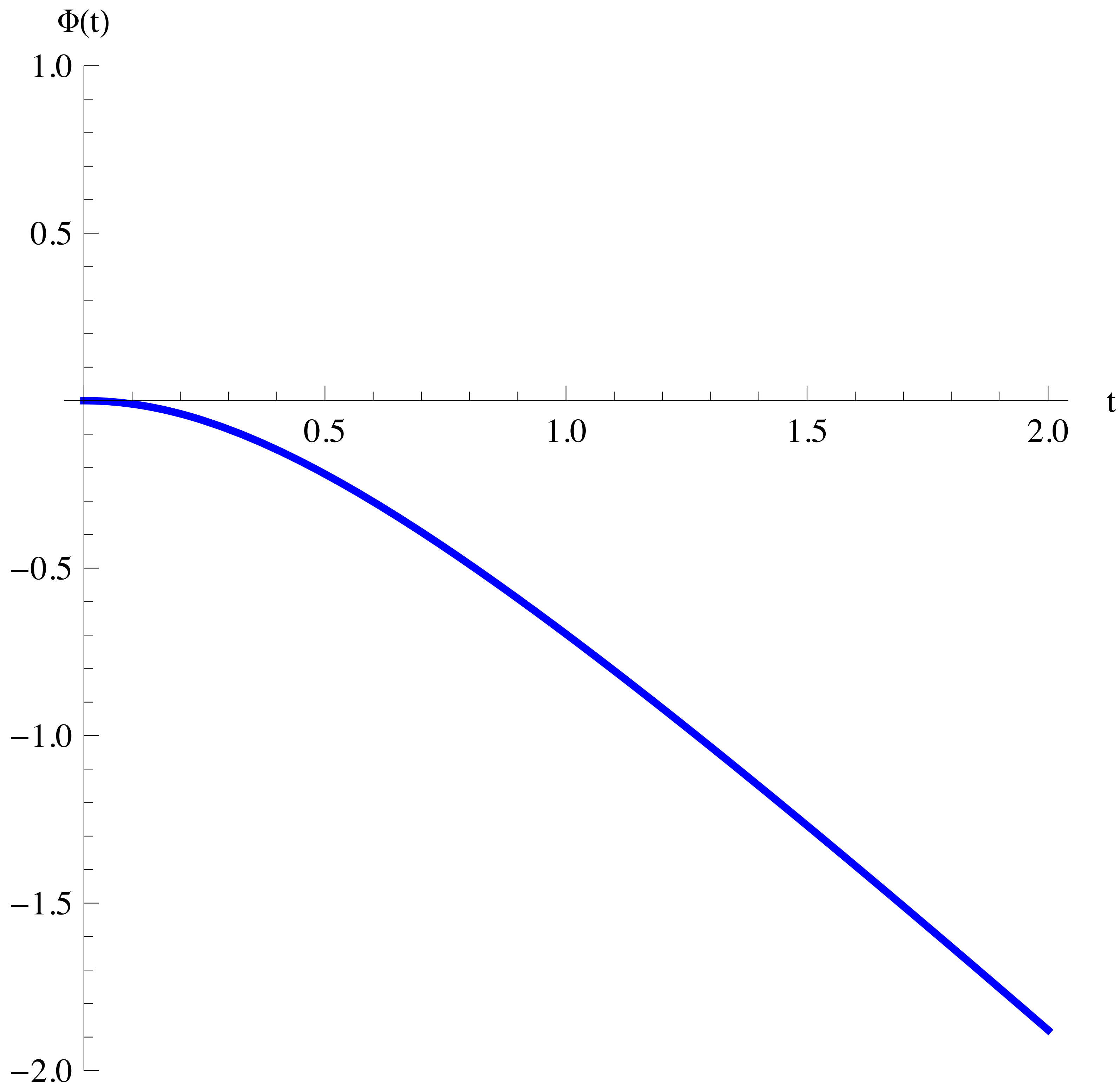


small t and Z=1 behavior

$$M(t,z) = 1 + (1-z)\,\Phi(t) + \cdots$$

$$\Phi(t) = 1 - \,_1F_2\left(-\tfrac{1}{2}; 1, 2; -4t^2\right)$$



III - Evolution of coherent and incoherent states

We can easily study analytically and illustrate the evolution on the matrix density of the spin, starting from a pure spin state $|\psi\rangle$

$$|\psi\rangle \to \rho = |\psi\rangle\langle\psi| \to W^{(l,m)} \to W(\vec{n}) = \sum_{l,m} W^{(l,m)}\, Y_l^m(\vec{n})$$

