Black Holes
in General Relativity

Nathalie Deruelle

Six Lectures

CEA, January-February 2009

CEA-Direction des Sciences de la Matière

Institut de Physique Théorique

unité de recherche associée au CNRS

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Cours de Physique Théorique de l'IPhT, année 2008-2009

Organisé en commun avec l'Ecole Doctorale de Physique de la Région Parisienne (ED 107)

Black Holes in General Relativity

Nathalie Deruelle

APC, Université Paris 7

Les vendredis 9/1, 16/1, 23/1, 30/1, 6/2 et 13/2/2009.

1 – From Schwarzschild to Kerr: the development of a concept

2 – The geometry of black holes

3 – The energetics of black holes

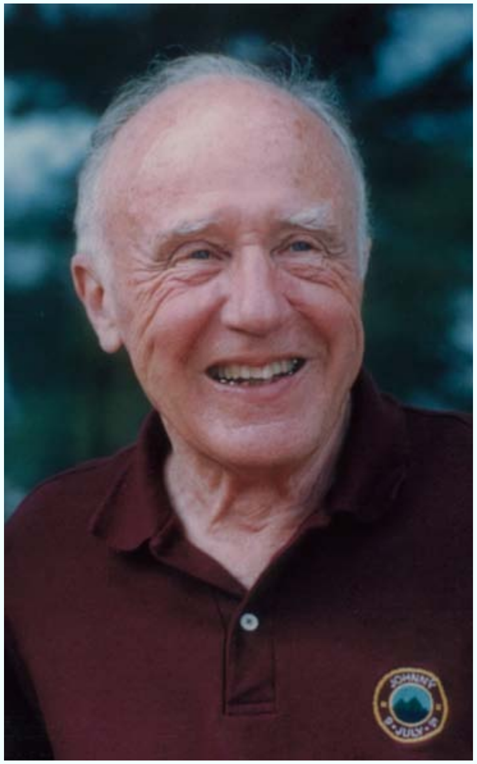
4 – Black holes and astrophysics

5 – Hawking's radiation and black hole thermodynamics

6 – Hairy and higher-dimensional black holes

Horaires : *les **vendredis de 10h15 à 12h15.***
Lieu : *IPhT, CEA Saclay, Orme des Merisiers, Bât. 774, p.1A Salle C. Itzykson.*
Accès : *Par lignes de bus publiques **(269.02** et **91.06)** ou*
- navettes CEA: RER B Le Guichet vers CEA Orme Bât. 774, toutes les 15min de 8h30 à 9h45;
- navette CEA: CEA Orme Bât. 774 vers RER B Le Guichet à 12h36.

Renseignements : ***http://ipht.cea.fr** ou **ipht-lectures@cea.fr***



John Archibald Wheeler (1912-2008) who coined the word “Black Hole” in 1967

Lecture One

From Schwarzschild to Oppenheimer :

the DEVELOPMENT of a CONCEPT

References

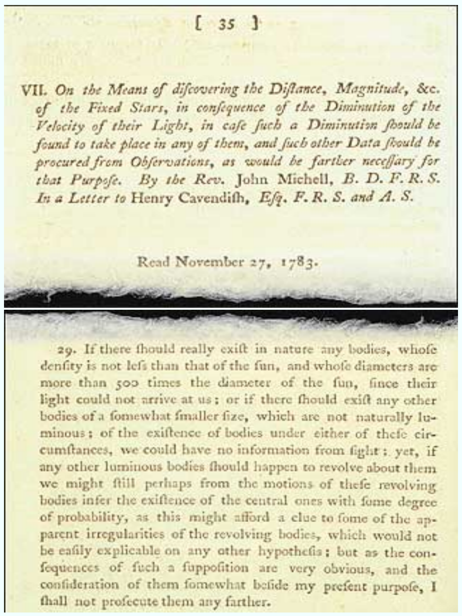
Einstein et la relativité générale, Jean Eisenstaedt, Cnrs-Eds, 2002
English translation : Oxford and Princeton U-Press, 2006

Black holes and time warps, Kip Thorne, Norton Publ., 1994

Dark stars : the evolution of an idea, Werner Israel
in 300 years of Gravitation, CUP, 1987

Prehistory

John Michell (1783), Pierre-Simon de Laplace (1796), Robert Blair (1786)



[35]

VII. *On the Means of discovering the Distance, Magnitude, &c. of the Fixed Stars, in consequence of the Diminution of the Velocity of their Light, in case such a Diminution should be found to take place in any of them, and such other Data should be procured from Observations, as would be farther necessary for that Purpose. By the Rev.* John Michell, *B. D. F. R. S. In a Letter to* Henry Cavendish, *Esq. F. R. S. and A. S.*

Read November 27, 1783.

29. If there should really exist in nature any bodies, whose density is not less than that of the sun, and whose diameters are more than 500 times the diameter of the sun, since their light could not arrive at us; or if there should exist any other bodies of a somewhat smaller size, which are not naturally luminous; of the existence of bodies under either of these circumstances, we could have no information from sight; yet, if any other luminous bodies should happen to revolve about them we might still perhaps from the motions of these revolving bodies infer the existence of the central ones with some degree of probability, as this might afford a clue to some of the apparent irregularities of the revolving bodies, which would not be easily explicable on any other hypothesis; but as the consequences of such a supposition are very obvious, and the consideration of them somewhat beside my present purpose, I shall not prosecute them any farther.

Michell, 1724-1793

Laplace, 1749-1827

Blair, 1748-1828

see Jean Eisenstaedt, "Avant Einstein", Seuil, 2005

(box1.) Newtonian "Dark bodies"

$$F = m_{\rm in} a \quad , \quad F = -m_{\rm grav} \nabla U \quad , \quad U = -\frac{GM}{r} \quad , \quad m_{\rm in} = m_{\rm grav}$$

hence $\quad \dot{r}^2 = 2E + \frac{2GM}{r} - \frac{L^2}{r^2} \quad , \quad \dot{\phi} = \frac{L}{r^2} \qquad$ (1)

- escape velocity is $\; c \;$ if $\; r = \frac{2GM}{c^2} \qquad$ ($E = 0, \; L = 0$ in (1))
(Michell, Laplace)

- deviation of "light" : $\; r = \frac{p}{1 + e\cos\phi} \;$ with velocity c at $\; r = r_{\rm min}$:

$$\Delta\phi = 2(\phi_\infty - \pi/2) \quad ; \quad \cos\phi_\infty = -1/e \quad ; \quad 1 + e = \frac{r_{\rm min} c^2}{GM}$$

if $\; e \gg 1 \;$: $\; \Delta\phi \approx \frac{2GM}{c^2 r_{\rm min}} \qquad$ (Soldner, 1801)

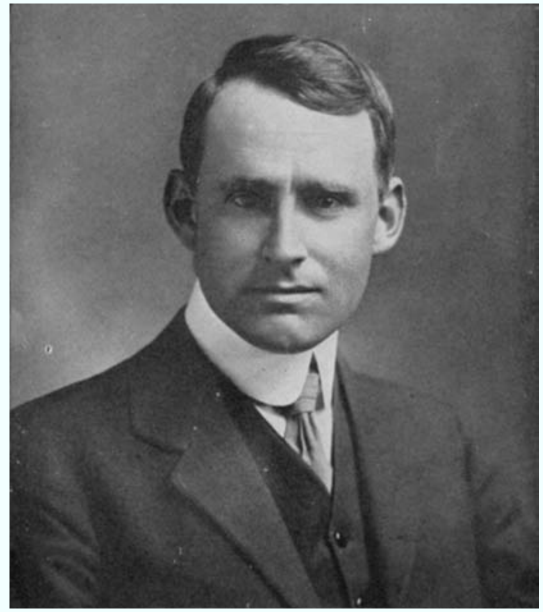
- "collapse" : $L = 0$, $E = \frac{GM}{r_0}$ in (1) ; hence :

$t = \frac{r_0}{2}\sqrt{\frac{r_0}{2GM}}(\eta + \sin\eta)$; $r = \frac{r_0}{2}(1 + \cos\eta)$; and : $t_{\rm collapse} = \frac{\pi}{2} r_0 \sqrt{\frac{r_0}{2GM}}$

All forgotten for 150 years

Revived by Oliver Lodge (1921) and Arthur Eddington (1924)

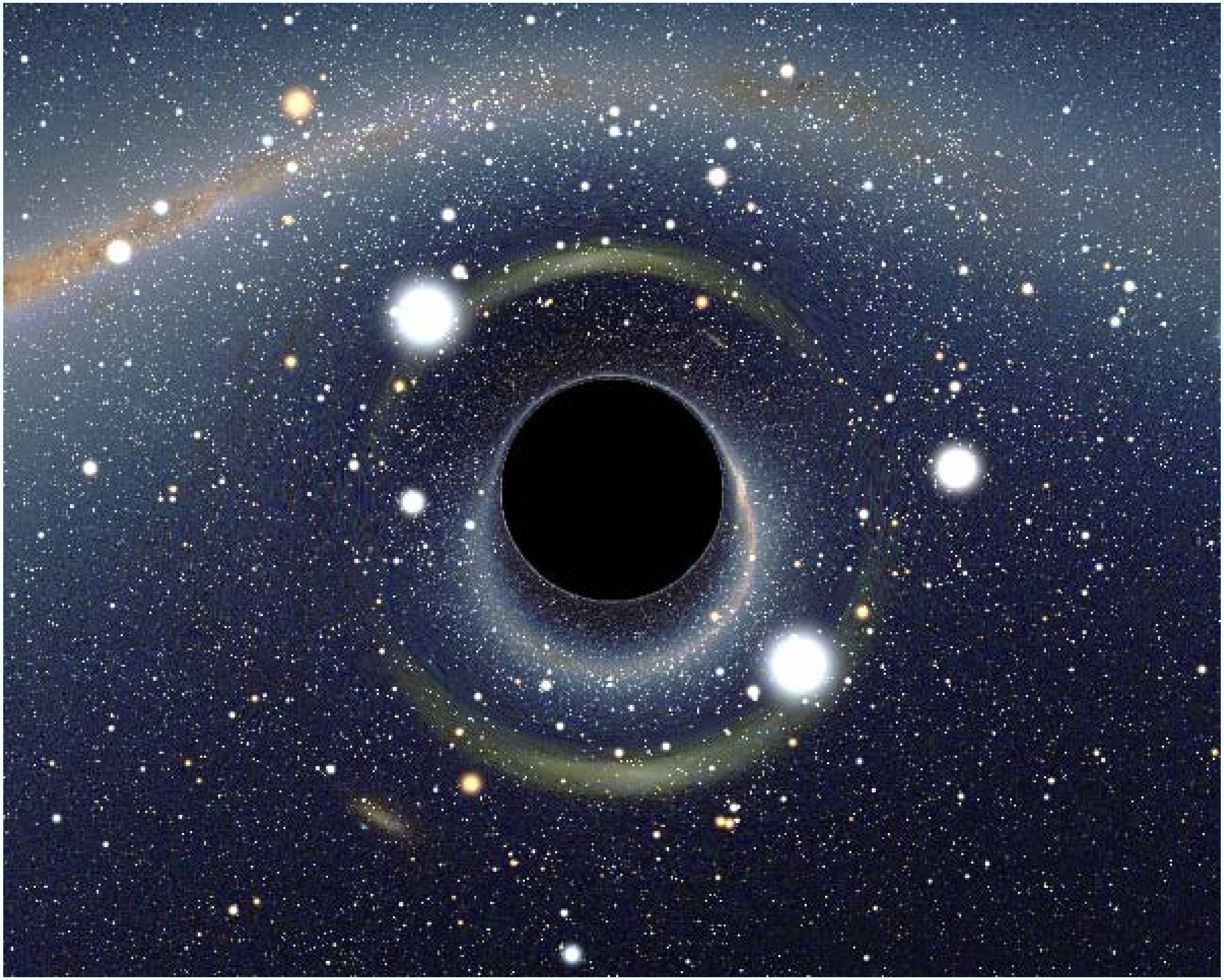
Lodge, 1851-1940



Eddington, 1882-1944

“M.T.W.” and Hawking-Ellis textbooks (1973); Gibbons/Schaffer (1979)

reason : **Dark Bodies have nothing much to do with Black holes...**



Alain Riazuelo (2007)   A Modern View of a Black Hole

Fifty-odd years of struggle

Early years of discoveries and debates

1915 : Einstein's equations of General Relativity
1916 : Schwarzschild solution
1922 : Einstein at the Collège de France

Input from Quantum Physics and Cosmology

1930 : Chandrasekhar's maximum mass of white dwarfs
1932 : Lemaître's insights
1939 : Oppenheimer-Snyder's collapsing star

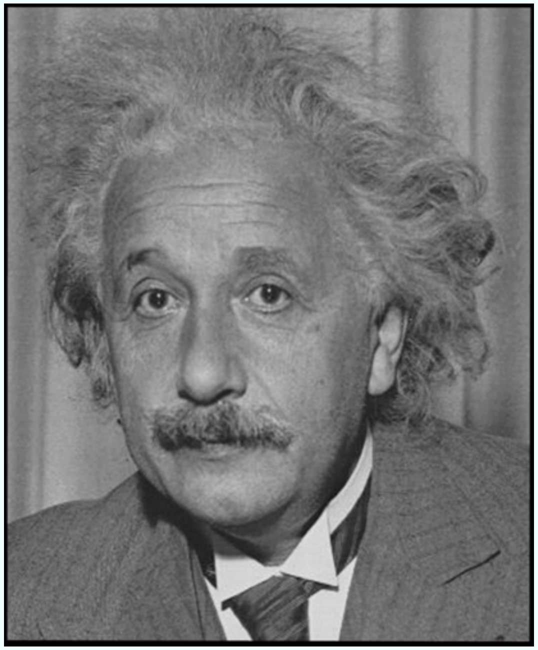
"Low-water mark" (1940-1955)

Renaissance

1960 : Kruskal diagram
1963 : Discovery of quasars
1963 : Kerr solution

Early Years : Einstein's equations of General Relativity

November 1915 : Einstein and Hilbert's "race" to $G_{\mu\nu} = \kappa T_{\mu\nu}$



Albert Einstein (1879-1955)

David Hilbert (1862-1943)

see, e.g., Todorov, arXiv:physics/0504179

(box2.) **Einstein's equations in a nutshell** (beginning)

Special Relativity

- Space and Time as M_4 : $ds^2 = \eta_{ij} dX^i dX^j = -dT^2 + d\vec{X}^2$
- (T, X^α) represent a "clock" at rest and "position" in inertial frame $\mathcal{S}$
- $X^i \to X'^i = \Lambda^i_j X^j$ such that $ds^2 = -dT'^2 + d\vec{X}'^2$. T' : time in $\mathcal{S}'$
- If $U^i \equiv \frac{dX^i}{d\tau}$ s.t. $U^i U_i = -1$, then τ is "proper time" along $X^i(\tau)$.
- equation of motion of a free particle : $\frac{dU^i}{d\tau} = 0$.

