

A general and solvable random matrix model for spin decoherence

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Abstract. We propose and solve a simple but very general quantum model of a $SU(2)$ spin interacting with a large external system with N states. The coupling is described by a random Hamiltonian in a new general Gaussian $SU(2) \times U(N)$ random matrix ensemble, that we introduce in this paper. We solve the model in the large N limit, for any value of the spin j and for any choice of the coupling matrix element distributions in the different possible angular momentum channels l (and provided that the internal dynamics of the spin is slow). Besides its mathematical interest as a non-trivial random matrix model, it allows us to study and illustrate in a simple framework various phenomena: the decoherence dynamics, the conditions of emergence of the classical phase space for the spin, and the properties of quantum diffusion in phase space. The large time evolution for the spin is shown to be non-Markovian in general, the Markov property emerging in some specific cases for the dynamics and the initial conditions.

Keywords: matrix models, exact results, entanglement in extended quantum systems (theory), diffusion

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1. Introduction

In this paper we introduce and solve a simple but general model of decoherence for a very simple system: an $SU(2)$ quantum spin coupled in a generic way to a large external system, which plays the role of the environment (external bath, other microscopic degrees of freedom, etc). The purpose of this model is to provide a simple but quite general and exactly solvable model, in order to discuss and illustrate in a pedagogical way some of the basic aspects of decoherence and of the emergence of classical degrees of freedom in a simple quantum system. There is of course already a very large scientific literature on these problems and this study will not provide revolutionary insights into these subjects. There are numerous excellent reviews and textbooks on decoherence [18, 28, 11], and more generally on irreversibility and dissipation in open quantum systems [25, 5]. Many studies have focused on simple but realistic physical systems like a single free massive particle (Brownian walk) or a harmonic oscillator. A quantum spin is in fact simpler than a particle, since its phase space is compact and since the $SU(2)$ symmetry induces many simplifications. The spin model presented here does not aim at being a realistic one for some specific physical system in relation to experiments. It incorporates features already introduced and studied by previous authors: quantum spin, coherent states representation, random matrix ensembles. But it also presents novel features and allows us to obtain new and exact results.

Firstly our model describes a general spin, and the results that we obtain are valid for any value of the spin j , allowing us to study the whole range going from the case $j = 1/2$ (the most studied case of the two-level system, i.e. of the q -bit) to $j = \infty$ (the classical limit where the spin becomes a classical top). The study of dissipation and decoherence in spin systems goes back to [19, 22], but since then most studies have focused on the spin $1/2$ case, i.e. the two-level system, both for simplicity and for the obvious connection with experiments. See however section VII of [21] for a discussion of decoherence for a large spin j .

Secondly we try to describe the interaction between the spin \mathcal{S} and the external system \mathcal{E} in the most general situation, by considering an interaction Hamiltonian belonging to a random matrix ensemble. The idea of using random matrix theory (RMT) for such problems is of course not new. It can be found already in [15, 14]. However, as already stated, most studies involve simpler systems (for instance the two-state system corresponding to $j = 1/2$), and some simple random matrix ensembles like a single GUE ensemble or a Gaussian ensemble of band matrices. In addition, in almost all studies, the interaction Hamiltonian is taken to be of the form $H_{\text{int}} = U_{\mathcal{S}} \otimes V_{\mathcal{E}}$, $U_{\mathcal{S}}$ being a well chosen operator for \mathcal{S} (the coupling agent), and $V_{\mathcal{E}}$ the random Hamiltonian for \mathcal{E} . Here we look for the most general random matrix model which can describe the coupling of an $SU(2)$ spin to an external N -state system, irrespective of the value of the spin j . This leads us to ask the question of what are the most general random matrix ensembles which are invariant under the global unitary group $U(N)$ for the external system \mathcal{E} , and the $SU(2)$ symmetry group for the spin \mathcal{S} . We solve this problem and define a class of Gaussian $SU(2) \times U(N)$ matrix ensemble, that we denote as $\text{GU}_{2 \times N}\text{E}$. This class of random matrix ensembles is new (as far as we know) and has interesting properties. In particular they can be formulated and visualized in terms of random Wigner and Husimi distributions on the Riemann sphere.

Thirdly, in the limit $N \rightarrow \infty$ (i.e. when the external system becomes large) we are able to write closed equations, and to write in a fully closed form the evolution functional for the density matrix $\rho_S(t)$ of the spin \mathcal{S} (once the trace over the external system \mathcal{E} has been taken). In other words we do not need to make any approximation and to write any master equation for the evolution of the density matrix. Our calculations rely on the classical methods developed in the physics literature in [7, 6, 27] for studying sums and products of random matrices, and are closely related to the techniques of free probabilities in mathematics (see [24] and references therein). They are also in fact closely related to the methods used by [15] and more recently by [13, 12] for studying analytically the dissipation in two-level systems.

One important limitation of the model discussed here is that the internal dynamics of the quantum spin \mathcal{S} is completely neglected. This means that our results apply in the specific case where the internal dynamics of \mathcal{S} is much slower than the dynamics of the external system \mathcal{E} and than the dynamics on the whole system $\mathcal{S} + \mathcal{E}$ induced by the coupling. We are thus studying the case of ‘interaction dominated decoherence’ already put forward by [21]. This case is relevant for the study of decoherence and of quantum diffusion effects, but not for dissipation.

Rather than discussing the whole literature in this introduction (this would make it much too long), we prefer to present our model now. Our results will be discussed and compared with previous ones at the end of the paper in section 6.2 (the interested reader can go directly to this section). The paper is organized as follows.

In section 2 we introduce our model, define the random matrix ensembles that we shall use, and discuss the connection of these ensembles with the theory of Wigner and Husimi representation for spin operators.

In section 3 we study the evolution functional for the spin density matrix, for a general choice of random coupling Hamiltonian, characterized by a set of variances $\Delta(l)$ for the couplings in the different angular momentum channels l . We first write recursion relations for the evolution functional and the associated resolvent operators in the large N limit in section 3.2. We show that these relations take a simple closed form in each l channel when expressed in angular momentum components (their Wigner transform double Fourier components). The general solution is given in section 3.3. It involves a universal decoherence function $M(t, Z(l))$ which depends on the time t and also a parameter $Z(l)$ which is some linear transform of the variance distribution $\Delta(l')$, involving the $SU(2)$ spin structure through the Racah $6j$ -coefficients. The structure of this decoherence function as a function of the parameters $\Delta(l)$ and various scaling limits is discussed at length in section 3.5, and illustrated in the appendix. The relation between the parameters of the decoherence function and the norm of operators built out from the Hamiltonians and the spin operators is discussed in section 3.6.

In section 4 we study explicitly the decoherence of spin states in our model. We concentrate on the case where the dynamics of the external system \mathcal{E} is also given by a GUE Hamiltonian (for simplicity) and when the initial state for \mathcal{E} is generic. For large values of the spin j this is of course naturally discussed in terms of spin coherent states, whose properties are recalled in section 4.1. In section 4.2 we compute the various time scales for the system (which corresponds to τ_{dyn} a dynamical time scale of the whole system, τ_{dec} a decoherence time scale and τ_{diss} a dissipation time scale) as a function of the parameters of the model. We then compare the evolution of generic non-coherent states (random

states), coherent states and simple superpositions of coherent states (cat states). Our solution shows explicitly under which conditions (choices of the $\Delta(l)$) the spin coherent states emerge as robust states under decoherence, and therefore are the pointer states which allow us to describe the spin system and its dynamics in semiclassical terms. Our explicit solutions allow us to illustrate the dynamics of decoherence very easily through the dynamics of the Wigner representation of the spin quantum states. This is done in section 4.3 with several pictures and animations. In section 4.4 we study the large time evolution of the semiclassical coherent spin states. We show that through the coupling with \mathcal{E} , they undergo a slow quantum diffusion process on the classical phase space (the sphere \mathcal{S}_2). We show that this diffusion process exhibits universal self-similar properties, but is non-Markovian at all time scales. Hence in our case the evolution functional cannot be approximated by a master equation that is local in time.

In section 5 we extend our calculations to the more general case where the dynamics of the external system \mathcal{E} is given by an arbitrary Hamiltonian with a continuum energy spectrum, and when the initial state for \mathcal{E} is not taken to be a random state, but is an energy eigenstate $|E\rangle$. The coupling between the spin and \mathcal{E} is still described by a $\text{GU}_{2 \times N} \mathbb{E}$ Hamiltonian. In section 5.2 we show that we can still compute the influence functional explicitly, but that it depends on more parameters, in particular on the energy E on the initial external state. The discussion of decoherence in the simple case when the spectrum density $\rho(E)$ for \mathcal{E} is a semi-circle (the explicit results are simpler in this case) is provided in section 5.3. Finally in section 5.4 we concentrate on the case where the internal dynamics of the external system \mathcal{E} is much faster than the dynamics induced by the coupling between the spin \mathcal{S} and the \mathcal{E} . In this case we show explicitly that the large time evolution of coherent states is still described by a quantum diffusion process, which depends on the initial state for \mathcal{E} that we start from. The quantum diffusion process has the property that if we start from an energy eigenstate $|E\rangle$ for \mathcal{E} (times a coherent state for \mathcal{S} of course), the quantum diffusion process is now Markovian, and coincides with a diffusion process local in time and space (i.e. the Wiener process). The effective diffusion constant in phase space depends on E , and takes a simple form related to the Fermi golden rule, indicating a fluctuation dissipation relation. When we start from a general state $|\psi\rangle$ for \mathcal{E} , the diffusion process is not Markovian, but is argued to be described by a randomized Markov process. We propose a simple explanation for this randomization, through the decoherence for the energy states of the environment induced by the now large spin.

In section 6 we first summarize the main results for our model, then put it in the context of the existing literature (section 6.2), and finally discuss open problems and possible generalizations of our model (section 6.3).

2. The model

2.1. A spin coupled to its environment

We start from the standard paradigmatic system: a small system \mathcal{S} coupled to a large system \mathcal{E} (the environment). We take for \mathcal{S} a single $SU(2)$ spin with value j . This has several advantages; we can use the coherent states representation and the techniques of group representation theory, and we can easily study the emergence of the classical

degrees of freedom in the large spin limit $j \rightarrow \infty$. The coupling between \mathcal{S} and \mathcal{E} is formulated and studied with random matrix models. The advantages are that we can use the mathematical tools of random matrix theory and that the interactions that we consider have are generic.

The Hilbert spaces for the subsystems \mathcal{S} and \mathcal{E} , and the whole system $\mathcal{S} + \mathcal{E}$ are of course

$$\mathcal{H}_{\mathcal{S}} = \mathbb{C}^{2j+1}, \quad \mathcal{H}_{\mathcal{E}} = \mathbb{C}^N, \quad \mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}}. \quad (1)$$

We take as an orthonormal state basis for $\mathcal{H}_{\mathcal{S}}$ the S^3 eigenstates $\{|m\rangle, m = -j, \dots, j\}$; the $S^\mu, \mu = 1-3$, are the spin operators (the generators of $su(2)$). An orthonormal basis for $\mathcal{H}_{\mathcal{E}}$ is taken to be $\{|\alpha\rangle, \alpha = 1, \dots, N\}$. The Hamiltonian of the whole system is written (although this separation is partly arbitrary) as a sum of three terms:

$$H = H_{\mathcal{S}} \otimes \mathbf{1}_{\mathcal{E}} + H_{\mathcal{S}\mathcal{E}} + \mathbf{1}_{\mathcal{S}} \otimes H_{\mathcal{E}}. \quad (2)$$

$H_{\mathcal{S}}$ and $H_{\mathcal{E}}$ describe the internal dynamics of \mathcal{S} and \mathcal{E} respectively. $H_{\mathcal{S}\mathcal{E}}$ describes the coupling between \mathcal{S} and \mathcal{E} and is usually taken to be small (fast internal dynamics).

We are mostly interested in the decoherence of spin and we therefore assume that the dynamics of the spin is slow compared to the other dynamics. This important simplification will be discussed later. Therefore we simply set

$$H_{\mathcal{S}} = 0. \quad (3)$$

Moreover we first assume that the internal dynamics of \mathcal{E} does not play a special role and is at most as fast as the dynamics of the coupling $\mathcal{S} + \mathcal{E}$. We shall therefore recast $H_{\mathcal{S}}$ as $H_{\mathcal{S}\mathcal{E}}$, thus setting simply

$$H_{\mathcal{E}} = 0. \quad (4)$$

We shall go beyond this simplification in section 5. In the rest of this section and in the following section 4 we shall simply denote $H_{\mathcal{S}\mathcal{E}}$ by H_{int} or simply by H .

2.2. Gaussian $SU(2) \times U(N)$ random matrix ensembles

We are interested in the generic situation where H_{int} is taken to be a typical (or generic) Hamiltonian allowing the emergence of the spin as a classical degree of freedom. Therefore we assume that H_{int} is a random Hamiltonian belonging to some $M \times M$ random matrix ensemble, with

$$M = (2j + 1)N. \quad (5)$$

As explained in section 1, in most models H_{int} is taken to be of the form

$$H_{\text{int}} = U_{\mathcal{S}} \otimes V_{\mathcal{E}}, \quad (6)$$

$U_{\mathcal{S}}$ being an appropriately chosen operator for \mathcal{S} (the coupling agent), and $V_{\mathcal{E}}$ a random Hamiltonian for \mathcal{E} (sometimes it is a sum of a few terms like this). Here we look for a general random Hamiltonian, simply assuming (mostly for simplicity, but there are some physical motivations too in some specific models) that it belongs to some Gaussian ensemble. The question is thus: can we characterize the most general Gaussian ensembles (random matrix probability distributions) which are invariant under: (1) the $SU(2)$ group acting on $\mathcal{H}_{\mathcal{S}}$; (2) generic unitary transformations $U(N)$ acting on $\mathcal{H}_{\mathcal{E}}$? This does not mean that H is invariant, simply that the distribution of the H s is invariant.

2.2.1. *Spin decomposition of operators.* To answer this question, we simply use standard tools of group representation theory. We refer the reader to [23] as a clear reference on the theory of $SU(2)$ representations, coherent states and the theory of Wigner functions, that we shall use heavily. Let us first concentrate on the spin part, i.e. on the subspace \mathcal{H}_S . Any operator A acting on \mathcal{H}_S , i.e. $A \in \mathcal{B}(\mathcal{H}_S) = M_{2j+1}(\mathbb{C})$, can be considered as an element of the tensor product (or Kronecker product) of two spin j representations of $SU(2)$. Of course such a product is a sum of the irreducible representations

$$\mathbf{j} \otimes \mathbf{j} = \mathbf{0} \oplus \mathbf{1} \oplus \dots \oplus \mathbf{2j} = \bigoplus_{l=0}^{2j} \mathbf{l}. \quad (7)$$

Hence the operator A can be decomposed into its spin l components $A^{(l)}$:

$$A = \sum_{l=0}^{2j} A^{(l)}. \quad (8)$$

The matrix elements of the $A^{(l)}$ are

$$A_{rs}^{(l)} = \langle r | A^{(l)} | s \rangle = \sum_{m=-j}^j (-1)^m \sqrt{\frac{2l+1}{2j+1}} \left\langle \begin{matrix} j & l \\ s & -m \end{matrix} \middle| \begin{matrix} j \\ r \end{matrix} \right\rangle W_A^{(l,m)} \quad (9)$$

where the coefficients $W_A^{(l,m)}$ are given in terms of the matrix elements $A_{rs} = \langle r | A | s \rangle$ of A by the inverse transform

$$W_A^{(l,m)} = \sum_{r,s=-j}^j \sqrt{\frac{2l+1}{2j+1}} \left\langle \begin{matrix} j & l \\ r & m \end{matrix} \middle| \begin{matrix} j \\ s \end{matrix} \right\rangle A_{rs}. \quad (10)$$

The $\left\langle \begin{matrix} j & l \\ m_1 & m_2 \end{matrix} \middle| \begin{matrix} j \\ m_3 \end{matrix} \right\rangle$ are the $SU(2)$ Clebsch–Gordan coefficients. Of course only the single terms such that $m = s - r$ contribute in the sums.

The spin l component of A satisfies

$$[\vec{S}, [\vec{S}, A^{(l)}]] = \sum_{\mu=1}^3 [S^\mu, [S^\mu, A^{(l)}]] = l(l+1)A^{(l)}. \quad (11)$$

The coefficients $W_A^{(lm)}$ are complex but satisfy the conjugation constraint

$$W_{A^\dagger}^{(l,m)} = (-1)^m \overline{W_A^{(l,-m)}}. \quad (12)$$

2.2.2. *The relation to the Wigner and the Husimi distributions.*

The Wigner representation. The coefficients $W_A^{(l,m)}$ are nothing but the (l, m) coefficients of the Wigner distribution $W_A(\vec{n})$ associated with the operator A in the basis of spherical harmonics $Y_l^m(\vec{n})$ on the unit sphere \mathcal{S}_2 . The Wigner distribution is

$$W_A(\vec{n}) = \sum_{l=0}^{2j} \sum_{m=-j}^j W_A^{(l,m)} Y_l^m(\vec{n}). \quad (13)$$

With the normalization for the Y_l^m , we have

$$\int_{\mathcal{S}_2} d^2\vec{n} Y_l^m(\vec{n}) \overline{Y_{l'}^{m'}(\vec{n})} = \delta_{l,l'} \delta_{m,m'}, \quad d^2\vec{n} = d\theta d\phi \sin(\theta). \quad (14)$$

With these normalizations we have

$$\text{tr}(AB^\dagger) = \int_{\mathcal{S}_2} d^2\vec{n} W_A(\vec{n}) \overline{W_B(\vec{n})} = \sum_{l,m} W_A^{(l,m)} \overline{W_B^{(l,m)}} \quad (15)$$

and for the unit operator and the trace

$$W_{\mathbf{1}}^{(l,m)} = \sqrt{2j+1} \delta_{l,0} \delta_{m,0}, \quad \text{tr}(A) = \sqrt{2j+1} W_A^{(0,0)}. \quad (16)$$

Note that in the literature the normalization for the W_A is often such that

$$\text{tr}(AB^\dagger) = \frac{2j+1}{4\pi} \int_{\mathcal{S}_2} d^2\vec{n} \mathbf{W}_A(\vec{n}) \overline{\mathbf{W}_B(\vec{n})}. \quad (17)$$

It corresponds to the change of normalization in the definition of the Wigner distribution

$$\mathbf{W}_A(\vec{n}) = \sqrt{\frac{4\pi}{2j+1}} W_A(\vec{n}). \quad (18)$$

The Husimi representation. Note that we can also construct the Q -symbol or Husimi representation

$$Q_A(\vec{n}) = \sum_{l,m} \left\langle \begin{matrix} j & l \\ j & 0 \end{matrix} \middle| \begin{matrix} j \\ j \end{matrix} \right\rangle W_A^{(l,m)} Y_l^m(\vec{n}) \quad (19)$$

and the P -symbol

$$P_A(\vec{n}) = \sum_{l,m} \left\langle \begin{matrix} j & l \\ j & 0 \end{matrix} \middle| \begin{matrix} j \\ j \end{matrix} \right\rangle^{-1} W_A^{(l,m)} Y_l^m(\vec{n}). \quad (20)$$

They are such that

$$\text{tr}(AB^\dagger) = \int_{\mathcal{S}_2} d^2\vec{n} Q_A(\vec{n}) \overline{P_B(\vec{n})}. \quad (21)$$

The Husimi function $Q_A(\vec{n})$ corresponds to the ‘physical’ probability distribution of spin in phase space, since with the proper normalization, we have

$$\langle \vec{n} | A | \vec{n} \rangle = \mathbf{Q}_A(\vec{n}), \quad \mathbf{Q}_A(\vec{n}) = \sqrt{\frac{4\pi}{2j+1}} Q_A(\vec{n}), \quad |\vec{n}\rangle \text{ coherent state.} \quad (22)$$

2.2.3. Gaussian SU_2 random matrix ensembles. In order to construct the most general Gaussian $SU(2)$ invariant ensemble on the self-adjoint matrices in $M_{2j+1}(\mathbb{C})$, we simply have to take separately each matrix component $A^{(l)}$ as Gaussian independent random (non-commutative) variables with zero mean and variance depending on l . Namely, we take the $W^{(l,m)}$ to be independent Gaussian variables, subject only to the Hermiticity constraint

$$W^{(l,m)} = (-1)^m \overline{W^{(l,-m)}} \quad (23)$$

which ensures that the A are Hermitian operators, and to the condition that the variance depends on l but not on m , which ensures $SU(2)$ invariance of the distribution. More precisely, we take the W s to be for $m = 0$

$$W^{(l,0)} = A^{(l)} \quad (24)$$

and for $m > 0$

$$W^{(l,m)} = B^{(l,m)} + iC^{(l,m)}, \quad W^{(l,-m)} = (-1)^m (B^{(l,m)} - iC^{(l,m)}) \quad \text{for } 0 < m \leq l \quad (25)$$

and take for the $A^{(l)}$, $B^{(l,m)}$ and $C^{(l,m)}$ random Gaussian independent variables with zero mean and mean square extent $\Delta(l)$ depending only on l . The cumulants are

$$\mathbb{E}[A^{(l)}] = \mathbb{E}[B^{(l,m)}] = \mathbb{E}[C^{(l,m)}] = 0 \quad (26)$$

$$\mathbb{E}[A^{(l)} A^{(l)}] = \mathbb{E}[B^{(l,m)} B^{(l,m)}] = \mathbb{E}[C^{(l,m)} C^{(l,m)}] = \Delta(l). \quad (27)$$

All the others cumulants are zero. The $\Delta(l)$ are a collection $\mathbf{\Delta}$ of $2j + 1$ positive numbers

$$\Delta(l) \geq 0, \quad l = 0, \dots, 2j \quad (28)$$

which completely characterize the $SU(2)$ Gaussian ensemble. This distribution is given by the Gaussian probability measure on self-adjoint $(2j + 1) \times (2j + 1)$ matrices

$$\mathcal{D}_{\mathbf{\Delta}}[A] \propto dA \exp \left[- \sum_{l=0}^{2j} \frac{1}{2\Delta(l)} \text{tr}[A^{(l)2}] \right] \quad (29)$$

where dA is the standard flat measure, such that $\mathbb{E}[F[A]] = \int \mathcal{D}_{\mathbf{\Delta}}[A] F[A]$.

