Random matrices and decoherence models for spin

François David
IPhT Saclay & CNRS

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Purpose of this talk:

- Present a very simple toy model
- Based on very standard ideas:
  - spin and coherent states *(Takahashi & Shibata, 1975)*
  - random matrix hamiltonians *(Mello, Pereyra & Kumar, 1988)*
  - which have been much applied for the spin 1/2 case ($j = 1/2$, Q-bit, 2 level system)
- But some (relatively) novel aspects
  - general spin j (from quantum to classical spin)
  - generic interaction (novel random matrix ensembles)
- It allows to study analytically several aspects decoherence
- and the dynamics of the emergence of a classical degree of freedom (here a very simple one: a single spin) in a quantum system

I - The model

A quantum SU(2) spin $\mathcal{S}$ + an external system $\mathcal{E}$

spin $= j \qquad \dim(\mathcal{H}_\mathcal{S}) = 2j+1 \qquad \dim(\mathcal{H}_\mathcal{E}) = N \gg j$

Single spin:

For large spin $j \to \infty$ the spin becomes a classical object

Classical phase space is the 2-sphere

The coherent states behave as quasi classical states

$$|\vec{n}\rangle \quad , \quad (\vec{n}\cdot\vec{\mathbf{S}})|\vec{n}\rangle = j|\vec{n}\rangle$$

Dynamics of the coupled spin:

$$H = H_\mathcal{S} \otimes \mathbf{1}_\mathcal{E} + H_{\mathcal{S}\mathcal{E}} + \mathbf{1}_\mathcal{S} \otimes H_\mathcal{E}$$

The Hamiltonians:

- Slow spin dynamics $\qquad H_\mathcal{S} = 0$
  (no dissipative & thermalisation effects)
- Dynamic of the external system generic $\qquad H_\mathcal{E} \to H_{\mathcal{S}\mathcal{E}}$

The interaction Hamiltonian

The interaction hamiltonian is given by a Gaussian random matrix ensemble, with the only constraint that the ensemble in invariant under

$$SU(2) \times U(N)$$

spin

external system

For this, go to Wigner representation of spin operators

$$\langle r\alpha | H | s\beta \rangle = H^{rs}_{\alpha\beta} \to W^{(lm)}_{\alpha\beta} \qquad \mathbf{j} \otimes \mathbf{j} = \mathbf{0} \oplus \mathbf{1} \oplus \cdots \oplus \mathbf{2j}$$

$$A_{rs} = \langle r | A | s \rangle \qquad W^{(l,m)}_{A} = \sum_{r,s=-j}^{j} \sqrt{\frac{2l+1}{2j+1}} \left\langle \begin{matrix} j & l \\ r & m \end{matrix} \middle| \begin{matrix} j \\ s \end{matrix} \right\rangle A_{rs}$$

It is enough to take for the $W^{(lm)}_{\alpha\beta}$ independent gaussian random variables with zero mean and variance depending only on l and with the Hermiticity constraint.

$$\text{Var}\left(W^{(lm)}_{\alpha\beta} \right) = \Delta(l) \qquad W^{(l,m)}_{\alpha\beta} = (-1)^m \overline{W}^{(l,-m)}_{\beta\alpha}$$

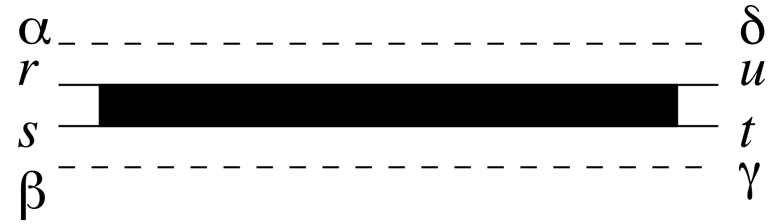
We thus get a matrix ensemble characterized by the variances

$$\Delta = \{\Delta(l)\ ,\ l = 0, 1, \cdots 2j\}$$

NB: The *l=m=0* term represents the $H_{\mathcal{E}}$ Hamiltonian

With this GU(2)xU(*N*) ensemble, the 2-points correlator is

$$\overline{H^{rs}_{\alpha\beta} H^{tu}_{\gamma\delta}} = \delta_{\alpha\delta}\delta_{\beta\gamma}\mathcal{D}_{rs,tu}$$



$$\mathcal{D}_{rs,tu} = \quad \delta_{s-r,t-u} \sum_{l=0}^{2j} \Delta(l) \frac{2l+1}{2j+1} \left\langle \begin{matrix} j & l \\ s & r-s \end{matrix} \middle| \begin{matrix} j \\ r \end{matrix} \right\rangle \left\langle \begin{matrix} j & l \\ t & u-t \end{matrix} \middle| \begin{matrix} j \\ u \end{matrix} \right\rangle$$

Standard ribbon propagator for the *N* indices, more complicated structure for the spin indices, but still planar.

II - The evolution functional

separable state $\to$ entangled state $\to$ mixed state for $\mathcal{S}$

$$|\psi_0\rangle \otimes |\phi_0\rangle \to |\Phi(t)\rangle \ , \quad \rho_{\mathcal{S}}(t) = \mathrm{tr}_{\mathcal{E}}(|\Phi(t)\rangle\langle\Phi(t)|$$

Evolution functional

$$\rho_{\mathcal{S}}(t) = \mathcal{M}(t) \cdot \rho_{\mathcal{S}}(0) \ , \quad \mathcal{M}(t) = \mathrm{tr}_{\mathcal{E}} \left(e^{-itH} \left(\cdot \otimes \rho_{\mathcal{E}}(0) \right) e^{itH} \right)$$

For simplicity, start from a random state $|\psi_{\mathrm{E}}\rangle$

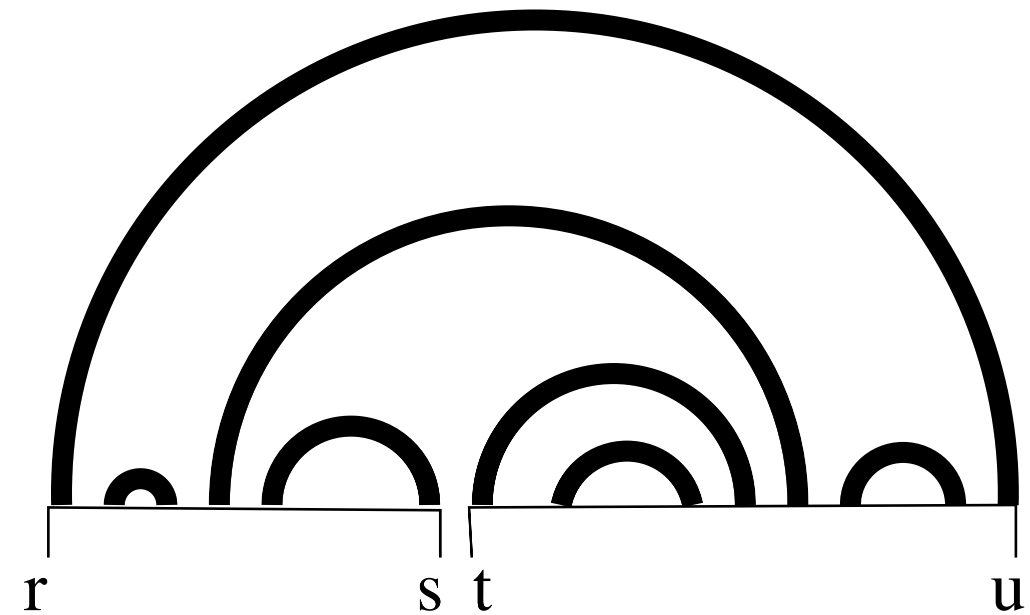
Then the evolution functional is

$$\mathcal{M}(t) = \oint \frac{dx}{2i\pi} \oint \frac{dy}{2i\pi} \, e^{it(x-y)} \, \mathcal{G}(x,y)$$

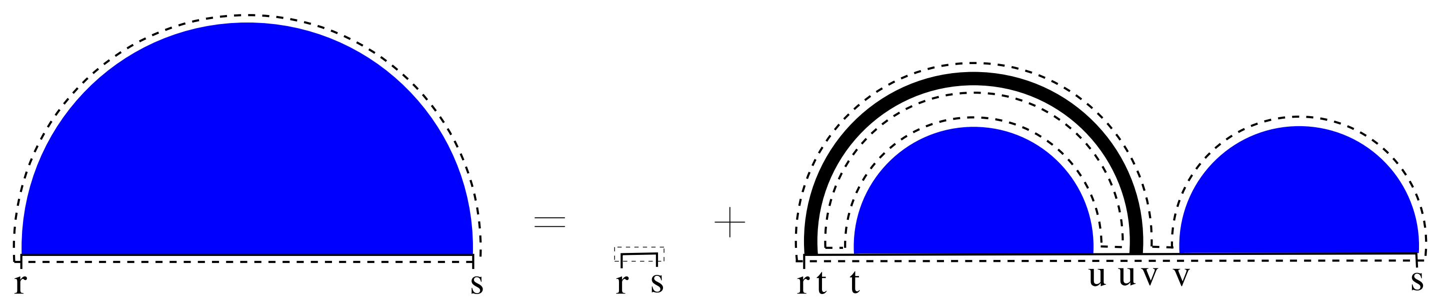
$$\mathcal{G}(x,y) = \frac{1}{N} \mathrm{tr}_{\mathcal{E}} \left[\frac{1}{x-H} \otimes_{\mathcal{S}} \frac{1}{y-H} \right]$$

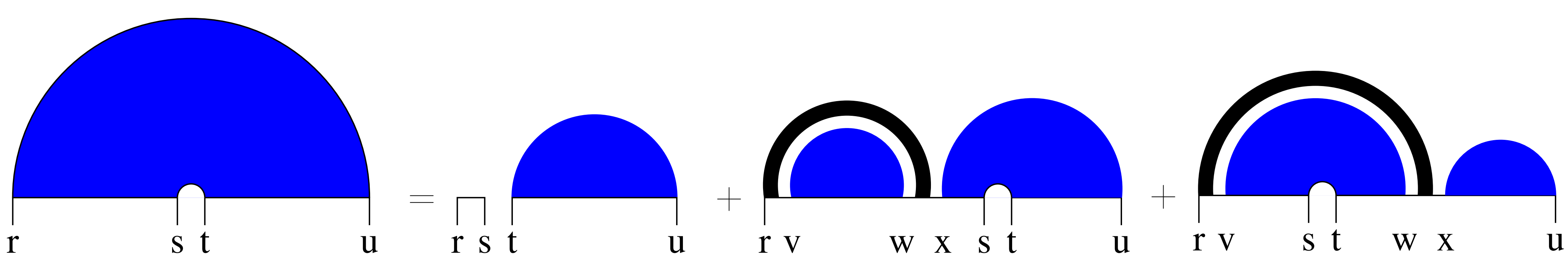
We take the large N limit (large external system) and make the average over the H, assuming self averaging as usual.

