# Anderson model on Bethe lattices: density of states, localization properties and isolated eigenvalue 

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#### Abstract

We revisit the Anderson localization problem on Bethe lattices, putting in contact various aspects which have been previously only discussed separately. For the case of connectivity 3 we compute by the cavity method the density of states and the evolution of the mobility edge with disorder. Furthermore, we show that below a certain critical value of the disorder the smallest eigenvalue remains delocalized and separated by all the others (localized) ones by a gap. We also study the evolution of the mobility edge at the center of the band with the connectivity, and discuss the large connectivity limit.


## §1. Introduction

The effect of quenched disorder on non-interacting electrons can be dramatic. The wave-function completely changes form going from an extended plane wave at zero disorder to a localized wave at large disorder. Actually, even an infinitesimal disorder is enough to induce localization of all eigenstates in one and two dimensions. ${ }^{(1)}$ Localized electrons do not lead to electronic transport except if the system is coupled to a thermal bath, as a consequence conduction properties changes completely in presence of localization.

This phenomenon, called Anderson localization, was discovered in 1958 by P.W. Anderson. (2) Since then a huge amount of work has been devoted to it, see for instance the brief review "Fifty years of Anderson Localization" ${ }^{(1)}$ and the monographs. ${ }^{(3),(4),(5),(6)}$ Exactly solvable models naturally played an important role in its understanding. One dimensional systems have been studied thoroughly ${ }^{(4)}$ However, in these systems all states are localized and hence some aspects of Anderson localization, such as the transition between localized and extended states, and the critical properties at the mobility edge could not be studied. All these properties can instead be analyzed on Bethe lattices (see the next section for a precise definition), which therefore provide a very useful benchmark since they are still simple enough to be studied without making any approximation.

The first analysis of Anderson localization on Bethe lattices was performed by Abou Chacra, Anderson and Thouless ${ }^{(7),(8)}$ and many other studies followed both in the physics (see for instance the works ${ }^{(9),[10),[11,, 12)}$ and references therein) and mathematics ${ }^{(13),(14),(15),(16)}$ communities. Our main goal in this work is to combine various point of views on the problem, and to underline a particular feature of Bethe lattices. In fact, besides the properties of the density of states, the evolution of the mobility
edge with disorder and with the connectivity $k+1$, we shall discuss the presence of isolated eigenvalues. Without disorder, the smallest eigenvalue for the tight-binding Hamiltonian (or the discrete Laplacian) on a Bethe lattice is isolated from all the others by a gap. ${ }^{17)}$ This eigenvalue plays a very important role in determining the physical properties of some statistical mechanics models displaying a condensation transition at low temperature, such as the spherical mode ${ }^{18)}$ or non-interacting bosons on a Bethe lattice. The evolution of the isolated eigenvalue in presence of disorder is interesting in particular in connection with studies of disordered bosonic models since it may lead to somewhat different behaviors than the ones obtained for finite dimensional systems, where no isolated eigenvalue is found (even though the relevance of the spectrum of the one-particle kinetic energy is less obvious in the presence of interactions).

## §2. Model and iterative equations

The Anderson model corresponds to the Hamiltonian:

$$
\mathcal{H}=-\sum_{\langle i, j\rangle} t\left(c_{i}^{\dagger} c_{j}+c_{j}^{\dagger} c_{i}\right)+\sum_{i=1}^{N} \epsilon_{i} c_{i}^{\dagger} c_{i}
$$

where the first sum runs over the nearest neighbors couples of the lattice, the second sum runs over all $N$ sites and $c_{i}^{\dagger}, c_{i}$ are fermionic creation and annihilation operators. For simplicity we consider spinless fermions. The on-site energies $\epsilon_{i}$ are independent and identically distributed random variables, that we shall take in the following uniformly distributed in the interval $[-W / 2, W / 2]$. As the fermions have no interactions, the study of this model amounts to determine the spectrum of the $N \times N$ random matrix $H_{i j}$ such that $H_{i j}=-t$ if $i \neq j$ are neighbors on the lattice, $H_{i i}=\epsilon_{i}$ on the diagonal. In the following we will take $t=1$, which means that $\epsilon_{i}$ are measured in units of $t$.