Wigner distribution = function on the sphere

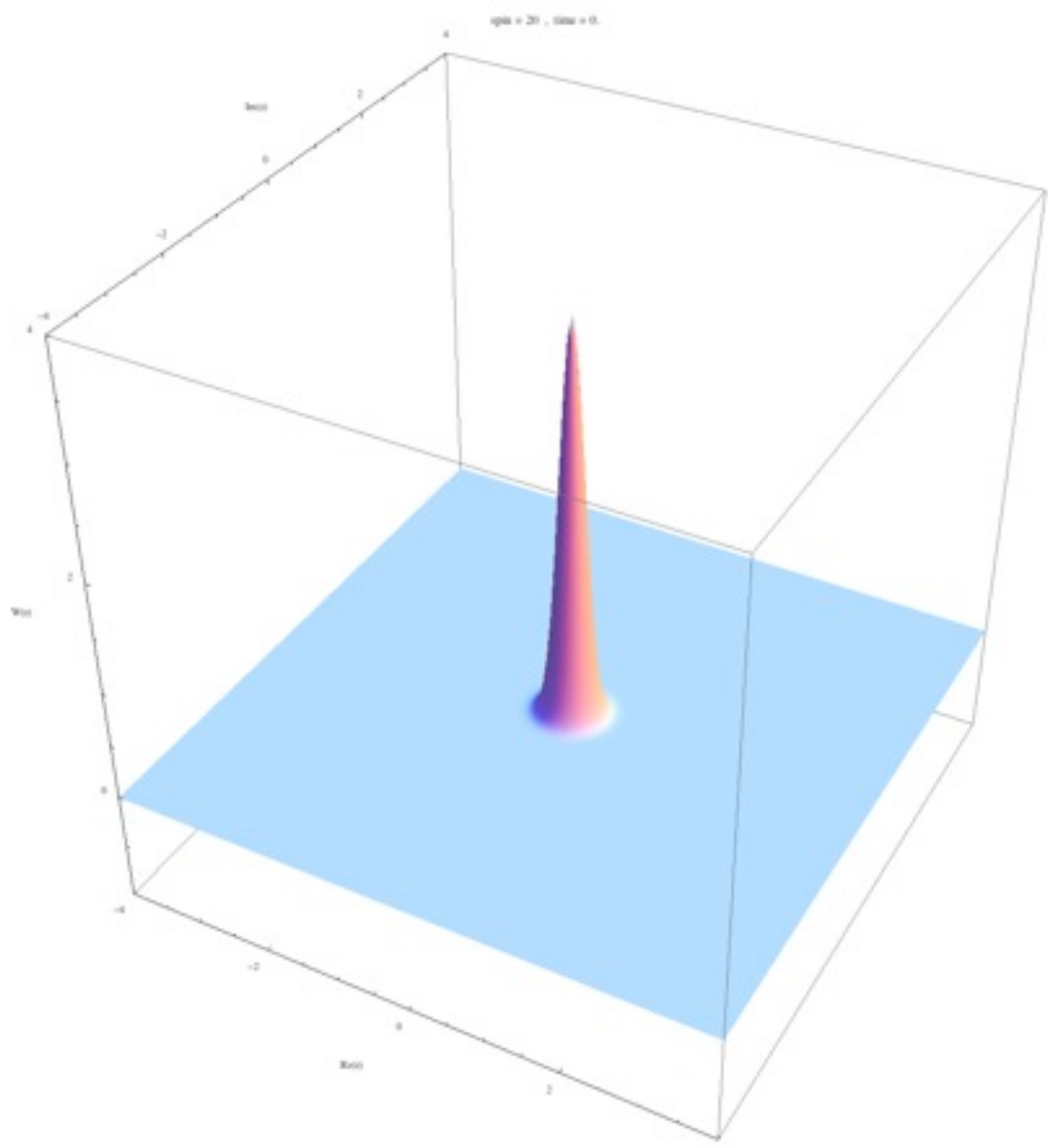
Coherent state

$$|\vec{n}\rangle = \sum_{m=-j}^{j} \sqrt{\frac{(2j)!}{(j+m)!\,(j-m)!}} \cos(\theta/2)^{j+m} \sin(\theta/2)^{j-m} \mathrm{e}^{-\mathrm{i}m\phi} |m\rangle$$

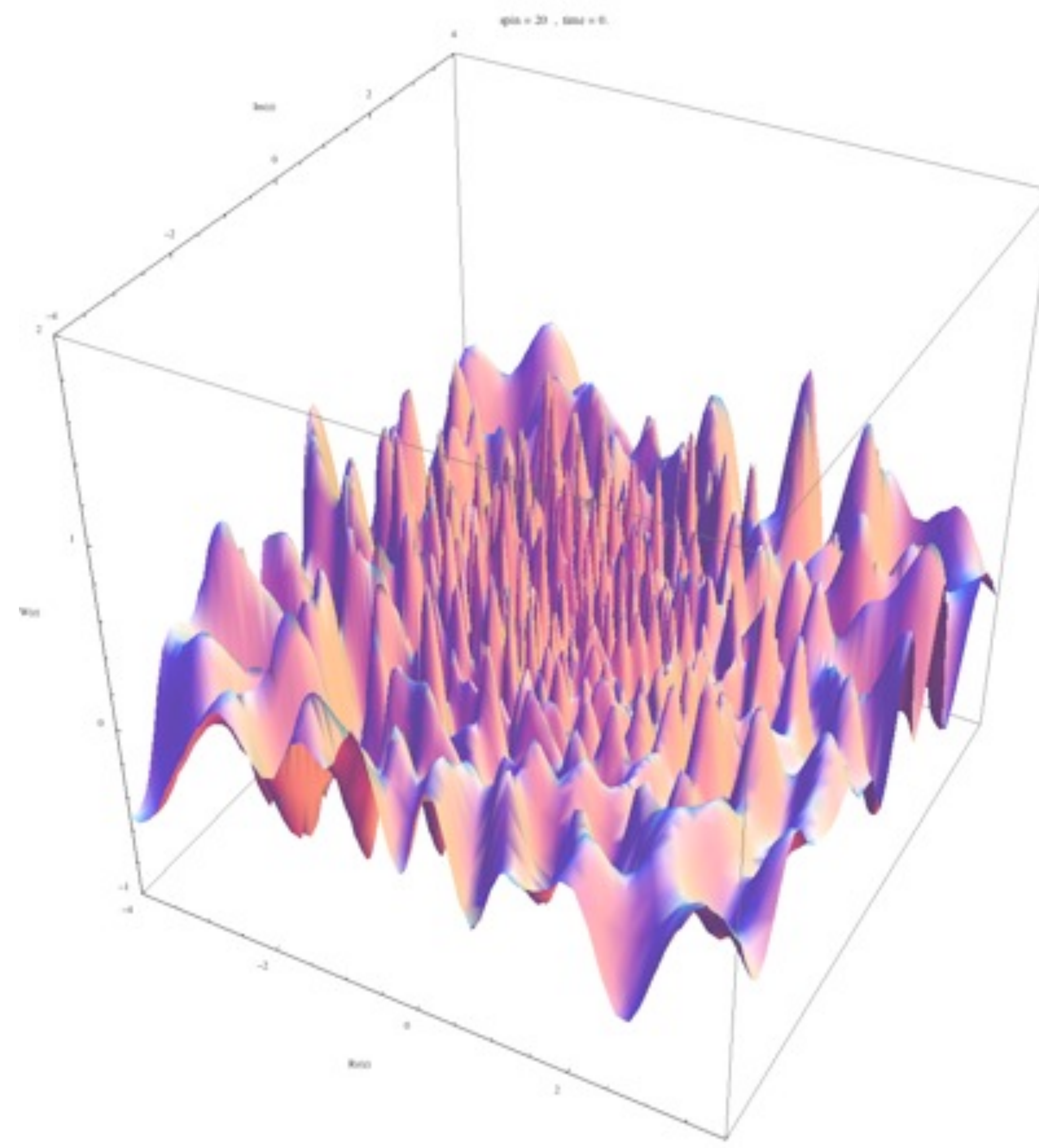
$$W^{(l)}_{\text{c.s.}} = \frac{2l+1}{\sqrt{2j+1}} \exp\left(-\frac{l^2}{2j}\right) \qquad l \sim \sqrt{j}$$