Special Relativity in accelerated frames $(X^i \to x^i = x^i(X^j))$

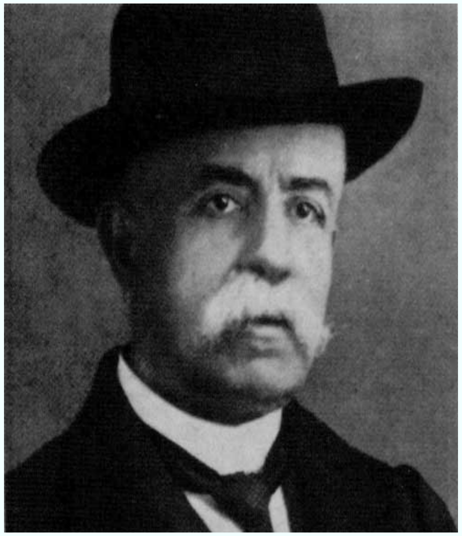
- $ds^2 = \ell_{ij} dx^i dx^j$ with $\ell_{ij} = \frac{\partial X^k}{\partial x^i}\frac{\partial X^l}{\partial x^j}\eta_{kl}$
  direct correspondence between coordinates and clock/position is lost
- eom of a free particle : $\frac{\tilde{D}u^i}{d\tau} = 0 \iff \frac{du^i}{d\tau} + \tilde{\Gamma}^i_{jk} u^j u^k = 0$
  where $u^i \equiv \frac{\partial x^i}{\partial X^j} U^j$ and $\tilde{\Gamma}^i_{jk} = \frac{1}{2}\ell^{ip}(\partial_j \ell_{kp} + \partial_k \ell_{pj} - \partial_p \ell_{jk})$
  inertial accelerations are "encoded" in the "Christoffel symbols" $\tilde{\Gamma}^i_{jk}$

(box2.) **Einstein's equations in a nutshell** (continued)

Principle of equivalence

- $m_{\text{inertial}} = m_{\text{gravitational}}$:
  all particles fall the same way in a gravitational field
  just like free particles "fall" the same way in an accelerated frame
  hence : gravity is inertia ("Weak equivalence principle", 1907).
  Constant acceleration = Constant gravitational field ; hence :
  $\ell_{00} \equiv g_{00} = -(1+2U)$ where $U = -\frac{GM}{r}$.
- Source of gravitational field is the stress-energy tensor of matter
  e.g. $T_{ij} = (\epsilon + p) u_i u_j + p g_{ij}$
  where ϵ and p are its energy density and pressure
  as measured in a local inertial frame ("Einstein equivalence principle")
- gravity cannot be globally "effaced", that is :
  there is no global coordinate transformation which turns g_{ij} into η_{ij}
  hence : spacetime must be curved (1912)

The Mathematicians of curved spaces

Bernhard Riemann (1826-1866) ; Gregorio Ricci Curbastro (1853-1925) ;

Tullio Levi-Civita (1873-1941) ; Marcel Grossmann (1878-1936)

(box2.) **Einstein's equations in a nutshell** (continued)

Riemannian geometry

- Spacetime as an ensemble of points labelled by 4 coordinates x^i. At each point one can define vectors t^i, forms t_i, tensors t_{ij} etc which transform as $t^i \to t'^i = \frac{\partial x'^i}{\partial x^j} t^j$ etc when $x^i \to x'^i$
- One introduces a "connexion" that is 40 (symmetric) functions $\Gamma^i_{jk}(x^l)$ which define "autoparallels" : $\frac{Du^i}{d\tau} = 0 \iff \frac{du^i}{d\tau} + \Gamma^i_{jk} u^j u^k = 0$ where $u^i \equiv \frac{dx^i}{d\tau}$ and "parallel transport" of tensors from point to point : $D_k t_{ij} = 0$...
- One also introduces a "metric" that is 10 (symmetric) functions $g_{ij}(x^l)$ which define "geodesics" $\frac{du^i}{d\tau} + \left\{ {i \atop jk} \right\} u^j u^k = 0$ where $\left\{ {i \atop jk} \right\} = \frac{1}{2} g^{ip}(\partial_j g_{kp} + \partial_k g_{pj} - \partial_p g_{jk})$
- Levi-Civita connexion : autoparallels=geodesics, that is $\Gamma^i_{jk} \equiv \left\{ {i \atop jk} \right\}$

(box2.) **Einstein's equations in a nutshell** (end)

The Riemann tensor and Bianchi identities

- The Riemann tensor $R^i_{jkl} \equiv \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{pk}\Gamma^p_{jl} - \Gamma^i_{pl}\Gamma^p_{jk}$
  if is zero in one coordinate system, is zero in all, and $V_4 = M_4$
  if not, spacetime is "curved"
- Ricci tensor : $R_{ij} \equiv R^p_{ipj}$; scalar curvature $R \equiv g^{ij} R_{ij}$
- Einstein's tensor : $G_{ij} \equiv R_{ij} - \frac{1}{2} g_{ij} R$
- Bianchi identity : $D_i G^i_j \equiv 0$ (1902)
  (unknown to Hilbert and Einstein in 1915)

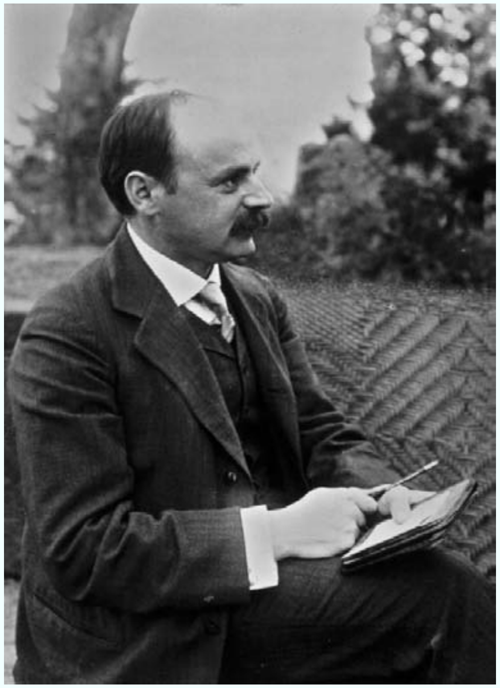
Einstein's equations

$$G_{ij} = \kappa T_{ij} \quad \text{with} \quad \kappa \equiv \frac{8\pi G}{c^4} = 8\pi$$

25 November 1915

Early Years : The Schwarzschild solution

Winter 1915-1916



Karl Schwarzschild
(1873-1916)

$$ds^2 = -f_0 dx_0^2 + f_1 dx_1^2 + f_2 \left[\frac{dx_2^2}{1 - x_2^2} + (1 - x_2^2) dx_3^2 \right]$$

$$f_i = f_i(x_1) \quad , \quad f_0 f_1 f_2^2 = 1 \quad (\sqrt{-g} = 1)$$

Impose asymptotical flatness and continuity of f_i ; introduce "auxiliary quantity" $r = (3x_1 + \alpha^3)^{1/3}$, (keeping $x_1 = 0$ as the "center"), and obtain

$$ds^2 = -(1 - \alpha/r)\, dt^2 + \frac{1}{1-\alpha/r} dr^2 + r^2 d\Omega^2$$

Newtonian limit : $\alpha = \frac{2GM}{c^2}$ $(\alpha_\odot = 3kms)$
(Einstein annoyed : sol. is "unMachian" $\Longrightarrow$ Λ!)

(box3.) **First exact solutions**

staticity and spherical symmetry : exists coordinates (t, r, θ, ϕ) such that
$ds^2 = -e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ (Droste, 1916)

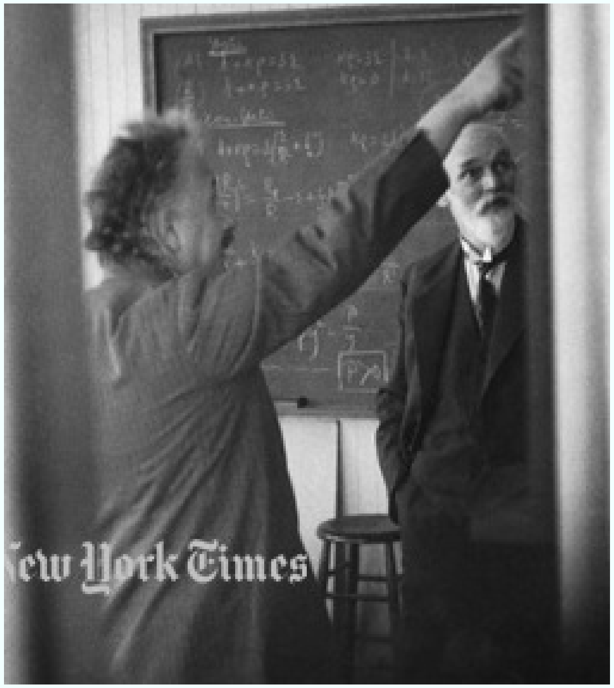
$$G_{00} = \frac{e^\nu}{r^2}\left[r(1-e^{-\lambda})\right]' \quad , \quad G_{rr} = -\frac{e^\lambda}{r^2}\left(1-e^{-\lambda}\right) + \frac{\nu'}{r}$$

• Schwarzschild : $G_{ij} = 0$ hence : $e^\nu = e^{-\lambda} = 1 - \frac{2m}{r}$ $\left(m \equiv \frac{GM}{c^2}\right)$
(other equations redundant because of Bianchi's identities.)

• de Sitter (1917) : $G_{ij} = \Lambda g_{ij}$ hence : $e^\nu = e^{-\lambda} = 1 - \frac{\Lambda r^2}{3}$
Kottler (1918) : $e^\nu = e^{-\lambda} = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}$, $r_{\rm deSitter} = \sqrt{\Lambda/3}$

• Reissner (1916) and Nordström (1918) : $G_{ij} = \kappa T_{ij}$
with $T_{ij} = F_{ik}F_j{}^k - \frac{1}{4}\eta_{ij}F_{kl}F^{kl}$
and $F_{ij} = \partial_i A_j - \partial_j A_i$ with $A^0 = \Phi(r)$; $A^\alpha = 0$
hence : $\Phi = \frac{q}{r}$ and $e^\nu = e^{-\lambda} = 1 - \frac{2m}{r} + \frac{q^2}{r^2}$ $\left(r_\pm = m \pm \sqrt{m^2 - q^2}\right)$
(NB : $q/m \simeq 2 \times 10^{21}$ for an electron)

Some exact solution pioneers



1917
Willem de Sitter
(1872-1934)

1916
Hans Reissner
(1874-1967)

1918
Gunnar Nordström
(1881-1923)

(box4.) Schwarzschild interior solution

Einstein's EOM : $G_{ij} = 8\pi T_{ij}$ with $ds^2 = -e^{2U(r)}dt^2 + \frac{dr^2}{1-2m(r)/r} + r^2 d\Omega^2$

where $T_{ij} = (\epsilon + p)u_i u_j + p g_{ij}$ and $u^i u_i = -1$

and Bianchi identities : $D_i T^{ij} = 0$ yield

$$\frac{dm}{dr} = 4\pi r^2 \epsilon \quad , \quad \frac{dU}{dr} = \frac{1}{1-2m/r}\left(\frac{m}{r^2} + 4\pi p r\right) \quad , \quad \frac{dp}{dr} = -(\epsilon + p)\frac{dU}{dr}$$

(under this form : Tolman ; Oppenheimer-Volkoff, 1939)

if $\epsilon = Const.$ the solution is (Schwarzschild, 1916) :

$m(r) = \frac{mr^2}{R^3}$; $\quad m \equiv \frac{GM}{c^2}$; $\quad$ (M is the mass of the star, R its radius)

$e^{U(r)} = \frac{3}{2}\sqrt{1 - \frac{2m}{R}} - \frac{1}{2}\sqrt{1 - \frac{2mr^2}{R^3}}$ and $p(r) = \epsilon\left(\sqrt{1 - \frac{2m}{R}}e^{-U} - 1\right)$

central pressure $p(0) = \frac{\epsilon}{3\sqrt{1-2m/R}}\left(1 - \sqrt{1 - \frac{2m}{R}}\right)$ is finite if $R > \frac{9m}{4}$

since $m_\odot = 1.5km$ and $R_\odot = 7 \times 10^5 kms$... : $r = 2m$ is "unphysical".

Early Years : The Schwarzschild “singularity”

Geodesics “avoid” $r = 2m$...

Schwarzschild (1915); Johannes Droste (1916) ; Max von Laue (1921) ;
C. de Jans (1923) ; Yusuke Hagihara (1931)...

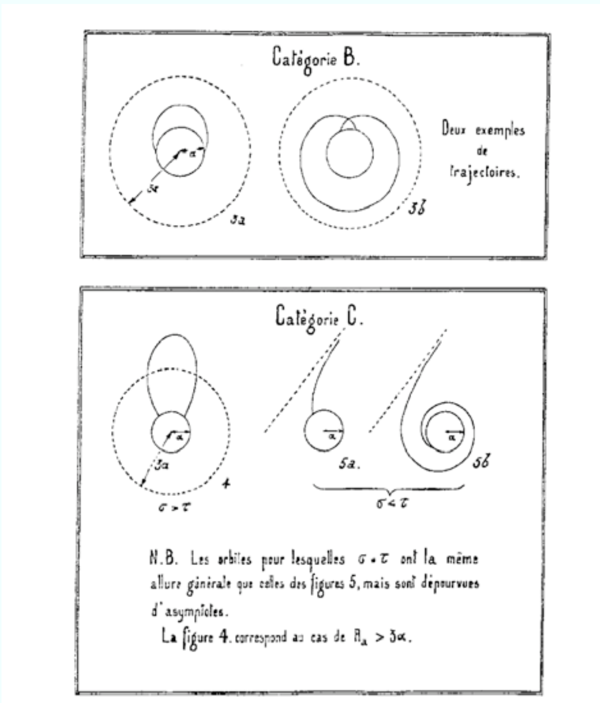


Fig. 1. Trajectoires matérielles d'après DE JANS, 1923

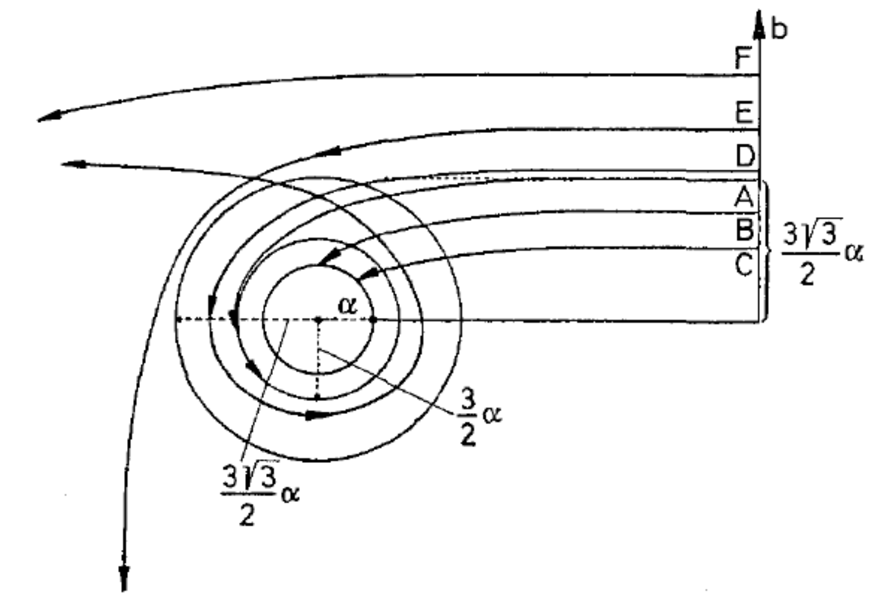


Fig. 2. Trajectoires lumineuses d'après VON LAUE, 1921

(from J. Eisenstaedt, Archive for History of Exact Sciences, 1987)

(box5.) Schwarzschild geodesics

$$ds^2 = -(1 - 2m/r)dt^2 + dr^2/(1 - 2m/r) + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

particle wordline : $x^i = x(\tau)$; 4-velocity: $u^i \equiv \frac{dx^i}{d\tau}$, $u_i \equiv g_{ij}u^j$.

Geodesic equation : $\frac{Du_i}{d\tau} = 0 \iff \frac{du_i}{d\tau} = \frac{u^k u^l}{2}\partial_i g_{kl}$.

hence $r^2\frac{d\phi}{d\tau} = L$, $\left(1 - \frac{2m}{r}\right)\frac{dt}{d\tau} = E$; (L and E : integration constants)

$g_{ij}u^i u^j = -\epsilon \implies \left(\frac{dr}{d\tau}\right)^2 = E^2 - U_{\rm eff}$ with $U_{\rm eff} = \left(1 - \frac{2m}{r}\right)\left(\epsilon + \frac{L^2}{r^2}\right)$.