As anticipated in the introduction, the lattice we will focus on is a Bethe lattice. It is possible, and this is the point of view of almost all mathematical works, to define it as an infinite regular tree, i.e. a graph without loop where every vertex has the same degree (that we shall denote $k+1$ in the following). It is however necessary for the study of thermodynamical properties of statistical mechanics model (e.g. to define a free-energy) to consider finite-size versions of the model, with a given number $N$ of vertices, before taking the thermodynamic limit $N \rightarrow \infty$. There are at least two possible ways to define a finite-size Bethe lattice. The first one is to consider a finite tree of depth $n$, that is a tree in which the root vertex (at generation 0 ) has $k+1$ offsprings, each of the vertices in the generation from 1 to $n-1$ has $k$ offsprings, and finally the leaves of the $n$-th generation have no descendent. The second way is to consider a random $k+1$-regular graph, i.e. a graph chosen uniformly at random among all the graphs on $N$ vertices where each of the vertex has degree $k+1$. The properties of such random graphs have been extensively studied (see ref. 19) for a review). It is known in particular that any finite portion of such a graph is a tree with a probability going to one as $N \rightarrow \infty$. The advantage of
the second construction is the absence of any boundary effect: all sites, for any finite $N$, have $k+1$ neighbors, whereas in the trees of the first construction the sites on the boundary, i.e. on the leaves of the $n$-th generation, are asymptotically as numerous as the bulk (volume) sites and have connectivity one, a pathological situation compared to finite dimensions. Random regular graphs can be thought of as regular trees wrapped onto themselves. This absence of boundary is particularly important for frustrated models, as discussed notably in ref. (20).

The spectral properties of $H$ are easily studied in terms of the resolvent matrix, which is defined as:

$$
G_{i j}(z)=\left(\frac{1}{H-z \mathcal{I}}\right)_{i j}
$$

where $\mathcal{I}$ is the identity matrix. For example, the density of eigenvalues $\lambda_{\alpha}, \rho(E)=$ $\sum_{\alpha} \delta\left(E-\lambda_{\alpha}\right) / N$, can be obtained as:

$$
\rho(E)=\lim _{\eta \rightarrow 0} \frac{1}{N \pi} \operatorname{Im} \operatorname{Tr} G(E+i \eta) .
$$

Moreover the localization properties of the eigenvectors of $H$ can be deduced from the behavior of $\lim _{\eta \rightarrow 0} \eta\left|G_{i i}(E+i \eta)\right|^{2}$ (see for instance refs. (21),(22)).

When the non-zero off-diagonal elements of $H$ have the structure of a finite tree it is easy to obtain the diagonal elements of the resolvent from

$$
G_{i i}(z)=\frac{1}{H_{i i}-z-\sum_{j \in \partial i} H_{i j}^{2} G_{j \rightarrow i}(z)} .
$$

Here $\partial i$ denotes the set of neighbors of $i$ and $G_{j \rightarrow i}(z)$ is the $G_{j j}(z)$ resolvent of a modified matrix in which the edge between $i$ and $j$ has been removed. These new variables verify the following recursive equations:

$$
G_{i \rightarrow j}(z)=\frac{1}{H_{i i}-z-\sum_{j^{\prime} \in \partial i \backslash j} H_{i j^{\prime}}^{2} G_{j^{\prime} \rightarrow i}(z)} .
$$

These equations were first obtained in ref. 7) using perturbation theory. Another simple way to obtain these equations is to consider a Gaussian model with a kernel $H-z \mathcal{I}$ (we shall come back on this in Sec. [5). More formally one can use resolvent identities to derive them; this last method has the advantage of being also valid directly on the infinite Bethe lattice, the matrix $H$ being turned in an operator acting on square-normalizable functions on the sites of the infinite tree. Note that it has been proven in ref. (23) that the random regular graph case can be studied using these recursion equations in the thermodynamic limit, thanks to the local convergence of random regular graphs to trees.