In this GU_2E ensemble, characterized by $\mathbf{\Delta}$, the ‘propagator’ $\mathcal{D}_{rs,tu}$ is

$$\mathcal{D}_{rs,tu} = \mathbb{E}[A_{rs} A_{tu}] = \delta_{s-r,t-u} \sum_{l=0}^{2j} \Delta(l) \frac{2l+1}{2j+1} \left\langle \begin{matrix} j & l \\ s & r-s \end{matrix} \middle| \begin{matrix} j \\ r \end{matrix} \right\rangle \left\langle \begin{matrix} j & l \\ t & u-t \end{matrix} \middle| \begin{matrix} j \\ u \end{matrix} \right\rangle. \quad (30)$$

Of course, if all the $\Delta(l)$ are equal to the same Δ , one recovers the standard GUE ensemble for $(2j + 1) \times (2j + 1)$ matrices, with

$$\mathbb{E}[A_{rs} A_{tu}]_{\text{GUE}} = \Delta \delta_{r,u} \delta_{s,t}. \quad (31)$$

2.2.4. Gaussian $SU_2 \times U_N$ random matrix ensembles. We now take into account the external system \mathcal{E} and wish to characterize the most general Gaussian ensemble of self-adjoint matrices $H \in \mathcal{B}(\mathcal{H}) = M_{(2j+1)N}(\mathbb{C})$ which is invariant under $SU(2)$ (acting on $\mathcal{H}_{\mathcal{S}}$ as the spin j representation) and under $U(N)$ (acting on $\mathcal{H}_{\mathcal{E}}$ as the fundamental representation).

The solution is simple. We make the same decomposition w.r.t. the spin sector $\mathcal{H}_{\mathcal{S}}$, keeping the sector $\mathcal{H}_{\mathcal{E}}$ untouched. H is decomposed as

$$H = \sum_l H^{(l)}. \quad (32)$$

The matrix elements of H are now denoted as

$$H_{rs}^{\alpha\beta} = \langle r\alpha | H | s\beta \rangle, \quad |r\alpha\rangle = |r\rangle \otimes |\alpha\rangle \quad (33)$$

and can be written as

$$H_{rs}^{\alpha\beta} = \sum_{l=0}^{2j} \sum_{m=-j}^j (-1)^m \sqrt{\frac{2l+1}{2j+1}} \left\langle \begin{matrix} j & l \\ s & -m \end{matrix} \middle| \begin{matrix} j \\ r \end{matrix} \right\rangle W_{\alpha\beta}^{(l,m)}. \quad (34)$$

The hermiticity constraint $H = H^\dagger$ for H reads now

$$W_{\alpha\beta}^{(l,m)} = (-1)^m \overline{W_{\beta\alpha}^{(l,-m)}}. \quad (35)$$

The $\text{GU}_{2,N}\text{E}$ ensemble is obtained by taking the $W_{\alpha,\beta}^{(l,m)}$ to be Gaussian independently distributed random variables, subject to the constraint of equation (35), with zero mean and a variance depending only on l , but not on m , α and β . This ensures the invariance under the group $SU(2) \times U(N)$ of the probability distribution on H . The ‘propagator’ \mathcal{D} is now the product of the propagator \mathcal{D} for the GU_2E ensemble (given by equation (30)) and the propagator for the standard GUE model:

$$\mathcal{D}_{rs,tu}^{\alpha\beta,\gamma\delta} = \mathbb{E}[A_{rs}^{\alpha\beta} A_{tu}^{\gamma\delta}] = \mathcal{D}_{rs,tu} \delta_{\alpha,\delta} \delta_{\beta,\gamma}. \quad (36)$$

This ensemble is characterized by the same ensemble $\Delta = \{\Delta(l), l = 0, \dots, 2j\}$ of positive parameters.

Let us note that the sector $l = 0$ gives a Hamiltonian $H^{(0)}$ independent of the spin, since its matrix elements are of the form

$$H_{rs}^{(0)\alpha\beta} = \delta_{r,s} H_{(0)}^{\alpha\beta}. \quad (37)$$

Hence $H^{(0)}$ can be written as

$$H^{(0)} = \mathbf{1}_S \otimes H_{(0)} \quad (38)$$

where $H_{(0)}$ is a random Hamiltonian for \mathcal{E} , whose distribution is given by the GUE ensemble with $(\text{variance})^2 = \Delta(0)$.

Finally we note that the Hamiltonian can of course be rewritten in the general form

$$H = \sum_{(l,m)} D_S^{(l,m)} \otimes W_{\mathcal{E}}^{(l,m)} \quad (39)$$

where the $D_S^{(l,m)}$ are some fixed spin operators (related to the Wigner D -matrices) and the $W_{\mathcal{E}}^{(l,m)}$ are some random operators on \mathcal{E} . Previous studies of spin decoherence have dealt with simpler interaction Hamiltonians $H_{S\mathcal{E}}$ with only one $D \otimes W$ term or a few (typically three terms when dealing with an $l = 1$ interaction). Here we keep all the possible terms in the decomposition (39). But we shall stay with the explicit form (34) for the decomposition of H .

2.3. Rescaled distributions with N and j

We thus take for our model a single spin j , with for simplicity no internal dynamics, $H_S = 0$, and for the Hamiltonian $H_{S\mathcal{E}}$ describing the coupling between the spin and its environment a random Hamiltonian in the $SU(2) \times U(N)$ random Gaussian ensemble characterized by the family of coupling amplitudes $\Delta = \{\Delta(l), l = 0, \dots, 2j\}$ in the different spin channels l . In this paper we are interested in the limit $N \rightarrow \infty$ (large environment) and $j \rightarrow \infty$ (classical spin). In these limits it is adequate to rescale these amplitudes (i.e. the time scale for the evolution of the system). For clarity we define here these rescaled parameters as

$$\tilde{\Delta}(l) = N\Delta(l), \quad \bar{\Delta}(l) = \frac{N}{2j+1}\Delta(l). \quad (40)$$

As we shall see later, the large N limit is obtained when keeping the $\tilde{\Delta}(l)$ of order $\mathcal{O}(1)$, while the large N and large j limit is obtained when keeping the $\bar{\Delta}(l)$ of order $\mathcal{O}(1)$.

3. The evolution functional

3.1. The general framework

We start at time $t = 0$ from a separable quantum state

$$\rho(0) = \rho^{\mathcal{S}}(0) \otimes \rho^{\mathcal{E}}(0). \tag{41}$$

We do not specify at that stage whether the system \mathcal{S} is in a pure or in a mixed quantum state. Since the ensemble for the Hamiltonian H is chosen to be invariant under arbitrary unitary transformations $U \in U(N)$ acting on $\mathcal{H}_{\mathcal{S}}$, and since we shall compute only averages w.r.t. the distributions of H , the precise initial quantum state $\rho^{\mathcal{E}}(0)$ is not important for what we are interested in. We may choose it to be the maximal entropy state

$$\rho^{\mathcal{E}}(0) = \frac{1}{N} \mathbf{1}_{\mathcal{E}}. \tag{42}$$

After evolution from time 0 to $t > 0$ of the whole system under the time independent Hamiltonian H , the reduced density matrix for the system \mathcal{S} is

$$\rho^{\mathcal{S}}(t) = \frac{1}{N} \text{tr}_{\mathcal{E}}(e^{-itH}(\rho^{\mathcal{S}}(0) \otimes \mathbf{1}_{\mathcal{E}})e^{itH}). \tag{43}$$

We are interested in the evolution of the model for a ‘typical’ Hamiltonian H in our $\text{GU}_{2 \times N} \text{E}$ ensemble, and we shall therefore take averages over H in this ensemble. The average reduced density matrix

$$\overline{\rho^{\mathcal{S}}}(t) = \mathbb{E}[\rho^{\mathcal{S}}(t)] \tag{44}$$

is sufficient for studying the observables of the subsystem. However in order to study decoherence and the evolution of the entanglement between the systems \mathcal{S} and \mathcal{E} , we must study functions of the reduced density matrix such as the von Neumann entropy S or the purity P (or whichever quantities you prefer):

$$S = -\text{tr}_{\mathcal{S}}[\rho^{\mathcal{S}} \log \rho^{\mathcal{S}}], \quad P = \text{tr}_{\mathcal{S}}[\rho^{\mathcal{S}} \rho^{\mathcal{S}}]. \tag{45}$$

It is well known that simplifications occur for such observables in the limit $N \rightarrow \infty$ where the number of degrees of freedom of the external system \mathcal{E} becomes infinite, with an adequate rescaling of time and of the couplings. Indeed, as we shall discuss in section 3.2, for ‘reasonable’ functions F of the density matrix $\rho^{\mathcal{S}}$ (and at least for polynomial functions), in the large N limit we have the factorization of the expectation of products of traces with respect to \mathcal{E} due to the fact that in this limit, only planar diagrams contribute in the perturbative expansions. This means that we have in fact¹

$$\overline{F(\rho^{\mathcal{S}})} = \mathbb{E}[F(\rho^{\mathcal{S}})] = F(\overline{\rho^{\mathcal{S}}})(1 + \mathcal{O}(N^{-2})). \tag{46}$$

This phenomenon is known in physics as ‘factorization’ or emergence of a large N master field. In mathematics it is known as the phenomenon of ‘concentration of measures’. It leads to the formulation of the $N \rightarrow \infty$ limit in terms of free probabilities.

It turns out that these kinds of Gaussian ensembles are self-averaging (this is discussed at length in the mathematical literature; see for instance [12]), but we shall not elaborate further on this point.

¹ In the rest of this paper we shall use the physics notation \overline{F} for the average of F rather than the probabilistic notation $\mathbb{E}[F]$ for the expectation of F , but this is of course the same quantity.

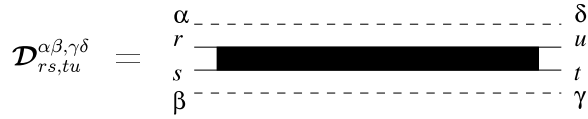


Figure 1. Diagrammatic representation of the propagator $\mathcal{D}_{rs,tu}^{\alpha\beta,\gamma\delta}$.

Thus we have to consider the evolution operator $\mathcal{M}(t)$ for the density matrix

$$\overline{\rho}_{ru}^{\mathcal{S}}(t) = \sum_{st} \mathcal{M}_{ru,st}(t) \rho_{st}^{\mathcal{S}}(0). \tag{47}$$

$\mathcal{M}(t)$ corresponds to the Feynman–Vernon influence functional; it is of course a strictly positive linear trace preserving application on $M_{2j+1}(\mathbb{C})$, and thus it is also called a quantum channel, or a POVM. Its matrix elements are given by

$$\mathcal{M}_{ru,st}(t) = \frac{1}{N} \overline{\sum_{\alpha,\beta} \langle r\alpha | e^{-itH} | s\beta \rangle \langle t\beta | e^{itH} | u\alpha \rangle}. \tag{48}$$

3.2. Perturbation theory and the planar limit

3.2.1. Resolvents. It is more convenient to consider the double resolvent

$$\mathcal{G}_{ru,st}(x, y) = \frac{1}{N} \overline{\sum_{\alpha,\beta} \langle r\alpha | (x - H)^{-1} | s\beta \rangle \langle t\beta | (y - H)^{-1} | u\alpha \rangle}. \tag{49}$$

Formally (integration paths to be discussed later),

$$\mathcal{M}_{ru,st}(t) = \oint \frac{dx}{2i\pi} \oint \frac{dy}{2i\pi} e^{-it(x-y)} \mathcal{G}_{ru,st}(x, y). \tag{50}$$

We consider first the single resolvent

$$\mathcal{H}_{rs}(x) = \frac{1}{N} \overline{\sum_{\alpha} \langle r\alpha | (x - H)^{-1} | s\alpha \rangle}. \tag{51}$$

To compute these generating functions, we use the standard diagrammatic techniques (see for instance [15, 27] and references therein). We represent the propagator $\mathcal{D}_{rs,tu}^{\alpha\beta,\gamma\delta}$ as a double fat line, depicted in figure 1. The dashed lines represent the two external tensors $\delta_{\alpha,\delta}$ and $\delta_{\beta,\gamma}$, and indicate that the \mathcal{E} indices (Greek letters) are conserved. The black ribbon represents the spin tensor $\mathcal{D}_{rs,tu}$. It indicates that the spin indices (italic letters) are mixed. But the difference between the left and right indices is conserved:

$$s - t = r - u. \tag{52}$$

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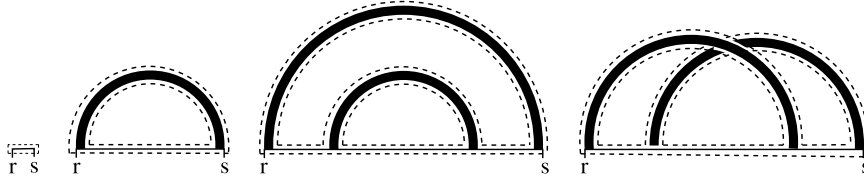


Figure 2. The first diagrams for $\mathcal{H}_{rs}(x)$.

3.2.2. *The recursion equation for the single resolvent.* To compute $\mathcal{H}_{rs}(x)$ we expand in a power series in x^{-1} :

$$\mathcal{H}_{rs}(x) = \sum_{k=0}^{\infty} x^{-1-k} \frac{1}{N} \overline{\sum_{\alpha} \langle r\alpha | H^k | s\alpha \rangle} \tag{53}$$

and use the Wick theorem to compute the average $\overline{H^k}$. We get a sum of contributions associated with diagrams of the form depicted in figure 2. The propagators form arches above a line going from r to s . Each arch gives a term \mathcal{D} proportional to the $\Delta(l)$, and each closed dashed loop gives a factor N (the sum over the \mathcal{E} indices α). So each diagram is of order

$$x^{-1-2\#\text{arches}} [\Delta]^{\#\text{arches}} N^{\#\text{loops}-1} = x^{-1} (x^{-2}[\Delta] N)^{\#\text{arches}} N^{-\chi}, \tag{54}$$

χ being the Euler characteristic of the fat diagram. $[\Delta]$ means any $\Delta(l)$. Thus in the large N limit, only planar diagrams (rainbow like) survive, provided we rescale the variances with N :

$$\tilde{\Delta}(l) = N\Delta(l) = \mathcal{O}(1) \quad N \rightarrow \infty. \tag{55}$$

$\mathcal{H}_{rs}(x)$ then satisfies the recursion equation

$$\mathcal{H}_{rs}(x) = x^{-1}\delta_{r,s} + x^{-1} \sum_{t,u,v} \tilde{\mathcal{D}}_{rt,uv} \mathcal{H}_{tu}(x) \mathcal{H}_{vs}(x) \tag{56}$$

with

$$\tilde{\mathcal{D}}_{rs,tu} = N \mathcal{D}_{rs,tu} = \delta_{s-r,t-u} \sum_{l=0}^{2j} \tilde{\Delta}(l) \frac{2l+1}{2j+1} \left\langle \begin{matrix} j & l \\ s & r-s \end{matrix} \middle| \begin{matrix} j \\ r \end{matrix} \right\rangle \left\langle \begin{matrix} j & l \\ t & u-t \end{matrix} \middle| \begin{matrix} j \\ u \end{matrix} \right\rangle. \tag{57}$$

This recursion equation is depicted graphically in figure 3. The solution is of the form, in fact required by $SU(2)$ invariance,

$$\mathcal{H}_{rs}(x) = \delta_{r,s} \mathcal{H}(x). \tag{58}$$

Inserting this ansatz (58) into (56) and (57) we obtain a simple recursion equation for $\mathcal{H}(x)$:

$$\mathcal{H}(x) = x^{-1} + x^{-1} \mathcal{H}(x)^2 \hat{\Delta} \quad \text{with } \hat{\Delta} = \sum_{l=0}^{2j} \frac{2l+1}{2j+1} \tilde{\Delta}(l). \tag{59}$$

Hence

$$\mathcal{H}(x) = \frac{1}{2\hat{\Delta}} \left(x - \sqrt{x^2 - 4\hat{\Delta}} \right) = x^{-1} \text{Cat}(\hat{\Delta}x^{-2}) \tag{60}$$

where $\text{Cat}(z) = \sum z^n C_n$ is the generating function of the Catalan numbers C_n .

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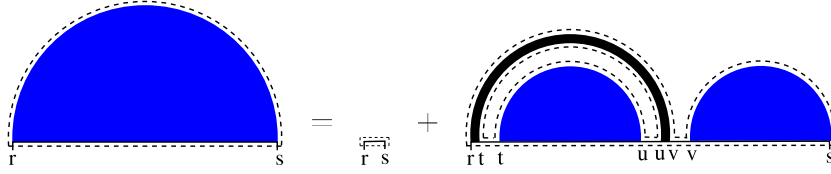


Figure 3. Graphical formulation of the recursion relation, equation (56), for $\mathcal{H}_{rs}(x)$.

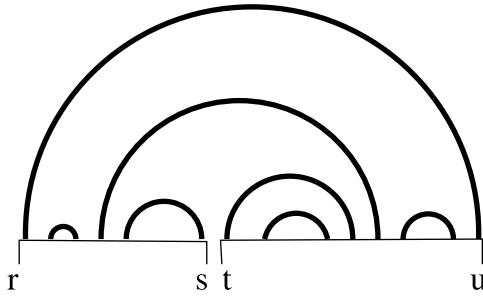


Figure 4. The planar diagrams of $\mathcal{G}_{ru,st}(x, y)$.

3.2.3. *The recursion equation for the double resolvent.* Now we can compute the function $\mathcal{G}_{ru,st}(x, y)$. It is given by the sum of the planar diagrams of the form given in figure 4. It thus obeys the recursion relation depicted in figure 5,

$$\mathcal{G}_{ru,st}(x, y) = x^{-1}\delta_{r,s}\mathcal{H}_{tu}(y) + x^{-1}\tilde{\mathcal{D}}_{rv,wx}\mathcal{H}_{vw}(x)\mathcal{G}_{xu,st}(x, y) + y^{-1}\tilde{\mathcal{D}}_{rv,wx}\mathcal{G}_{vw,st}(x, y)\mathcal{H}_{xu}(y). \quad (61)$$

To solve this equation, it is better to use its $SU(2)$ invariance properties, and to rewrite it with its ‘double-Wigner-transform’ coefficients

$$W_{\mathcal{G}}^{(l_1, m_1), (l_2, m_2)}(x, y) = \sum_{r, u=-j}^j \sum_{s, t=-j}^j \sqrt{\frac{2l_1+1}{2j+1}} \left\langle \begin{matrix} j & l_1 \\ u & m_1 \end{matrix} \middle| \begin{matrix} j \\ r \end{matrix} \right\rangle \times \sqrt{\frac{2l_2+1}{2j+1}} \left\langle \begin{matrix} j & l_2 \\ s & m_2 \end{matrix} \middle| \begin{matrix} j \\ t \end{matrix} \right\rangle \mathcal{G}_{ru,st}(x, y). \quad (62)$$

Indeed, we re-express the initial propagator, $\tilde{\mathcal{D}}_{rs,tu}$ (given by (57)) in the $(s, t) \rightarrow (u, r)$ channel, as

$$\mathcal{D}_{rs,tu} = \hat{\mathcal{D}}_{ru,st} \quad \text{i.e.} \quad \begin{array}{c} r & & u \\ & \diagdown & / \\ & \text{---} & \\ & / & \diagdown \\ s & & t \end{array} \longrightarrow \begin{array}{c} r & & u \\ & / & \diagdown \\ & \text{---} & \\ & \diagdown & / \\ s & & t \end{array}. \quad (63)$$

The corresponding double Wigner transform of $\hat{\mathcal{D}}_{ru,st}$ is

$$W_{\hat{\mathcal{D}}}^{(l_1, m_1), (l_2, m_2)} = \sum_{r, u=-j}^j \sum_{s, t=-j}^j \sqrt{\frac{2l_1+1}{2j+1}} \left\langle \begin{matrix} j & l_1 \\ u & m_1 \end{matrix} \middle| \begin{matrix} j \\ r \end{matrix} \right\rangle \sqrt{\frac{2l_2+1}{2j+1}} \left\langle \begin{matrix} j & l_2 \\ s & m_2 \end{matrix} \middle| \begin{matrix} j \\ t \end{matrix} \right\rangle \hat{\mathcal{D}}_{ru,st}. \quad (64)$$

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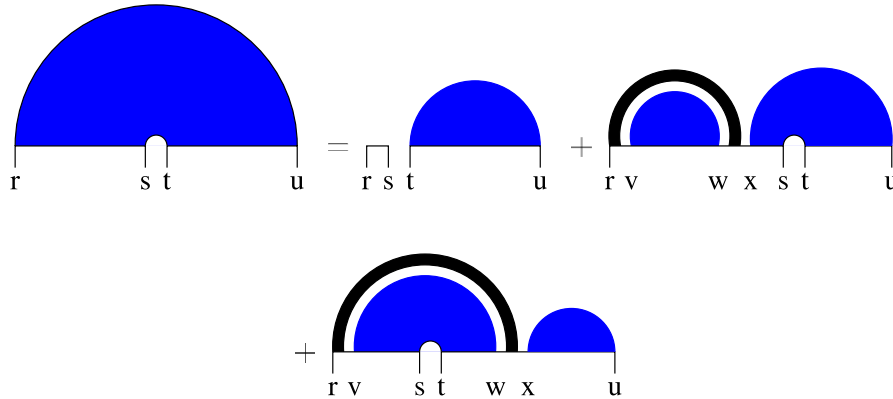


Figure 5. Graphical formulation of the recursion relation of equation (61) for $\mathcal{G}_{ru,st}(x, y)$.

Using the original expression (30) for $\mathcal{D}_{rs,tu}$, this sum is rewritten as a multiple sum over products of four Clebsch–Gordan coefficients. After some $SU(2)$ algebra, it can be reduced to the simple form

$$W_{\mathcal{D}}^{(l_1, m_1), (l_2, m_2)} = \delta_{l_1, l_2} \delta_{m_1 + m_2, 0} (-1)^{m_1} \hat{\Delta}(l_1) \tag{65}$$

with $\hat{\Delta}(l_1)$ given by

$$\hat{\Delta}(l_1) = \sum_{l'=0}^{2j} \tilde{\Delta}(l') (2l' + 1) (-1)^{2j+l'+l_1} \left\{ \begin{matrix} j & j & l' \\ j & j & l_1 \end{matrix} \right\} \tag{66}$$

where $\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}$ is the Racah 6 j symbol. In particular, $\hat{\Delta}(0)$ is nothing but the $\hat{\Delta}$ of equation (59):

$$\hat{\Delta}(0) = \hat{\Delta} = \sum_{l=0}^{2j} \frac{2l + 1}{2j + 1} \tilde{\Delta}(l). \tag{67}$$

Remember that the $\tilde{\Delta}$ s are just the original Δ s rescaled by a factor of N , $\tilde{\Delta}(l) = N\Delta(l)$.