$\overline{\mathcal{G}(x,y)}$ is given by a sum of planar diagrams of the standard form (rainbow diagrams)



These resolvents obey recursion relations





Thanks to the SU(2) invariance, the evolution functional for the density matrix of the spin $\rho_{\mathcal{S}}(t)$ takes a simple diagonal form in the Wigner representation basis

$$\rho_{\mathcal{S}\,rs}(t) \;\rightarrow\; W_{\mathcal{S}}^{(l,m)}(t) = \widehat{\mathcal{M}}^{(l)}(t) \cdot W_{\mathcal{S}}^{(l,m)}(0)$$

with the kernel given by a universal decoherence function

$$\widehat{\mathcal{M}}^{(l)}(t) = M(t/\tau_0, Z(l))$$

depending on a rescaled time $t' = t/\tau_0$ and a factor $Z(l)$

$$\tau_0 = 1/\sqrt{\widehat{\Delta}(0)} \qquad\qquad Z(l) = \frac{\widehat{\Delta}(l)}{\widehat{\Delta}(0)}$$

τ_0 is the dynamical time scale of the system (more later)

The parameter $Z(l)$ depends on the spin sector considered.

The $Z(l)$ function

The l dependence of the factor $Z(l)$ depends on the initial variances of the $\mathrm{GU}(2)$ ensemble for the Hamiltonian.

$$\widehat{\Delta}(l) = N \sum_{l'=0}^{2j} \Delta(l')(2l'+1)(-1)^{2j+l'+l} \begin{Bmatrix} j & j & l' \\ j & j & l \end{Bmatrix} \longleftarrow \text{6-j symbol}$$

$$Z(l) = \widehat{\Delta}(l)/\widehat{\Delta}(0) \qquad\qquad Z(l) \in [-1, 1]$$

$Z(l)$ is maximal for $l=0$

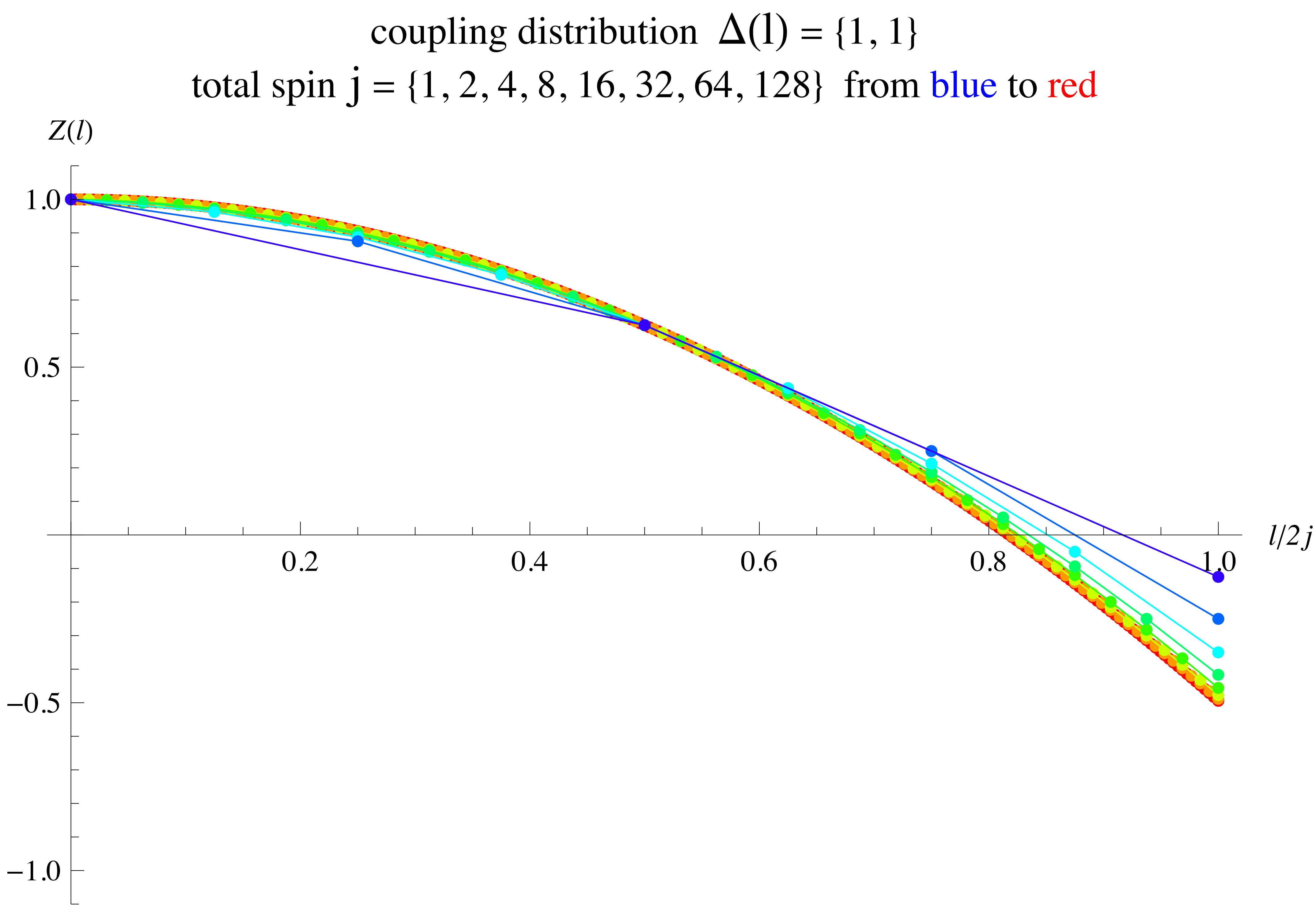
$Z(l)$ takes a scaling form in the large spin limit

$$Z(l) = \widehat{\Delta}(l)/\widehat{\Delta}(0) \to Y(x) \text{ with } x = l/2j$$

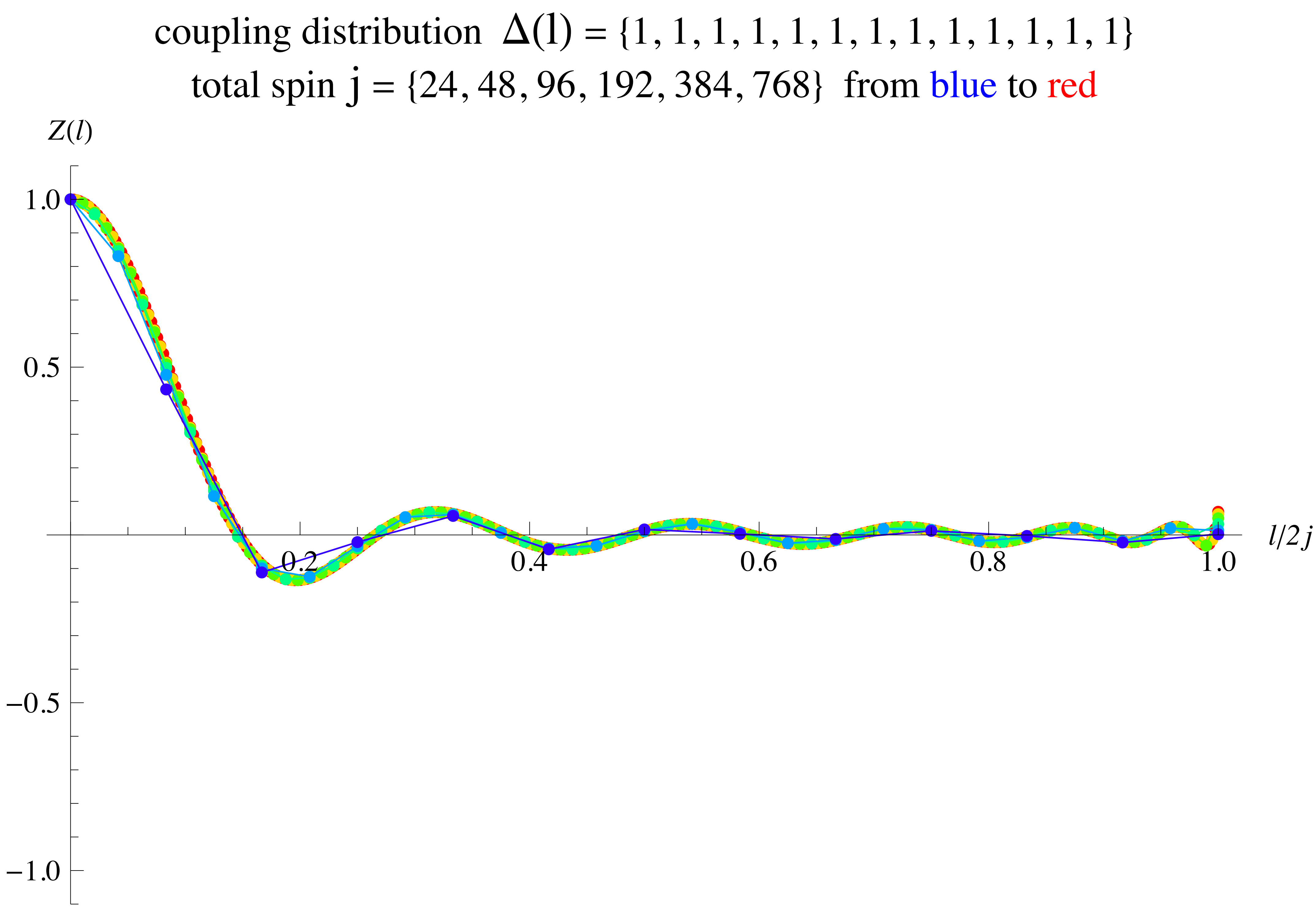
Its small l behavior is quadratic in l

$$Z(l) = 1 - l(l+1)\,\frac{1}{4}\frac{D_0}{j(j+1)} + \cdots \quad , \quad D_0 = \frac{\sum_{l'=1}^{l_0} \bar{\Delta}(l')\,(2l'+1)\,l'(l'+1)}{\sum_{l'=0}^{l_0} \bar{\Delta}(l')(2l'+1)}$$

