Symmetries of generating functionals of Langevin processes with colored multiplicative noise

Camille Aron[*], **Giulio Biroli**[†] **and Leticia F. Cugliandolo**[*]

[*]Laboratoire de Physique Théorique et Hautes Énergies,
Université Pierre et Marie Curie - Paris VI,
4 Place Jussieu, Tour 24, 5ème étage, 75252 Paris Cedex 05, France
[†]Institut de Physique Théorique, CEA Saclay, 91191 Gif-sur-Yvette, France

Abstract. We present a comprehensive study of the symmetries of the generating functionals of generic Langevin processes with multiplicative colored noise. We treat both Martin-Siggia-Rose-Janssen-deDominicis and supersymmetric formalisms. We summarize the relations between observables that they imply including fluctuation relations, fluctuation-dissipation theorems, and Schwinger-Dyson equations. Newtonian dynamics and their invariances follow in the vanishing friction limit.

1. Introduction

The stochastic evolution of a classical system coupled to a quite generic environment can be described with the Langevin formalism [1, 2] and its generating functional, the Martin-Siggia-Rose-Janssen-deDominicis (MSRJD) path-integral [3, 4, 5]. In many cases of practical interest the effect of the environment is captured by an additive white noise and its memory-less friction, Brownian motion being the paradigmatic example [1]. Nevertheless, there are many other interesting instances in which the noise is multiplicative and colored, and the friction effect is consistently described by a memory kernel coupled to a non-linear function of the state variable. Such Langevin equations appear in many different branches of physics (as well as chemistry and other sciences). In magnetism, the motion of the classical magnetic moments of small particles is phenomenologically described by the Landau-Lifshitz-Gilbert equation in which the fluctuations of the magnetic field are coupled multiplicatively to the magnetic moment [7]. Many other examples pertain to soft condensed matter; two of these are confined diffusion, in which the diffusion coefficient of the particle depends on the position *via* hydrodynamic interactions [8], and the stochastic partial differential equation that rules the time-evolution of the density of an ensemble of N Brownian particles [9]. In a cosmological framework, these equations arise as effective equations of motion for the out of (although close to) equilibrium evolution of self-interacting quantum fields in which the short-wave length modes serve as thermal baths for longer wave-length modes with slower dynamics [10]. Such type of fluctuations may yield *a priori* unexpected results such as noise induced phase transitions in systems in which the associated deterministic potential does not exhibit any symmetry breaking [11].

In order to better understand these processes it is useful to distinguish cases in which sources of fluctuations and dissipation can be different. On the one hand, the noise and friction term can have an 'internal' origin, such as in diffusion problems. On the other hand, the stochastic fluctuations can be due to an 'external' source [12]. In the former cases one usually assumes that the variables generating the noise and friction are in equilibrium and the terms in the Langevin equation associated to them are linked by a fluctuation-dissipation theorem. In the absence of non-conservative external forces the Boltzmann measure of the system of interest is a steady state of its dynamics. In the latter cases noise and dissipation are not forced to satisfy any equilibrium condition and this translates into the possibility of having any kind of noise and friction terms. For concreteness we shall focus on the first type of problems and only mention a few results concerning the latter.

In treatments of the examples mentioned in the first paragraph, the delicate double limit of vanishing fast variables relaxation time and noise correlation time is often taken.

These lead to a first order stochastic differential equation with multiplicative white noise. Its interpretation in the Itô, Stratonovich or other sense requires a very careful analysis of the order of limits, see e.g. [13] and references therein. In the main part of this paper we shall keep both time-scales finite and thus avoid the subtleties encountered in the double vanishing limit.

In this manuscript, we identify a number of symmetries of the MSRJD generating functional of inertial Langevin processes with multiplicative colored noise. One symmetry is only valid in equilibrium. The corresponding Ward-Takahashi identities between the correlation functions of the field theory lead to various equilibrium relations such as stationarity, fluctuation-dissipation theorems [14, 15] or Onsager relations. Away from equilibrium, the symmetry is broken giving rise to various out of equilibrium fluctuation relations [16]-[28]. Another symmetry holds for generic out of equilibrium set-ups and implies dynamic equations coupling correlations and linear responses. It allows to express, in particular, the linear response in terms of correlations without applying a perturbing field [30]-[41].

We are aware of the fact that some of the results in this manuscript – especially, in the limit of additive noise – were already known and we do our best to attribute them to the authors of the original papers or review articles. Still, the presentation that we gradually develop in this article allows one to go beyond the simple cases and treat the multiplicative non-Markov processes with the same level of difficulty. Moreover, we discuss in greater detail than previously done the transformation of the measure and several Jacobians, and the domain of integration of the fields in the path-integral. The importance of dealing with colored noise, and to treat the transformation of the fields in the complex plane, is enhanced by our purpose to extend this analysis to quantum dissipative problems. These results will be presented in a separate publication [42].

The organization of the paper is the following. In Sect. 2 we give a short review of Langevin equations with additive and multiplicative noise. Section 3 presents the MSRJD functional representation of Langevin equations. In Sect. 4 we deal with the equilibrium symmetries while in Sect. 5 we treat the out of equilibrium ones. In both Sections we discuss supersymmetric formulations. We conclude in Sect. 6.

2. The Langevin equation

We consider a 0-dimensional field ψ (*e.g.* a particle at position ψ) with mass m driven by a force F and in contact with a thermal bath in equilibrium at inverse temperature β. The initial time, t_0, is the instant at which the particle is set in contact with the bath and the stochastic dynamics 'starts'. We call it $t_0 = -T$ and without loss of generality we work within a symmetric time-interval $t \in [-T, T]$. The extension to

higher dimensional cases is straightforward. Our conventions are given in Appendix A.

2.1. Additive noise

The Langevin equation with additive noise is given by

$$\text{EQ}([\psi], t) \equiv m\ddot{\psi}(t) - F([\psi], t) + \int_{-T}^{T} du\, \gamma(t, u)\, \dot{\psi}(u) = \xi(t)\,, \tag{1}$$

with $\dot{\psi}(t) = d\psi(t)/dt$ and $\ddot{\psi}(t) = d^2\psi(t)/dt^2$. The force can be decomposed into conservative and non-conservative parts: $F([\psi], t) = -V'(\psi(t), \lambda(t)) + f([\psi], t)$. V is a local potential the time-dependence of which is controlled externally through a protocol $\lambda(t)$. V' denotes the partial derivative of V with respect to ψ. $f([\psi], t)$ collects all the non-conservative forces that are externally applied. $f([\psi], t)$ is assumed to be causal in the sense that it does not depend on the future states of the system, $\psi(t')$ with $t' > t$. Furthermore, we suppose that $f([\psi], t)$ does not involve second – nor higher – order time-derivatives of the field $\psi(t)$. The last term in the left-hand-side (LHS) and the right-hand-side (RHS) of the equation model the interaction with the bath. These two heuristic terms can be derived using a model [43, 44] in which the bath consists in a set of non-interacting harmonic oscillators of coordinates q_i that are bilinearly coupled to the state variable of the system of interest ψ. The function γ is the retarded friction $[\gamma(t, t') = 0$ for $t' > t]$ and the noise ξ is a random force taken to be a Gaussian process. This assumption is quite reasonable, for instance, for a Brownian particle much bigger than the surrounding particles of the bath, its motion being the result of a large number of successive collisions, which is a condition for the central limit theorem to apply. Since we assume the environment to be in equilibrium, $\gamma(t, t')$ is a function of $t - t'$ and the bath obeys the fluctuation-dissipation theorem of the 'second kind' [45]:

$$\langle \xi(t) \rangle_\xi = 0\,, \qquad \langle \xi(t)\xi(t') \rangle_\xi = \beta^{-1}\, \Gamma(t - t')\,, \tag{2}$$

where $\langle \dots \rangle_\xi$ denotes the average over the noise history. We introduced the symmetric kernel $\Gamma(t - t') \equiv \gamma(t - t') + \gamma(t' - t) = \Gamma(t' - t)$. The white noise limit, in which the bath has no memory, is achieved by setting $\gamma(t - t') = \gamma_0 \delta(t - t')$ with $\gamma_0 > 0$ the friction coefficient. The Langevin equation then takes the more familiar form

$$\text{EQ}([\psi], t) \equiv m\ddot{\psi}(t) - F([\psi], t) + \gamma_0\, \dot{\psi}(t) = \xi(t)\,. \tag{3}$$

Newtonian dynamics, for which the system is not in contact with a thermal bath, are recovered by simply taking $\gamma(t) = \Gamma(t) = 0$ at all t. Out of equilibrium environments can be taken into account by relaxing the condition between the noise statistics and the friction kernel $\Gamma(t - t') = \gamma(t - t') + \gamma(t' - t)$.

2.2. Multiplicative noise

We generalize our discussion to the multiplicative noise case in which the Gaussian noise ξ is coupled to a state-dependent function $M'(\psi)$. The Langevin equation reads

$$\mathrm{E}_{\mathrm{Q}}([\psi], t) \equiv m\ddot{\psi}(t) - F([\psi], t) + M'(\psi(t)) \int_{-T}^{T} \mathrm{d}u\; \gamma(t-u) M'(\psi(u))\dot{\psi}(u)$$
$$= M'(\psi(t))\, \xi(t)\;. \tag{4}$$

This equation can also be shown by using the oscillator model for the bath and a nonlinear coupling of the form $M(\psi)\sum_i c_i q_i$ where c_i are coefficients that depend on the details of the coupling and $M(\psi)$ is a smooth function of the state variable with $M(0) = 0$. By a suitable renormalization of γ, one can always achieve $M'(0) = 1$. For reasons that will soon become clear, we need to assume that $M'(\psi) \neq 0\; \forall\, \psi$. These assumptions can be realized with functions of the type $M(\psi) = \psi + L(\psi)$ where L is a smooth and increasing function satisfying $L(0) = L'(0) = 0$. The complicated structure of the friction term takes its rationale from the fluctuation-dissipation theorem of the second kind that expresses the equilibrium condition of the bath. This equation models situations in which the friction between the system and its bath is state-dependent. ξ has the same statistics as in the additive case, see eq. (2). The Langevin equation for the additive noise problem is recovered by taking $M(\psi) = \psi$.

2.3. Initial conditions

The Langevin equation is a second order differential equation that needs two initial values, say the field and its derivative at time $-T$. We shall use initial conditions drawn from an initial probability distribution $P_{\mathrm{i}}\left(\psi(-T), \dot{\psi}(-T)\right)$ and average over them. The initial conditions are not correlated with the thermal noise ξ. In the particular case in which the system is prepared in an equilibrium state, P_{i} is given by the Boltzmann measure.

2.4. Markov limit

Langevin equations are often given in the Markov limit in which they appear to be first order stochastic differential equations. Second and higher order time-derivatives as well as non-local terms such as memory kernels are not allowed. In other words, the effect of inertia is neglected (Smoluchowski limit) and the bath is taken to be white. This is justified in situations in which the two associated time scales are sufficiently small compared to all other time scales involved. Concretely, the resulting equation is derived by using an adiabatic elimination procedure that consists in integrating over the fast

variables of the system (the velocities) and of the bath. However, this double limiting procedure requires a careful analysis.

The physics of the resulting equation may depend on how the relaxation time associated to inertia compares with the correlation time of the noise before sending the two of them to zero. In cases in which the latter is much larger than the former, the limiting stochastic equation should be interpreted in the sense of Stratonovich [47]. The RHS of eq. (4) is given a meaning by stating that ψ in $M'(\psi(t))$ is evaluated at half the sum of its values before and after the kick. Conversely, when the inertia relaxation time is much larger than the noise correlation time, the limiting equation should be interpreted in the Itô sense [46]. In this scenario, the rule is that $M'(\psi(t))$ is evaluated just before the kick $\xi(t)$. When the noise is additive the two conventions are equivalent (see Appendix B.2) for all practical purposes. However, they are not for processes with multiplicative noise [12]. For these it is possible to rewrite the Itô stochastic equation in terms of a Stratonovich stochastic equation by adding an adequate drift term to the deterministic force – and be allowed to use the rules of conventional calculus. The Fokker-Planck equation associated to the Markov process does not depend on the scenario and the Boltzmann distribution is a steady state independently of the convention used. However, the action of the generating functional acquires extra terms depending on the discretization prescription [8, 49].

In this article, we decide not to cope with the Markov limit and, unless otherwise stated, we keep the inertia of the system in our equations ($m \neq 0$) and we use a colored noise with a finite relaxation time.

3. The generating functional

The generating functionals associated to the equations of motion (1) and (4) are given by the Martin-Siggia-Rose-Janssen-deDominicis (MSRJD) path-integral. In this Section we recall its construction for additive noise [4] and we extend it to multiplicative noise by using a continuous time formalism. In Appendix B and Appendix C we develop a careful construction in the discretized formulation.

3.1. Action in the additive noise case

The Langevin equation (1) is a second order differential equation with source ξ. The knowledge of the history of the field ξ and the initial conditions $\psi(-T)$ and $\dot{\psi}(-T)$ is sufficient to construct $\psi(t)$. Therefore, the probability $P[\psi]$ of a given ψ history between $-T$ and T is linked to the probability of the noise history $P_{\mathrm{n}}[\xi]$ through

$$P[\psi]\mathcal{D}[\psi] = P_{\mathrm{n}}[\xi]\,\mathcal{D}[\xi]\,P_{\mathrm{i}}\left(\psi(-T),\dot{\psi}(-T)\right)\mathrm{d}\psi(-T)\,\mathrm{d}\dot{\psi}(-T) \qquad (5)$$

implying

$$P[\psi] = P_{\mathrm{n}}[\mathrm{EQ}[\psi]] \; |\mathcal{J}[\psi]| \; P_{\mathrm{i}}\left(\psi(-T), \dot{\psi}(-T)\right) , \tag{6}$$

where $\mathcal{J}[\psi]$ is the Jacobian which reads, up to some constant factor,

$$\mathcal{J}[\psi] \equiv \det_{uv}\left[\frac{\delta\xi(u)}{\delta\psi(v)}\right] = \det_{uv}\left[\frac{\delta\mathrm{EQ}([\psi], u)}{\delta\psi(v)}\right] \equiv \mathcal{J}_0[\psi] . \tag{7}$$

det [...] stands for the functional determinant. We introduced the notation $\mathcal{J}_0[\psi]$ for future convenience and we shall discuss it in Sect. 3.3. After a Hubbard-Stratonovich transformation that introduces the auxiliary real field $\hat{\psi}$, the Gaussian probability for a given noise history to occur reads

$$P_{\mathrm{n}}[\xi] = \mathcal{N}^{-1} \int \mathcal{D}[\hat{\psi}] \; \mathrm{e}^{-\int\mathrm{d}u \, \mathrm{i}\hat{\psi}(u)\xi(u) + \frac{1}{2}\iint\mathrm{d}u \, \mathrm{d}v \, \mathrm{i}\hat{\psi}(u)\beta^{-1}\Gamma(u-v)\mathrm{i}\hat{\psi}(v)} , \tag{8}$$

with the boundary conditions $\hat{\psi}(-T) = \hat{\psi}(T) = 0$ and where all the integrals over time run from $-T$ to T. In the following, unless otherwise stated, we shall simply denote them $\int$. $\mathcal{N}$ is a infinite constant prefactor that we absorb in a re-definition of the measure $\mathcal{D}[\hat{\psi}]$. Back in eq. (6) one has

$$P[\psi]\mathcal{D}[\psi] = \mathcal{D}[\psi] \int \mathcal{D}[\hat{\psi}] \; \mathrm{e}^{S[\psi,\hat{\psi}]} , \tag{9}$$

with the MSRJD action functional

$$S[\psi, \hat{\psi}] \equiv \ln P_{\mathrm{i}}\left(\psi(-T), \dot{\psi}(-T)\right) - \int\mathrm{d}u \, \mathrm{i}\hat{\psi}(u)\mathrm{EQ}([\psi], u)$$
$$+ \frac{1}{2}\iint\mathrm{d}u \, \mathrm{d}v \, \mathrm{i}\hat{\psi}(u) \, \beta^{-1}\Gamma(u-v) \, \mathrm{i}\hat{\psi}(v) + \ln|\mathcal{J}_0[\psi]| . \tag{10}$$

The latter is the sum of a deterministic, a dissipative and a Jacobian term,

$$S[\psi, \hat{\psi}] \equiv S^{\mathrm{det}}[\psi, \hat{\psi}] + S^{\mathrm{diss}}[\psi, \hat{\psi}] + \ln|\mathcal{J}_0[\psi]| ,$$

with

$$S^{\mathrm{det}}[\psi, \hat{\psi}] \equiv \ln P_{\mathrm{i}}\left(\psi(-T), \dot{\psi}(-T)\right)$$
$$- \int\mathrm{d}u \, \mathrm{i}\hat{\psi}(u)\left[m\ddot{\psi}(u) - F([\psi], u)\right], \tag{11}$$

$$S^{\mathrm{diss}}[\psi, \hat{\psi}] \equiv \int\mathrm{d}u \, \mathrm{i}\hat{\psi}(u) \int\mathrm{d}v \, \gamma(u-v)\left[\beta^{-1}\mathrm{i}\hat{\psi}(v) - \dot{\psi}(v)\right] . \tag{12}$$

S^{det} takes into account inertia and the forces exerted on the field, as well as the measure of the initial condition. S^{diss} has its origin in the coupling to the dissipative bath. In the white noise limit, $\gamma(t-t') = \gamma_0\delta(t-t')$, the dissipative action naively simplifies to $S^{\mathrm{diss}}[\psi, \hat{\psi}] = \gamma_0 \int \mathrm{d}u \, \mathrm{i}\hat{\psi}(u)\left[\beta^{-1}\mathrm{i}\hat{\psi}(u) - \dot{\psi}(u)\right]$ (see Sect. 2.4 for additional details on this limit).