The constraints $l_1 = l_2$ and $m_1 + m_2 = 0$ are very important! They just express the $SU(2)$ invariance of $\mathcal{D}_{rs,tu}$, i.e. the initial constraint $s - t = r - u$. But when we take the Wigner transform of the recursion equation (61) for $\mathcal{G}_{ru,st}(x, y)$, one sees that they are preserved by the equation and that the solution for $W_{\mathcal{G}}^{(l_1, m_1), (l_2, m_2)}(x, y)$ must be of the same form as $W_{\mathcal{D}}$:

$$W_{\mathcal{G}}^{(l_1, m_1), (l_2, m_2)}(x, y) = \delta_{l_1, l_2} \delta_{m_1 + m_2, 0} (-1)^{m_1} \hat{\mathcal{G}}^{(l_1)}(x, y). \tag{68}$$

We can now come back to the recursion equation (61). Using (62) and (68) it factorizes into independent equations for each $\hat{\mathcal{G}}^{(l_1)}(x, y)$:

$$\hat{\mathcal{G}}^{(l)}(x, y) = x^{-1} \mathcal{H}(y) + x^{-1} \hat{\Delta}(0) \mathcal{H}(x) \hat{\mathcal{G}}^{(l)}(x, y) + y^{-1} \hat{\Delta}(l) \hat{\mathcal{G}}^{(l)}(x, y) \mathcal{H}(y). \tag{69}$$

The solution is simply (using the explicit form (60) for \mathcal{H})

$$\hat{\mathcal{G}}^{(l)}(x, y) = \frac{\mathcal{H}(x) \mathcal{H}(y)}{1 - \hat{\Delta}(l) \mathcal{H}(x) \mathcal{H}(y)}. \tag{70}$$

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3.3. The solution for the evolution functional

3.3.1. *The general form.* We can now obtain the influence functional $\mathcal{M}(t)$. As we shall see, the functions $\hat{\mathcal{G}}^{(l)}(x, y)$ are analytic in x and y around ∞ , and have a cut in the x and y planes along $[-2\sqrt{\hat{\Delta}}, -2\sqrt{\hat{\Delta}}]$. We can integrate in x and y along a closed anticlockwise curve around the cut to obtain the double Wigner transform of the influence functional $\mathcal{M}_{ru,st}(t)$:

$$W_{\mathcal{M}}^{(l_1, m_1), (l_2, m_2)}(t) = \delta_{l_1, l_2} \delta_{m_1 + m_2, 0} (-1)^{m_1} \hat{\mathcal{M}}^{(l_1)}(t) \quad (71)$$

where

$$\hat{\mathcal{M}}^{(l)}(t) = \oint \frac{dx}{2i\pi} \oint \frac{dy}{2i\pi} e^{-it(x-y)} \hat{\mathcal{G}}^{(l)}(x, y). \quad (72)$$

Therefore, the evolution of the reduced density matrix $\rho^S(t)$ becomes a separate simple linear evolution in each (l, m) sector when one considers the components of its Wigner transform. More precisely, if $W_{\rho^S}^{(l, m)}(t)$ is the (l, m) harmonic, given by (10), we have simply

$$W_{\rho^S}^{(l, m)}(t) = \hat{\mathcal{M}}^{(l)}(t) W_{\rho^S}^{(l, m)}(0). \quad (73)$$

Then using (9) we can reconstruct $\rho^S(t)$ in the $|r\rangle\langle s|$ basis.

3.3.2. *The general decoherence function.* Thanks to the $SU(2)$ invariance, the evolution functional reduces to a single function in each l sector. This function depends on time and on the distributions of the $\Delta(l)$ which measure the strength of the coupling between the spin and the environment in the different angular momentum sectors l . This function depends in fact only on two parameters, since it can be rewritten as

$$\hat{\mathcal{M}}^{(l)}(t) = M(t/\tau_0, Z(l)), \quad (74)$$

where τ_0 is a time scale:

$$\tau_0 = 1/\sqrt{\hat{\Delta}(0)}, \quad \hat{\Delta}(0) = \sum_{l=0}^{2j} \frac{2l+1}{2j+1} \tilde{\Delta}(l) \quad (75)$$

and $Z(l)$ a parameter depending on the angular momentum l :

$$Z(l) = \frac{\hat{\Delta}(l)}{\hat{\Delta}(0)} = \frac{\sum_{l'=0}^{2j} (2l'+1) (-1)^{2j+l'+l_1} \left\{ \begin{matrix} j & j & l' \\ j & j & l_1 \end{matrix} \right\} \tilde{\Delta}(l')}{\sum_{l'=0}^{2j} (2l'+1)/(2j+1) \tilde{\Delta}(l')} \quad (76)$$

and the decoherence function $M(t, Z)$ is simply

$$M(t, Z) = \oint \frac{dx}{2i\pi} \oint \frac{dy}{2i\pi} e^{-it(x-y)} \frac{H(x)H(y)}{1 - Z H(x)H(y)}, \quad H(x) = \frac{1}{2}(x - \sqrt{x^2 - 4}). \quad (77)$$

$H(x)$ is nothing but the resolvent of the standard normalized GUE ensemble. It is the Hilbert–Stieltjes transform of the Wigner–Dyson semi-circle density distribution. It has a cut along the interval $[-2, 2]$, behaves as x^{-1} at ∞ and its modulus is $|H(x)| < 1$ for

$x \in \mathbb{C} \setminus [-2, 2]$. Hence the function $M(t, Z)$ is well defined for any real t , and analytic in the disc $|Z| \leq 1$. We shall discuss its properties below.

Thus we have a completely closed and simple formula for the evolution functional of a spin coupled to a large environment via a random coupling Hamiltonian which belongs to an $SU(2) \times SU(N)$ invariant ensemble. Our formula is valid for any value of the spin j , going from $j = 1/2$ (the q -bit or two-level system) to $j \rightarrow \infty$ (the classical spin), and for any distribution $\Delta(l)$ of the strength of the couplings as a function of the total spin l exchanged via the interaction.

Our result separates into two parts: (1) the universal decoherence function $M(t, z)$ which comes from the RMT part of the calculation; (2) the parameters $\hat{\Delta}(l)$ which depend linearly from the initial distribution $\Delta(l)$ of the couplings as a function of the angular momentum l , which come from the $SU(2)$ group theory part of the calculation. They give simply the time scale τ_0 and the parameter $Z(l)$.

3.4. Properties of the decoherence function $M(t, Z)$

3.4.1. Analytic representation. Making the standard inversion of variables $x \rightarrow H$ as in [27] (i.e. going from the Green function $H(x)$ to the so-called ‘Blue function’ $B(w)$):

$$w = H(x) \iff x = B(w) = w + w^{-1} \tag{78}$$

we rewrite M as

$$M(t, z) = \oint \frac{dw_1}{2i\pi} \oint \frac{dw_2}{2i\pi} e^{-itB(w_1)} e^{itB(w_2)} \frac{(w_1 - w_1^{-1})(w_2 - w_2^{-1})}{1 - zw_1w_2} \tag{79}$$

where integrating along the cut $[-2, 2]$ in (77) amounts to integrating along the unit circle in (79). We can use it to obtain the (t, x) double-series expansion of $M(t, z)$ which is found (after a bit of algebra) as

$$M(t, z) = \sum_{m=0}^{\infty} \sum_{n=0}^m t^{2m} z^n (-1)^{m+n} \frac{2(2m+1)(n+1)^2(2m)!}{m!(m+1)!(m-n)!(m+n+2)!}. \tag{80}$$

Thus $M(t, z)$ is a generalized hypergeometric function of the two variables t^2 and z . It is depicted in figure 6.

3.4.2. The small t limit. The small t behavior of $M(t, z)$ is

$$M(t, z) = 1 + t^2(z - 1) + \mathcal{O}(t^4). \tag{81}$$

3.4.3. The large t limit. The large time behavior of $M(t, z)$ is most easily calculated from the integral representation (79) by using the steepest descent method at the saddle points $w_1 = \pm 1, w_2 = \pm 1$. We obtain an algebraic decay as t^{-3} , with an oscillatory term negligible when $z \rightarrow 1$ and dominant when $z \rightarrow -1$:

$$M(t, z) = \frac{1}{2\pi} t^{-3} \left(\frac{1+z}{(1-z)^3} - \frac{1-z}{(1+z)^3} \sin(4t) \right) (1 + \mathcal{O}(t^{-1})). \tag{82}$$

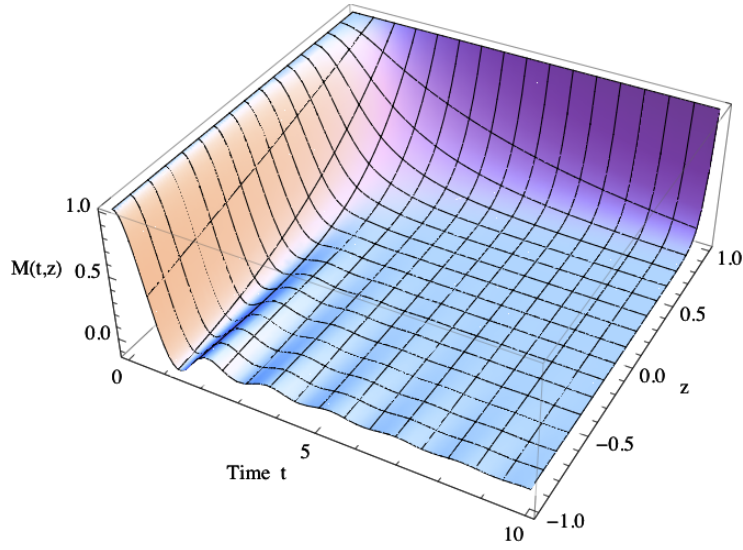


Figure 6. The decoherence function $M(t, z)$ as a function of the rescaled time t and the parameter $z \in [-1, 1]$. Decoherence is fast when $z < 1$ but very slow when $z \simeq 1$.

3.4.4. *The $z \rightarrow 1$ and $t(1 - z) = \mathcal{O}(1)$ scaling.* When $z = 1$, we have in fact

$$\lim_{z \rightarrow 1_-} M(t, z) = 1 \tag{83}$$

but the function $M(t, z)$ takes a scaling form when $z \rightarrow 1$ while t is large. In fact

$$M(t, z) = \Psi(t') \quad \text{with } t' = t(1 - z) \quad \text{in the limit } t' = \mathcal{O}(1), \quad z \rightarrow 1_-. \tag{84}$$

Indeed, in this limit, the measure in equation (79) concentrates around $w_2 = w_1$ and the integral becomes a single-integral representation. We get a simple hypergeometric function

$$\begin{aligned} \Psi(t') &= \oint \frac{dw}{2i\pi} e^{-t'B(w)} \frac{1}{2w} B(w)^2 = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} d\theta e^{-2t' \cos(\theta)} \cos(\theta)^2 \\ &= \frac{1}{2\pi} \int_{-2}^2 dx \sqrt{4 - x^2} e^{-t' \sqrt{4 - x^2}} \end{aligned} \tag{85}$$

whose series expansion is explicitly

$$\Psi(t') = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} (-2t')^k \frac{\Gamma((3 + k)/2)}{k! \Gamma(2 + k/2)}. \tag{86}$$

This function is depicted in figure 7. Its asymptotic behavior is

$$\Psi(t') = 1 - \frac{16}{3\pi} t' + \mathcal{O}(t'^2) \quad t' \rightarrow 0, \quad \psi(t') = \frac{1}{\pi} t'^{-3} + \mathcal{O}(t'^{-4}) \quad t' \rightarrow \infty. \tag{87}$$

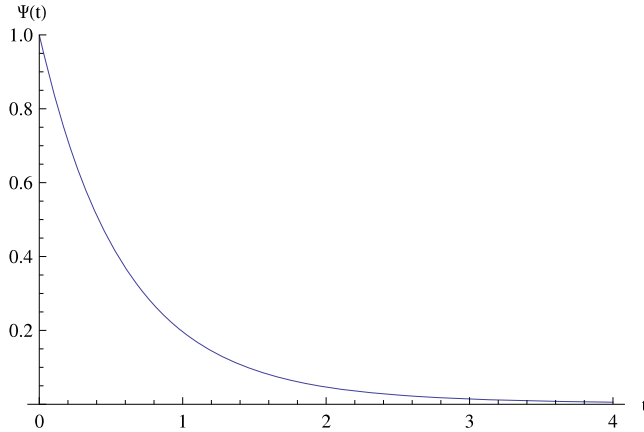


Figure 7. The scaling function $\Psi(t)$ for the decoherence function when $z \rightarrow 1_-$.

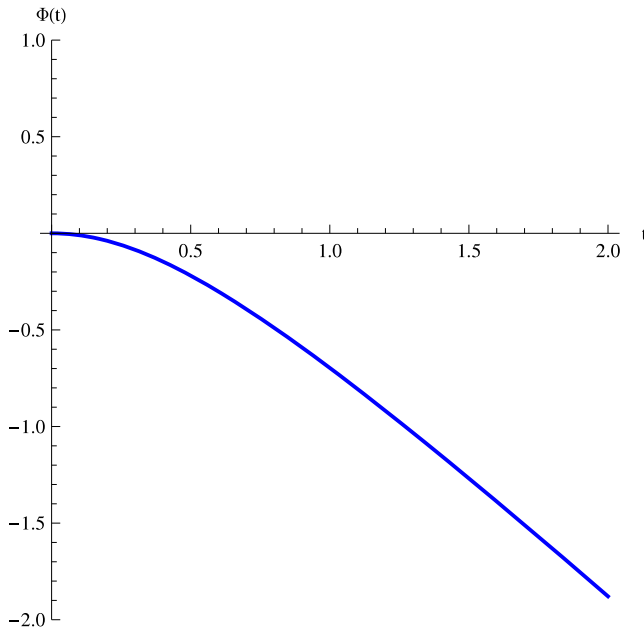


Figure 8. The function $\Phi(t)$ of equations (88) and (89) that describes the small t and $z \sim 1$ (and hence $l \ll j$) behavior of the function $M(t, z)$.

3.4.5. *The $z \rightarrow 1$ and $t = \mathcal{O}(1)$ scaling.* Note that $\Psi(t)$ is linear in t at small time, not quadratic in t like $M(t, z)$ for $z < 1$. For $z = 1 - \epsilon$ close to 1 ($\epsilon \ll 1$) but t of order 1, the function $M(t, z)$ behaves as

$$M(t, z) = 1 + \epsilon \Phi(t) + \mathcal{O}(\epsilon^2) \tag{88}$$

with $\Phi(t) = 1 - {}_1F_2(-\frac{1}{2}; 1, 2; -4t^2)$ a universal nonlinear function depicted in figure 8 which behaves as

$$\Phi(t) = -t^2 + \mathcal{O}(t^4) \quad \text{when } t \rightarrow 0, \quad \Phi(t) = -\frac{1}{\pi}t + \mathcal{O}(1) \quad \text{when } t \rightarrow \infty. \tag{89}$$

Therefore, the crossover between the nonlinear regime (88) for small t and the linear regime (84) for large $t \sim \epsilon^{-1}$ occurs in a domain of t of size $\mathcal{O}(1)$, and hence in a very small interval in t' of size $\mathcal{O}(\epsilon)$.

3.5. The dependence of $Z(l)$ on the choice of couplings $\Delta(l)$ and on the total spin j .

3.5.1. *Various examples.* Now we study how the time scale τ_0 and the parameters $Z(l)$ (which govern the dynamics in each angular momentum sector l) depend on the choice of the original distribution of couplings $\Delta(l)$ and of the total spin j . We recall that

$$\tau_0 = 1/\sqrt{\hat{\Delta}(0)}, \quad \hat{\Delta}(0) = \sum_{l=0}^{2j} \frac{2l+1}{2j+1} \tilde{\Delta}(l), \quad \tilde{\Delta}(l) = N \Delta(l) \quad (90)$$

and that $Z(l)$ is as follows:

$$Z(l) = \frac{\hat{\Delta}(l)}{\hat{\Delta}(0)}, \quad \hat{\Delta}(l) = \sum_{l'=0}^{2j} (2l'+1)(-1)^{2j+l'+l} \left\{ \begin{matrix} j & j & l' \\ j & j & l \end{matrix} \right\} \tilde{\Delta}(l'). \quad (91)$$

In the trivial case where all the Δ s are equal we have

$$\Delta(l) = \Delta \quad \text{for all } l \implies \hat{\Delta}(0) = (2j+1)N\Delta \quad \text{and} \quad \hat{\Delta}(l) = 0 \quad \text{if } 0 < l \leq 2j. \quad (92)$$

Hence $Z(l) = 0$ for all $l > 0$. We recover the GUE ensemble.

We are interested in the situation where only a finite number l_0 of interaction channels are activated, and where this number is much smaller than the total spin

$$\Delta(l) = 0 \quad \text{if } l > l_0, \quad l_0 \ll j. \quad (93)$$

We first assume that all the $\Delta(l)$ for $l \leq l_0$ are of the same order Δ . Then $\hat{\Delta}(0)$ is of order $\Delta N l_0^2 / j$. We thus rescale $\Delta \rightarrow \bar{\Delta}$ both with N (as before) and with the spin j , having in mind having both a large environment ($N \rightarrow \infty$) and a semiclassical spin ($j \rightarrow \infty$):

$$\Delta(l) = (2j+1)N^{-1} \bar{\Delta}(l) = (2j+1) \tilde{\Delta}(l) \quad (94)$$

and we are now interested in the limit of a large environment ($N \rightarrow \infty$) and a semiclassical spin ($j \rightarrow \infty$):

$$N \rightarrow \infty, \quad j \ll 1 \quad \text{large but finite or infinite, } \bar{\Delta}(l) \text{ of order } \mathcal{O}(1). \quad (95)$$

We now have

$$\hat{\Delta}(0) = \tau_0^{-2} = \sum_{l=0}^{l_0} (2l+1) \bar{\Delta}(l) \quad \text{independent of the spin } j.$$

In the figures presented in the appendix we plot for several choices of distributions of $\bar{\Delta}$ s the resulting $Z(l)$, and then discuss the results and prove some of the properties of the function $Z(l)$.

- In figure A.1 we plot $Z(l)$ as a function of l in the case where all the $\Delta(l)$ are equal for $0 \leq l \leq l_0$, and zero otherwise.
- In figure A.2 we plot $Z(l)$ as a function of l in the case where $\Delta(0) = 0$, and all the $\Delta(l)$ are equal for $1 \leq l \leq l_0$, and zero otherwise.
- In figure A.3 only a single and odd l contributes.
- In figure A.4 all odd $l \leq l_0$ contribute.

- In figure A.5 all even $l \leq l_0$ contribute.
- In figure A.6 we compare different cases, letting $\Delta(0)$ become large.
- In figure A.7 we consider some random distributions of $\Delta(l)$ for $l \leq l_0$.

First we make experimental observations:

- (1) For $0 < l \leq 2j$, $Z(l)$ is always in the interval $] -1, 1[$.
- (2) For a fixed coupling distribution $\bar{\Delta}$, when the total spin j is large, $Z(l)$ takes a limit scaling form Y (which depends of course of the $\bar{\Delta}$ s):

$$Z(l) = Y(l/2j) \quad \text{when } j \rightarrow \infty, \ l/j \text{ fixed.} \quad (96)$$

- (3) By normalization, $Z(0) = 1$. Otherwise $Z(l)$ can be close to $+1$ (respectively to -1) only in the limit $j \rightarrow \infty$ when $l \simeq 2j$ and when the $\bar{\Delta}(l)$ are zero for all odd values of l (respectively all even values of l).

3.5.2. *The limit $j \rightarrow \infty, l/j$ fixed.* The existence of a limit distribution $Y(x)$, $x = l/2j$, when $j \rightarrow \infty$ is easily explained. We use Racah formula of the $6j$ -symbols to rewrite the formula (91) for $\hat{\Delta}(l)$ as

$$\hat{\Delta}(l) = (2j + 1) \sum_{l'=0}^{2j} \bar{\Delta}(l') (2l' + 1) \sum_{k=0}^{\min(l,l')} \frac{(-1)^k}{(k!)^2} \frac{(l' + k)! (l + k)! (2j - k)!}{(l' - k)! (l - k)! (2j + k + 1)!}. \quad (97)$$

We use the Stirling formula to take the limit

$$j \rightarrow \infty, \quad x = \frac{l}{2j} \text{ fixed} \quad (98)$$

to obtain

$$\hat{\Delta}(l) \rightarrow \sum_{l'=0}^{l_0} \bar{\Delta}(l') (2l' + 1) F_{l'}(x) \quad (99)$$

with the $F_{l'}(x)$ the polynomials

$$F_{l'}(x) = \sum_{k=0}^{l'} \frac{(-1)^k (l' + k)!}{(k!)^2 (l' - k)!} x^{2k} = {}_2F_1(1 + l', -l', 1, x^2). \quad (100)$$

Hence the explicit polynomial form for the limit scaling function Y in equation (96) is

$$Y(x) = \frac{\sum_{l'=0}^{l_0} \bar{\Delta}(l') (2l' + 1) F_{l'}(x)}{\sum_{l'=0}^{l_0} \bar{\Delta}(l') (2l' + 1)}. \quad (101)$$

3.5.3. *The limit $j \rightarrow \infty, l \ll j$.* Apart from in some very special cases, $Z(l)$ is close to 1 only if l is small. This case is needed for the study of decoherence. When $l \ll j$ we need only to keep the terms $k = 0$ and 1 in the explicit form (97) for $\hat{\Delta}(l)$. We obtain

$$\hat{\Delta}(l) = \hat{\Delta}(0) - \frac{l(l+1)}{4j(j+1)} \sum_{l'=1}^{l_0} \bar{\Delta}(l') (2l'+1) l'(l'+1) + \mathcal{O}((l/j)^4). \quad (102)$$

Hence

$$Z(l) = 1 - l(l+1) \frac{1}{4j(j+1)} \frac{D_0}{j(j+1)} + \dots, \quad D_0 = \frac{\sum_{l'=1}^{l_0} \bar{\Delta}(l') (2l'+1) l'(l'+1)}{\sum_{l'=0}^{l_0} \bar{\Delta}(l') (2l'+1)}. \quad (103)$$

This approximation is valid at the top of the curve $Y(x)$ near $x = 0$, i.e. provided that

$$l(l+1) \ll j(j+1). \quad (104)$$

It is in particular valid when $l \propto \sqrt{j}$, which is the case to consider when studying the coherent states.

Note that if all the $\bar{\Delta}(l)$ are of the same order $\bar{\Delta}$ when $l \leq l_0$ and zero otherwise, the coefficient D_0 is of order

$$D_0 \sim \frac{1}{2} l_0^2. \quad (105)$$

But note also that the numerator in D_0 involves only the $\bar{\Delta}(l)$ for $l > 0$. In particular, if the $l = 0$ coupling $\bar{\Delta}(0)$ is much larger than the others, $\bar{\Delta}(l) \sim \bar{\Delta}$ for $0 < l \leq l_0$,

$$\bar{\Delta}(0) \gg l_0^2 \bar{\Delta} \implies D_0 \sim \frac{1}{4} \frac{l_0^4 \bar{\Delta}}{\bar{\Delta}(0)} \ll l_0^2. \quad (106)$$

3.5.4. *The case of even l or odd l .* One sees from figure A.5 that when there are no couplings for odd l , i.e. when

$$l_{\text{odd}} \implies \Delta(l) = 0, \quad (107)$$

then for very large spin $j \rightarrow \infty$ the condition of slow decoherence $Z(l) \simeq 1$ is satisfied for l close to the maximal value $l = 2j$. In fact in this case the scaling function $Y(x)$ defined by equation (96) is for $x \simeq 1$

$$Y(x) \simeq 1 - \text{cst}(1-x). \quad (108)$$

The reason why there is a very slow decoherence between opposite states such as $|j\rangle$ and $| -j\rangle$ is of course that the coupling Hamiltonian $H_{\mathcal{SE}}$ has an additional \mathbb{Z}_2 parity symmetry, the inversion of spin, which protects the states with this symmetry from decoherence.