Let us denote, with a slight abuse of notation, $G(z)$ the diagonal element of the resolvent for an arbitrary site in the Bethe lattice (the infinite tree or the thermodynamic limit of random regular graphs), and $\widehat{G}(z)$ the similar quantity in the absence of one incident edge. These are random variables, because of the randomness in the
choice of the $H_{i i}=\epsilon_{i}$. The above written equations implies that their distribution follows from

$$
G(z) \stackrel{\mathrm{d}}{=} \frac{1}{\epsilon-z-\sum_{i=1}^{k+1} \widehat{G}_{i}(z)}, \quad \widehat{G}(z) \stackrel{\mathrm{d}}{=} \frac{1}{\epsilon-z-\sum_{i=1}^{k} \widehat{G}_{i}(z)}
$$

where $\stackrel{\mathrm{d}}{=}$ denotes the equality in distribution of two random variables. In the r.h.s. the various $\widehat{G}_{i}$ are i.i.d. copies of $\widehat{G}$, and $\epsilon$ is independently drawn uniformly on $[-W / 2, W / 2]$. The spectral properties of the model can be obtained from the solution of these distributional equations, by computing

$$
\rho(E)=\lim _{\eta \rightarrow 0} \frac{1}{\pi} \operatorname{Im} \mathbb{E}[G(E+i \eta)], \quad L(E)=\lim _{\eta \rightarrow 0} \frac{1}{\pi} \eta \mathbb{E}\left[|G(E+i \eta)|^{2}\right]
$$

where the expectation $\mathbb{E}[\cdot]$ is with respect the distribution of $G$. The first quantity gives the density of states of energy $E$, while the second one corresponds to the average inverse participation ratio of the eigenstates of energy close to $E$ (for a finite graph, the inverse participation ratio of the eigenstate $\alpha$ is $L^{\alpha}=\sum_{i}\left|\psi_{i}^{\alpha}\right|^{4}$, where $\psi_{i}^{\alpha}$ is the component of the normalized $\alpha$-th eigenvector on site $i$ ): it is non-zero only if $E$ falls in an interval of energy comprising localized states (pure-point spectrum) and vanishes for extended states (absolutely continuous spectrum). Indeed the density of states alone is smooth at the mobility edge separating localized and extended states ${ }^{(24)}$ and cannot be used to distinguish the two regimes. An alternative and numerically more precise method to compute the mobility edge was proposed in ref. 7) and consists in investigating the stability of a real solution of the equations (2.6) for real $z=E$ to the insertion of a small imaginary part; this solution is stable whenever $E$ belongs to the localized regime. Physically, for a finite and large tree, this amounts to couple boundary sites to a thermal bath and test whether the site at the center can dissipate energy at very low temperature when the Fermi energy is close to $E$. If $E$ belongs to the localized regime, energy is not transported across the tree and no dissipation is possible far from the boundary, i.e. the imaginary part of the self energy vanishes for sites very distant from the boundary.

A numerical procedure to solve the distributional equations (2.6) was proposed in ref. 77), and revivified more recently in the context of finite-connectivity mean-field disordered systems under the name of population dynamics, ${ }^{[20}$ also called the pool method (11) The idea is to approximate the distribution of a random variable, say $\widehat{G}$, by the empirical distribution of a sample of a large number $\mathcal{N}$ of representants $\left\{\widehat{G}_{1}, \ldots, \widehat{G}_{\mathcal{N}}\right\}$. Starting from an arbitrary initial condition a sequence of samples is produced. The elements of the new sample $\widehat{G}_{j}^{\prime}$ are generated by, identically and independently for each $j$, selecting $k$ elements uniformly at random from the current sample $\left\{\widehat{G}_{1}, \ldots, \widehat{G}_{\mathcal{N}}\right\}$, drawing an energy $\epsilon$ uniformly in $[-W / 2, W / 2]$, and computing the value of $\widehat{G}_{j}^{\prime}$, according to $(2 \cdot 6)$. Repeating these steps, one reaches a sample approximating the fixed point solution of $(2 \cdot 6)$, and then the observables in $(2 \cdot 7)$ can be obtained by computing empirical averages over the representants of the distribution. The numerical accuracy is controlled by the size $\mathcal{N}$ of the samples.

