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Monodromy and Jacobi-like Relations for Color-Ordered Amplitudes

N. E. J. Bjerrum-Bohr, Poul H. Damgaard and Thomas Søndergaard

Niels Bohr International Academy and DISCOVERY Center,
The Niels Bohr Institute, Blegdamsvej 17,
DK-2100 Copenhagen Ø, Denmark,
email: {*bjbohr;phdamg;tsonderg*}*@nbi.dk*

Pierre Vanhove

Institut des Hautes Etudes Scientifiques, Le Bois-Marie,
F-91440 Bures-sur-Yvette, France
and
CEA, DSM, Institut de Physique Théorique, IPhT, CNRS, MPPU,
URA2306, Saclay, F-91191 Gif-sur-Yvette, France,
email: *pierre.vanhove@cea.fr*

ABSTRACT: We discuss monodromy relations between different color-ordered amplitudes in gauge theories. We show that Jacobi-like relations of Bern, Carrasco and Johansson can be introduced in a manner that is compatible with these monodromy relations. The Jacobi-like relations are not the most general set of equations that satisfy this criterion. Applications to supergravity amplitudes follow straightforwardly through the KLT-relations. We explicitly show how the tree-level relations give rise to non-trivial identities at loop level.

KEYWORDS: Amplitudes, Field Theory, String Theory.

Contents

1. Introduction

One of the most striking aspects of string theory is the manner in which it reorganizes the perturbative calculation of amplitudes in the field theory limit. Perhaps the most remarkable example of this is found in the Kawai-Lewellen-Tye (KLT) relations [1] that link gauge field tree-level amplitudes based on a non-Abelian gauge group to tree-level amplitudes in perturbative gravity. As it is based on a relationship between closed and open

strings [2], it immediately yields an even larger class of relations when considered in the context of superstring theory: a whole set of relations between supergravity and supersymmetry multiplets at tree level. For a comprehensive discussion, see, *e.g.*, the review by Bern [3]. These relations are puzzling from the point of view of field theory itself, although there are attempts to see their origin at the Lagrangian level [4].

Recently, three of the present authors have provided another example of how string theory can be used to derive non-trivial amplitude relations that hold even in the field theory limit, although their origin remains mysterious there [5]. The relations were conjectured earlier by Bern-Carrasco-Johansson [6], and we shall call them BCJ-relations in what follows. The peculiar aspect in this case is that these BCJ-relations seemed to follow from a new principle of Jacobi-like relations among tree-level amplitudes [6], relations that hold on-shell for four-point amplitudes [7], but which do *not* hold off-shell. Nevertheless, imposing these Jacobi-like relations even above four-point amplitudes yields correct amplitude relations. It was subsequently shown that analogous amplitude relations can be derived for external particles of the full $\mathcal{N}=4$ hypermultiplet [8], a result that indeed also follows directly from the proof using superstring theory [5].

To understand the significance of a new set of amplitude relations one needs to consider the factorial growth in n for color-ordered n-point amplitudes. For a tree-level n-point amplitude $\mathcal{A}_n$ with legs in the adjoint representation of, say, $SU(N)$ gauge group, one defines the color-ordered n-point amplitude $A_n(1,\ldots,n)$ through

$$\mathcal{A}_n = g_{\rm YM}^{n-2} \sum_{\sigma\in S_n/\mathbb{Z}_n} {\rm Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}})\, A_n(\sigma(1,\ldots,n))\,, \tag{1.1}$$

where g_{YM} is the coupling constant, and the T's are group generators of $SU(N)$. The relations we shall discuss all concern the color-ordered amplitudes $A_n(1,\ldots,n)$. Of course, to obtain cross sections, these must be "dressed" with the appropriate color factors and summed. The shorter the sum, the faster will routines work that do this sum automatically. It is therefore not only of theoretical interest, but also of great practical value to have exact relations available among the color-order amplitudes. Because of cyclicity of the amplitudes, the basis is not of size $n!$ but of size $(n-1)!$ Additional non-trivial generic relations known before the BCJ-relations were the following. Reflections:

$$A_n(1,\ldots,n) = (-1)^n A_n(n,n-1,\ldots,2,1)\,, \tag{1.2}$$

the photon decoupling relation

$$0 \;=\; \sum_\sigma A_n(1,\sigma(2,\ldots,n))\,, \tag{1.3}$$

and the Kleiss-Kuijf relations [9]

$$A_n(\beta_1,\ldots,\beta_r,1,\alpha_1,\ldots,\alpha_s,n) = (-1)^r \sum_{\sigma\subset {\rm OP}\{\alpha\}\cup\{\beta^T\}} A_n(1,\sigma,n)\,, \tag{1.4}$$

where the sum runs over the *ordered set of permutations* that preserves the order within each set. Transposition on the set $\{\beta\}$ means that order is reversed.

It was shown in ref. [9, 10] that these relations reduce the basis of amplitudes from $(n-1)!$ to $(n-2)!$ The BCJ-relations reduce the basis down to $(n-3)!$ As follows from the proof based on monodromy [5], no further reduction for arbitrary n will be possible. After imposition of the BCJ-relations one has thus reached the minimal basis of amplitudes.

In this paper we confront some of the questions that are raised by the apparently valid imposition of Jacobi-like relations among tree-level amplitudes. Given that the BCJ-relations have now been *proven* based on monodromy [5] a natural question is whether the Jacobi-like relations, conversely, follow from the BCJ-relations. Not unexpectedly, we find that this is not the case. In fact, we find that a huge extension of these Jacobi-like relations is possible [1], still leaving invariant the BCJ-relations.

The paper is organised as follows. In section 2, we briefly review monodromy relations in string theory, and show how they give rise to string theory generalizations of both the Kleiss-Kuijf and BCJ-relations. Section 3 contains a discussion of the connection between monodromy and Jacobi-like relations. There are clearly some issues related to gauge symmetry, and we choose in section 4 to consider this from the point of view of string theory, which automatically imposes a specific gauge choice. In section 5, we turn to gravity, and consider the extended Jacobi-like identities in the light of KLT-relations. All of these issues concern tree-level amplitudes only. In section 6, we explore what these by now established tree-level identities imply for loop amplitudes. A straightforward way to attack this is through the use of cuts. We illustrate this in the most simple case of one-loop amplitudes in $\mathcal{N}=4$ super Yang-Mills theory and comment on applications to theories with less, or no, supersymmetry. Finally, section 7 contains our conclusions. Some details about hypergeometric functions are relegated to an appendix.

2. Monodromy relations

In this section we will briefly recall how to derive monodromy relations for amplitudes through string theory. The color-ordered amplitudes on the disc are given by [2]

$$\mathcal{A}_n(a_1,\ldots,a_n) = \int \prod_{i=1}^{n} dz_i \, \frac{|z_{ab}\, z_{ac}\, z_{bc}|}{dz_a dz_b dz_c} \prod_{i=1}^{n-1} H(x_{a_{i+1}} - x_{a_i}) \prod_{1\le i<j\le n} |x_i - x_j|^{2\alpha' k_i\cdot k_j}\, F_n \,, \tag{2.1}$$

with

$$\begin{array}{llll} dz_i = dx_i & \text{and} & z_{ij} = x_i - x_j & \text{for the bosonic case and} \\ dz_i = dx_i d\theta_i & \text{and} & z_{ij} = x_i - x_j + \theta_i\theta_j & \text{for the supersymmetric case}\,. \end{array} \tag{2.2}$$

[1] In the process of completing this manuscript a paper by H. Tye and Y. Zhang [11] appeared. They consider amplitude relations from the viewpoint of heterotic string models. Some of their results overlap with ours, in particular regarding the existence of *extended* (or *generalized*) Jacobi identities, which we discuss in sections 3 and 4.

The ordering of the external legs is enforced by the product of Heaviside functions such that

$$H(x) = \begin{cases} 0 & x < 0\,, \\ 1 & x \geq 0\,. \end{cases} \tag{2.3}$$

The Möbius $SL(2,\mathbb{R})$ invariance requires one to fix the position of three points denoted z_a, z_b and z_c. A traditional choice is $x_1 = 0$, $x_{n-1} = 1$ and $x_n = +\infty$, supplemented by the condition $\theta_{n-1} = \theta_n = 0$ in the superstring case.

The helicity dependence of the external states is contained in the F_n factor. For tachyons $F_n = 1$. For n gauge bosons with polarization vectors h_i one has

$$F_n = \exp\left(-\sum_{i\neq j}\left(\frac{\sqrt{\alpha'}(h_i\cdot k_j)}{(x_i - x_j)} - 2\frac{(h_i\cdot h_j)}{(x_i-x_j)^2}\right)\right)\Bigg|_{\text{multilinear in } h_i}\,, \tag{2.4}$$

for the bosonic string. For the superstring F_n reads (the η_i are anticommuting variables)

$$F_n = \int\prod_{i=1}^{n} d\eta_i\,\exp\left(-\sum_{i\neq j}\left(\frac{\eta_i\sqrt{\alpha'}(\theta_i-\theta_j)(h_i\cdot k_j) - \eta_i\eta_j(h_i\cdot h_j)}{(x_i - x_j + \theta_i\theta_j)}\right)\right)\,. \tag{2.5}$$

We start with a review of the monodromy relations that appear at four points [5,12,13]. For simplicity, we phrase the discussion in terms of tachyon amplitudes. With the choice $x_1 = 0$, $x_3 = 1$ and $x_4 = +\infty$, all three different color-ordered amplitudes $\mathcal{A}(i,j,k,l)$ are given by the same integrand

$$|x_2|^{2\alpha'\,k_1\cdot k_2}|1-x_2|^{2\alpha'\,k_2\cdot k_3}\,,$$

but with x_2 integrated over different domains:

$$\mathcal{A}_4(1,2,3,4) = \int_0^1 dx\;x^{2\alpha'\,k_1\cdot k_2}(1-x)^{2\alpha'\,k_2\cdot k_3}\,, \tag{2.6}$$

$$\mathcal{A}_4(1,3,2,4) = \int_1^\infty dx\;x^{2\alpha'\,k_1\cdot k_2}(x-1)^{2\alpha'\,k_2\cdot k_3}\,, \tag{2.7}$$

$$\mathcal{A}_4(2,1,3,4) = \int_{-\infty}^0 dx\,(-x)^{2\alpha'\,k_1\cdot k_2}(1-x)^{2\alpha'\,k_2\cdot k_3}\,. \tag{2.8}$$

We indicate the contour integration from 1 to $+\infty$ in fig. 1.