Trajectories (Binet method) : express $\left(\frac{dr}{d\tau}\right)^2 / \left(\frac{d\phi}{d\tau}\right)^2$ in terms of $u \equiv 1/r$:

$$\left(\frac{du}{d\phi}\right)^2 = \frac{E^2 - 1}{L^2} + \frac{2\epsilon m}{L^2}u - u^2 + 2mu^3 \qquad \text{(Einstein 1915)}$$

$r = 2m$ is a "magic circle" which cannot be crossed. (Eddington, 1924)

N.B. : Radial geodesic equation : $\left(\frac{dr}{d\tau}\right)^2 = E^2 - 1 + \frac{2m}{r}$
same as in Newtonian gravity with $t \to \tau$ (noticed in 1936, Drumeaux)

The Flamm diagram : “conforts” the impenetrability of $r = 2m$

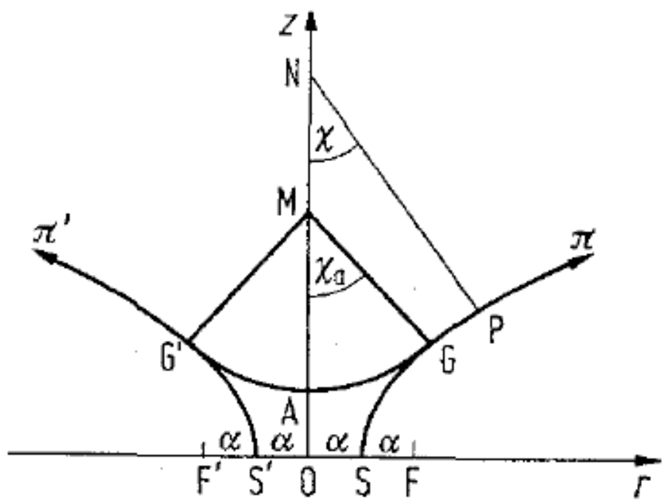


Fig. 1

Ludwig Flamm (1916)

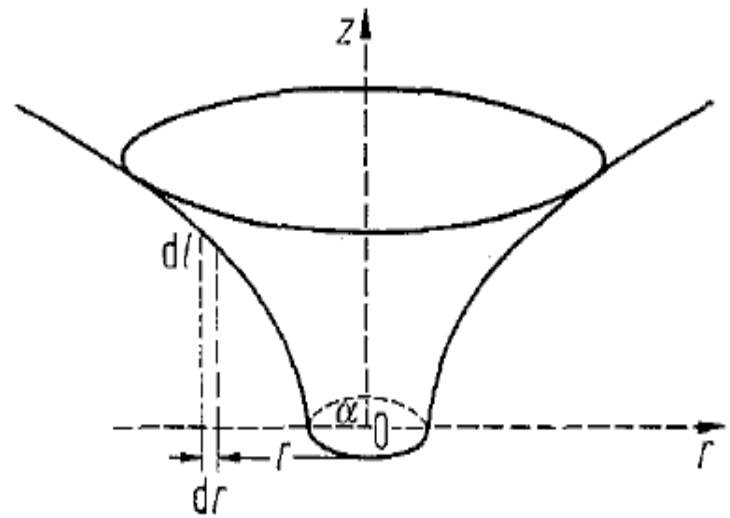


Fig. 2

Jean Becquerel (1923)

(from J. Eisenstaedt, Archive for History of Exact Sciences, 1982)

(box6.) Embedding of Schwarzschild spacetime

$$r > R \;:\; ds^2 = -(1-2m/r)dt^2 + dr^2/(1-2m/r) + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$r < R \;:\; ds^2 = -\left(\frac{3}{2}\sqrt{1-\frac{2m}{R}} - \frac{1}{2}\sqrt{1-\frac{2mr^2}{R^3}}\right)^2 dt^2 + \frac{dr^2}{1-\frac{2mr^2}{R^3}} + r^2 d\Omega^2$$

Sections $t = Const$ and $\theta = \pi/2$: $d\sigma^2_{\rm int} = \frac{dr^2}{1-2mr^2/R^3} + r^2 d\phi^2$: 2-sphere

$$d\sigma^2_{\rm ext} = dr^2/(1-2m/r) + r^2 d\phi^2)$$

Consider the paraboloid : $z^2 = 8m(r-2m)$ in E_3
The induced metric on this surface is $d\sigma^2_{\rm ext}$ (for $r > 2m$!).

Hermann Weyl (1885-1955) introduces isotropic coordinates (1917) $r = \bar{r}(1+m/2\bar{r})^2$

$$ds^2 = -\left(\frac{1-m/2\bar{r}}{1+m/2\bar{r}}\right)^2 dt^2 + \left(1+\frac{m}{2\bar{r}}\right)^4 d\vec{x}^2$$

to make clearer the double representation of $r > 2m$ obtained from the embedding

(Mis)understanding general covariance

Hilbert (1917) : A metric is regular if exists an invertible and bijective coordinate transformation which makes g_{ij} regular...

• Painlevé-Gullstrand (1921) coordinates :

$$ds^2 = -(1 - 2m/r)d\tilde{t}^2 + 2\sqrt{2m/r}\, drd\tilde{t} - d\vec{x}^2$$

$$\tilde{t} = t + 4m\left(\sqrt{\tfrac{r}{2m}} + \tfrac{1}{2}\log\left(\sqrt{\tfrac{r/2m-1}{r/2m+1}}\right)\right)$$

(Authors conclude to the "ambiguity" of the Schwarzschild metric)

• Eddington coordinates (1924) : $\bar{t} = t + 2m\log(r/2m - 1)$

$$ds^2 = -(1 - 2m/r)d\bar{t}^2 + (4m/r)drd\bar{t} + (1 + 2m/r)dr^2 + r^2 d\Omega^2$$

(used to discuss Whitehead theory)

Nobody sees at the time that the metric coefficients no longer diverge...

Early Years : Debates at the Collège de France, April 1922

The closing of an era

Paul Painlevé,
(1863-1933)

Paul Langevin,
(1872-1946)

Jacques Hadamard,
(1865-1963)

Théophile de Donder, Marcel Brillouin, Jean Becquerel, Henri Bergson...

Summary of “mental blocks”

1. Schwarzschild interior solution : $p(0)$ is finite if $R > 9m/4 > 2m$. Known stars are such that $R \gg 9m/4$; hence Schwarzschild's exterior solution “never” extends to $r = 2m$. (“Neo-newtonian” bias, Eisenstaedt)

2. Even if one accepts to consider Schwarzschild's exterior solution up to $r = 2m$: a metric coefficient diverges there (“Hadamard catastrophy”), and must signal a “magic circle” (Eddington), that is, a thin shell of matter which cannot be crossed (no definition of “singular spacetime”)

3. The embedding of the $t = const$, $\theta = \pi/2$ slices of Schwarzschild's spacetime in E_3 (Flamm diagram) seems to indicate that $r < 2m$ is unaccessible; and that space may be doubly connected (Weyl) (no understanding of “maximally extended manifold”)

(for the record : Einstein sails to Japan in December 1922; learns during the trip that he is awarded the Nobel Prize... but not for Relativity !)

Fifty-odd years of struggle

Early years of discoveries and debates

Input from Quantum Physics and Cosmology

1930 : Chandrasekhar's maximum mass of white dwarfs
1932 : Lemaître's insights
1939 : Oppenheimer-Snyder's collapsing star

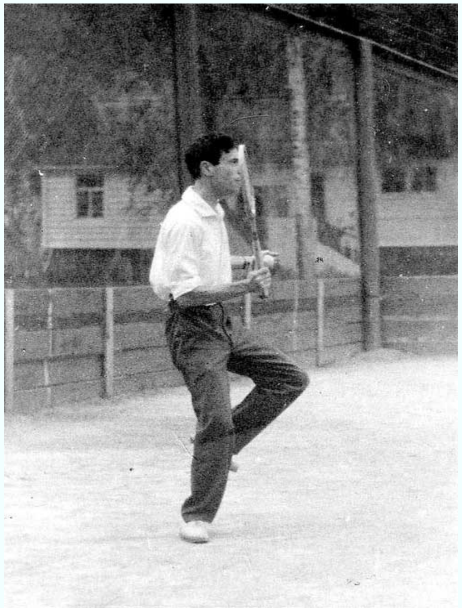
Low-water mark (1940-1955)

Renaissance

Input from Quantum Physics : GR and critical masses

or : beyond the "neo-newtonian" interpretation

Subrahmanyan Chandrasekhar (1910-1995)



Lev Davidovich Landau (1908-1968)

Robert Oppenheimer (1904-1967)

(box7). **Elementary notion of critical mass**

Recall that Schwarzschild's interior solution for a constant density star implies that $p(0)$ is finite if $R > 9m/4 > 2m$ with $m = \frac{4\pi\epsilon R^3}{3}$.

However, instead of concluding, like relativists and astronomers for 25 years, that R "must be" large enough (and "is" for all known stars),

rewrite the inequality as $m < \frac{4}{9\sqrt{\pi\epsilon}}$

and conclude (with Oppenheimer-Snyder, 1939) that if
$\epsilon = \epsilon_{\text{nuclear}} \sim 10^{15} g/cm^3$

then a star with mass $M > 4M_{\odot}$ is unstable and...

predict gravitational collapse !

(First "mental block" blows up) (for more : see lecture 4)

Input from Cosmology : Crossing "singularities"

Insights from de Sitter's solution

$$ds^2 = -(1 - \Lambda r^2/3)dt^2 + dr^2/(1 - \Lambda r^2/3) + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

• 1917 : Einstein fails to find a coordinate transformation which eliminates the singularity at $r = R \equiv \sqrt{3/\Lambda}$. He concludes that probably there is a shell of matter there.

• 1918 : Felix Klein describes de Sitter's spacetime as a hyperboloid in M_5 (hence regular everywhere) and writes the metric as

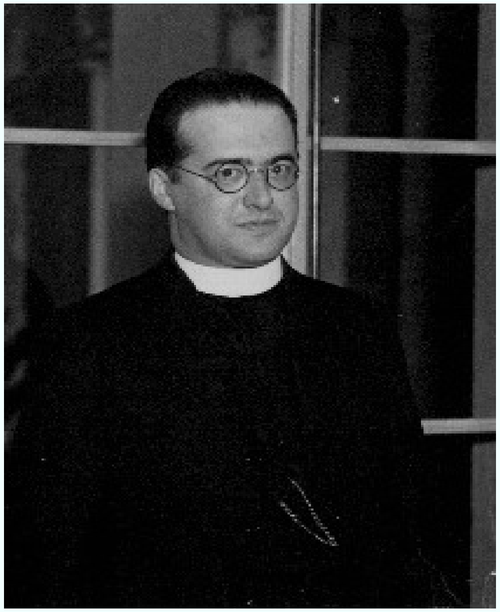
$$ds^2 = -R^2 dT^2 + R^2 \cosh^2 T d\Omega_3^2$$

Weyl and Einstein object that the coordinate transformation is not invertible and that the "Klein-de Sitter" spacetime is not stationary.

• 1925 : Lemaître writes de Sitter's metric as : $ds^2 = -d\bar{T}^2 + e^{2\bar{T}/R} d\vec{x}^2$ and notes that it "gives a possible interpretation of the mean receding motion of spiral nebulae"

box8. **Lemaître coordinates (1932)** (beginning)

or : how to describe the collapse of a "nebula" in an expanding universe



Georges Lemaître
(1894-1966)

- $ds^2 = -d\tau^2 + e^{\lambda(\rho,\tau)} d\rho^2 + r^2(\rho,\tau) d\Omega_2^2$
- "nebula" surface $\rho = Const.$ is a geodesic
- Einstein's equations $G^i_j = 8\pi T^i_j$

$T^\tau_\tau = \epsilon(\rho,\tau)$ $(p \approx 0)$

$G^0_\rho = \frac{e^{-\lambda}}{r}(-2\dot{r}' + \dot{\lambda} r')$, $G^\rho_\rho = ...$, $G^\tau_\tau = ...$

$e^\lambda = \frac{r'^2}{(1+2E(\rho))}$, $\dot{r}^2 = 2E(\rho) + \frac{2m(\rho)}{r}$

$4\pi\epsilon = \frac{m'}{r' r^2}$ "Tolman-Bondi" (1934,1947)

- Outside nebula $\epsilon = 0$; choose $E(\rho) = 0$

hence $r = \left(\frac{9m}{2}\right)^{1/3} (\tau_0(\rho) - \tau)$; choose $\tau_0 = \rho$
so that

$$ds^2 = -d\tau^2 + \frac{2m d\rho^2}{\left[\frac{3}{2}\sqrt{2m}(\rho-\tau)\right]^{2/3}} + \left[\frac{3}{2}\sqrt{2m}(\rho - \tau)\right]^{4/3} d\Omega^2$$

box8. **Lemaître coordinates (1932)** (end)

• aside : All spherically symmetric solutions of Einstein's vacuum equations reduce to Schwarzschild's (Birkhoff theorem, 1923). Hence Lemaître's metric must be Schwarzschild's in disguise.

• explicit coordinates transformation (Lemaître, 1932) :

$$\rho = t + \frac{4m}{3}\sqrt{\frac{r}{2m}}\left(3 + \frac{r}{2m}\right) + 2m\log\left(\frac{\sqrt{r/2m}-1}{\sqrt{r/2m}+1}\right)$$

$$\tau = t + 4m\left(\sqrt{\frac{r}{2m}} + \frac{1}{2}\log\left(\frac{\sqrt{r/2m}-1}{\sqrt{r/2m}+1}\right)\right)$$ (as in Painlevé, 1921)

• hence $r = \left[\frac{3}{2}\sqrt{2m}(\rho - \tau)\right]^{2/3}$ and $r = 2m$ is a regular point

"The singularity of the Schwarzschild field then is a fictitious singularity, analogous to the one appearing on the horizon of the center in the original form of the De Sitter universe."

(Second "mental block" blows up) (for more : see lecture 2)

Synthesis : Gravitational collapse

Oppenheimer Snyder paper (1939)

"When all thermonuclear sources of energy are exhausted a sufficiently heavy star will collapse...the radius of the star approaches asymptotically its gravitational radius; light from the surface of the star is progressively reddened, and can escape over a progressively narrower range of angles ... The total time of collapse for an observer comoving with the stellar matter is finite... An external observer sees the star asymptotically shrinking to its gravitational radius."

(Birth date of "black hole" concept)

(box9.) **Gravitational redshift of a collapsing star**

- radial geodesics (see box5) : $\quad (\dot{f} \equiv \frac{df}{d\tau}$, τ being proper time)

$$(1) \quad \left(1 - \frac{2m}{r}\right) \dot{t} = \sqrt{1 - \frac{2m}{r_0}} \quad , \quad (2) \quad \dot{r} = -\sqrt{\frac{2m}{r} - \frac{2m}{r_0}}$$

- motion of collapsing material (see box1)

$\tau = \frac{r_0}{2}\sqrt{\frac{r_0}{2GM}}(\eta + \sin\eta)$; $r = \frac{r_0}{2}(1 + \cos\eta)$; and : $\tau_{\text{collapse}} = \frac{\pi}{2} r_0 \sqrt{\frac{r_0}{2GM}}$

- redshift calculation

(a) $\Delta t_{\text{em}} = \Delta\tau_{\text{em}} \frac{\sqrt{1-2m/r_0}}{1-2m/r_{\text{em}}}$ (from (1))

(b) nul geodesics : $\frac{dr}{dt} = 1 - \frac{2m}{r} \Longrightarrow \Delta t_{\text{rec}} = \Delta t_{\text{rec}} = \Delta t_{\text{em}} - \frac{\Delta r_{\text{em}}}{1-\frac{2m}{r_{\text{em}}}}$

(c) $\Delta r_{\text{em}} = -\Delta\tau_{\text{em}}\sqrt{2m/r_{\text{em}} - 2m/r_0}$ (from (2)), hence :

$\Delta\tau_{\text{rec}} = \frac{\Delta\tau_{\text{em}}}{\sqrt{1-\frac{2m}{r_{\text{em}}}}} \left(\sqrt{1 - \frac{2m}{r_0}} + \sqrt{\frac{2m}{r_{\text{em}}} - \frac{m}{r_0}}\right) \quad \to \infty$ when $r_{\text{em}} \to 2m$

Fifty-odd years of struggle

Early years of discoveries and debates

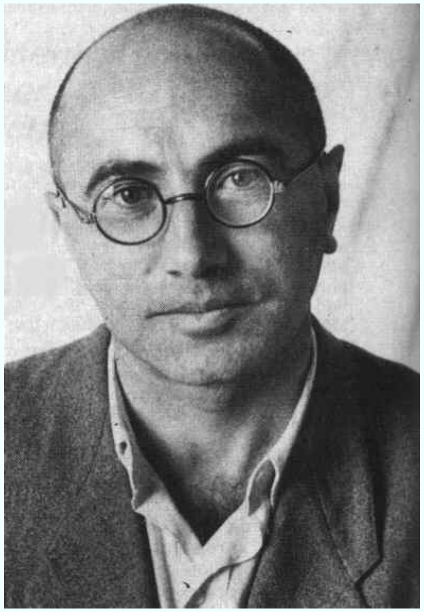
Input form Quantum Physics and Cosmology

Low-water mark (1940-1955)

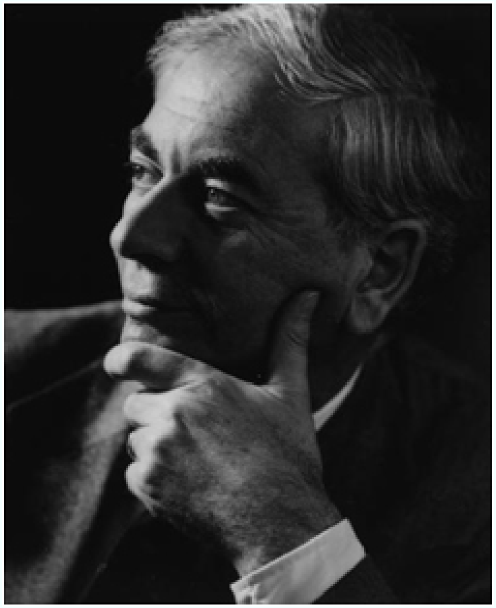
Renaissance

Renaissance : Three Leading Figures

John Wheeler
(1912-2008)



Yakov Zel'dovich
(1914-1987)



Dennis Sciama
(1926-1999)

Renaissance

1960 : Extension of Schwarzschild spacetime (Kruskal) (see lecture 2)

1963 : Discovery of quasars (see lecture 4)

1963 : Kerr solution (see lecture 3)

Complement to lecture one : Black Hole chronology 1 : up to 1939

Six Lectures on Black Holes in General Relativity
IPhT-CEA, Saclay, January-February 2009

Nathalie Deruelle

Black Hole Chronology

I. up to 1939

Sources :
–"Einstein et la relativité générale, Jean Eisenstaedt, Cnrs-Eds, 2002
(English translation : Oxford and Princeton U-Press, 2006)
–"Black holes and time warps", Kip Thorne, Norton Publ., 1994
–"Dark stars" : the evolution of an idea, Werner Israel *in* 300 years of Gravitation, CUP, 1987
– Wikipedia !