Coherent states are the most localised states on the sphere

- Coherent states look like a Gaussian on the unit sphere with width $\Delta_\theta = 1/\sqrt{j}$
- Random states look like random functions on the unit sphere

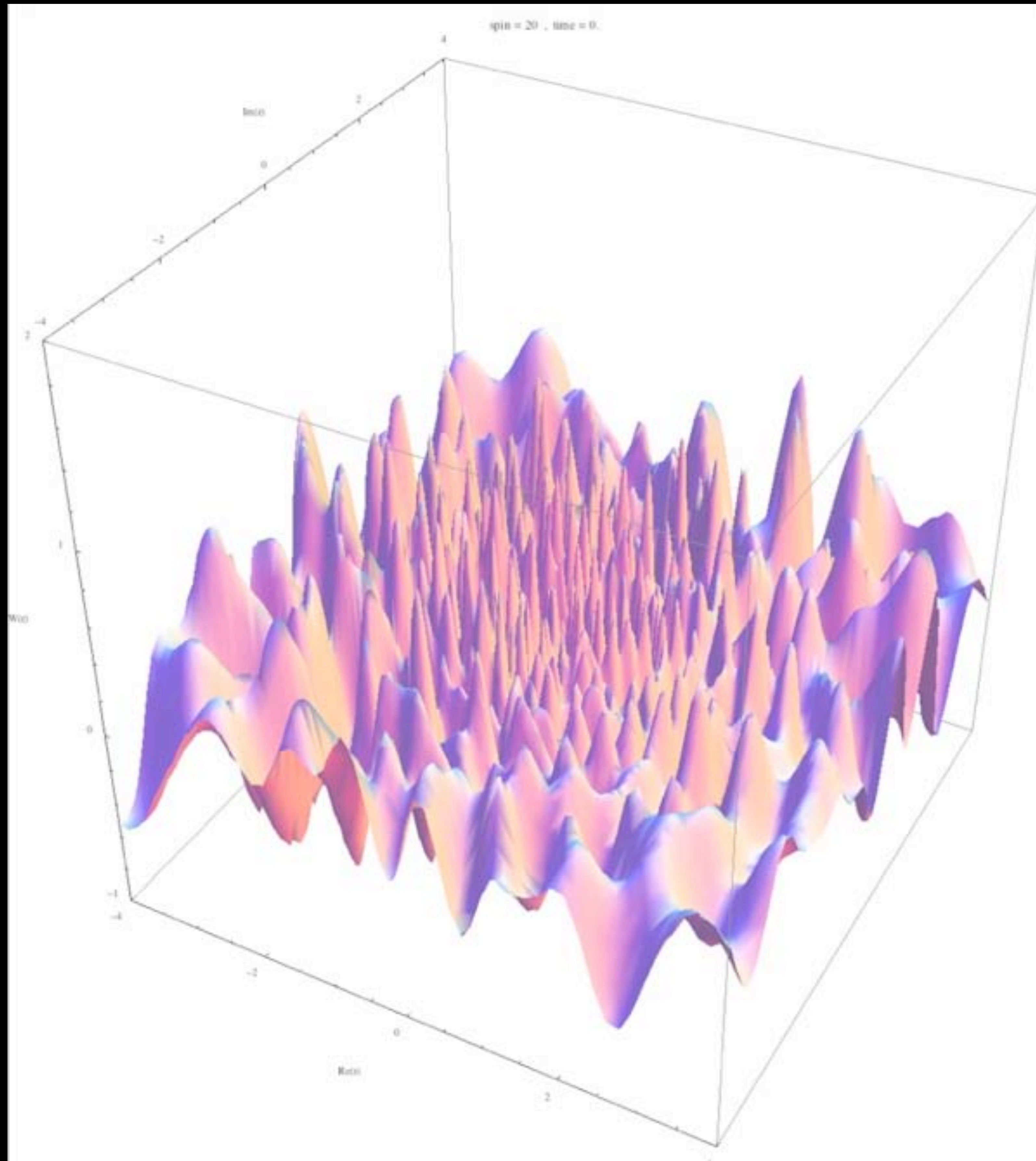


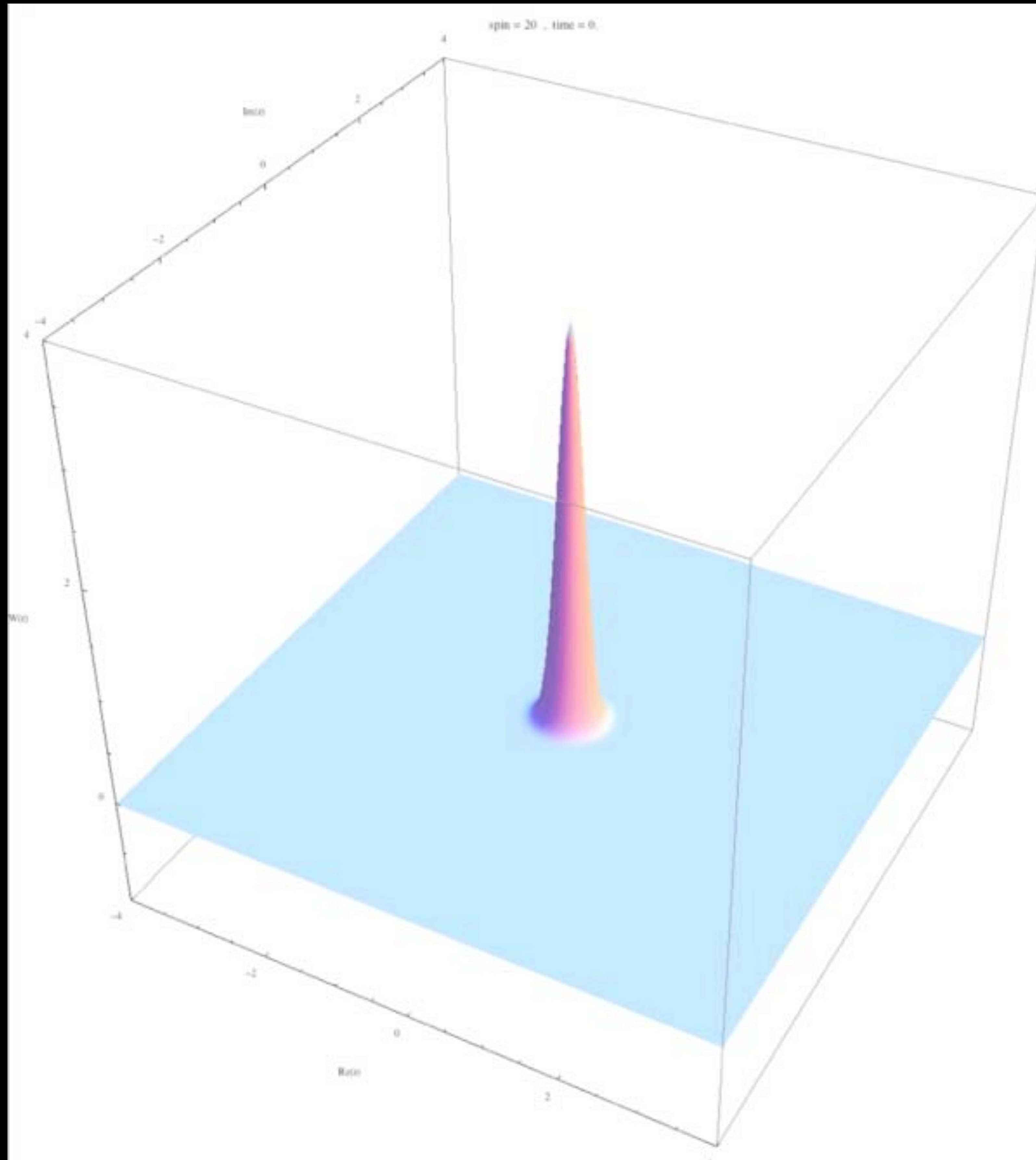
coherent state

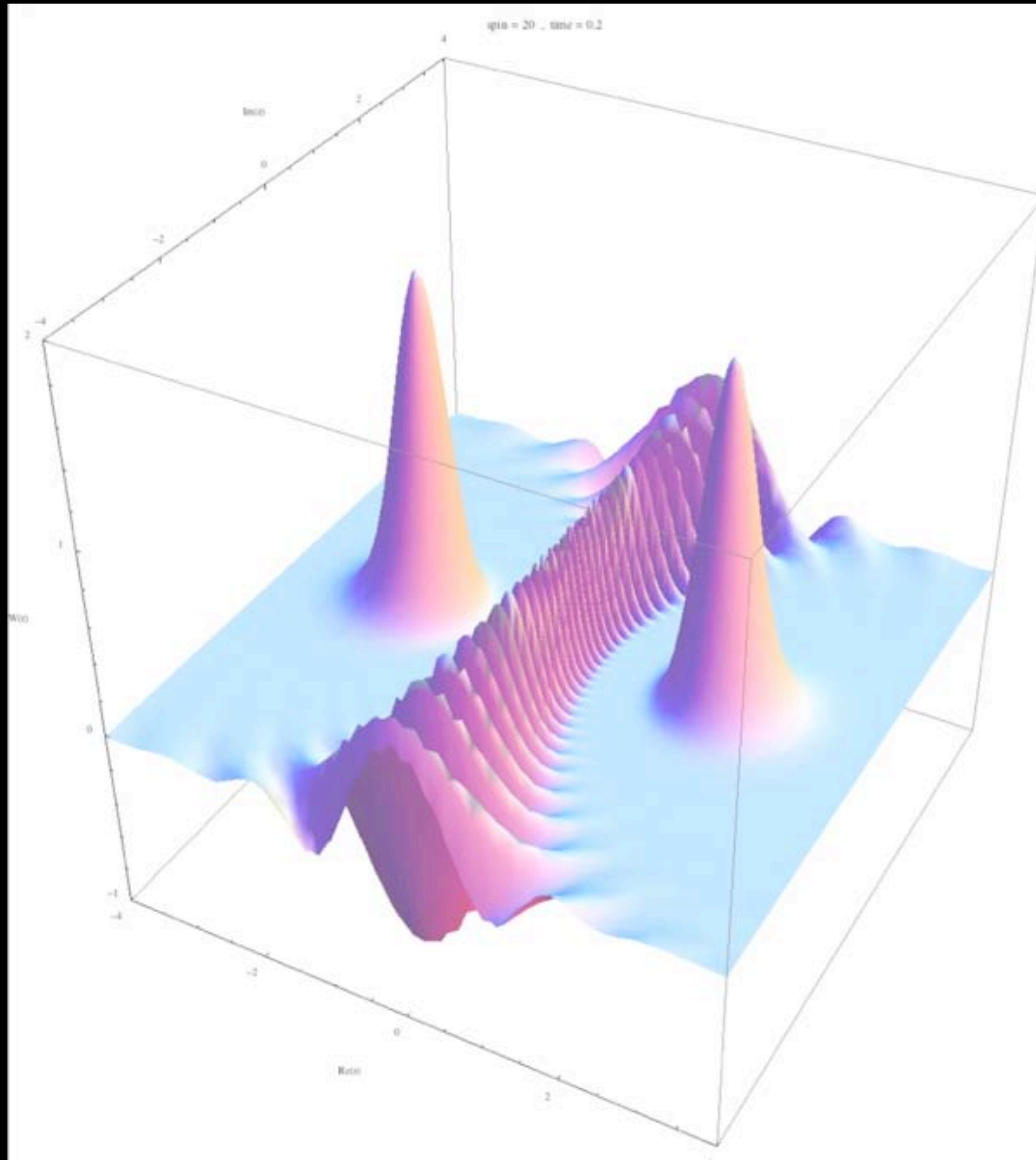