## §3. Phase diagram, density of states and mobility edges

We have numerically solved the equations (2.5) with the method described above for the connectivity $k+1=3$ and using samples with sizes up to $\mathcal{N}=10^{7}$. The results are displayed in Fig. $⿴ 囗$. The innermost solid line is the mobility edge separating the regimes of extended and localized states. It crosses the $E=0$ axis for a critical value of the disorder $W_{\mathrm{c}} \approx 17.4$, above which no extended states exist. This value is in agreement with the one found in refs. (7), (8), (11). Moreover for small disorder the whole interval $[-2 \sqrt{k}, 2 \sqrt{k}]$ of the density of states of the pure model $(W=0)$ corresponds to extended states, in other words the mobility edge does not enter the band of the pure model up to a strictly positive value of $W$. This fact has indeed been proven rigorously! (14), 16 (16) The dashed lines show the numerically determined edge for the density of states, that is the limit of the interval for which $\rho(E)$ of Eq. (2.7) is found to be non-vanishing. The edge of the spectrum has been proved rigorously to be equal to $E= \pm(2 \sqrt{k}+W / 2)$ (solid lines in Fig. (1)) by ergodicity arguments. ${ }^{(14),(15),(16)}$ The discrepancy between the numerically computed value and the true one is due to the extreme smallness of the density of states close to the edge. As we shall discuss in the next section, the decrease of $\rho(E)$ is extremely fast. Therefore, one would need extremely large sample sizes $\mathcal{N}$ in order to reproduce the density of states close to the edge. We have checked that in agreement with this interpretation the location of the observed band edge depends on the sample size $\mathcal{N}$ used. The data presented in the figure for the band edge corresponds to $\mathcal{N}=210^{6}$.

The shape of the density of states and its evolution with the disorder is shown in Fig. 2, where we plotted $\rho(E)$ for $W=0.3$ and $W=12$. We also show the density of states obtained by exact diagonalization for matrices of sizes 8192 . The agreement between the two methods is very good. However, for strong disorder finite size effects become important close to the edge, as expected from the discussion above. In Fig. 3 we show the inverse participation ratio $L^{\alpha}$ obtained by exact diagonalization (we recall that for a finite system $L^{\alpha}=\sum_{i}\left|\psi_{i}^{\alpha}\right|^{4}$, where $\psi_{i}^{\alpha}$ is the value of the normalized $\alpha$-th eigenvector at site $i$ ). The evolution of the inverse participation ratio with the system size points to a mobility edge $E_{c} \simeq 4$ for $W=12$, which is in agreement with the prediction of the cavity method, see Fig. [1

We have also studied the evolution of these results with the connectivity of the Bethe lattice. The phase diagrams are qualitatively similar. The main issue is the evolution of the mobility edge with $k$. In particular we have focused on the critical value of the disorder, $W_{\mathrm{c}}$, above which only localized states exist. This is interesting for several reasons. First, it is relevant for recent works studying the effect of electronelectron interaction on Anderson localization. In this case one is interested in the problem of localization in the Fock space, which has been argued to be related to the localization problem on a Bethe lattice with very large connectivity! ${ }^{(25)}$ Second, the limit of large connectivity might be the only case where a complete analytical solution is within reach. Abou-Chacra et al ${ }^{(7)}$ developed two different approximation schemes to obtain the mobility edges analytically. Both methods lead to an asymptotic form of $W_{\mathrm{c}}$ of the form $c k \ln k$. The constants $c$ are equal to 4 and $2 e \simeq 5.43$, depending on the approximation. We have found that the stability analysis used to determine


Fig. 1. Phase diagram for the Bethe lattice with connectivity $k+1=3$. The innermost solid line indicates the mobility edge between extended and localized states, the outermost solid line being the edge of the density of states $E= \pm(2 \sqrt{k}+W / 2)$. The dashed line is the numerically observed edge, see the text for details.


Fig. 2. Density of states, $\rho(E)$, obtained using the cavity method and exact diagonalization of the connectivity matrix of random regular graphs of 8192 sites for $W=0.3$ (left panel) and $W=12$ (right panel). The data are averaged over 16 different realizations of the disorder and of the graph.


Fig. 3. Inverse participation ratio, $L$, as a function of the energy obtained via exact diagonalization for $W=12$ and for different sizes of the adjacency matrix $(1024,2048,4096)$. In the localized regime $L$ does not depend on the size of the matrix, whereas the inverse participation ratio of the extended states decrease as the inverse of the system size. The mobility edge is found in $E \simeq 4$, in agreement, within the numerical accuracy, with the numerical prediction of the cavity method.
the mobility edge can be interpreted, in the large $k$ limit, as an iterative equation on Levy random variables with tail exponent $\mu=1 / 2$. This leads to the result $c=2 \sqrt{2 \pi} \simeq 5.01$. Whether this is still an approximation or an exact result is unclear at this stage and will be investigated and detailed elsewhere. ${ }^{(26)}$