Prehistory

1783 : John Michell (1724-1793) computes the escape velocity of a "particle" of light and introduces the concept of dark body in Newtonian gravity. "The paper communicated to the Royal Society by Cavendish on November 27, 1783 (...) caused such a stir in London circles that it overshadowed exciting news coming from Paris about Coulomb's electrical experiments" (Israel 1987).

1790 : Pierre Simon de Laplace (1749-1827) reproduces Michell's results in his first edition of "Système du Monde" (with no reference to Michell).

1799 : Second edition of Laplace's "Système du Monde" where the section on dark bodies is reproduced without changes.

1801 : Johann Georg von Soldner, a German astronomer, calculates light deviation in Newtonian theory.

1801 : Christian Huygens discovers light interferences.

1808 : Third edition of Laplace's "Système du Monde" : the section on dark bodies is suppressed.

1905-1915

1905 : Albert Einstein (1879-1955) invents Special Relativity.

1907 : "I was sitting in a chair at the patent office in Bern when all of the sudden a thought occurred to me ; if a person falls freely he will not feel his own weight" (November) ; formulation of the geodesic principle; prediction of gravitational redshift.

1912 : Einstein in Prague in the Summer; arrives at ETH, Zurich, in August; understands that space-time must be represented by a curved space (10-16 August).

1915 : Invited by David Hilbert (1862-1943) Einstein gives six lectures at Gottingen in June. in September Arnold Sommerfed writes to Einstein to tell him that Hilbert is working on his theory of gravitation; November : Einstein and Hilbert exchange postcards about their respective work; Einstein presents four communications to the Berlin Academy, one each Thursday; on November 20th Hilbert presents the "Hilbert action" at a seminar in Gottingen; on November 25th Einstein communicates the final equations of General Relativity to the Berlin Academy. (see, e.g., Todorov, arXiv:physics/0504179).

1915 : Karl Schwarzschild (1876-1916), who is on the Russian front, finds (between November 8th and the end of the year) the "Schwarzschild solution" (in coordinates such that det g=-1 ; however he rewrites its in terms of the standard r coordinates that he calls "auxiliary quantity" ($g_{tt} = 1 - 2m/r$ form). In the coordinates such that det g=-1 the center is located at $r = 2m$). He also gives the equation for the trajectories of test particles (that is, for $u(\phi)$ where $u = 1/r$).

1916-1922

1916 : Einstein acknowledges receiving Schwarzschild's paper (January 9th) : "I would not have thought one could find so easily the exact solution to the problem". January 13th : Einstein presents Schwarzschild's solution to the Prussian Academy of Sciences in Berlin.

1916 : In March Schwarzschild finds the interior solution ($\rho = Const$, still in coordinates such that det g=-1). He remarks that the "auxiliary quantity" r has a geometrical meaning (radius of the 2-spheres of symmetry). He notes that the radius R of the star of mass M must be such that $R > 9m/4$ (where $m \equiv GM/c^2$) for the pressure at the center ($r = 0$) to remain finite. He gives the numerical values of $2m$ for the Sun (3kms) and for 1 gram (1.5×10^{-28} cm). On May 11th Schwarzschild dies from a skin disease contracted on the front. Einstein presents Schwarzschild's interior solution to the Academy in June (letter to Michele Besso).

1916 : in May, Johannes Droste (30 years old, student of Lorentz in Leiden where is also Wilhem de Sitter) obtains Schwarzschild's solution without imposing det g=-1 in standard (t, r, θ, ϕ) coordinates. In December he defends his thesis ("The field of a single centre in Einstein's theory of gravitation, and the motion of a particle in that field", written in Dutch). He studies geodesics, recovers the equations for the trajectories ($u(\phi)$) and integrates then in terms of Weiestrass functions. He studies dr/dt and concludes that $r = 2m$ is never reached. He finds that light follows circular geodesics at $r = 3m$.

1916 : in September, Ludwig Flamm (in Viena) describes the $t = Const, \theta = \pi/2$ sections of the Schwarzschild exterior and interior solution as a paraboloid connected to a 2-phere embedded in E_3. ("Beiträge zur Einsteinschen Gravitationstheorie", Physik Z. 17 (1916) p 448-454).

1916 : Hans Reissner (1874-1923) extends the Schwarzschild solution to the case of a charged mass : "Über die Eigengravitation des elektrischen Feldes nach Einsteinschen Theorie." Ann. Phys. 59, 106-120, 1916.

1917 : Einstein writes to Felix Klein to express his dissatisfaction with the Schwarzschild solution which does not embody Mach's principle. He modifies his field equations and introduces the "cosmological constant" Λ ("Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie." Sitzungsber. Preuss. Akad. Wiss., 142-152, 1917. Reprinted in English in Lorentz, H. A.; Einstein, A.; Minkowski, H.; and Weyl, H. The Principle of Relativity: A Collection of Original Memoirs on the Special and General Theory of Relativity. New York: Dover, 1952.)

1917 : Hilbert writes (in "Grunlagen der Physik" p 70) : "A line element or a gravitational field g_{ij} is regular at a point if it is possible to introduce by a reversible, one-to-one transformation a coordinate system, such that in this system the corresponding functions g'_{ij} are regular at that point, i.e. they are continuous and arbitrarily differentiable at the point and in a neighbourhood of the point and the determinant g' is different from 0". (quoted by Earman and Eisenstaedt, 1999)

1917 : Willem de Sitter (1872-1934) finds the "de Sitter solution" ($g_{tt} = 1 - \Lambda r^2/3$) : "On the Relativity of Inertia: Remarks Concerning Einstein's Latest Hypothesis." Proc. Kon. Ned. Akad. Wet. 19, 1217-1225, 1917 (and : de Sitter, W. "The Curvature of Space." Proc. Kon. Ned. Akad. Wet. 20, 229-243, 1917. Also : de Sitter, W. Proc. Kon. Ned. Akad. Wet. 20, 1309, 1917. and : de Sitter, W. Mon. Not. Roy. Astron. Soc. 78, 3, 1917.) NB : de Sitter introduces his solution as an hyperboloid embedded in M5 (See footnote 23 of Earman and Eisenstaedt 1999). Einstein studies the metric under the form : $ds^2 = -\cos^2(\bar{r}/R)dt^2 + d\bar{r}^2 + \sin^2(\bar{r}/R)d\Omega_2^2$ (so that $\sqrt{\Lambda/3}r = \sin(\bar{r}/R)$ with $R \equiv \sqrt{3/\Lambda}$ where r is the standard radial coordinate). He sees that the origin ($\bar{r} = 0$) is a coordinate singularity. He sees that $\bar{r}/R = \pi R/2$ (that is, $r = \sqrt{\Lambda/3}$) is also a singularity but fails to find a transformation which would eliminate it (March 1918 in "Kritisches zu einer von Hr De Sitter..."). He concludes "until proof to the contrary" that de Sitter solution must be regarded as having a genuine singularity and that it corresponds "to a universe whose matter has been entirely concentrated on the surface $\bar{r} = \pi R/2$..." (quoted by Earman and Eisenstaedt 1999).

1917 : Hermann Weyl (1885-1955) introduces the isotropic coordinates (so called by Eddington in 1924) (Annalen der Phyzik 54 p 117-145). He finds that this form makes clearer the double representation of

$r > 2m$ in the Flamm diagram. $\bar{r} < m/2$ represents, according to him, the interior of the massive point; $\bar{r} > m/2$ the exterior, matter being concentrated on $r = 2m$. He presents the result in his 1918 book "Raum Zeit Materie" where he also writes that "the complete realisation of this solution would imply that [...] space is doubly connected, that is, contains not one but two boundaries accessible at infinity". Note that this remark appears only in the first edition of his book. (see J Eisenstaedt, 1982)

1918 : Gunnar Nordström (1881-1923) rediscovers Reissner solution for a charged mass. "On the Energy of the Gravitational Field in Einstein's Theory." Proc. Kon. Ned. Akad. Wet. 20, 1238-1245, 1918.

1918 : Felix Klein (1849-1925) (in a letter to Einstein dated 16 June 1918; details published in 1919) describes the de Sitter solution as a hyperboloid in M5. (This had already been done by de Sitter himself). He shows that the de Sitter coordinates do not cover the full hyperboloid; and that the full hyperboloid ST is not stationary; and that the coordinate transformation from the de Sitter coordinates to those covering the whole hyperboloid are not regular. Einstein dismisses the coordinates which cover the whole hyperboloid ($ds^2 = -R^2 dT^2 + R^2 \cosh^2 T d\Omega_3^2$) because the metric is no-longer stationary and the metric coefficients are time dependent. (see Earman and Eisenstaedt 1999).

1918 : Kottler gives the Schwarzschild-de Sitter metric ($g_{tt} = 1 - 2m/r + \Lambda r^2/3$).

1919 : Hermann Weyl in a correspondence with Felix Klein argues that Klein's non-stationary hyperboloid and de Sitter's static solution are separated by "an abyss".

1919 : Arthur Stanley Eddington (1882-1944) measures the deflection of light

1920 : Eddington's book on GR. "There is a magic circle [$r = 2m$] which no measurement can bring us inside. It is not unnatural that we should picture something obstructing our closer approach and say that a particle of matter is filing up the interior".

1920 : A. Anderson of University College, Galway speculates in the Philosophical Magazine : "We may remark, though perhaps the assumption is very violent, that if the mass of the sun were concentrated in a sphere of diameter 1.47 kms... it [would] be shrouded in darkness". (quoted by Israel, 1987)

1921 : Wolfgang Pauli (1900-1958) writes a review article on GR (he is 21). He is one of the few to have a clear idea about the significance of proper time and the distinction between proper and coordinate time.

1921 : Max von Laue studies the Schwarzschild null geodesics (he had shown in 1920 that electromagnetic waves followed null geodesics). They all "stop" at $r = 2m$. (He argues that the velocity of light is zero at $r = 2m$.) He finds the capture radius $b = 3\sqrt{3}m$ below which all light rays will asymptotically reach $r = 3m$. (see Eisenstaedt 1987.)

1921 : Oliver Lodge rediscovers Michell's and Laplace's results on dark bodies : He writes "A stellar system–say a super spiral nebula– of aggregate mass equal to 10^{15} suns... might have a group radius of 300 parsecs... with a corresponding average density of 10^{-15} cgs, without much light being able to escape from it. This does not seem an utterly impossible concentration of matter" and concludes : "In regions where our ignorance is great, occasional guesses are permissible"(quoted by Israel 1987).

1921 : Paul Painlevé (1863-1933) and Allvar Gullstrand coordinates : $ds^2 = -(1 - 2m/r)d\tilde{t}^2 + 2\sqrt{2m/r}\, dr d\tilde{t} - d\vec{x}^2$ where $\tilde{t} = t + 4m\left(\sqrt{\frac{r}{2m}} + \frac{1}{2}\log\left(\sqrt{\frac{r/2m-1}{r/2m+1}}\right)\right)$. The authors conclude to the "ambiguity" of the Schwarzschild metric. On December 7th Einstein writes to Painlevé to tell him that coordinates have no physical significance. (Langevin about Painlevé : "Painlevé studied very carefully Einstein's theory but, unfortunately, after have written about it". Allvar Gullstrand : ophtalmologist, Nobel prize 1911 in physiology; member of the Nobel committee.)

1922 : Cornelius Lanczos (1993-1974) eliminates the de Sitter singularity at $r = \sqrt{\Lambda/3}$ by writing the metric as $ds^2 = -dt^2 + \cosh^2 Ht d\Omega_3^2$. He has doubts about the reality of the $r = 2m$ Schwarzschild's singularity (that he writes in harmonic coordinates, as de Donder had already done in 1921).

1922 : In April colloquium at the Collège de France where Einstein delivers a series of lectures about GR (in the audience : Jean Becquerel (who publishes in 1922 the first book in French about GR), Marcel Brillouin, Elie Cartan, Théophile de Donder, Jacques Hadamard, Paul Langevin, Paul Painlevé, Henri Bergson and Charles Nordmann–who took notes). Hadamard worries about the g_{ij} becoming infinite (the "Hadamard catastrophy" as Einstein pleasantly puts it). Einstein comes back the following day with Schwarzschild's interior solution (that he rederives without mentioning Schwarzschild) and argues, like Schwarzschild in 1916, that the Hadamard catastrophy at $r = 2m$ cannot occur as the radius of the star must be bigger than $9m/4$, otherwise the pressure at the center becomes infinite. Brillouin will remark, after the colloquium, that the region $r < 2m$ is unphysical since the roles of t and r are interchanged.

1922 : On his way to Japan (where he arrives in December) Einstein learns that he is awarded the Nobel prize (but NOT for Relativity, as explicitely stated in the telegram).

1923-1939

1922-1924 : Alexander Friedmann finds the "Friedmann solution".

1923 : de Jans in Brussels (a student of de Donder) studies the Schwarzschild geodesics and defends his thesis in 1924. All trajectories terminate at $r = 2m$. He discovers that everything had already been done by Droste.

1923 : Birkhoff (a Harvard Mathematician) proves his theorem (suggested by Jebsen in 1921) : The unique spherically symmetric solution of Einstein's vacuum equations is Schwarzschild's (that is : there is a extra Killing vector field so that the solution for $r > 2m$ is static whatever the (radial) motion of matter).

1923 : new edition of Eddington's book ("Mathematical theory of relativity"). He compares the Schwarzschild and de Sitter "singularities" at $r = 2m$ and $r = 1/H$ and thinks there is a concentration of matter there. However he also presents Klein's embedding of de Sitter spacetime as a (regular) hyperboloid in M5. According to Israel (1987) Eddington on p 121 gives an unorthodox definition of particle density (which will be at the core of his rejection of the Chandrasekhar mass). Modern view : n is the zero component (in the rest frame) of the conserved vector n^μ. Eddington proposes $nm = T$ (m rest-mass of particles; T trace of the stress energy tensor). Plainly wrong since implies that the internal energy ($nm - T^0_0$) be 3 times the pressure, which is too large by a factor 2 in the case of a non-relativistic gas.

1924-26 : Publications of books on Riemannian geometry by Levi-Civita, Elie Cartan, Schouten, Eisenhart.

1924 : In the paper "A comparison of Whitehead's and Einstein's formulae", Nature, 2832, Eddington introduces the "Eddington" coordinates and writes the Schwarzschild metric as $ds^2 = ds^2_{M_4} + (2m/r)(dt - dr)^2$. He does not comment on the fact that the metric coefficients no longer diverge.

1925 : George Lemaître (1894-1966), in Cambrigde, writes the de Sitter metric as $ds^2 = -d\tilde{t}^2 + e^{2\tilde{t}/R}d\vec{x}^2$ (a form rediscovered independently by Robertson in 1928). Contrary to Einstein the non-static character of the line element is taken to speak in favour of it because it "gives a possible interpretation of the mean receding motion of spiral nebulae". He finds however that the fact that the spatial sections are flat to be "completely inadmissible". In his thesis Lemaître also studies the Schwarzschild interior solutions (Eddington had told him that the equation of state $p - 3\rho = const$ was more realistic that $\rho = Const$). Lemaître understands that the minimim radius of a star coming form the limit $p(0) \to \infty$ found by Schwarzschild depends on the equation of state.