Integrating away the auxiliary field $\hat{\psi}$ yields the Onsager-Machlup action functional [50]. However, we prefer to work with the action written in terms of ψ and $i\hat{\psi}$ as this is the form that arises as the classical limit of the Schwinger-Keldysh action used to treat interacting out of equilibrium quantum systems [44, 51], that we shall analyze along the same lines in [42].

3.2. Action in the multiplicative noise case

To shorten expressions, we adopt a notation in which the arguments of the fields and functions appear as subindices, $\psi_u \equiv \psi(u)$, $\gamma_{u-v} \equiv \gamma(u-v)$, and so on and so forth, and the integrals over time expressed as $\int_u \equiv \int_{-T}^{T} du$.

In the case of the Langevin equation (4) with multiplicative noise, the relation (6) is modified and reads

$$P[\psi] = P_{\rm n}\left[\frac{{\rm Eq}[\psi]}{M'(\psi)}\right] \ |\mathcal{J}[\psi]| \ P_{\rm i}(\psi_{-T}, \dot{\psi}_{-T}) \, , \tag{13}$$

with the Jacobian

$$\mathcal{J}[\psi] = \det_{uv}\left[\frac{\delta {\rm Eq}_u[\psi]/M'(\psi_u)}{\delta \psi_v}\right] = \det_{uv}\left[\frac{\delta_{u-v}}{M'(\psi_u)}\right] \mathcal{J}_0[\psi] \tag{14}$$

and the generalization of the definition of $\mathcal{J}_0$ in eq. (7) to the multiplicative case:

$$\mathcal{J}_0[\psi] \equiv \det_{uv}\left[\frac{\delta {\rm Eq}_u[\psi]}{\delta \psi_v} - \frac{M''(\psi_u)}{M'(\psi_u)} {\rm Eq}_u[\psi] \, \delta_{u-v}\right] \, . \tag{15}$$

The construction of the MSRJD action follows the same steps as in the additive noise case, complemented by a further transformation of the field $i\hat{\psi} \mapsto i\hat{\psi} \, M'(\psi)$, the Jacobian of which cancels the first determinant factor in the RHS of eq. (14). Therefore, the MSRJD action reads

$$S[\psi, \hat{\psi}] \equiv \ln P_{\rm i}(\psi_{-T}, \dot{\psi}_{-T}) - \int_u i\hat{\psi}_u {\rm Eq}_u[\psi]$$

$$+ \frac{1}{2}\int_u \int_v i\hat{\psi}_u M'(\psi_u) \, \beta^{-1}\Gamma_{u-v} \, M'(\psi_v) i\hat{\psi}_v + \ln|\mathcal{J}_0[\psi]| \, , \quad (16)$$

with $\mathcal{J}_0$ defined in eq. (15) . The deterministic part of the action is unchanged compared to the additive noise case and the dissipative part is now

$$S^{\rm diss}[\psi, \hat{\psi}] \equiv \int_u i\hat{\psi}_u \int_v M'(\psi_u) \, \gamma_{u-v} \, M'(\psi_v) \left[\beta^{-1}i\hat{\psi}_v - \dot{\psi}_v\right] \, . \tag{17}$$

3.3. The Jacobian

In Appendix C we prove that the Jacobian $\mathcal{J}_0$ is a field-independent positive constant for Langevin equations with inertia and multiplicative colored noise. One can therefore

safely include the Jacobian contribution in the normalization. However, we decide to keep track of this term and represent it as a Gaussian integral over Grassmann conjugate fields c and c^*,

$$\mathcal{J}_0[\psi] = \int \mathcal{D}[c, c^*] \, e^{S^{\mathcal{J}}[c, c^*, \psi]} \ , \tag{18}$$

with

$$S^{\mathcal{J}}[c, c^*, \psi] \equiv \int_u \int_v c_u^* \frac{\delta \mathrm{EQ}_u[\psi]}{\delta \psi_v} c_v - \int_u c_u^* \frac{M''(\psi_u)}{M'(\psi_u)} \mathrm{EQ}_u[\psi] \, c_u \ , \tag{19}$$

and the boundary conditions: $c(-T) = \dot{c}(-T) = c^*(T) = \dot{c}^*(T) = 0$. Plugging in the Langevin equation (4), we arrive at

$$S^{\mathcal{J}}[c, c^*, \psi] = \int_u \int_v c_u^* \left[m \partial_u^2 \delta_{u-v} - \frac{\delta F_u[\psi]}{\delta \psi_v} + M'(\psi_u) \partial_u \gamma_{u-v} M'(\psi_v) \right] c_v$$

$$- \int_u c_u^* \frac{M''(\psi_u)}{M'(\psi_u)} \left[m \partial_u^2 \psi_u - F_u[\psi] \right] c_u \ . \tag{20}$$

The Grassmann fields c and c^* that enter the integral representation of the determinant are known as Faddeev-Popov ghosts and can be interpreted as spinless fermions. The two-time fermionic Green function defined as

$$\langle c_t^* c_{t'} \rangle_{S^{\mathcal{J}}} \equiv \int \mathcal{D}[c, c^*] \, c_t^* c_{t'} \, e^{S^{\mathcal{J}}[c, c^*, \psi]} \ , \tag{21}$$

is related, by use of Wick's theorem, to the inverse operator of $\frac{\delta \mathrm{EQ}_{t'}[\psi]}{\delta \psi_t} - \frac{M''(\psi_t)}{M'(\psi_t)} \mathrm{EQ}[\psi_t] \delta_{t-t'}$. $\langle c_t^* c_{t'} \rangle_{S^{\mathcal{J}}}$ inherits the causality structure of the latter and it vanishes at equal times as long as the Markov limit is not taken (*i.e.* all fermionic tadpole contributions cancel): $\langle c_t^* c_{t'} \rangle_{S^{\mathcal{J}}} = 0$ for $t \geq t'$. The last statement can be easily verified by considering the discretized version of $S^{\mathcal{J}}$ (see Appendix B.3.3 and Appendix C) and by checking that the diagonal terms of the inverse operator vanish in the continuous limit. $S^{\mathcal{J}}$ only involves combinations of the form $c^* c$, *i.e.* it conserves the fermionic charge and $\langle c_t \rangle_{S^{\mathcal{J}}} = \langle c_t^* \rangle_{S^{\mathcal{J}}} = 0$. This implies that $S^{\mathcal{J}}[c, c^*, \psi]$ and more generally the MSRJD generating functional (at zero sources) are invariant under the following field transformation

$$\mathcal{T}_{\mathcal{J}}(\alpha) \equiv \begin{cases} c_t & \mapsto & \alpha \, c_t \ , \\ c_t^* & \mapsto & \alpha^{-1} \, c_t^* \ , \end{cases} \qquad \forall \, \alpha \in \mathbb{C}^* \ . \tag{22}$$

The Jacobian of the transformation is trivially equal to one and the measure $\mathcal{D}[c, c^*]$ is left unchanged. One has $\mathcal{T}_{\mathcal{J}}(\alpha) \mathcal{T}_{\mathcal{J}}(\beta) = \mathcal{T}_{\mathcal{J}}(\alpha\beta)$.

The total MSRJD action given in eq. (16) can be written equivalently as a functional of ψ, $\hat{\psi}$, c and c^* provided that the path-integral measure is extended to the newly introduced fermionic fields:

$$S[\psi, \hat{\psi}, c, c^*] \equiv S^{\mathrm{det}}[\psi, \hat{\psi}] + S^{\mathrm{diss}}[\psi, \hat{\psi}] + S^{\mathcal{J}}[c, c^*, \psi] \ . \tag{23}$$

3.4. Observables

3.4.1. Measure. We denote $\langle \dots \rangle$ the average over the thermal noise and the initial conditions. Within the MSRJD formalism, the average is evaluated with respect to the action functional $S[\psi, \hat{\psi}]$ or $S[\psi, \hat{\psi}, c, c^*]$ and we use the notation $\langle \dots \rangle_S$:

$$\langle \dots \rangle_S \equiv \int \mathcal{D}[\psi, \hat{\psi}] \dots e^{S[\psi, \hat{\psi}]} = \int \mathcal{D}[\psi, \hat{\psi}, c, c^*] \dots e^{S[\psi, \hat{\psi}, c, c^*]} . \tag{24}$$

3.4.2. Local observable. The value of a generic local observable A at time t is a function of the field and its time-derivatives evaluated at time t, *i.e.* a functional of the field ψ around t, $A([\psi], t)$. Unless otherwise specified we assume it does not depend explicitly on time and denote it $A[\psi(t)]$. Its mean value is

$$\langle A[\psi(t)] \rangle = \langle A[\psi(t)] \rangle_S . \tag{25}$$

3.4.3. Time-reversal. Since it will be used in the rest of this work, we introduce the time-reversed field $\bar{\psi}$ by $\bar{\psi}(t) \equiv \psi(-t)$ for all t. The time-reversed observable is defined as

$$A_{\rm r}([\psi], t) \equiv A([\bar{\psi}], -t). \tag{26}$$

It has the effect of changing the sign of all odd time-derivatives in the expression of local observables, *e.g.* if $A[\psi(t)] = \partial_t \psi(t)$ then $A_{\rm r}[\psi(t)] = -\partial_t \psi(t)$. As an example for non-local observables, the time-reversed Langevin equation (1) reads

$$\mathrm{EQ}_{\rm r}([\psi], t) = m\ddot{\psi}(t) - F_{\rm r}([\psi], t) - \int_{-T}^{T} \mathrm{d}u \, \gamma(u - t)\dot{\psi}(u) . \tag{27}$$

Notice the change of sign in front of the friction term that is no longer dissipative in this new equation.

3.4.4. Two-time correlation. We define the two-time self correlation function as

$$C(t, t') \equiv \langle \psi(t)\psi(t') \rangle = \langle \psi(t)\psi(t') \rangle_S . \tag{28}$$

Given two local observables A and B, we similarly introduce the two-time generic correlation as

$$C_{\{AB\}}(t, t') \equiv \langle A[\psi(t)]B[\psi(t')] \rangle_S , \tag{29}$$

The curly brackets are here to stress the symmetry that underlies this definition: $C_{\{AB\}}(t, t') = C_{\{BA\}}(t', t)$.

3.4.5. Linear response. If we slightly modify the potential according to $V(\psi) \mapsto V(\psi) - f_\psi \psi$, the self linear response at time t to an infinitesimal perturbation linearly coupled to the field at a previous time t' is defined as

$$R(t, t') \equiv \left.\frac{\delta \langle \psi(t) \rangle}{\delta f_\psi(t')}\right|_{f_\psi=0} = \left.\frac{\delta \langle \psi(t) \rangle_{S[f_\psi]}}{\delta f_\psi(t')}\right|_{f_\psi=0} . \tag{30}$$

It is clear from causality that if t' is later than t, $\langle \psi(t) \rangle_{S[f_\psi]}$ cannot depend on the perturbation $f_\psi(t')$ so $R(t, t') = 0$ for $t' > t$. At equal times, the linear response $R(t, t)$ also vanishes as long as inertia is not neglected $(m \neq 0)$‡. More generally, the linear response of A at time t to an infinitesimal perturbation linearly applied to B at time $t' < t$ is

$$R_{AB}(t, t') \equiv \left.\frac{\delta \langle A[\psi(t)] \rangle}{\delta f_B(t')}\right|_{f_B=0} = \left.\frac{\delta \langle A[\psi(t)] \rangle_{S[f_B]}}{\delta f_B(t')}\right|_{f_B=0} , \tag{31}$$

with $V(\psi) \mapsto V(\psi) - f_B B[\psi]$, where B is local.

3.5. Classical Kubo formula

By computing explicitly the functional derivative $\delta/\delta f_\psi$ in the path integral generating functional, we deduce

$$\left.\frac{\delta \langle \dots \rangle_{S[f_\psi]}}{\delta f_\psi(t)}\right|_{f_\psi=0} = \left\langle \dots \left.\frac{\delta S[\psi, \hat\psi, c, c^*; f_\psi]}{\delta f_\psi(t)}\right|_{f_\psi=0}\right\rangle_S$$

$$= \langle \dots \mathrm{i}\hat\psi(t) \rangle_S + \left\langle \dots \frac{M''(\psi(t))}{M'(\psi(t))} c^*(t)c(t)\right\rangle_S . \tag{32}$$

The first term in the RHS comes from the functional derivative of S^{det}. The second term comes from the Jacobian term expressed with the fermionic ghosts, $S^{\mathcal{J}}$, and vanishes identically (see the discussion on the equal-time fermionic Green function in Sect. 3.3). One has

$$\langle \mathrm{i}\hat\psi(t) \rangle_S = \left.\frac{\delta \langle 1 \rangle_{S[f_\psi]}}{\delta f_\psi(t)}\right|_{f_\psi=0} = 0 , \tag{33}$$

$$\langle \mathrm{i}\hat\psi(t)\mathrm{i}\hat\psi(t') \rangle_S = \left.\frac{\delta^2 \langle 1 \rangle_{S[f_\psi]}}{\delta f_\psi(t)\,\delta f_\psi(t')}\right|_{f_\psi=0} = 0 . \tag{34}$$

From the definition of the linear response, eq. (30), we deduce the 'classical Kubo formula' [45]

$$R(t, t') = \langle \psi(t)\mathrm{i}\hat\psi(t') \rangle_S . \tag{35}$$

‡ In the double limit of a white noise and $m \to 0$, the equal-time response can slightly violate the causality principle depending on the order in which the limits are taken. In the Itô scenario it vanishes whereas in the Stratonovich one it has a finite value.

The linear response is here written within the MSRJD formalism as a correlation computed with an unperturbed action. The causality of the response is not explicit, nevertheless following the lines in Ref. [8] one can check it is built-in§. Because of this expression, the auxiliary field $\hat{\psi}$ is often called the response field. Observe that we have not specified the nature of the initial probability distribution P_i nor the driving forces; eq. (35) holds even out of equilibrium.

Similarly, by plugging eq. (25) into eq. (31), we obtain the classical Kubo formula for generic local observables:

$$
\begin{aligned}
R_{AB}(t,t') &= \langle A[\psi(t)] \left. \frac{\delta S[\psi,\hat{\psi},c,c^*;f_B]}{\delta f_B(t')} \right|_{f_B=0} \rangle_S \\
&= \langle A[\psi(t)] \int \mathrm{d}u \, \mathrm{i}\hat{\psi}(u) \frac{\delta B[\psi(t')]}{\delta \psi(u)} \rangle_S \\
&= \langle A[\psi(t)] \sum_{n=0}^{\infty} \partial_{t'}^n \mathrm{i}\hat{\psi}(t') \frac{\partial B[\psi(t')]}{\partial\, \partial_{t'}^n \psi(t')} \rangle_S .
\end{aligned}
\tag{36}
$$

This formula is valid in and out of equilibrium and allows us to write the response functions associated to generic observables (*e.g.* functions of the position, velocity, acceleration, kinetic energy, etc.) as correlators of ψ, $\hat{\psi}$ and their time derivatives. For example if B is just a function of the field (and not of its time-derivatives), only the $n=0$-term subsists in the above sum, yielding

$$
R_{AB}(t,t') = \langle A[\psi(t)]\mathrm{i}\hat{\psi}(t') \frac{\partial B[\psi(t')]}{\partial \psi(t')} \rangle_S .
\tag{37}
$$

As another example, if one is interested in the response of the acceleration $A[\psi(t)] = \partial_t^2 \psi(t)$ to a perturbation of the kinetic energy $B[\psi(t)] = \frac{1}{2}m(\partial_t \psi(t))^2$ one should compute

$$
R_{AB}(t,t') = m\langle \partial_t^2 \psi(t) \partial_{t'} \mathrm{i}\hat{\psi}(t') \partial_{t'} \psi(t') \rangle_S .
\tag{38}
$$

Furthermore, it is straightforward to see that within the MSRJD formalism we can extend all the previous definitions and formulæ to A being a local functional of the auxiliary field: $A[\hat{\psi}(t)]$. For example, if $A[\hat{\psi}(t)] = \mathrm{i}\hat{\psi}(t)$ and $B[\psi(t)] = \psi(t)$, we obtain the mixed response

$$
R_{\mathrm{i}\hat{\psi}\psi}(t,t') = \langle \mathrm{i}\hat{\psi}(t)\mathrm{i}\hat{\psi}(t') \rangle_S = 0 ,
\tag{39}
$$

where we used eq. (34).

§ In general, a multi-time correlator involving $\mathrm{i}\hat{\psi}(t_1)$ vanishes if t_1 is the largest time involved.

4. Equilibrium

In this Section we focus on situations in which the system is in equilibrium. We identify a field transformation that leaves the MSRJD generating functional (evaluated at zero sources) invariant. The corresponding Ward-Takahashi identities between the expectation values of different observables imply a number of model independent equilibrium properties including stationarity, Onsager relations and the fluctuation-dissipation theorem (FDT). These proofs are straightforward in the generating functional formalism, demonstrating its advantage with respect to the Fokker-Planck or master equation ones, when the environment acts multiplicatively and has a non-vanishing correlation time. We shall report soon [42] on the extension to the quantum case where the Keldysh action also exhibits a non-trivial symmetry for equilibrium dynamics. Similarly to the classical case, this symmetry leads to the quantum FDT.