Our numerical results for $W_{\mathrm{c}}$ obtained by solving the distributional equation (2.6) are presented in table [1 None of the analytical predictions cited above fit well the data. Presumably, very strong sub-leading corrections are present, possibly scaling like $k$ with potentially $\ln \ln$ corrections. In this situation one would need very large values of $k$ to find the true asymptotic form. A best fit of our data of the form $W_{\mathrm{c}}=c k \ln k+b k$ leads to $c=4.7$ and $b=5.9$. Imposing one of the three values of $c$ quoted above and adjusting the parameter $b$ leads to fits of comparable quality, these numerical data are hence insufficient to discriminate between the proposed values of c.

| $k+1$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $W_{\mathrm{c}}$ | 17.4 | 33.2 | 50.1 | 67.7 | 87.3 | 105 | 125.2 |

Table I. Critical value of the disorder as a function of the connectivity.

## §4. Lifshits tail and the edge of the spectrum.

As discussed in the previous section, it was proved rigorously that the edge of the density of state is located in $E= \pm(2 \sqrt{k}+W / 2) \cdot{ }^{[14,(15), 16)}$ Numerically we found that the density of states becomes extremely small close to the edges. In the following we shall develop simple arguments à la Lifshits that allows one to understand the result $E= \pm(2 \sqrt{k}+W / 2)$ and to obtain the form of $\rho(E)$ close to the edge.

Let us first recall how the Lifshits' argument ${ }^{[27]}$ works on finite dimensional lattices. Let us focus on the left edge (the argument for the right edge is the same). The key observation is that there is a finite probability to have all the on-site random energies inside a sphere of radius $R$ taking values very close to $-W / 2$, say $-W / 2+\epsilon$. Using that such a sphere, when disconnected from the rest, is characterized by states with energies arbitrarily close to the exact band edge in absence of disorder shifted by $-W / 2+\epsilon$, one can construct variational states for the original Hamiltonian that have the same energies. This allows one to show that the left and right edges in presence of disorder are just the ones without disorder shifted respectively by $-W / 2$ and $W / 2$, and also to find the decrease of $\rho(E)$ at the edges. Note that all this can be put on a rigorous ground, as done for instance in ref. (5).

In this section we shall develop similar arguments to study the behaviour of the tail of the spectrum of the Anderson model on Bethe lattices (a related computation was presented in ref. (28) for off-diagonal disorder). Following the strategy developed for finite dimensional systems, the first thing to analyze is the eigenvalue problem for an infinite tree of connectivity $k+1$ without disorder, i.e. where the adjacency matrix $H$ is equal to -1 if $i$ and $j$ are nearest neighbours and zero otherwise. We look for eigenvectors of $H$ of energy $E$, i.e., we want to find states $|\psi\rangle$ such that $H|\psi\rangle=E|\psi\rangle$. We consider spherically symmetric states for which the components of $|\psi\rangle$ depend only on the distance from the central site of the tree and must satisfy the following set of equations:

$$
\begin{align*}
& E \psi_{n}=-k \psi_{n+1}-\psi_{n-1} \\
& E \psi_{0}=-(k+1) \psi_{1}
\end{align*}
$$

where $\psi_{n}$ is the component of $|\psi\rangle$ on any given site at distance $n$ from the origin. For energies in the range $-2 \sqrt{k}<E<2 \sqrt{k}$, the solution of the equations above can be written in the form:

$$
\psi_{n}=k^{-n / 2}\left(\frac{1-i c}{2} e^{i n \theta}+\frac{1+i c}{2} e^{-i n \theta}\right)
$$

with $e^{i \theta}=\left(-E+i \sqrt{4 k-E^{2}}\right) / 2 \sqrt{k}$. The value of $c$ must be chosen in order to satisfy the equation for the central site. This yields:

$$
c=-\frac{(k-1) E}{(k+1) \sqrt{4 k-E^{2}}} .
$$

Finally, the component of the eigenvector $|\psi\rangle$ at distance $n$ from the origin can be rewritten as:

$$
\psi_{n}=\frac{\cos (n \theta)+c \sin (n \theta)}{k^{n / 2}}
$$

Thus $\left|\psi_{n}\right|^{2}$ decrease with the distance from the central site as $k^{-n}$, while the number of sites at distance $n$ from the origin grows as $k^{n}$. As a consequence, such radial (quasi-)eigenvectors are not normalizable:

$$
\langle\psi \mid \psi\rangle=1+\frac{k+1}{k} \sum_{n \geq 1} k^{n}\left|\psi_{n}\right|^{2}=1+\frac{k+1}{k} \sum_{n \geq 1}|\cos (n \theta)+c \sin (n \theta)|^{2},
$$

which diverges for an infinite tree.
One can however define a modified variational wave-function that is normalizable and has variational energy equal to $E$. The idea is the following: define $n_{\star}(E)$ as the smallest integer $n$ such that $\psi_{n}=0$, and consider the truncated wave-function $\psi_{n}^{\prime}=$ $\psi_{n}$ if $n \leq n_{\star}(E), \psi_{n}^{\prime}=0$ for $n \geq n_{\star}(E)$. This wave-function is not a solution of the equations (4•1), yet it is normalizable and verifies $\frac{\left\langle\psi^{\prime}\right| H\left|\psi^{\prime}\right\rangle}{\left\langle\psi^{\prime} \mid \psi^{\prime}\right\rangle}=E$. The determination of the radius $n_{\star}(E)$ is simplified when $E$ tends to one of the band-edge. Let us focus on the left band edge (the argument for the right edge is almost identical) and write $E=-2 \sqrt{k}+\delta$. At the lowest order in $\delta$ one finds that $\theta \simeq \sqrt{\frac{\delta}{\sqrt{k}}}$ and $c \simeq \frac{k-1}{k+1} \sqrt{\frac{\sqrt{k}}{\delta}}$. Solving the equation $\psi_{n_{\star}}=0$ in this limit one obtains $n_{\star} \sim \pi \sqrt{\sqrt{k} / \delta}$.

Let us now go back to the disordered case, where the on-site energies $H_{i i}$ are independent and identically distributed random variables in the interval $[-W / 2, W / 2]$, and let us use the variational states derived above to determine the position of the edge of the spectrum in presence of disorder. There is a finite probability that inside a spherical region of radius $n_{\star}$ all the on-site energies $\epsilon_{i}$ are arbitrarily close to $-W / 2$. The variational state constructed above has now a variational energy arbitrarily close to $-2 \sqrt{k}-W / 2$. The same argument for the right edge leads to the $2 \sqrt{k}+W / 2$. As a consequence, the edges of the Anderson model on the Bethe lattice are $\pm(2 \sqrt{k}+W / 2)$, in agreement with the rigorous results based on ergodicity theorem. ${ }^{(14), 15)}$

One can make this argument more quantitative and estimate the behaviour of the density of states around the left edge of the band, in presence of disorder (again the argument for the right edge is essentially identical). Indeed, the probability that all the energies in the ball of radius $n_{\star}$ are in the interval $[-W / 2,-W / 2+\delta]$ scales as $(\delta / W)^{\left(\frac{k+1}{k}\right) k^{n_{\star}}}$, since the number of sites inside the ball is $\left(\frac{k+1}{k}\right) k^{n_{\star}}$. Combining this relation with the scaling of $n_{\star}$ with $\delta$ obtained above for the ordered model yields the following scaling for the density of states

$$
\rho(-2 \sqrt{k}-W / 2+\delta) \simeq \exp \left[-\left(\frac{k+1}{k}\right) k^{\pi k^{1 / 4} \delta^{-1 / 2}} \log (W / \delta)\right]
$$

This function vanishes indeed extremely fast with $\delta$, as mentioned previously when discussing the numerical results of Sec.33. Note that, strictly speaking, this expression is only a lower bound to the density of states. A form similar to Eq. (4.6) has already been found for models with off-diagonal disorder on Bethe lattices. ${ }^{[28),(29)}$

## §5. Isolated eigenvalue

The last point we shall address is whether the smallest eigenvalue $\lambda_{1}$ is at a finite distance from the edge of the density of states at the thermodynamic limit. Note that one needs an extensive number of eigenvalues to obtain a $\rho(E)$ different from zero. A single eigenvalue gives a contribution to $\rho(E)$ of the order of $1 / N$, which is negligible in the thermodynamic limit. As a consequence, it is in principle possible to have the smallest eigenvalue separated from all the others by a gap and the left edge of $\rho(E)$ at a finite distance from $\lambda_{1}$. Although this might seem counterintuitive at first sight, at least based on the intuition developed for finite dimensional system, it is what happens in the case without disorder when $H$ is (minus) the adjacency matrix of a random regular graph*). It is easy to check in this case that the constant vector is an eigenvector of $H$ with eigenvalue $-(k+1)$. It appears thus a gap between $\lambda_{1}=-(k+1)$ and the second smallest eigenvalue $\lambda_{2}$, which concentrates around $-2 \sqrt{k}$, the left edge of the band of the Bethe lattice model, as proved in ref. (17).