1925 : W.S. Adams at Mount Wilson measures the gravitational redshift at the surface of a white dwarf (Sirius B). His observations agree with the predictions from GR but both were wrong : his observations, and the predictions, which both were 5 times too small because of errors on Sirius B mass and radius.

1926 : Eddington publishes "The internal Constitution of stars". Eddington is the first since the 19th century to mention Laplace's "dark bodies". In this book "he poses the mystery of white dwarfs and attacks the reality of "BH"" (Thorne p 538)

1926 : Ralph H. Fowler (Cambridge) publishes (10 dec) : "On dense matter" in MNRAS : the pressure in white dwarfs comes from electron degeneracy.

1927 : Lemaître, independently of Friedmann, discovers a cosmological solution which starts as the Einstein static universe and ends up as de Sitter's.

1928 : Rainich studies gravitational redshifts in Schwarzschild's spacetime.

1928 : Sommerfeld visits Madras and meets Chandra (who is 17).

1930 : Arthur Milne shows that the zero-point pressure of a degenerate gaz of cold electrons can balance gravity at a given radius (a decreasing function of the mass) a result that corresponds well to the actual size of white dwarfs.

1929 : (and 1930). Edmund C. Stoner (Leeds, best known for application of Fermi-Dirac statistics to para and ferro-magnetism) finds that the density of white dwarfs goes like the square of the mass and finds a density an order of magnitude larger than observed. Wilhem Anderson of Tartu Estonia tells him that the electrons are relativistic and that the density is considerably lower. In fact Anderson finds a critical mass (when density becomes infinite) but does not comment on this. Stoner gives then what is now known as the

Anderson-Stoner equation of state for white dwarfs (that is, the relativistic equation of state for a degenerate (zero temperature) electron gas with the expression for the adiabatic index $\gamma(\rho)$ which varies from 5/3 (no relativistic limit) to 4/3). Stoner confirms the existence of a limiting mass (1.7 solar mass); no comments however, apart from remarking that known white dwarf masses are below this value. (NB : Stars idealized as constant density.) (see Israel 1987 and Thorne.)

1930 : Subramanyan Chandrasekhar (1910-1995) embarks a steamer to go to England in July (he is 19). During the 18 day trip he finds (independently from Stoner and simultaneously) that the adiabatic index of a white dwarf (sustained by electron degeneracy) is 5/3 in a Newtonian description. He then finds that the speed of electons is 0.57 c. Hence Special Relativity is required. He takes SR into account by saying that the electrons behave as if their mass was larger than their rest mass. He finds that this implies that the adiabatic index becomes 4/3. However this implies the existence of a maximum mass, that he estimates to be 1.4 solar mass beyond which the star cannot sustain gravity. In September the first paper (about the newtonian description) is published in the Philo. Magazine via Fowler. Fowler and Milne do not understand the second paper on the critical mass. Chandra submits it to the Astrophysical Phys. J.. The referee is Carl Eckart. The paper is eventually published 1 year later : "The maximum mass of ideal white dwarfs", Astrophys. J. 74 (1931) 81

1931: Ernest Rutherford (Cambridge) postulates the existence of the neutron.

1931 : in February Lev Davidovich Landau (1908-1968) is in Zurich where he visits Pauli. Independently from Chandrasekhar (and Stoner) he derives a mass limit for white dwarfs : for stars heavier than 1.5 solar mass "the density of matter becomes so great that atomic nuclei come in close contact forming one gigantic nucleus". This work is published in 1932 : "On the theory of stars", Zs. Phys. Sowjetunion 1 (1932) 285 ("Thus we get an equilibrium state only for masses greater [sic!] than a critical mass M_0... about 1.5 solar mass (for m=2 protonic masses). For $M > M_0$ there exists in the whole quantum theory no cause preventing the system for collapsing to a point... As in reality such masses exist quietly as stars and do not show any such ridiculous tendencies we must conclude that all stars heavier then 1.5 solar mass certainly possess regions in which the laws of quantum mechanics (and therefore quantum statistics) are violated" (quoted by Israel 1987)

1931 : Hagihara (in Tokyo) studies Schwarzschild's geodesics (a 120 pages long "treatise" where trajectories crossing $r = 2m$ are rejected.) He quotes Stoner's work on white dwarf densities and uses the result to argue that the radius of solar mass objects is larger than $2m$.

1931: Einstein visits Caltech and discusses in particular with Fritz Zwicky (1898-1974), Robert Millikan and Richard Chace Tolman. It seems that at that time Zwicky and Walter Baade (1893-1960) had invented the idea of supernovae.

1932 : in his paper "L'univers en expansion" (Publication du Laboratoire d'Astronomie et de Géodésie de l'Université de Louvain 9: 171-205) Lemaître, who wants to describe the collapse of a nebula, finds what is usually known as the Tolman (1934) and Bondi (1947) solution. He notes that the Friedmann-Lemaître cosmological solution can be an interior spherically symmetric solution. From Birkhoff's theorem he deduces that his solution (with no matter) must be the Schwarzschild metric in disguise. He gives the explicit coordinate transformation between his coordinates and the standard Schwarzschild coordinates. Schwarzschild's metric in his coordinates is well-behaved at $r = 2m$: "The singularity of the Schwarzschild field then is a fictitious singularity, analogous to the one appearing on the horizon of the center in the original form of the De Sitter universe". Lemaître also understands that the region $r < 2m$ is not static. He influenced Howard Percy Robertson.

1932 : James Chadwick (from Rutherford's team) finds the neutron in February. Israel, 1987, says p 224 that he heard form Rosenfeld that Landau discussed the possibility of neutrons stars the same evening in Copenhagen with Bohr. But this was a slip of memory as neither Bohr, nor Landau, nor Rosenfeld were in Copenhagen at that time (see note added in proof p 276).

1932 : Karl Jansky (ingeneer at Bell in Holndel New Jersey) finds cosmic radio waves.

1933 : Baade and Zwicky at the meeting of the American Phys. Soc. in Stanford (15 December) announce : "With all reserve we advance the view that supernovae represent the transitions from ordinary stars to neutron stars, consisting mainly of neutrons. Such a star may possess a very small radius and an extremely high density. As neutrons can be packed much more closely than ordinary nuclei and electrons, the 'gravitational packing' energy in a cold neutron star may become very large, and, under certain circumstances, may far exceed the ordinary nuclear packing fractions. A neutron star would therefore represent the most stable configuration of matter as such..." Good idea but not "substantiated" (according to Kip Thorne).

1934 : Tolman publishes his book on GR.

1934 : Chandra, using Eddington's computer at Trinity, computes $M(R)$ for white dwarfs using the Stoner-Anderson equation of state. and confirms the existence of a critical mass.

1935 : at the Januray 11th meeting of the Royal Astronomical Soc. at Burlington House Chandra presents his results on the maximum mass of white dwarfs.Then Eddington speaks and rejects the idea of a maximal mass (and Stoner-Anderson equation of state). His argument is that the mass of Sirius (not the white dwarf Sirius B) is higher than Chandra's critical mass, and that stars must evolve... and cannot end up in collapse ! "I think that there should be a law of Nature to prevent a star to behave in such an absurd way". He thinks that Chandra's meshing of SR and quantum mechanics is not trustworthy ("I do not regard the offspring of such a union as born in lawful wedlock"). (Israel 1987 argues that Eddington's reasoning came from his wrong definition of particle density, see his book 1923, which exaggerates the stability of massive stars). The next day Chandra seeks for help. Bohr in Copenhagen is on his side.

1935 : Einstein and his assistant Rosen (who is 26) publish "The particle problem in GR", Phys. Rev. 48 p 73-77. First they consider M4 in"Rindler" coordinates (introduced by Lanczos) and remove the (polar-like) singularity by going back to Minkowski coordinates. However they insist that there is a concentration of matter at the origin (in Rindler coordinates). They notice that by going from Rindler to Minkowski coordinates spacetime is extended ("Einstein-Rosen bridge"). They then turn to the Schwarzschild metric that they try to put under a Rindler-like form by introducing a coordinate $u^2 = r - 2m$, with, "hence" a concentration of matter at $r = 2m$. (see Earman and Eisenstaedt 1999).

1936 : Drumeaux is the first to note that the equation for the radial geodesics in Schwarzschild spacetime is identical to the equation of motion in Newtonian gravity, if one trades universal time for proper time. (Eisenstaedt 1987)

1937 : Landau sends a manuscript on "neutron cores" to Doklady and to Niels Bohr (according to Kip Thorne he hopes thus to avoid being put in jail as a German spy; despite help from Bohr he will spend one year in jail in 38-39). The paper is published in Nature (19 feb 1938). Landau estimates the neutron core to be about 10^{-3} solar mass (he uses Fowler's non relativist equation of state.)

1938 : J. Robert Oppenheimer (1904-1967) and Serber publish a paper in the September 1st issue of Phys. Rev. and argue that Landau is wrong ; there cannot be a neutron core in the Sun (or, rather, according to Israel 1987, they say that the core mass is much higher, about 1/10 solar mass).

1938 : In November-December Oppenheimer and George Volkov work on the maximum mass of a neutron star. They redo what Chandra had done but replace electrons by neutrons and Newtonian gravity by GR, that is, they use the "Tolman-Oppenheimer-Volkov" equation. (Volkoff is in Berkeley; Tolman in Caltech; Oppie at both). They play with hypothetical equations of state (soft to stiff) and find (numerically) a maximum mass between 0.7 and 6 solar masses. (From Israel 1987 : It was obvious that Newton gravity would give an upper limit of 4 times the Chandra mass (that is about 6 solar masses) since the star pressure would be carried by neutrons instead of electrons stripped off from Helium nuclei, thus reducing the mean mass per pressure-producing particle by a factor 2.) Oppenheimer-Volkoff publish their paper: "On Massive Neutron Cores" in the February 15th issue of Phys Rev 55 (1939) 374-381(with references to Landau but not to Zwicky).

1939 : Robertson (who is aware of Lemaître's work) gives a lecture in Toronto and argues that $r = 2m$ is not singular. John Lighton Synge is in the audience.

1939 : Richard Chace Tolman publishes : "Static solutions of Einstein's field equations for spheres of fluid", Phys. Rev. 55, 364-373

1939 : Einstein writes a paper on circular light trajectories in Schwarzschild geometry : "The essential result of this investigation is a clear understanding as to why the "Schwarzschild singularities" do not exist in physical reality".

1939 : Grote Reber (an excentric bachelor and ham radio operator, see Thorne) finds the radio emission of Cyg A.

1939 : André Lichnérowicz (1915-1998, a student of Elie Cartan)) defends his thesis :"Sur certains problèmes globaux relatifs au système des équations d'Einstein" where, in particular, he shows that there can be no non-singular (spatially) asymptotically flat stationary solutions to the exterior equations of GR (exept M4), see J Eisenstaedt 1993.

1938 : In the winter 38-39 Hartland Snyder and Oppenheimer study gravitational collapse. Oppenheimer decides that what is important for the calculation is GR and not the details of the collapse (spin, non sphericity, etc). They get help for the calculations ($p = 0$) from Tolman in Caltech (in fact they essentially give the "Tolman-Bondi" solution, discovered in 1932 by Lemaître). They describe collapse for an external observer as well as in terms of the proper time of the infalling material. They calculate the redshift and find that the star blackens. There is no description of the final crunch however. According to Israel 1987 : "To a reader of the paper it was left obscure how the interior comoving picture, in which the collapse proceeds to zero radius in a finite proper time, is to be reconciled with the contracting "freezing" at the gravitational radius. The authors do not even make clear their views on the status of the gravitational radius, which was still generally considered to be some kind of singularity." Israel goes on saying that Oppenheimer had probably discussed with Robertson about the fictitious character of the Schwarzschild radius "and may even have supposed that this was common knowledge in relativistic circles".

1939 : Oppenheimer and Snyder submit their paper ("On continued gravitational contraction") in July. "When all thermonuclear sources of energy are exhausted a sufficiently heavy star will collapse...the radius of the star approaches asymptotically its gravitational radius; light from the surface of the star is progressively reddened, and can escape over a progressively narrower range of angles ... The total time of collapse for an observer comoving with the stellar matter is finite...an external observer sees the star asymptotically shrinking to its gravitational radius." The paper is published in the September 1st issue of Phys. Rev. (One finds in the same issue the paper by Bohr and Wheeler on fission. Recall too that Hitler invades Poland the same day.)

BH in GR
Lecture 2

IPhT-CEA 09
N. Deruelle

(1

The geometry of black holes

Introductory remarks

As we saw in lecture 1 the early years of the development of the BH concept were plagued by a number of "mental blocks":

- The "neo-newtonian bias" (Eisenstaedt): stars such that $2m/R<1$ cannot exist.

Recall Schwarzschild argument: $p(0)$ is finite if $R > \frac{9m}{4}$ ($> 2m$).
This "mental block" blows up in the 30's with the understanding of the $\exists$ of critical masses (Chandrasekhar - Landau - Oppenheimer) which allows a rewriting of $R > 9m/4$ as (using $m = \frac{4}{3}\pi\rho R^3$):
$m < \frac{4}{9\sqrt{3\pi}} \frac{1}{\sqrt{\rho}}$. For $\rho \sim 10^{15}\,g/cm^3$ $m \sim 4 M_\odot$
hence: if $m \gtrsim$ a few solar masses neutron stars collapse.

- The "metric potentials" are the fundamental quantities which describe gravity.

general covariance is understood but the question is: which coordinate transformations are allowed (Hilbert: they must be one to one).
Illustration: the Kretshmann curvature invariant (1917)
$R_{ijkl}R^{ijkl} = \frac{48m^2}{r^6}$ is regular at $r=2m$ but does not mean that the gravitational field is regular there if one must also exhibit a regular coordinate transformation which makes the metric coefficients there.

This "mental block" blows up in the 30's thanks to the input of cosmology and the understanding of the de Sitter solution. Indeed it was known (de Sitter; Klein 1917-18) that de Sitter ST was a hyperboloid embedded in Π_5 and hence, "clearly" regular everywhere. Hence the (non regular) coordinate transformations which eliminated the singularity at $r = \sqrt{3/\Lambda}$ became acceptable; eg:

$$ds^2_{dS} = -\left(1-\frac{\Lambda r^2}{3}\right)dt^2 + \frac{dr^2}{1-\Lambda r^2/3} + r^2 d\Omega^2 = -d\tau^2 + e^{2\sqrt{\frac{\Lambda}{3}}\tau} d\vec{x}^2$$

(Lemaître 1925)

Similar treatment of Schwarzschild ST (Lemaître 1932)

$$ds^2_{Schw} = -d\tau^2 + \frac{2m\, d\rho^2}{\left[\frac{3}{2}\sqrt{2m}\,(\rho-\tau)\right]^{2/3}} + \left[\frac{3}{2}\sqrt{2m}\,(\rho-\tau)\right]^{4/3} d\Omega^2$$

(however: the metrics are no longer static (that is: the Killing vector is no longer time like when $r < 2m$).)

These breakthroughs allowed Oppenheimer - Snyder to invent the concept of BH in 1939.

- The "topological problem". It was known (Klein 1918) that the static de Sitter coordinates did not cover the whole hyperboloïd. Those which do [e.g: $ds^2 = -dT^2 + \cosh^2\sqrt{\Lambda/3}\,T\, d\Omega_3^2$] are not static.

In the case of Schwarzschild's ST the Flamm diagram indicated too that the Schwarzschild static coordinates may not describe the whole ST (Weyl 1917).

The purpose of this lecture is to describe a clearer picture of the geometry of BH spacetimes that emerged during the period 1950 (Synge) to 1972 (Les Houches school).