Example 1: $l=0$ and 1 channels only



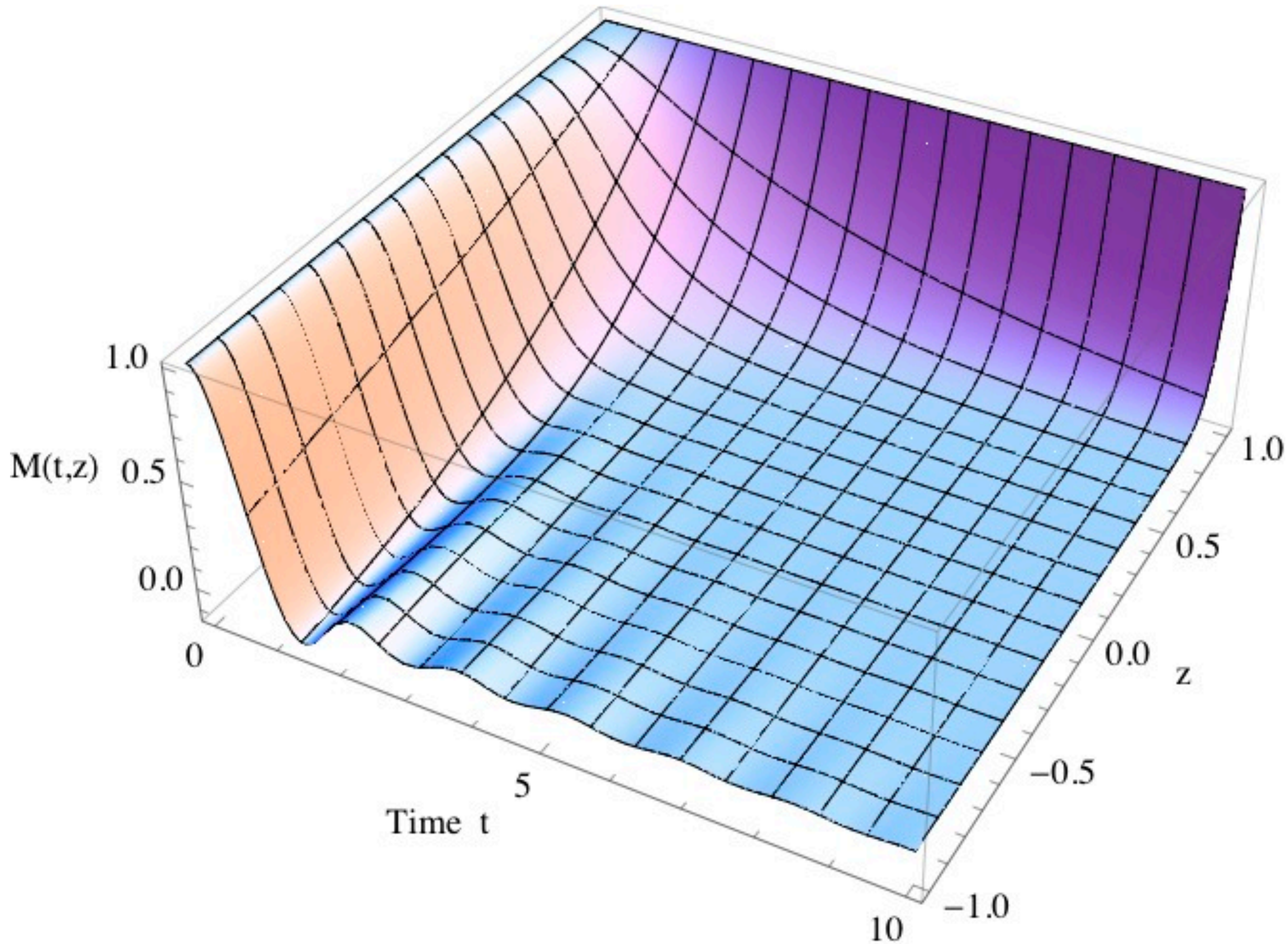
Example 2: l=0 to 12 channels



The decoherence function is a generalized hypergeometric function

$$M(t,Z) = \oint \frac{dx}{2\mathrm{i}\pi} \oint \frac{dy}{2\mathrm{i}\pi}\, \mathrm{e}^{-\mathrm{i}t(x-y)}\, \frac{H(x)H(y)}{1 - Z\,H(x)H(y)} \quad , \qquad H(x) = \frac{1}{2}(x - \sqrt{x^2-4})$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{m} t^{2m}\, z^n\, (-1)^{m+n}\, \frac{2(2m+1)(n+1)^2(2m)!}{m!(m+1)!(m-n)!(m+n+2)!}$$



large time limit: fast algebraic decay with t except for Z close to unity

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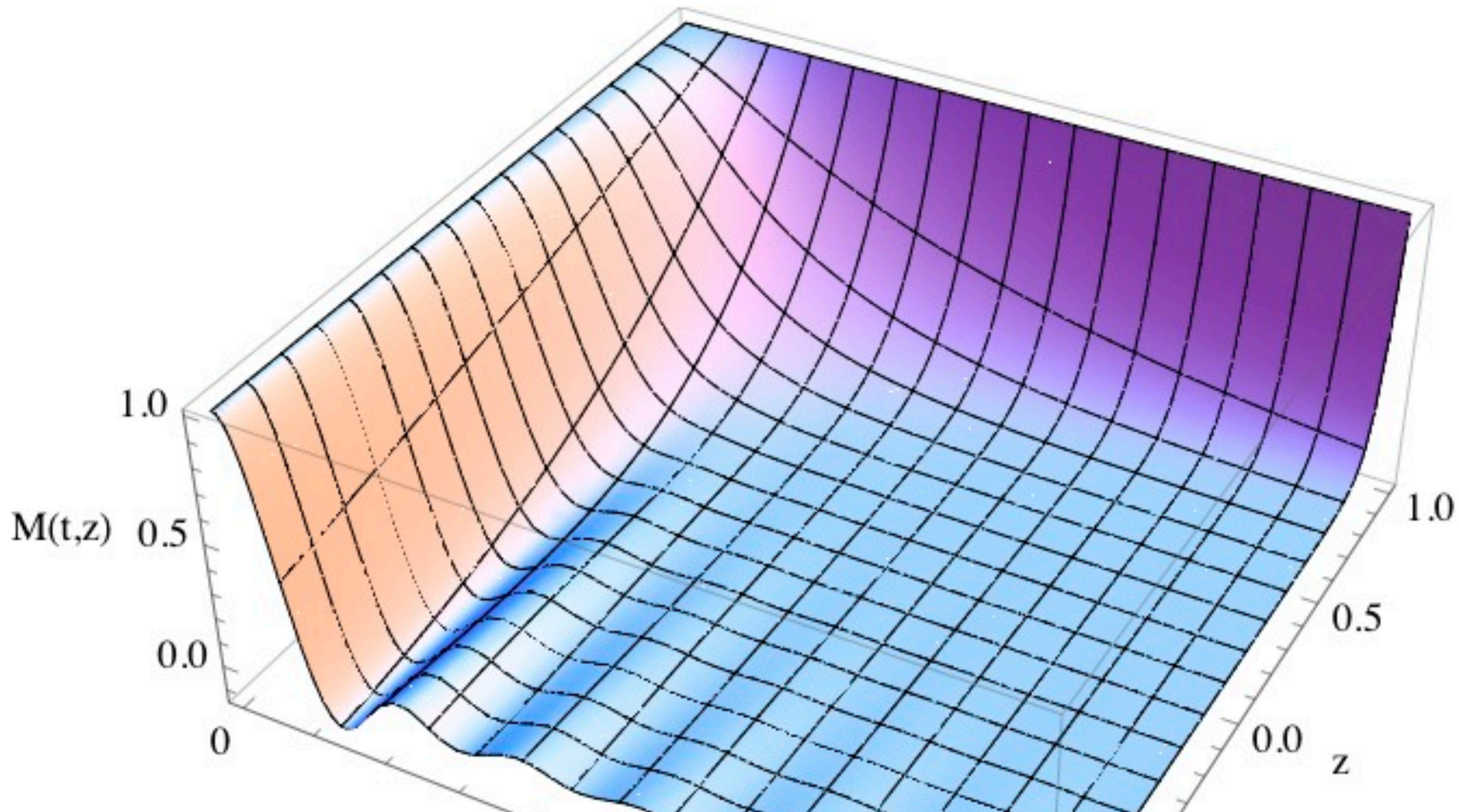
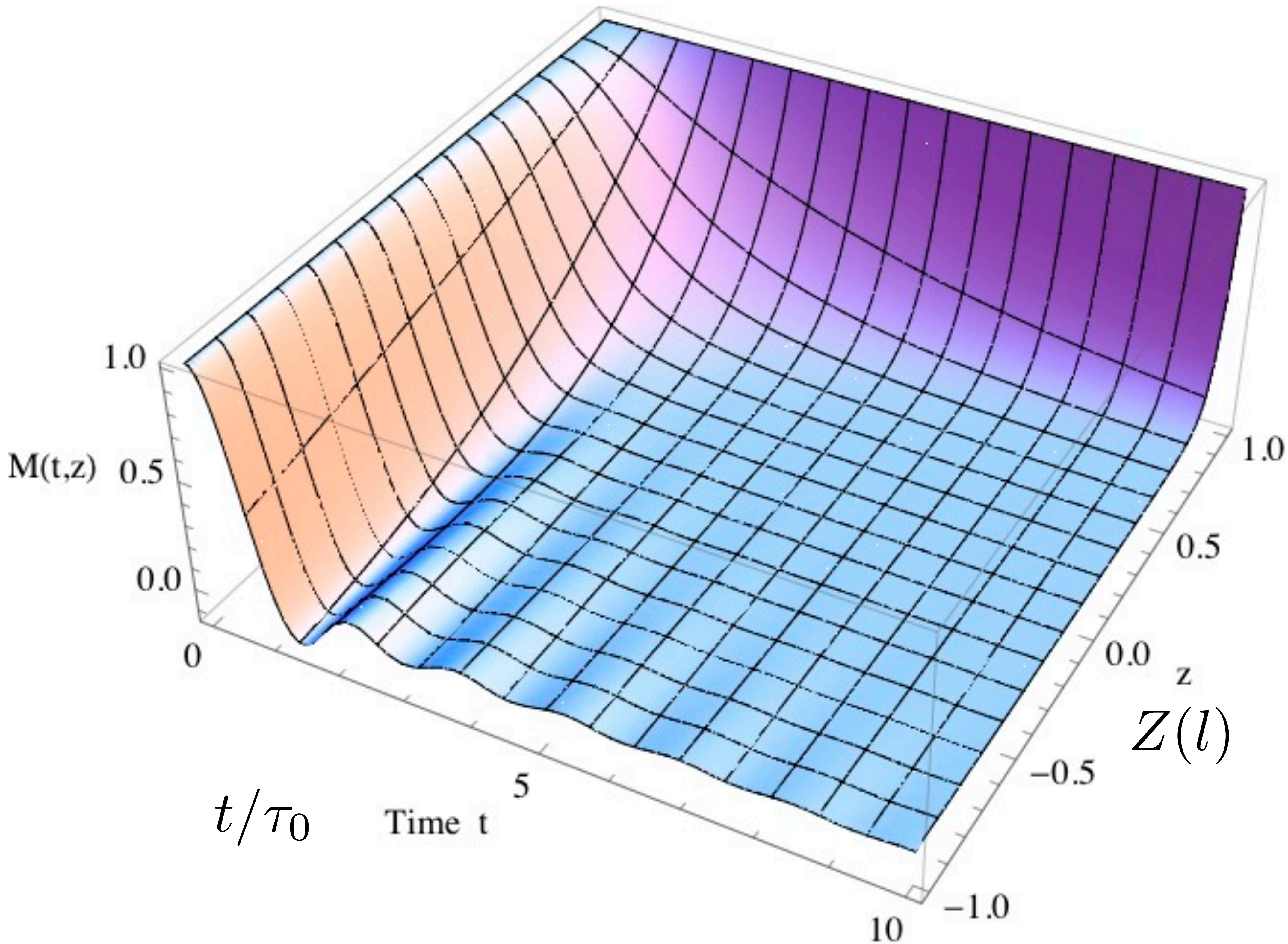