A natural question is therefore what happens to the isolated smallest eigenvalue when disorder is introduced. It is also interesting to investigate the properties of the corresponding eigenvector, which it is completely delocalized without disorder. We shall follow an approach first proposed in ref. 30) for off-diagonal disorder, and begin by rederiving it with a slightly different presentation, related to the discussion of ref. 18). Let us consider a Gaussian probability measure on $N$ real variables $\phi_{i}$ defined by the averages

$$
\langle\bullet\rangle=\frac{1}{\mathcal{N}} \int \mathrm{~d} \phi_{1} \ldots \mathrm{~d} \phi_{N} \bullet \exp \left[-\frac{1}{2} \sum_{i, j} \phi_{i}\left(H_{i, j}-E \delta_{i, j}\right) \phi_{j}\right]
$$

where $\mathcal{N}$ is a normalizing factor. It is clear that the integral is convergent only if $E<\lambda_{1}$, and that by symmetry $\left\langle\phi_{i}\right\rangle=0$ for all $i$. If however the limit $E \rightarrow \lambda_{1}^{-}$is taken then an infinitesimal linear term in the action above is enough to produce a spontaneous magnetization $\left\langle\phi_{i}\right\rangle \neq 0$ in the direction of the eigenvector associated to $\lambda_{1}$, i.e. $\left\langle\phi_{i}\right\rangle$ is proportional to its $i$-th component. This corresponds to a condensation transition on the first eigenvector. Using the replica symmetric cavity method ${ }^{(20)}$ (which corresponds here to the Gaussian Belief Propagation ${ }^{(31)}$ ) one finds that computing the averages with respect to this weight, on a given graph, amounts to solve the following recursion equations

$$
\mu_{i \rightarrow j}\left(\phi_{i}\right)=\frac{1}{\mathcal{N}_{i \rightarrow j}} e^{-\frac{1}{2}\left(H_{i i}-E\right) \phi_{i}^{2}} \prod_{j^{\prime} \in \partial i \backslash j} \int \mathrm{~d} \phi_{j^{\prime}} \mu_{j^{\prime} \rightarrow i}\left(\phi_{j^{\prime}}\right) e^{-H_{i j^{\prime}} \phi_{i} \phi_{j^{\prime}}}
$$

where $\mathcal{N}_{i \rightarrow j}$ ensures the normalization and $\mu_{i \rightarrow j}\left(\phi_{i}\right)$ is the probability measure for $\phi_{i}$ when the link between $i$ and $j$ is removed*). We parametrize this Gaussian

[^0]probability as:
$$
\mu_{i \rightarrow j}\left(\phi_{i}\right)=\frac{1}{\mathcal{N}_{i \rightarrow j}^{\prime}} e^{-\frac{1}{2} \frac{1}{G_{i \rightarrow j}} \phi_{i}^{2}-y_{i \rightarrow j} \phi_{i}}
$$

Plugging this expression in (5•2) we find:

$$
G_{i \rightarrow j}=\frac{1}{H_{i i}-E-\sum_{j^{\prime} \in \partial i \backslash j} H_{i j^{\prime}}^{2} G_{j^{\prime} \rightarrow i}}, \quad y_{i \rightarrow j}=-\sum_{j^{\prime} \in \partial i \backslash j} H_{i j^{\prime}} G_{j^{\prime} \rightarrow i} y_{j^{\prime} \rightarrow i}
$$

which are indeed the same as derived in ref. (30). The equations linking the 'messages' $G$ on the various edges of the graph are decoupled from the $y$ 's, and corresponds to the recursion equations for the resolvents stated in (2.5). The linear term coefficients $y$ are solutions of a homogeneous set of equations, hence invariant with respect to a common multiplication. The smallest eigenvalue $\lambda_{1}$ can thus be found as the largest value of $E$ such that the fixed point $y=0$ is stable.