Eddington's coordinates (1924) revived : Finkelstein 1958

Eddington 1924 : $\begin{cases} \bar{t} = t + 2m \ln |r/2m - 1| \\ ds^2 = ds_4^2 + \frac{2m}{r}(d\bar{t} + dr)^2 \end{cases}$; $ds_4^2 = \eta_{ij} dX^i dX^j$

that is : $ds^2 = -\left(1 - \frac{2m}{r}\right) d\bar{t}^2 + \frac{4m}{r} d\bar{t}\, dr + dr^2\left(1 + \frac{2m}{r}\right) + r^2 d\Omega^2$

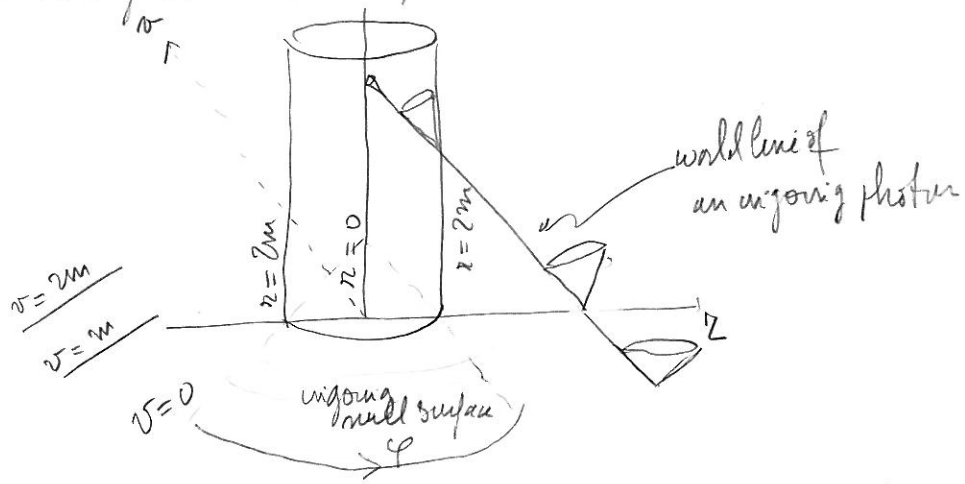
or, introducing $v = \bar{t} + r = t + r^*$, $r^* = r + 2m \ln |r/2m - 1|$:

$$\boxed{ds^2 = -\left(1 - \frac{2m}{r}\right) dv^2 + 2\, dv\, dr + r^2 d\Omega^2}$$

(The importance of writing the Schwarzschild metric under this form became apparent in the late 50's when it was understood (Synge 1950) that to probe the geometry of a ST one should study the structure of the light cones, that is, propagation of light.)

eom of radial photons : $ds^2 = 0 \iff \frac{dv}{dr} = 0$; $\frac{dv}{dr} = \frac{2}{1 - 2m/r}$

hence the diagram (MTW p 829)



(pictorial representation of the redshift found by Oppenheimer - Snyder)

This transformation however does not exhibit any double representation of spatial infinity (as suggested by the Flamm diagram and the extension of the de Sitter static space time to the whole Weyl hyperboloïd).

Moreover : the metric is still ill-behaved at $r = 2m$.

An example of "maximal extension": from Rindler ST to M_4

(see B. Wald p149)

(done by Martin Kruskal, plasma physicist at Princeton, when trying to understand BH with a small study group in the late 50's)

- Consider the metric: $ds^2 = -x^2 dt^2 + dx^2 + dy^2 + dz^2$ AND ; $\begin{pmatrix} t \in [-\infty, +\infty] \\ x \in [0, +\infty] \end{pmatrix}$

(This metric is flat (Riemann $= 0$). The coordinates may be seen as representing space & time in a uniformly accelerated frame. This metric had already been introduced by Lanczos (in the 20's) and Einstein-Rosen (1938). "popularized" by Rindler in 1956)

- Find the null radial geodesics: $ds^2 = 0 \iff t = \pm \ln x + \text{cte}$

and introduce new "null" coordinates. $\begin{cases} u = t - \ln x \\ v = t + \ln x \end{cases}$ $(u, v \in [-\infty, +\infty])$

("Eddington-Finkelstein" coordinates).

in these coordinates $ds_2^2 = -e^{v-u} du\, dv$ (ignore y & z coordinates)

- introduce then: $U = -e^{-u}$; $V = e^{v}$, $U \in [-\infty, 0]$; $V \in [0, +\infty]$.

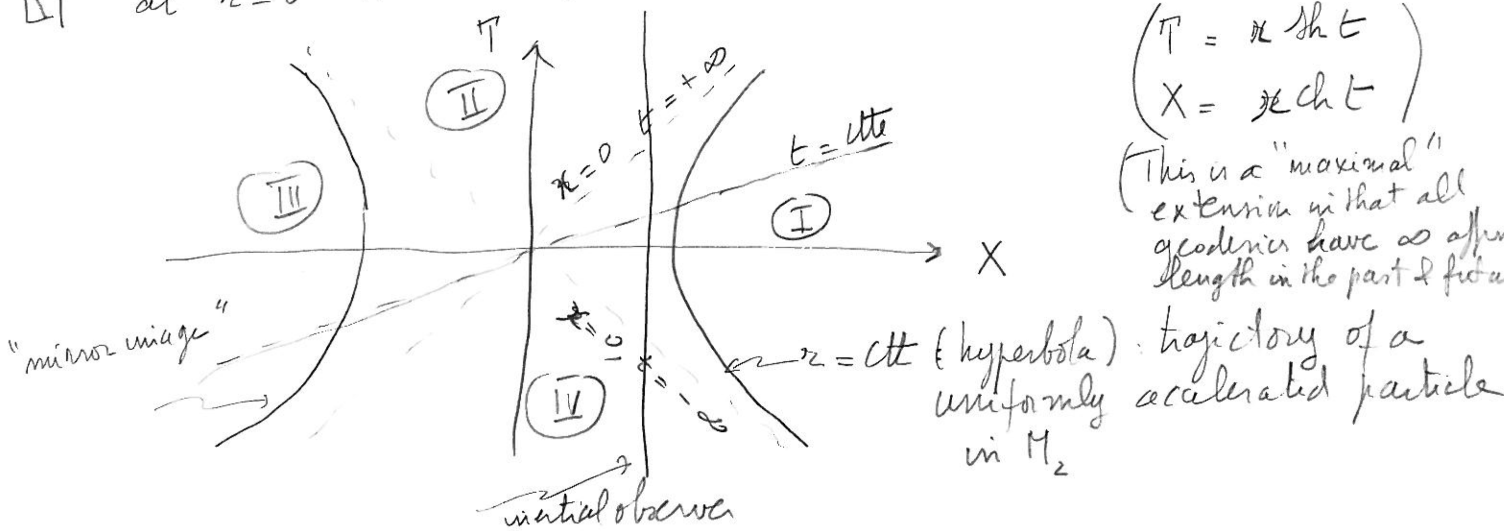
the metric becomes: $ds^2 = -dU dV$

The metric is regular for ALL values of U & V; hence EXTEND (U, V) to the whole real line.

- Finally: $T = \frac{U+V}{2}$; $X = \frac{V-U}{2}$ so that $ds_2^2 = -dT^2 + dX^2$: M_2

- Hence: in going from the Rindler coordinates (t, x) to the Minkowski coordinates $(T\, X)$ we have eliminated the coordinate singularity at $x = 0$ AND extended the (2D) Rindler ST to M_2

$\begin{pmatrix} T = x \,\mathrm{sh}\, t \\ X = x \,\mathrm{ch}\, t \end{pmatrix}$



(This is a "maximal" extension in that all geodesics have ∞ affine length in the past & future)

The Kruskal - Szekeres extension of Schwarzschild ST

(MTW p828) (Wald p152)

Consider the section $\theta = \pi/2$, $\varphi = $ cst of Schwarzschild's metric:

$$d\sigma^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - 2m/r} \quad ; \quad t \in [-\infty, +\infty]; \; r \in [0, +\infty]$$

the null geodesics are $t = \pm r_* + \text{Const.}$ with $r^* = r + 2m \ln|r/2m - 1|$

("tortoise" coordinate - Wheeler)

- Introduce the Eddington-Finkelstein coordinates $\begin{cases} u = t - r_*; & v = t + r_* \\ (u, v \in [-\infty, \infty]). \end{cases}$

so that $d\sigma^2 = -\left(1 - \frac{2m}{r}\right) du\, dv$

$= -\frac{2m}{r} e^{-r/2m} e^{\frac{v-u}{4m}} du\, dv$ } which is still singular at $r = 2m$.

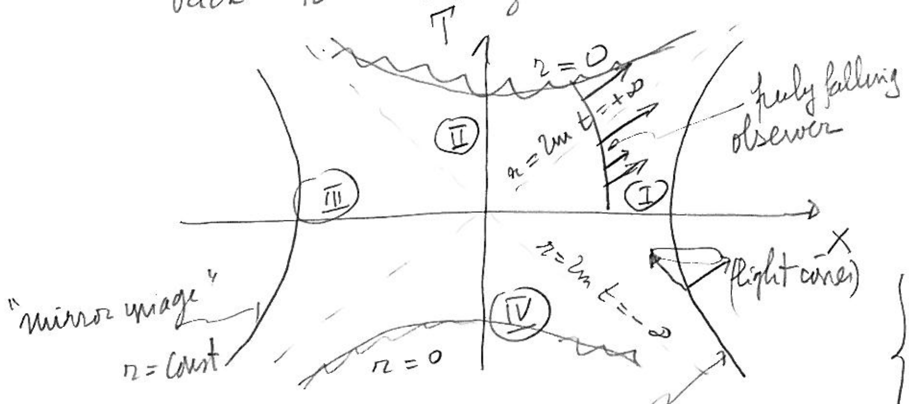
- Introduce $U = -e^{-u/4m}$; $V = e^{v/4m}$ $\quad (U \in [-\infty, 0],\; V \in [0, +\infty])$

so that $d\sigma^2 = -\frac{32 m^3}{r} dU dV$ and EXTEND (U, V) to the whole plane

- finally set: $T = \frac{U+V}{2}$; $X = \frac{V-U}{2}$ to obtain the Kruskal - Szekeres metric:

$$\boxed{ds^2 = \frac{32 m^3}{r} e^{-r/2m} \left(-dT^2 + dX^2\right) + r^2 d\Omega_2^2}$$

since one has extended the manifold care must be exercised when "going back" to Schwarzschild's coordinates:



$\begin{cases} T = \pm\sqrt{r/2m - 1}\; e^{r/4m} \cosh t/4m & \text{in (I)} \\ X = \pm\sqrt{r/2m - 1}\; e^{r/4m} \sinh t/4m & \text{and (III)} \end{cases}$

$\begin{cases} T = \pm\sqrt{1 - r/2m}\; e^{r/4m} \sinh t/4m & \text{in (II)} \\ X = \pm\sqrt{1 - r/2m}\; e^{r/4m} \cosh t/4m & \text{and (IV)} \end{cases}$

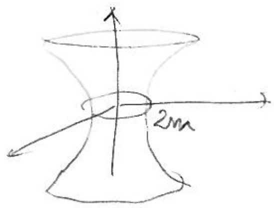
$\begin{cases} (r/2m - 1) e^{r/2m} = T^2 - X^2 & \text{in (I - IV)} \\ t = \begin{cases} 4m \tanh^{-1}(X/T) & \text{in I \& III} \\ 4m \tanh^{-1}(T/X) & \text{in II \& IV} \end{cases} \end{cases}$

(Found by Kruskal in the late 50's - Wheeler not interested at first. Then publicizes it at the Royaumont conference in 1959. Eventually Wheeler writes up the paper & publishes it under Kruskal's name. Meanwhile Szekeres finds the same transformation (paper published in hungarian Journal) but gives credit to Synge (1950).)

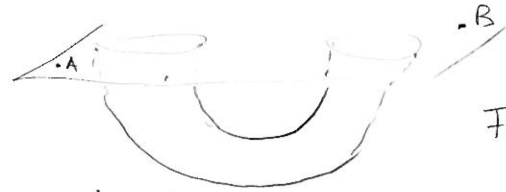
Einstein-Rosen bridges & "wormholes"

Recall Flamm diagram: $t = Ct$; $\theta = \pi/2$ slices of Schwarzschild metric:

$$d\sigma^2 = \frac{dr^2}{1-2m/r} + r^2 d\varphi^2$$ are a paraboloid embedded in E_3 (for $r > 2m$)



; distort it into:



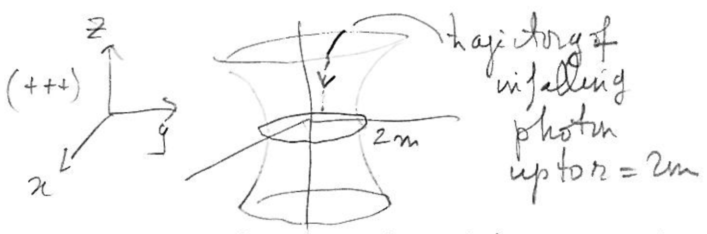
Einstein-Rosen 1935

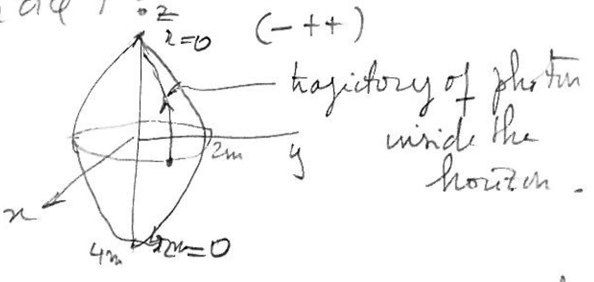
Fuller-Wheeler 1962

Question: if we identify the 2 asymptotically flat region can A send signals to B through the wormhole faster that by using the asymptotically flat route?

The answer can be read off from Kruskal's diagram and is NO since region I & III are causally disconnected -

One can also answer the question by studying how the wormhole (which is a snapshot of Schwarzschild spacetime at various instants t) changes when a light ray crosses the horizon. As long as $r > 2m$ the light ray path is represented by a curve on the Flamm paraboloid. But for $r < 2m$ the geometry of $d\sigma^2 = -\frac{dr^2}{2m/r - 1} + r^2 d\varphi^2$ is no longer that of a paraboloid. Rather it is that of the closed surface $z^2 = 8m(2m - r)$ embedded in M_3 $(dS^2 = -dz^2 + dr^2 + r^2 d\varphi^2)$ $(-++)$





NB: this result can be generalized: "Any asympt. flat ST with a non-simply connected Cauchy surface has singular time evolution if it satisfies the weak energy condition" $(\rho \geq 0;\ \rho + p \geq 0)$

hence the demise of Wheeler's "geons".

however: revival of wormholes & time machines with K. Thorne (1988)

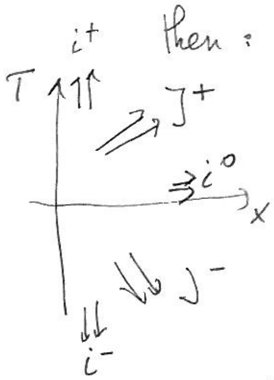
(need for "exotic material to keep mouth open").

(see recent papers by Matt Visser) see also Frolov-Novikov (1998)

see also review by F. Lobo gr-qc 0710.4474

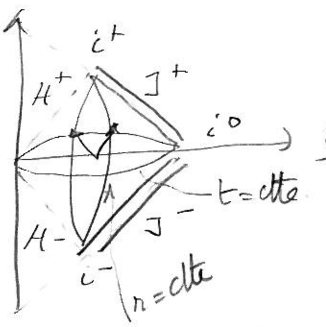
Penrose conformal diagram of Kruskal's spacetime

- Consider region I of Kruskal's diagram.
  go from (T,X) coordinates to (Ψ,ξ) such that $\begin{cases} T+X = \frac{1}{2}\,tg(\Psi+\xi) \\ T-X = \frac{1}{2}\,tg(\Psi-\xi) \end{cases}$
  $\Rightarrow ds^2 = \Omega^2\left[-d\Psi^2+d\xi^2+r^2\Omega^{-2}d\Omega_2^2\right]$; $\Omega^2 = \dfrac{32m^3}{r}\dfrac{e^{-r/2m}}{4\cos^2\left[\frac{1}{2}(\Psi+\xi)\right]\cos^2\left[\frac{1}{2}(\Psi-\xi)\right]}$



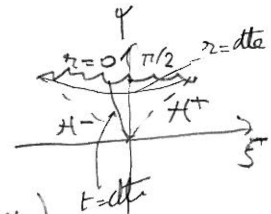
then: $\begin{cases} \mathcal{H}^+, \text{ that is the surface } T=X, \text{ that is } r=2m,\ t=+\infty \text{ is such that } \Psi-\xi=0 \\ \mathcal{H}^-, \quad T=-X, \quad r=2m,\ t=-\infty \quad \Psi+\xi=0 \\ \mathcal{J}^+, \quad T+X\to\infty \text{ with } T-X \text{ finite} \quad \Psi+\xi=\pi/2 \\ \mathcal{J}^-, \quad T-X\to\infty \text{ with } T+X \text{ finite} \quad \Psi-\xi=-\pi/2 \end{cases}$

hence region I is represented by:
$\mathcal{J}^\pm$ future/past null infinity
i^0: spacelike infinity
$i^\pm$: future/past timelike infinity



NB: other coordinates "better behaved" at $\mathcal{J}^\pm$ are introduced in Frolov-Novikov (98)

- similarly region II of Kruskal's diagram is represented by:
  (see Les Houches 1972 for the details of the coord. transformation)

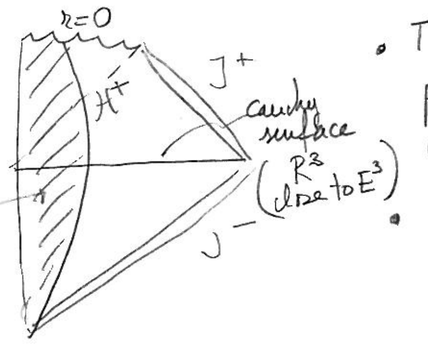


- Hence the complete Penrose diagram:

Cauchy surface
(ie: a spacelike hypersurface which every non-spacelike curve intersects once).

remarks: such a ST is not "black": light rays & particle can emerge from $r=0$ through $\mathcal{H}^-$
- any point in region II represent a 2-sphere which is a "closed trapped surface" in that both light rays propagate toward smaller r regions.