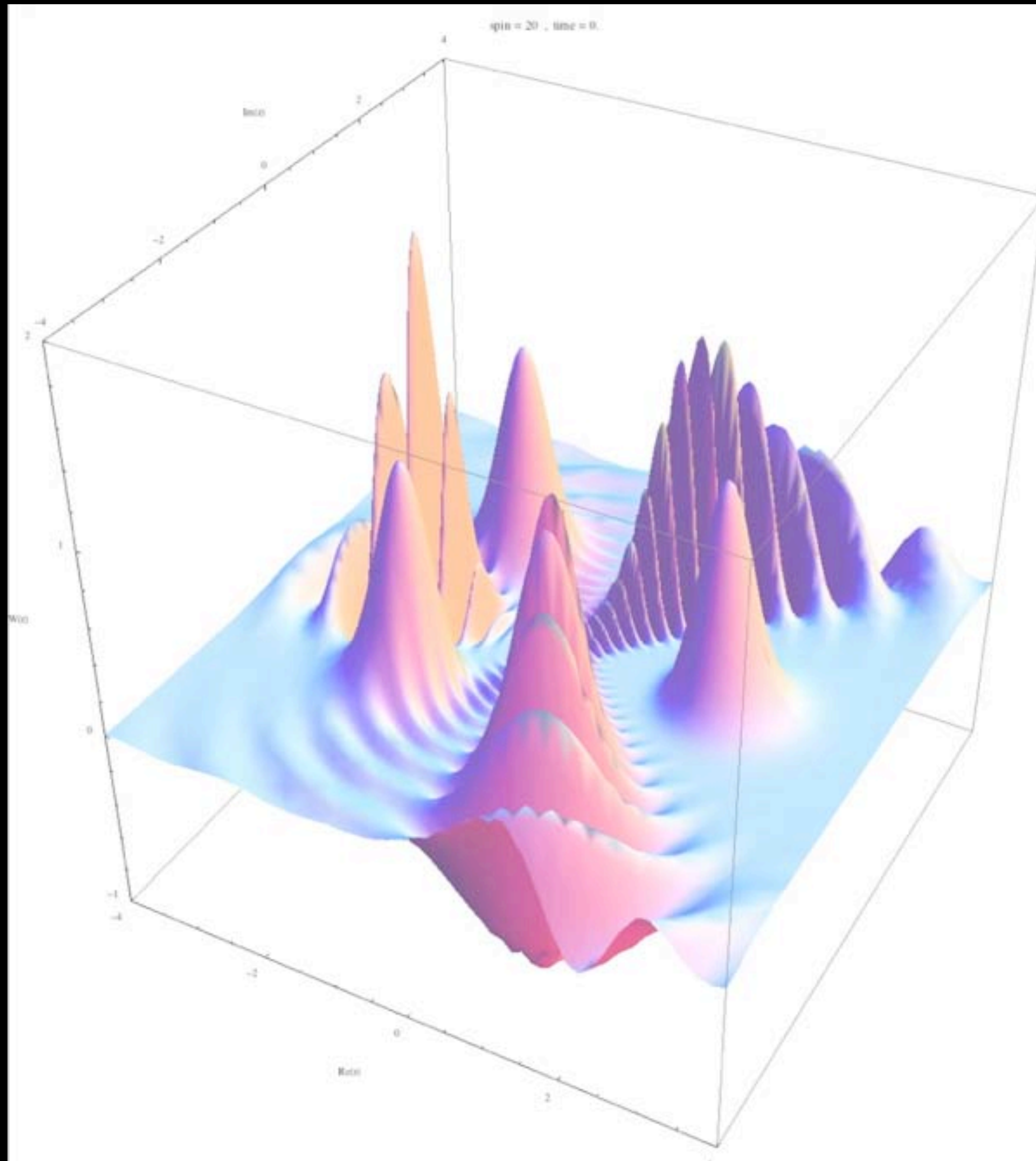
random state

stereographic projection and *j=20*


spin = 20 , time = 0.


spin = 20 , time = 0.


spin = 20., time = 0.2
Im(z)
Re(z)
W(z)


spin = 20 , time = 0.