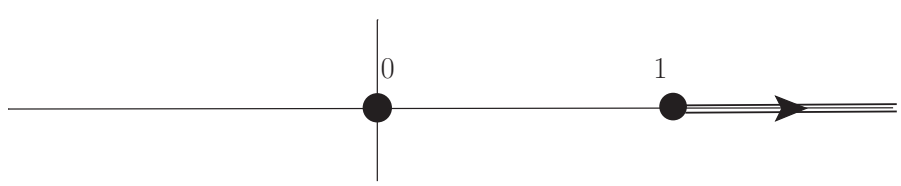

Figure 1: *The contour of integration from 1 to $+\infty$.*

Under the assumption that $\alpha'\,k_i\cdot k_j$ is complex and has a negative real part, we are allowed to deform the region of integration so that instead of integrating between from 1 to $+\infty$

on the real axis we integrate either on a contour slightly above or below the real axis. By a deformation of each of the contours, one can convert the expression into an integration from $-\infty$ to 1. One needs to include the appropriate phases each time x passes through $y = 0$ or $y = 1$ (when rotating the contours),

$$(x-y)^\alpha = (y-x)^\alpha \times \begin{cases} e^{+i\pi\,\alpha} & \text{for clockwise rotation}\,, \\ e^{-i\pi\,\alpha} & \text{for counterclockwise rotation}\,. \end{cases}$$

One can thus deform the integration region in two equivalent ways $\mathcal{I}^+$ and $\mathcal{I}^-$, see fig. 2.

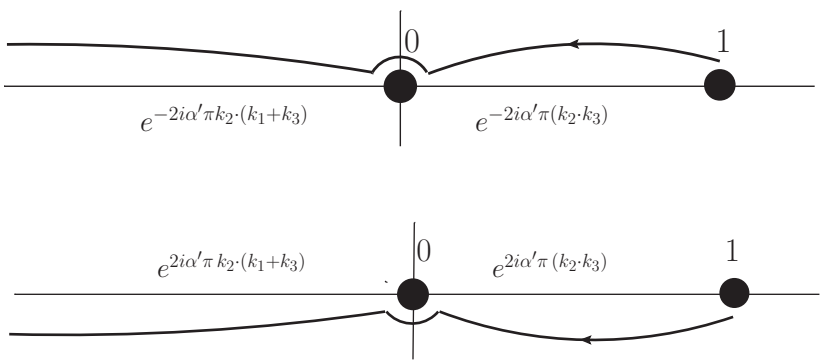

Figure 2: The contours $\mathcal{I}^+$ and $\mathcal{I}^-$.

We have $\mathcal{I}^+ = \mathcal{I}^- = \mathcal{A}_4(1,3,2,4)$. If now $\mathcal{I}^+$ is multiplied by $e^{2i\alpha'\pi k_2\cdot(k_1+k_3)}$ and $\mathcal{I}^-$ by $e^{-2i\alpha'\pi k_2\cdot(k_1+k_3)}$ we get for the contours as illustrated in fig. 3. We thus have $\mathcal{I}^+\, e^{2i\alpha'\pi k_2\cdot(k_1+k_3)} - \mathcal{I}^-\, e^{-2i\alpha'\pi k_2\cdot(k_1+k_3)} = 2i\,\mathcal{A}_4(1,3,2,4)\,\sin(2\alpha'\pi\, k_2\cdot(k_1+k_3))$. However, the contour obtained after subtracting these two contours can also be interpreted as in fig. 4. This is equal to $-2i\,\mathcal{A}_4(1,2,3,4)\,\sin(2\alpha'\pi k_1\cdot k_2)$. In this way we arrive at the following monodromy relation: $\sin(2\pi\alpha' k_1\cdot k_2)\mathcal{A}_4(1,2,3,4) = \sin(2\pi\alpha' k_2\cdot k_4)\mathcal{A}_4(1,3,2,4)$ where we have used momentum conservation and the on-shell condition. For other external states of higher spin, the integrals change appropriately to restore the identities (including sign factors for the fermionic statistics of half-integer spins).

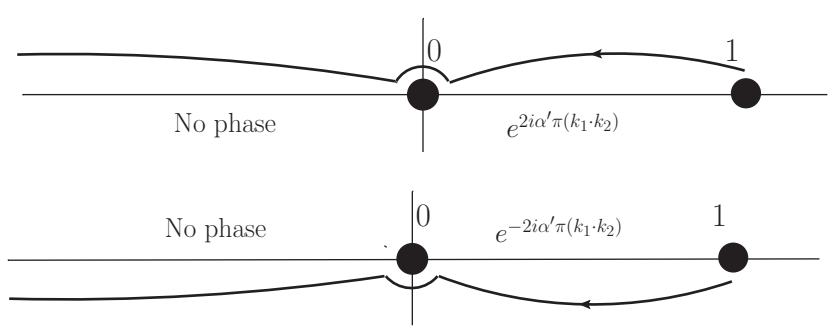

Figure 3: The contours $\mathcal{I}^+$ and $\mathcal{I}^-$ after multiplying with phases $e^{2i\alpha'\pi k_2\cdot(k_1+k_3)}$ and $e^{-2i\alpha'\pi k_2\cdot(k_1+k_3)}$.

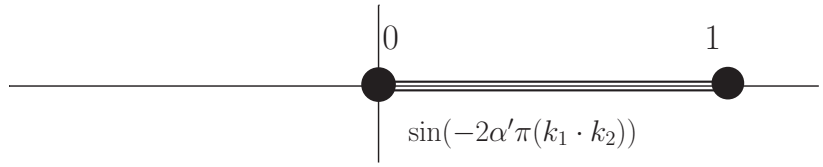

Figure 4: Another interpretation of the two contours.

By deforming the contour of integration of $\mathcal{A}_4(2,1,3,4)$ one finds in an equivalent fashion: $\sin(2\pi\alpha' k_2\cdot k_3)\mathcal{A}_4(1,2,3,4)=\sin(2\pi\alpha' k_2\cdot k_4)\mathcal{A}_4(2,1,3,4)$. This implies that all the amplitudes can be related to the $\mathcal{A}_4(1,2,3,4)$

$$\begin{aligned}\mathcal{A}_4(1,3,2,4)&=\frac{\sin(2\pi\alpha' k_1\cdot k_2)}{\sin(2\pi\alpha' k_2\cdot k_4)}\,\mathcal{A}_4(1,2,3,4)\,,\\ \mathcal{A}_4(1,3,4,2)=\mathcal{A}_4(2,1,3,4)&=\frac{\sin(2\pi\alpha' k_2\cdot k_3)}{\sin(2\pi\alpha' k_2\cdot k_4)}\,\mathcal{A}_4(1,2,3,4)\,.\end{aligned}\tag{2.9}$$

Taking the limit $\alpha'\to 0$, we get the following relations between the field theory amplitudes

$$\begin{aligned}A_4(1,3,2,4)&=\frac{k_1\cdot k_2}{k_2\cdot k_4}\,A_4(1,2,3,4)\,,\\ A_4(1,3,4,2)=A_4(2,1,3,4)&=\frac{k_2\cdot k_3}{k_2\cdot k_4}\,A_4(1,2,3,4)\,.\end{aligned}\tag{2.10}$$

The string theory relations can immediately be checked to hold based on the explicit string amplitude expression. In the low energy limit, the corresponding relations (2.10) coincide with those of ref. [6].

As shown in ref. [5], one has the following n-point amplitude relations:

$$\begin{aligned}&\mathcal{A}_n(\beta_1,\ldots,\beta_r,1,\alpha_1,\ldots,\alpha_s,n)=(-1)^r\\ &\qquad\times\Re\mathrm{e}\Big[\prod_{1\le i<j\le r}e^{2i\pi\alpha'(k_{\beta_i}\cdot k_{\beta_j})}\sum_{\sigma\subset\mathrm{OP}\{\alpha\}\cup\{\beta^T\}}\prod_{i=0}^{s}\prod_{j=1}^{r}e^{(\alpha_i,\beta_j)}\mathcal{A}_n(1,\sigma,n)\Big]\,,\end{aligned}\tag{2.11}$$

$$0=\Im\mathrm{m}\Big[\prod_{1\le i<j\le r}e^{2i\pi\alpha'(k_{\beta_i}\cdot k_{\beta_j})}\sum_{\sigma\subset\mathrm{OP}\{\alpha\}\cup\{\beta^T\}}\prod_{i=0}^{s}\prod_{j=1}^{r}e^{(\alpha_i,\beta_j)}\mathcal{A}_n(1,\sigma,n)\Big]\,,\tag{2.12}$$

with

$$e^{(\alpha,\beta)}\equiv\begin{cases}e^{2i\pi\alpha'(k_\alpha\cdot k_\beta)} & \text{if}\quad x_\beta>x_\alpha\,,\\ 1 & \text{otherwise.}\end{cases}$$

In these equations α_0 denotes the leg 1 at point 0.

These string theory amplitude relations reduce in the field theory limit $\alpha'\to 0$ to the Kleiss-Kuijf [9, 10] and BCJ-relations [6], respectively.

Explicitly, using (2.12) as well as momentum conservation, the five-point amplitude gives rise to the following four independent relations

$$
\begin{aligned}
0 &= \mathcal{S}_{k_3,k_1+k_2}\mathcal{A}_5(1,2,3,4,5) - \mathcal{S}_{k_3,k_5}\mathcal{A}_5(1,2,4,3,5) + \mathcal{S}_{k_1,k_3}\mathcal{A}_5(1,3,2,4,5)\,,\\
0 &= \mathcal{S}_{k_3,k_2+k_5}\mathcal{A}_5(1,4,3,2,5) - \mathcal{S}_{k_1,k_3}\mathcal{A}_5(1,3,4,2,5) + \mathcal{S}_{k_3,k_5}\mathcal{A}_5(1,4,2,3,5)\,,\\
0 &= \mathcal{S}_{k_4,k_2+k_5}\mathcal{A}_5(1,3,4,2,5) - \mathcal{S}_{k_1,k_4}\mathcal{A}_5(1,4,3,2,5) + \mathcal{S}_{k_4,k_5}\mathcal{A}_5(1,3,2,4,5)\,,\\
0 &= \mathcal{S}_{k_2,k_4+k_5}\mathcal{A}_5(1,3,2,4,5) - \mathcal{S}_{k_1,k_2}\mathcal{A}_5(1,2,3,4,5) + \mathcal{S}_{k_2,k_5}\mathcal{A}_5(1,3,4,2,5)\,.
\end{aligned}
\tag{2.13}
$$

Here we have used the notation $\mathcal{S}_{p,q} \equiv \sin(2\alpha'\pi\, p\cdot q)$. There are of course various ways of writing these monodromy relations, but they reduce to just four independent equations. One can immediately verify these relations from the explicit form of the tree amplitudes in string theory given by [14–17]. In the field theory limit they reduce to relations that are equivalent to those discussed in ref. [6].

3. Jacobi-like identities

The field theory limit of the monodromy relations were originally conjectured on the basis of an observation for the four-point gluon amplitudes [6]. We start by briefly reviewing the argument.

3.1 The four-point case

At four points, the photon decoupling identity reads

$$A_4(1,2,3,4) + A_4(2,1,3,4) + A_4(2,3,1,4) = 0\,. \tag{3.1}$$

It holds independently of polarization and external on-shell momenta. The natural way this identity can be satisfied is through

$$A_4(1,2,3,4) + A_4(2,1,3,4) + A_4(2,3,1,4) = \chi(s+t+u) = 0\,, \tag{3.2}$$

with χ being a common factor [2].