- Collapsing star vs maximally extended ST:



interior of collapsing star

- This diagram is firmly believed to represent physical space & time outside a collapsing (spherically symmetric) star.
- $\exists$ doubts about the physical relevance of the full diagram. (see eg Wald p 155 (1984))
  (and: Dafermos - Rodnianski gr-qc 0811.0354)

because "the initial configuration is unphysical: the Cauchy surface has 2 ends and topology $R\times S^2$"

Maximal extension of the Reissner-Nordström spacetime

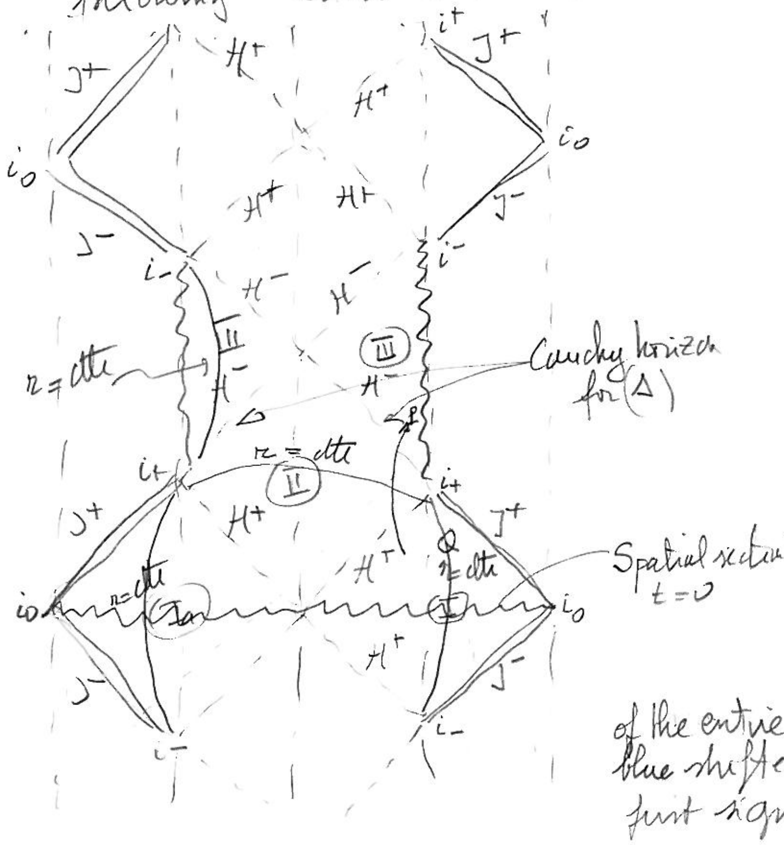
(Graves & Brill 1960. Carter 1966)

$$\begin{cases} ds^2 = -\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)dt^2 + \dfrac{dr^2}{1-\frac{2m}{r}+q^2/r^2} + r^2 d\Omega_2^2 \\ \text{radial light rays eom: } \quad \frac{dr}{dt} = \pm\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right) \iff t = \pm r_* + \text{cte} \\ \text{with} \quad r^* = \displaystyle\int \frac{dr}{1-2m/r+q^2/r^2} \end{cases}$$

(see Hawking-Ellis p157 for explicit expressions; note also that $\dot{E} = \frac{E}{1-2m/r+q^2/r^2}$; $\dot{r} = \pm E$)

- Eddington-Finkelstein coordinates: $u = t - r_*$; $v = t + r_*$
  so that $ds^2 = -\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right) du\,dv + r^2 d\Omega_2^2$
- then introduce (case $q^2<m^2$): $U = -e^{-\frac{r_+-r_-}{4r_+^2}u}$; $V = e^{+\frac{r_+-r_-}{4r_+^2}v}$
  followed by: $T = \frac{U+V}{2}$; $X = \frac{V-U}{2}$
- finally compactify à la Penrose: $T+X = \frac{1}{2}\operatorname{tg}(\psi+\xi)$; $T-X = \frac{1}{2}\operatorname{tg}(\psi-\xi)$
  (details of the transformation in Hawking-Ellis p157 (1973)
  or in Chandrasekhar p213 (1984))
- Thus obtain 3 fundamental "blocks": (cas $q^2<m^2$)

- To obtain the maximal extension "glue" these blocks in such a way that geodesics all end up either at ∞ or on the singularity. The result is the following "Jacob ladder"



A few intriguing features:

- the singularity is time-like
- hence timelike curve can avoid $r=0$
- there is no Cauchy surface in the sense that light rays outgoing from a surface do not reach the whole ST; they cannot go beyond a "Cauchy horizon"; hence the future of this Cauchy horizon cannot be predicted from data on (Δ).
- a particle P crossing H^+ sends signals to Q which are ∞^{ty} redshifted (as in Schwarzschild ST) ($\dot{E} \xrightarrow{r\to r_+^{(+)}} +\infty$)

but "witnesses, in a flash, a panorama of the entire history of the external world, in ∞^{tly} blue shifted ($\dot{E}\to-\infty$) null rays" (Chandra)
first sign of the instability of Cauchy horizon.

Kerr solution

- Introductory remarks

Finding an axisymmetric solution of the 4D-vacuum Einstein equation representing the gravitational field of a compact gravitating body took almost 50 years (1915 – 1963)!

see eg { Kerr's account of the story : arXiv 0706.1109 gr-qc
{ see also the account by Dautcourt arXiv 0807.3473 physics.

The solution was found by Kerr (Roy Kerr b. 1934) in 1963.
(not in the manner described below for which see Anabalon ND et al Dec 08)

- Kerr-Schild metrics

Einstein's eqns are non linear in the metric coefficients. There $\exists$ however classes of metrics which make them linear. Consider metrics whose coefficients in some coordinate system read:

$$\begin{cases} g_{ij} = \bar{g}_{ij} + l_{ij} \quad \text{where } \bar{g}_{ij} \text{ is a given "background" metric} \\ \text{and where } l_{ij} = f(x^k)\, l_i l_j \quad \text{with } l^i \equiv \bar{g}^{ij} l_j \text{ null \& geodesic} \\ \text{That is, such that } \quad \bar{g}_{ij} l^i l^j = 0 \quad \text{and } l^i \bar{D}_i l^j = 0 \end{cases}$$

It is an exercise to compute the Ricci tensor for such metrics. The result is:

$$\begin{cases} R^i_j = \bar{R}^i_j - l^{ik} \bar{R}_{kj} + \bar{D}_k (\bar{g}^{il} \Delta^k_{jl}) \quad (*) \\ \text{with } \Delta^i_{jk} = \frac{1}{2} \left(\bar{D}_j l^i_k + \bar{D}_k l^i_j - \bar{D}^i l_{jk} \right) \end{cases}$$

Note that this (exact) expression is linear in the "perturbation" l_{ij}.

As for the scalar curvature $R \equiv R^i_i$ it reduces to:

$$R = \bar{R} - l^{ij} \bar{R}_{ij} + \begin{cases} \bar{D}_k (l^k \bar{D}_l (f l^l)) \\ \frac{1}{\sqrt{-\bar{g}}} \partial_k \left[l^k \partial_l (\sqrt{-\bar{g}} f l^l) \right] \end{cases} \quad (**)$$

example: $\begin{cases} ds^2 = -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) \quad ; \; \bar{R}_{ij} = 0 \\ l^i = (1,1,0,0) \quad\quad (\bar{g}_{ij} l^i l^j = 0 ; \; l^i \bar{D}_i l^j = 0) \end{cases}$

trace of Einstein's vacuum equations: $R = 0 \overset{(**)}{\Longleftrightarrow} \frac{1}{r^2} \frac{d^2}{dr^2} (r^2 f) = 0$

$$\Longleftrightarrow \quad f = \frac{a + br}{r^2} \quad ;$$

One must then check if this solution is indeed Ricci flat:

now, from (*):
$$\begin{cases} R^0_0 = R^1_1 = \frac{1}{2r^2}(r^2 f')' \\ R^2_2 = R^3_3 = \frac{f^2}{r^2}(rf)' \end{cases}$$ (exercise)

hence: $R^2_2 = R^3_3 = 0$ if $a=0$ and then (Bianchi) $R^0_0 = R^1_1 = 0$.

Thus we recover Schwarzschild's metric in Kerr-Schild coordinates ($b \equiv 2m$):
$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{4m}{r}dt\,dr + \left(1+\frac{2m}{r}\right)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$
(found by Eddington in 1924). (Recall that the coordinate change
$$t \to t + 2m \ln|r/2m - 1|$$ brings the metric to the standard form)

- Choosing the M_4 background coordinates to describe rotating bodies

or: the Newtonian insight.

- rotating gravitationally bound bodies are spheroids (MacLaurin 1742) or ellipsoids (Jacobi 1834) because of centrifugal force (see Chandra's "Ellipsoidal figures of equilibrium" book for details). Hence the gravitational potential outside also has the same symmetry: the equipotentials are spheroids [also the surface of the body is not an equipotential].
- A way to find solutions of Laplace equation $\Delta U = 0$ whose equipotentials are spheroids (ie surfaces of eqn $X^2 + Y^2 + \frac{Z^2}{1-e^2} = a^2$) is to write in spheroidal coordinates $(\rho\,\theta\,\varphi)$ defined as: (XYZ being cartesian)
$$X = \sqrt{\rho^2+a^2}\sin\theta\cos\varphi\,;\quad Y = \sqrt{\rho^2+a^2}\sin\theta\sin\varphi\,;\quad Z = \rho\cos\theta.$$
(since $\frac{X^2+Y^2}{\rho^2+a^2} + \frac{Z^2}{\rho^2} = 1$ the surfaces $\rho = $ cte are spheroids)

also:
$$\begin{cases} \text{the surface } \rho = 0 \text{ is the disk of radius } a \text{ in the } Z=0 \text{ plane} \\ \text{the origin } X=Y=Z=0 \iff \rho = 0,\ \theta = 0 \\ \rho = 0,\ \theta = \pi/2 \text{ is the } \binom{\text{ring}}{\text{circle}}\ X^2+Y^2 = a^2 \text{ in the } Z=0 \text{ plane.} \end{cases}$$

in the $(\rho\theta\varphi)$ coordinates the euclidean metric reads:
$$dl^2 = \frac{\rho^2+a^2\cos^2\theta}{\rho^2+a^2}d\rho^2 + (\rho^2+a^2\cos^2\theta)\,d\theta^2 + (\rho^2+a^2)\sin^2\theta\, d\varphi^2$$

the Laplace operator is $\Delta = \frac{1}{\sqrt{e}}\partial_\alpha \sqrt{e}\, e^{\alpha\beta}\partial_\beta$; $\sqrt{e} = (\rho^2+a^2\cos^2\theta)\sin\theta$

if $U = U(\rho)$: $\Delta U = 0 \iff [(\rho^2+a^2)U']' = 0 \Rightarrow U = \frac{GM}{a}\left[-\frac{\pi}{2} + \operatorname{arctg}\frac{\rho}{a}\right]$

it describes the gravitational potential of a "ring singularity" at $\rho = 0$.

- hence the idea to try to find a Kerr-Schild type solution of the vacuum equations with the background being M_4 in spheroidal coordinates, that is:

$$\boxed{ds^2 = -d\tilde{t}^2 + \frac{\rho^2+a^2\cos^2\theta}{\rho^2+a^2}\,d\rho^2 + (\rho^2+a^2\cos^2\theta)d\theta^2 + (\rho^2+a^2)\sin^2\theta\, d\varphi^2}$$

- as for the null vector, choose $\boxed{l_i = \left(1, \frac{\rho^2+a^2\cos^2\theta}{\rho^2+a^2}, 0, a\sin^2\theta\right)}$

- solving the Einstein eqns

The trace $R=0$ is easily solved (Mathematica helps!) and yields:

$$f = \frac{a(\theta)+\rho\, b(\theta)}{\rho^2+a^2\cos^2\theta}$$

One must then ensures that the metric is indeed Ricci flat (not at all guaranteed a priori!). One checks that yes it is if $a(\theta)=0$; $b(\theta)=2m$

hence: $\boxed{f = \frac{2m\rho}{\rho^2+a^2\cos^2\theta}}$

The solution thus found ($ds^2 = d\tilde{s}^2 + f\, l_i l_j dx^i dx^j$) is Kerr's solution (1963) (in Kerr-Schild coordinates (1964)).

- Boyer-Lindquist coordinates (1967).

Kerr's metric can be written in a more amenable form as:

$$\boxed{\begin{aligned} ds^2 = &-\frac{\Delta}{\Sigma^2}(dt - a\sin^2\theta\, d\varphi)^2 + \frac{\sin^2\theta}{\Sigma^2}\left[(r^2+a^2)d\varphi - a\,dt\right]^2 + \\ &+ \frac{\Sigma}{\Delta}dr^2 + \Sigma^2 d\theta^2 \ ; \quad \Delta \equiv r^2 - 2mr + a^2 \ ; \quad \Sigma^2 \equiv r^2 + a^2\cos^2\theta \end{aligned}}$$

for the explicit transformation from the Kerr-Schild coordinates $(\tilde{t}, \rho, \theta, \varphi)$ to the Boyer-Lindquist ones $(t\, r\, \theta\, \varphi)$ see eg $\left\{\begin{array}{l}\text{M. Visser arXiv: 0706.0622 gr-qc} \\ \text{Gibbons et al hep-th/0404008}\end{array}\right.$

- remarks : - when $m=0$ the Kerr metric is flat (as it obvious from the way it was derived here).

- when $r\to 0$ $ds^2 \sim -\left(1-\frac{2m}{r}+O\left(\frac{1}{r^2}\right)\right)dt^2 - \left[\frac{4ma\sin^2\theta}{r} + O\left(\frac{1}{r^3}\right)\right]dt\,d\varphi$
$+\left[1+\frac{2m}{r}+O\left(\frac{1}{r^2}\right)\right]\left[dr^2 + r^2 d\Omega_2^2\right].$

identification with the post-newtonian metric of a rotating bodies (see eg Weinberg's book "gravitation") yields: $J = ma$ as the angular momentum of the rotating body.

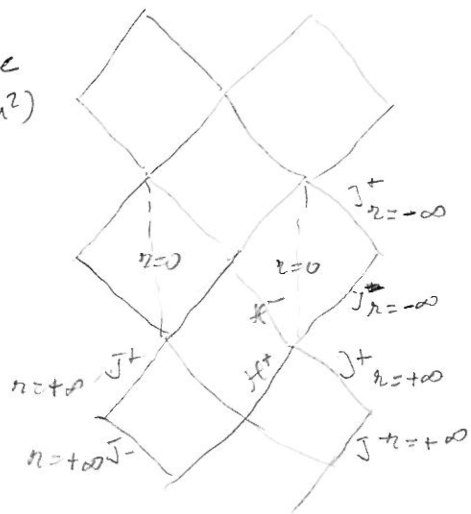
- no Birkhoff theorem: hence not clear (in 1963) if can be matched to an interior solution. NO realistic interior solution known (OK with Newtonian insight).