In contrast, one sees in figures A.3 and A.4 that when there are no couplings for even l , the condition of maximally fast decoherence $Z(l) = -1$ is satisfied for l close to the maximal value $l = 2j$.

3.6. Decoherence parameters and norms of operators

3.6.1. $\hat{\Delta}(0)$ and the norm of H . When all the $\Delta(l)$ are equal to the same Δ (GUE ensemble) all the matrix elements of H are of the same order, $\sqrt{\Delta}$, and the whole Hamiltonian H is a random $M \times M$ matrix in a GUE ensemble, with $m = (2j + 1)N$. The normalization is such that

$$\Delta(l) = \Delta \implies \overline{\text{tr}(H^2)} = (2j + 1)^2 N^2 \Delta. \quad (109)$$

In the general case where the $\Delta(l)$ are different, we have

$$\overline{\text{tr}(H^2)} = (2j + 1)N \hat{\Delta}(0) \quad \text{with as before } \hat{\Delta}(0) = \frac{N}{2j + 1} \sum_l (2l + 1)\Delta(l) \quad (110)$$

so $\sqrt{\hat{\Delta}(0)}$ is the typical size of an eigenvalue of H , that we call its norm $\|H\|_2$, the norm of an operator A being defined as²

$$\|A\|_2^2 = \frac{\text{tr}(A^\dagger A)}{\text{tr}(\mathbf{1})}. \quad (111)$$

In our modified GUE ensembles, if we take for $|m, \alpha\rangle$ a basis of the whole Hilbert space $H_{S\mathcal{E}} = \mathbb{C}^{(2j+1)N}$, since

$$\|H\|_2^2 = \frac{1}{(2j + 1)N} \sum_{m,n} \sum_{\alpha,\beta} |\langle m, \alpha | H | n, \beta \rangle|^2 \quad (112)$$

we see that our norm is

$$\|H\|_2 \simeq \sqrt{(2j + 1)N} \times \text{‘average value’ of } |\langle m, \alpha | H | n, \beta \rangle|. \quad (113)$$

3.6.2. *The spectrum of H .* It is also easy to see from the solution (58) and (60) of the single resolvent $\mathcal{H}(x)$ that, although the Hamiltonian H belongs not to a GUE ensemble but to the modified $\text{GU}_{2,N}\text{E}$ ensemble, in the large N limit its density of states (DOS) $\rho(\lambda)$, i.e. the density distribution of the eigenvalues of H , is still given by a Wigner semi-circle law, on the interval $[-\hat{E}, \hat{E}]$ with

$$\rho(\lambda) \propto \sqrt{\hat{E}^2 - \lambda^2} \quad \text{with } \hat{E} = 2\sqrt{\hat{\Delta}}. \quad (114)$$

3.6.3. *The mean $\hat{\Delta}_{\text{av}}$ for the $\hat{\Delta}(l)$.* It will be convenient to consider the mean value of the $\hat{\Delta}(l)$ when averaged over all possible modes (l, m) , $\hat{\Delta}_{\text{av}}$. It is defined as

$$\hat{\Delta}_{\text{av}} = \frac{1}{(2j + 1)^2} \sum_{l=0}^{2j} (2l + 1)\hat{\Delta}(l) \quad (115)$$

and in found to be nothing but

$$\hat{\Delta}_{\text{av}} = \frac{N}{2j + 1} \Delta(0) = \bar{\Delta}(0). \quad (116)$$

² This norm is the Hilbert–Schmidt norm divided by $\sqrt{\dim(\mathcal{H})}$, and is not the C^* norm $\|A\|^2 = \sup_{|\psi\rangle} \langle \psi | A^\dagger A | \psi \rangle / \langle \psi | \psi \rangle$ which corresponds to the modulus of the largest eigenvalue of the operator.

The average value of $Z(l)$ is

$$Z_{\text{av}} = \frac{1}{(2j+1)^2} \sum_{l=0}^{2j} (2l+1) Z(l) = \frac{\bar{\Delta}(0)}{\hat{\Delta}(0)} \quad (117)$$

and in particular

$$1 - Z_{\text{av}} = \sum_{l=1}^{2j} (2l+1) \frac{\bar{\Delta}(l)}{\hat{\Delta}(0)} = \left(\frac{\|H'\|_2}{\|H\|_2} \right)^2, \quad H' = H - H^{(0)}, \quad (118)$$

H' being the purely $\mathcal{S} + \mathcal{E}$ interaction part of the Hamiltonian H , and $H^{(0)}$ being the purely external part \mathcal{E} of H .

4. Decoherence and the emergence of coherent states

It is easy now to study the dynamics of decoherence and the emergence of the coherent states for spin as the semiclassical states robust against the interaction with the environment.

4.1. Coherent states

4.1.1. Pure coherent states. Coherent states are the pure states which minimize the uncertainty relations for spin, i.e. the states with a maximally localized Wigner distribution. They read explicitly

$$|\vec{n}\rangle = \sum_{m=-j}^j \sqrt{\frac{(2j)!}{(j+m)!(j-m)!}} \cos(\theta/2)^{j+m} \sin(\theta/2)^{j-m} e^{-im\phi} |m\rangle \quad (119)$$

with (θ, ϕ) the spherical coordinates of the unit vector \vec{n} . Coherent states are formed by a coherent superposition of modes such that $l \sim \sqrt{j} \ll j$. Indeed for the single pure state

$$|\vec{e}_z\rangle = |j\rangle \quad (120)$$

the matrix density components are

$$W_{|j\rangle\langle j|}^{(l,m)} = \delta_{m,0} W_{\text{c.s.}}^{(l)}, \quad W_{\text{c.s.}}^{(l)} = \sqrt{\frac{((2j)!)^2 (2l+1)}{(2j+l+1)!(2j-l)!}} \quad (121)$$

and for large j and small l Stirling formula gives

$$W_{\text{c.s.}}^{(l)} = \frac{2l+1}{\sqrt{2j+1}} \exp\left(-\frac{l^2}{2j}\right) \left(1 + \mathcal{O}\left(\frac{l^3}{j^2}\right)\right). \quad (122)$$

The Wigner representation of the coherent state $|\vec{n}\rangle$ is a Gaussian-like positive distribution with width $1/\sqrt{j}$ centered at \vec{n} on the unit sphere.

4.1.2. *Random pure states.* At variance with coherent states, a random pure spin state $|\psi\rangle$ is such that its density matrix components are independent equally distributed random variables

$$W_{|\psi\rangle\langle\psi|}^{(l,m)} \sim \frac{1}{2j+1} \quad (123)$$

and its Wigner representation is a random function on the sphere (analogous to a random polynomial with zeros obeying Wigner statistics with short distance cut-off $1/\sqrt{j}$).

4.1.3. *Superpositions of coherent states.* Quantum superpositions of coherent states correspond to more complicated functions. For instance it is well known that the ‘Schrödinger cat-like’ state

$$|\uparrow+\downarrow\rangle = \frac{1}{\sqrt{2}}(|+\vec{e}_z\rangle + |-\vec{e}_z\rangle) = \frac{1}{\sqrt{2}}(|j\rangle + |-j\rangle) \quad (124)$$

corresponds to modes (l even, $m = 0$) and ($l = 2j, m = \pm 2j$).

$$W_{\text{CAT}}^{(l,m)} = \frac{1}{2}\delta_{m,0}(1+(-1)^l)\sqrt{\frac{((2j)!)^2(2l+1)}{(2j+l+1)!(2j-l)!}} + \frac{1}{2}\delta_{l,2j}(\delta_{m,2j}(-1)^{2j} + \delta_{m,-2j})\sqrt{\frac{1+2j}{1+4j}}. \quad (125)$$

In general, the density matrix of a superposition of two coherent states will have large components $W^{(lm)}$ describing the quantum correlations for

$$W^{(lm)} \simeq \mathcal{O}(1) \quad \text{for } l \sim j. \quad (126)$$

In particular, if $|\psi\rangle$ is a superposition of two coherent states $|\vec{n}_1\rangle$ and $|\vec{n}_2\rangle$ which are at distance θ_{12} on the sphere \mathcal{S}_2 ,

$$|\psi\rangle = c_1|\vec{n}_1\rangle + c_2|\vec{n}_2\rangle \quad \text{with } \vec{n}_1 \cdot \vec{n}_2 = \cos\theta_{12}, \quad (127)$$

the largest elements $W^{(lm)}$ of the ‘off-diagonal’ part of the density matrix $\rho_{\text{off}} = |\vec{n}_2\rangle\langle\vec{n}_1| + |\vec{n}_1\rangle\langle\vec{n}_2|$ occur for

$$W_{\rho_{\text{off}}}^{(lm)} \simeq \mathcal{O}(1) \quad \text{for } l \simeq 2j \sin(\theta_{12}/2). \quad (128)$$

4.2. The time scales of the system

From the above discussion, one sees that in our model the evolution of the spin is characterized by at least three time scales, which are such that

$$\tau_0 < \tau_1 \ll \tau_2. \quad (129)$$

4.2.1. *The dynamical time scale for the whole system* $\tau_0 = \tau_{\text{dyn}}$. The first characteristic time is

$$\tau_0 = (\hat{\Delta}(0))^{-1/2} = (\|H\|_2)^{-1}. \quad (130)$$

It is the typical evolution time scale for a generic state of the whole system $\mathcal{E} + \mathcal{S}$. Thus we denote it also as τ_{dyn} .

4.2.2. *The decoherence time scale for spin states* $\tau_1 = \tau_{\text{dec}}$. The second characteristic time is

$$\tau_1 = \frac{\tau_0}{1 - Z_{\text{av}}} = \frac{\|H\|_2}{(\|H'\|_2)^2}. \quad (131)$$

It is the time scale for the decay of the $W^{(lm)}$ coefficients of the density matrix for typical $l \sim j$. Hence it corresponds to the decoherence time scale for random pure states $|\psi\rangle$ for the spin, with no particular spin properties. Hence we denote it also by τ_{dec} .

For $t \gg \tau_1$, a pure random state $|\psi\rangle$ has become a complete statistical mixture $\rho \propto \mathbf{1}$. We have of course $\tau_1 > \tau_0$ and in general τ_1 is of the same order as τ_0 . But we have seen that this decoherence time τ_1 can be much larger than τ_0 in the special case where $\Delta(0) \gg \Delta(l)$, $l > 0$, i.e., when the internal dynamics of \mathcal{E} , given by $H_{\mathcal{E}} = H^{(0)}$, is much faster than the dynamics of \mathcal{S} induced by the coupling $\mathcal{S} + \mathcal{E}$, and given by $H_{\mathcal{S}\mathcal{E}} = H'$.

4.2.3. *The time scale for evolution of the coherent states* $\tau_2 = \tau_{\text{diff}}$. The third characteristic time is the evolution time for a single coherent state. It is the decay time scale in sectors such that

$$l \sim \sqrt{j} \ll j. \quad (132)$$

In this case the evolution of the density matrix is given by the regime $Z(l) \simeq 1$ studied in section 3.5.3. Using equation (103), this time τ_2 is given by

$$\tau_2 = \tau_0 \frac{j}{D_0} \quad (133)$$

with D_0 given by equation (103). Note that on general grounds,

$$\frac{j}{l_0(l_0 + 1)} \tau_1 \leq \tau_2 \leq \frac{j}{2} \tau_1 \quad \text{where } l_0 = \sup\{l : \Delta(l) > 0\}. \quad (134)$$

As we shall see in section 4.4, for $t > \tau_2$ the evolution of coherent states is described by a quantum diffusion process. Hence we denote it also by τ_{diff} .

4.2.4. *The conditions for decoherence.* The dynamics of coherent states is much slower than the evolution of non-coherent states when the spin j is large and when l_0 is of order $\mathcal{O}(1)$; hence $l_0 \ll j$. In fact using equation (103) for D_0 , and equation (11), the ratio of the two characteristic times τ_1 and τ_2 can easily be rewritten in terms of the ratio of operator norms:

$$\frac{\tau_1}{\tau_2} = \frac{(\|\vec{S}, H'\|_2)^2}{(\|\vec{S}\|_2)^2 (\|H'\|_2)^2}. \quad (135)$$

H' is the interaction Hamiltonian $H' = \sum_{l>0} H^{(l)}$. Of course the norm of the spin operator \vec{S} is simply $\|\vec{S}\|_2 = \sqrt{j(j+1)}$. The numerator is the squared norm of the commutator of the spin operators with H' :

$$(\|\vec{S}, H'\|_2)^2 = \sum_{\mu=1}^3 (\|[S^\mu, H']\|_2)^2 = \frac{1}{(2j+1)N} \sum_{\mu=1}^3 \text{tr}([S^\mu, H']^2). \quad (136)$$

One sees explicitly that it is when the commutator of the spin with the interaction Hamiltonian is small compared to the product of these two operators

$$[\vec{S}, H'] \ll \vec{S} \times H' \quad (137)$$

(here in the sense of the $\|\cdot\|_2$ norm) that the single spin coherent states $|\vec{n}_1\rangle$ are much more robust than quantum superpositions of macroscopically distinct coherent states $|\psi\rangle = c_1|\vec{n}_1\rangle + c_2|\vec{n}_2\rangle$ with $\theta_{12} \gg \sqrt{j}$. It is in the regime

$$\tau_1 \ll t \ll \tau_2 \quad (138)$$

that coherent states behave as classical states (pointer states), and that one can use a semiclassical picture.

In general, a pure state of the form $|\psi\rangle = c_1|\vec{n}_1\rangle + c_2|\vec{n}_2\rangle$ becomes a statistical mixture of $|\vec{n}_1\rangle$ and $|\vec{n}_2\rangle$ after a time of order τ_1 . But this decoherence time scale depends on the distance between the two states, i.e. the angle θ between the vectors \vec{n}_1 and \vec{n}_2 . In the semiclassical regime when the spin is large, $j \gg 1$, and where this distance is small, but still large compared to the width of a coherent state, $1/\sqrt{j} \ll \theta \ll 1$, using equations (128), (103) and the scaling equation (84) discussed in section 3.4.4, we have for the typical decoherence time scale

$$\tau_{\text{dec}}(\theta) \sim \tau_1 / \sin(\theta/2)^2 \quad \text{such that } \tau_1 \ll \tau_{\text{dec}}(\theta) \ll \tau_2. \quad (139)$$

Thus the closer two coherent states are in phase space, the longer it takes for decoherence to wash out quantum correlations between these two states. This is a well known effect for harmonic oscillators or free particles discussed for instance in [21, 20, 28].

Beyond the regime discussed here, i.e. for $t \gg \tau_2$, coherent states start to evolve, and in fact becomes statistical mixtures according to a quantum diffusion process that we discuss in section 4.4.

4.3. Illustrative examples of evolutions

4.3.1. The cases considered. We have a complete explicit solution for the evolution of the (l, m) components of the density matrix for the spin, starting from arbitrary initial conditions. The (l, m) components are nothing but the components of the decomposition of the Wigner transform of the density matrix in spherical harmonics. Therefore it is both tempting and easy to illustrate our results by explicitly plotting the time evolution of the Wigner transform for various initial states, and various choices of dynamics (given by the couplings $\Delta(l)$), and various values of the spin j .

In the following we take a large (but not tremendously large) value of the spin: $j = 20$. We chose for the coupling distributions the simplest case

$$\Delta(l) = 0 \quad \text{unless } l = 1 \quad (140)$$

and express the time evolution in units such that $\tau_0 = 1$. This corresponds for the various time scales to

$$\tau_0 = \tau_1 = 1, \quad \tau_2 = j/2 = 10. \quad (141)$$

We represent the Wigner transform $W(\vec{n})$ which is a real function over the unit sphere \mathcal{S}_2 as a real function $W(z)$ over the complex plane \mathbb{C} using the stereographic projection

$$\vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \rightarrow z = x + iy = r e^{i\phi}, \quad r = 2 \arctan(\theta/2). \quad (142)$$

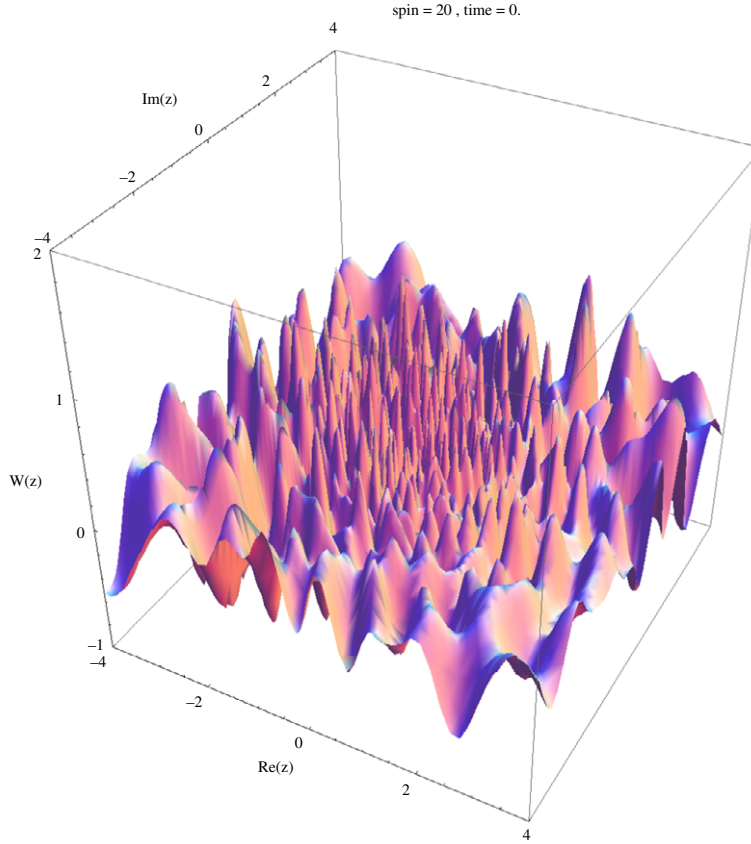


Figure 9. Evolution of a random state for spin $j = 20$. The Wigner distribution $W(\vec{n})$ is represented using stereographic projection of the complex plane. An mp4 video file of this figure is available from the online journal, see stacks.iop.org/JSTAT/2011/P01001/mmedia.

4.3.2. *The evolution of random states.* We start with a random pure state $|\psi\rangle \propto \sum_{m=-j}^j a_m |m\rangle$, with the a_m complex Gaussian independent random variables. In this case the initial $W^{(lm)}$ are (not independent) random variables of the same order $(2j + 1)^{-1}$, independently of l and m (except for $W^{(0,0)} = (2j + 1)^{-1/2}$). For $0 < t < \tau_1$, all the high angular frequency modes $W^{(lm)}(t)$ with $l \sim j$ undergo a fast decay, and for $t > \tau_1$, only the low frequency modes with $l \sim \sqrt{j}$ are still there; these decay at a much slower pace. The evolution of the Wigner function is depicted in figure 9, from $t = 0$ to 8, that is approximately for $0 \leq t \leq \tau_2$. One does indeed see the behavior discussed above.

In other words, one starts from the pure quantum state $|\psi\rangle$ which can be written as a quantum superposition of $\sim 2j + 1$ semiclassical coherent states. This state evolves and becomes for $t > \tau_1$ a statistical mixture of these coherent states, with approximately equal probabilities of order $1/(2j + 1)$ (with small fluctuations of relative order $1/\sqrt{2j + 1}$ which are responsible for the long-lived remnant wiggles).

4.3.3. *The evolution of a coherent state.* In figure 10 we plot the evolution of a coherent state $|\vec{n}\rangle = |\theta, \phi\rangle$, here the one centered at the origin:

$$|\vec{e}_z\rangle = |\theta = 0, \phi\rangle = |j\rangle. \tag{143}$$

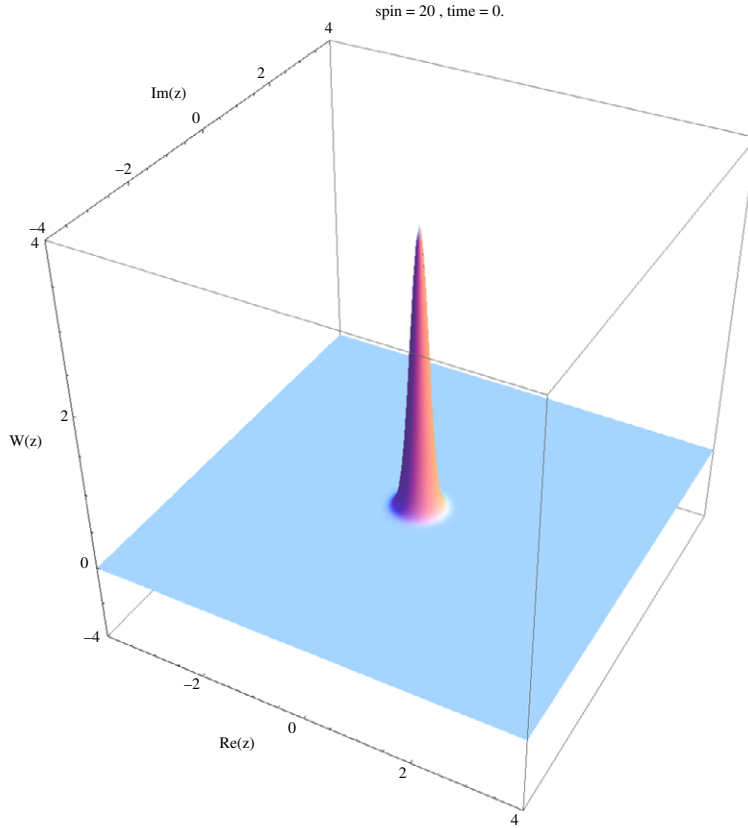


Figure 10. Evolution of a coherent state for spin $j = 20$. An mp4 video file of this figure is available from the online journal, see stacks.iop.org/JSTAT/2011/P01001/mmedia.

One sees immediately that its evolution is much slower and that it stays localized in phase space, as expected. We are working for finite j , so τ_2 is larger than τ_1 but finite, which explains the fact that the Wigner transform widens with time. Its evolution will be discussed in more detail in section 4.4.

4.3.4. *The evolution of superpositions of coherent states.* We now look at the evolution of quantum superpositions of coherent states. These kinds of states are often dubbed ‘Schrödinger cat states’ or ‘cat states’ in the literature.