In the amplitude $A_4(1,2,3,4)$ both pairs of legs (1,2) and (1,4) are adjacent, and we should thus treat the s and t factors on the same footing. The contribution of this color ordering to eq. (3.2) must therefore be

$$A_4(1,2,3,4) = -\chi(s+t) = \chi u\,. \tag{3.3}$$

Likewise, one is led to

$$A_4(2,1,3,4) = \chi t, \qquad A_4(2,3,1,4) = \chi s\,. \tag{3.4}$$

[2] We will discuss the explicit expression for χ in the case of vector particles in section 4.

Eliminating χ one obtains

$$tA_4(1,2,3,4) = uA_4(2,1,3,4), \qquad sA_4(1,2,3,4) = uA_4(2,3,1,4),$$
$$sA_4(2,1,3,4) = tA_4(2,3,1,4)\,. \tag{3.5}$$

These are of course just the monodromy relations eq. (2.10). To proceed further, one can parameterize the three subamplitudes in terms of their possible pole structures and unspecified numerators

$$A_4(1,2,3,4) = \frac{n_s}{s} + \frac{n_t}{t}\,, \tag{3.6}$$

$$A_4(2,1,3,4) = -\frac{n_u}{u} - \frac{n_s}{s}\,, \tag{3.7}$$

$$A_4(2,3,1,4) = -\frac{n_t}{t} + \frac{n_u}{u}\,. \tag{3.8}$$

It follows from (3.5) that $n_u - n_s + n_t = 0$. This resembles the Jacobi identity for the associated color factors. Bern, Carrasco and Johansson [6] took as hypothesis that this can be extended iteratively for general n-point amplitudes. This is equivalent to assuming that one can choose a parametrization in which Jacobi relations for numerator factors can be imposed in one-to-one correspondence with the genuine Jacobi identities for the color factors. Imposing this hypothesis gets quite involved as n grows, but it can be carried through systematically; for details see ref. [6]. This leads to the BCJ-relations [6]. The same principle can be used to generate relations for scalar and fermionic matter in the adjoint representation [8]. We of course now understand that this is because the monodromy relations hold for the full $\mathcal{N} = 4$ supermultiplet in four dimensions [5].

Since the BCJ-relations have been proven [5], one would like to understand the meaning of these Jacobi-like identities for the numerators. In the four-point case the identities are exact, but only on-shell [7]. Even if the theory in question had only three-point vertices (which it does not) so that all n-point tree-level amplitudes for $n \geq 5$ could be constructed by gluing three-point vertices on a four-point function (thus having at least one leg off-shell on all four-point sub-diagrams), this would represent a puzzle. How can this starting point then lead to correct amplitude identities?

3.2 Generalized Jacobi-like relations

To see what is going on it suffices to focus on the 5-point case. We will simply derive exactly what follows directly from the field theory BCJ-relations when expressed in terms of the pertinent set of poles for each color-ordered amplitude. We use the parametrization

$$A_5(1,2,3,4,5) = \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}}\,, \tag{3.9}$$

$$A_5(1,4,3,2,5) = \frac{n_6}{s_{14}s_{25}} + \frac{n_5}{s_{43}s_{51}} + \frac{n_7}{s_{32}s_{14}} + \frac{n_8}{s_{25}s_{43}} + \frac{n_2}{s_{51}s_{32}}\,, \tag{3.10}$$

$$A_5(1,3,4,2,5) = \frac{n_9}{s_{13}s_{25}} - \frac{n_5}{s_{34}s_{51}} + \frac{n_{10}}{s_{42}s_{13}} - \frac{n_8}{s_{25}s_{34}} + \frac{n_{11}}{s_{51}s_{42}}\,, \tag{3.11}$$

$$A_5(1,2,4,3,5) = \frac{n_{12}}{s_{12}s_{35}} + \frac{n_{11}}{s_{24}s_{51}} - \frac{n_3}{s_{43}s_{12}} + \frac{n_{13}}{s_{35}s_{24}} - \frac{n_5}{s_{51}s_{43}}\,, \tag{3.12}$$

$$A_5(1,4,2,3,5) = \frac{n_{14}}{s_{14}s_{35}} - \frac{n_{11}}{s_{42}s_{51}} - \frac{n_7}{s_{23}s_{14}} - \frac{n_{13}}{s_{35}s_{42}} - \frac{n_2}{s_{51}s_{23}}\,, \tag{3.13}$$

$$A_5(1,3,2,4,5) = \frac{n_{15}}{s_{13}s_{45}} - \frac{n_2}{s_{32}s_{51}} - \frac{n_{10}}{s_{24}s_{13}} - \frac{n_4}{s_{45}s_{32}} - \frac{n_{11}}{s_{51}s_{24}}\,. \tag{3.14}$$

This can be easily illustrated by diagrams involving only anti-symmetric three-vertices. However, since the coefficients n_i may depend on the kinematic variables (and thus cancel poles) there is no assumption of only three-vertices here. The listed subamplitudes are related through the monodromy relations in the field limit of (2.13), *i.e.*,

$$0 = (s_{13}+s_{23})A_5(1,2,3,4,5) - s_{35}A_5(1,2,4,3,5) + s_{13}A_5(1,3,2,4,5)\,, \tag{3.15}$$

$$0 = (s_{23}+s_{35})A_5(1,4,3,2,5) - s_{13}A_5(1,3,4,2,5) + s_{35}A_5(1,4,2,3,5)\,, \tag{3.16}$$

$$0 = (s_{24}+s_{45})A_5(1,3,4,2,5) - s_{14}A_5(1,4,3,2,5) + s_{45}A_5(1,3,2,4,5)\,, \tag{3.17}$$

$$0 = (s_{24}+s_{25})A_5(1,3,2,4,5) - s_{12}A_5(1,2,3,4,5) + s_{25}A_5(1,3,4,2,5)\,. \tag{3.18}$$

Plugging the expressions for the amplitudes in terms of the n_i's into (3.15)–(3.18) we immediately obtain:

1. From (3.15)

$$0 = \frac{n_4 - n_1 + n_{15}}{s_{45}} - \frac{n_{10} - n_{11} + n_{13}}{s_{24}} - \frac{n_3 - n_1 + n_{12}}{s_{12}} - \frac{n_5 - n_2 + n_{11}}{s_{51}}\,, \tag{3.19}$$

2. From (3.16)

$$0 = \frac{n_7 - n_6 + n_{14}}{s_{14}} - \frac{n_{10} - n_{11} + n_{13}}{s_{24}} - \frac{n_8 - n_6 + n_9}{s_{25}} - \frac{n_5 - n_2 + n_{11}}{s_{51}}\,, \tag{3.20}$$

3. From (3.17)

$$0 = \frac{n_{10} - n_9 + n_{15}}{s_{13}} + \frac{n_5 - n_2 + n_{11}}{s_{51}} - \frac{n_4 - n_2 + n_7}{s_{23}} + \frac{n_8 - n_6 + n_9}{s_{25}}\,, \tag{3.21}$$

4. From (3.18)

$$0 = \frac{n_4 - n_1 + n_{15}}{s_{45}} - \frac{n_{10} - n_9 + n_{15}}{s_{13}} - \frac{n_5 - n_2 + n_{11}}{s_{51}} - \frac{n_3 - n_5 + n_8}{s_{34}}\,. \tag{3.22}$$

We thus see that the BCJ-relations can be written as kind of extended Jacobi identities when expressed in terms of the numerators. Let us simplify the notation a bit by denoting the nine numerator combinations as

$$\begin{aligned} &X_1 \equiv n_3 - n_5 + n_8\,, \quad X_2 \equiv n_3 - n_1 + n_{12}\,, \quad X_3 \equiv n_4 - n_1 + n_{15}\,, \\ &X_4 \equiv n_4 - n_2 + n_7\,, \quad X_5 \equiv n_5 - n_2 + n_{11}\,, \quad X_6 \equiv n_7 - n_6 + n_{14}\,, \\ &X_7 \equiv n_8 - n_6 + n_9\,, \quad X_8 \equiv n_{10} - n_9 + n_{15}\,, \quad X_9 \equiv n_{10} - n_{11} + n_{13}\,. \end{aligned} \tag{3.23}$$

Our four equations then take the form

$$0 = \frac{X_3}{s_{45}} - \frac{X_9}{s_{24}} - \frac{X_2}{s_{12}} - \frac{X_5}{s_{51}}\,, \tag{3.24}$$

$$0 = \frac{X_6}{s_{14}} - \frac{X_9}{s_{24}} - \frac{X_7}{s_{25}} - \frac{X_5}{s_{51}}\,, \tag{3.25}$$

$$0 = \frac{X_8}{s_{13}} + \frac{X_5}{s_{51}} - \frac{X_4}{s_{23}} + \frac{X_7}{s_{25}}\,, \tag{3.26}$$

$$0 = \frac{X_3}{s_{45}} - \frac{X_8}{s_{13}} - \frac{X_5}{s_{51}} - \frac{X_1}{s_{34}}\,. \tag{3.27}$$

These four equations describe the general constraints on the numerator factors dictated by the monodromy relations at five points. As long as these equations are satisfied we have numerator identities leading to eq. (3.15)–(3.18). Of course, the simplest solution is to put all $X_i = 0$, but this is clearly not the most general solution.

3.3 Reparametrization invariance

To make the amount of freedom one has in the above parametrization of subamplitudes more clear, let us write the most general solution by means of five arbitrary functions f_1, f_2, f_3, f_4 and f_5

$$X_1 \equiv s_{34} f_1, \qquad X_2 \equiv s_{12} f_2\,, \qquad X_3 \equiv s_{45} f_3, \qquad X_4 \equiv s_{23} f_4\,, \qquad X_5 \equiv s_{15} f_5\,, \tag{3.28}$$

i.e. from eq. (3.24)–(3.27)

$$\begin{aligned} &X_1 \equiv s_{34} f_1\,, && X_2 \equiv s_{12} f_2\,, && X_3 \equiv s_{45} f_3\,, \\ &X_4 \equiv s_{23} f_4\,, && X_5 \equiv s_{15} f_5\,, && X_6 = s_{14}(f_1 - f_2 + f_4)\,, \\ &X_7 = s_{25}(f_1 - f_3 + f_4)\,, && X_8 = s_{13}(f_3 - f_1 - f_5)\,, && X_9 = s_{24}(f_3 - f_2 - f_5)\,. \end{aligned} \tag{3.29}$$

Note that we have used the canonical set of kinematic variables (generalized Mandelstam variables for the 5-point case) $s_{12}, s_{23}, s_{34}, s_{45}, s_{51}$ in our definition of the f_i. The s_{ij} occuring in the expression for X_6, X_7, X_8 and X_9 are related to this canonical set by

$$\begin{aligned} s_{14} &= s_{23} - s_{15} - s_{45}, \qquad & s_{25} &= s_{34} - s_{12} - s_{15}\,, \\ s_{13} &= s_{45} - s_{12} - s_{23}, \qquad & s_{24} &= s_{15} - s_{23} - s_{34}\,. \end{aligned} \tag{3.30}$$

The freedom we have to generalize the solution, *i.e.* eq. (3.29), is not just related to gauge degrees or the freedom to absorb contact terms. It can be seen as the trivial freedom to add a "zero" to the subamplitude and forcing it into a parametrization of the form eq. (3.9)–(3.14).