4.1. The action

Equilibrium dynamics are guaranteed provided that, apart from its interactions with the bath, the system is prepared and subjected to the same time-independent and conservative forces ($F = -V'$). In such situations, the initial state is taken from the Boltzmann probability distribution

$$\ln P_{\mathrm{i}}(\psi_{-T}, \dot{\psi}_{-T}) = -\beta \mathcal{H}[\psi_{-T}] - \ln \mathcal{Z} , \tag{40}$$

where $\mathcal{H}[\psi_t] \equiv \frac{1}{2}m\dot{\psi}_t^2 + V(\psi_t)$ is the internal energy of the system, and $\mathcal{Z}$ is the partition function. The Langevin evolution of the system in contact with the bath can be put in the form

$$-\int_u \frac{\delta \mathcal{L}[\psi_u]}{\delta \psi_t} + M'(\psi_t) \int_u \gamma_{t-u} M'(\psi_u) \dot{\psi}_u = M'(\psi_t)\xi_t , \tag{41}$$

with $\mathcal{L}[\psi_u] \equiv \frac{1}{2}m\dot{\psi}_u^2 - V(\psi_u)$ being the Lagrangian of the system. In this equilibrium set-up, the deterministic part of the MSRJD action functional reads

$$S^{\mathrm{det}}[\psi, \hat{\psi}] = -\beta \mathcal{H}[\psi_{-T}] - \ln \mathcal{Z} + \int_u \int_v i\hat{\psi}_u \frac{\delta \mathcal{L}[\psi_v]}{\delta \psi_u}$$

$$= -\beta \left(\frac{1}{2}m\dot{\psi}_{-T}^2 + V(\psi_{-T}) \right) - \ln \mathcal{Z} - \int_u i\hat{\psi}_u \left[m\ddot{\psi}_u + V'(\psi_u) \right] . \tag{42}$$

The dissipative part of the MSRJD action functional remains the same, see eq. (17). As discussed in Sect. 3.3, the Jacobian $\mathcal{J}_0$ enters the action through the constant term $\ln \mathcal{J}_0$ or it can be expressed in terms of a Gaussian integral over the ghosts fields c and c^*. In this case, its contribution to the action reads

$$S^{\mathcal{J}}[c, c^*, \psi] = \int_u \int_v c_u^* \left[m\partial_u^2 \delta_{u-v} + M'(\psi_u)\partial_u \gamma_{u-v} M'(\psi_v) \right] c_v$$

$$- \int_u c_u^* \left[-V''(\psi_u) + \frac{M''(\psi_u)}{M'(\psi_u)} \partial_u^2 \psi_u + \frac{M''(\psi_u)}{M'(\psi_u)} V'(\psi_u) \right] c_u . \qquad (43)$$

4.2. Symmetry of the MSRJD generating functional

We shall prove that $\int \mathcal{D}[\psi, \hat{\psi}, c, c^*] \ e^{S[\psi, \hat{\psi}, c, c^*]}$ is invariant under the equilibrium field transformation:

$$\mathcal{T}_{\mathrm{eq}} \equiv \begin{cases} \psi_u & \mapsto & \psi_{-u} , & c_u & \mapsto & c_{-u}^* , \\ \mathrm{i}\hat{\psi}_u & \mapsto & \mathrm{i}\hat{\psi}_{-u} + \beta \partial_u \psi_{-u} , & c_u^* & \mapsto & -c_{-u} . \end{cases} \qquad (44)$$

This transformation is involutary, $\mathcal{T}_{\mathrm{eq}} \mathcal{T}_{\mathrm{eq}} = 1$, when applied to the fields ψ or $\mathrm{i}\hat{\psi}$ and the composite field $c^* c$. It does not involve the kernel γ and includes a time-reversal. It is interesting to reckon that the invariance is achieved independently by the deterministic (S^{det}), the dissipative (S^{diss}) and the Jacobian ($S^{\mathcal{J}}$) contributions to the action. This means that it is still valid in the Newtonian limit ($\gamma = 0$). The detailed proof that we develop here consists of two parts: we first show that the Jacobian of the transformation is unity, then that the integration domain of the transformed fields is unchanged. Afterwards we show that the action functional $S[\psi, \hat{\psi}, c, c^*]$ is invariant under $\mathcal{T}_{\mathrm{eq}}$.

4.2.1. Invariance of the measure. The equilibrium transformation $\mathcal{T}_{\mathrm{eq}}$ acts separately on the fields ψ and $\mathrm{i}\hat{\psi}$ on the one hand, and the fields c and c^* on the other. The Jacobian $\mathcal{J}_{\mathrm{eq}}$ thus factorizes into a bosonic part and a fermionic part. The bosonic part is the determinant of a triangular matrix:

$$\mathcal{J}_{\mathrm{eq}}^{\mathrm{b}} \equiv \det \left[\frac{\delta(\psi, \hat{\psi})}{\delta(\mathcal{T}_{\mathrm{eq}}\psi, \mathcal{T}_{\mathrm{eq}}\hat{\psi})} \right] = \det_{uv}^{-1} \begin{bmatrix} \frac{\delta \psi_{-u}}{\psi_v} & 0 \\ \frac{\delta \hat{\psi}_{-u}}{\psi_v} & \frac{\delta \hat{\psi}_{-u}}{\hat{\psi}_v} \end{bmatrix} = \left(\det_{uv}^{-1} [\delta_{u+v}] \right)^2 = 1 ,$$

and it is thus identical to one [53]. It is easy to verify that the fermionic part $\mathcal{J}_{\mathrm{eq}}^{\mathrm{f}} = 1$ as well.

4.2.2. Invariance of the integration domain. Before and after the transformation, the functional integration on the field ψ is performed for values of ψ_t on the real axis. However, the new domain of integration for the field $\hat{\psi}$ is complex. For a given time t, $\hat{\psi}_t$ is now integrated over the complex line with a constant imaginary part $-\mathrm{i}\beta \partial_t \psi_t$. One can return to an integration over the real axis by closing the contour at both infinities. Indeed, the integrand, e^S, goes to zero sufficiently fast at $\psi_t \to \pm\infty$ for neglecting the vertical ends of the contour thanks to the term $\beta^{-1}\gamma_0(\mathrm{i}\hat{\psi}_t)^2$ in the action. Furthermore, the new field is also integrated with the boundary conditions $\hat{\psi}(-T) = \hat{\psi}(T) = 0$.

The equilibrium transformation leaves the measure $\mathcal{D}[c, c^*]$ unchanged together with the set of boundary conditions $c(-T) = \dot{c}(-T) = c^*(T) = \dot{c}^*(T) = 0$.

4.2.3. *Invariance of the action functional.* The MSRJD action functional $S[\psi, \hat{\psi}, c, c^*] = S^{\mathrm{det}}[\psi, \hat{\psi}] + S^{\mathrm{diss}}[\psi, \hat{\psi}] + S^{\mathcal{J}}(c, c^*, \psi)$ is invariant term by term. The deterministic contribution given in eq. (42) satisfies

$$S^{\mathrm{det}}[\mathcal{T}_{\mathrm{eq}}\psi, \mathcal{T}_{\mathrm{eq}}\hat{\psi}] = \ln P_{\mathrm{i}}(\psi_T, \dot{\psi}_T) - \int_u [i\hat{\psi}_{-u} + \beta\partial_u\psi_{-u}][m\partial_u^2\psi_{-u} + V'(\psi_{-u})]$$

$$= \ln P_{\mathrm{i}}(\psi_T, \dot{\psi}_T) - \int_u i\hat{\psi}_u[m\ddot{\psi}_u + V'(\psi_u)] + \beta\int_u \dot{\psi}_u[m\ddot{\psi}_u + V'(\psi_u)]$$

$$= \ln P_{\mathrm{i}}(\psi_T, \dot{\psi}_T) - \int_u i\hat{\psi}_u[m\ddot{\psi}_u + V'(\psi_u)] + \beta\int_u \partial_u\mathcal{H}[\psi_u]$$

$$= S^{\mathrm{det}}[\psi, \hat{\psi}] \,, \tag{45}$$

where we used the initial equilibrium measure $\ln P_{\mathrm{i}}(\psi, \dot{\psi}) = -\beta\mathcal{H}[\psi] - \ln \mathcal{Z}$. In the first line we just applied the transformation, in the second line we made the substitution $u \mapsto -u$, in the third line we wrote the last integrand as a total derivative the integral of which cancels the first term and creates a new initial measure.

Secondly, we show that the dissipative contribution $S^{\mathrm{diss}}[\psi, \hat{\psi}]$, defined in eq. (12), is also invariant under the equilibrium transformation. We have

$$S^{\mathrm{diss}}[\mathcal{T}_{\mathrm{eq}}\psi, \mathcal{T}_{\mathrm{eq}}\hat{\psi}] = \int_u [i\hat{\psi}_{-u} + \beta\partial_u\psi_{-u}] \int_v \beta^{-1}M'(\psi_{-u})\, \gamma_{u-v}\, M'(\psi_{-v})\, i\hat{\psi}_{-v}$$

$$= \int_u [i\hat{\psi}_u - \beta\dot{\psi}_u] \int_v M'(\psi_u)\, \gamma_{v-u}\, M'(\psi_v)\beta^{-1}i\hat{\psi}_v$$

$$= S^{\mathrm{diss}}[\psi, \hat{\psi}] \,. \tag{46}$$

In the first line we just applied the transformation, in the second line we made the substitution $u \mapsto -u$ and in the last step we exchanged u and v.

Finally, we show that the Jacobian term in the action is invariant once it is expressed in terms of a Gaussian integral over conjugate Grassmann fields (c and c^*). We start from eq. (43)

$$S^{\mathcal{J}}[\mathcal{T}_{\mathrm{eq}}c, \mathcal{T}_{\mathrm{eq}}c^*, \mathcal{T}_{\mathrm{eq}}\psi]$$

$$= -\int_u \int_v c_{-u} \left[m\partial_u^2\delta_{u-v} + M'(\psi_{-u})\partial_u\gamma_{u-v}M'(\psi_{-v}) \right] c^*_{-v}$$

$$+ \int_u c_{-u} \left[-V''(\psi_{-u}) + \frac{M''(\psi_{-u})}{M'(\psi_{-u})}\partial_u^2\psi_{-u} + \frac{M''(\psi_{-u})}{M'(\psi_{-u})}V'(\psi_{-u}) \right] c^*_{-u}$$

$$= \int_u \int_v c^*_v \left[m\partial_u^2\delta_{v-u} - M'(\psi_u)\partial_u\gamma_{v-u}M'(\psi_v) \right] c_u$$

$$- \int_u c^*_u \left[-V''(\psi_u) + \frac{M''(\psi_u)}{M'(\psi_u)}\partial_u^2\psi_u + \frac{M''(\psi_u)}{M'(\psi_u)}V'(\psi_u) \right] c_u$$

$$= S^{\mathcal{J}}[c, c^*, \psi] \,. \tag{47}$$

In the first line we just applied the transformation, in the second line we exchanged the

anti-commuting Grassmann variables and made the substitutions $u \mapsto -u$ and $v \mapsto -v$, in the last step we used $\partial_v \gamma_{v-u} = -\partial_v \gamma_{u-v}$ and exchanged u and v.

4.3. Ward-Takahashi identities

We just proved that equilibrium dynamics manifest themselves as a symmetry of the MSRJD action and more generally at the level of the generating functional. This symmetry has direct consequences at the level of correlation functions. If A is a generic functional of ψ and $\hat{\psi}$, it implies the following Ward-Takahashi identity

$$\langle A[\psi, \hat{\psi}] \dots \rangle_S = \langle A[\mathcal{T}_{\mathrm{eq}}\psi, \mathcal{T}_{\mathrm{eq}}\hat{\psi}] \dots \rangle_S . \tag{48}$$

This identity leads to all the possible equilibrium relations between observables as we shall now describe in the following. These relations can be proven without using the MSRJD path integral formalism, however our point is to show that the symmetry is able to generate all the equilibrium relations without using any other ingredient.

4.4. Stationarity

In equilibrium, one expects noise-averaged observables to be independent of the time t_0 at which the system was prepared (in our case $t_0 = -T$). One-time dependent noise-averaged observables are expected to be constant, $\langle A[\psi_t] \rangle = \mathrm{ct}$, and two-time correlations to be time-translational invariant: $\langle A[\psi_t]B[\psi_{t'}] \rangle = f_{t-t'}$. Similarly, one argues that multi-time correlations can only depend upon all possible independent time-differences between the times involved. These statements have been proven for additive white noise processes using the Fokker-Planck [54] formalism. The use of the transformation $\mathcal{T}_{\mathrm{eq}}$ allows one to show these properties very easily for generic Langevin processes.

One-time observables. Taking $A = 1$ and letting B be a generic local observable, the equal-time linear response vanishes, $R_{AB}(t, t) = 0$. Using the classical Kubo formula (36),

$$R_{AB}(t, t) = \langle \sum_{n=0}^{\infty} \partial_t^n \mathrm{i}\hat{\psi}_t \frac{\partial B[\psi_t]}{\partial \, \partial_t^n \psi_t} \rangle_S = 0 . \tag{49}$$

Applying the transformation $\mathcal{T}_{\mathrm{eq}}$, we find

$$R_{AB}(t, t) = \langle \sum_{n=0}^{\infty} \partial_t^n \mathrm{i}\hat{\psi}_{-t} \frac{\partial B_{\mathrm{r}}[\psi_{-t}]}{\partial \, \partial_t^n \psi_{-t}} \rangle_S + \beta \langle \sum_{n=0}^{\infty} \partial_t^{n+1} \psi_{-t} \frac{\partial B_{\mathrm{r}}[\psi_{-t}]}{\partial \, \partial_t^n \psi_{-t}} \rangle_S . \tag{50}$$

The LHS and the first term in the RHS vanish identically at all times. One is left with the second term in the RHS that simply reads $\langle \partial_t B_{\mathrm{r}}[\psi_{-t}] \rangle = \partial_t \langle B_{\mathrm{r}}[\psi_{-t}] \rangle = 0$, proving that all one-time local observables are constant in time.

Two-time observables. Because we just showed that $\langle A[\psi(t)]\rangle$ is constant in equilibrium, the response $R_{AB}(t,t')$, see its formal definition in eq. (31), can only be a function of the time-difference between the observation time and the time at which the perturbation is applied. Therefore it can be written in the form $R_{AB}(t,t') = f(t-t')\theta(t-t')$. We shall see in Sect. 4.7 that the fluctuation-dissipation theorem relates, in equilibrium, the linear response $R_{AB}(t,t')$ to the two-time correlation $C_{\{AB\}}(t,t')$ implying that this last quantity is also time-translational invariant.

Similarly, $(n+1)$-time correlators can be proven to be functions of n independent time-differences because they are related, in equilibrium, to responses of n-time correlators that are time-translational invariant.

4.5. Equipartition theorem

Let us consider the local observables $A[\psi(t)] = \partial_t\psi(t)$ and $B[\psi(t)] = \psi(t)$. The linear response is $R_{AB}(t,t') = \langle\partial_t\psi_t\mathrm{i}\hat{\psi}_{t'}\rangle_S = \partial_t\langle\psi_t\mathrm{i}\hat{\psi}_{t'}\rangle_S$ and we recognize $\partial_t R(t,t')$. Using the field transformation $\mathcal{T}_{\mathrm{eq}}$, we find

$$\partial_t R(t,t') = \partial_t\langle\psi_{-t}\mathrm{i}\hat{\psi}_{-t'}\rangle_S + \beta\langle\partial_t\psi_{-t}\partial_{t'}\psi_{-t'}\rangle_S$$
$$= \partial_t\langle\psi_{-t}\mathrm{i}\hat{\psi}_{-t'}\rangle_S + \beta\langle\partial_t\psi_t\partial_{t'}\psi_{t'}\rangle_S \ . \tag{51}$$

If $t > t'$, the first term in the RHS vanishes by causality. Considering moreover the limit $t' \to t^-$ the LHS is $1/m$ as we shall show in Sect. 5.2.3. Finally, we recover the equipartition theorem for the kinetic energy

$$\beta m\langle(\partial_t\psi_t)^2\rangle = 1 \ . \tag{52}$$

4.6. Reciprocity relations

If we use $\mathcal{T}_{\mathrm{eq}}$ in the expression (29) of generic two-time correlation functions, we have

$$\langle A[\psi_t]B[\psi_{t'}]\rangle_S = \langle A_{\mathrm{r}}[\psi_{-t}]B_{\mathrm{r}}[\psi_{-t'}]\rangle_S \ , \tag{53}$$

reading

$$C_{\{AB\}}(t,t') = C_{\{A_{\mathrm{r}}B_{\mathrm{r}}\}}(-t,-t') \ . \tag{54}$$

In cases in which A and B have a definite parity under time-reversal:

$$C_{\{AB\}}(\tau) = C_{\{AB\}}(|\tau|) \text{ if } A \text{ and } B \text{ have the same parity,}$$
$$C_{\{AB\}}(\tau) = -C_{\{AB\}}(-\tau) \text{ otherwise.}$$

4.7. Fluctuation-dissipation theorem (FDT)

4.7.1. Self FDT. Applying the transformation to the expression (35) of the self response $R(t, t')$ we find

$$\langle \psi_t \mathrm{i} \hat{\psi}_{t'} \rangle_S = \langle \mathcal{T}_{\mathrm{eq}} \psi_t \mathcal{T}_{\mathrm{eq}} \mathrm{i} \hat{\psi}_{t'} \rangle_S = \langle \psi_{-t} \mathrm{i} \hat{\psi}_{-t'} \rangle_S + \beta \langle \psi_{-t} \partial_{t'} \psi_{-t'} \rangle_S \ , \tag{55}$$

and we read

$$R(t, t') = R(-t, -t') + \beta \partial_{t'} C(-t, -t') \tag{56}$$

that, using the equilibrium time-translational invariance, becomes

$$R(\tau) - R(-\tau) = -\beta \partial_\tau C(-\tau) \ , \tag{57}$$

where we set $\tau \equiv t - t'$. Since $C(\tau)$ is symmetric in τ by definition, this expression can be rewritten, once multiplied by $\Theta(\tau)$, as

$$R(\tau) = -\Theta(\tau) \beta \partial_\tau C(\tau) \ . \tag{58}$$

Equation (58) is the well-known fluctuation-dissipation theorem. It allows one to predict the slightly out of equilibrium behavior of a system – such as the irreversible dissipation of energy into heat – from its reversible fluctuations in equilibrium.