The time scales of decoherence dynamics

There are 4 time scales* $\tau_0 \leq \tau_1 \ll \tau_2 \ll \tau_3$

* not correlation times since algebraic decay

τ_0 dynamical time scale for the whole system $\tau_0 \simeq 1/ \parallel H_{\mathcal{S}\mathcal{E}} + H_{\mathcal{E}} \parallel$

τ_1 decoherence time scale for generic states $l \gg \sqrt{j}$

τ_2 evolution time scale for coherent states (onset of quantum diffusion)

τ_3 equilibration time for quantum diffusion

For our simple model with Gaussian Hamiltonian ensembles

$$\tau_0 \simeq 1/ \parallel H_{\mathcal{S}\mathcal{E}} + H_{\mathcal{E}} \parallel \qquad \frac{\tau_0}{\tau_1} \simeq \left(\frac{\parallel H_{\mathcal{S}\mathcal{E}} \parallel}{\parallel H_{\mathcal{S}\mathcal{E}} + H_{\mathcal{E}} \parallel} \right)^2$$

$$\frac{\tau_1}{\tau_2} \simeq \left(\frac{\parallel [\vec{\mathbf{S}}, H_{\mathcal{S}\mathcal{E}}] \parallel}{\parallel \vec{\mathbf{S}} \parallel \, \parallel H_{\mathcal{S}\mathcal{E}} \parallel} \right)^2 \qquad \frac{\tau_2}{\tau_3} = \frac{1}{j}$$

$H_{\mathcal{E}} \leftarrow l = 0$ term

$H_{\mathcal{S}\mathcal{E}} \leftarrow l \neq 0$ terms

with the «L₂ norm» for operators $\parallel A \parallel^2 = \dfrac{\mathrm{tr}(A^\dagger A)}{\mathrm{tr}(1)}$

The ratio $\tau_2 \gg \tau_1$ is large iff the commutator $\left[\vec{S}, H_{\mathcal{SE}}\right]$ is «small»

$$\left[\vec{S}, H_{\mathcal{SE}}\right] \ll \vec{S} \times H_{\mathcal{SE}}$$

Coherent states are robust against decoherence and play the role of pointer states if

$$\Delta(l) \neq 0 \text{ for } l \leq l_0 \text{ and } j \gg l_0^2$$

The dynamics of decoherence depends on the details of the Hamiltonian ensemble

$$\Delta = \{\Delta(l),\ l = 0, \cdots l_0\}$$

Beyond the decoherence time scale τ_1 , the dynamics of coherent states is much simpler and exhibit some universal features.

IV - Quantum diffusion and «Markovianity»

For $\tau_1 \ll t \ll \tau_2$ only semiclassical coherent states survive

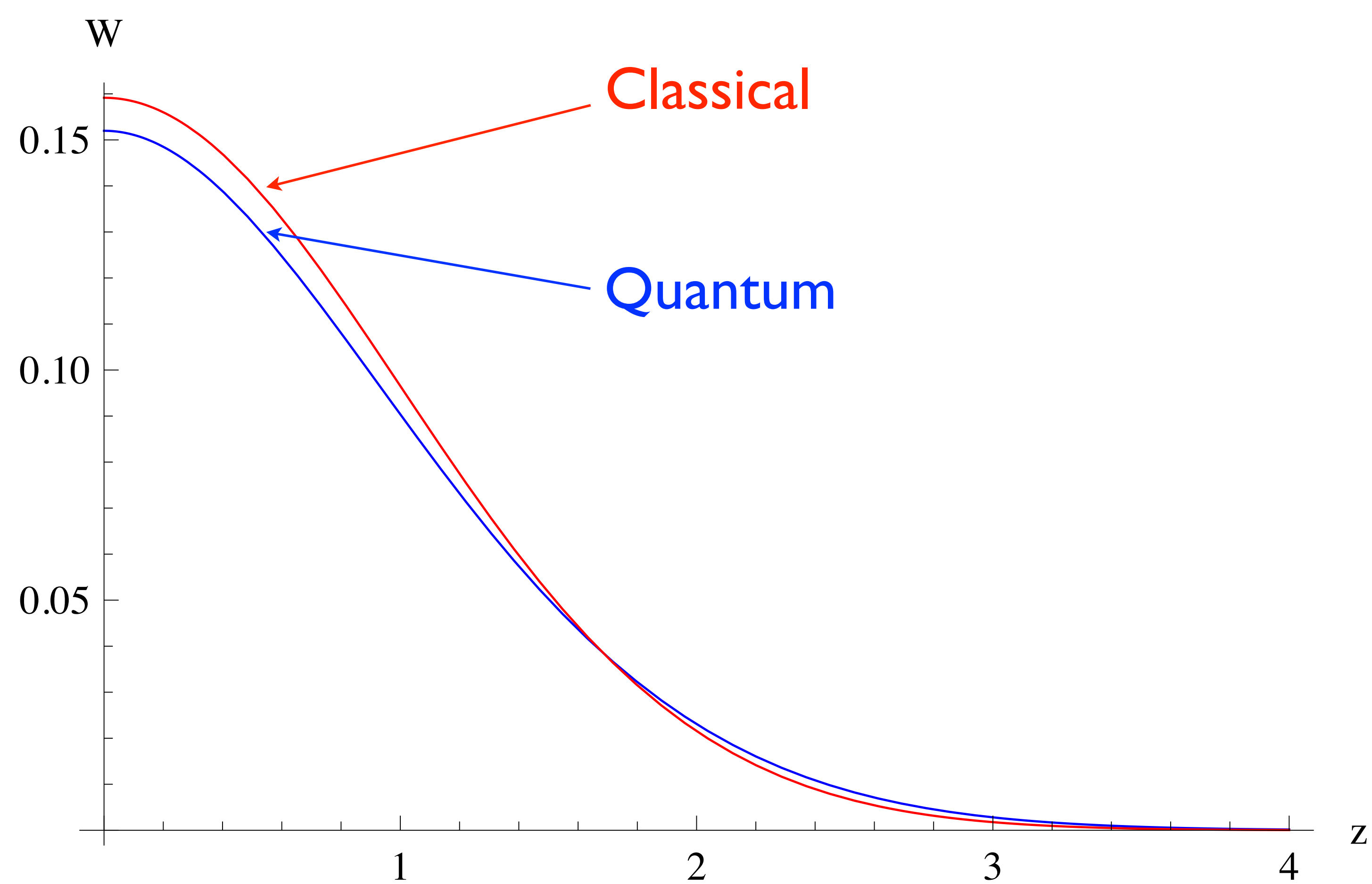
For $\tau_2 < t$ coherent states start to become mixed states $\quad j \gg 1$