As a simple example, imagine that we add $0 = g - g$ to eq. (3.9), with g being an arbitrary function. We can then absorb the g's in n_1 and n_3, *i.e.*

$$A_5(1,2,3,4,5) = \frac{(n_1 + s_{12}s_{45}g)}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{(n_3 - s_{34}s_{12}g)}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}}\,. \tag{3.31}$$

In no other amplitude than $A_5(1,2,3,4,5)$ does n_1 appear, however, n_3 appears in eq. (3.12) so we add $0 = g - g$ to the amplitude, and absorb in the following way:

$$A_5(1,2,4,3,5) = \frac{(n_{12} - s_{12}s_{35}g)}{s_{12}s_{35}} + \frac{n_{11}}{s_{24}s_{51}} - \frac{(n_3 - s_{34}s_{12}g)}{s_{43}s_{12}} + \frac{n_{13}}{s_{35}s_{24}} - \frac{n_5}{s_{51}s_{43}} \,. \tag{3.32}$$

We have thereby redefined n_1, n_3 and n_{12}

$$n_1 \quad \to \quad n_1 + s_{12}s_{45}g \,, \tag{3.33}$$

$$n_3 \quad \to \quad n_3 - s_{34}s_{12}g \,, \tag{3.34}$$

$$n_{12} \quad \to \quad n_{12} - s_{12}s_{35}g \,, \tag{3.35}$$

which changes X_1, X_2 and X_3

$$X_1 = s_{34}f_1 \quad \to \quad s_{34}(f_1 - s_{12}g) \equiv s_{34}f_1' \,, \tag{3.36}$$

$$X_2 = s_{12}f_2 \quad \to \quad s_{12}(f_2 - (s_{45} + s_{34} + s_{35})g) = s_{12}(f_2 - s_{12}g) \equiv s_{12}f_2' \,, \tag{3.37}$$

$$X_3 = s_{45}f_3 \quad \to \quad s_{45}(f_3 - s_{12}g) \equiv s_{45}f_3' \,, \tag{3.38}$$

and we now have

$$\begin{array}{lll} X_1 = s_{34}f_1' \,, & X_2 = s_{12}f_2' \,, & X_3 = s_{45}f_3' \,, \\ X_4 = s_{23}f_4 \,, & X_5 = s_{15}f_5 \,, & X_6 = s_{14}(f_1' - f_2' + f_4) \,, \\ X_7 = s_{25}(f_1' - f_3' + f_4) \,, & X_8 = s_{13}(f_3' - f_1' - f_5) \,, & X_9 = s_{24}(f_3' - f_2' - f_5) \,. \end{array} \tag{3.39}$$

This trivial addition of zeros to the amplitudes illustrates the fact that we can find many different representations of the numerators, all of which are perfectly consistent with the monodromy relations. The freedom is that of general reparametrizations of the amplitude and not just gauge symmetry.

4. String amplitudes

Let us consider tree-level open string amplitudes in superstring theory. We have already given the needed formulas in section 2. We first focus on the color-ordered four-point amplitude for vector particles

$$\mathcal{A}_4^\sigma = \int_{D_\sigma} dz_2 \, |z_2|^{2\alpha' \, k_1 \cdot k_2} |1 - z_2|^{2\alpha' \, k_2 \cdot k_3} \, \tilde{F}_4(z_2) \,, \tag{4.1}$$

where the domain of integration D_σ for each color ordering are given by $D_{1234} = \{0 \leq z_2 \leq 1\}$, $D_{1324} = \{1 \leq z_2\}$, $D_{2134} = \{z_2 \leq 0\}$. Expanding the function $\tilde{F}_4$ in (2.5) leads[3] to

$$\tilde{F}_4(y) = \frac{a_1}{y} + \frac{b_1}{y-1} \,, \tag{4.2}$$

[3]This can be derived with a very tedious expansion [18] of the expression in eq. (2.5). The simplicity of the expansion appears naturally in the pure spinor formalism [19,20]. The tilde on F_n indicates that we have fixed the three conformal points in the expression.

where a_1 and b_1 are expressed in terms of the polarizations and the momenta. Their expressions are particularly long but there is a relation between the two coefficients

$$s\, b_1 - t\, a_1 = \alpha'\, t_8^{m_1\cdots m_8} F^1_{m_1 m_2} F^2_{m_3 m_4} F^3_{m_5 m_6} F^4_{m_7 m_8}\,, \tag{4.3}$$

where F^i are the field-strengths corresponding to the external legs. The tensor t_8 is contracting the Lorentz indices as defined in appendix 9.A of [2] (it is common to define $\chi = t_8^{m_1\cdots m_8} F^1_{m_1 m_2} F^2_{m_3 m_4} F^3_{m_5 m_6} F^4_{m_7 m_8}/(stu)$). The quantity a_1 and b_1 are not gauge invariant but the combination in (4.3) is gauge invariant.

For the four-point color-ordered amplitudes we find

$$\mathcal{A}_4(1,2,3,4) = \Phi_{2,1}(\alpha'\, s, \alpha'\, t)\, \left(-\frac{a_1}{\alpha'\, s} + \frac{b_1}{\alpha'\, t}\right)\,, \tag{4.4}$$

$$\mathcal{A}_4(1,3,2,4) = \Phi_{2,1}(\alpha'\, u, \alpha'\, t)\, \left(-\frac{a_1+b_1}{\alpha'\, u} - \frac{b_1}{\alpha'\, t}\right)\,, \tag{4.5}$$

$$\mathcal{A}_4(2,1,3,4) = \Phi_{2,1}(\alpha'\, s, \alpha'\, u)\, \left(\frac{a_1}{\alpha'\, s} + \frac{a_1+b_1}{\alpha'\, u}\right)\,, \tag{4.6}$$

where we introduced the hypergeometric functions

$$\Phi_{2,1}(\alpha' s, \alpha' t) \equiv {}_2\mathrm{F}_1(-\alpha'\, s, \alpha'\, t; 1-\alpha'\, s; 1) = \frac{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)}{\Gamma(1+\alpha' u)}\,. \tag{4.7}$$

In the convention of BCJ [6],

$$n_s = -a_1/\alpha', \qquad n_t = -b_1/\alpha', \qquad n_u = -(a_1+b_1)/\alpha'\,, \tag{4.8}$$

we immediately obtain the exact relation $n_u = n_t - n_s$.

4.1 Five points

Let us now consider the five point amplitude. Having fixed the position vertex operators at positions $z_1 = 0$, $z_4 = 1$ and $z_5 = \infty$, the integrand takes the compact form [20]

$$\mathcal{A}_5^\sigma = \int_{D_\sigma} dz_2 dz_3 \prod_{i<j} |z_{ij}|^{2\alpha' k_i\cdot k_j} \left[\frac{A}{z_{12}z_{13}} + \frac{B}{z_{23}z_{24}} + \frac{C}{z_{12}z_{34}} + \frac{D}{z_{24}z_{34}} + \frac{E}{z_{23}z_{13}} + \frac{F}{z_{24}z_{13}} + \frac{G}{z_{23}^2}\right]. \tag{4.9}$$

In this parametrization A to F are of order $\mathcal{O}(\alpha'^2)$ and G is of order $\mathcal{O}(\alpha')$. The twelve domains of integration are given in eq. (4.13).

There is some freedom in which the OPEs leading to the expression (4.9) are performed [20] that can give an equivalent form of the integrand of the amplitude. Let us define the quantity

$$C^z_{x,y} = \frac{1}{(x-z)(z-y)}\,. \tag{4.10}$$

Clearly this function satisfies the Jacoby identity

$$J(x,y,z) = C^z_{x,y} + C^y_{z,x} + C^x_{y,z} = 0\,. \tag{4.11}$$

The freedom in parameterizing the amplitude in (4.9) is given by the possibility of having

$$J(1,2,3) = 0\,, \qquad J(4,2,3) = 0\,. \tag{4.12}$$

In the amplitude (4.9) we have made explicit the poles $C^1_{2,3}$ and $C^3_{1,2}$ and $C^2_{3,4}$ and $C^4_{2,3}$.

This freedom corresponds to local monodromy transformations exchanging the position of neighboring vertex operators. There are as well global monodromy transformations given by moving vertex operators from one side of the line to the other side which are not captured by these local transformations.

The 12 color-ordered five-point amplitudes are given by specifying the range of integration over z_2 and z_3 over the following domains[4] of integrations D_σ

$$\begin{aligned}
D_{12345} &= \{0 \le z_2 \le z_3 \le 1\}\,,\\
D_{13245} &= \{0 \le z_3 \le z_2 \le 1\}\,,\\
D_{12435} &= \{0 \le z_2 \le 1 \le z_3\}\,,\\
D_{13425} &= \{0 \le z_3 \le 1 \le z_2\}\,,\\
D_{14235} &= \{0 \le 1 \le z_2 \le z_3\}\,,\\
D_{14325} &= \{0 \le 1 \le z_3 \le z_2\}\,,\\
D_{21345} &= \{z_2 \le 0 \le z_3 \le 1\}\,,\\
D_{31245} &= \{z_3 \le 0 \le z_2 \le 1\}\,,\\
D_{23145} &= \{z_2 \le z_3 \le 0\}\,,\\
D_{32145} &= \{z_3 \le z_2 \le 0\}\,,\\
D_{21435} &= \{z_2 \le 0 \le 1 \le z_3\}\,,\\
D_{31425} &= \{z_3 \le 0 \le 1 \le z_2\}\,.
\end{aligned} \tag{4.13}$$

We now use the result for $I(a,b,c,d,e)$ which is given in the appendix A. The integrals are explicitly evaluated in appendix A. We here quote the field theory results. In the field theory limit $\alpha' \to 0$ we get

$$\mathcal{A}_5(1,2,3,4,5) = \frac{A}{s_{12}s_{45}} + \frac{B - Gs_{34}}{s_{23}s_{51}} + \frac{C}{s_{34}s_{12}} + \frac{E + Gs_{13}}{s_{45}s_{23}} + \frac{D - Gs_{34}}{s_{51}s_{34}}\,, \tag{4.14}$$

[4]We have $(n-1)!/2$ such domains corresponding to the different $(n-1)!$ color-ordered amplitudes divided by 2 by reflection.

$$\mathcal{A}_5(1,3,4,2,5) = \frac{A-E-F}{s_{13}s_{25}} - \frac{D-Gs_{34}}{s_{34}s_{51}} + \frac{-F}{s_{42}s_{13}} - \frac{D-C}{s_{25}s_{34}} + \frac{B-D}{s_{51}s_{42}}\,, \tag{4.15}$$

$$\mathcal{A}_5(1,2,4,3,5) = \frac{A-C}{s_{12}s_{35}} + \frac{B-D}{s_{24}s_{51}} - \frac{C}{s_{43}s_{12}} + \frac{F+B-D}{s_{35}s_{24}} - \frac{D-Gs_{34}}{s_{51}s_{43}}\,, \tag{4.16}$$

$$\mathcal{A}_5(1,3,2,4,5) = \frac{A-E-Gs_{13}}{s_{13}s_{45}} - \frac{B-Gs_{34}}{s_{32}s_{51}} - \frac{-F}{s_{24}s_{13}} - \frac{E+Gs_{13}}{s_{45}s_{32}} - \frac{B-D}{s_{51}s_{24}}\,, \tag{4.17}$$

$$\begin{aligned}\mathcal{A}_5(1,4,3,2,5) = &\frac{D-C+A-E-F}{s_{14}s_{25}} + \frac{D-Gs_{34}}{s_{43}s_{51}} + \frac{B-E+Gs_{35}}{s_{32}s_{14}} \\ &+ \frac{D-C}{s_{25}s_{43}} + \frac{B-Gs_{34}}{s_{51}s_{32}}\,,\end{aligned} \tag{4.18}$$

$$\begin{aligned}\mathcal{A}_5(1,4,2,3,5) = &\frac{D-C+A-F-B-Gs_{35}}{s_{14}s_{35}} - \frac{B-D}{s_{42}s_{51}} - \frac{B-E+Gs_{35}}{s_{23}s_{14}} \\ &- \frac{F+B-D}{s_{35}s_{42}} - \frac{B-Gs_{34}}{s_{51}s_{23}}\,.\end{aligned} \tag{4.19}$$

It is interesting to note that we could use monodromy relations for integrals on the individual A, B, C etc. terms in (4.9). Thereby one would obtain the same relations as for the full subamplitudes, but now just for the individual terms. Hence, the OPEs provide us with expressions for the subamplitudes in which the relations are very explicitly reduced to relations in the pole structure. This can also be checked explicitly for the five-point case by use of (4.14)–(4.19).