4.7.2. Generic two-time FDTs. We generalize the previous FDT relation to the case of generic local observables A and B. Applying the equilibrium transformation $\mathcal{T}_{\mathrm{eq}}$ to expression (36) of the linear response $R_{AB}(t, t')$

$$\langle A[\psi_t] \sum_{n=0}^{\infty} \partial_{t'}^n \mathrm{i} \hat{\psi}_{t'} \ \frac{\partial B[\psi_{t'}]}{\partial \ \partial_{t'}^n \psi_{t'}} \rangle_S = \langle A_{\mathrm{r}}[\psi_{-t}] \sum_{n=0}^{\infty} \partial_{t'}^n \mathrm{i} \hat{\psi}_{-t'} \ \frac{\partial B_{\mathrm{r}}[\psi_{-t'}]}{\partial \ \partial_{t'}^n \psi_{t'}} \rangle_S$$

$$+ \beta \ \langle A_{\mathrm{r}}[\psi_{-t}] \sum_{n=0}^{\infty} \partial_{t'}^{n+1} \psi_{-t'} \ \frac{\partial B_{\mathrm{r}}[\psi_{-t'}]}{\partial \ \partial_{t'}^n \psi_{t'}} \rangle_S$$

$$= \langle A_{\mathrm{r}}[\psi_{-t}] \sum_{n=0}^{\infty} \partial_{t'}^n \mathrm{i} \hat{\psi}_{-t'} \ \frac{\partial B_{\mathrm{r}}[\psi_{-t'}]}{\partial \ \partial_{t'}^n \psi_{t'}} \rangle_S + \beta \ \partial_{t'} \langle A_{\mathrm{r}}[\psi_{-t}] B_{\mathrm{r}}[\psi_{-t'}] \rangle_S \ .$$

Applying once again the transformation to the last term in the RHS yields

$$\langle A[\psi_t] \sum_{n=0}^{\infty} \partial_{t'}^n \mathrm{i} \hat{\psi}_{t'} \ \frac{\partial B[\psi_{t'}]}{\partial \ \partial_{t'}^n \psi_{t'}} \rangle_S = \langle A_{\mathrm{r}}[\psi_{-t}] \sum_{n=0}^{\infty} \partial_{t'}^n \mathrm{i} \hat{\psi}_{-t'} \ \frac{\partial B_{\mathrm{r}}[\psi_{-t'}]}{\partial \ \partial_{t'}^n \psi_{t'}} \rangle_S + \beta \partial_{t'} \langle A[\psi_t] B[\psi_{t'}] \rangle_S, \tag{59}$$

which reads

$$R_{AB}(\tau) - R_{A_{\mathrm{r}} B_{\mathrm{r}}}(-\tau) = -\beta \partial_\tau C_{\{AB\}}(\tau) \ . \tag{60}$$

By multiplying both sides by $\Theta(\tau)$ we obtain the FDT for any local A and B

$$R_{AB}(\tau) = -\Theta(\tau) \beta \partial_\tau C_{\{AB\}}(\tau) \ . \tag{61}$$

4.8. Higher-order FDTs: e.g. 3-time observables

We give a derivation, *via* the symmetry of the MSRJD formalism, of relations shown and discussed in, *e.g.* [54], within the Fokker-Planck formalism for stochastic processes with white noise.

4.8.1. Response of a two-time correlation.

We first look at the response of a two-time correlator to a linear perturbation applied at time t_1

$$R(t_3, t_2; t_1) \equiv \left. \frac{\delta \langle \psi_{t_3} \psi_{t_2} \rangle}{\delta f_{\psi_{t_1}}} \right|_{f_\psi = 0} . \tag{62}$$

In the MSRJD formalism, it can be expressed as the 3-time correlator

$$R(t_3, t_2; t_1) = \langle \psi_{t_3} \psi_{t_2} i \hat{\psi}_{t_1} \rangle_S . \tag{63}$$

Causality ensures that the response vanishes if the perturbation is posterior to the observation times: $R(t_3, t_2; t_1) = 0$ if $t_1 > \max(t_2, t_3)$. We assume without loss of generality that $t_2 < t_3$. Under equilibrium conditions, the response transforms under $\mathcal{T}_{eq}$ as

$$R(t_3, t_2; t_1) = \langle \psi_{-t_3} \psi_{-t_2} i \hat{\psi}_{-t_1} \rangle_S + \beta \partial_{t_1} \langle \psi_{-t_3} \psi_{-t_2} \psi_{-t_1} \rangle_S . \tag{64}$$

Multiplying both sides by $\Theta(t_3 - t_1)$ and transforming once again the last term in the RHS, the last equation can be written in the form

$$R(t_3, t_2; t_1) = \begin{cases} \beta \partial_{t_1} \langle \psi_{t_3} \psi_{t_2} \psi_{t_1} \rangle_S \text{ if } t_1 < t_2 < t_3 , \\ R(-t_3, -t_2; -t_1) + \beta \partial_{t_1} \langle \psi_{t_3} \psi_{t_2} \psi_{t_1} \rangle_S \text{ if } t_2 < t_1 < t_3 , \\ 0 \text{ if } t_2 < t_3 < t_1 . \end{cases} \tag{65}$$

4.8.2. Second order response.

Let us now look at the response to a perturbation at time t_1 of the linear response $R(t_3, t_2)$:

$$R(t_3; t_2, t_1) \equiv \left. \frac{\delta^2 \langle \psi_{t_3} \rangle}{\delta f_{\psi_{t_1}} \, \delta f_{\psi_{t_2}}} \right|_{f_\psi = 0} . \tag{66}$$

In the MSRJD formalism, it can be expressed as the 3-time correlator

$$R(t_3; t_2, t_1) = \langle \psi_{t_3} i \hat{\psi}_{t_2} i \hat{\psi}_{t_1} \rangle_S . \tag{67}$$

It is clear from causality that the response vanishes if the observation time is before the two perturbations: $R(t_3; t_2, t_1) = 0$ if $t_3 < \min(t_1, t_2)$. The response transforms under $\mathcal{T}_{eq}$ as

$$\begin{aligned} R(t_3; t_2, t_1) = R(-t_3; -t_2, -t_1) + \beta \partial_{t_1} R(-t_3, -t_1; -t_2) \\ + \beta \partial_{t_2} R(-t_3, -t_2; -t_1) + \beta^2 \partial_{t_1} \partial_{t_2} \langle \psi_{-t_3} \psi_{-t_2} \psi_{-t_1} \rangle_S . \end{aligned} \tag{68}$$

Let us assume without loss of generality that $t_1 < t_2$. Using causality arguments and applying once more the equilibrium transformation to the remaining terms,

$$R(t_3; t_2, t_1) = \begin{cases} 0 & \text{if } t_3 < t_1 < t_2 \text{ ,} \\ +\beta \partial_{t_1} R(t_3, t_1; t_2) & \text{if } t_1 < t_3 < t_2 \text{ ,} \\ \beta \partial_{t_1} R(t_3, t_1; t_2) & \text{if } t_1 < t_2 < t_3 \text{ .} \end{cases} \tag{69}$$

4.9. Onsager reciprocal relations

Rewriting twice eq. (60) as

$$R_{AB}(\tau) - R_{A_r B_r}(-\tau) = -\beta \partial_\tau C_{\{AB\}}(\tau) \text{ ,} \tag{70}$$

$$R_{BA}(-\tau) - R_{B_r A_r}(\tau) = \beta \partial_\tau C_{\{BA\}}(-\tau) = \beta \partial_\tau C_{\{AB\}}(\tau) \text{ ,} \tag{71}$$

and summing up these two equations with $\tau > 0$

$$R_{AB}(\tau) = R_{B_r A_r}(\tau) \text{ .} \tag{72}$$

These equilibrium relations, known as the Onsager reciprocal relations, express the fact that the linear response of an observable A to a perturbation coupled to another observable B can be deduced by the response of B_r to a perturbation coupled to A_r.

4.10. Supersymmetric formalism

4.10.1. Generating functional. The generating functional of stochastic equations with conservative forces admits a supersymmetric formulation. This has been derived and discussed for additive noise in a number of publications [55, 56, 57]. We extend it here to multiplicative non-Markov Langevin processes (see [58] for a study of the massless and white noise limits). To this end, let us introduce θ and θ^*, two anticommuting Grassmann coordinates, and the superfield

$$\Psi(t, \theta, \theta^*) \equiv \psi(t) + c^*(t)\,\theta + \theta^*\,c(t) + \theta^*\theta\left(\mathrm{i}\hat{\psi}(t) + c^*(t)\,c(t)\,\frac{M''(\psi(t))}{M'(\psi(t))}\right) \text{ .}$$

The MSRJD action S [see eq. (23)] has a compact representation in terms of this superfield:

$$S = S_{\text{susy}}^{\text{det}} + S_{\text{susy}}^{\text{diss}} \text{ ,} \tag{73}$$

with

$$S_{\text{susy}}^{\text{det}}[\Psi] \equiv -\beta \int \mathrm{d}\theta\,\mathrm{d}\theta^*\,\theta^*\theta\,\mathcal{H}[\Psi(-T, \theta, \theta^*)] - \ln \mathcal{Z} + \int \mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\theta^*\,\mathcal{L}[\Psi] \text{ ,} \tag{74}$$

$$S_{\text{susy}}^{\text{diss}}[\Psi] \equiv \frac{1}{2} \iint \mathrm{d}\Upsilon'\,\mathrm{d}\Upsilon\,M(\Psi(\Upsilon'))\,\mathbf{D}^{(2)}(\Upsilon', \Upsilon)\,M(\Psi(\Upsilon)) \text{ ,} \tag{75}$$

$\mathcal{H}[\Psi] \equiv \frac{1}{2}m\dot{\Psi}^2 + V(\Psi)$ and $\mathcal{L}[\Psi] \equiv \frac{1}{2}m\dot{\Psi}^2 - V(\Psi)$. In the second equation above we used the notation $\Upsilon \equiv (t, \theta, \theta^*)$ and $d\Upsilon \equiv dt\, d\theta\, d\theta^*$. The 'dissipative' differential operator is defined as

$$\mathbf{D}^{(2)}(\Upsilon', \Upsilon) \equiv \gamma(t' - t)\delta(\theta^{*'} - \theta^*)\delta(\theta' - \theta)\left(2\beta^{-1}\frac{\partial^2}{\partial\theta\,\partial\theta^*} + \overrightarrow{\mathrm{sig}}_\theta \frac{\partial}{\partial t}\right) , \quad (76)$$

where $\overrightarrow{\mathrm{sig}}_\theta$ is a short notation for $2\theta\frac{\partial}{\partial\theta} - 1$. It is equal to 1 if there is a θ factor in the right and to -1 otherwise. $\mathbf{D}^{(2)}$ can be written as

$$\mathbf{D}^{(2)}(\Upsilon', \Upsilon) = \gamma(t' - t)\delta(\theta^{*'} - \theta^*)\delta(\theta' - \theta)\left(\bar{\mathbf{D}}\mathbf{D} - \mathbf{D}\bar{\mathbf{D}}\right) , \quad (77)$$

with the (covariant‖) derivatives acting on the superspace:

$$\bar{\mathbf{D}} \equiv \frac{\partial}{\partial\theta} , \qquad \mathbf{D} \equiv \beta^{-1}\frac{\partial}{\partial\theta^*} - \theta\frac{\partial}{\partial t} , \quad (78)$$

that obey¶ $\{\bar{\mathbf{D}}, \mathbf{D}\} = -\frac{\partial}{\partial t}$ and $\{\mathbf{D}, \mathbf{D}\} = \{\bar{\mathbf{D}}, \bar{\mathbf{D}}\} = 0$. In the white noise limit the dissipative part of the action simplifies to

$$S_{\mathrm{susy}}^{\mathrm{diss}}[\Psi] = \frac{1}{2}\int d\Upsilon\, M(\Psi(\Upsilon))\, \mathbf{D}^{(2)}(\Upsilon)\, M(\Psi(\Upsilon)) , \quad (79)$$

with the 'dissipative' differential operator

$$\mathbf{D}^{(2)}(\Upsilon) \equiv \gamma_0\left(2\beta^{-1}\frac{\partial^2}{\partial\theta\,\partial\theta^*} + \overrightarrow{\mathrm{sig}}_\theta \frac{\partial}{\partial t}\right) = \gamma_0\left(\bar{\mathbf{D}}\mathbf{D} - \mathbf{D}\bar{\mathbf{D}}\right) . \quad (80)$$

This formulation is only suitable in situations in which the applied forces are conservative. The Jacobian term $S^{\mathcal{J}}$ contributes to both the deterministic $(S_{\mathrm{susy}}^{\mathrm{det}})$ and the dissipative part $(S_{\mathrm{susy}}^{\mathrm{diss}})$ of the action.

4.10.2. Symmetries. In terms of the superfield, the transformation $\mathcal{T}_{\mathcal{J}}(\alpha)$ defined in eq. (22) acts as

$$\mathcal{T}_{\mathcal{J}}(\alpha) \equiv \Psi(t, \theta, \theta^*) \mapsto \Psi(t, \alpha^{-1}\theta, \alpha\theta^*) \quad \forall\, \alpha \in \mathbb{C}^* , \quad (81)$$

and leaves the action $S[\Psi]$, see eq. (73), invariant. The transformation $\mathcal{T}_{\mathrm{eq}}$ given in eq. (44) acts as

$$\mathcal{T}_{\mathrm{eq}} \equiv \Psi(t, \theta, \theta^*) \mapsto \Psi(-t - \beta\theta^*\theta, -\theta^*, \theta) , \quad (82)$$

and leaves the action $S[\Psi]$, see eq. (73), invariant.

The action $S[\Psi]$ given in (73) has an additional supersymmetry generated by

$$\mathbf{Q} \equiv \frac{\partial}{\partial\theta^*} , \qquad \bar{\mathbf{Q}} \equiv \beta^{-1}\frac{\partial}{\partial\theta} + \theta^*\frac{\partial}{\partial t} , \quad (83)$$

‖ Covariant in the sense that the derivative of a supersymmetric expression is still supersymmetric.
¶ Therefore the $\dot{\Psi}^2$ term in $\mathcal{L}[\Psi]$ can be written in terms of covariant derivatives as $\left(\{\bar{\mathbf{D}}, \mathbf{D}\}\Psi\right)^2$.

that obey $\{\bar{\mathbf{Q}}, \mathbf{Q}\} = \frac{\partial}{\partial t}$ and $\{\mathbf{Q}, \mathbf{Q}\} = \{\bar{\mathbf{Q}}, \bar{\mathbf{Q}}\} = \{\mathbf{D}, \mathbf{Q}\} = \{\mathbf{D}, \bar{\mathbf{Q}}\} = \{\bar{\mathbf{D}}, \mathbf{Q}\} = \{\bar{\mathbf{D}}, \bar{\mathbf{Q}}\} = 0$. Both operators $\mathbf{Q}$ and $\bar{\mathbf{Q}}$ are thus nilpotent and $\{\bar{\mathbf{Q}}, \mathbf{Q}\}$ is the generator of the Lie sub-group. They act on the superfield as

$$e^{\epsilon^* \mathbf{Q}} \Psi = \Psi + \epsilon^* \mathbf{Q} \Psi , \qquad e^{\epsilon \bar{\mathbf{Q}}} \Psi = \Psi + \epsilon \bar{\mathbf{Q}} \Psi , \tag{84}$$

where ϵ and ϵ^* are two extra independent[+] Grassmann constants and

$$\mathbf{Q}\Psi = c + \theta \left(i\hat{\psi} + c^* c \frac{M''(\psi)}{M'(\psi)} \right) , \tag{85}$$

$$\bar{\mathbf{Q}}\Psi = -\beta^{-1} c^* - \theta^* \left(\beta^{-1} i\hat{\psi} - \partial_t \psi + \beta^{-1} c^* c \frac{M''(\psi)}{M'(\psi)} \right) - \theta^* \theta \, \partial_t c^* . \tag{86}$$

Expressed in terms of superfield transformations, $S[\Psi]$ is invariant under both

$$\Psi(t, \theta, \theta^*) \mapsto \Psi(t, \theta, \theta^* + \epsilon^*) \tag{87}$$

and

$$\Psi(t, \theta, \theta^*) \mapsto \Psi(t + \epsilon \theta^*, \theta + \beta^{-1} \epsilon, \theta^*) . \tag{88}$$

Here again, the invariance of the action is achieved independently by the deterministic (S^{det}) and the dissipative (S^{diss}) contributions. We would like to stress the fact that the presence of the boundary term accounting for the initial equilibrium measure of the field ψ as well as the boundary conditions for the fields $i\hat{\psi}$, c and c^* are necessary to obtain a full invariance of the action.