large time limit: fast algebraic decay with t except for Z close to unity

$$\boxed{M(t,z) = \frac{1}{2\pi}\, t^{-3} \left(\frac{1+z}{(1-z)^3} - \frac{1-z}{(1+z)^3} \sin(4t) \right) \left(1 + \mathcal{O}(t^{-1})\right)}$$

$z \to 1$ scaling



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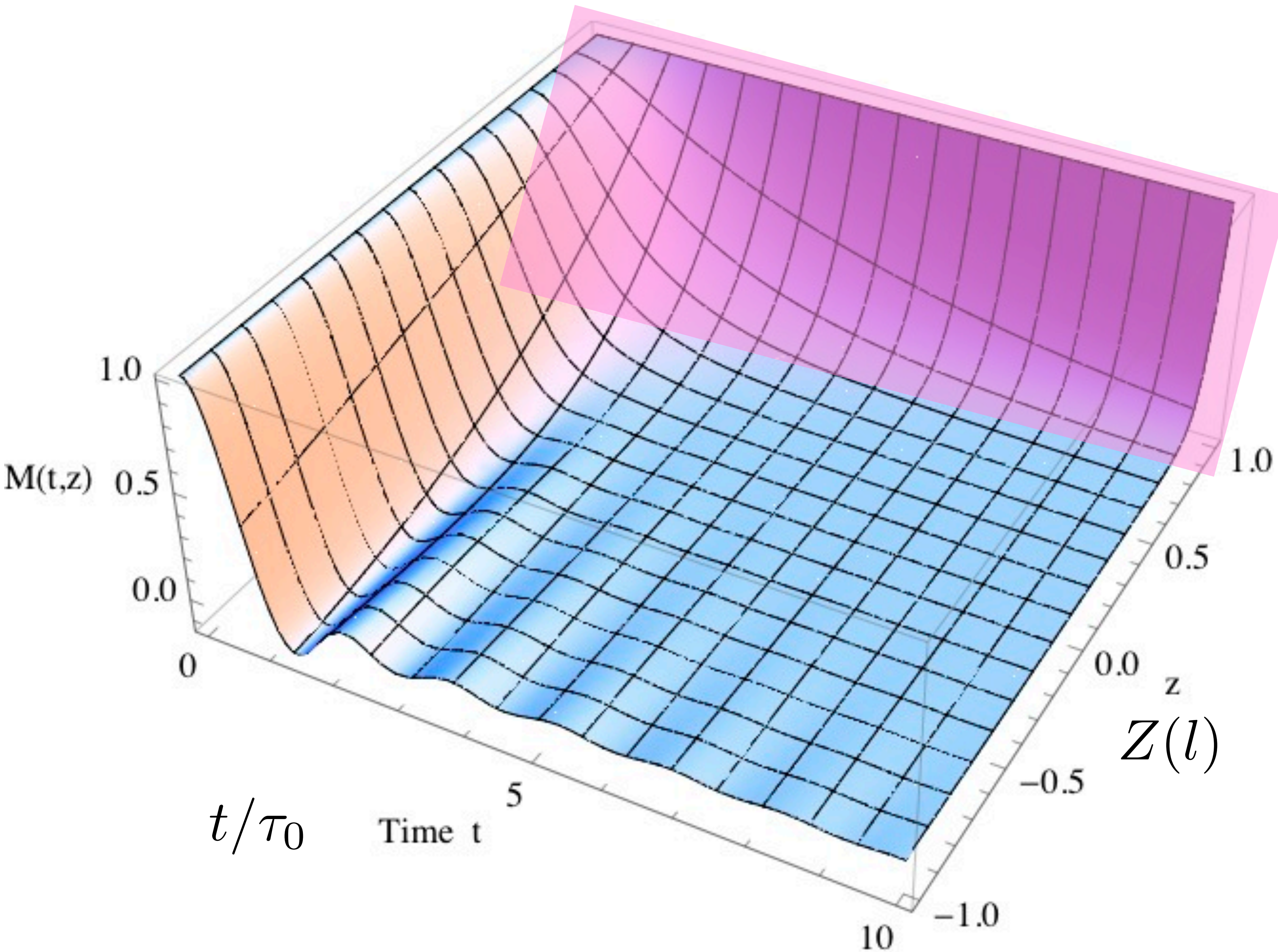
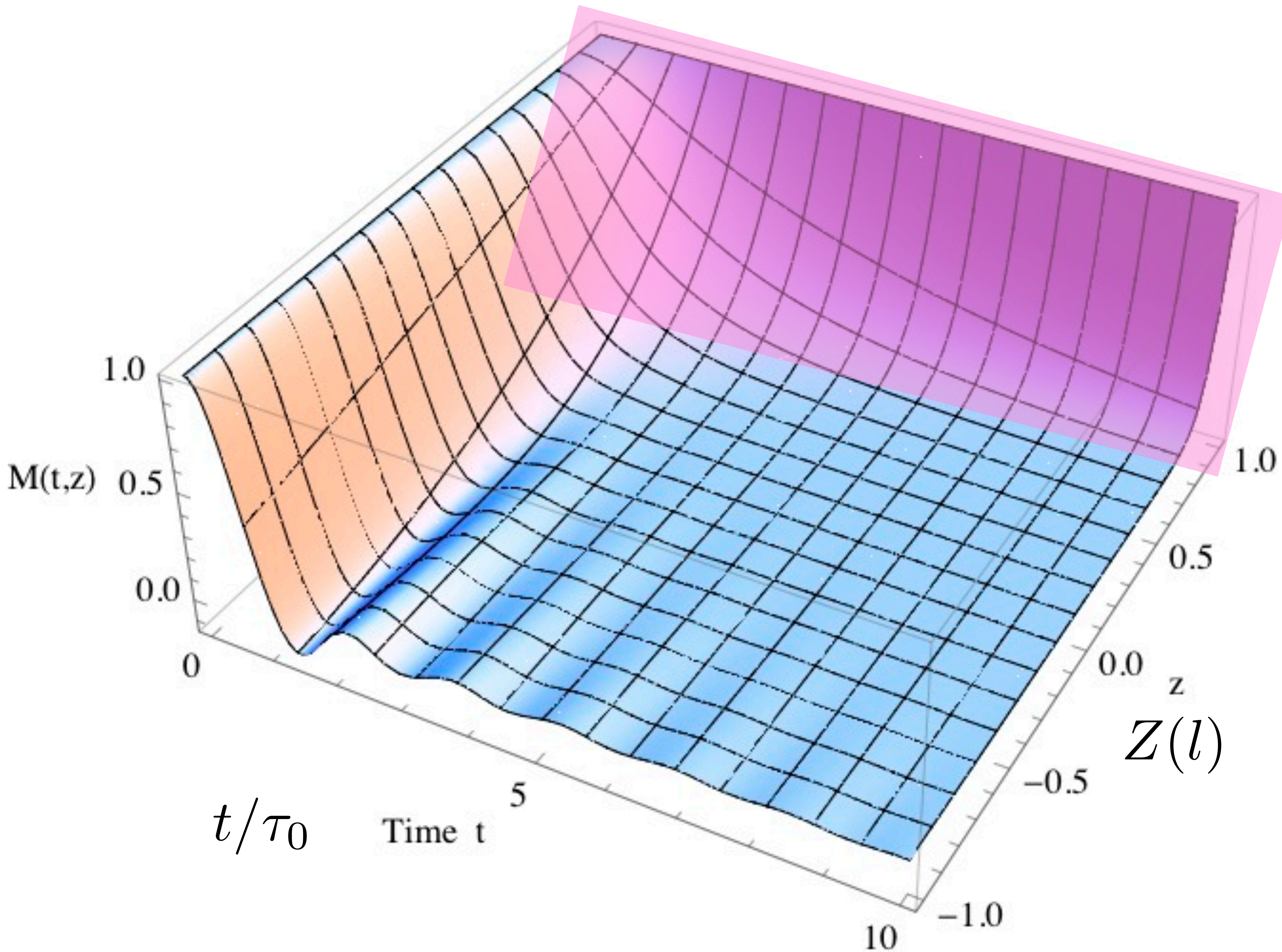
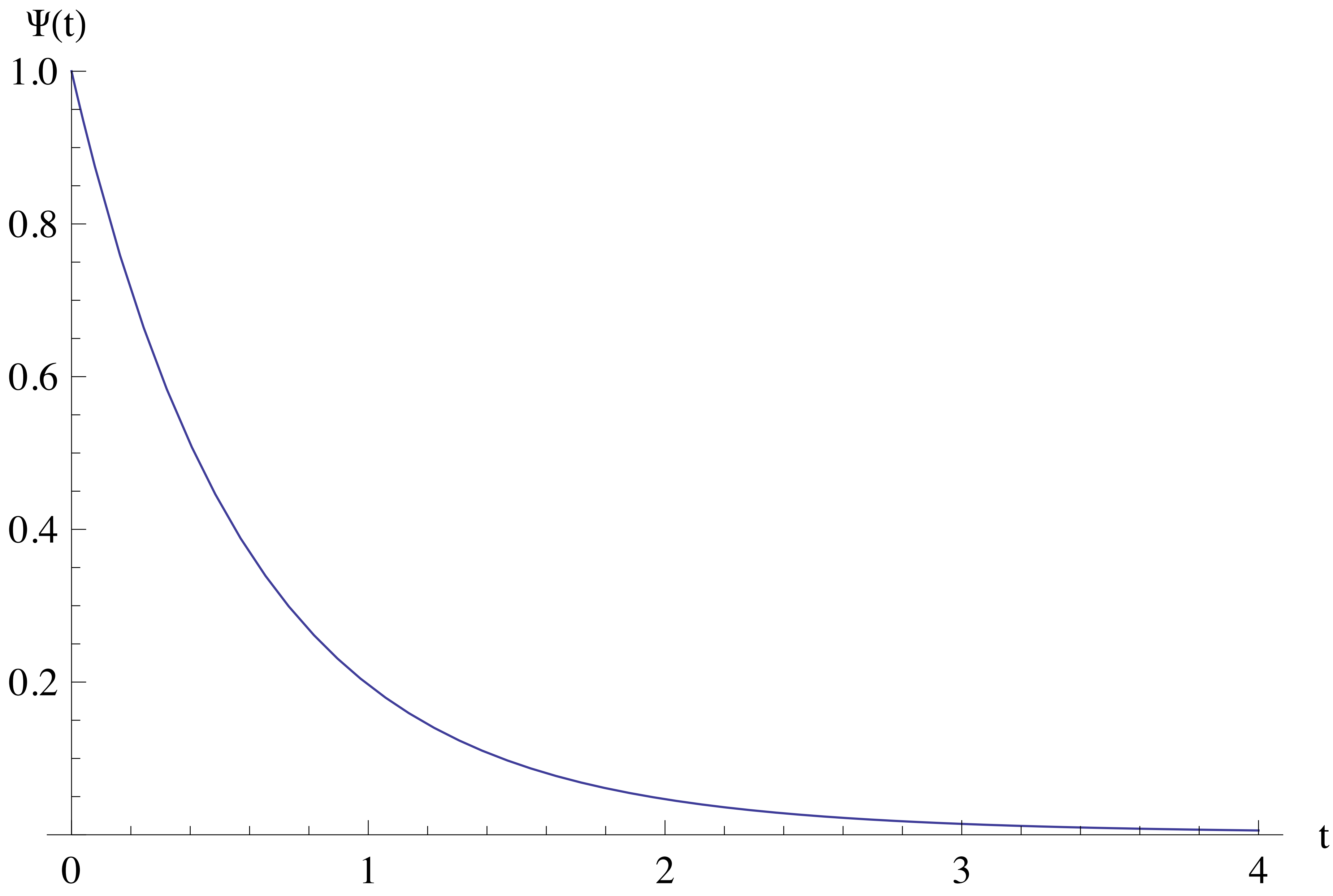