Two-state cats. In figure 11 we start from a simple superposition of the two opposite (hence orthogonal) coherent states:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\pi/2, 0\rangle + |\pi/2, \pi\rangle). \quad (144)$$

At time $t = 0$ the Wigner transform shows the two peaks associated with the two coherent states and the strong interference fringes along the $\phi = \pm\pi/2$ line (a great circle on the sphere), which contains the information about the superposition between the two coherent states.

As time increases the two coherent peaks evolve slowly, while the interference fringes disappear very quickly with a characteristic time of order τ_1 . We thus see explicitly the

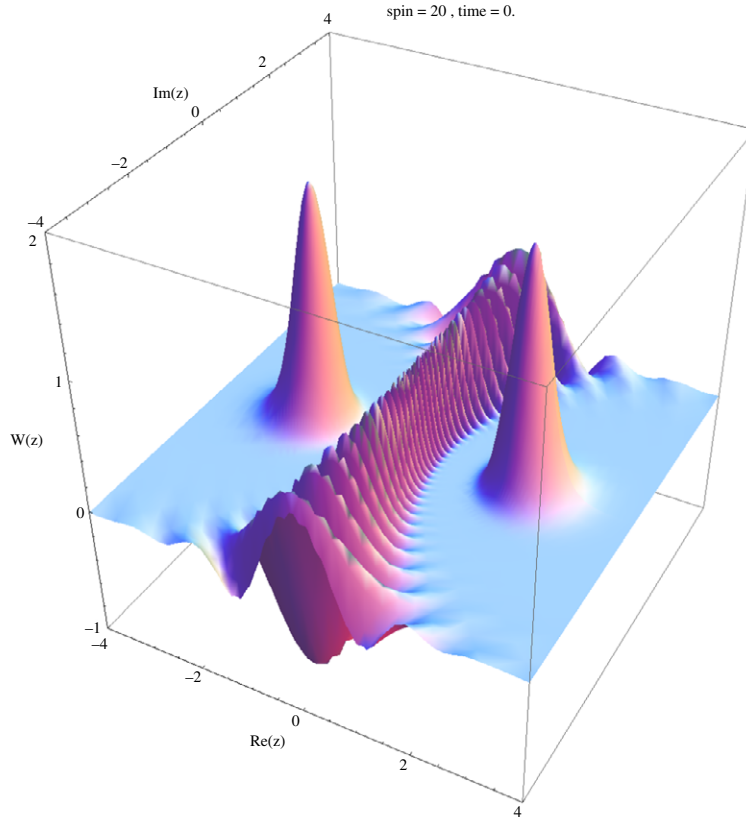


Figure 11. Evolution of a ‘two-state cat’ for $j = 20$. An mp4 video file of this figure is available from the online journal, see stacks.iop.org/JSTAT/2011/P01001/mmedia.

decoherence induced by the coupling of the spin with the environment, which transforms the quantum superposition of the two coherent states into a statistical mixture (here with equal probabilities) of these two states.

Three-state cats. In figure 12 we start from a more complex superposition of three coherent states:

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|\pi/2, 0\rangle + |\pi/2, 3\pi/4\rangle + |\pi/2, 5\pi/4\rangle). \quad (145)$$

At time $t = 0$ the Wigner transform shows the three peaks associated with the three coherent states and the interference fringes in between, which contains the information about the superposition between the three coherent states. The rightmost peak corresponds to the first coherent state $|\pi/2, 0\rangle$, and the leftmost pair of peaks corresponds to the second and third coherent states $|\pi/2, 3\pi/4\rangle$ and $|\pi/2, 5\pi/4\rangle$. These two states are closer to each other than to the first one.

As time increases the three coherent peaks evolve slowly, and the interference fringes disappear quickly. One sees that the coherences between the close second and third states decay more slowly than the coherences between these states and the first one, as expected from equation (139).

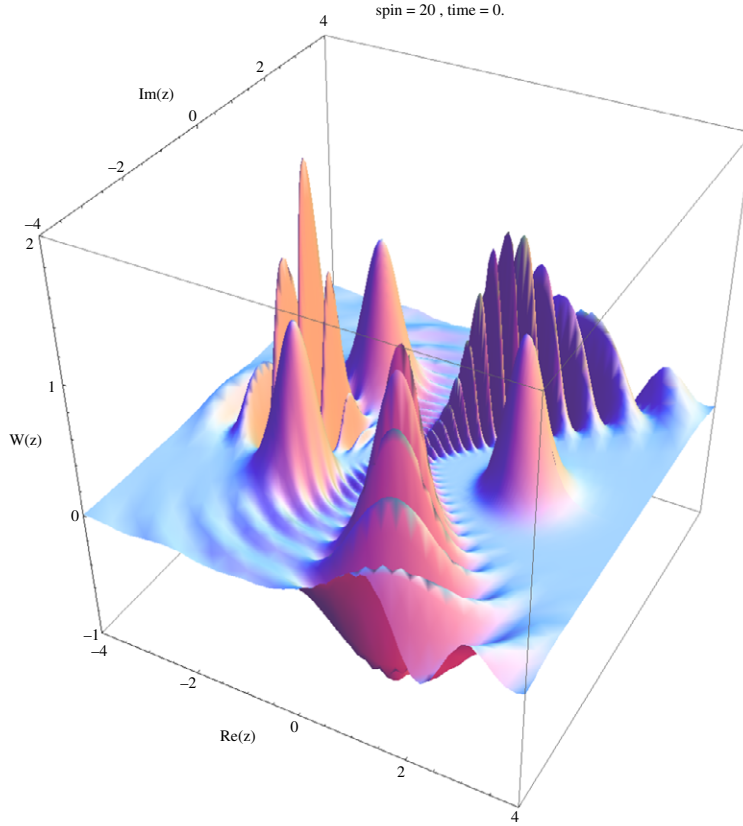


Figure 12. Evolution of a ‘three-state cat’ for $j = 20$. An mp4 video file of this figure is available from the online journal, see stacks.iop.org/JSTAT/2011/P01001/mmedia.

4.4. The evolution of coherent states and quantum diffusion

We now discuss in more detail the dynamics of a single coherent state at large times $t > \tau_1$. We concentrate on the semiclassical limit where $j \gg 1$. Then the modes for a single coherent state are such that $l \sim \sqrt{j} \ll j$, the corresponding function $Z(l)$ is always $Z(l) \simeq 1$ and we may approximate the evolution function $\hat{\mathcal{M}}^{(l)}(t) = M(t, Z(l))$ by its universal form when $1 - Z(l)$ is small:

$$\hat{\mathcal{M}}^{(l)}(t) \simeq \Psi \left(t \frac{4}{3\pi} \frac{D_0}{\tau_0} \frac{l(l+1)}{j(j+1)} \right) \quad (146)$$

with $\Psi(t)$ given by equations (85) and (86).

Using the asymptotic equation (122) for the modes of a coherent state at large j , and starting at time $t = 0$ from the coherent state $|j\rangle$ of equation (143), the harmonics of the Wigner transform of the density matrix read explicitly

$$W^{(l)}(t) = \frac{2l+1}{\sqrt{2j+1}} \exp \left(-\frac{l^2}{2j} \right) \Psi \left(t \frac{4}{3\pi} \frac{D_0}{\tau_0} \frac{l(l+1)}{j(j+1)} \right). \quad (147)$$

There are two very different time regimes.

4.4.1. *The small time $t < \tau_2 = \tau_{\text{diff}}$ regime.* For the relevant modes $l \sim \sqrt{j}$, the variable $t' = t(4/3\pi)(D_0/\tau_0)(l(l+1)/j(j+1))$ for ψ is still very small, $t' \ll 1$, and in equation (147), $\Psi(t') \simeq 1$,

$$W^{(l)}(t) = W^{(l)}(0) = \frac{2l+1}{\sqrt{2j+1}} \exp\left(-\frac{l^2}{2j}\right). \quad (148)$$

The coherent state does not evolve yet. The Wigner distribution in phase space $W(\vec{n}, t)$ given by (13) is a Gaussian distribution with width

$$\Delta_\theta \simeq 1/\sqrt{j}.$$

When $t \sim \tau_2$ the two terms $W^{(l)}(0)$ and $\Psi(t')$ are of the same order and the Wigner transform $W(\vec{n}, t)$ starts to widen in phase space. This is the regime that we are seeing in figure 10 since j is not very large.

4.4.2. *The large time $t \gg \tau_2 = \tau_{\text{diff}}$ regime.* When $t \gg \tau_2$ and l is small $l \sim \sqrt{j}$ the $\Psi(t')$ function dominates over the initial $\exp(-l^2/2j)$ term. The coefficients of the Wigner distribution take the form

$$W^{(l)}(t) \simeq \frac{2l+1}{\sqrt{2j+1}} \Psi\left(t \frac{4}{3\pi} \frac{D_0}{\tau_0} \frac{l(l+1)}{j(j+1)}\right). \quad (149)$$

This gives a Wigner distribution $W(\vec{n}, t)$ in phase space (the unit sphere) with a width

$$\Delta_\theta(t) \simeq \sqrt{\frac{t}{\tau_2 j}}. \quad (150)$$

The width of the distribution grows with the time t as \sqrt{t} ; thus the evolution of the spin in this semiclassical regime is a diffusion-like process. But as we shall see, the distribution in phase space is not exactly a Gaussian.

4.4.3. *The final time $t \gg \tau_{\text{equ}} = j\tau_2$ regime.* Finally, and as expected, there is a final equilibration time

$$\tau_3 = j\tau_2 = \tau_{\text{equ}} \quad (151)$$

which corresponds to the equilibration time for the spin. When $t \gg \tau_3 = j\tau_2$, only the $l = 0$ mode survives, the quantum diffusion has completely averaged the spin over the phase space and we obtain a rotationally invariant complete statistical mixture of states

$$W(\vec{n}, t) = \frac{1}{\sqrt{4\pi(2j+1)}}. \quad (152)$$

This corresponds to the canonical distribution since the Hamiltonian for the spin is $H_S = 0$. We recall that, from section 2.2.2 in our normalizations for the Wigner distribution,

$$\int_{S_2} d^2\vec{n} W(\vec{n}, t) = \sqrt{\frac{4\pi}{2j+1}}.$$

4.5. Quantum diffusion and non-Markovianity

4.5.1. *The spin density profile.* We now look more closely at the diffusive regime $\tau_2 \ll t \ll \tau_3$. Since the spin j is assumed to be large, a coherent state evolves into a mixed state whose Wigner distribution $W(\vec{n}, t)$ is still localized in phase space, with a width larger than the size of a quantum coherent state \sqrt{j} , but much smaller than the size of phase space, of order unity.

We perform the rescaling of the spin components

$$\vec{n} = (\vec{u}, \sqrt{1 - \vec{u}^2}), \quad \vec{u} = \sqrt{2j + 1} \vec{z} \quad (153)$$

so that the curvature of phase space becomes negligible, and phase space becomes the flat complex plane \mathbb{C} with coordinate $z = x + iy$ where $\vec{z} = (x, y)$. We perform a similar rescaling for the time

$$t = t' \tau_2 = \frac{\tau_2 j}{D_0} t' \quad (154)$$

and we normalize the Wigner distribution so that

$$\int_{\mathbb{C}} d^2 \vec{z} W(\vec{z}, t') = 1. \quad (155)$$

In this regime of large spin j and not too large time t , the phase space is similar to that of a harmonic oscillator and our results can be compared to those of decoherence models for a harmonic oscillator; see in particular [3, 21, 20].

From (149) we find that in this limit the Wigner distribution is simply the Fourier transform of the universal decoherence function Ψ :

$$\begin{aligned} W(\vec{z}, t') &= (2\pi)^{-2} \int d^2 \vec{\ell} \Psi \left(\frac{8}{3\pi} t' |\vec{\ell}|^2 \right) e^{i\vec{\ell}\vec{z}} \\ &= \frac{3}{8\pi} \frac{1}{t'} \int_{-\pi/2}^{\pi/2} d\theta \cos(\theta) \exp \left(-\frac{|\vec{z}|^2}{t' \cos(\theta)} \frac{3\pi}{16} \right). \end{aligned} \quad (156)$$

This result is valid of course as long as the width of the distribution is large but smaller than \sqrt{j} (the radius of the rescaled sphere), i.e. for time scales such that

$$\tau_2 \ll t \ll j\tau_2 = \tau_3. \quad (157)$$

In this quantum diffusive regime the Wigner distribution takes the universal scaling form

$$W(\vec{z}, t) = \frac{1}{t} W_{\text{quantum}}(r), \quad r = |\vec{z}|/\sqrt{t}. \quad (158)$$

This universal function W_{quantum} is plotted in figure 13, together with the standard Gaussian which represents the density profile associated with a classical Markovian local diffusion process (Wiener process) in phase space:

$$W_{\text{classical}}(\vec{z}, t) = \frac{1}{2\pi t} \exp(-|\vec{z}|^2/(2t)). \quad (159)$$

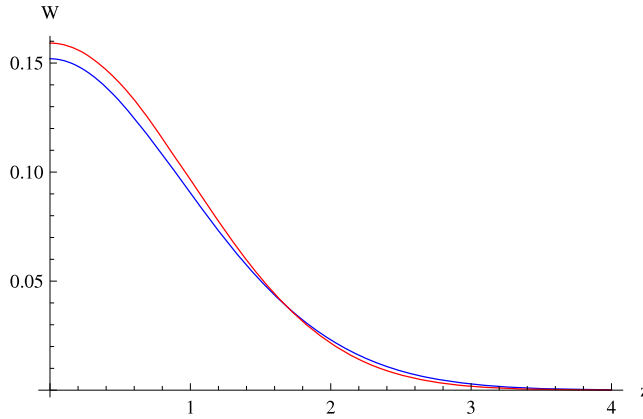


Figure 13. The quantum phase space density profile $W_{\text{quantum}}(z)$ (blue curve) versus the classical one $W_{\text{classical}}(z)$ (red curve) in the large j limit. The normalization is such that the variance is the same, $\langle |z|^2 \rangle = 1$. There is a small but definite difference between the classical and the quantum cases.

4.5.2. Non-Markovianity. It is interesting to note that there is a small but definite difference between the two curves. Thus, once the decoherence caused by the coupling of the spin to the external system \mathcal{E} has absorbed the quantum correlations between the coherent states (and reduced in practice the quantum phase space to the semiclassical phase space, the sphere \mathcal{S}_2), the coupling between the spin \mathcal{S} and \mathcal{E} still affects dynamics of the spin. This effective dynamics is a quantum diffusion process, not exactly a classical diffusion. Non-trivial quantum correlations and quantum memory effects induced by the dynamics of \mathcal{E} and the couplings are present at all time scales, resulting in the non-trivial and non-classical diffusion profile given by equation (156).

Indeed, the self-similar profile $W(\vec{z}, t)$ given by equation (156) can never be the solution of a homogeneous evolution equation local in time of the form

$$\frac{d}{dt}W(\vec{z}, t) = \mathcal{M}_{\vec{z}}W(\vec{z}, t), \quad \mathcal{M}_{\vec{z}} \text{ differential operator in } \vec{z} \quad (160)$$

(as would be the case if it could be written (at least in this limit $t \gg \tau_2$) as a quantum master equation of a Lindblad form). The evolution of the spin at times larger than the decoherence time $\tau_1 = \tau_{\text{dec}}$ is described by a universal self-similar but non-Markovian quantum diffusion process. As will be discussed more in section 6.2, such a non-Markovian behavior is not so surprising. Non-Markovian relaxation processes have been already observed and discussed for two-level systems.

This diffusion is universal in the following sense. It does not depend on the precise choice of the parameters $\Delta(l)$ for the amplitude of the couplings in the different spin channels l , only on the fact that the couplings belong to an $SU(2) \times U(N)$ GUE ensemble, and that they are important only for small l (in our case for a finite number of channels $l \leq l_0$). However the diffusion process cannot really depend on the internal dynamics of the environment \mathcal{E} (since this dynamics is given by the $l = 0$ channel and hence is GUE), nor of the initial state taken for \mathcal{E} , since we take a random state $|\psi_{\text{random}}\rangle \in H_{\mathcal{E}}$.

As we shall show in section 5.4, a Markovian behavior emerges only in the limit where the dynamics for \mathcal{E} becomes very fast, and when the initial state is chosen to be an energy eigenstate of $H^{(0)}$ for \mathcal{E} .

4.5.3. *Spin relaxation.* Finally, as a simple application, let us compute the long time evolution of the spin component along one axis (here the axis z), assuming that one starts from an initial spin state $|\psi_S\rangle$ such that $S_z(0) = \langle\psi_S|S^3|\psi_S\rangle \neq 0$. From

$$S_z(t) = \langle S^3 \rangle(t) = \text{tr}(\rho_S(t)S^3) \quad (161)$$

and averaging over the H_{int} ensemble, we get simply the decoherence function for the $l = 1$ mode:

$$\overline{S_z}(t) = \hat{\mathcal{M}}^{(1)}(t) S_z(0) \quad (162)$$

where (for large j)

$$\hat{\mathcal{M}}^{(1)}(t) \simeq \Psi\left(t \frac{4}{3\pi} \frac{D_0}{\tau_{\text{dyn}}} \frac{2}{j(j+1)}\right) \simeq \Psi\left(\frac{t}{\tau_{\text{equ}}} \frac{8}{3\pi}\right) \quad (163)$$

with $\Psi(t)$ the function defined in section 3.4.4. So the magnetization relaxes to its equilibrium value 0 with a typical time scale $\tau_{\text{equ}} = \tau_3$, as expected. However, the relaxation is not exponential (as would be the case if the quantum diffusion process were Markovian), but only algebraic, as

$$\overline{S_z}(t) \propto \frac{1}{t^3}, \quad t \rightarrow \infty. \quad (164)$$

5. The external dynamics for \mathcal{E}

Up to now, and for simplicity, we have treated the internal dynamics of \mathcal{E} on exactly the same footing as the coupling between \mathcal{S} and \mathcal{E} . In particular, the Hamiltonian $H^{(0)}$ (corresponding to the $l = 0$ sector of H) was chosen to be a random Hamiltonian in a GUE ensemble; its distribution had the same $U(N)$ invariance as for the $l > 0$ sectors. No particular attention was given to the initial state $|\psi_{\text{init}}\rangle$ for \mathcal{E} , which was taken to be a random state.

5.1. The external Hamiltonian for \mathcal{E}

Now the Hamiltonian $H_{\mathcal{E}}$ for the system \mathcal{E} , i.e. the $l = 0$ component $H^{(0)}$ of the Hamiltonian $H_{\mathcal{S}\mathcal{E}}$,³ is taken to be a fixed (not necessarily GUE) Hamiltonian with a regular normalized state density in the large N limit. We define the density of states $\rho(E)$ and the normalized density of states $\nu(E)$ by

$$\nu(E) = \lim_{N \rightarrow \infty} \frac{1}{N} \rho(E), \quad \rho(E) = \sum_{a=1}^N \delta(E - E_a) \quad (165)$$

and assume that $\nu(E)$ is finite and regular. The E_α are the eigenvalues of the Hamiltonian $H^{(0)}$, considered as an $N \times N$ matrix acting on $\mathcal{H}_{\mathcal{E}} = \mathbb{C}^N$. In the calculations we choose as a basis $\{|\alpha\rangle, \alpha = 1, \dots, N\}$ of $\mathcal{H}_{\mathcal{E}}$ the eigenstates of $H^{(0)}$:

$$|\alpha\rangle = |E_\alpha\rangle, \quad H^{(0)}|E_\alpha\rangle = E_\alpha|E_\alpha\rangle. \quad (166)$$

For the $\mathcal{S} + \mathcal{E}$ interaction (the $l \neq 0$ sectors) we keep a random Hamiltonian H' in the $SU(2) \times U(N)$ GUE ensemble, characterized by the variances

$$\Delta' = \{\Delta(l), l = 1, \dots, 2j + 1\}. \quad (167)$$

³ Strictly speaking $H^{(0)} = \mathbf{1}_{\mathcal{S}} \otimes H_{\mathcal{E}}$.

5.2. The general form of the influence functional

We can then repeat the calculation for the influence functional. We shall be interested in the evolution of the spin \mathcal{S} , starting from an initial separable state of the form

$$\rho(0) = \rho^{\mathcal{S}}(0) \otimes |E_\alpha\rangle\langle E_\alpha|, \tag{168}$$

i.e. an arbitrary spin state for \mathcal{S} times a given pure energy eigenstate for \mathcal{E} . One can choose a more general fixed initial state $|E\rangle$ for \mathcal{E} , neither random nor a pure eigenstate of $H_{\mathcal{E}}$, but thanks to the U_N invariance of the ensemble of the $H_{\mathcal{S}\mathcal{E}}$ it is in fact sufficient to consider the previously introduced case.

The influence functional is now a tensor $\mathcal{M}(t, E)$ depending on t and E :

$$\rho_{ru}^{\mathcal{S}}(t) = \overline{\text{tr}_{\mathcal{E}}(e^{-itH}(\rho^{\mathcal{S}}(0) \otimes |\alpha\rangle\langle\alpha|)e^{itH})}_{ru} = \mathcal{M}_{ru,st}(t, E_\alpha)\rho_{st}^{\mathcal{S}}(0) \tag{169}$$

with $H = H^{(0)} + H'$. We shall compute this functional in the large N planar limit.