4.10.3. BRS symmetry. The symmetry generated by $\mathbf{Q}$ is the BRS symmetry that generically arises when a system has dynamical constraints (here we impose the system to obey the Langevin equation of motion). Applying the corresponding superfield transformation in $\langle \Psi(t, \theta, \theta^*) \rangle_S$ gives

$$\langle \Psi(t, \theta, \theta^*) \rangle_S = \langle \Psi(t, \theta, \theta^*) + \epsilon^* \mathbf{Q} \Psi(t, \theta, \theta^*) \rangle_S , \tag{89}$$

and therefore $\langle \mathbf{Q}\Psi(t, \theta, \theta^*) \rangle_S = 0$. This leads to

$$\langle c_t \rangle_S = 0 , \qquad \langle i\hat{\psi}_t + c_t^* c_t \frac{M''(\psi_t)}{M'(\psi_t)} \rangle_S = 0 . \tag{90}$$

Applying the transformation inside the two-point correlator $\langle \Psi(t, \theta, \theta^*) \Psi(t', \theta', \theta^{*\prime}) \rangle_S$, we find $\langle \mathbf{Q}\Psi(t, \theta, \theta^*) \, \Psi(t', \theta', \theta^{*\prime}) \rangle_S + (t, \theta, \theta^*) \leftrightarrow (t', \theta', \theta^{*\prime}) = 0$. This leads to identify the two-time fermionic correlator as being the (bosonic) linear response:

$$R(t, t') \equiv \langle \psi_t \left[i\hat{\psi}_{t'} + c_{t'}^* c_{t'} \frac{M'(\psi_{t'})}{M''(\psi_{t'})} \right] \rangle_S = \langle c_{t'}^* c_t \rangle_S . \tag{91}$$

Corroborating the discussion in Sect. 3.3, this tells us that $\langle c_t^* c_{t'} \rangle_S$ (and more generally the fermionic Green function $\langle c_t^* c_{t'} \rangle_{S^{\mathcal{J}}}$) vanishes for $t > t'$ and also for $t = t'$ provided

[+] ϵ and ϵ^* are independent of the coordinates θ and θ^*.

that the Markov limit is not taken. Using this result, the second relation in (90) now yields $\langle i\hat{\psi}_t \rangle_S = 0$.

4.10.4. FDT. The use of the symmetry generated by $\bar{\mathbf{Q}}$ on $\langle \Psi(t, \theta, \theta^*) \rangle_S$ gives,

$$\langle c_t^* \rangle_S = 0 , \qquad \langle i\hat{\psi}_t - \beta \partial_t \psi_t \rangle_S = 0 . \tag{92}$$

By use of $\langle i\hat{\psi}_t \rangle_S = 0$ (which was a consequence of the BRS symmetry), the second relation becomes $\partial_t \langle \psi_t \rangle_S = 0$. This expresses the stationarity and can be easily generalized to more complicated one-time observables, $A(\psi)$, by use of the supersymmetry in $\langle A(\Psi) \rangle_S$.

The use of the symmetry generated by $\bar{\mathbf{Q}}$ on a two-point correlator of the superfield reads

$$\langle \Psi(t, \theta, \theta^*) \Psi(t', \theta', \theta^{*'}) \rangle_S = \langle \Psi(t + \epsilon \theta^*, \theta + \beta \epsilon, \theta^*) \Psi(t' + \epsilon \theta^{*'}, \theta' + \beta \epsilon, \theta^{*'}) \rangle_S ,$$

implying, amongst other relations,

$$\left\langle \psi_t \left[i\hat{\psi}_{t'} - \beta \partial_{t'} \psi_{t'} + c_t^* c_t \frac{M''(\psi_t)}{M'(\psi_t)} \right] - c_t^* c_{t'} \right\rangle_S = 0 . \tag{93}$$

As discussed in Sect. 4.10.3, $\langle c_t^* c_{t'} \rangle_{S\mathcal{J}}$ vanishes for $t \geq t'$. Therefore, the term in $c_t^* c_t$ disappears from eq. (93) and the FDT is obtained by multiplying both sides of the equation by $\Theta(t - t')$

$$R(t, t') = \beta \partial_{t'} C(t, t') \Theta(t - t') . \tag{94}$$

4.11. Link between $\mathcal{T}_{\mathrm{eq}}$ and the supersymmetries

It is interesting to remark that both supersymmetries (the one generated by $\mathbf{Q}$ and the one generated by $\bar{\mathbf{Q}}$) are needed to derive equilibrium relations such as stationarity or the FDT. All the Ward-Takahashi identities generated by the combined use of these supersymmetries can be generated by $\mathcal{T}_{\mathrm{eq}}$ but the inverse is not true. The supersymmetries do not yield relations in which a time-reversal appears explicitly such as the Onsager reciprocal relations.

It is clear from its expression in terms of the superfield, eq. (82), that the equilibrium transformation $\mathcal{T}_{\mathrm{eq}}$ cannot be written using the generator of a continuous supersymmetry. However, the transformation $\mathcal{T}_{\mathrm{eq}}$ can be formally written in terms of the supersymmetry generators as

$$\mathcal{T}_{\mathrm{eq}} \equiv \Psi \mapsto \Pi \, \Xi \, e^{\tilde{\mathbf{Q}}} \, \Psi , \tag{95}$$

where Π is the time-reversal operator ($t \mapsto -t$), Ξ exchanges the extra Grassmann coordinates ($\theta \mapsto -\theta^*$ and $\theta^* \mapsto \theta$) and the generator $\tilde{\mathbf{Q}}$ is defined in terms of $\mathbf{Q}$ and $\bar{\mathbf{Q}}$

as

$$\tilde{\mathbf{Q}} \equiv -\beta \theta^* \theta \left\{ \bar{\mathbf{Q}}, \mathbf{Q} \right\} = -\beta \theta^* \theta \frac{\partial}{\partial t} \ . \tag{96}$$

5. Out of equilibrium

We now turn to more generic situations in which the system does no longer evolve in equilibrium. This means that it can now be prepared with an arbitrary distribution and it can evolve with time-dependent and non-conservative forces f.

We first show that the way in which the symmetry $\mathcal{T}_{\mathrm{eq}}$ is broken gives a number of so-called transient* fluctuations relations [16]-[28, 29]. Although fluctuation theorems in cases with additive colored noise were studied in several publications [23]-[26], we are not aware of similar studies in cases with multiplicative noise.

We then exhibit another symmetry of the MSRJD generating functional, valid in and out of equilibrium. This new symmetry implies out of equilibrium relations between correlations and responses and generalizes the formulæ in [30]-[37] obtained for additive white noise. Finally, we come back to the equilibrium case to combine the two symmetries and deduce other equilibrium relations.

5.1. Non-equilibrium fluctuation relations

5.1.1. Work fluctuation theorems. Let us assume that the system is initially prepared in thermal equilibrium with respect to the potential $V(\psi, \lambda_{-T})\sharp$. The expression for the deterministic part of the MSRJD action functional [see eq. (11)] is

$$\begin{aligned} S^{\mathrm{det}}[\psi, \hat{\psi}; \lambda, \mathrm{f}] = & -\beta \mathcal{H}([\psi_{-T}], \lambda_{-T}) - \ln \mathcal{Z}(\lambda_{-T}) \\ & - \int_u \mathrm{i}\hat{\psi}_u \left[m\ddot{\psi}_u + V'(\psi_u, \lambda_u) - \mathrm{f}_u[\psi] \right] \ , \end{aligned} \tag{97}$$

where $\mathcal{H}([\psi_t], \lambda_t) \equiv \frac{1}{2}m\dot{\psi}_t^2 + V(\psi_t, \lambda_t)$. The external work done on the system along a given trajectory between times $-T$ and T is the sum of the work induced by the non-conservative forces and the one performed through the external protocol λ:

$$W[\psi; \lambda, \mathrm{f}] \equiv \int_u \dot{\psi}_u \, \mathrm{f}_u[\psi] + \int_u \partial_u \lambda_u \, \partial_\lambda V(\psi_u, \lambda_u) \ . \tag{98}$$

The equilibrium transformation $\mathcal{T}_{\mathrm{eq}}$ does not leave S^{det} invariant but yields

$$S^{\mathrm{det}}[\psi, \hat{\psi}; \lambda, \mathrm{f}] \mapsto S^{\mathrm{det}}[\psi, \hat{\psi}; \bar{\lambda}, \mathrm{f}_{\mathrm{r}}] + \beta \Delta \mathcal{F}_{\mathrm{r}} - \beta W[\psi; \bar{\lambda}, \mathrm{f}_{\mathrm{r}}] \ , \tag{99}$$

---

* As opposed to *steady-state* fluctuation relations the validity of which is only asymptotic, in the limit of long averaging times.

$\sharp$ This is in fact a restriction on the initial velocities, $\dot{\psi}_{-T}$, that are to be taken from the Boltzmann distribution with temperature β^{-1}, independently of the positions ψ_{-T}. The distribution of the latter can be tailored at will through the λ dependence of V.

or equivalently

$$S^{\text{det}}[\psi, \hat{\psi}; \lambda, f] + \beta \Delta \mathcal{F} - \beta W[\psi; \lambda, f] \mapsto S^{\text{det}}[\psi, \hat{\psi}; \bar{\lambda}, f_r] . \tag{100}$$

$S^{\text{det}}[\psi, \hat{\psi}; \bar{\lambda}, f_r]$ corresponds to the MSRJD action of the system that is prepared (in equilibrium) and evolves under the time-reversed protocol $\bar{\lambda}(u) \equiv \lambda(-u)$ and external forces $f_r([\psi], u) \equiv f([\bar{\psi}], -u)$. $\Delta \mathcal{F}_r$ is the change in free energy associated to this time-reversed protocol: $\beta \Delta \mathcal{F}_r = - \ln \mathcal{Z}(\bar{\lambda}(T)) + \ln \mathcal{Z}(\bar{\lambda}(-T)) = -\beta \Delta \mathcal{F}$ between the initial and the final 'virtual' equilibrium states. The dissipative part of the action, S^{diss}, is still invariant under $\mathcal{T}_{\text{eq}}$. This means that, contrary to the external forces F, the interaction with the bath is time-reversal invariant: the friction is still dissipative after the transformation. This immediately yields

$$e^{\beta \Delta \mathcal{F}} \langle A[\psi, \hat{\psi}] e^{-\beta W[\psi; \lambda, f]} \rangle_{S[\lambda, f]} = \langle A[\mathcal{T}_{\text{eq}} \psi, \mathcal{T}_{\text{eq}} \hat{\psi}] \rangle_{S[\bar{\lambda}, f_r]} \tag{101}$$

for any functional A of ψ and $\hat{\psi}$. In particular, for a local functional of the field, $A[\psi(t)]$, it leads to the relation [21]

$$e^{\beta \Delta \mathcal{F}} \langle A[\psi(t)] e^{-\beta W[\psi; \lambda, f]} \rangle_{S[\lambda, f]} = \langle A_r[\psi(-t)] \rangle_{S[\bar{\lambda}, f_r]} , \tag{102}$$

or also

$$e^{\beta \Delta \mathcal{F}} \langle A[\psi(t)] B[\psi(t')] e^{-\beta W[\psi; \lambda, f]} \rangle_{S[\lambda, f]}$$
$$= \langle A_r[\psi(-t)] B_r[\psi(-t')] \rangle_{S[\bar{\lambda}, f_r]}. \tag{103}$$

Setting $A[\psi, \hat{\psi}] = 1$, we obtain the Jarzynski equality [18]

$$e^{\beta \Delta \mathcal{F}} \langle e^{-\beta W[\psi; \lambda, f]} \rangle_{S[\lambda, f]} = 1 . \tag{104}$$

Setting $A[\psi, \hat{\psi}] = \delta(W - W[\psi; \lambda, f])$ we deduce the Crooks fluctuation theorem [20, 19]

$$P(W) = P_r(-W) \, e^{\beta(W - \Delta \mathcal{F})} , \tag{105}$$

where $P(W)$ is the probability for the external work done between $-T$ and T to be W given the protocol $\lambda(t)$ and the non-conservative force $f([\psi], t)$. $P_r(W)$ is the same probability, given the time-reversed protocol $\bar{\lambda}$ and time-reversed force f_r. The previous Jarzynski equality is the integral version of this theorem.

5.1.2. Fluctuation theorem. Let us now relax the condition that the system is prepared in thermal equilibrium and allow for any initial distribution P_i. We recall the corresponding deterministic part of the MSRJD action functional given in Sect. 3, eq. (11)

$$S^{\text{det}}[\psi, \hat{\psi}] \equiv \ln P_i \left(\psi(-T), \dot{\psi}(-T) \right)$$
$$- \int du \, i\hat{\psi}(u) \left[m\ddot{\psi}(u) + V'(\psi(u), \lambda(u)) - f([\psi], u) \right]. \tag{106}$$

The transformation $\mathcal{T}_{\mathrm{eq}}$ does not leave S^{det} invariant but one has

$$S^{\mathrm{det}}[\psi,\hat{\psi};\lambda,\mathrm{f}] - \mathcal{S} \mapsto S^{\mathrm{det}}[\psi,\hat{\psi};\bar{\lambda},\mathrm{f_r}] \,, \tag{107}$$

with the stochastic entropy $\mathcal{S} \equiv - \left[\ln P_{\mathrm{i}}(\psi(T),-\dot{\psi}(T)) - \ln P_{\mathrm{i}}(\psi(-T),\dot{\psi}(-T))\right] - \beta\mathcal{Q}$. The first term is the Shannon entropy whereas the second term is the exchange entropy defined through the heat transfer $\mathcal{Q} \equiv \Delta\mathcal{H} - W[\psi;\lambda,\mathrm{f}]$. $\Delta\mathcal{H} \equiv \mathcal{H}([\psi(T)],\lambda(T)) - \mathcal{H}([\psi(-T)],\lambda(-T))$ is the change of internal energy. The dissipative part of the action, S^{diss}, is still invariant under $\mathcal{T}_{\mathrm{eq}}$. This immediately yields

$$\langle A[\psi,\hat{\psi}]\mathrm{e}^{-\mathcal{S}}\rangle_{S[\lambda,\mathrm{f}]} = \langle A[\mathcal{T}_{\mathrm{eq}}\psi,\mathcal{T}_{\mathrm{eq}}\hat{\psi}]\rangle_{S[\bar{\lambda},\mathrm{f_r}]} \tag{108}$$

for any functional A of ψ and $\hat{\psi}$. Setting $A[\psi,\hat{\psi}] = 1$, we obtain the integral fluctuation theorem (sometimes referred as the Kawasaki identity)

$$1 = \langle\mathrm{e}^{-\mathcal{S}}\rangle_{S[\psi,\hat{\psi};\lambda,\mathrm{f}]} \,, \tag{109}$$

which using the Jensen inequality gives $\langle\mathcal{S}\rangle_{S[\psi,\hat{\psi};\lambda,\mathrm{f}]} \geq 0$, expressing the second law of thermodynamics. Setting $A[\psi,\hat{\psi}] = \delta(\zeta - \mathcal{S})$ we derive the fluctuation theorem [17, 19]

$$P(\zeta) = P_{\mathrm{r}}(-\zeta)\,\mathrm{e}^{\zeta} \,, \tag{110}$$

where $P(\zeta)$ is the probability for the entropy created between $-T$ and T to be ζ given the protocol $\lambda(t)$ and the non-conservative force $\mathrm{f}([\psi],t)$. $P_{\mathrm{r}}(\zeta)$ is the same probability, given the time-reversed protocol $\bar{\lambda}$ and time-reversed force $\mathrm{f_r}$.

Similar results can be obtained for isolated systems by switching off the interaction with the bath, *i.e.* by taking $\gamma = 0$. It is also straightforward to obtain extended relations when the bath is taken to be out of equilibrium, for example by using $\Gamma(t-t') \neq \gamma(t-t') + \gamma(t'-t)$, and the contribution of the change in the dissipative action is taken into account. This kind of fluctuation relation may be specially important in quantum systems.

5.2. Generic relations between correlations and linear responses

A number of generic relations between linear responses and the averages of other observables have been derived for different types of stochastic dynamics: Langevin with additive white noise [30], Ising variables with Glauber updates [31], or the heat-bath algorithm [32, 33, 34, 35], and even molecular dynamics of hard spheres or Lennard-Jones particle systems [36]. Especially interesting are those in which the relation is established with functions of correlations computed with the unperturbed dynamics [30, 33] as explained in [37]. The main aim of the studies in [31]-[37] was to give the most efficient computational method to obtain the linear response in the theoretical limit of no applied field. Another set of recent articles discusses very similar relations with the goal of

giving a thermodynamic interpretation to the various terms contributing to the linear response [38]-[40].

In the concrete case of Langevin processes this kind of relations can be very simply derived by multiplying the equation by the field or the noise and averaging over the noise in the way done in [30]. We derive here the same relations within the MSRJD formalism, using a symmetry property that is more likely to admit an extension to systems with quantum fluctuations.