4.2 The generalized parametrization (from strings)

In (4.14)–(4.19) we already wrote the amplitudes in terms of double poles. The quantities A to F were naturally put into the double-pole form, but the G term, a single-pole term, was forced into this representation by making a specific choice. Later we will come back to the freedom in absorbing the G terms, but for now we just consider the form given above.

Comparing with Bern, Carrasco and Johansson's [6] parametrization (*i.e.* (3.9)–(3.14)) we identify from (4.14)–(4.19)

$$\begin{aligned}
&n_1 = A\,, && n_6 = D-C+A-E-F\,, && n_{11} = B-D\,,\\
&n_2 = B-Gs_{34}\,, && n_7 = B-E+Gs_{35}\,, && n_{12} = A-C\,,\\
&n_3 = C\,, && n_8 = D-C\,, && n_{13} = F+B-D\,,\\
&n_4 = E+Gs_{13}\,, && n_9 = A-E-F\,, && n_{14} = D-C+A-F-B-Gs_{35}\,,\\
&n_5 = D-Gs_{34}\,, && n_{10} = -F\,, && n_{15} = A-E-Gs_{13}\,.
\end{aligned} \tag{4.20}$$

The Jacobi-like identities then take the form

$$
\begin{aligned}
X_1 &= n_3 - n_5 + n_8 = G s_{34}\,, \\
X_2 &= n_3 - n_1 + n_{12} = 0\,, \\
X_3 &= n_4 - n_1 + n_{15} = 0\,, \\
X_4 &= n_4 - n_2 + n_7 = -G s_{32}\,, \\
X_5 &= n_5 - n_2 + n_{11} = 0\,, \\
X_6 &= n_7 - n_6 + n_{14} = 0\,, \\
X_7 &= n_8 - n_6 + n_9 = 0\,, \\
X_8 &= n_{10} - n_9 + n_{15} = -G s_{13}\,, \\
X_9 &= n_{10} - n_{11} + n_{13} = 0\,.
\end{aligned}
\tag{4.21}
$$

And from (3.24)–(3.27) it is easy to see that these amplitudes do indeed satisfy the BCJ-relations. Moreover not all X_i's vanish.

Note that the BCJ-relations could also be derived from (4.14)–(4.19) by expressing, for instance, A and B in terms of two subamplitudes and the C to G terms. Using these expressions for A and B in the remaining amplitudes leads directly to BCJ-relations (the C to G terms vanish after the substitution).

4.3 Distributing the single-pole terms

There are many ways of arranging the G terms into the numerators of double poles. The expressions given above correspond to just one specific choice. To see this more clearly let us begin by defining $\tilde{n}_i$'s

$$
\begin{array}{lll}
\tilde{n}_1 = A\,, & \tilde{n}_6 = D - C + A - E - F\,, & \tilde{n}_{11} = B - D\,, \\
\tilde{n}_2 = B\,, & \tilde{n}_7 = B - E\,, & \tilde{n}_{12} = A - C\,, \\
\tilde{n}_3 = C\,, & \tilde{n}_8 = D - C\,, & \tilde{n}_{13} = F + B - D\,, \\
\tilde{n}_4 = E\,, & \tilde{n}_9 = A - E - F\,, & \tilde{n}_{14} = D - C + A - F - B\,, \\
\tilde{n}_5 = D\,, & \tilde{n}_{10} = -F\,, & \tilde{n}_{15} = A - E\,.
\end{array}
\tag{4.22}
$$

The amplitudes can then, in all generality, be represented like

$$
\mathcal{A}_5(1,2,3,4,5) \equiv \frac{\tilde{n}_1 + G g_1}{s_{12} s_{45}} + \frac{\tilde{n}_2 + G g_2}{s_{23} s_{51}} + \frac{\tilde{n}_3 + G g_3}{s_{34} s_{12}} + \frac{\tilde{n}_4 + G g_4}{s_{45} s_{23}} + \frac{\tilde{n}_5 + G g_5}{s_{51} s_{34}}\,,
\tag{4.23}
$$

$$\mathcal{A}_5(1,4,3,2,5) \equiv \frac{\tilde{n}_6 + Gg_6}{s_{14}s_{25}} + \frac{\tilde{n}_5 + Gg_5}{s_{43}s_{51}} + \frac{\tilde{n}_7 + Gg_7}{s_{32}s_{14}} + \frac{\tilde{n}_8 + Gg_8}{s_{25}s_{43}} + \frac{\tilde{n}_2 + Gg_2}{s_{51}s_{32}}\,, \tag{4.24}$$

$$\mathcal{A}_5(1,3,4,2,5) \equiv \frac{\tilde{n}_9 + Gg_9}{s_{13}s_{25}} - \frac{\tilde{n}_5 + Gg_5}{s_{34}s_{51}} + \frac{\tilde{n}_{10} + Gg_{10}}{s_{42}s_{13}} - \frac{\tilde{n}_8 + Gg_8}{s_{25}s_{34}} + \frac{\tilde{n}_{11} + Gg_{11}}{s_{51}s_{42}}\,, \tag{4.25}$$

$$\mathcal{A}_5(1,2,4,3,5) \equiv \frac{\tilde{n}_{12} + Gg_{12}}{s_{12}s_{35}} + \frac{\tilde{n}_{11} + Gg_{11}}{s_{24}s_{51}} - \frac{\tilde{n}_3 + Gg_3}{s_{43}s_{12}} + \frac{\tilde{n}_{13} + Gg_{13}}{s_{35}s_{24}} - \frac{\tilde{n}_5 + Gg_5}{s_{51}s_{43}}\,, \tag{4.26}$$

$$\mathcal{A}_5(1,4,2,3,5) \equiv \frac{\tilde{n}_{14} + Gg_{14}}{s_{14}s_{35}} - \frac{\tilde{n}_{11} + Gg_{11}}{s_{42}s_{51}} - \frac{\tilde{n}_7 + Gg_7}{s_{23}s_{14}} - \frac{\tilde{n}_{13} + Gg_{13}}{s_{35}s_{42}} - \frac{\tilde{n}_2 + Gg_2}{s_{51}s_{23}}\,, \tag{4.27}$$

$$\mathcal{A}_5(1,3,2,4,5) \equiv \frac{\tilde{n}_{15} + Gg_{15}}{s_{13}s_{45}} - \frac{\tilde{n}_2 + Gg_2}{s_{32}s_{51}} - \frac{\tilde{n}_{10} + Gg_{10}}{s_{24}s_{13}} - \frac{\tilde{n}_4 + Gg_4}{s_{45}s_{32}} - \frac{\tilde{n}_{11} + Gg_{11}}{s_{51}s_{24}}\,, \tag{4.28}$$

where the g_i's are new parameters representing the fractions of the G terms absorbed into the specific double poles. Since these expressions must equal (4.14)–(4.19) in order to express the actual amplitudes, we get six equations constraining the g_i parameters

$$\frac{s_{13}}{s_{45}s_{23}} - \frac{s_{34}}{s_{23}s_{51}} - \frac{1}{s_{51}} = \frac{g_1}{s_{12}s_{45}} + \frac{g_2}{s_{23}s_{51}} + \frac{g_3}{s_{34}s_{12}} + \frac{g_4}{s_{45}s_{23}} + \frac{g_5}{s_{51}s_{34}}\,, \tag{4.29}$$

$$\frac{s_{35}}{s_{14}s_{23}} - \frac{s_{34}}{s_{23}s_{51}} - \frac{1}{s_{51}} = \frac{g_6}{s_{14}s_{25}} + \frac{g_5}{s_{43}s_{51}} + \frac{g_7}{s_{32}s_{14}} + \frac{g_8}{s_{25}s_{43}} + \frac{g_2}{s_{51}s_{32}}\,, \tag{4.30}$$

$$\frac{1}{s_{51}} = \frac{g_9}{s_{13}s_{25}} - \frac{g_5}{s_{34}s_{51}} + \frac{g_{10}}{s_{42}s_{13}} - \frac{g_8}{s_{25}s_{34}} + \frac{g_{11}}{s_{51}s_{42}}\,, \tag{4.31}$$

$$\frac{1}{s_{51}} = \frac{g_{12}}{s_{12}s_{35}} + \frac{g_{11}}{s_{24}s_{51}} - \frac{g_3}{s_{43}s_{12}} + \frac{g_{13}}{s_{35}s_{24}} - \frac{g_5}{s_{51}s_{43}}\,, \tag{4.32}$$

$$\frac{s_{34}}{s_{51}s_{23}} - \frac{s_{35}}{s_{23}s_{41}} - \frac{1}{s_{41}} = \frac{g_{14}}{s_{14}s_{35}} - \frac{g_{11}}{s_{42}s_{51}} - \frac{g_7}{s_{23}s_{14}} - \frac{g_{13}}{s_{35}s_{42}} - \frac{g_2}{s_{51}s_{23}}\,, \tag{4.33}$$

$$\frac{s_{34}}{s_{15}s_{23}} - \frac{s_{13}}{s_{23}s_{45}} - \frac{1}{s_{45}} = \frac{g_{15}}{s_{13}s_{45}} - \frac{g_2}{s_{32}s_{51}} - \frac{g_{10}}{s_{24}s_{13}} - \frac{g_4}{s_{45}s_{32}} - \frac{g_{11}}{s_{51}s_{24}}\,. \tag{4.34}$$

Any solution to these equations give a valid distribution of the G terms, *i.e.* provide us with a representation of the form (3.9)–(3.14) that satisfy (3.24)–(3.27).