We find it better to consider the double resolvent

$$\mathcal{G}_{ru,st}^{\alpha,\beta}(x, y) = \overline{\langle r\alpha | \frac{1}{x-H} |s\beta\rangle \langle t\beta | \frac{1}{y-H} |u\alpha\rangle} \tag{170}$$

and the single resolvent

$$\mathcal{H}_{r,s}^{\alpha,\beta}(x) = \overline{\langle r\alpha | \frac{1}{x-H} |s\beta\rangle}. \tag{171}$$

Obviously, when there is no coupling between \mathcal{S} and \mathcal{E} , this resolvent is simply

$$\langle r\alpha | \frac{1}{x-H^{(0)}} |s\beta\rangle = \delta_{r,s} \delta_{\alpha,\beta} \frac{1}{x-E_\alpha}. \tag{172}$$

When there is a coupling, i.e. when the $\Delta(l)$ are non-zero, thanks to the $SU(2)$ invariance of the distribution ensemble for the H' and the fact that the $|\alpha\rangle$ are eigenstates of $H^{(0)}$ we can show that $\mathcal{H}_{r,s}^{\alpha,\beta}(x)$ is still of this form:

$$\mathcal{H}_{r,s}^{\alpha,\beta}(x) = \delta_{r,s} \delta_{\alpha,\beta} \tilde{C}_\alpha(x). \tag{173}$$

In the large N limit, $\tilde{C}_\alpha(x)$ satisfies the recurrence equation (which generalizes the recurrence equation (59) for $\mathcal{H}(x)$)

$$\tilde{C}_\alpha(x) = \frac{1}{x-E_\alpha} + \frac{\hat{D}}{x-E_\alpha} \tilde{C}_\alpha(x) \left(\sum_{\beta=1}^N \tilde{C}_\beta(x) \right) \tag{174}$$

where

$$\hat{D} = \sum_{l=1}^{2j} \frac{2l+1}{2j+1} \Delta(l). \tag{175}$$

Thanks to the $SU(2)$ invariance, we can easily show that the double resolvent \mathcal{G} is diagonal. Its double-Wigner-transform decomposition takes the form

$$W_{\hat{D}(l_1, m_1), (l_2, m_2)}^{\alpha,\beta} = \delta_{l_1, l_2} \delta_{m_1+m_2, 0} (-1)^{m_1} \hat{\mathcal{G}}_{(l_1)}^{\alpha,\beta}(x, y). \tag{176}$$

$\hat{\mathcal{G}}_{(l)}^{\alpha,\beta}(x, y)$ is given by a planar recursion equation similar to that of equation (69) for $\hat{\mathcal{G}}_{(l)}(x, y)$. Its solution is

$$\hat{\mathcal{G}}_{(l)}^{\alpha,\beta}(x, y) = \delta_{\alpha,\beta} \tilde{C}_\alpha(x) \tilde{C}_\alpha(y) + \frac{\hat{D}(l) \tilde{C}_\alpha(x) \tilde{C}_\alpha(y) \tilde{C}'_\beta(x) \tilde{C}'_\beta(y)}{1 - \hat{D}(l) (\sum_{\gamma=1}^N \tilde{C}_\gamma(x) \tilde{C}_\gamma(y))} \quad (177)$$

where the $\hat{D}(l)$ are defined, as in equation (66) (the same $SU(2)$ structure), by

$$\hat{D}(l_1) = \sum_{l'=1}^{2j} \Delta(l')(2l' + 1) (-1)^{2j+l'+l_1} \left\{ \begin{matrix} j & j & l' \\ j & j & l_1 \end{matrix} \right\}. \quad (178)$$

Note however that in equation (178) the sum over l excludes the $l = 0$ case, in contrast to the sum in equation (66) which defines the $\hat{\Delta}(l)$.

Since we are only interested in taking the trace over the final states $|\beta\rangle$, we simply have to consider

$$\hat{\mathcal{F}}_{(l)}^\alpha(x, y) = \sum_{\beta=1}^N \hat{\mathcal{G}}_{(l)}^{\alpha,\beta}(x, y) = \frac{\tilde{C}_\alpha(x) \tilde{C}_\alpha(y)}{1 - \hat{D}(l) (\sum_{\gamma=1}^N \tilde{C}_\gamma(x) \tilde{C}_\gamma(y))}. \quad (179)$$

It is of course natural in the large N limit to re-express the sum over states of \mathcal{E} as a continuum integral over the spectrum of $H^{(0)}$:

$$\sum_{\alpha=1}^N \rightarrow N \int dE \nu(E), \quad \tilde{C}_\alpha(x) \rightarrow \tilde{C}(x, E_\alpha), \quad \tilde{F}_{(l)}^\alpha(x, y) \rightarrow \tilde{F}_{(l)}(x, y, E_\alpha). \quad (180)$$

Equation (174) becomes

$$\tilde{C}(x, E) = \frac{1}{x - E - \hat{\Delta}' \tilde{C}(x)}, \quad \tilde{C}(x) = \int dE \nu(E) \tilde{C}(x, E) \quad (181)$$

with

$$\hat{\Delta}' = N \hat{D} = \sum_{l=1}^{2j} \frac{2l+1}{2j+1} \tilde{\Delta}(l), \quad \tilde{\Delta}(l) = N \Delta(l) \quad (182)$$

while equation (179) becomes

$$\tilde{\mathcal{F}}_{(l)}(x, y, E) = \frac{\tilde{C}(x, E) \tilde{C}(y, E)}{1 - \hat{\Delta}'(l) \int dE' \nu(E') \tilde{C}(x, E') \tilde{C}(y, E')} \quad (183)$$

where

$$\hat{\Delta}'(l) = N \hat{D}(l) = \sum_{l'=1}^{2j} \tilde{\Delta}(l')(2l' + 1) (-1)^{2j+l'+l_1} \left\{ \begin{matrix} j & j & l' \\ j & j & l_1 \end{matrix} \right\}. \quad (184)$$

The notation $\hat{\Delta}'$ and $\hat{\Delta}'(l)$ (with a tilde) in the definitions of equations (182) and (184) is here to recall that there is no $l = 0$ contribution in the sum over l , in contrast to the case for the definition for $\hat{\Delta}(l)$ and $\hat{\Delta}(0)$ given by equations (66) and (67). We have obviously

$$\hat{\Delta}'(0) = \hat{\Delta}'. \quad (185)$$

As in the previous situation, the evolution of the reduced density matrix factorizes into each (l, m) sector, and the influence functional becomes a single function of the time t , of the angular momentum l , and now of the initial state energy E . Thanks to the $SU(2)$ invariance it is still independent of m . It is given by the integral

$$\hat{\mathcal{M}}^{(l)}(t, E) = \oint \frac{dx}{2i\pi} \oint \frac{dy}{2i\pi} e^{-it(x-y)} \tilde{\mathcal{F}}_{(l)}(x, y, E). \quad (186)$$

This can be rewritten in a simpler form by means of the change of variables (and its inverse)

$$w = W(x) = x - \hat{\Delta}'\tilde{C}(x), \quad x = X(w). \quad (187)$$

Indeed equation (181) becomes

$$\tilde{C}(x) = \int dE \frac{\nu(E)}{w - E}, \quad \tilde{C}(x, E) = \frac{1}{w - E} \quad (188)$$

so $\tilde{C}(X(w))$ is the Hilbert transform of $\nu(N)$, i.e. the resolvent of $H^{(0)}$. Then in equation (183) we rewrite the integral as

$$\begin{aligned} \int dE' \nu(E') \tilde{C}(x_1, E') \tilde{C}(x_2, E') &= \int dE' \nu(E') \frac{1}{W(x_1) - E'} \frac{1}{W(x_2) - E'} \\ &= -\frac{\tilde{C}(W(x_1)) - \tilde{C}(W(x_2))}{W(x_1) - W(x_2)} = \frac{1}{\hat{\Delta}'} \left(-\frac{x_1 - x_2}{W(x_1) - W(x_2)} + 1 \right) \end{aligned} \quad (189)$$

and after some algebra the evolution kernel $\hat{\mathcal{M}}^{(l)}(t, E)$ is written in a form similar to that of equation (77), as an integral representation involving the variances $\Delta(l)$ for the interaction Hamiltonians $H_{SE} = \sum H^{(l,m)}$ and the Hamiltonian for E , $H_E = H^{(0)}$, through the function $W(x)$ (related to the inverse of the resolvent for \mathcal{H}_S), and a parameter $Z'(l)$:

$$\begin{aligned} \hat{\mathcal{M}}^{(l)}(t, E) &= \oint \frac{dx_1}{2i\pi} \oint \frac{dx_2}{2i\pi} \frac{e^{-it(x_1-x_2)}}{(W(x_1) - E)(W(x_2) - E)} \\ &\quad \times \frac{1}{(1 - Z'(l)) + Z'(l)((x_1 - x_2)/(W(x_1) - W(x_2)))} \end{aligned} \quad (190)$$

where

$$Z'(l) = \frac{\hat{\Delta}'(l)}{\hat{\Delta}'(0)} \quad (191)$$

with $\hat{\Delta}'(l)$ defined through equations (178) and (183).

5.3. Application to the Wigner ensemble

5.3.1. The general form of the solution. As a simple and illustrative ensemble, let us treat the case where the density spectrum of H_E is the Wigner semi-circle distribution with width $2E_0$:

$$\nu(E) = \frac{2}{\pi E_0^2} \sqrt{E_0^2 - E^2}. \quad (192)$$

This corresponds to taking for $H^{(0)}$ a sample in a GUE distribution, as in section 5.2, with the identification

$$E_0 = 2\sqrt{\bar{\Delta}(0)} \quad \text{where } \bar{\Delta}(0) = \frac{N}{2j+1}\Delta(0) = \hat{\Delta}_{\text{av}}. \quad (193)$$

But now we are able to study the dependence of the final state on the choice of initial state $|E\rangle$.

To compare our calculations with the results of section 5.2, it is convenient to perform the same rescalings. The first time scale τ_0 is now

$$\tau_0 = \frac{1}{\sqrt{\hat{\Delta}(0)}}, \quad \hat{\Delta}(0) = \hat{\Delta}_{\text{av}} + \hat{\Delta}'(0) = \frac{E_0^2}{4} + \sum_{l=1}^{2j} \frac{2l+1}{2j+1} \tilde{\Delta}(l). \quad (194)$$

The $Z(l)$ parameters of equation (76) (not to be confused with the $Z'(l)$ of equation (191) and the average Z_{av} of equation (117) are

$$Z(l) = \frac{\hat{\Delta}_{\text{av}} + \hat{\Delta}'(l)}{\hat{\Delta}_{\text{av}} + \hat{\Delta}'(0)}, \quad Z_{\text{av}} = \frac{\hat{\Delta}_{\text{av}}}{\hat{\Delta}(0)} = \frac{\hat{\Delta}_{\text{av}}}{\hat{\Delta}_{\text{av}} + \hat{\Delta}'(0)}. \quad (195)$$

After performing the rescaling

$$\bar{t} = t/\tau_0, \quad \bar{E} = E\tau_0 \quad (196)$$

the evolution kernel $\hat{\mathcal{M}}^{(l,m)}(t, E)$ takes the form

$$\hat{\mathcal{M}}^{(l,m)}(t, E) = M(\bar{t}, \bar{E}, Z(l), Z_{\text{av}}). \quad (197)$$

The function $M(t, E, Z, Z_{\text{av}})$ is given by an integral representation, which generalizes equation (77):

$$M(\bar{t}, \bar{E}, Z(l), Z_{\text{av}}) = \oint \frac{dX_1}{2i\pi} \oint \frac{dX_2}{2i\pi} \frac{e^{-i\bar{t}(X_1-X_2)}}{(W_1 - \bar{E})(W_2 - \bar{E})} \frac{1 - Z_{\text{av}}H_1H_2}{1 - Z(l)H_1H_2} \quad (198)$$

where as above $H_{1,2}$ are given by the function $H(X)$ (the resolvent for the Wigner ensemble)

$$H_{1,2} = H(X_{1,2}), \quad H(X) = \frac{1}{2}(X - \sqrt{X^2 - 4}) \quad \text{i.e. } X = H + \frac{1}{H} \quad (199)$$

while $W_{1,2}$ are given by the function $W(X)$ defined from $H(X)$ and Z_{av} through

$$W_{1,2} = W(X_{1,2}), \quad \sqrt{Z_{\text{av}}}H(X) = H\left(\frac{W(X)}{\sqrt{Z_{\text{av}}}}\right) \quad \text{i.e. } W(X) = Z_{\text{av}}H(X) + \frac{1}{H(X)}. \quad (200)$$

As in section 5.2, $H(X)$ is analytic in $\mathbb{C} \setminus [-2, 2]$, with the cut along the interval $X \in I_1 = [-2, 2]$ corresponding to the spectrum of the total Hamiltonian $H = H_{\mathcal{E}} + H_{\mathcal{SE}}$. Away from this cut, one has $X \in \mathbb{C} \setminus I_1 \implies |H(X)| > 1$. In addition, $H(W)$ is analytic in $\mathbb{C} \setminus [-2\sqrt{Z_{\text{av}}}, 2\sqrt{Z_{\text{av}}}]$, with the cut along the interval $W \in I_2 = [-2\sqrt{Z_{\text{av}}}, 2\sqrt{Z_{\text{av}}}]$, included in the previous cut since $Z_{\text{av}} < 1$. Away from this cut, one has $W \in \mathbb{C} \setminus I_2 \implies |H| > 1/\sqrt{Z_{\text{av}}}$. Remember also that \bar{E} is the (rescaled) energy of the initial state $|E\rangle \in \mathcal{H}_{\mathcal{E}}$, chosen to be an eigenstate of $H_{\mathcal{E}}$; hence $|\bar{E}| \leq 2\sqrt{Z_{\text{av}}}$. Therefore the integral form for $M(\bar{t}, \bar{E}, Z(l), Z_{\text{av}})$ given in equation (198) is defined for a contour in X_1 and X_2 encircling the cut $I_1 = [-2, 2]$ and the poles $W(X) = \bar{E}$, or equivalently for a contour in H_1 and H_2 such that $|H| > 1/\sqrt{Z_{\text{av}}}$.

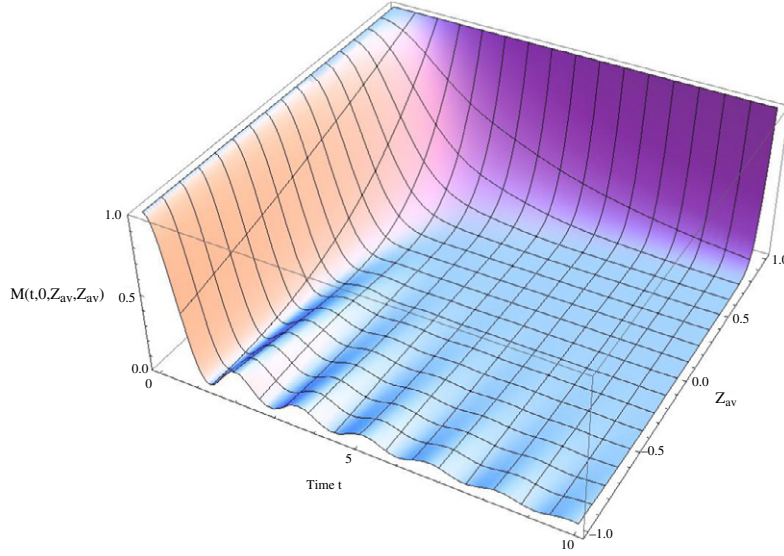


Figure 14. The decoherence function $M(t, E, Z, Z_{\text{av}})$ plotted as a function of t and of Z for $E = 0$ and $Z = Z_{\text{av}}$. Its behavior as a function of time t and of Z_{av} is similar to that of the function $M(t, Z)$.

5.3.2. Discussion of the solution. Although it depends on more parameters, the function $M(\bar{t}, \bar{E}, Z, Z_{\text{av}})$ has a large t behavior similar to the behavior of $M(t, Z)$ depicted in figure 6. As long as $Z < 1$, it decays when $t \rightarrow \infty$ (fast decoherence).

In particular, if we choose $Z_{\text{av}} = 0$ and $E = 0$, we recover the function $M(t, z)$ given by equation (77), depicted in figure 6 and studied above:

$$M(t, E = 0, Z(l), Z_{\text{av}} = 0) = M(t, Z(l)). \quad (201)$$

For a typical l (i.e. $l \sim j$), $Z(l) \sim Z_{\text{av}}$. If we choose

$$Z(l) = Z_{\text{av}} \quad \text{and} \quad \bar{E} = 0 \quad (202)$$

we get

$$M(\bar{t}, 0, Z_{\text{av}}, Z_{\text{av}}) = |N(t, Z_{\text{av}})|^2, \quad N(t, Z_{\text{av}}) = \oint \frac{dX}{2i\pi} \frac{e^{-itX}}{W(X)}. \quad (203)$$

The function $N(t, Z)$ is easily calculated, for instance from its small t and small Z series expansion. The function $M(t, 0, Z, Z)$ is depicted in figure 14. Although it is not exactly the same as the decoherence function $M(t, Z)$ obtained in section 5.2 (see equation (74)) and depicted in figure 6, one sees that it is very similar. In particular one sees that when Z_{av} is close to 1, the decoherence becomes very slow.

5.3.3. Large t asymptotics. This is a general feature: for fixed $Z_{\text{av}} < 1$ and fixed E , when l is small (i.e. $l \ll j$), $Z(l) \simeq 1$ and the decoherence becomes always very slow. This can be shown by re-expressing the integral representation (198) for the function $M(\bar{t}, \bar{E}, Z(l), Z_{\text{av}})$ as a double contour integral over H_1 and H_2 . The contours can be chosen to be the unit circle $H_{1,2} = e^{i\theta_{1,2}}$, and at large time \bar{t} the integral can be evaluated by the saddle point

method. The relevant saddle points are the extrema of the functions $X(H) = H + 1/H$, situated at H_1 and $H_2 = \pm 1$. We shall not detail the full calculation here, but one can show that the final large \bar{t} behavior of $M(\bar{t}, \bar{E}, Z(l), Z_{\text{av}})$ is of a similar form to that for the function $M(t, Z(l))$ given by equation (82):

$$M(\bar{t}, \bar{E}, Z(l), Z_{\text{av}}) = t^{-3}(A - B \sin(4t))(1 + \mathcal{O}(t^{-1})). \quad (204)$$

The coefficients $A = A(\bar{E}, Z(l), Z_{\text{av}})$ and $B = B(\bar{E}, Z(l), Z_{\text{av}})$ depend in a complicated way on the spin l through $Z(l)$, but now also on the energy \bar{E} and on Z_{av} .

Let us just give the form of these coefficients in the limit of small angular momentum l . This is the limit where the decoherence becomes slow and where coherent states emerge as semiclassical states. In this case we have seen that $Z(l)$ is very close to 1, and we find that in this limit

$$A \simeq \frac{1}{\pi} \frac{(1 + Z_{\text{av}})^2 + E^2}{(1 + Z_{\text{av}})^2 - E^2} \frac{(1 - Z_{\text{av}})}{(1 - Z(l))^3} \quad (205)$$

$$B \simeq \mathcal{O}(1). \quad (206)$$

One sees in particular that in this regime the function is a simple scaling function of the variable $t(1 - Z(l))$.

5.3.4. Small $Z(l)$ scaling. One can study the regime (similar to the regime of section 3.4.4) where

$$Z(l) \rightarrow 1, \quad \bar{t}(1 - Z(l)) = t' = \mathcal{O}(1). \quad (207)$$

In this limit the dominant contribution in the integral (198) is given by $H_2 \simeq 1/(Z(l)H_1)$. Noting that $H_1 = e^{-i\theta}$ and integrating over H_2 we find the scaling function

$$M_{\text{scaling}}(\bar{t}, \bar{E}, Z(l), Z_{\text{av}}) = \int_0^\pi \frac{d\theta}{\pi} \frac{(1 - \cos(2\theta))(1 - Z_{\text{av}})}{((1 + Z_{\text{av}})\cos\theta - \bar{E})^2 + (1 - Z_{\text{av}})^2(\sin\theta)^2} e^{-2t' \sin\theta}. \quad (208)$$

For large t' this integral is dominated by the contribution of the endpoints $\theta = 0, \pi$, and this function does indeed scale as

$$M_{\text{scaling}} \simeq \frac{1}{\pi} \frac{(1 + Z_{\text{av}})^2 + E^2}{(1 + Z_{\text{av}})^2 - E^2} \frac{(1 - Z_{\text{av}})}{t'^3}. \quad (209)$$

However, when $1 - Z_{\text{av}}$ is small there is an interesting regime where this integral is dominated by the point θ_c where the denominator is small, namely

$$(1 + Z_{\text{av}})\cos\theta_c - \bar{E} = 0. \quad (210)$$

The integral can be approximated by a Lorentzian integral and one finds that the function M_{scaling} scales as

$$M_{\text{scaling}} \underset{Z_{\text{av}} \rightarrow 1}{\simeq} \exp\left(-t' \sqrt{4 - \bar{E}^2}\right). \quad (211)$$

The crossover between the exponential decay (211) (valid for $t' < t'_{\text{cross}}$) and the algebraic decay (209) (valid for $t' > t'_{\text{cross}}$) occurs for

$$t'_{\text{cross}} \sim \log(1/(1 - Z_{\text{av}})) = \log(\tau_1/\tau_0). \quad (212)$$

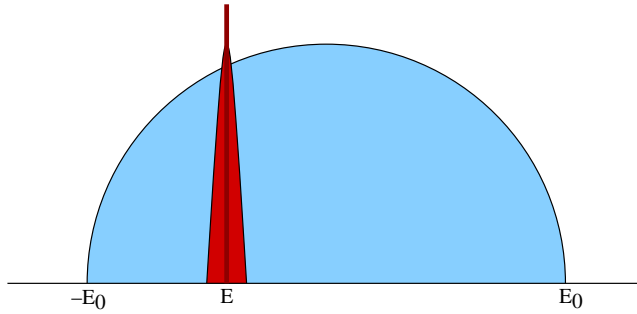


Figure 15. The case of fast \mathcal{E} dynamics. The energy spectrum density $\nu(E)$ of $H_{\mathcal{E}}$ is depicted in blue. E is the energy of the initial energy eigenstate $|E\rangle$ of \mathcal{E} that we couple with the spin \mathcal{S} at time $t = 0$. The energy perturbation of this state and its mixing with other energy eigenstates $|E'\rangle$ of \mathcal{E} induced by the coupling Hamiltonian $H_{\mathcal{S}\mathcal{E}}$ is represented in red.

5.4. The limit of fast \mathcal{E} dynamics ($\tau_1 \gg \tau_0$)

5.4.1. Fast dynamics. These results can be used to study the regime where the dynamics of \mathcal{E} is much faster than the dynamics of \mathcal{S} induced by the coupling $\mathcal{S} + \mathcal{E}$. In the situation considered here (the Wigner ensemble for $H_{\mathcal{E}}$), this occurs if the width of the spectrum of $H_{\mathcal{E}}$, namely $2E_0$, is much larger than the width of the spectrum of $H_{\mathcal{S}\mathcal{E}}$, which is of the order of $\sqrt{\hat{\Delta}'(0)}$:

$$E_0 = 2\sqrt{\hat{\Delta}_{\text{av}}} \gg \sqrt{\hat{\Delta}'(0)}. \quad (213)$$

This situation is depicted in figure 15.