5.2.1. A symmetry of the MSRJD generating functional valid also out of equilibrium.

We consider the most generic out of equilibrium situation. We allow for any initial preparation (P_i) and any evolution of the system (F). $\int \mathcal{D}[\psi, \hat\psi] \ e^{S[\psi, \hat\psi]}$ is invariant under the involutary field transformation $\mathcal{T}_{\rm eom}$, given by

$$\mathcal{T}_{\rm eom} \equiv \begin{cases} \psi_u & \mapsto & \psi_u \,, \\ i\hat\psi_u & \mapsto & -i\hat\psi_u + \dfrac{2\beta}{M'(\psi_u)} \displaystyle\int_v \Gamma^{-1}_{u-v} \dfrac{{\rm EQ}_v[\psi]}{M'(\psi_v)} \,, \end{cases} \tag{111}$$

The meaning of the subscript referring to 'equation of motion' will become clear in the following. For additive noise $[M'(\psi) = 1]$ the transformation becomes

$$i\hat\psi_u \mapsto -i\hat\psi_u + 2\beta \int_v \Gamma^{-1}_{u-v} \left[m\ddot\psi_v - F_v[\psi] + \int_w \gamma_{v-w}\dot\psi_w \right] \,,$$

and in the white noise limit it simplifies to

$$i\hat\psi_u \mapsto -i\hat\psi_u + \beta\gamma_0^{-1} \left[m\ddot\psi_u - F_u[\psi] + \gamma_0\dot\psi_u \right] \,. \tag{112}$$

The proof of invariance is similar to the one developed in Sect. 4.2 when dealing with the equilibrium symmetry. The Jacobian of this transformation is unity since its associated matrix is block triangular with ones on the diagonal. The integration domain of ψ is unchanged while the one of $\hat\psi$ can be chosen to be the real axis by a simple complex analysis argument. In the following lines we show that the action S evaluated in the transformed fields remains identical to the action evaluated in the original fields. We give the proof using an additive noise but the generalization to a multiplicative noise is straightforward. We start from the expression (10) and evaluate

$$
\begin{aligned}
S[\mathcal{T}_{\rm eom}\psi, \mathcal{T}_{\rm eom}\hat\psi] &= \ln P_i(\psi_{-T}, \dot\psi_{-T}) + \int_u \left[i\hat\psi_u - 2\beta \int_v \Gamma^{-1}_{u-v}{\rm EQ}_v[\psi] \right] \\
&\quad \times \left[{\rm EQ}_u[\psi] - \frac{1}{2}\int_w \beta^{-1}\Gamma_{u-w} \left(-i\hat\psi_w + 2\beta\int_z \Gamma^{-1}_{w-z}{\rm EQ}_z[\psi] \right) \right] \\
&= \ln P_i(\psi_{-T}, \dot\psi_{-T}) + \int_u \left[i\hat\psi_u - 2\beta \int_v \Gamma^{-1}_{u-v}{\rm EQ}_v[\psi] \right] \left[\frac{1}{2}\int_w \beta^{-1}\Gamma_{u-w}\ i\hat\psi_w \right] \\
&= S[\psi, \hat\psi] \,.
\end{aligned}
\tag{113}
$$

Contrary to the equilibrium transformation $\mathcal{T}_{\text{eq}}$, it does not include a time-reversal and is not defined in the Newtonian limit ($\gamma = 0$).

5.2.2. Supersymmetric version. In Sect. 4.10, we encoded the fields ψ, $\mathrm{i}\hat{\psi}$, c and c^* in a unique superfield Ψ. In this fashion, the transformation $\mathcal{T}_{\text{eom}}$ given in eq. (111) acts as

$$\Psi(t, \theta, \theta^*) \mapsto \Psi\left(t + \theta^*\theta \frac{2\beta \int_u \Gamma^{-1}_{t-u} M'(\Psi(u, \theta, \theta^*)) \mathrm{EQ}_u[\Psi]}{\partial_t M(\Psi(t, \theta, \theta^*))}, \theta, \theta^*\right) , \quad (114)$$

and leaves the equilibrium action $S[\Psi]$, see eq. (73), invariant.

5.2.3. Out of equilibrium relations. We first derive some relations in the additive case $[M'(\psi) = 1]$ and then we generalize the results to the case of multiplicative noise.

Additive noise. Using $\mathcal{T}_{\text{eom}}$ in the expression (35) of the self response $R(t, t')$ we find

$$\langle \psi_t \mathrm{i}\hat{\psi}_{t'} \rangle_S = \langle \mathcal{T}_{\text{eom}}\psi_t\, \mathcal{T}_{\text{eom}}\mathrm{i}\hat{\psi}_{t'} \rangle_S = -\langle \psi_t \mathrm{i}\hat{\psi}_{t'} \rangle_S + 2\beta \int_v \Gamma^{-1}_{t'-v} \langle \psi_t \mathrm{EQ}_v[\psi] \rangle_S ,$$

giving an explicit formula for computing the linear response without perturbing field:

$$R(t, t') = \beta \int \mathrm{d}v\, \Gamma^{-1}(t' - v)$$

$$\times \left[m\partial_v^2 C(t, v) + \int \mathrm{d}u\, \gamma(v - u)\partial_u C(t, u) - \langle \psi(t) F([\psi], v) \rangle \right] . \quad (115)$$

Once multiplied by $\Gamma_{t''-t'}$ and integrated over t' yields

$$m\partial_{t'}^2 C(t, t') + \int \mathrm{d}u\, \gamma(t' - u)\partial_u C(t, u)$$

$$- \langle \psi(t) F([\psi], t') \rangle = \beta^{-1} \int \mathrm{d}u\, \Gamma(t' - u) R(t, u) , \quad (116)$$

with no assumption on the initial $P_{\mathrm{i}}(\psi_{-T}, \dot{\psi}_{-T})$.

Using now $\mathcal{T}_{\text{eom}}$ in $\langle \mathrm{EQ}_t[\psi]\mathrm{i}\hat{\psi}_{t'} \rangle_S$, we get

$$\langle \mathrm{EQ}_t[\psi]\mathrm{i}\hat{\psi}_{t'} \rangle_S = \langle \mathrm{EQ}_t[\mathcal{T}_{\text{eom}}\psi]\, \mathcal{T}_{\text{eom}}\mathrm{i}\hat{\psi}_{t'} \rangle_S$$

$$= -\langle \mathrm{EQ}_t[\psi]\mathrm{i}\hat{\psi}_{t'} \rangle_S + 2\beta \int_u \Gamma^{-1}_{t'-u} \langle \mathrm{EQ}_t[\psi]\mathrm{EQ}_u[\psi] \rangle_S .$$

Since $\langle \mathrm{EQ}_t[\psi]\mathrm{EQ}_u[\psi] \rangle_S = \beta^{-1}\Gamma_{t-u}$, this simplifies in

$$\langle \mathrm{EQ}_t[\psi]\mathrm{i}\hat{\psi}_{t'} \rangle_S = \delta_{t-t'} ,$$

that yields

$$m\partial_t^2 R(t, t') + \int \mathrm{d}v\, \gamma(t - v)\partial_v R(v, t') - \langle \mathrm{i}\hat{\psi}(t') F([\psi], t) \rangle_S = \delta(t - t') \quad (117)$$

with no assumption on the initial P_i. One can trade the last term in the LHS for $\beta \int_u \Gamma_{t'-u}^{-1} \langle \xi(u) F_t[\psi] \rangle_\xi$ by use of Novikov's theorem.

Integrating both eqs. (116) and (117) around $t = t'$ we find the equal-time conditions

$$m\, \partial_{t'} C(t,t')|_{t'=t} = 0, \quad m\, \partial_t R(t,t')|_{t'\to t^-} = 1, \quad m\, \partial_t R(t,t')|_{t'\to t^+} = 0 \ . \quad (118)$$

The last two relations imply that the first derivative of the response function is discontinuous at equal times††.

The use of this symmetry is an easy way to get a generalization of eq. (115) for a generic response R_{AB}. Indeed, applying this transformation to expression (36) of the linear response we obtain

$$R_{AB}(t,t') = \beta \int du\ \Gamma^{-1}(t'-u) \sum_{n=0}^{\infty} \left\{ m\, \partial_u^{n+2} \langle A[\psi(t)]\psi(u) \frac{\partial B[\psi(t')]}{\partial\, \partial_{t'}^n \psi(t')} \rangle_S \right.$$

$$- \partial_u^n \langle A[\psi(t)] F([\psi], u) \frac{\partial B[\psi(t')]}{\partial\, \partial_{t'}^n \psi(t')} \rangle_S$$

$$\left. + \int dv\ \gamma(u-v) \partial_v^{n+1} \langle A[\psi(t)]\psi(v) \frac{\partial B[\psi(t')]}{\partial\, \partial_{t'}^n \psi(t')} \rangle_S \right\} \ . \quad (119)$$

This formula gives the linear response as an explicit function of multiple-time correlators of the field ψ. For example, if B is a function of the field only (and not of its time-derivatives), just the $n=0$-term subsists in the above sum:

$$R_{AB}(t,t') = \beta \int du\ \Gamma^{-1}(t'-u) \left\{ m\, \partial_u^2 \langle A[\psi(t)]\psi(u) \frac{\partial B[\psi(t')]}{\partial\, \psi(t')} \rangle_S \right.$$

$$- \langle A[\psi(t)] F([\psi], u) \frac{\partial B[\psi(t')]}{\partial\, \psi(t')} \rangle_S$$

$$\left. + \int dv\ \gamma(u-v) \partial_v \langle A[\psi(t)]\psi(v) \frac{\partial B[\psi(t')]}{\partial\, \psi(t')} \rangle_S \right\} \ . \quad (120)$$

As another example if one is interested in the self-response of the velocity, $A[\psi(t)] = B[\psi(t)] = \partial_t \psi(t)$, one obtains

$$R_{AB}(t,t') = \beta \int du\ \Gamma^{-1}(t'-u) \left\{ m\, \partial_t \partial_u^3 C(t,u) - \partial_t \partial_u \langle \psi(t) F([\psi], u) \rangle_S \right.$$

$$\left. + \int dv\ \gamma(u-v) \partial_v^2 C(t,v) \right\} \ . \quad (121)$$

Multiplicative noise. Similar results can be obtained for multiplicative noise. Applying the transformation to the correlator $\int_u \Gamma_{t'-u} \langle \psi_t M'(\psi_{t'}) M'(\psi_u) i\hat{\psi}_u \rangle_S$ we obtain

$$\langle \psi_t \mathrm{EQ}_{t'}[\psi] \rangle_S = \beta^{-1} \int_u \Gamma_{t'-u} \langle \psi_t M'(\psi_{t'}) M'(\psi_u) i\hat{\psi}_u \rangle_S \ ,$$

††It is clear from the expressions given in (118) that the overdamped $m \to 0$ limit allows for a sudden discontinuity of the response function as well as a finite slope of the correlation function at equal times.

implying

$$m\partial_{t'}^2 C(t,t') + \int_u \gamma_{t'-u}\langle\psi_t M'(\psi_{t'})M'(\psi_u)\partial_u\psi_u\rangle_S$$

$$- \langle\psi_t F_{t'}[\psi]\rangle_S = \beta^{-1}\int_u \Gamma_{t'-u}\langle\psi_t M'(\psi_{t'})M'(\psi_u)\mathrm{i}\hat\psi_u\rangle_S . \qquad (122)$$

Applying now the transformation to the correlator $\langle \mathrm{EQ}_t[\psi]\mathrm{i}\hat\psi_{t'}\rangle_S$, one obtains

$$\langle \mathrm{EQ}_t[\psi]\mathrm{i}\hat\psi_{t'}\rangle_S = \delta_{t-t'} + \beta^{-1}\int_u \Gamma_{t-u}\langle M'(\psi_t)M'(\psi_u)\mathrm{i}\hat\psi_u\mathrm{i}\hat\psi_{t'}\rangle_S \qquad (123)$$

yielding

$$m\partial_t^2 R(t,t') + \int_u \gamma_{t-u}\langle M'(\psi_{t'})M'(\psi_u)\partial_u\psi_u\mathrm{i}\hat\psi_{t'}\rangle_S$$

$$- \langle F_t[\psi]\mathrm{i}\hat\psi_{t'}\rangle_S = \delta_{t-t'}\beta^{-1}\int_u \Gamma_{t-u}\langle\psi_t M'(\psi_{t'})M'(\psi_u)\mathrm{i}\hat\psi_u\rangle_S . \qquad (124)$$

One can check from eqs. (122) and (124) that the equal-time conditions given in eqs. (118) are still valid in the multiplicative case.

5.3. Composition of $\mathcal{T}_{\mathrm{eom}}$ and $\mathcal{T}_{\mathrm{eq}}$

For an equilibrium situation, the MSRJD action functional is fully invariant under the composition of $\mathcal{T}_{\mathrm{eom}}$ and $\mathcal{T}_{\mathrm{eq}}$,

$$\mathcal{T}_{\mathrm{eq}}\circ\mathcal{T}_{\mathrm{eom}} = \begin{cases} \psi_u \;\mapsto\; \psi_{-u} , \\ \mathrm{i}\hat\psi_u \;\mapsto\; -\mathrm{i}\hat\psi_{-u} - \beta\partial_u\psi_{-u} + \dfrac{2\beta}{M'(\psi_{-u})}\displaystyle\int_v \Gamma_{u-v}^{-1}\dfrac{\mathrm{EQ}_v[\bar\psi]}{M'(\psi_{-v})} , \end{cases}$$

that simply reads in the white noise limit

$$\mathcal{T}_{\mathrm{eq}}\circ\mathcal{T}_{\mathrm{eom}} = \begin{cases} \psi_u \;\mapsto\; \psi_{-u} , \\ \mathrm{i}\hat\psi_u \;\mapsto\; -\mathrm{i}\hat\psi_{-u} + \dfrac{\beta}{\gamma_0 M'(\psi_{-u})^2}\left[m\partial_u^2\psi_{-u} + V'(\psi_{-u})\right] . \end{cases}$$

For simplicity we only show the implication of this symmetry in this limit and in the additive noise case:

$$R(t,t') = -R(-t,-t') + \frac{\beta}{\gamma_0}\left[m\partial_{t'}^2 C(-t,-t') + \langle\psi(-t)V'(\psi(-t'))\rangle_S\right] .$$

Using equilibrium properties, *i.e.* time-translational invariance of all observables and time-reversal symmetry of two-time correlation functions of the field ψ (shown in Sect. 4.2), and causality of the response,

$$R(\tau) = \Theta(\tau)\frac{\beta}{\gamma_0}\left[m\partial_\tau^2 C(\tau) + \langle\psi(t)V'(\psi(t'))\rangle_S\right] , \qquad (125)$$

with $\tau \equiv t - t'$ which is eq. (116) after cancellation of the LHS with the last term in the RHS when FDT between R and C holds [also eq. (117) after a similar simplification].

Here again, one can easily obtain a generalization of this last relation for a generic response R_{AB} by plugging the transformation into the expression (36) of the linear response.

6. Conclusions

In this paper we recalled the path-integral approach to classical stochastic dynamics with generic multiplicative colored noise. The action has three terms: a deterministic (Newtonian dynamics) contribution, a dissipative part and a Jacobian. We identified a number of symmetries of the generating functional when the sources are set to zero. The invariance of the action is achieved by the three terms independently.

One of these symmetries applies only when equilibrium dynamics are assumed. Equilibrium dynamics are ensured whenever the system is prepared with equilibrium initial conditions at temperature β^{-1} (a statistical mixture given by the Gibbs-Boltzmann measure), evolves with the corresponding time-independent conservative forces, and is in contact with an equilibrium bath at the same temperature β^{-1}. The invariance also holds in the limit in which the contact with the bath is suppressed, *i.e.* under deterministic (Newtonian) dynamics, but the initial condition is still taken from the Gibbs-Boltzmann measure. This symmetry yields all possible model-independent fluctuation-dissipation theorems as well as stationarity and Onsager reciprocal relations. When the field-transformation is applied to driven problems, the symmetry no longer holds, but it gives rise to different kinds of fluctuation theorems.

We identified another more general symmetry that applies to equilibrium and out of equilibrium set-ups. It holds for any kind of initial conditions – they can be any statistical mixture or even deterministic, and the evolution can be dictated by time-dependent and/or non-conservative forces as long as the system is coupled to an equilibrium bath. The symmetry implies exact dynamic equations that couple generic correlations and linear responses. These equations are model-dependent in the sense that they depend explicitly on the applied forces. They are the starting point to derive Schwinger-Dyson-type approximations and close them on two-time observables. Although the symmetry is ill-defined in the Newtonian limit, the dynamic relations it yields can nevertheless be evaluated in the Newtonian case.

Finally, we gave a supersymmetric expression of the path-integral for problems with multiplicative colored noise and conservative forces. We expressed all the previous symmetries in terms of superfield transformations and we discussed the relationship between supersymmetry and other symmetries.

We intended to present a self-contained presentation of some symmetry properties of classical deterministic and stochastic dynamics. We focused on the so-called model

A dynamics (with a non-conserved order parameter) for a 0-dimensional field. The generalization to a vector field is straightforward. Extensions to this work include the study of other dynamics such as the so-called model B dynamics (with a conserved order parameter). Higher dimensional fields would also allow the study of forces that do not work, such as Larmor precession around a magnetic vector field [6]. This article should serve as an introduction to and motivation for the study of quantum problems that we shall develop in [42]. Although some of these results were known, notably those associated to additive noise processes, they were scattered and somehow hidden in different publications. The close relation between all these properties was not always fully appreciated either. The multiplicative noise results are, as far as we know, new.

Appendix A.   Conventions and notations

Θ is the Heaviside step function. When dealing with Markov Langevin equations, the choice of the value of the Heaviside step function $\Theta(t)$ at $t = 0$ is imposed by the choice of the Itô $[\Theta(0) = 0]$ or the Stratonovich convention $[\Theta(0) = 1/2]$. However, away from the Markov case, *i.e.* as long as both inertia and the color of the bath are not neglected simultaneously, the choice of $\Theta(0)$ is unconstrained and the physics should not depend on it. We recall the identities

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{2\pi} \mathrm{e}^{ixy} = \delta(y) \qquad \text{and} \qquad \int_{-\infty}^{y} \mathrm{d}x\, \delta(x) = \Theta(y)\,, \quad \text{(A.1)}$$

where δ is the Dirac delta function.