The representation written out explicitly in (4.14)–(4.19) corresponds to the solution

$$\begin{aligned}
&g_1 = 0\,, && g_6 = 0\,, && g_{11} = 0\,,\\
&g_2 = -s_{34}\,, && g_7 = s_{35}\,, && g_{12} = 0\,,\\
&g_3 = 0\,, && g_8 = 0\,, && g_{13} = 0\,,\\
&g_4 = s_{13}\,, && g_9 = 0\,, && g_{14} = -s_{35}\,,\\
&g_5 = -s_{34}\,, && g_{10} = 0\,, && g_{15} = -s_{13}\,.
\end{aligned} \tag{4.35}$$

A numerical check have shown that there *do* exits solutions for g_i such that the nine Jacobi identities ($n_i - n_j + n_k = 0$) are satisfied, and in such a way that four of the g_i's can be chosen arbitrarily. This correspond to the freedom Bern, Carrasco and Johansson find in choosing their α_1, α_2, α_3 and α_4 arbitrarily.

An example of a (valid) choice of g_i's which generate n_i's that satisfy the Jacobi identities is

$$\begin{aligned}
&g_1 = -s_{12}\,, && g_6 = -s_{25}\,, && g_{11} = 0\,,\\
&g_2 = -s_{12} - s_{25}\,, && g_7 = -s_{25}\,, && g_{12} = 0\,,\\
&g_3 = -s_{12}\,, && g_8 = -s_{25}\,, && g_{13} = 0\,,\\
&g_4 = -s_{12}\,, && g_9 = 0\,, && g_{14} = 0\,,\\
&g_5 = -s_{12} - s_{25}\,, && g_{10} = 0\,, && g_{15} = 0\,,
\end{aligned} \tag{4.36}$$

with, *e.g.*

$$\begin{aligned}
n_3 - n_5 + n_8 &= (\tilde{n}_3 - \tilde{n}_5 + \tilde{n}_8) + G(g_3 - g_5 + g_8)\\
&= (C - D + D - C) + G(-s_{12} - (-s_{12} - s_{25}) - s_{25})\\
&= 0, \qquad etc\ldots
\end{aligned} \tag{4.37}$$

From the expansion given by the OPE this might not be the most simple or natural way of absorbing the G terms into double-poles, but it does show that the assumption of Bern, Carrasco and Johansson is allowed for (at least) the five-point case.

5. Monodromy and KLT relations

As a direct application of the monodromy relations in Yang-Mills theory, we can rewrite the Kawai-Lewellen-Tye relations at four-point level in the following manner

$$\mathcal{M}_4 = \frac{\kappa_{(4)}^2}{\alpha'}\,\frac{\mathcal{S}_{k_1,k_2}\mathcal{S}_{k_1,k_4}}{\mathcal{S}_{k_1,k_3}}\,\mathcal{A}_4^L(1,2,3,4)\mathcal{A}_4^R(1,2,3,4)\,. \tag{5.1}$$

The field theory limit of the string amplitude (5.1), $\alpha' \to 0$ gives the symmetric form of the gravity amplitudes of [6]

$$M_4 = \kappa_{(4)}^2\,\frac{st}{u}\left(\frac{n_s}{s} + \frac{n_t}{t}\right)\left(\frac{\tilde{n}_s}{s} + \frac{\tilde{n}_t}{t}\right) = -\kappa_{(4)}^2\left(\frac{n_s\tilde{n}_s}{s} + \frac{n_t\tilde{n}_t}{t} + \frac{n_u\tilde{n}_u}{u}\right)\,. \tag{5.2}$$

Here we have made use of the on-shell relation $s+t+u=0$ and the four-point Jacobi relation $n_u = n_s - n_t$.

At five point order Bern, Carrasco and Johansson [6] showed that if the subamplitudes are parameterized by numerators like in eqs. (3.9)–(3.14), and we assume the numerators satisfy the Jacobi-like identities, then the KLT relation

$$\begin{aligned}-iM_5(1,2,3,4,5) = s_{12}s_{34}A_5(1,2,3,4,5)\widetilde{A}_5(2,1,4,3,5) \\ + s_{13}s_{24}A_5(1,3,2,4,5)\widetilde{A}_5(3,1,4,2,5)\,,\end{aligned} \tag{5.3}$$

implies the following form of M_5

$$\begin{aligned}-iM_5(1,2,3,4,5) = {} & \frac{n_1\tilde{n}_1}{s_{12}s_{45}} + \frac{n_2\tilde{n}_2}{s_{23}s_{51}} + \frac{n_3\tilde{n}_3}{s_{34}s_{12}} + \frac{n_4\tilde{n}_4}{s_{45}s_{23}} + \frac{n_5\tilde{n}_5}{s_{51}s_{34}} \\ & + \frac{n_6\tilde{n}_6}{s_{14}s_{25}} + \frac{n_7\tilde{n}_7}{s_{32}s_{14}} + \frac{n_8\tilde{n}_8}{s_{25}s_{43}} + \frac{n_9\tilde{n}_9}{s_{13}s_{25}} + \frac{n_{10}\tilde{n}_{10}}{s_{42}s_{13}} \\ & + \frac{n_{11}\tilde{n}_{11}}{s_{51}s_{42}} + \frac{n_{12}\tilde{n}_{12}}{s_{12}s_{35}} + \frac{n_{13}\tilde{n}_{13}}{s_{35}s_{24}} + \frac{n_{14}\tilde{n}_{14}}{s_{14}s_{35}} + \frac{n_{15}\tilde{n}_{15}}{s_{13}s_{45}}\,.\end{aligned} \tag{5.4}$$

If we instead use the more general solution for A_5 and $\widetilde{A}_5$, *i.e.*

$$\begin{array}{lll} X_1 \equiv s_{34}f_1\,, & X_2 \equiv s_{12}f_2\,, & X_3 \equiv s_{45}f_3\,, \\ X_4 \equiv s_{23}f_4\,, & X_5 \equiv s_{15}f_5\,, & X_6 = s_{14}(f_1 - f_2 + f_4)\,, \\ X_7 = s_{25}(f_1 - f_3 + f_4)\,, & X_8 = s_{13}(f_3 - f_1 - f_5)\,, & X_9 = s_{24}(f_3 - f_2 - f_5)\,, \end{array} \tag{5.5}$$

and

$$\begin{array}{lll} \widetilde{X}_1 \equiv s_{34}g_1\,, & \widetilde{X}_2 \equiv s_{12}g_2\,, & \widetilde{X}_3 \equiv s_{45}g_3\,, \\ \widetilde{X}_4 \equiv s_{23}g_4\,, & \widetilde{X}_5 \equiv s_{15}g_5\,, & \widetilde{X}_6 = s_{14}(g_1 - g_2 + g_4)\,, \\ \widetilde{X}_7 = s_{25}(g_1 - g_3 + g_4)\,, & \widetilde{X}_8 = s_{13}(g_3 - g_1 - g_5)\,, & \widetilde{X}_9 = s_{24}(g_3 - g_2 - g_5)\,. \end{array} \tag{5.6}$$

Here $X_1 = n'_3 - n'_5 + n'_8$ and $\widetilde{X}_1 = \tilde{n}'_3 - \tilde{n}'_5 + \tilde{n}'_8$, see eq. (3.23), and we obtain

$$\begin{aligned}-iM_5(1,2,3,4,5) = {} & \frac{n'_1\tilde{n}'_1}{s_{12}s_{45}} + \frac{n'_2\tilde{n}'_2}{s_{23}s_{51}} + \frac{n'_3\tilde{n}'_3}{s_{34}s_{12}} + \frac{n'_4\tilde{n}'_4}{s_{45}s_{23}} + \frac{n'_5\tilde{n}'_5}{s_{51}s_{34}} \\ & + \frac{n'_6\tilde{n}'_6}{s_{14}s_{25}} + \frac{n'_7\tilde{n}'_7}{s_{32}s_{14}} + \frac{n'_8\tilde{n}'_8}{s_{25}s_{43}} + \frac{n'_9\tilde{n}'_9}{s_{13}s_{25}} + \frac{n'_{10}\tilde{n}'_{10}}{s_{42}s_{13}} \\ & + \frac{n'_{11}\tilde{n}'_{11}}{s_{51}s_{42}} + \frac{n'_{12}\tilde{n}'_{12}}{s_{12}s_{35}} + \frac{n'_{13}\tilde{n}'_{13}}{s_{35}s_{24}} + \frac{n'_{14}\tilde{n}'_{14}}{s_{14}s_{35}} + \frac{n'_{15}\tilde{n}'_{15}}{s_{13}s_{45}} \\ & - \big[f_1g_1 + f_2g_2 + f_3g_3 + f_4g_4 + f_5g_5 \\ & \quad + f_1(g_4 - g_3) + g_1(f_4 - f_3) \\ & \quad + f_2(g_5 - g_4) + g_2(f_5 - f_4) \\ & \quad - f_3g_5 - g_3f_5\big]\,.\end{aligned} \tag{5.7}$$

This representation of the gravity is of course guaranteed to be exact due to the KLT-construction. We obtain the simple factorized form (5.4) only when we choose

$$f_1g_1 + f_2g_2 + f_3g_3 + f_4g_4 + f_5g_5 + f_1(g_4 - g_3) + g_1(f_4 - f_3) \\ + f_2(g_5 - g_4) + g_2(f_5 - f_4) - f_3g_5 - g_3f_5 = 0\,. \tag{5.8}$$

This is evidently satisfied when the numerators fulfill the simple Jacobi-like relations. However, more general parameterizations are consistent with this equation as well. For instance, eq. (4.14)–(4.19) implies

$$f_1 = G, \quad f_4 = -G, \quad \text{and} \quad f_2 = f_3 = f_5 = 0\,, \tag{5.9}$$

and using the same parametrization for $\widetilde{A}_5$, eq. (5.8) is seen to be satisfied:

$$f_1g_1 + f_4g_4 + f_1g_4 + g_1f_4 = G^2 + G^2 - G^2 - G^2 = 0\,. \tag{5.10}$$

Again, the freedom in choosing different representations of the KLT-relations arise from the freedom to pick parameterizations of the gauge invariant amplitudes in terms of different pole structures. These pole structures are not gauge invariant by themselves and we see that this arbitrariness in the gauge theory is inherited in the gravity amplitude.

6. One-loop coefficient relations

We end this paper with an obvious application of the monodromy relations in the field theory limit. We illustrate how these relations can imply relations between coefficients of integrals in one-loop gluon amplitudes. For simplicity we will focus on amplitudes in $\mathcal{N} = 4$ super Yang-Mills, but it will be evident that most of the considerations here will apply also to the case of less supersymmetric or even non-supersymmetric amplitudes.

6.1 Preliminaries

Our starting point will be the one-loop gluon amplitudes which can be color decomposed [21] as follows

$$\mathcal{A}_n^{1-loop} = g^n \sum_{c=1}^{[n/2]+1} \sum_{\sigma \in S_n/S_{n;c}} \mathrm{Gr}_{n;c}(\sigma) A_{n;c}(\sigma)\,. \tag{6.1}$$

Here $[x]$ is the largest integer less than or equal to x. The leading color factor is

$$\mathrm{Gr}_{n;1}(\sigma) = N_c \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}})\,, \tag{6.2}$$

and the subleading color factors ($c > 1$) are

$$\mathrm{Gr}_{n;c}(\sigma) = \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(c-1)}}) \mathrm{Tr}(T^{a_{\sigma(c)}} \cdots T^{a_{\sigma(n)}})\,. \tag{6.3}$$

S_n here denotes the set of all permutations of n objects. $S_{n;c}$ is the subset leaving $\mathrm{Gr}_{n;c}$ invariant.