In this case we use the results of section 4.2. The time scale τ_0 for the internal dynamics of the whole system is from equation (194)

$$\tau_0 = \frac{1}{\sqrt{\hat{\Delta}(0)}} \simeq \frac{1}{\sqrt{\hat{\Delta}_{\text{av}}}} = \frac{2}{E_0}. \quad (214)$$

The decoherence time scale τ_1 for a typical spin state is from equation (131)

$$\tau_1 = \frac{\tau_0}{1 - Z_{\text{av}}} \simeq \tau_0 \frac{\hat{\Delta}_{\text{av}}}{\hat{\Delta}'(0)} \quad (215)$$

and hence it is much larger than τ_0 :

$$\tau_1 \gg \tau_0. \quad (216)$$

We recall that from equation (194), $\hat{\Delta}'(0)$ is defined by

$$\hat{\Delta}'(0) = \sum_{l=1}^{l_0} (2l+1) \bar{\Delta}(l) \quad (217)$$

where the $\bar{\Delta}(l)$ are of order $\mathcal{O}(1)$. The factor $Z(l)$ behaves for small l according to equation (103):

$$Z(l) = 1 - \frac{l(l+1)}{j(j+1)} \frac{D_0}{4} + \dots \quad (218)$$

with now

$$D_0 \simeq \frac{1}{\bar{\Delta}_{\text{av}}} \sum_{l=1}^{l_0} \bar{\Delta}(l) l(l+1)(2l+1) \ll 1. \quad (219)$$

We have seen that the decoherence time scale for a coherent state is from equation (133)

$$\tau_2 = \tau_0 \frac{j}{D_0}. \quad (220)$$

Therefore when the spin j is large it is always much larger than τ_1 since

$$\frac{\tau_2}{\tau_1} = \frac{j \sum_{l=1}^{l_0} (2l+1) \bar{\Delta}(l)}{\sum_{l=1}^{l_0} l(l+1)(2l+1) \bar{\Delta}(l)} \sim \frac{j}{l_0^2} \gg 1. \quad (221)$$

5.4.2. *The evolution of coherent states.* In the limit of large E_0 , from equations (213) and (195) we have in any case

$$Z_{\text{av}} \simeq 1 \quad \text{and} \quad Z(l) \simeq 1 \quad (222)$$

and we can apply the results of the section 5.3.4. Let us consider as an initial state $|\Psi\rangle$ of the system $\mathcal{S} + \mathcal{E}$ the single spin coherent state $|\vec{e}_z\rangle$ times the $H_{\mathcal{E}}$ eigenstate $|E\rangle$:

$$|\Psi(t=0)\rangle = |\vec{e}_z\rangle \otimes |E\rangle. \quad (223)$$

We study the evolution of the reduced spin matrix density for times $t \gg \tau_0$, as was done in section 4.4 for the case of a spin coherent state $|\vec{e}_z\rangle$ times a random \mathcal{E} state $|\psi\rangle$. The harmonics of the Wigner transform of the density matrix are now, using equation (211) for the scaling of the evolution functional $M(t, E, Z(l), Z_{\text{av}})$,

$$W^{(l)}(t) = \frac{2l+1}{\sqrt{2j+1}} \exp\left(-\frac{l^2}{2j}\right) \exp\left(-t \frac{l(l+1)}{j(j+1)} \frac{D_0}{4} \sqrt{E_0^2 - E^2}\right) \quad (224)$$

to be compared with equation (147). This gives a Wigner distribution $W(\vec{n}, t)$ on the unit sphere with width proportional to \sqrt{t} when $t \gg \tau_2$; hence the evolution of the spin is again a quantum diffusion process.

5.4.3. *The Markovian limit for quantum diffusion in phase space.* For fast \mathcal{E} dynamics, initial state $|E\rangle$ and large spin $j \gg 1$, the spin diffusion at $t \gg \tau_2$ becomes Markovian, with no memory effects. Indeed, at not too large times, when we can approximate the phase space \mathcal{S}_2 by the tangent complex plane \mathbb{C} , the Wigner distribution $W(\vec{u}, t)$, which is the Fourier transform of the decoherence function

$$M^{(l,m)}(t) = \exp\left(-t \frac{l(l+1)}{j(j+1)} \frac{D_0}{4} \sqrt{E_0^2 - E^2}\right),$$

is exactly a Gaussian distribution of the form (159), given by equation (156):

$$W(\vec{u}, t; D) = \frac{1}{4\pi D t} \exp(-|\vec{u}|^2 / (4D t)), \quad (225)$$

instead of the non-classical distribution function W_{quantum} of equation (156). This corresponds to a classical diffusion process (Wiener process) on the unit sphere⁴ \mathcal{S}_2 with a diffusion coefficient D which depends on the energy E of the initial state through

$$D = \frac{D_0 \sqrt{E_0^2 - E^2}}{4j(j+1)}. \quad (226)$$

This Markovian regime where quantum diffusion is classical is expected to hold only when the exponential scaling (211) of M_{scaling} is valid. Since it holds only for $t(1 - Z(l)) < t'_{\text{cross}}$ given by (212), we can show that non-Markovian deviations from the distribution function $W(\vec{Z}, t)$ given by (225) occur when (this is a very rough estimate)

$$\frac{|\vec{u}|^2}{t} > 4D \log(\tau_1/\tau_0). \quad (227)$$

This means that departures from Markovian behavior for finite τ_1/τ_0 are more easily observable in the large distance/small time diffusive regime.

5.4.4. A golden rule formula for the diffusion coefficient. It is interesting to note that the diffusion coefficient D can be rewritten in terms of the normalized density of states $\nu(E)$ for the system \mathcal{E} , given by equation (192):

$$\nu(E) = \frac{2}{\pi E_0^2} \sqrt{E_0^2 - E^2}, \quad (228)$$

and of the norm of the commutator

$$\vec{C} = i[\vec{S}, H_{S\mathcal{E}}] \quad (229)$$

between the spin operator \vec{S} and the coupling Hamiltonian $H_{S\mathcal{E}}$. Indeed, using equation (219) for D_0 , equation (193) for E_0 , and the equations (133)–(137) of section 4.2.3, we have

$$D = 2\pi\nu(E) \frac{1}{N(2j+1)^2} \text{tr}(\vec{C}^2) = 2\pi\nu(E) \frac{1}{2j+1} \|\vec{C}\|^2. \quad (230)$$

Using equation (165) we can rewrite the diffusion coefficient D in terms of the full density of states $\rho(E) = N\nu(E)$ for the Hamiltonian $H_{\mathcal{E}}$, acting on the full system, and of the average squared norm of a matrix element of the commutator \vec{C} :

$$\mathbb{E}[|\langle \Psi | \vec{C} | \Psi' \rangle|^2] = \frac{1}{N(2j+1)} \|\vec{C}\|^2 = \frac{1}{(N(2j+1))^2} \sum_{\Psi, \Psi'} |\langle \Psi | \vec{C} | \Psi' \rangle|^2 \quad (231)$$

where $\{|\Psi\rangle\}$ is a basis of $H_{S\mathcal{E}} = \mathbb{C}^{N(2j+1)}$. Indeed we rewrite equation (230) as the general form for the spin diffusion coefficient D :

$$D = 2\pi\rho(E) \mathbb{E}[|\langle \Psi | \vec{C} | \Psi' \rangle|^2]. \quad (232)$$

This form for the diffusion coefficient for the dynamics of coherent states, as a function of the energy E of the initial state for the environment, is similar to the Fermi golden

⁴ Important: we do not perform the rescaling (153) by a factor $\sqrt{2j+1}$ on the spin here.

rule for the transition rate of a state ψ of a quantum system into a continuum of states Ψ' (with density of final states ρ_{final}), induced by a perturbation Hamiltonian H_{pert} :

$$T_{\Psi \rightarrow \Psi'} = 2\pi \rho_{\text{final}} |\langle \psi | H_{\text{pert}} | \Psi' \rangle|^2. \quad (233)$$

This is not so surprising since we are in a regime where the effective dynamics is Markovian, and it is known that there is a relation between the Markov approximation and the Fermi golden rule for simple systems coupled to a bath (see [2] and references therein). The form (232) for the diffusion coefficient can therefore be viewed as a fluctuation dissipation relation for the quantum dynamics of the spin in the regime $t \gg \tau_2$. What is interesting is that it involves the matrix elements of the commutator \vec{C} between the spin operator \vec{S} and the interaction Hamiltonian $H_{\mathcal{SE}}$.

We therefore expect that the relation (232) for D is general and independent of the explicit form of the density of states $\rho(E)$ for the external Hamiltonian $H_{\mathcal{E}}$. It should be easy to study the limit $\tau_1 \gg \tau_2$ using the general form (190) for the evolution functional $M(t, E, Z(l), Z_{\text{av}})$ instead of the explicit form (198) for a semi-circle state density law. In particular, we remark that the derivation of the exponential scaling (211) starting from the general form (208) of the scaling function M_{scaling} at small l uses a Lorentzian approximation very similar to the one used in the textbook derivation of the Fermi golden rule.

5.4.5. The dependence on the initial state $|\psi_{\mathcal{E}}\rangle$ and randomized Markovian processes. It remains to understand why, even in the limit of fast dynamics $\tau_1 \gg \tau_0$, if for \mathcal{E} as initial state $|\psi_{\text{init}}\rangle = |\psi(0)\rangle$ in $H_{\mathcal{E}}$ one starts from a random state $|\psi_{\text{random}}\rangle$, in the quantum diffusion regime the distribution profile in phase space $W(\vec{u}, t)$ for the spin is not a Gaussian, but is given by the non-Gaussian quantum distribution W_{quantum} of equation (156) given by the Fourier transform of the scaling function Ψ obtained through equation (85).

This can be understood easily. First let us come back to the case where $H_{\mathcal{E}}$ is a GUE random Hamiltonian, and so its normalized density of states $\nu(E)$ is given by the Wigner semi-circle law (228)

$$\nu(E) = \frac{2}{\pi E_0^2} \sqrt{E_0^2 - E^2}. \quad (234)$$

In the fast \mathcal{E} dynamics limit $\tau_1/\tau_0 \rightarrow \infty$ and large spin limit $j \rightarrow \infty$, if one starts from a separable initial state product of a coherent spin state $|\vec{n}\rangle$ and a random \mathcal{E} state $|\psi_{\text{random}}\rangle$, in the quantum diffusive regime $\tau_2 \ll t \ll j\tau_2$ the probability distribution for the spin (156) can be written as

$$W(\vec{u}, t) = \int dE \nu(E) W(\vec{u}, t; D(E)) \quad (235)$$

with $\nu(E)$ given by (234), $W(\vec{u}, t; D)$ the Gaussian density profile (225) and $D(E)$ the spin diffusion coefficient (226) when starting from an initial energy eigenstate $|E\rangle$:

$$W(\vec{u}, t; D(E)) = \frac{1}{4\pi D(E)t} \exp\left(-\frac{1}{4D(E)} \frac{|\vec{u}|^2}{t}\right), \quad D(E) = \frac{D_0 \sqrt{E_0^2 - E^2}}{4j(j+1)}. \quad (236)$$

Equation (235) has a simple probabilistic interpretation. The spin diffusion process is simply a statistical superposition, with probability distribution $\nu(E)$, of the Wiener processes with the different diffusion constants $D(E)$. In other words, starting from a random state for \mathcal{E} , to estimate the probability (density) of observing the spin at position \vec{u} in phase space at time t , for all practical purpose we may assume that:

- (1) at time 0 we choose at random an energy eigenstate $|E\rangle$ for \mathcal{E} (therefore with probability law given by the density of states $\nu(E)$),
- (2) then we let the spin diffuse according to a Wiener process with diffusion constant $D(E)$.

This kind of non-Markovian process is built by the operation of randomization of a family of Markovian processes (see the mathematical literature, for instance [9]). Here the family of Wiener processes depending on the parameter E through the diffusion constant $D(E)$ is randomized with the probability distribution $\nu(E)$.

To explain the physical origin of this randomization, we remark that a random state $|\psi_{\text{random}}\rangle$ of \mathcal{E} is nothing but a quantum superposition of energy eigenstates of $H_{\mathcal{E}}$ with random complex coefficients c_E , chosen independently according to the Gaussian normal distribution

$$|\psi_{\text{random}}\rangle = \sum_E c_E |E\rangle. \quad (237)$$

Hence the initial step (1) amounts to projecting the initial state $|\psi_{\text{random}}\rangle$ onto a random energy eigenstate $|E\rangle$, i.e. to performing an ideal (von Neumann) measurement of the energy of the external system \mathcal{E} .

This is to be expected. Indeed, we are in the regime where the interaction Hamiltonian $H_{S\mathcal{E}}$ is a small perturbation compared to the internal $H_{\mathcal{E}}$ Hamiltonian for \mathcal{E} . But we are also in the large spin regime where $j \gg 1$, so the spin \mathcal{S} can be considered as a large quantum system, with many states, weakly coupled to \mathcal{E} . In the same way as spin decoherence occurs with the time scales τ_1 with (from (130) and (131))

$$\frac{\tau_0}{\tau_1} = \left(\frac{\|H_{S\mathcal{E}}\|}{\|H\|} \right)^2 \simeq \left(\frac{\|H_{S\mathcal{E}}\|}{\|H_{\mathcal{E}}\|} \right)^2 \ll 1 \quad (238)$$

with $H = H_{S\mathcal{E}} + H_{\mathcal{E}}$, we expect the decoherence between (not too close on the spectrum E line) energy eigenstates $|E\rangle$ of $H_{\mathcal{E}}$ to occur with a time scale τ_e given by

$$\frac{\tau_0}{\tau_e} = \left(\frac{\|[H, H_{\mathcal{E}}]\|}{\|H\| \cdot \|H_{\mathcal{E}}\|} \right)^2 \simeq \left(\frac{\|[H_{S\mathcal{E}}, H_{\mathcal{E}}]\|}{\|H_{\mathcal{E}}\|^2} \right)^2 \ll 1. \quad (239)$$

In other words, the interaction between \mathcal{S} and \mathcal{E} induces both decoherence for the spin and decoherence in energy for the external system. As a consequence, at the time scale τ_2 , where quantum diffusion starts, the initial coherent state $|\vec{n}\rangle$ for the spin can be considered as a classical spin state \vec{n} , while the initial random state $|\psi_{\text{random}}\rangle$ for \mathcal{E} can be considered as a classical equidistributed statistical mixture of energy states E .

This suggests the general form of the distribution profile $W(\vec{n}, t)$ as a function of the \mathcal{E} initial state $|\psi_{\text{init}}\rangle$. We consider a general $H_{\mathcal{E}}$ with a continuous density of states

$\rho(E) = N\nu(E)$, and we normalize the energy eigenstates $|\mathbf{E}\rangle = (1/\sqrt{\rho(E)})|E\rangle$ such that

$$\langle \mathbf{E} | \mathbf{E}' \rangle = \delta(E - E'). \quad (240)$$

If we start from a general initial state $|\psi\rangle$ for \mathcal{E} which has a smooth decomposition in energy eigenstates, that is to say

$$\langle \mathbf{E} | \psi \rangle = \psi(E) \quad \text{is a regular function of } E, \quad (241)$$

at least when averaged over small energy intervals $\Delta E \simeq 1/\tau_2$, then the probability distribution for the spin at large time $t \gg \tau_{\text{diff}}$ should be given by

$$W(\vec{u}, t; \psi) = \int dE |\langle \mathbf{E} | \psi \rangle|^2 \frac{1}{4\pi D(E) t} \exp\left(-\frac{|\vec{u}|^2}{4 D(E) t}\right), \quad (242)$$

with the effective diffusion constant $D(E)$ given by equation (232)

$$D(E) = 2\pi\rho(E)\mathbb{E}, \quad \mathbb{E} = \mathbb{E}[|\langle \Psi | [H_{\mathcal{SE}}, \vec{S}] | \Psi' \rangle|^2] = \frac{\text{tr}(\text{i}[H_{\mathcal{SE}}, \vec{S}])^2}{(N(2j+1))^2}. \quad (243)$$

6. Discussion and conclusions

6.1. Summary

We have proposed and solved a model of a general quantum spin j interacting with a large external system (with N states). The interaction Hamiltonian H_{int} is described by a new random matrix model, relying on a $\text{GU}_{2 \times N} \mathbb{E}$ ensemble, which takes into account the interactions in all possible angular momentum channels. The only constraint is that the probability distribution for H is Gaussian and invariant under the spin symmetry $SU(2)$ and the external $U(N)$ group. The dynamics of the model is solved exactly in the large N limit, in the case where the internal dynamics of the spin is trivial. We can thus study in full generality the decoherence of spin states in various dynamical regimes, and as a function of the initial condition and of the dynamics of the external system.

For the semiclassical $j \rightarrow \infty$ limit, we are able to characterize quantitatively the interaction Hamiltonians which are such that coherent spin states have a much longer coherence time than any other states, and emerge effectively as the semiclassical ‘pointer states’ of a classical spin. The large time dynamics of these coherent states is found to be a quantum diffusion process, in general non-Markovian.

When the dynamics of the external system is much faster than the dynamics induced by its coupling with the spin, and when the initial state of the external system is an energy eigenstate, the quantum spin diffusion is found to be Markovian. The diffusion coefficient takes a golden rule form, but involving matrix elements of the commutator between the spin and the interaction Hamiltonian. When the initial external state is a general state, the quantum spin diffusion is described by a randomization of Markov processes. We argue that this is explained by the decoherence of energy states (for the external system) induced by the coupling with the spin.

6.2. Comparison with previous studies of decoherence in spin systems

We finally discuss the relation of the present work with some previous studies of open quantum spin systems.

In the pioneering work [22], the relaxation of a large spin j in an external magnetic field and coupled to a heat bath was already studied. A coherent state representation for the spin states different from ours is used. The Born approximation and some methods of [1] are used. Therefore their results are only valid for large time. A relaxation towards a classical distribution is obtained and the corresponding diffusion coefficients are calculated. In [19] the spin relaxation in the specific case of the $j = 1/2$ spin is studied within the Markovian approximation. In both papers the spin has a non-trivial dynamics since it is coupled to an external field; this is not taken into account in our study.

In [15] the relaxation of a $j = 1/2$ spin coupled to an external large system (bath) is studied. The spin has no internal dynamics ($H_{\text{spin}} = 0$); the bath has its own dynamics characterized by some H_{B} with its density of states. The interaction is of the simple form $H_{\text{int}} = \sigma_x \otimes V_{\text{ext}}$, where V_{ext} is a random operator for the external system, chosen in a GOE ensemble local in energy E (for instance the matrix elements of V_{ext} are non-zero only between energy eigenstates such that the energy difference is finite, $|E_1 - E_2| \leq \Delta E$; thus it is a finite width band GOE). They use the RMT methods which will be used in most subsequent studies based on RMT, including the present one. With the initial condition considered they are able to study the relaxation of the spin at all time, and find an exponential relaxation (already with a golden rule form for the relaxation time).

In [13, 12] the study of relaxation for this spin $j = 1/2$ model with a band GOE H_{int} has been extended to the non-trivial case where the spin has its own dynamics $H_{\text{spin}} \propto \sigma_z$. Special attention is paid to the case where the external bath is formed of a large number of independent large subsystems, in some particular limits where an effective temperature can be associated with the large bath. Explicit expressions are obtained for the evolution of the spin $1/2$ density matrix, but in practice the regime of relaxation towards an equilibrium distribution is studied, in particular in the so-called van Hove limit where the coupling spin bath is small, which corresponds to our limit $\tau_{\text{dyn}} \ll \tau_{\text{dec}}$. Thus the study of [13, 12] is more limited than our work, since only the spin $j = 1/2$ case is considered and the GOE ensemble for H_{int} is not $SU(2)$ invariant, but also more general, since H_{int} is a band matrix in the energy states for the bath, and, most importantly, the spin is coupled to an external magnetic field, so relaxation and thermalization effects can be studied.

Another study involving RMT techniques is [14]. General system + bath couplings with band random matrix Hamiltonians are studied, both for a quantum harmonic oscillator and for a two-level system. In the case studied, Markovian equations can be derived.

In [8] an extensive study of the relaxation of a two-level system (spin $1/2$) coupled to a bath is performed. The coupling is given by a $H_{\text{int}} = \sigma_x \otimes V_{\text{bath}}$ Hamiltonian with V_{bath} a GUE random matrix, H_{bath} also given by a random matrix, and $H_{\text{spin}} \propto \sigma_z$. Most of the study is based on numerical simulations. The finite N effects, where N is the size (number of states) of the bath, the statistics of the eigenvalues, and the crossover between the Poisson and Wigner behavior for the spacing of the eigenvalues,

are very thoroughly studied. Explicit checks of the self-averaging property for different realizations of the random Hamiltonian are made. The relaxation and equilibrium effects are considered. Interesting analytical results are obtained in the regime of weak coupling (which corresponds again to the regime $\tau_{\text{dec}} \gg \tau_{\text{dyn}}$ in the present paper). In particular, the fact that the relaxation can be non-Markovian is observed and discussed. The fact that in the regime of strong coupling the decay of the magnetization (evolution of $\langle \sigma_z \rangle$) with time can be non-exponential but algebraic with some oscillations is also observed.

The relaxations of a two-level system (and systems with a small but >2 number of levels) coupled by RMT Hamiltonians to a bath have also been studied numerically and by the so-called HAM method in [16, 10]. The two-level system has been studied more extensively and precisely by the TCL method in [4]. Here also, it is found that depending on the initial states for the system and the bath, the relaxation dynamics can be non-Markovian.

Finally, let us quote some papers which discuss more specifically the problem of decoherence in spin systems. There are of course many studies of the dynamics of several coupled spins $1/2$, and of their equilibration dynamics; see for instance [17]. In a recent numerical study [26] it is confirmed for instance that decoherence is a much faster process than thermal relaxation.

In [3, 21, 20] decoherence is studied for various systems (mostly a particle or an oscillator) coupled to a bath through a simple $H_{\text{int}} = Q_{\text{sys}} \otimes B_{\text{bath}}$ Hamiltonian (sometimes two), in the so-called interaction dominated regime, which should correspond in the present paper to the regime $\tau_{\text{sys}} \ll \tau_{\text{dec}}$. Of particular interest for the present work is section VII of [21], where the system considered is one with large spin j , studied by coherent states techniques. However the difference is that the coupling agent is just $Q_{\text{sys}} = \sigma_x$, while the bath Hamiltonian B_{bath} is a sum of independent B_i for small independent sub-baths (so random limit theorems may be applied). Thus it is not clear how to compare this model to our model. Some small time expansions are used and with these approximation the decay of the off-diagonal elements of the density matrix is obtained (decoherence) but with a time dependence which is very different from the exact results that we have obtained.

6.3. Generalizations and open questions

Many interesting questions remain to be addressed for these kinds of models of a spin coupled to a large external system. Firstly, we have seen that the non-Markovian dynamics that we obtain for large spin j has been already observed in some two-level systems. In order to better understand this effect, and how the non-Markovianity is related to the dynamics of the coupling and to the initial states, one must generalize the random interaction Hamiltonians that we have considered to the more general case of an Gaussian ensemble with a law depending on the initial and final energy states for the external system, as in [15, 13]. It should not be difficult to extend the study of section 5 to this more general case, but this will be discussed in further work.

In order to compare more precisely our results with those of previous studies, one would like to study simpler coupling Hamiltonians of the form $H_{\text{int}} = Q_{\text{sys}} \otimes B_{\text{bath}}$ with for instance $Q_{\text{sys}} = S_x$. However, our model is solvable and its solution takes a simple form precisely because the random interaction Hamiltonian belongs to an ensemble H_{int} which

is $SU(2)$ invariant. This is clearly not the case for the simple H_{int} s considered previously, since there is a favored x direction. Another simplification would be to consider a coupling of the large spin j with a large collection of independent systems, as is done in [21]. However, here again, to do so while preserving the $SU(2)$ invariance of the ensemble of random interaction Hamiltonians is not that easy and does not lead to models simple enough to be discussed here.