To investigate the behaviour of $\lambda_{1}$ for large regular graphs one turns these equations in distributional equations,

$$
(\widehat{G}, y) \stackrel{\mathrm{d}}{=}\left(\frac{1}{\epsilon-E-\sum_{i=1}^{k} \widehat{G}_{i}}, \sum_{i=1}^{k} \widehat{G}_{i} y_{i}\right)
$$

As explained in Sec. 2 this kind of distributional equation is easily solved numerically by representing the distribution of the random variable $(\widehat{G}, y)$ as a sample of couples $\left\{\left(\widehat{G}_{1}, y_{1}\right), \ldots,\left(\widehat{G}_{\mathcal{N}}, y_{\mathcal{N}}\right)\right\}$. These are updated according to a generalization of the method explained in Sec. 2, where now the update is made on the couple $\left(\widehat{G}_{j}, y_{j}\right)$ and not on $\widehat{G}_{j}$ alone. The stability of the $y=0$ fixed point can then be determined by monitoring the norm $\sum_{j} y_{j}^{2}$ after each iteration, and one identifies the location of $\lambda_{1}$ as the largest value for which this norm decreases upon iterating Eq. (5.5).

The numerical results of this procedure are presented in Fig. 4 for the connectivity $k+1=3$. The smallest eigenvalue $\lambda_{1}$ sligthly decreases upon increasing the disorder from the value -3 it has at $W=0$. We find that even in presence of disorder the smallest eigenvalue is separated from the others by a gap. Moreover it remains delocalized as shown by its participation ratio that can be obtained from the $y_{i \rightarrow j}$ when $E=\lambda_{1}^{-}$. There is a critical value of the disorder, $W_{1}$, where $\lambda_{1}$ becomes equal to the edge of the density of states. For larger values of $W$ the smallest eigenvalue equals the edge of the density of states*). In Fig. 4 we show the numerical result obtained by the cavity method for the isolated eigenvalue in the case $k+1=3$, which crosses the left edge of the band for a value of the disorder $W_{1} \simeq 0.36$.
except through their coupling to $i$. This is justified in the present case and directly leads to the equations (5•2).
${ }^{*)}$ For $W>W_{1}$ our method still finds a value of $\lambda_{1}$, which is now larger then the left band edge and that approaches the mobility edge when increasing the disorder strength. The interpretation of the replica symmetry cavity method in this regime is unclear. It might be that between the extensive number of localized states, there is one extended state at least for $W$ not much larger than $W_{1}$ and that the replica symmetry cavity method misses all the other localized states because looking at the stability with respect to $y_{i \rightarrow j}$ is like studying the stability to a delocalized external field, which does not couple to localized states. This point certainly deserves further investigation.


Fig. 4. Blow-up of the phase diagram of Fig. 1 the supplementary line starting from -3 is the isolated eigenvalue $\lambda_{1}$.

## §6. Conclusion

In this work we have revisited the Anderson localization problem on a Bethe lattice. We have shown detailed results on the density of states and the mobility edge for connectivity equal to 3 . We have also discussed the large connectivity limit, whose exact solution remains an open question. Finally, we have shown that for not too large disorder the smallest eigenvalue is extended and has a gap from the subsequent (localised) eigenvalues. This property may play an important role for the disordered Bose Hubbard model on Bethe lattice. Indeed, the usual argument used to claim that a Bose glass phase always intervenes between the superfluid and the Mott phase may fail. In finite dimensional lattices, when the disorder strength $W$ is only slightly larger than the gap of the Mott insulator, it was argued ${ }^{(32),(33)}$ that it is favorable to insert particles (or vacancies) but that these are not delocalized and hence do not lead to superfluidity since the lowest eigenvalue of the corresponding Anderson problem is localized. In the Bethe lattice case, depending on the effective Anderson model one finds for these quasi-particles, the smallest eigenvalue may correspond to a completely delocalized state with a finite gap from the localized ones. Thus, the possibility of a direct quantum phase transition from the superfluid to the Mott state is not completely excluded on Bethe lattices.

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[^0]:    ${ }^{*)}$ For this phenomenon, there is a difference between an infinite random tree and a very large random regular graph. Only for the latter the smallest eigenvalue is separated from the others by a gap.
    ${ }^{* *)}$ The replica symmetric cavity method consists in assuming that the $\phi_{j^{\prime}}$ are not correlated