It is sufficient to consider the subamplitude $A_{n;1}$ which is leading in color counting, since the remaining $A_{n;c}$ subamplitudes with $c > 1$ can be obtained as a sum over different permutations of $A_{n;1}$ [21, 22].

In $\mathcal{N} = 4$ super Yang-Mills theory we can always write the one-loop gluon amplitude (using a Passarino-Veltman reduction [23]) as a linear combination of scalar box integrals with rational coefficients [22, 24]. For the leading subamplitude the expression becomes

$$A_{n;1} = \sum \left(\widehat{b} I^{1m} + \widehat{c} I^{2m\,e} + \widehat{d} I^{2m\,h} + \widehat{g} I^{3m} + \widehat{f} I^{4m} \right) . \tag{6.4}$$

Here the sum runs over color-ordered box diagrams, and the integrals (defined in dimensional regularization) are given by

$$I = -i(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon} l}{(2\pi)^{4-2\epsilon}} \frac{1}{l^2 (l - K_1)^2 (l - K_1 - K_2)^2 (l + K_4)^2} . \tag{6.5}$$

The external momenta K_i are given by the sum of momenta of consecutive external legs, and all momenta are taken to be outgoing. The labels $1m$, $2m$, $3m$ and $4m$ refer to the number of "massive" corners, *i.e.* the number of $K_i^2 \neq 0$. This is equivalent to the number of corners with more than one external gluon. The $2m$ case is separated into adjacent massive corners $I^{2m\,h}$ (h for hard), and diagonally opposite massive corners $I^{2m\,e}$ (e for easy).

Since the scalar box integrals are all known explicitly [24], calculation of one-loop amplitudes is reduced to finding the coefficients. From that general setting the existence of relations between coefficients of different one-loop amplitudes is surprising. The indication of such structures does not appear until we introduce unitarity cuts [22,25]. Working in complex momenta it is possible to do quadruple cuts and derive formulas for general coefficients [26]

$$\widehat{a}_\alpha = \frac{1}{2} \sum_{S,J} n_J A_1^{\mathrm{tree}} A_2^{\mathrm{tree}} A_3^{\mathrm{tree}} A_4^{\mathrm{tree}} . \tag{6.6}$$

Here α represent a specific ordering of external legs, J the spin of a particle (running in the loop) in the $\mathcal{N} = 4$ multiplet, n_J the number of particles in the multiplet with spin J and S is the set of the two solutions to the on-shell conditions

$$S = \{ \; l \; | \; l^2 = 0, \quad (l - K_1)^2 = 0, \quad (l - K_1 - K_2)^2 = 0, \quad (l + K_4)^2 = 0 \; \} . \tag{6.7}$$

It turns out that for many amplitudes eq. (6.6) simplifies significantly. The helicity configuration often kills the sum over non-gluonic states and one of the S solutions. These coefficients are therefore only given by a single term of four tree-level gluon amplitudes multiplied together. Monodromy relations on these tree amplitudes then leads to relations among coefficients for one-loop amplitudes. Most interesting is probably the possibility of relating coefficient for split-helicity loop amplitudes to mixed-helicity loop amplitudes. For some reviews of the work at tree and loop level involving helicity amplitudes for gluons, see *e.g.* refs. [27–29].

6.2 Six-point examples

In the following section we give two explicit examples of how the monodromy relations, in combination with unitarity cuts, can be used to obtain relations between scalar box integral coefficients of different one-loop amplitudes. These should be sufficient to get the idea for more general one-loop amplitudes.

6.2.1 Two-mass (easy) coefficient relation

Let us begin by considering the $\widehat{c}_1$ coefficient to the $A_{6;1}(1^+, 2^-, 3^-, 4^+, 5^+, 6^+)$ one-loop amplitude, *i.e.* the coefficient to the $I^{2m\,e}$ integral for a specific ordering of the legs. Here we choose the one illustrated in fig. 5. Note that with this helicity configuration fig. 5 is the only diagram that contributes to $\widehat{c}_1$. Any other assignment of helicities to the loop-legs makes at least one of the corners vanish. In addition, only gluons can run in the loop for this helicity configuration – fermions and scalars would make the two corners with equal helicity vanish.

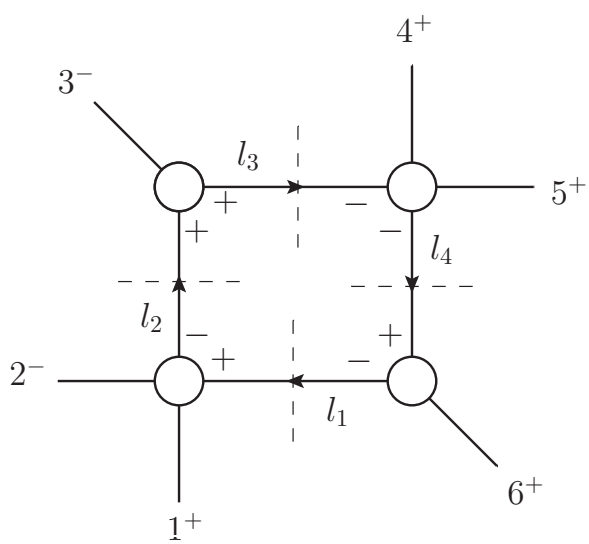

Figure 5: Two-mass (easy) cut diagram.

Since the four corners are just given by the appropriate (on-shell) tree-level amplitudes, we can use the four-point monodromy relations to flip the legs around. One of the advantages of the monodromy relations is that we can always keep two of the legs fixed. This is important here since we do not want to change the position of legs in the loop. The diagram in fig. 5, which we denote $\mathcal{D}_{12}^{2m\,e}$, is therefore related to the diagram of same type, but with legs 1 and 2 interchanged, through

$$\mathcal{D}_{21}^{2m\,e} = \frac{s_{(-l_1)1}}{s_{l_2 1}} \mathcal{D}_{12}^{2m\,e}\,. \tag{6.8}$$

The helicity configuration $(+\,+\,-)$ of the two three-point corners is only consistent with one of the S solutions [26], and the coefficient is simply given by $\widehat{c}_1 = \mathcal{D}_{12}^{2m\,e}/2$. The same

is of course true in the case of leg 1 and 2 interchanged, which imply that

$$\widehat{c}_1 = \frac{s_{(-l_1)1}}{s_{l_2 1}} \widehat{c}_1' \,, \tag{6.9}$$

where $\widehat{c}_1'$ is the coefficient to the $I^{2m\,e}$ scalar box integral for the one-loop amplitude $A_{6;1}(2^-, 1^+, 3^-, 4^+, 5^+, 6^+)$. This is a very simple relation between coefficients for split-helicity and mixed-helicity loop amplitudes.

For completeness, we show how to solve for the loop-momenta and express the fraction in front of $\widehat{c}_1'$ solely in terms of external momenta. For this we will be using the spinor helicity formalism. From momentum conservation and on-shell conditions we have

$$\begin{aligned} &l_2 = l_1 - p_1 - p_2, && (l_1 - p_1 - p_2)^2 = 0, \\ &l_3 = l_2 - p_3 = l_1 - p_1 - p_2 - p_3, && (l_1 - p_1 - p_2 - p_3)^2 = 0, \\ &l_4 = l_3 - p_4 - p_5 = l_1 + p_6, && (l_1 + p_6)^2 = 0\,, \end{aligned} \tag{6.10}$$

and in terms of spinor products

$$\frac{s_{(-l_1)1}}{s_{l_2 1}} = \frac{s_{(-l_1)1}}{s_{(-l_1)2}} = \frac{\langle 1 l_1\rangle [l_1 1]}{\langle 2 l_1\rangle [l_1 2]}\,. \tag{6.11}$$

Since the three-point corners have helicity configuration $(++-)$ we must take the holomorphic spinors at these corners to be proportional and hence having vanishing $\langle\bullet\rangle$ product (remember, we are working with complex momenta, so the $[\bullet]$ product can be non-vanishing). In particular we get

$$\langle l_1 6\rangle = 0 \quad \Longrightarrow \quad |l_1\rangle = \alpha |6\rangle\,. \tag{6.12}$$

The proportionality factor α can be obtained from

$$(l_1 - p_1 - p_2)^2 = 0 \quad \Longrightarrow \quad 2 l_2 \cdot (p_1 + p_2) = (p_1 + p_2)^2\,, \tag{6.13}$$

and since $2 l_2 \cdot (p_1 + p_2) = \langle l_1 | 1 + 2 | l_1]$ $= \alpha \langle 6 | 1 + 2 | l_1]$,

$$\alpha = \frac{(p_1 + p_2)^2}{\langle 6 | 1 + 2 | l_1]}\,. \tag{6.14}$$

To express the anti-holomorphic spinor of l_1 we use

$$(l_1 - (p_1 + p_2 + p_3))^2 = 0 \quad \Longrightarrow \quad 2 l_1 \cdot (p_1 + p_2 + p_3) = (p_1 + p_2 + p_3)^2\,, \tag{6.15}$$

and

$$2 l_1 \cdot (p_1 + p_2 + p_3) = \langle l_1 | 1 + 2 + 3 | l_1] = \alpha \langle 6 | 1 + 2 + 3 | l_1]\,, \tag{6.16}$$

from which follows

$$\begin{aligned} &(p_1 + p_2)^2 \langle 6 | 1 + 2 + 3 | l_1] = \langle 6 | 1 + 2 | l_1] (p_1 + p_2 + p_3)^2 \quad \Longleftrightarrow \\ &\big[\underbrace{(p_1 + p_2)^2 \langle 6 | (1 + 2 + 3) - (p_1 + p_2 + p_3)^2 \langle 6 | (1 + 2)}_{\equiv [\gamma|} \big] | l_1] = 0\,, \end{aligned} \tag{6.17}$$

i.e. $|l_1] = \beta|\gamma]$. We are not interested in the proportionality factor β since it cancels out from eq. (6.11) anyway. Using these expressions for the spinors of l_1, we get, after a bit of rewriting,

$$\frac{s_{(-l_1)1}}{s_{l_2 1}} = -\frac{\langle 16\rangle\langle 23\rangle}{\langle 26\rangle\langle 13\rangle}\,. \tag{6.18}$$

6.2.2 One-mass coefficient relation

Let us now consider a one-mass box integral coefficient. As in the example above we just use the $A_{6;1}(1^+,2^-,3^-,4^+,5^+,6^+)$ one-loop amplitude to illustrate the idea. The diagram is given in fig. 6, which we denote as $\mathcal{D}^{1m}_{612}$. Again this helicity configuration kills all other diagrams and allow only gluons to run in the loop.