This is an effect of quantum diffusion, i.e. the remaining weak effect of the external system on the coherent states.

The width of the distribution function in phase space is found to grow like $\Delta_\theta(t) \propto \sqrt{t}$

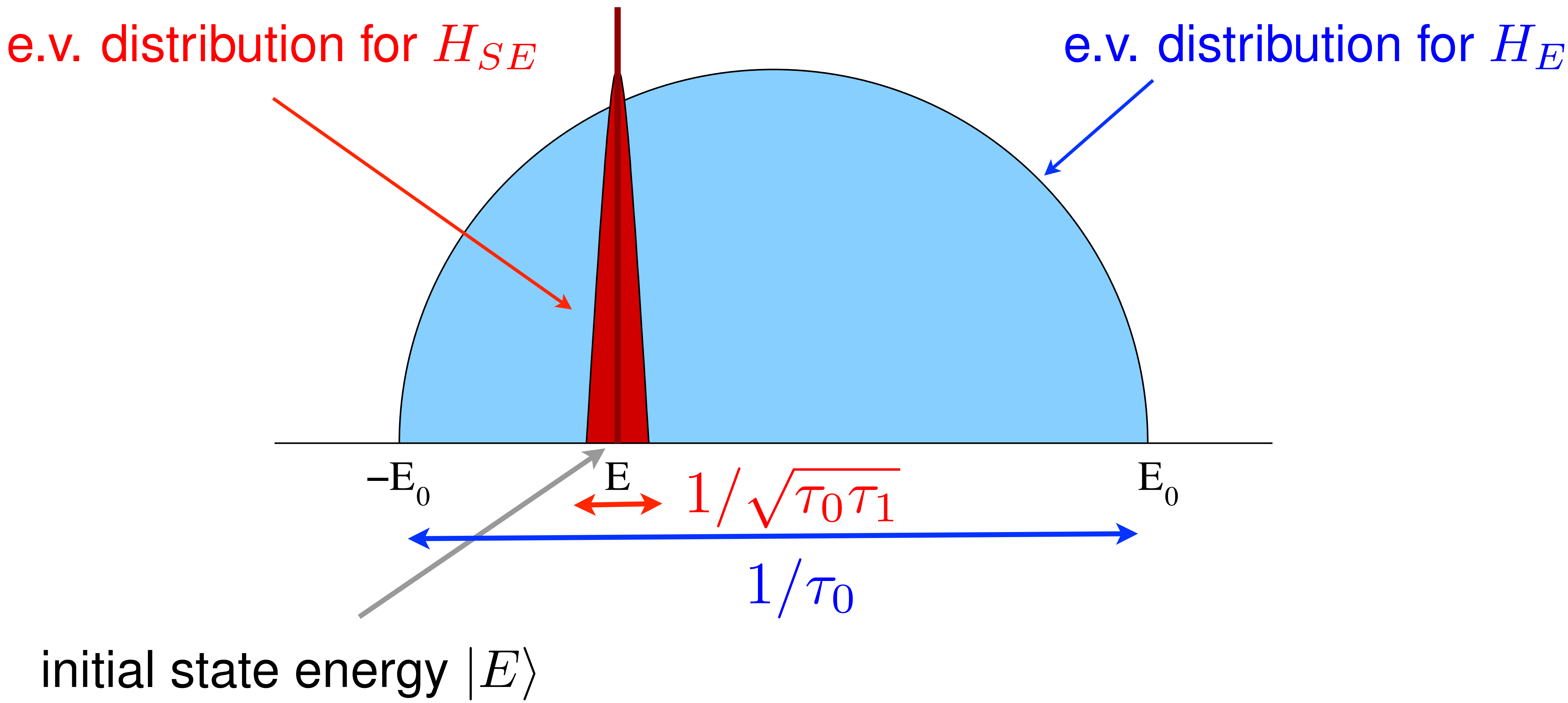
This suggests a random walk in phase space

But the probability profile can be computed and is not a Gaussian ! This is a signal that the evolution is not a Markovian short range process, even at large times!