We have not studied the case where the internal dynamics for the spin is non-trivial ($H_S \neq 0$). This has been done analytically for the spin $j = 1/2$ case in [13, 12], and they have obtained explicit expressions (of course more complex than our results for $j = 1/2$). It should not be difficult to extend their results to our class of interaction Hamiltonians in the specific $j = 1/2$ case (then only the $l = 1$ sector contributes). However the problem becomes increasingly complicated as the spin j increases, and no general solution valid for generic spin j has been obtained yet. This is clearly an interesting problem, relevant for studying the interplay between decoherence and dissipation. If the internal spin dynamics is slow, i.e. when H_S is small, it should be possible to study the dynamics by standard approximation methods (short time expansion, TCL).

We have studied the matrix model in the planar large N limit, where N corresponds to the size (number of states) of the external system. It should be interesting to study the case of finite but large N , as has been done already for $j = 1/2$ in [8]. Especially interesting should be the case where N (the dimension of the external system) is of the same order as j (the spin).

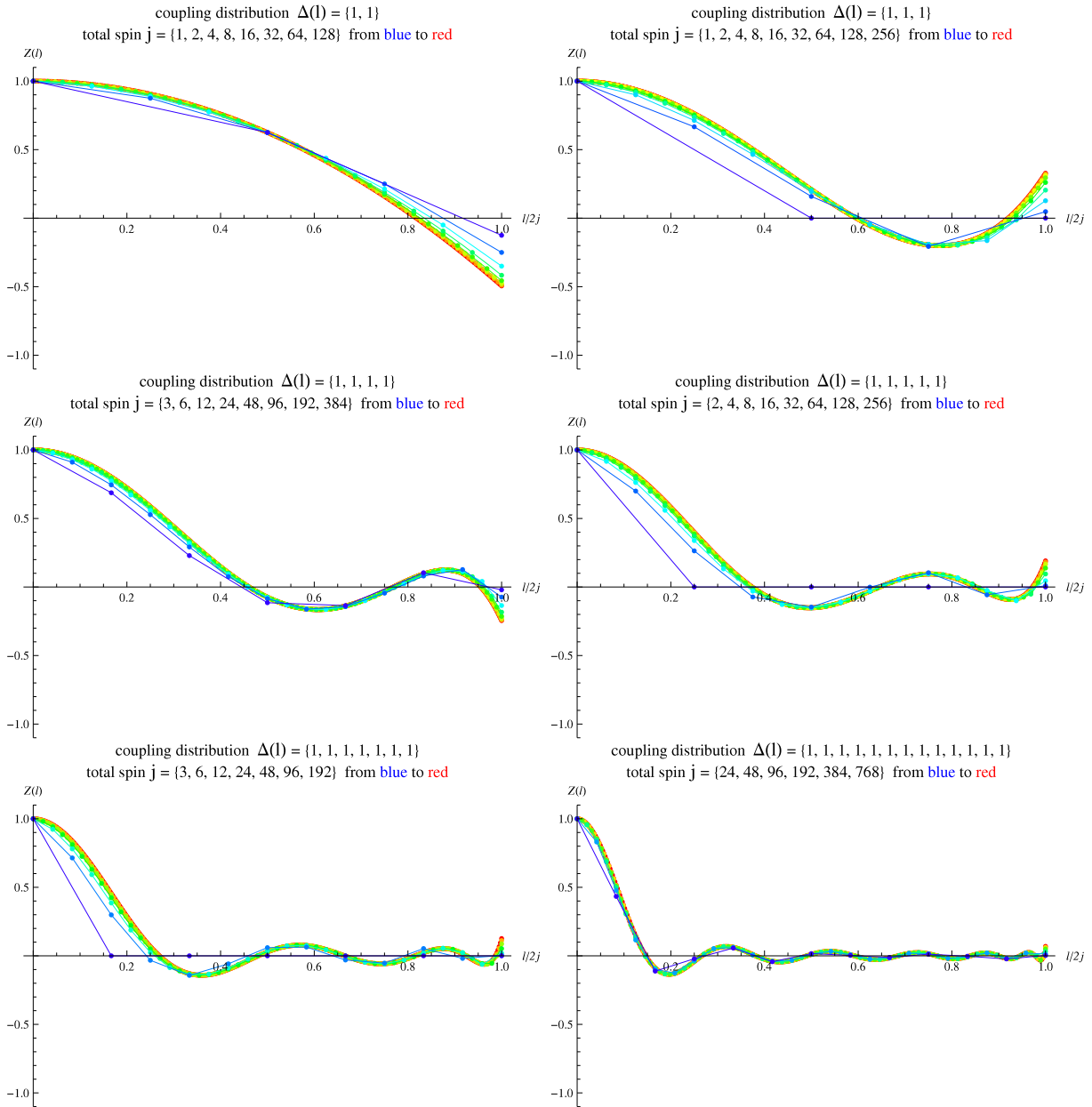
In our model we have considered a random $M \times M$ matrix model of the GUE type, but such that the distribution probability is not invariant under the whole $U(M)$ group, but only under the action of a subgroup G acting on the space (in our case $G = SU(2) \times U(N)$ with $M = (2j + 1)N$). This class of models belongs of course to the very general class of block random matrix models, but in our case the presence of the symmetry group G and of its spin representation adds a lot of structure and simplifications. It should be interesting (at least mathematically) to study these kinds of models for more general representations of the group $G = SU(2) \times U(N)$, and for general groups and representations.

Finally, the model presented here is very idealized and is mostly of pedagogical and mathematical value. But it should be a first step in studying general but more realistic models of spin decoherence, in particular for systems where the classical macroscopic degrees of freedom do not come from a *per se* large quantum spin, but emerge from the interaction between many small quantum spins.

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Appendix. The function $Z(l)$ for various coupling distributions $\Delta(l)$ and spin j



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Figure A.1. $Z(l)$ for various l_0 and j ; all $\Delta(l)$ are equal.

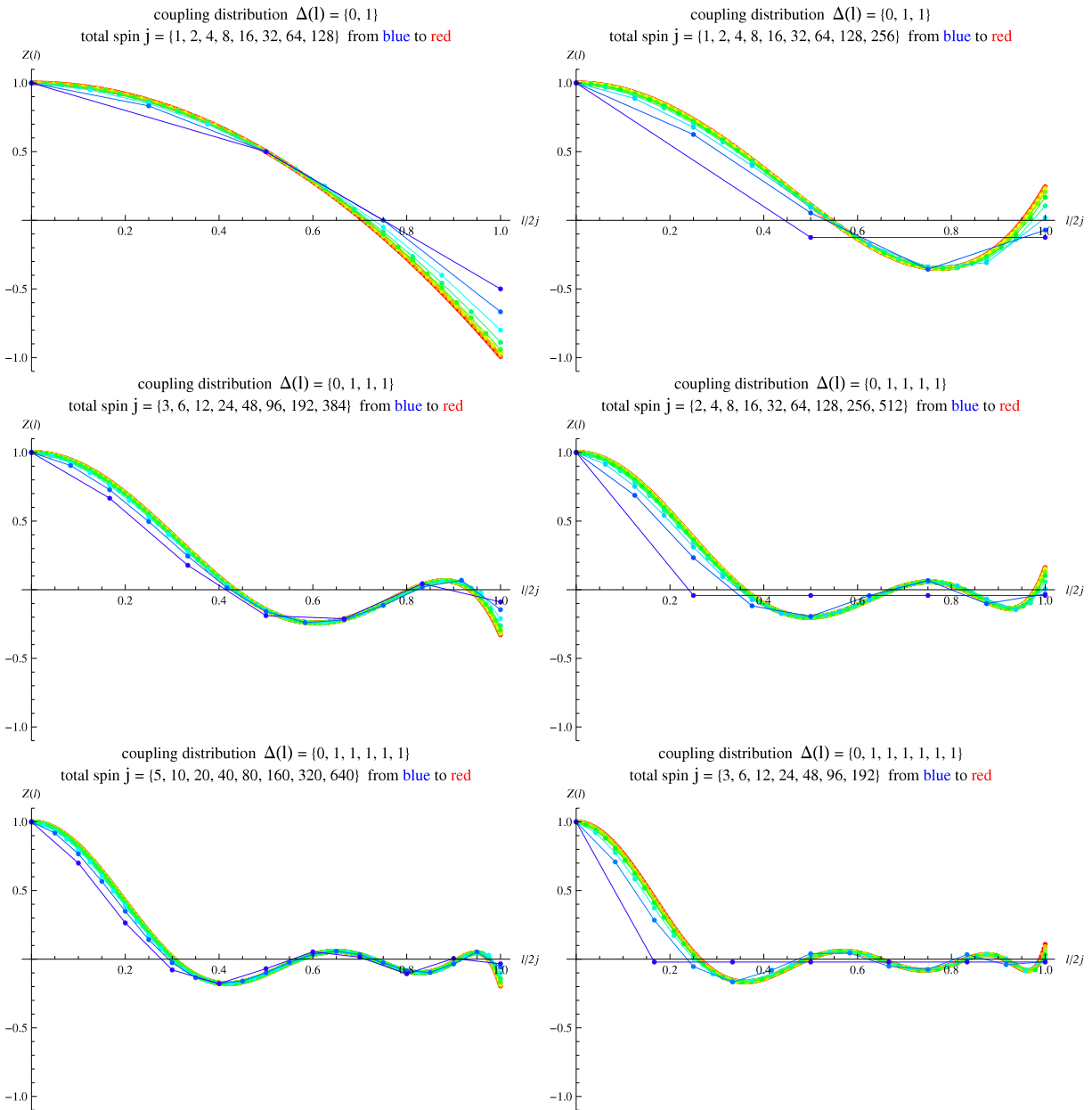


Figure A.2. $Z(l)$ for various l_0 and j ; all $\Delta(l)$ are equal but $\Delta(0) = 0$.

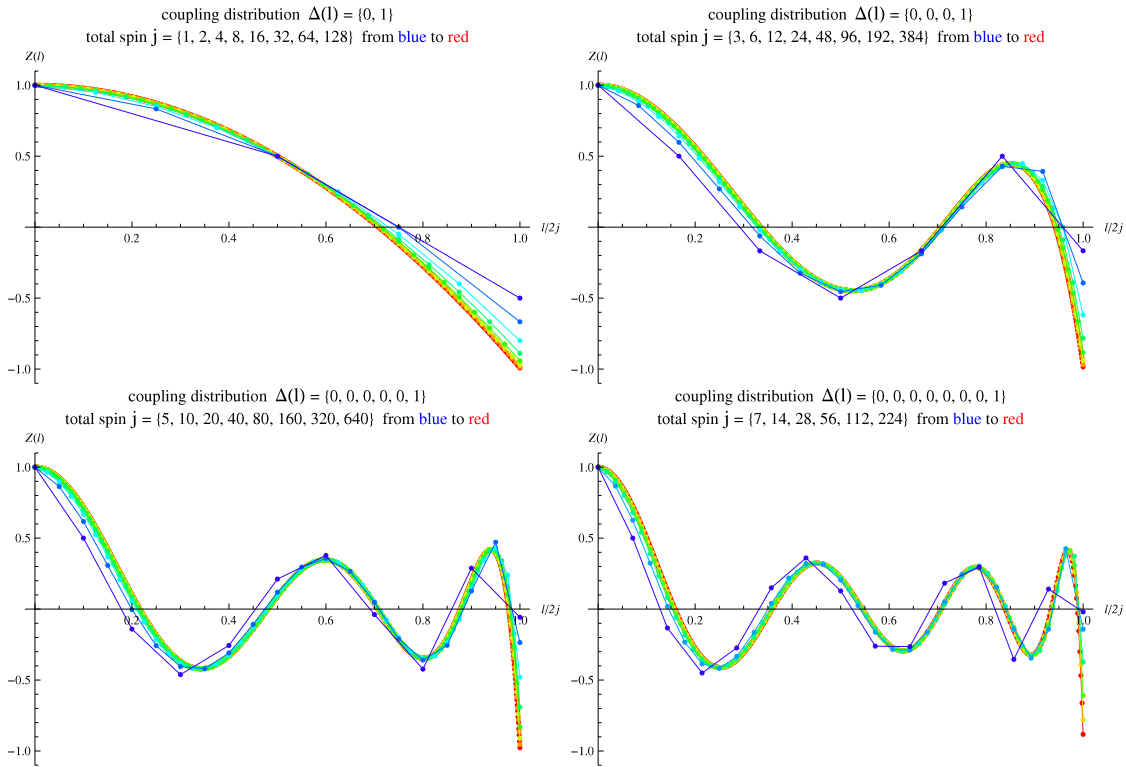


Figure A.3. $Z(l)$ for various l_0 and j ; one odd l only.

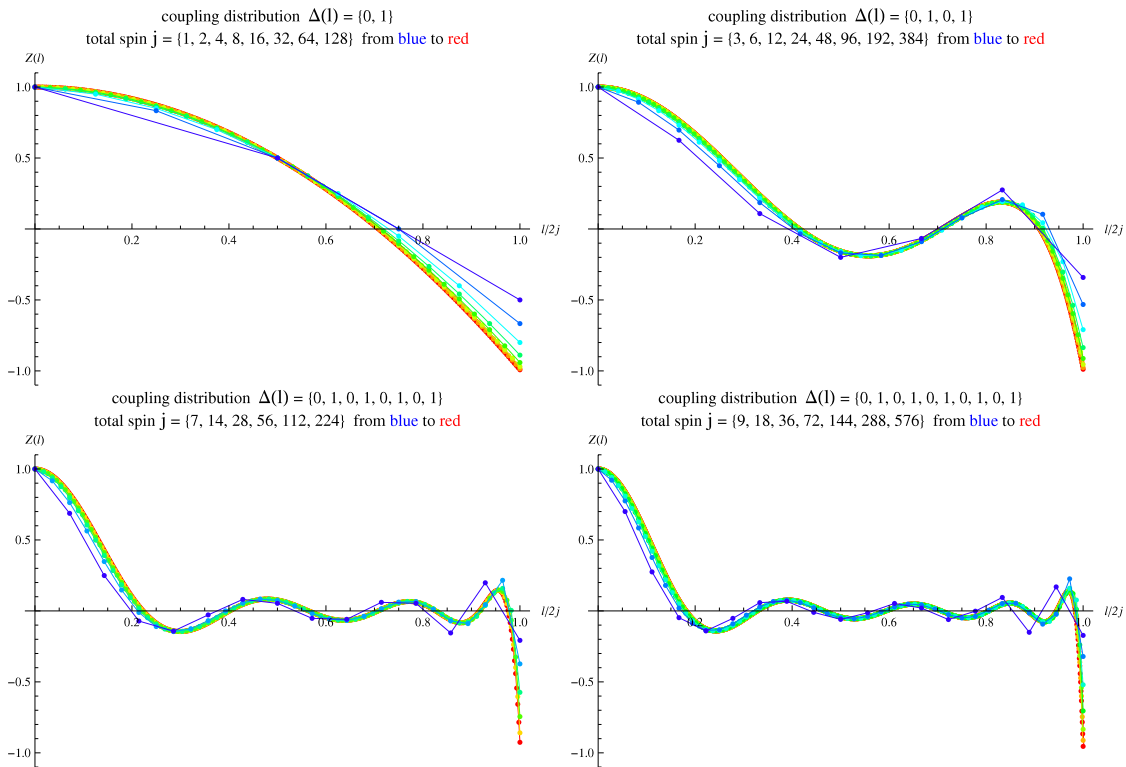


Figure A.4. $Z(l)$ for various l_0 and j ; only odd l , all equal.

A general and solvable random matrix model for spin decoherence

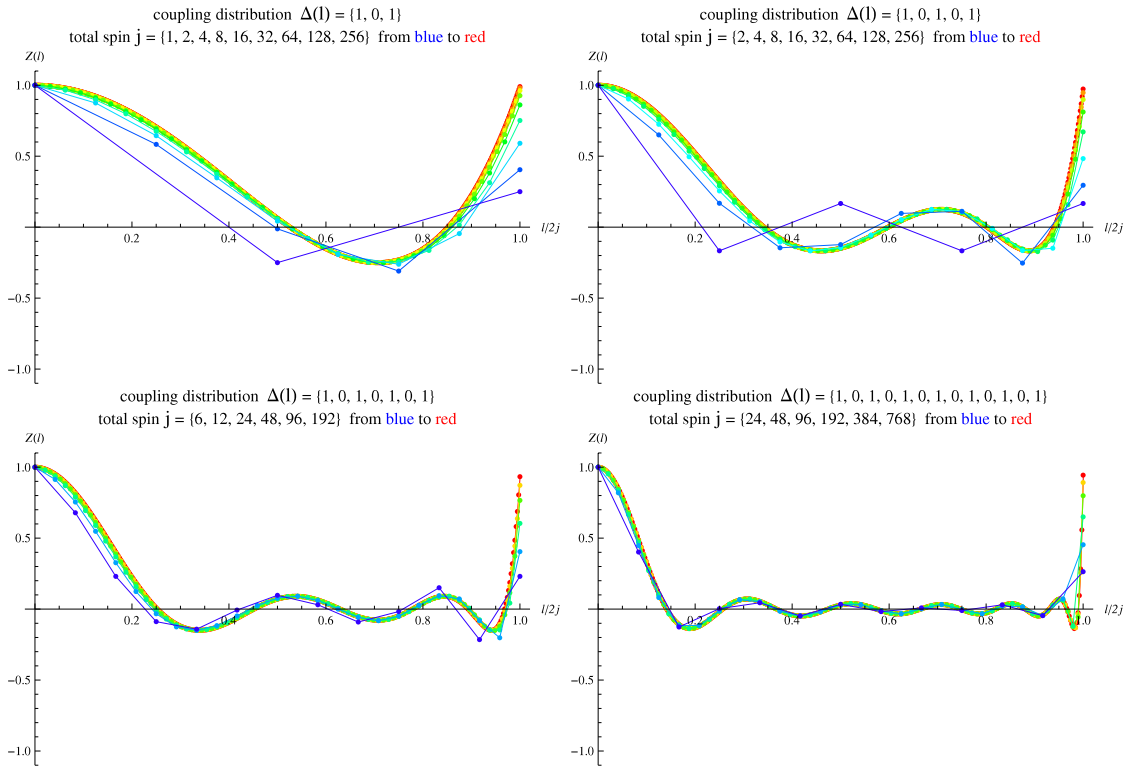


Figure A.5. $Z(l)$ for various l_0 and j ; only even l , all equal.

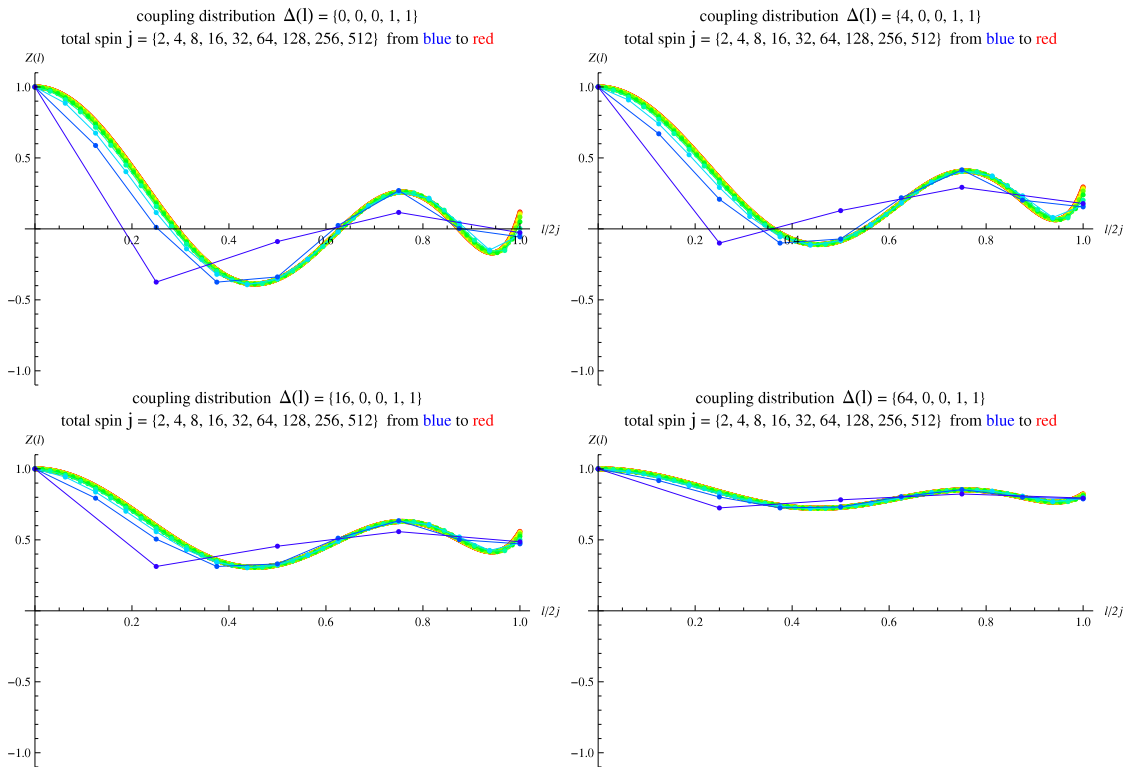


Figure A.6. $\Delta(0)$ becomes large; the others are fixed.

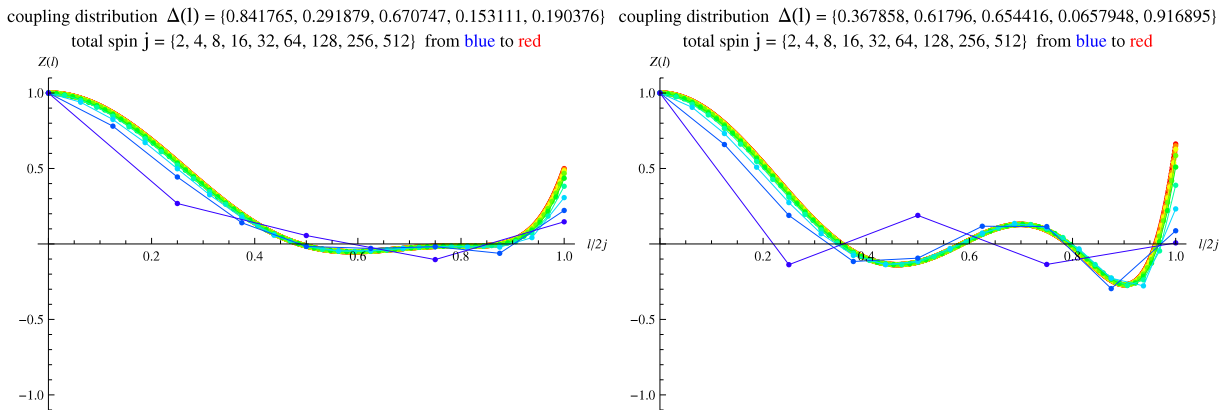


Figure A.7. Random $\Delta(l)$.

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