$z \to 1$ scaling $\quad M(t', z) = \Psi(t'')$ with $t'' = t'(1-z)$



$z \to 1$ scaling function $\qquad \Psi(t'') \;=\; \dfrac{1}{2\pi}\displaystyle\int_{-2}^{2} dx\,\sqrt{4-x^2}\; e^{-t''\,\sqrt{4-x^2}}$



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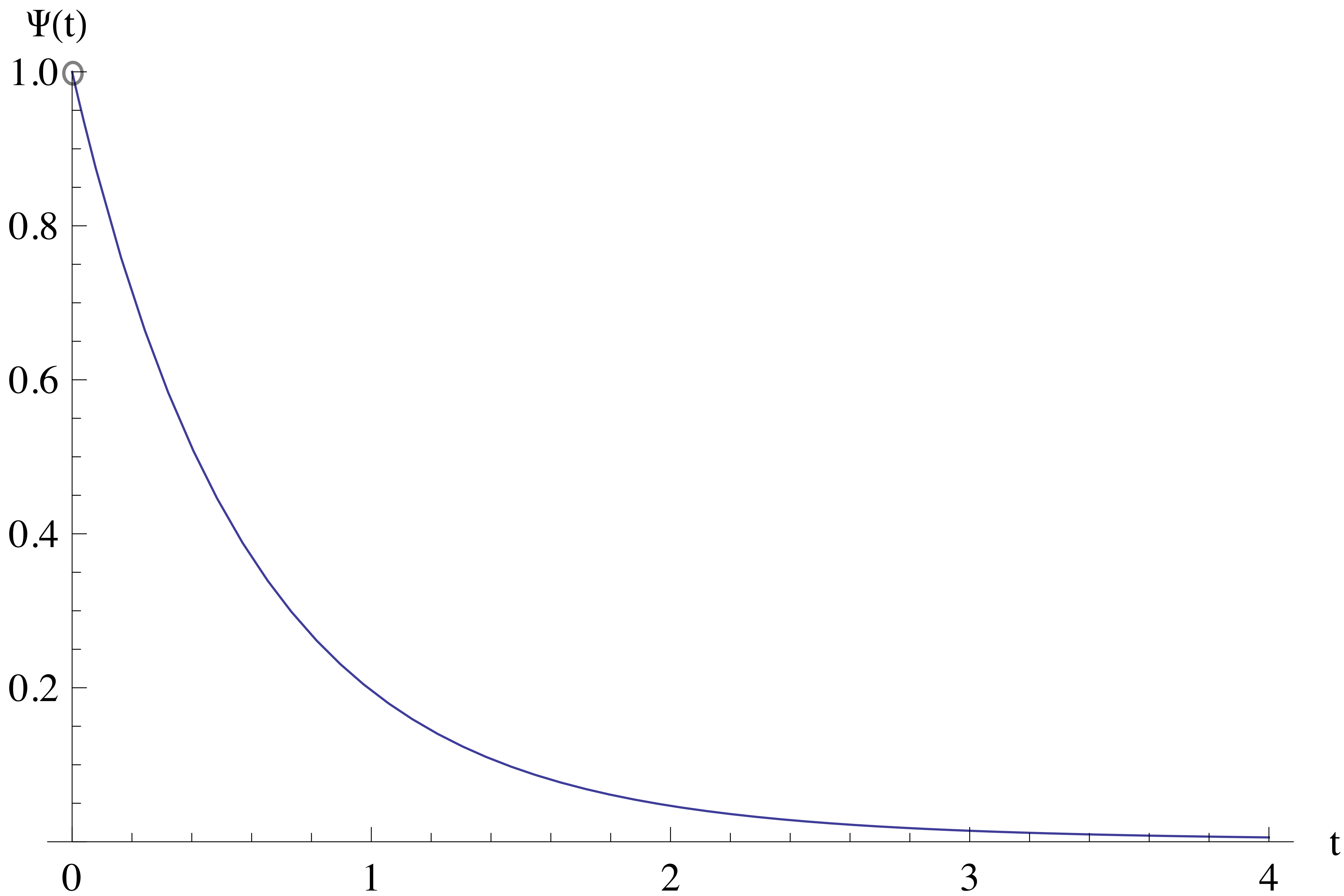