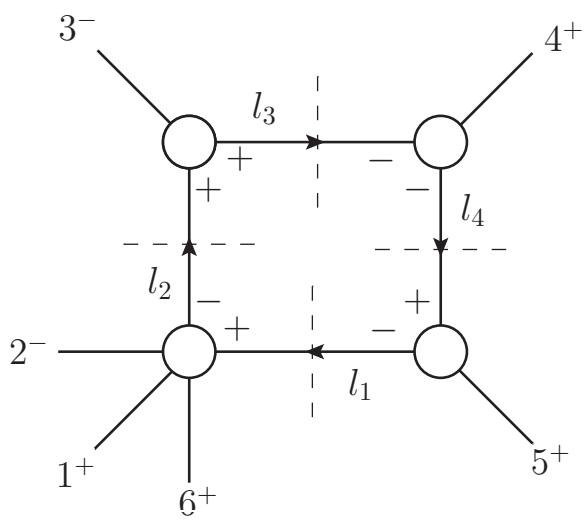

Figure 6: One-mass cut diagram.

This time we can use the five-point monodromy relations to connect a diagram of mixed helicity to two diagrams of split helicities

$$\mathcal{D}^{1m}_{621} = \frac{(s_{16}+s_{(-l_1)1})\mathcal{D}^{1m}_{612} + s_{(-l_1)1}\mathcal{D}^{1m}_{162}}{s_{l_2 1}}\,, \tag{6.19}$$

with obvious notation for the different diagrams. Like above, the coefficients related to these diagrams only consist of these single terms, and we can therefore equally well write it as

$$\widehat{b}_{621} = \frac{(s_{16}+s_{(-l_1)1})\widehat{b}_{612} + s_{(-l_1)1}\widehat{b}_{162}}{s_{l_2 1}}\,, \tag{6.20}$$

where we have a one-mass integral coefficient belonging to the mixed-helicity amplitude $A_{6;1}(2^-,1^+,3^-,4^+,5^+,6^+)$ related to one-mass coefficients of the split-helicity amplitudes $A_{6;1}(1^+,2^-,3^-,4^+,5^+,6^+)$ and $A_{6;1}(6^+,2^-,3^-,4^+,5^+,1^+)$.

Using very similar methods as for the two-mass case we could again express the kinematic invariants in terms of external momenta. However, this is not our focus here.

7. Conclusion

We have reconsidered the BCJ-relations in gauge theories from several points of view. Based on the monodromy proof, we have explored the extent to which Jacobi-like relations for residues of poles (and multiple poles) can be *derived*. We have found that Jacobi-like relations can be introduced consistently with the constraints of the monodromy relations. But extended Jacobi-like identities are also perfectly consistent with the gauge invariant relations. We have demonstrated this explicitly from both field and string theoretic angles.

We have also considered the implications for gravity amplitudes. Very symmetric forms follows in a simple manner through using the KLT-relations together with the link posed by monodromy in the gauge theory side. This direction appears worthwhile to pursue in the future.

As an application of monodromy relations, we have explicitly illustrated how these tree-level relations give rise to non-trivial identities at loop level. The simplest case is that of $\mathcal{N} = 4$ super Yang-Mills theory where relations between one-loop box functions are directly derivable through quadruple cut techniques. Similar considerations are valid for less supersymmetric or non-supersymmetric amplitudes as well, although in such cases the relations are rather more complicated. There are thus clearly several interesting directions for future work that will exploit these relations.

8. Acknowledgments

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A. Evaluation of the five-point integrals

In this appendix we evaluate the five point amplitudes (4.9) for the ordering $(1, 2, 3, 4, 5)$. We use the result

$$\begin{aligned} I(a,b,c,d,e) &= \int_0^1 dz_3 \int_0^{z_3} dz_2\, z_2^a (z_3 - z_2)^b (1-z_2)^c (1-z_3)^d z_3^e \\ &= \frac{\Gamma(a+1)\Gamma(b+1)\Gamma(d+1)\Gamma(a+b+e+2)}{\Gamma(a+b+2)\Gamma(a+b+d+e+3)} \\ &\quad \times {}_3F_2(a+1, -c, a+b+e+2; a+b+2, a+b+d+e+3; 1)\,, \end{aligned} \tag{A.1}$$

that expresses the integral in terms of the hypergeometric function ${}_3F_2$. We introduce the notation

$$\begin{aligned} I_5(a,b,c,d,e) &= \frac{\Gamma(\alpha' s_{12} + a + 1)\Gamma(\alpha' s_{23} + b + 1)\Gamma(\alpha' \alpha' s_{34} + d + 1)\Gamma(\alpha' s_{45} + a + b + e + 2)}{\Gamma(s_{2,13} + a + b + 2)\Gamma(\alpha' s_{4,35} + a + b + d + e + 3)} \\ &\times\ {}_3F_2(\alpha' s_{12} + a + 1, -s_{24} - c, \alpha' s_{45} + a + b + e + 2; \alpha' s_{2,13} + a + b + 2, \alpha' s_{4,35} + a + b + d + e + 3; 1)\,, \end{aligned} \tag{A.2}$$

Setting $\hat{s}_{i,j} = \alpha' \, s_{i,}$ we have

Contribution A

The integral is

$$I_5(-1,0,0,0,-1) = \frac{1}{\hat{s}_{1,2}\hat{s}_{1,5}} \frac{\Gamma(\hat{s}_{1,2}+1)\Gamma(\hat{s}_{1,5}+1)\,\Gamma(\hat{s}_{2,3}+1)\Gamma(\hat{s}_{3,4}+1)}{\Gamma(\hat{s}_{1,2}+\hat{s}_{2,3}+1)\Gamma(\hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}+\hat{s}_{3,4}+1)}$$
$$\times {}_3\mathrm{F}_2(\hat{s}_{1,2}, -\hat{s}_{2,4}, \hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}; \hat{s}_{1,2}+\hat{s}_{2,3}+1, \hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}+\hat{s}_{3,4}+1; 1)\,, \tag{A.3}$$

Contribution B

$$I_5(0,-1,-1,0,0) = \frac{1}{\hat{s}_{2,3}\hat{s}_{3,4}}$$
$$\frac{\Gamma(\hat{s}_{1,2}+1)\Gamma(\hat{s}_{2,3}+1)\Gamma(\hat{s}_{3,4}+1)\Gamma(\hat{s}_{4,5}+1)}{\Gamma(\hat{s}_{1,2}+\hat{s}_{2,3}+1)\Gamma(\hat{s}_{3,4}+\hat{s}_{4,5}+1)}$$
$$\Big[{}_3\mathrm{F}_2(\hat{s}_{1,2}+1, -\hat{s}_{2,4}, \hat{s}_{4,5}+1; \hat{s}_{1,2}+\hat{s}_{2,3}+1, \hat{s}_{3,4}+\hat{s}_{4,5}+1; 1)$$
$$-\frac{\hat{s}_{2,3}(\hat{s}_{4,5}+1)}{(\hat{s}_{1,2}+\hat{s}_{2,3}+1)(\hat{s}_{3,4}+\hat{s}_{4,5}+1)}{}_3\mathrm{F}_2(\hat{s}_{1,2}+1, 1-\hat{s}_{2,4}, \hat{s}_{4,5}+2; \hat{s}_{1,2}+\hat{s}_{2,3}+2, \hat{s}_{3,4}+\hat{s}_{4,5}+2; 1)\Big]\,, \tag{A.4}$$

Contribution C

$$I_5(-1,0,0,-1,0) = \frac{1}{\hat{s}_{3,4}} \frac{\Gamma(\hat{s}_{1,2}+2)\Gamma(\hat{s}_{2,3}+1)\Gamma(\hat{s}_{3,4}+1)\Gamma(\hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}+3)}{\Gamma(\hat{s}_{1,2}+\hat{s}_{2,3}+3)\Gamma(\hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}+\hat{s}_{3,4}+3)}$$
$$\times {}_3\mathrm{F}_2(-\hat{s}_{2,4}, \hat{s}_{1,2}+2, \hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}+3; \hat{s}_{1,2}+\hat{s}_{2,3}+3, \hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}+\hat{s}_{3,4}+3; 1)\,, \tag{A.5}$$

Contribution D

$$I_5(0,0,-1,-1,0) = \frac{1}{\hat{s}_{3,4}}$$
$$\times \frac{\Gamma(\hat{s}_{1,2}+1)\Gamma(\hat{s}_{2,3}+1)\Gamma(\hat{s}_{3,4}+1)\Gamma(\hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}+2)}{\Gamma(\hat{s}_{1,2}+\hat{s}_{2,3}+2)\Gamma(\hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}+\hat{s}_{3,4}+2)}$$
$$\times {}_3\mathrm{F}_2(\hat{s}_{1,2}+1, \hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}+2, 1-\hat{s}_{2,4}; \hat{s}_{1,2}+\hat{s}_{2,3}+2, \hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}+\hat{s}_{3,4}+2; 1)\,, \tag{A.6}$$

Contribution E

$$I_5(0,-1,0,0,-1) = \frac{1}{\hat{s}_{2,3}\hat{s}_{4,5}}$$
$$\frac{\Gamma(\hat{s}_{1,2}+1)\Gamma(\hat{s}_{2,3}+1)\Gamma(\hat{s}_{3,4}+1)\Gamma(\hat{s}_{4,5}+1)}{\Gamma(\hat{s}_{1,2}+\hat{s}_{2,3}+1)\Gamma(\hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}+\hat{s}_{3,4}+1)}$$
$$\times {}_3\mathrm{F}_2(-\hat{s}_{2,4}, \hat{s}_{1,2}+1, \hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}; \hat{s}_{1,2}+\hat{s}_{2,3}+1, \hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}+\hat{s}_{3,4}+1; 1)\,, \tag{A.7}$$

Contribution F

$$I_5(0,0,-1,0,-1) = \frac{\Gamma(\hat{s}_{1,2}+1)\Gamma(\hat{s}_{2,3}+1)\Gamma(\hat{s}_{3,4}+1)\Gamma(\hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}+1)}{\Gamma(\hat{s}_{1,2}+\hat{s}_{2,3}+2)\Gamma(\hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}+\hat{s}_{3,4}+2)}$$
$$\times\, {}_3\mathrm{F}_2(\hat{s}_{1,2}+1, \hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}+1, 1-\hat{s}_{2,4}; \hat{s}_{1,2}+\hat{s}_{2,3}+2, \hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}+\hat{s}_{3,4}+2; 1)\,, \tag{A.8}$$

Contribution G

$$I_5(0,-2,0,0,0) = \frac{\hat{s}_{1,2}+\hat{s}_{2,3}}{(\hat{s}_{2,3}-1)\hat{s}_{2,3}\,\hat{s}_{4,5}}$$
$$\times \frac{\Gamma(\hat{s}_{1,2}+1)\Gamma(\hat{s}_{2,3}+1)\Gamma(\hat{s}_{3,4}+1)\Gamma(\hat{s}_{4,5}+1)}{\Gamma(\hat{s}_{1,2}+\hat{s}_{2,3}+1)\Gamma(\hat{s}_{4,5}+\hat{s}_{3,4}+1)}$$
$$\times\, {}_3\mathrm{F}_2(-\hat{s}_{2,4}, \hat{s}_{1,2}+1, \hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}; \hat{s}_{1,2}+\hat{s}_{2,3}, \hat{s}_{1,2}+\hat{s}_{1,3}+\hat{s}_{2,3}+\hat{s}_{3,4}+1; 1)\,,$$

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