Field theory notations. Let ψ be a real field. The integration over this field is denoted $\int \mathcal{D}[\psi]$. If A is a functional of the field, we denote it $A[\psi]$. If it also depends on one or several external parameters, such as the time t and a protocol λ, we denote it $A([\psi], \lambda, t)$. Whenever A is a local functional of the field at time t (*i.e.* a function of $\psi(t)$ and its first time-derivatives), we use the short-hand notation $A[\psi(t)]$. The time-reversed field constructed from ψ is denoted $\bar{\psi}$: $\bar{\psi}(t) \equiv \psi(-t)$. The time-reversed functional constructed from $A([\psi], \lambda, t)$ is called A_{r}: $A_{\mathrm{r}}([\psi], \lambda, t) \equiv A([\bar{\psi}], \lambda, -t)$. Applied on local observables of ψ, it has the effect of changing the sign of all odd time-derivatives in the expression of A.

To shorten expressions, we adopt a notation in which the arguments of the fields appear as subindices, $\psi_t \equiv \psi(t)$, $\gamma_{t-t'} \equiv \gamma(t - t')$, and so on and so forth, and the integrals over time as expressed as $\int_t \equiv \int dt$.

Grassmann numbers. Let θ_1 and θ_2 be two anticommuting Grassmann numbers and θ_1^* and θ_2^* their respective Grassmann conjugates. We adopt the following convention for the complex conjugate of a product of Grassmann numbers: $(\theta_1\theta_2)^* = \theta_2^*\theta_1^*$.

Appendix B. Discrete MSRJD for additive noise

In this Appendix we discuss the MSRJD action for processes with additive colored noise.

Appendix B.1. Discrete Langevin equation

The Langevin equation is a stochastic differential equation and one can give a rigorous meaning to it by specifying a particular discretization scheme.

Let us divide the time interval $[-T, T]$ into $N + 1$ infinitesimal slices of width $\epsilon \equiv 2T/(N + 1)$. The discretized times are $t_k = -T + k\epsilon$ with $k = 0, ..., N + 1$. The discretized version of $\psi(t)$ is $\psi_k \equiv \psi(t_k)$. The continuum limit is achieved by sending N to infinity and keeping $(N + 1)\epsilon = 2T$ constant. Given some initial conditions ψ_i and $\dot{\psi}_i$, we set $\psi_1 = \psi_i$ and $\psi_0 = \psi_i - \epsilon\dot{\psi}_i$ meaning that the first two times (t_0 and t_1) are reserved for the integration over the initial conditions whereas the N following ones correspond to the stochastic dynamics given by the discretized Langevin equation:

$$
\text{EQ}_{k-1} \equiv m\frac{\psi_{k+1} - 2\psi_k + \psi_{k-1}}{\epsilon^2} - F_k(\psi_k, \psi_{k-1}, ...) + \epsilon\sum_{l=1}^{k}\gamma_{kl}\frac{\psi_l - \psi_{l-1}}{\epsilon}
$$

$$
= \xi_k , \tag{B.1}
$$

defined for $k = 1, ..., N$. The force F_k typically depends on the state ψ_k but can have a memory kernel (*i.e.* it can depend on previous states ψ_{k-1}, ψ_{k-2}, etc.). The notation γ_{kl} stands for $\gamma_{kl} \equiv \epsilon^{-1}\int_{0-}^{\epsilon}du\,\gamma(t_k - t_l + u)$. The ξ_k are independent Gaussian random variables with variance $\langle\xi_k\xi_l\rangle = \beta^{-1}\Gamma_{kl}$ where $\Gamma_{kl} \equiv \gamma_{kl} + \gamma_{lk}$. Inspecting the equation above, we notice that the value of ψ_k depends on the realization of the previous noise realization ξ_{k-1} and there is no need to specify ξ_0 and ξ_{N+1}.

In the white noise limit, one has $\gamma_{kl} = \epsilon^{-1}\gamma_0\delta_{kl}$, $\langle\xi_k\xi_l\rangle = 2\gamma_0\beta^{-1}\epsilon^{-1}\delta_{kl}$ where δ is the Kronecker delta, and

$$
\text{EQ}_{k-1} \equiv m\frac{\psi_{k+1} - 2\psi_k + \psi_{k-1}}{\epsilon^2} - F_k(\psi_k, \psi_{k-1}, ...) + \gamma_0\frac{\psi_k - \psi_{k-1}}{\epsilon} = \xi_k .
$$

Appendix B.2.   *Construction of the MSRJD action*

The probability density P for a complete field history $(\psi_0, \psi_1, ..., \psi_{N+1})$ is set by the relation

$$P(\psi_0, \psi_1, ..., \psi_{N+1})\mathrm{d}\psi_0\mathrm{d}\psi_1...\mathrm{d}\psi_{N+1}$$
$$= P_\mathrm{i}(\psi_\mathrm{i}, \dot{\psi}_\mathrm{i})\mathrm{d}\psi_\mathrm{i}\mathrm{d}\dot{\psi}_\mathrm{i}\ P_\mathrm{n}(\xi_1, \xi_2, ..., \xi_N)\mathrm{d}\xi_1\mathrm{d}\xi_2...\mathrm{d}\xi_N\ .$$

P_i is the initial probability distribution of the field. The probability for a given noise history to occur between times t_1 and t_N is given by

$$P_\mathrm{n}(\xi_1, ..., \xi_N) = \mathcal{M}_N^{-1}\ \mathrm{e}^{-\frac{1}{2}\sum_{k,l=1}^N \xi_k\,\beta\Gamma_{kl}^{-1}\,\xi_l} \tag{B.2}$$

where Γ_{kl}^{-1} is the inverse matrix of Γ_{kl} (and not the discretized version of the inverse operator of Γ) and the normalization is given by $\mathcal{M}_N^2 \equiv \frac{(2\pi)^N}{\det_{kl}(\beta\Gamma_{kl}^{-1})}$ where $\det(...)$ stands for the matrix determinant. From eq. (B.2), one derives

$$P(\psi_0, \psi_1, ..., \psi_{N+1}) = |\mathcal{J}_N|P_\mathrm{i}(\psi_1, \frac{\psi_1 - \psi_0}{\epsilon})P_\mathrm{n}(\mathrm{EQ}_0, ..., \mathrm{EQ}_{N-1})\ , \tag{B.3}$$

with the Jacobian

$$\mathcal{J}_N \equiv \det\left(\frac{\partial\,(\psi_\mathrm{i}, \dot{\psi}_\mathrm{i}, \xi_1, \ldots, \xi_N)}{\partial\,(\psi_0, \psi_1, \ldots, \psi_{N+1})}\right) = \det\left(\frac{\partial\,(\psi_\mathrm{i}, \dot{\psi}_\mathrm{i}, \mathrm{EQ}_0, \ldots, \mathrm{EQ}_{N-1})}{\partial\,(\psi_0, \psi_1, \ldots, \psi_{N+1})}\right)\ ,$$

that will be discussed in Appendix B.3. The expression (B.2) for the noise history probability reads, after a Hubbard-Stratonovich transformation that introduces the auxiliary variables $\hat{\psi}_k$ ($k = 1, ..., N$),

$$\mathcal{N}_N P_\mathrm{n}(\xi_1, ..., \xi_N) = \int\mathrm{d}\hat{\psi}_1...\mathrm{d}\hat{\psi}_N\ \mathrm{e}^{-\epsilon\sum_k \mathrm{i}\hat{\psi}_k\xi_k + \frac{1}{2}\beta^{-1}\epsilon^2\sum_{kl}\mathrm{i}\hat{\psi}_k\Gamma_{kl}\mathrm{i}\hat{\psi}_l}$$

$$= \int\mathrm{d}\hat{\psi}_0...\mathrm{d}\hat{\psi}_{N+1}\ \delta(\hat{\psi}_0)\delta(\hat{\psi}_{N+1})\,\mathrm{e}^{-\epsilon\sum_k \mathrm{i}\hat{\psi}_k\mathrm{EQ}_{k-1} + \frac{1}{2}\beta^{-1}\epsilon^2\sum_{kl}\mathrm{i}\hat{\psi}_k\Gamma_{kl}\mathrm{i}\hat{\psi}_l}\ , \quad \text{(B.4)}$$

with $\mathcal{N}_N \equiv (2\pi/\epsilon)^N$. In the last step, we replaced ξ_k by EQ_{k-1} and we allowed for summations over $k = 0$ and $k = N + 1$ as well as integrations over $\hat{\psi}_0$ and $\hat{\psi}_{N+1}$ at the cost of introducing delta functions. The Hubbard-Stratonovich transformation allows for some freedom in the choice of the sign in front of $\mathrm{i}\hat{\psi}_k$ in the exponent (indeed P_n is real so $P_\mathrm{n} = P_\mathrm{n}^*$). Together with eq. (B.3) this gives

$$\mathcal{N}_N P(\psi_0, \psi_1, ..., \psi_{N+1}) = |\mathcal{J}_N|\int\mathrm{d}\hat{\psi}_0...\mathrm{d}\hat{\psi}_{N+1}\ \delta(\hat{\psi}_0)\delta(\hat{\psi}_{N+1})$$

$$\times\mathrm{e}^{-\sum_k \mathrm{i}\hat{\psi}_k\mathrm{EQ}_{k-1} + \frac{1}{2}\beta^{-1}\sum_{kl}\mathrm{i}\hat{\psi}_k\Gamma_{kl}\mathrm{i}\hat{\psi}_l + \ln P_\mathrm{i}\left(\psi_1, \frac{\psi_1-\psi_0}{\epsilon}\right)}$$

that in the continuum limit becomes

$$\mathcal{N}P[\psi] = |\mathcal{J}[\psi]|\,\mathrm{e}^{\ln P_\mathrm{i}}\int\mathcal{D}[\hat{\psi}]\,\mathrm{e}^{-\int\mathrm{d}u\,\mathrm{i}\hat{\psi}(u)\mathrm{EQ}([\psi],u) + \frac{1}{2}\iint\mathrm{d}u\,\mathrm{d}v\,\mathrm{i}\hat{\psi}(u)\beta^{-1}\Gamma(u-v)\mathrm{i}\hat{\psi}(v)}\ ,$$

with the boundary conditions $\hat{\psi}(-T) = \hat{\psi}(T) = 0$ and where all the integrals over time run from $-T$ to T. In the following, unless otherwise stated, we shall simply denote them by $\int$. The infinite prefactor $\mathcal{N} \equiv \lim_{N \to \infty} (2\pi/\epsilon)^N$ can be absorbed in the definition of the measure:

$$\mathcal{D}[\psi, \hat{\psi}] = \lim_{N \to \infty} \left(\frac{\epsilon}{2\pi}\right)^N \prod_{k=0}^{N+1} \mathrm{d}\psi_k \, \mathrm{d}\hat{\psi}_k \ . \tag{B.5}$$

Markov case. In the Markov limit, the Langevin equation is a first order differential equation, therefore only the first time t_0 should be reserved for integrating over the initial conditions. Moreover, one has to specify the discretization:

$$\mathrm{EQ}_{k-1} \equiv \gamma_0 \frac{\psi_k - \psi_{k-1}}{\epsilon} - F_k(\tilde{\psi}_k) = \xi_k \ , \tag{B.6}$$

where $\tilde{\psi}_k \equiv a\psi_k + (1-a)\psi_{k-1}$ with $a \in [0,1]$. $a = 0$ corresponds to the Itô interpretation whereas $a = 1/2$ corresponds to the Stratonovich one (see the discussion in Sect. 2.4). Following the steps in Appendix B.2, we upgrade eq. (B.6) to the following a-dependent action†:

$$S_N(a) = \epsilon \sum_k \left(\beta^{-1} \gamma_0 (i\hat{\psi}_k)^2 - i\hat{\psi}_k \left[\gamma_0 \frac{\psi_k - \psi_{k-1}}{\epsilon} - F_k(\tilde{\psi}_k) \right] - \frac{a}{\gamma_0} F'_k(\tilde{\psi}_k) \right) . \tag{B.7}$$

The last term in the RHS comes from the Jacobian:

$$\mathcal{J}_N = \det_{kl} \left(\frac{\partial \mathrm{EQ}_{k-1}}{\partial \psi_l} \right) = \prod_k \left(\frac{\gamma_0}{\epsilon} - aF'_k(\tilde{\psi}_k) \right) = \left(\frac{\gamma_0}{\epsilon} \right)^N e^{-\epsilon \sum_k \frac{a}{\gamma_0} F'_k(\tilde{\psi}_k)} \ .$$

In the Itô discretization scheme ($a = 0$) this Jacobian term disappears from the action. Although $S_N(a)$ seems to be a-dependent, we now prove that all discretization schemes yield the same physics by showing that the difference $S_N(a) - S_N(0)$ is negligible. The Taylor expansion of $F_k(\tilde{\psi}_k)$ around ψ_{k-1}, $F_k(\psi_{k-1}) + a\,(\psi_k - \psi_{k-1})\,F'(\psi_{k-1}) + O(\epsilon)$ [since $\psi_k - \psi_{k-1} = O(\sqrt{\epsilon})$], yields

$$S_N(a) - S_N(0) = a\epsilon \sum_k F'(\psi_{k-1}) \left[i\hat{\psi}_k (\psi_k - \psi_{k-1}) - \frac{1}{\gamma_0} \right] + O(\epsilon^2) . \tag{B.8}$$

Although the first term within the square brackets looks smaller than the second one, they are actually both $O(1)$ since $i\hat{\psi}_k = O(1/\sqrt{\epsilon})$. Thus, each term in the sum in the RHS is $O(\epsilon)$. We now compute the average of $S_N(a) - S_N(0)$ with respect to $S_N(0)$ by neglecting in the latter the term $\epsilon i\hat{\psi}_k F_k(\psi_{k-1})$ which is of order $\sqrt{\epsilon}$ whereas the others are of order 1. Since $\langle i\hat{\psi}_k (\psi_k - \psi_{k-1}) \rangle_{S_N(0)} = 1/\gamma_0$, it is easy to show that $\langle S_N(a) - S_N(0) \rangle_{S_N(0)} = 0$ and therefore all the $S_N(a)$ actions are equivalent to the simpler Itô one.

† We omit the initial measure which is not relevant in this discussion.

Appendix B.3.   *Jacobian*

Appendix B.3.1.   *Discrete evaluation of the Jacobian.*   In this section we take the continuum limit of the Jacobian defined in eq. (B.4). In the additive noise case, we start from

$$
\mathcal{J}_N = \det \left(\frac{\partial \left(\psi_{\mathrm{i}}, \dot{\psi}_{\mathrm{i}}, \mathrm{EQ}_0, \ldots, \mathrm{EQ}_{N-1} \right)}{\partial \left(\psi_0, \psi_1, \ldots, \psi_{N+1} \right)} \right)
$$

$$
= \det \begin{pmatrix}
0 & 1 & 0 \ldots \\
-1/\epsilon & 1/\epsilon & 0 \ldots \\
\frac{\partial \mathrm{EQ}_0}{\partial \psi_0} & \frac{\partial \mathrm{EQ}_0}{\partial \psi_1} & \frac{\partial \mathrm{EQ}_0}{\partial \psi_2} & 0 \ldots \\
\frac{\partial \mathrm{EQ}_1}{\partial \psi_0} & \frac{\partial \mathrm{EQ}_1}{\partial \psi_1} & \frac{\partial \mathrm{EQ}_1}{\partial \psi_2} & \frac{\partial \mathrm{EQ}_1}{\partial \psi_3} & 0 \ldots \\
\ldots & & & & & 0 \\
\frac{\partial \mathrm{EQ}_{N-1}}{\partial \psi_0} & & \ldots & & & \frac{\partial \mathrm{EQ}_{N-1}}{\partial \psi_{N+1}}
\end{pmatrix}
$$

$$
= \frac{1}{\epsilon} \det \begin{pmatrix}
\frac{\partial \mathrm{EQ}_0}{\partial \psi_2} & 0 \ldots \\
\frac{\partial \mathrm{EQ}_1}{\partial \psi_2} & \frac{\partial \mathrm{EQ}_1}{\partial \psi_3} & 0 \ldots \\
\ldots & & & 0 \\
\frac{\partial \mathrm{EQ}_{N-1}}{\partial \psi_2} & \ldots & & \frac{\partial \mathrm{EQ}_{N-1}}{\partial \psi_{N+1}}
\end{pmatrix} . \tag{B.9}
$$

Causality manifests itself in the lower triangular structure of the last matrix. One can evaluate the last determinant by plugging eq. (B.1). It yields

$$
\mathcal{J}_N = \frac{1}{\epsilon} \prod_{k=1}^{N} \frac{\partial \mathrm{EQ}_{k-1}}{\partial \psi_{k+1}} = \frac{1}{\epsilon} \left(\frac{m}{\epsilon^2} \right)^N .
$$

The Jacobian $\mathcal{J} \equiv \lim_{N \to \infty} \mathcal{J}_N$ is therefore a field-independent positive constant that can be absorbed in a redefinition of the measure:

$$
\mathcal{D}[\psi, \hat{\psi}] \equiv \lim_{N \to \infty} \frac{1}{\epsilon} \left(\frac{m}{2\pi\epsilon} \right)^N \prod_{k=0}^{N+1} \mathrm{d}\psi_k \, \mathrm{d}\hat{\psi}_k . \tag{B.10}
$$

We show that this result also holds for multiplicative noise in Appendix C.