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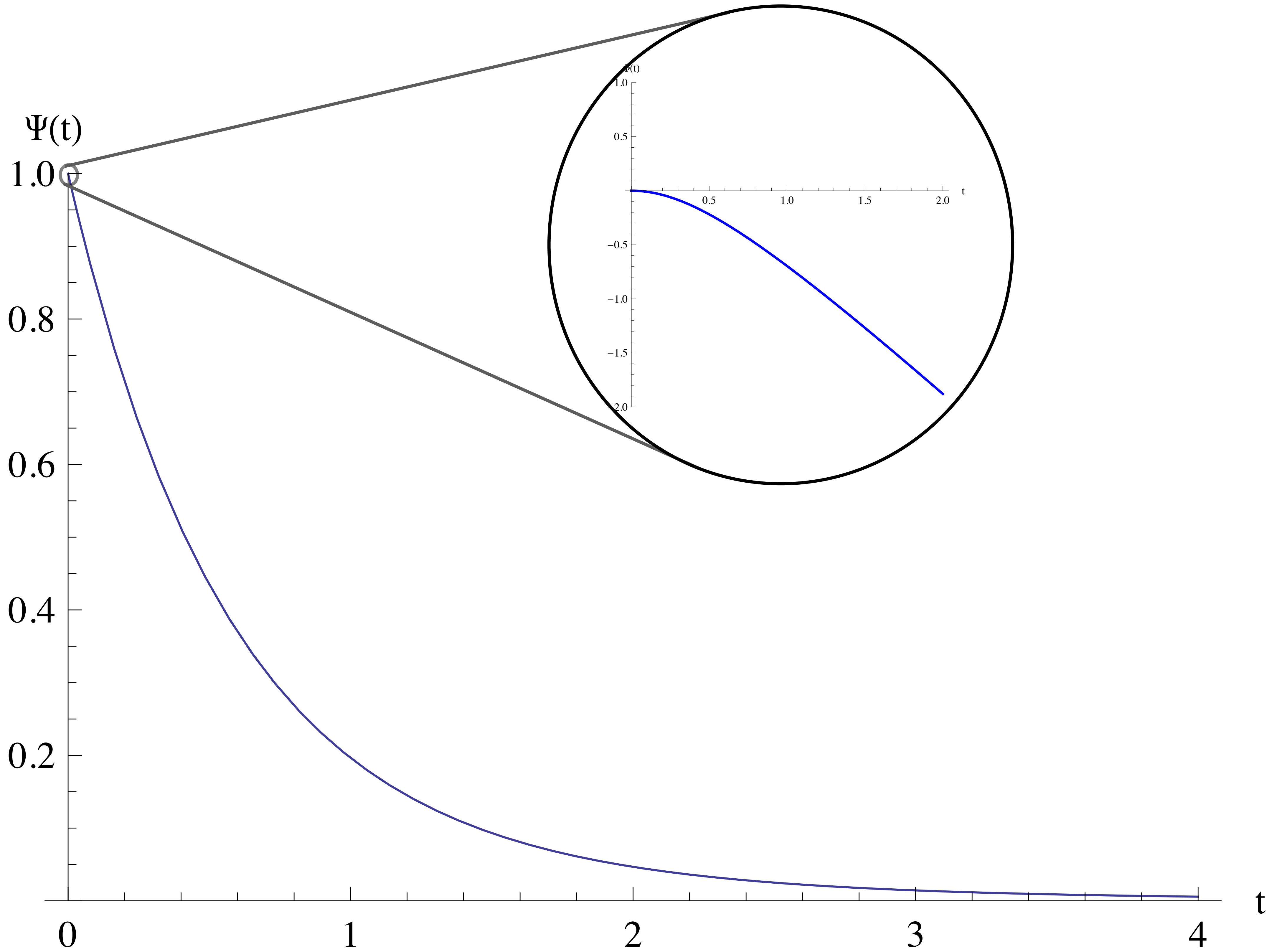
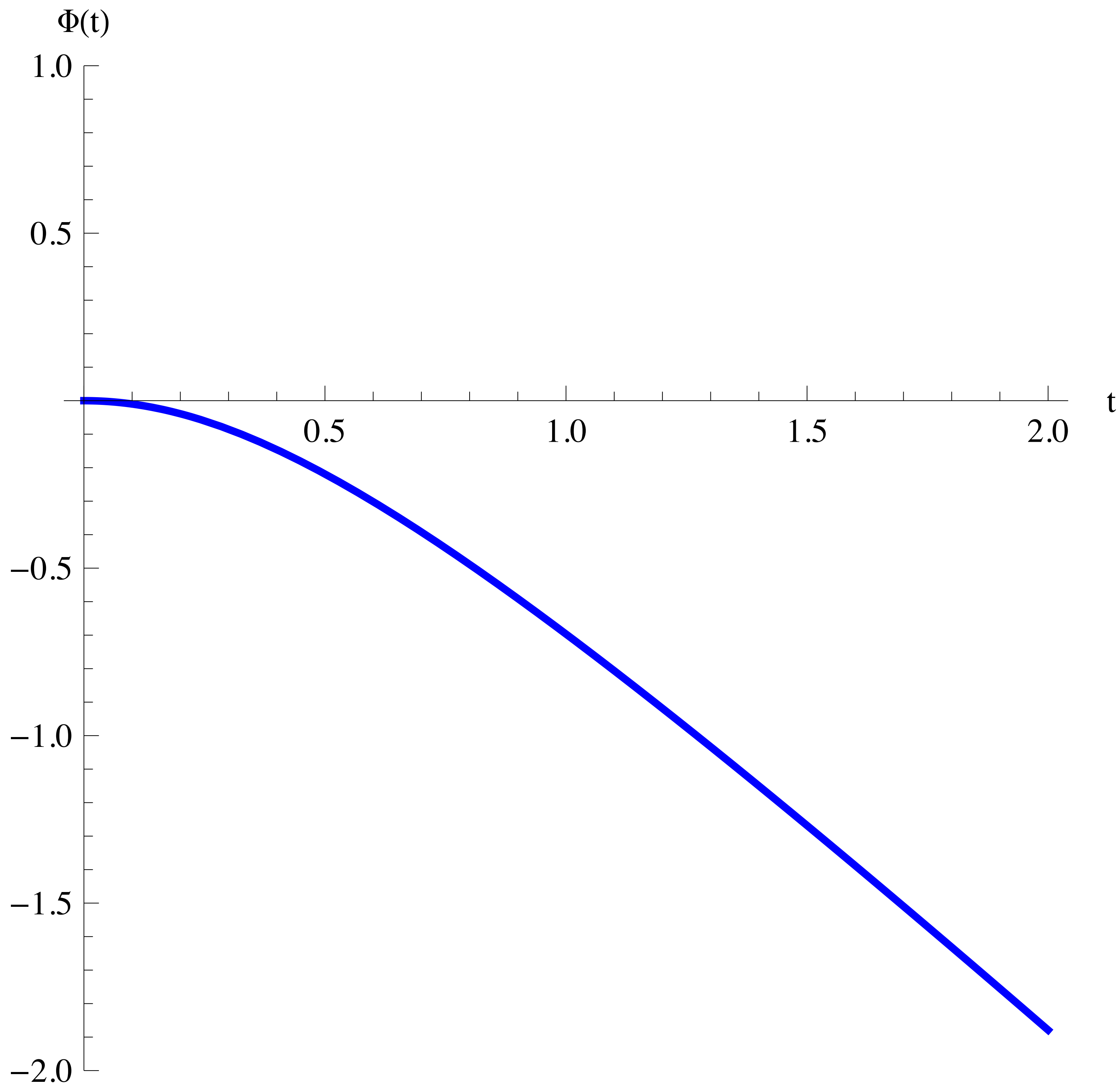


small t and $z \to 1$ behavior

$$M(t,z) = 1 + (1-z)\,\Phi(t) + \cdots$$

$$\Phi(t) = 1 - \,{}_1F_2\left(-\tfrac{1}{2}; 1, 2; -4t^2\right)$$



III - Evolution of coherent and incoherent states

We can easily study analytically and illustrate the evolution on the matrix density of the spin, starting from a pure spin state $|\psi\rangle$

$$|\psi\rangle \to \rho = |\psi\rangle\langle\psi| \to W^{(l,m)} \to W(\vec{n}) = \sum_{l,m} W^{(l,m)}\, Y_l^m(\vec{n})$$