Generic dynamics for $\mathcal{E}$ and initial conditions

The calculation can be extended to a general Hamiltonian for the external system with a general eigenvalue distribution, and to a given initial state $|\phi_{\mathcal{E}}\rangle$ such as an energy eigenstate



The calculations and the explicit solutions for the evolution functional $\mathbf{M}(t)$ for general interactions are then much more complicated (free probability calculus)

Question: What are the conditions for Markovian dynamics and classical diffusion?

Answer: Fast dynamics + initial energy eigenstate $|\phi_{\mathcal{E}}\rangle = |E\rangle$ for the external system $\mathcal{E}$

$$H_{\mathcal{E}} \gg H_{\rm int.} \quad \Longrightarrow \quad \tau_1 \gg \tau_0$$

Then the diffusion of coherent states on the sphere is Markovian and the diffusion coefficient is

$$D_{\rm diff} = 2\pi\, \rho_{\,\mathcal{E}}(E)\, \overline{\left|\langle\Phi|[\vec{\mathbf{S}}, H_{\mathcal{S}\mathcal{E}}]|\Phi'\rangle\right|^2}$$

d.o.s. of the external system

typical size of a matrix element of the commutator

A Fermi Golden Rule-like formula! This is not too surprising, one must be able to write a master equation for the evolution of the density matrix, and derive a fluctuation-dissipation theorem.

Evolution functional (general case)

$$\rho^{\mathcal{S}}_{ru}(t) = \overline{\mathrm{tr}_{\mathcal{E}}(\mathrm{e}^{-\mathrm{i}tH}(\rho^{\mathcal{S}}(0) \otimes |\alpha\rangle\langle\alpha|)\mathrm{e}^{\mathrm{i}tH})}_{ru} = \mathcal{M}_{ru,st}(t, E_\alpha)\rho^{\mathcal{S}}_{st}(0)$$

Wigner transform integral representation

$$\hat{\mathcal{M}}^{(l)}(t,E) = \oint \frac{\mathrm{d}x_1}{2\mathrm{i}\pi} \oint \frac{\mathrm{d}x_2}{2\mathrm{i}\pi} \frac{\mathrm{e}^{-\mathrm{i}t(x_1-x_2)}}{(W(x_1)-E)(W(x_2)-E)}$$

$$\times \frac{1}{(1-Z'(l)) + Z'(l)((x_1-x_2)/(W(x_1)-W(x_2)))}$$

where

$$Z'(l)) = \frac{\hat{\Delta}'(l)}{\hat{\Delta}'(0)} \qquad \hat{\Delta}'(0) = \hat{\Delta}'$$

$$\hat{\Delta}'(l) = N\hat{D}(l) = \sum_{l'=1}^{2j} \tilde{\Delta}(l')(2l'+1)(-1)^{2j+l'+l_1} \begin{Bmatrix} j & j & l' \\ j & j & l_1 \end{Bmatrix}$$

$$\tilde{C}(x) = \int \mathrm{d}E\, \frac{\nu(E)}{w-E} \qquad W(x) = x - \hat{\Delta}'\tilde{C}(x)$$

If the initial state is a quantum superposition of energy eigenstates

$$|\phi_{\mathcal{E}}\rangle = \sum_i c_i |E_i\rangle$$

then the quantum diffusion is a randomization of a collection of Markovian processes (random walks) on the sphere, with weight $W_i = |c_i|^2$ and diffusion constant $D_{\text{diff}}(E_i)$

This reflects the decoherence between the energy eigenstates (of the external system) induced by the coupling with the large spin (mutual decoherence between states of each large system ***S*** or ***E*** induced by the coupling)

V - Extensions and open problems

1. Take into account the dynamics of the spin, e.g. $H_{\mathcal{S}} = -\vec{\mathbf{S}} \cdot \vec{B}$
   - mathematically more difficult (known for low spin *j*=$1/2$)
   - should lead to solutions for dissipation and relaxation processes in non Markovian regimes, to quantum fluctuation-dissipation relations, etc.
2. Treat less generic Hamiltonians ensembles: examples

   $SU(2) \times U(N) \times \mathbb{Z}_M \qquad\qquad SU(2) \times G$

   (for instance, for interactions with finite energy range)
3. Compute multi-times correlations in non-Markovian regime.
   - Some results known in Markovian regimes (quantum stochastic processes)
   - The knowledge of the evolution functional M(t) is not enough!
   - An interesting planar algebraic structure seems to emerge

$$\mathrm{tr}(\rho(t_0)A(t_1)A_2(t_2)\dots A_N(t_N))$$

$$\mathcal{G}(z_0, z_1, z_2, z_N) = \mathrm{Tr}\left(\frac{1}{z_0 - H} \otimes \frac{1}{z_1 - H} \otimes \cdots \otimes \frac{1}{z_N - H}\right)$$

5. Define and study more realistic models.
Example: **clusters of quantum spins**
(measurement devices, decoherence in closed systems, etc..)

Start from N spins 1/2, N large

Ferromagnetic coupling + small random multi-spin Hamiltonian

$$H = -J \left(\sum_i \vec{S}_i \right)^2 + \epsilon H_{\mathrm{random}}(\vec{S}_1, \vec{S}_2, \cdots \vec{S}_N)$$

Classification of these ensembles of random Hamiltonians require understanding of matrix ensembles invariant under the symmetric (permutation) group S_N and tensor products of its representations

This gives interesting matrix ensembles and interesting dynamics (requires free probabilitie/random matrix general calculation techniques)

Work in progress!   Thank you!