Appendix B.3.2.   *Continuous evaluation of the Jacobian.*   One might also wish to check this result in the continuous notations. A very similar approach can be found in [26]. In the continuous notations, $\lim_{N \to \infty} \mathcal{J}_N$ reads up to some constant factor

$$
\mathcal{J}[\psi] = \det_{uv} \left[\frac{\delta \mathrm{EQ}([\psi], u)}{\delta \psi(v)} \right] .
$$

where $\det[...]$ stands for the functional determinant. Defining F'_{uv} as $\delta F_u[\psi]/\delta \psi_v$, the Jacobian reads

$$
\mathcal{J}[\psi] = \det_{uv} \left[m \partial_u^2 \delta_{u-v} + \int_w \gamma_{u-w} \partial_w \delta_{w-v} - F'_{uv}[\psi] \right]
$$

$$= \det_{uv} \left[m\partial_u^2 \delta_{u-v} + \int_w \gamma_{u-w} \, \partial_w \delta_{w-v} \right] \det_{uv} \left[\delta_{u-v} - \int_w G_{u-w} F'_{wv}[\psi] \right]$$

$$= \det_{uv} \left[m\partial_u^2 \delta_{u-v} + \int_w \gamma_{u-w} \, \partial_w \delta_{w-v} \right] \exp \mathrm{Tr}_{uv} \ln \left[\delta_{u-v} - M_{uv} \right]$$

$$= \det_{uv} \left[m\partial_u^2 \delta_{u-v} + \int_w \gamma_{u-w} \, \partial_w \delta_{w-v} \right] \exp - \sum_{n=1}^{\infty} \frac{1}{n} \int_u \left\{ \underbrace{M \circ M \circ \dots \circ M}_{n \text{ times}} \right\}_{uu} \quad \text{(B.11)}$$

where we used the notations $M_{uv} \equiv \{ G \circ F' \}_{uv} \equiv \int_w G_{u-w} F'_{wv}[\psi]$. G is the retarded Green function solution to

$$m\partial_u^2 G(u - v) + \int \mathrm{d}w \, \gamma(u - w)\partial_w G(w - v) = \delta(u - v) . \quad \text{(B.12)}$$

Since both G_{u-v} and F'_{uv} are causal, it is easy to see that the $n \geq 2$ terms do not contribute to the sum in eq. (B.11). If the force $F([\psi], t)$ does not have any local term (involving the value of ψ or $\dot{\psi}$ at time t) the $n = 1$ term is also zero. Otherwise the $n = 1$ term can still be proven to be zero provided that $G(t = 0) = 0$. This will be true, as we shall show in the next paragraph, unless the white noise limit is taken together with the Smoluchowski limit ($m = 0$). Away from this Markov limit we establish

$$\mathcal{J}[\psi] = \det_{uv} \left[m\partial_u^2 \delta(u - v) + \int_w \gamma_{u-w} \, \partial_w \delta_{w-v} \right] ,$$

meaning that the Jacobian is a constant that does not depend on the field ψ.

We now give a proof that $G(t = 0) = 0$. Taking the Fourier transform of eq. (B.12),

$$G(t = 0) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} G(\omega) = - \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \frac{1}{m\omega^2 + \mathrm{i}\omega\gamma(\omega)} . \quad \text{(B.13)}$$

$G(\omega)$ and $\gamma(\omega)$ are the Fourier transforms of the retarded Green function and friction. They are both analytic in the upper half plane (UHP) thanks to their causality structure. The convergence of the integrals around $|\omega| \to \infty$ in eq. (B.13) is ensured by either the presence of inertia or the colored noise. For a white noise $[\gamma(\omega) = \gamma_0]$, it is clear that the mass term renders the integrals in eq. (B.13) well defined. In the $m = 0$ limit the convergence is still guaranteed as long as the white noise limit is not taken simultaneously. Indeed, because $\gamma(\omega)$ is analytic in the UHP, it is hence either divergent on the boundaries of the UHP or constant everywhere $[\gamma(\omega) = \gamma_0]$. In the first case, which corresponds to a generic colored noise, this renders the integrals in eq. (B.13) well defined. In the second case, corresponding to a white noise limit, they are ill-defined and require a more careful treatment†. When the integrals in eq. (B.13) are well defined on the boundaries, the absence of poles (or branch cuts) in the UHP of

† In the white noise limit, $G(t) = \gamma_0^{-1} \left[1 - \mathrm{e}^{-\gamma_0 t/m} \right] \Theta(t)$ is a continuous function that vanishes at $t = 0$. If we take $m \to 0$ in the previous expression, we still have $G(0) = 0$ and $G(t) = \Theta(t)/\gamma_0$ for $t \gg m/\gamma_0$. By choosing $\Theta(0) = 0$, these two results can be collected in $G(t) = \Theta(t)/\gamma_0$ for all t. The

$G(\omega)$ gives, after a little deformation of the integration contour in eq. (B.13) above the $\omega = 0$ pole, the result $G(t = 0) = 0$.

Appendix B.3.3.   *Representation in terms of a fermionic field integral.* The determinant can be represented as a Gaussian integration over Grassmannian conjugate fields c and c^*. This formulation is a key ingredient to the supersymmetric representation of the MSRJD path integral. Let us first recall the discretized expression of the Jacobian obtained in eq. (B.9):

$$\mathcal{J}_N = \frac{1}{\epsilon} \det{}_{kl} \left(\frac{\partial \mathrm{EQ}_{k-1}}{\partial \psi_{l+1}} \right) \, ,$$

where k and l run from 1 to N. Introducing ghosts, it can be put in the form

$$\mathcal{J}_N = \frac{1}{\epsilon} \frac{1}{\epsilon^N} \int \mathrm{d}c_2 \mathrm{d}c_0^* ... \mathrm{d}c_{N+1} \mathrm{d}c_{N-1}^* \, \mathrm{e}^{\epsilon^2 \sum_{k=0}^{N-1} \sum_{l=2}^{N+1} c_k^* \frac{1}{\epsilon} \frac{\partial \mathrm{EQ}_k}{\partial \psi_l} c_l}$$

$$= \frac{1}{\epsilon} \frac{1}{\epsilon^N} \int \mathrm{d}c_0 \mathrm{d}c_0^* ... \mathrm{d}c_{N+1} \mathrm{d}c_{N+1}^* \, c_{N+1}^* c_N^* c_1 c_0 \, \mathrm{e}^{\epsilon^2 \sum_{k=0}^{N+1} \sum_{l=0}^{N+1} c_k^* \frac{1}{\epsilon} \frac{\partial \mathrm{EQ}_k}{\partial \psi_l} c_l} \, ,$$

where in the last step, we allowed integration over c_0, c_1, c_N^* and c_{N+1}^* at the cost of introducing delta functions (remember that for a Grassmann number c, the delta function is achieved by c itself). In the continuum limit, absorbing the prefactor into a redefinition of the measure,

$$\mathcal{D}[\psi, \hat{\psi}] = \lim_{N\to\infty} \frac{1}{(2\pi)^N} \frac{1}{\epsilon} \prod_{k=0}^{N+1} \mathrm{d}\psi_k \, \mathrm{d}\hat{\psi}_k \quad \text{and} \quad \mathcal{D}[c, c^*] = \lim_{N\to\infty} \prod_{k=0}^{N+1} \mathrm{d}c_k \, \mathrm{d}c_k^* \, , \quad \text{(B.14)}$$

this yields

$$\mathcal{J}[\psi] = \int \mathcal{D}[c, c^*] \, \mathrm{e}^{S^{\mathcal{J}}[c,c^*,\psi]}$$

with

$$S^{\mathcal{J}}[c, c^*, \psi] \equiv \int_u \int_v c_u^* \frac{\delta \mathrm{EQ}_u[\psi]}{\delta \psi_v} c_v \, ,$$

and the extra boundary conditions: $c(-T) = \dot{c}(-T) = c^*(T) = \dot{c}^*(T) = 0$. Plugging the Langevin equation (1), we have

$$\frac{\delta \mathrm{EQ}_u[\psi]}{\delta \psi_v} = m \partial_u^2 \delta_{u-v} - \frac{\delta F_u[\psi]}{\delta \psi_v} + \int_w \gamma_{w-v} \partial_w \delta_{w-v} \, .$$

Jacobian is still a constant. This limiting procedure where inertia has been sent to zero after the white noise limit was taken, is the so-called Itô convention. However if m is set to 0 from the beginning, in the so-called Stratonovich convention with $\Theta(0) = 1/2$, then $G(t) = \Theta(t)/\gamma_0$ for all t and $G(0) = 1/(2\gamma_0)$. This can lead to a so-called Jacobian extra-term in the action. If $F([\psi], t)$ is a function of $\psi(t)$ only (ultra-local functional), it reads $-1/(2\gamma_0) \int_u F_u'(\psi_u)$. It is invariant under time-reversal of the field $\psi_u \mapsto \psi_{-u}$ as long as F' is itself time-reversal invariant.

The kinetic term in $S^{\mathcal{J}}[c, c^*, \psi]$ can be re-written

$$\int_u \int_v c_u^* \partial_u^2 \delta_{u-v} \, c_v = \int_u c_u^* \partial_u^2 c_u + \Theta_0 \left[\dot{c}^* c - c^* \dot{c}\right]_{-T}^T + \Theta_0 \delta_0 \left[c^* c\right]_{-T}^T .$$

The last two terms in the RHS vanish by use of the boundary conditions ($c_{-T} = \dot{c}_{-T} = c_T^* = \dot{c}_T^* = 0$). The retarded friction can be re-written

$$\int_u \int_v c_u^* \partial_u \gamma_{u-v} \, c_v - \Theta_0 \int_u c_u^* \left[\gamma_{u+T} \, c_{-T} - \gamma_{u-T} \, c_T\right] ,$$

where the second term vanishes identically for two reasons: the boundary condition ($c_{-T} = 0$) kills the first part and the causality of the friction kernel ($\gamma_u = 0 \, \forall \, u < 0$) suppresses the second one. If there is a Dirac contribution to γ centered at $u = 0$ like in the white noise case, the other boundary condition ($c_{-T}^* = 0$) cancels the second part. Finally, we have

$$S^{\mathcal{J}}[c, c^*, \psi] = \int_u c_u^* \partial_u^2 c_u + \int_u \int_v c_u^* \left[\partial_u \gamma_{u-v} - \frac{\delta F_u[\psi]}{\delta \psi_v}\right] c_v . \qquad \text{(B.15)}$$

Appendix C.   Discrete MSRJD for multiplicative noise

The discretized Langevin equation reads:

$$\text{EQ}_{k-1} \equiv m \frac{\psi_{k+1} - 2\psi_k + \psi_{k-1}}{\epsilon^2} - F_k(\tilde{\psi}_k, \tilde{\psi}_{k-1}, ...)$$
$$+ M'(\tilde{\psi}_k) \, \epsilon \sum_{l=1}^k \gamma_{kl} M'(\tilde{\psi}_l) \frac{\psi_l - \psi_{l-1}}{\epsilon} = M'(\tilde{\psi}_k) \xi_k .$$

with $\tilde{\psi}_k \equiv a\psi_k + (1 - a)\psi_{k-1}$ and $k = 1, ..., N$. In the Markov limit ($m = 0$ and $\gamma_{kl} = \epsilon^{-1}\gamma_0 \delta_{kl}$) the results depend on a (see the discussion in Sect. 2.4). In the additive noise case, the choices $a = 0$ and $a = 1/2$ correspond to the Itô and Stratonovich conventions, respectively. However, we decide to stay out of the Markov limit: the results are then independent of a and we choose to work with $a = 1$. The probability for a field history is

$$P(\psi_0, \psi_1, ..., \psi_{N+1}) = |\mathcal{J}_N| P_\text{i}(\psi_1, \frac{\psi_1 - \psi_0}{\epsilon}) P_\text{n}(\widetilde{\text{EQ}}_0, ..., \widetilde{\text{EQ}}_{N-1}) , \qquad \text{(C.1)}$$

where we introduced the shorthand notation $\widetilde{\text{EQ}}_k \equiv \text{EQ}_k / M'(\psi_{k+1})$. The Jacobian is

$$\mathcal{J}_N \equiv \det\left(\frac{\partial \, (\psi_\text{i}, \dot{\psi}_\text{i}, \xi_1, \ldots, \xi_N)}{\partial \, (\psi_0, \psi_1, \ldots, \psi_{N+1})}\right) = \det\left(\frac{\partial \, (\psi_\text{i}, \dot{\psi}_\text{i}, \widetilde{\text{EQ}}_0, \ldots, \widetilde{\text{EQ}}_{N-1})}{\partial \, (\psi_0, \psi_1, \ldots, \psi_{N+1})}\right) . \text{(C.2)}$$

P_n is still given by expression (B.4) and $P_\text{n}(\widetilde{\text{EQ}}_0, ..., \widetilde{\text{EQ}}_{N-1})$ reads, after the substitution $\hat{\psi}_k \mapsto \hat{\psi}_k M'(\psi_k)$,

$$\mathcal{N}_N^{-1} \int d\hat{\psi}_0 ... d\hat{\psi}_{N+1} \, \delta(\hat{\psi}_0) \delta(\hat{\psi}_{N+1}) \, |\hat{\mathcal{J}}_N| \, e^{-\epsilon \sum_k i\hat{\psi}_k \text{EQ}_{k-1} + \frac{1}{2}\beta^{-1}\epsilon^2 \sum_{kl} i\hat{\psi}_k M'(\psi_k)\Gamma_{kl} M'(\psi_l) i\hat{\psi}_l} ,$$

where $\hat{\mathcal{J}}_N \equiv \det_{kl}\left(\delta_{kl}\, M'(\psi_k)\right)$ is the Jacobian of the previous substitution. The probability for a given history is therefore

$$P(\psi_0, \psi_1, ..., \psi_{N+1}) = \mathcal{N}_N^{-1} \int d\hat{\psi}_0 ... d\hat{\psi}_{N+1} \left|\mathcal{J}_N \hat{\mathcal{J}}_N\right|$$

$$\times e^{-\sum_k i\hat{\psi}_k \mathrm{EQ}_{k-1} + \frac{1}{2}\beta^{-1}\sum_{kl} i\hat{\psi}_k M'(\psi_k)\Gamma_{kl}M'(\psi_l)i\hat{\psi}_l + \ln P_i\left(\psi_1, \frac{\psi_1 - \psi_0}{\epsilon}\right)} .$$

The Jacobian $\mathcal{J}_N$ defined in eq. (C.2) reads

$$\mathcal{J}_N = \frac{1}{\epsilon}\det_{kl}\left(\frac{1}{M'(\psi_k)}\frac{\partial \mathrm{EQ}_{k-1}}{\partial \psi_{l+1}} - \frac{M''(\psi_k)}{M'(\psi_k)^2}\mathrm{EQ}_{k-1}\,\delta_{k\,l+1}\right)$$

$$= \frac{1}{\epsilon}\hat{\mathcal{J}}_N^{-1}\det_{kl}\left(\frac{\partial \mathrm{EQ}_{k-1}}{\partial \psi_{l+1}} - \frac{M''(\psi_k)}{M'(\psi_k)}\mathrm{EQ}_{k-1}\,\delta_{k\,l+1}\right) \qquad (C.3)$$

where k and l run from 1 to N. Causality is responsible for the triangular structure of the matrix involved in the last expression. The second term within the square brackets yields matrix elements below the main diagonal and these do not contribute to the Jacobian. Therefore, we find

$$\mathcal{J}_N \hat{\mathcal{J}}_N = \frac{1}{\epsilon}\prod_{k=1}^{N}\frac{\partial \mathrm{EQ}_{k-1}}{\partial \psi_{k+1}} = \frac{1}{\epsilon}\left(\frac{m}{\epsilon^2}\right)^N .$$

that is the same field-independent positive constant as in the additive noise case that can be dropped in the measure, see eq. (B.10).

A fermionic functional representation of the Jacobian can be obtained by introducing ghosts, expression (C.3) can be put in the form

$$\mathcal{J}_N \hat{\mathcal{J}}_N = \frac{1}{\epsilon}\frac{1}{\epsilon^N}\int dc_0 dc_0^* ... dc_{N+1}dc_{N+1}^*\, c_{N+1}^* c_N^* c_1 c_0\, e^{S_N^{\mathcal{J}}} ,$$

with

$$S_N^{\mathcal{J}} \equiv \epsilon^2 \sum_{k=0}^{N+1}\sum_{l=0}^{N+1} c_k^* \frac{1}{\epsilon}\frac{\partial \mathrm{EQ}_k}{\partial \psi_l} c_l - \epsilon \sum_{k=0}^{N+1} c_k^* \frac{M''(\psi_{k+1})}{M'(\psi_{k+1})}\mathrm{EQ}_k\, c_{k+1} .$$

In the continuum limit it becomes

$$S^{\mathcal{J}} \equiv \lim_{N \to \infty} S_N^{\mathcal{J}} = \int_u \int_v c_u^* \frac{\delta \mathrm{EQ}_u[\psi]}{\delta \psi_v} c_v - \int_u c_u^* \frac{M''(\psi_u)}{M'(\psi_u)}\mathrm{EQ}_u[\psi]\, c_u ,$$

with the boundary conditions $c(-T) = \dot{c}(-T) = 0$ and $c^*(T) = \dot{c}^*(T) = 0$ and the measure of the corresponding path integral is given in (B.14).

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