Wigner distribution = function on the sphere

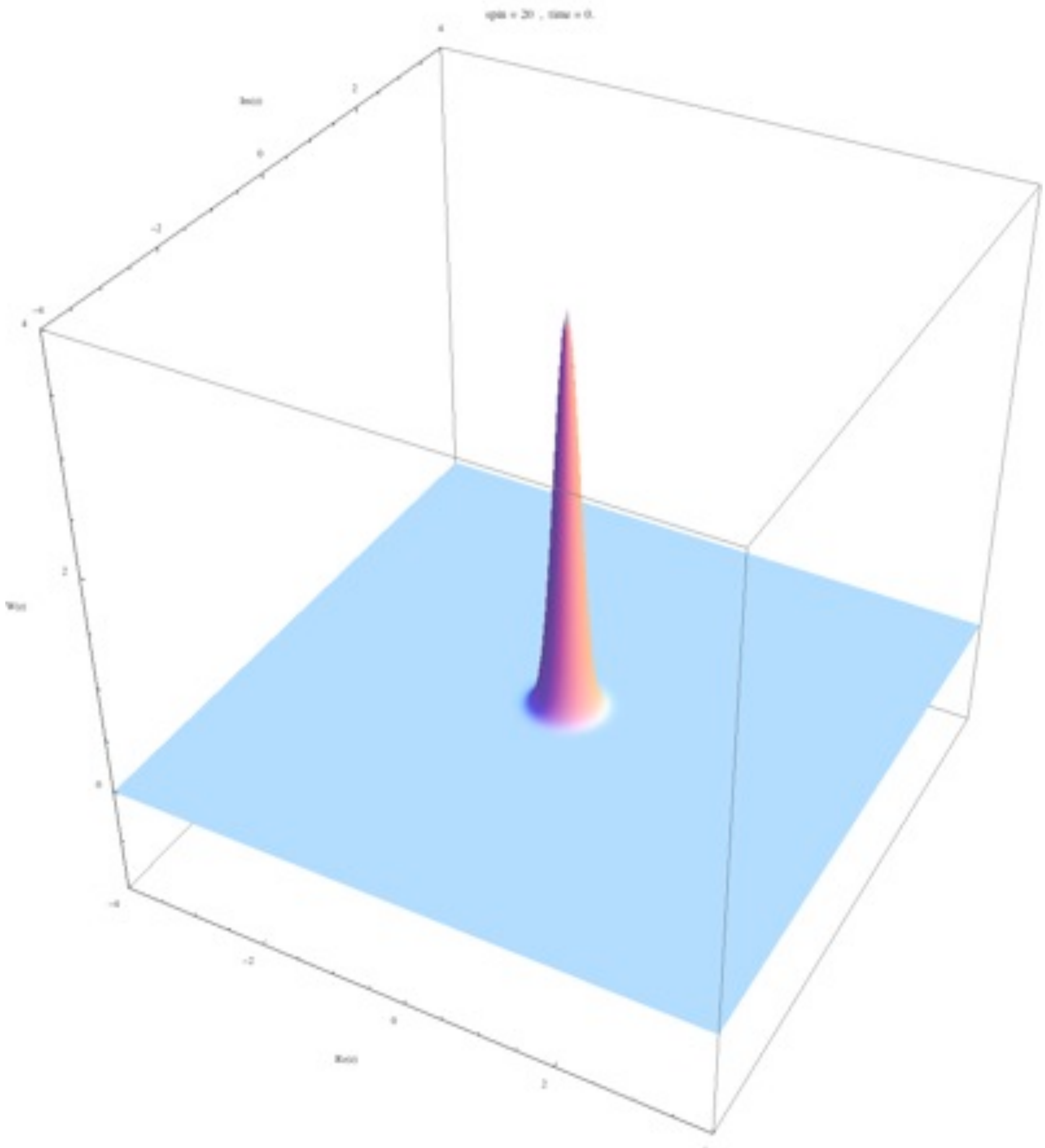
Coherent state

$$|\vec{n}\rangle = \sum_{m=-j}^{j} \sqrt{\frac{(2j)!}{(j+m)!\,(j-m)!}} \cos(\theta/2)^{j+m} \sin(\theta/2)^{j-m} \mathrm{e}^{-\mathrm{i}m\phi} |m\rangle$$

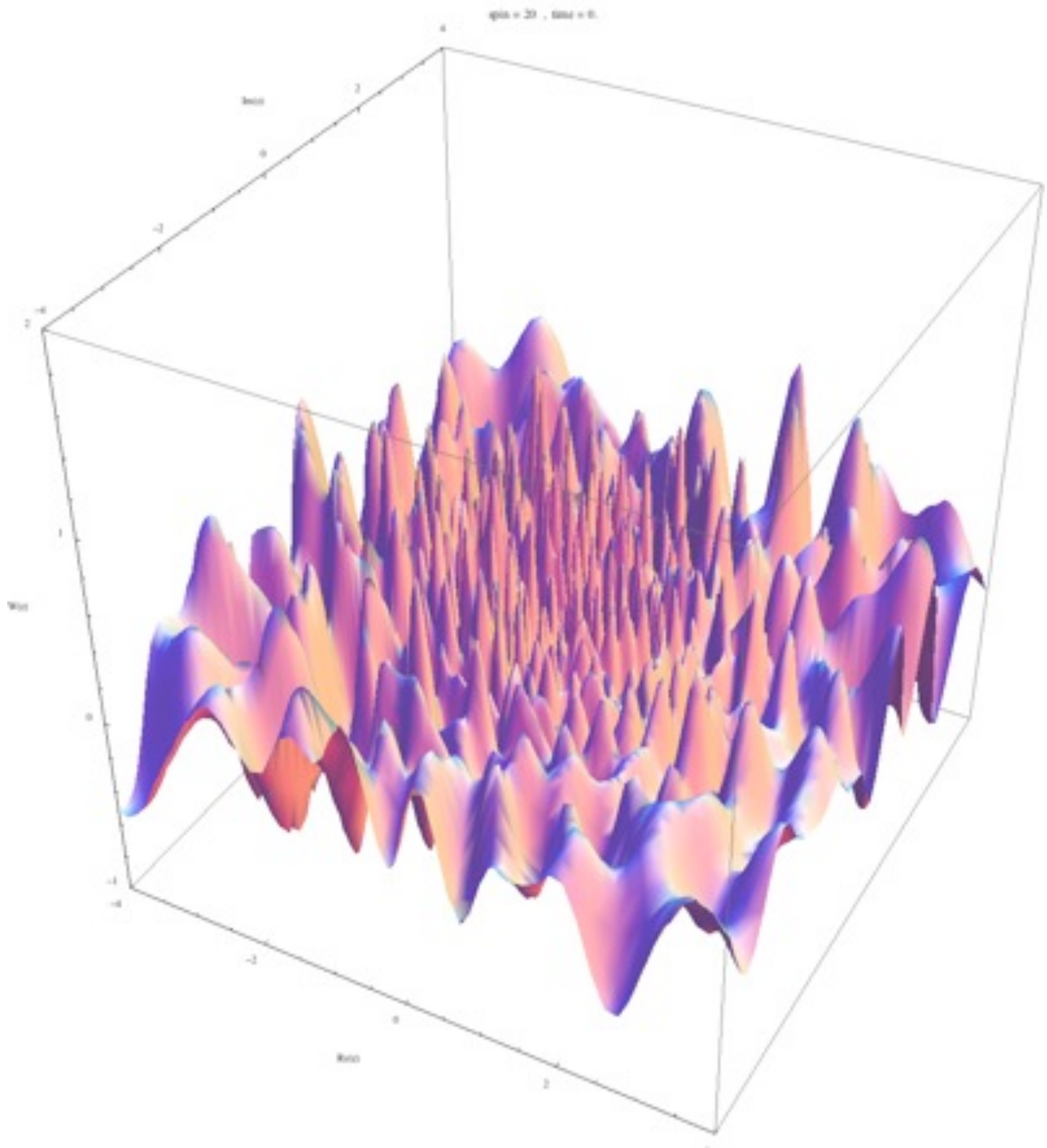
$$W^{(l)}_{\mathrm{c.s.}} = \frac{2l+1}{\sqrt{2j+1}} \exp\left(-\frac{l^2}{2j}\right) \qquad\qquad l \sim \sqrt{j}$$

Coherent states are the most localised states on the sphere

- Coherent states look like a Gaussian on the unit sphere with width $\Delta_\theta = 1/\sqrt{j}$
- Random states look like random functions on the unit sphere



coherent state



random state

stereographic projection and *j=20*

Spin decoherence

Random pure state

spin = 20 , couplings = (0,1,0,...)

Spin decoherence

Coherent state

spin = 20 , couplings = (0,1,0,...)

Spin decoherence

Cat state

j=20 , couplings = (0,1,0,...)

Spin decoherence

Superposition of 3 coherent states

spin = 20 , couplings = (0,1,0,...)

The time scales of decoherence dynamics

There are 4 time scales $\tau_0 \leq \tau_1 \ll \tau_2 \ll \tau_3$

τ_0 dynamical time scale for the whole system

τ_1 decoherence time scale for generic states $l \gg \sqrt{j}$

τ_2 evolution time scale for coherent states (onset of quantum diffusion)

τ_3 equilibration time for quantum diffusion

For our simple model with Gaussian Hamiltonian ensembles

$$\tau_0 = 1/\parallel H_{\mathcal{SE}} + H_{\mathcal{E}} \parallel \qquad \frac{\tau_0}{\tau_1} = \left(\frac{\parallel H_{\mathcal{SE}} \parallel}{\parallel H_{\mathcal{SE}} + H_{\mathcal{E}} \parallel} \right)^2$$

$$\frac{\tau_1}{\tau_2} = \left(\frac{\parallel [\vec{\mathbf{S}}, H_{\mathcal{SE}}] \parallel}{\parallel \vec{\mathbf{S}} \parallel \, \parallel H_{\mathcal{SE}} \parallel} \right)^2 \qquad \frac{\tau_2}{\tau_3} = \frac{1}{j}$$

$H_{\mathcal{E}} \leftarrow l = 0$ term

$H_{\mathcal{SE}} \leftarrow l \neq 0$ terms

with the «L2 norm» for operators $\parallel A \parallel^2 = \dfrac{\mathrm{tr}(A^\dagger A)}{\mathrm{tr}(1)}$

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with the «L₂ norm» for operators $\parallel A \parallel^2 = \dfrac{\mathrm{tr}(A^\dagger A)}{\mathrm{tr}(1)}$

The ratio $\tau_2 \gg \tau_1$ is large iff the commutator $\left[\vec{S}, H_{\mathcal{S}\mathcal{E}}\right]$ is small

$$\left[\vec{S}, H_{\mathcal{S}\mathcal{E}}\right] \ll \vec{S} \times H_{\mathcal{S}\mathcal{E}}$$

Coherent states are robust against decoherence and play the role of pointer states if

$$\Delta(l) \neq 0 \text{ for } l \leq l_0 \text{ and } j \gg l_0^2$$

The dynamics of decoherence depends on the details of the Hamiltonian ensemble

$$\Delta = \{\Delta(l),\ l = 0, \cdots l_0\}$$

Beyond the decoherence time scale τ_1 , the dynamics of coherent states is much simpler and exhibit some universal features.

IV - Quantum diffusion

For $\tau_1 \ll t \ll \tau_2$ only semiclassical coherent states survive

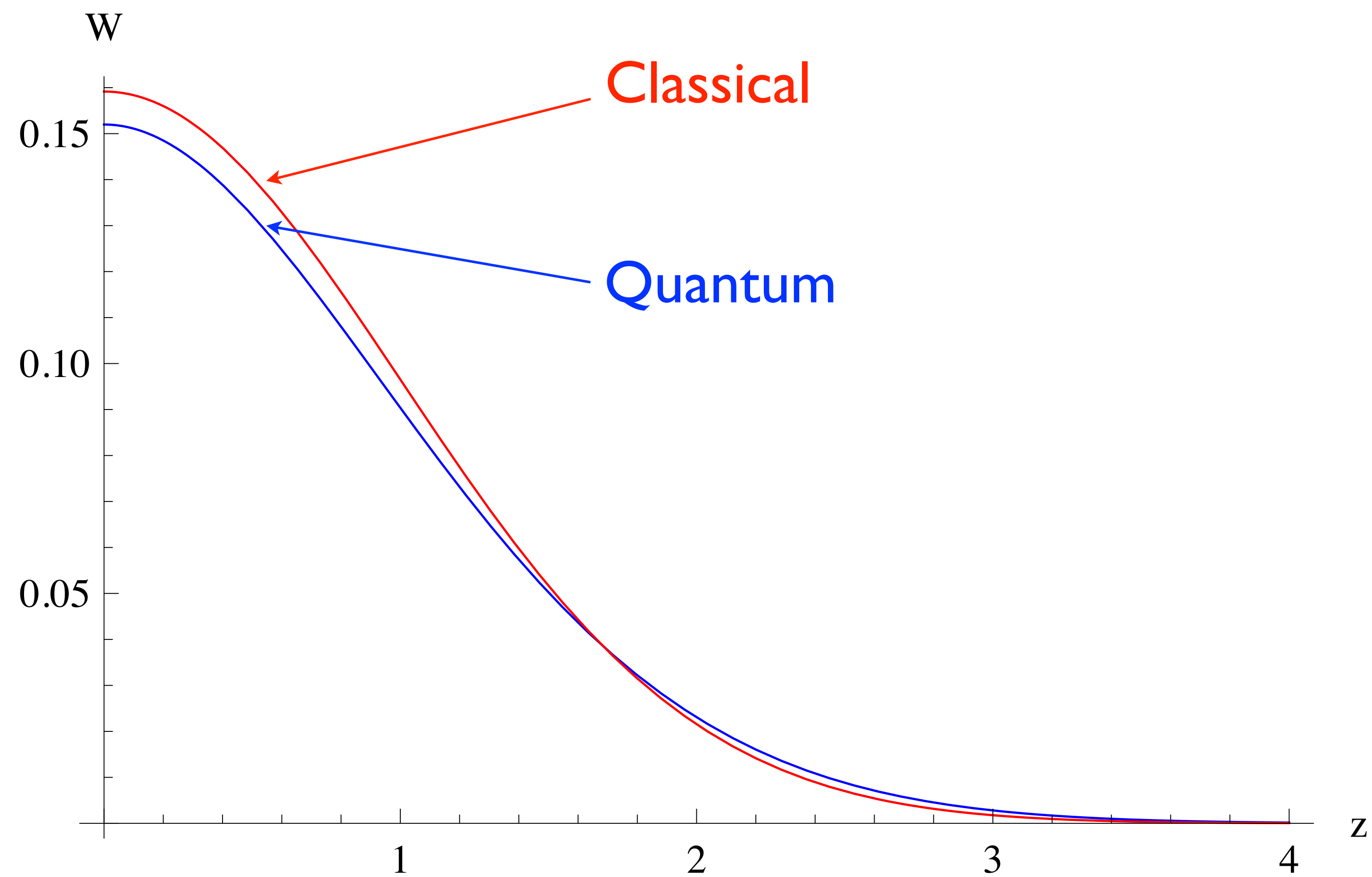
For $\tau_2 < t$ coherent states start to become mixed states $\quad j \gg 1$

This is an effect of quantum diffusion, i.e. the remaining weak effect of the external system on the coherent states.

The width of the distribution function in phase space is found to grow like $\Delta_\theta(t) \propto \sqrt{t}$

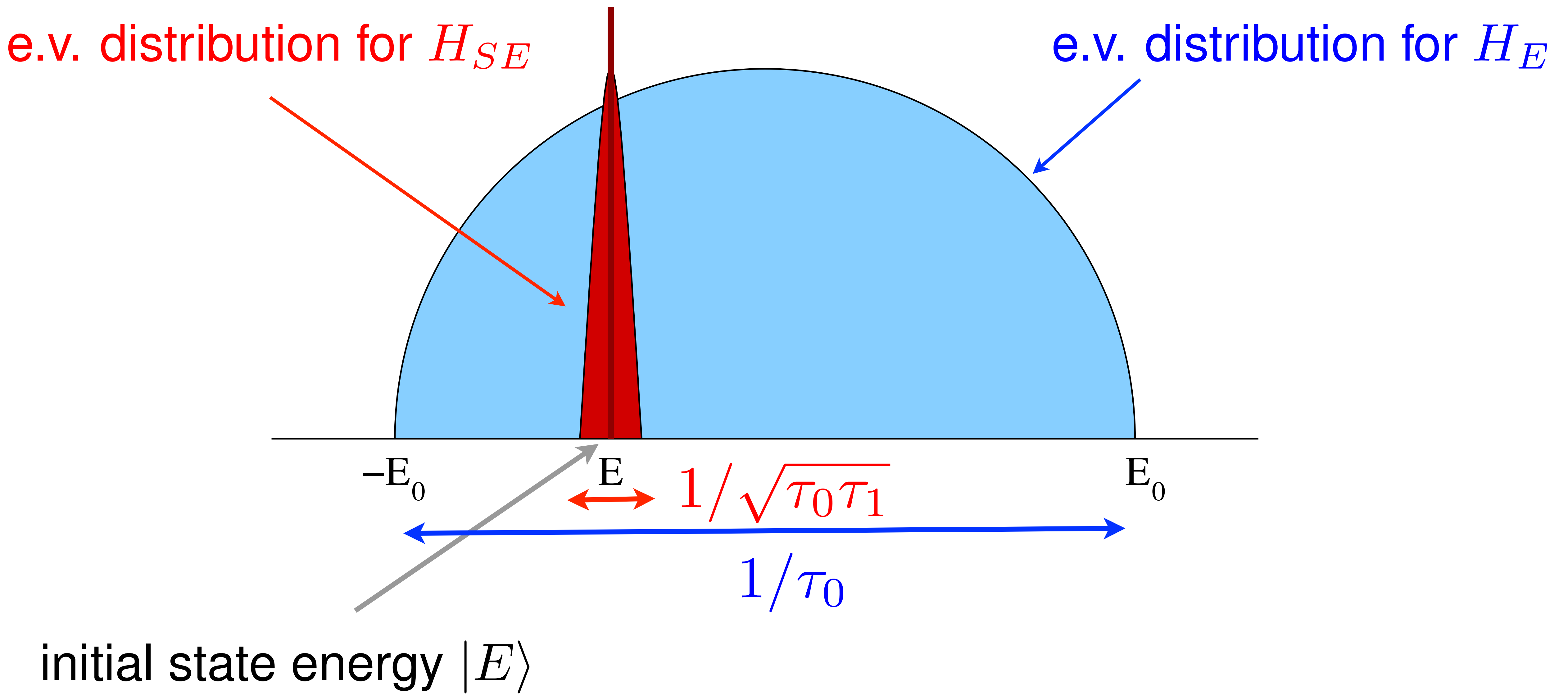
This suggests a random walk in phase space

But the probability profile can be computed and is not a Gaussian ! This is a signal that the evolution is not a Markovian short range process, even at large times!



V - Dynamics and initial conditions for $\mathcal{E}$

The calculation can be extended to a general Hamiltonian for the external system with a general eigenvalue distribution, and to a given initial state $|\phi_{\mathcal{E}}\rangle$ such as an energy eigenstate



The calculations and the explicit solutions are less simple and much longer.

If external system: fast dynamics + initial energy e.g. state

If $\tau_1 \gg \tau_0$ and if one starts from an energy eigenstate $|E\rangle$ then the diffusion is Markovian and the diffusion coefficient is

$$D_{\rm diff} = 2\pi\, \rho_{\mathcal{E}}(E)\, \overline{\left|\langle\Phi|[\vec{\mathbf{S}}, H_{\mathcal{S}\mathcal{E}}]|\Phi'\rangle\right|^2}$$

d.o.s. of the external system

typical size of a matrix element of the commutator

This is a Golden Rule formula

Not too surprising, one must be able to write a master equation for the evolution of the density matrix

If the initial state is a quantum superposition of energy eigenstates

$$|\phi_{\mathcal{E}}\rangle = \sum_E \phi(E)|E\rangle$$

the diffusive regime is found to be a randomisation of the collection of Markovian diffusion processes $\mathbb{P}(E)$

Each diffusion process $\mathbb{P}(E)$ is a RW with diffusion constant $D(E)$

$$|\vec{n}\rangle \otimes |E\rangle \xrightarrow{t} \sum_{\vec{n}'} \Psi(E,\vec{n}';t)\, |\vec{n}'\rangle \otimes |E,\vec{n}';\vec{n},t\rangle$$

The processes are taken with probability weight $W(E) = |\phi(E)|^2$

The $|E,\vec{n}';\vec{n},t\rangle$ are all (approximately) orthogonals

This reflects the decoherence between energy eigenstates (of the the external system) induced by the coupling with the large spin

The evolution of a superposition of energy eigenstates for the external system

$$c_1|E_1\rangle + c_1|E_2\rangle$$

can be studied by the same kind of planar diagrams resummation. The coupling to the spin induces decoherence between these states with the same decoherence time τ_1

In the non Markovian general case $\tau_1 \simeq \tau_0$
one can study multi-time functions as a function of the initial spin state (no Lindbladian allows to compute these functions)

$$\langle A_1(t_1)A_2(t_2)\cdots A_N(t_N)\rangle$$

There is an interesting planar diagrammatics (in progress)

Conclusion

A very simple but quite rich model

An entrance gate to study more realistic models with interesting dynamics

For condensed matter or quantum information systems

For studying the emergence of continuum classical degrees of freedom out of complicated fully quantum systems

Another aspect of random matrices....