- Maximal extension of Kerr solution (Carter 1966, 1968)
  - $\exists$ 2 $r =$ cte null surfaces ($\det g = 0$) : $\Delta = 0$ $\quad r_\pm = m \pm \sqrt{m^2 - a^2}$
  - hence 3 "building blocks" as in the case of RN spacetime
  - inspection of the Riemann tensor (not "too" painful when working with the Kerr-Schild form of the metric) (see Carter in Les Houches 72 & Chandra 1983) shows that $\exists$ a curvature singularity at $\Sigma = 0$ ie $r = 0 \oplus \theta = \frac{\pi}{2}$

    Now recall (see Newtonian analogue) that $\rho = 0$ $\theta = \pi/2$ is the "ring" $X^2 + Y^2 = a^2$ in the $Z = 0$ plane

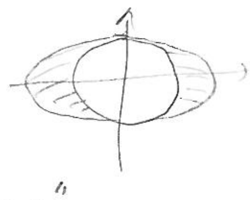
  we are hence entitled to extend r to negative values... $r \to -\infty$ yields the same metric with $m \to -m$.
  - the Kruskal-Szekeres coordinates are obtained as in the Reissner-Nordström or Schwarzschild cases along "the axis" $\theta = 0$. See Carter & Chandra.

case ($a^2 < m^2$)



and a similar one but truncated at $r = 0$ for the extension in the equatorial "plane" $\theta = \pi/2$.

- ergosphere



  in Boyer-Lindquist coordinates $g_{tt} = -\left(1 - \frac{2mr}{\Sigma^2}\right)$

  $g_{tt} = 0$ delineates the surface $r_e = m + \sqrt{m^2 - a^2\cos^2\theta}$

  The region between the horizon r_+ & r_e is the "ergosphere" (Christodoulou - Ruffini 1971)

$g_{tt} = 0$ is the "stationary limit"

Indeed all particles within the ergosphere must corotate with the hole.

Consider a light ray with $\theta = \pi/2$ & $\dot r = 0$ (circular orbit in some "plane")

eom : $0 = g_{tt}\dot t^2 + 2g_{t\varphi}\dot t\dot\varphi + g_{\varphi\varphi}\dot\varphi^2 \Leftrightarrow \frac{d\varphi}{dt} = \frac{-g_{t\varphi} \pm \sqrt{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}}{g_{\varphi\varphi}}$

if $g_{tt} = 0$ then $\left.\frac{d\varphi}{dt}\right|_+ = 0$ $\quad \left.\frac{d\varphi}{dt}\right|_- = \frac{-2g_{t\varphi}}{g_{\varphi\varphi}}$

now $g_{t\varphi} = -\frac{2mra\sin^2\theta}{\Sigma^2}$; $g_{\varphi\varphi} = (r^2+a^2)\Sigma^2 + 2mra^2\sin^2\theta$

hence $\left.\frac{d\varphi}{dt}\right|_- > 0$: the pro & retro-grade photons corotate with the hole.

The stability & energetics of BH

0. Kerr geometry (continued from lecture 2).

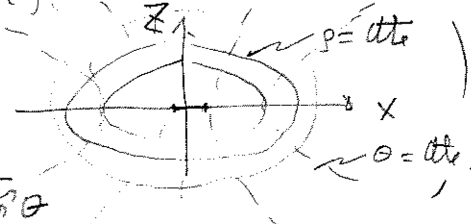
1. From Kerr-Schild to Boyer-Lindquist coordinates

• We showed in the previous lecture that the following metric solves $R_{ij}=0$:

$$ds^2 = d\sigma^2 + f\, l_i l_j\, dx^i dx^j \quad \text{with } x^i = (\tilde{t}, \rho, \theta, \varphi)$$

$$d\sigma^2 = -d\tilde{t}^2 + \frac{\rho^2+a^2\cos^2\theta}{\rho^2+a^2}\,d\rho^2 + (\rho^2+a^2\cos^2\theta)\,d\theta^2 + (\rho^2+a^2)\sin^2\theta\, d\varphi^2$$

(Minkowski metric in spheroidal coordinates)



$$l_i = \left(1, \frac{\rho^2+a^2\cos^2\theta}{\rho^2+a^2}, 0, a\sin^2\theta\right); \quad f = \frac{2m\rho}{\rho^2+a^2\cos^2\theta}$$

• to eliminate the cross-terms containing $d\rho$ perform the coord. change: ($\triangle$: $\varphi \neq \phi$!)

$(\tilde{t}, \rho, \theta, \varphi) \longrightarrow (t, r, \theta, \phi)$ such that:

$$\tilde{t} = t + 2m\int\frac{r\,dr}{r^2-2mr+a^2}; \quad \rho \equiv r; \quad \varphi = \phi - a\int\frac{dr}{r^2-2mr+a^2}$$

so that the metric becomes:

$$ds^2 = -\frac{\Delta}{\Sigma^2}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\sin^2\theta}{\Sigma^2}\left[(r^2+a^2)\,d\phi - a\,dt\right]^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\theta^2$$

$$= -\left(1-\frac{2mr}{\Sigma^2}\right)dt^2 - \frac{4mra\sin^2\theta}{\Sigma^2}\,dt\,d\phi + \sin^2\theta\left(r^2+a^2+\frac{2mra^2\sin^2\theta}{\Sigma^2}\right)d\phi^2 + \frac{\Sigma dr^2}{\Delta} + \Sigma d\theta^2$$

with $\Delta \equiv r^2 - 2mr + a^2$; $\Sigma^2 \equiv r^2 + a^2\cos^2\theta$ [NB: Kerr-Newman: $\Delta = r^2-2mr+a^2+e^2$]

• remarks:
- when $m=0$ $\quad ds^2 = d\sigma^2$: M_4 in spheroidal coordinates.
- when $r \to \infty$ $\quad ds^2 \sim -\left(1-\frac{2m}{r}+\dots\right)dt^2 - \left(\frac{4ma\sin^2\theta}{r}+\dots\right)dt\,d\varphi + \left(1+\frac{2m}{r}+\dots\right)dr^2 + (r^2+\dots)(d\theta^2+\sin^2\theta\, d\phi^2)$

identification with post-newtonian metric of a rotating body (see eg Weinberg's "gravitation") yields $\boxed{J = ma}$ as the angular momentum of the source.

- NO Birkhoff theorem. NO realistic interior solution known.

2. Singularity, horizons & ergosphere

- Calculation of the Riemann tensor (see Carter les Houches 72, Chandra 83) shows that $\exists$ a curvature singularity at $\Sigma^2 = 0$ ie $r = 0$ + $\theta = \pi/2$. In terms of the coordinates XYZ (such that $d\sigma^2 = \gamma_{ij} dX^i dX^j$), $\Sigma^2 = 0$ is a "ring" singularity $X^2 + Y^2 = a^2$ in the $Z = 0$ plane. (Hence continue to $r<0$) (PTO)

- In Boyer-Lindquist coord. $\begin{cases} \boxed{\Delta = 0} \\ r_\pm = m \pm \sqrt{m^2 - a^2} \end{cases}$ is a singularity of the metric. $(m^2 \geq a^2)$.

  - It is a coordinate singularity (as the expression of the metric in Kerr-Schild coordinates shows). $r_\pm$ are the "horizons". Indeed:
  - the 3-metric on the surfaces $r = \text{cte}$ has a determinant:
  $g_{(3)} = -\sin^2\theta\, \Sigma^2 \Delta$; hence $\Delta = 0$ ($r = r_\pm$) are null surfaces:
  ($\exists$ solutions to $\overset{(3)}{g}_{\alpha\beta} L^\alpha = 0$; L^α defined on $\Sigma_{(3)}$ only; L^α is null.)

  More precisely: if we introduce $\Omega_\pm \equiv \dfrac{a}{2mr_\pm} = \dfrac{a}{r_\pm^2 + a^2}$

  then $L_\pm^i = (1, 0, 0, \Omega_\pm)$ is null at $r = r_\pm$ $\left(g_{ij} L_\pm^i L_\pm^j \big|_{r=r_\pm} = 0 \right)$

  and are the tangent to null geodesics: $X(t) = (t, r_\pm, \theta_0, \phi_0 + \Omega_\pm t)$ describing photons orbiting at $r = r_\pm$ with an angular velocity $\Omega_\pm$ (as measured at infinity). $\boxed{\Omega_+ : \text{"angular velocity of the BH"}}$

  - the horizons are the surfaces $r = r_\pm$ but their intrinsic geometry is NOT that of 2-sphere! Indeed the metric of the 2-surfaces $t = \text{ctt}$, $r = r_\pm$ is $d\sigma_2^2 = \Sigma_\pm^2 d\theta^2 + \dfrac{4m^2 r_\pm^2}{\Sigma_\pm^2} \sin^2\theta\, d\phi^2$ $\quad (\Sigma_\pm^2 = r_\pm^2 + a^2\cos^2\theta)\ (\Delta = 0)$

  the area of this surface is $\underline{A_\pm = 4\pi (r_\pm^2 + a^2)}$ (NOT $4\pi r_\pm^2$)

  and the scalar curvature is $R_{(2)} = \dfrac{2(r_\pm^2 + a^2)(r_\pm^2 - 3a^2\cos^2\theta)}{(\Sigma_\pm^2)^3}$

  (at the pôles $\theta = 0$: $R_{(2)}$ is negative is $3a^2 > r_+^2$ ie $a > \frac{\sqrt{3}}{2} m$)

  Conclusion: the horizons $r = r_\pm$ are spheroids in terms of the background geometry; but their intrinsic geometry is more complicated see Visser arXiv:0706.0622 (gr-qc).

- maximal extension: see Carter, les Houches 1972 or Carter, gr-qc/0604064

• ergosphere : - the region where observers cannot be at rest wrt ∞^{ty}.
an "observer" is a timelike curve. an observer "at rest" wrt ∞^{ty} has a worldline $X^i(t) = (t, r_0, \theta_0, \phi_0)$ (since BL coord are asymptotically Minkowskian coordinates)
hence a tangent $T^i = \frac{dX^i}{dt} = (1,0,0,0)$.
which is time like if $g_{ij} T^i T^j < 0 \iff g_{tt} < 0 \iff 1 - \frac{2mr}{\Sigma^2} > 0$.
The surface $g_{tt} = 0 \iff \underline{r_e = m + \sqrt{m^2 - a^2 \cos^2\theta}}$ is the "ergosphere" or "stationary limit"
(and the region $r_+ < r < e$ is the "ergoregion")
(The intrinsic geometry of the ergosphere is described in Visser 0706.0622)
- All particles within the ergosphere must corotate with the BH

Consider light rays with $\theta = \pi/2$ & $\dot r = 0$ ("circular" orbit in some "plane")
com: $0 = g_{tt} \dot t^2 + 2 g_{t\phi} \dot t \dot\phi + g_{\phi\phi} \dot\phi^2 \iff \frac{d\phi}{dt} = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt} g_{\phi\phi}}}{g_{\phi\phi}}$
if $g_{tt} = 0$ then $\left.\frac{d\phi}{dt}\right|_+ = 0$; $\left.\frac{d\phi}{dt}\right|_- = -\frac{2 g_{t\phi}}{g_{\phi\phi}}$ $\Rightarrow \left.\frac{d\phi}{dt}\right|_- > 0$ $(a>0)$
now $g_{t\phi} = -\frac{2mra \sin^2\theta}{\Sigma^2}$; $g_{\phi\phi} = r^2 + a^2 + \frac{2mra^2}{\Sigma^2}$

3. Kerr geodesics : From Birkhoff's theorem to Carter's constant.

- Birkhoff theorem (1923): All spherically symmetric solutions of Einstein's 4D vacuum eqns reduce to Schwarzschild. Now the Schwarzschild solution is not only spher. symm but also static ($r \geq 2m$). In other words the Schwarzschild geometry possesses an extra Killing vector $\xi^i_{(t)} = (1, 0, 0, 0)$ with norm $\xi_{(t)} \cdot \xi_{(t)} = g_{tt} = -\left(1 - \frac{2m}{r}\right)$ (timelike for $r > 2m$).
- Something similar occurs for the Kerr geometry. Beside the Killing vectors corresponding to axi-symmetry & stationarity: $(D_i \xi_j + D_j \xi_i = 0)$

$$\begin{cases} \xi^i_{(t)} = (1,0,0,0) & \xi_{(t)} \cdot \xi_{(t)} = g_{tt} = -\left(1 - \frac{2mr}{\Sigma^2}\right) \\ \xi^i_{(\phi)} = (0,0,0,1) & \xi_{(\phi)} \cdot \xi_{(\phi)} = g_{\phi\phi} \end{cases}$$

(see Frolov-Novikov p 660 & Carter gr-qc/0604064)

there $\exists$ a "Killing tensor" ξ_{ij} such that $D_i \xi_{jk} + D_j \xi_{ki} + D_k \xi_{ij} = 0$
with components

$$\begin{cases} \xi_{00} = a^2\left(1 - \frac{2mr\cos^2\theta}{\Sigma^2}\right); \quad \xi_{rr} = -\frac{a^2\cos^2\theta\, \Sigma}{\Delta}; \quad \xi_{\theta\theta} = r^2 \Sigma^2; \\ \xi_{03} = -\frac{a r \sin^2\theta}{\Sigma^2}\left[\Delta a^2 \cos^2\theta + r^2(r^2 + a^2)\right]; \quad \xi_{33} = \frac{r\sin^2\theta}{\Sigma^2}\left[r^2(r^2+a^2)^2 + \frac{1}{4}\Delta a^4 \sin^2 2\theta\right] \end{cases}$$

(Penrose - Walker 1969).

– The ∃ of the Killing vectors & tensor allow to reduce the obtention of the geodesics to quadrature.

Recall: geodesic eqn: $u^j D_j u^i = 0$ $\quad (u^i = \frac{dx^i}{d\tau};\ u^i u_i = -1 \text{ or } 0)$

and: if ξ^i is a Killing vector (st $D_i\xi_j + D_j\xi_i = 0$) then:

$$\frac{d}{d\tau}(\xi.u) = u^i D_i(\xi^j u_j) = \underbrace{u^i u_j D_i\xi^j}_{=0\ (\text{Killing})} + \underbrace{\xi^j u^i D_i u_j}_{=0\ (\text{geodesic})} = 0 \Rightarrow \xi.u = \text{cte}$$

hence

$$\begin{cases} u_t = g_{ti}u^i = g_{tt}\frac{dt}{d\tau} + g_{t\phi}\frac{d\phi}{d\tau} = -E/\mu \\ u_\phi = g_{\phi i}u^i = g_{\phi t}\frac{dt}{d\tau} + g_{\phi\phi}\frac{d\phi}{d\tau} = L_z/\mu \end{cases}$$

(as expected). (*) $\xi_{(t)} = (1,0)$; $\xi_{(\phi)} = (0,0,0,1)$

As for the Killing tensor it yields another 1st integral; indeed:

$$\frac{d}{d\tau}(\Xi_{ij}u^i u^j) = u^k D_k(\Xi_{ij}u^i u^j) = \underbrace{u^k u^i u^j \cancel{D_k \Xi_{ij}}}_{=0\ (\text{Killing})} + \underbrace{\Xi_{ij}u^k(u^i D_k u^j + u^j D_k u^i)}_{=0\ (\text{geod.})}$$

$= 0$

hence $\Xi_{ij}u^i u^j = K$ $\quad (**)$

Combining (*) & (**) and $u_i u^i = -1$ one obtains (see eg MTW).

$$\begin{cases} \Sigma^2 \frac{dt}{d\tau} = a(L_z - aE\sin^2\theta) + \frac{r^2+a^2}{\Delta}\left[E(r^2+a^2) - L_z a\right] \\ \Sigma^2\frac{d\phi}{d\tau} = \frac{L_z}{\sin^2\theta} - aE + \frac{a}{\Delta}\left[E(r^2+a^2) - L_z a\right] \\ \Sigma^2\frac{d\theta}{d\tau} = \pm\sqrt{Q - \cos^2\theta\left[a^2(\mu^2 - E^2) + L_z^2/\sin^2\theta\right]} \quad (\equiv \pm\sqrt{\Theta}) \\ \Sigma^2\frac{dr}{d\tau} = \pm\sqrt{\left[E(r^2+a^2) - L_z a\right]^2 - \Delta\left[\mu^2 r^2 + (L_z - aE)^2 + Q\right]} \quad (\equiv \pm\sqrt{R}) \end{cases}$$

$\left(Q = K - (Ea - L_z)^2\right)$

NB: Carter (1968) found the constant of the motion K $(r\,\theta)$ by showing that the Hamilton-Jacobi eqn could be separated

Indeed: geod eqn $\iff \frac{dx^i}{d\tau} = \frac{\partial \mathcal{H}}{\partial p_i},\ \frac{dp_i}{d\tau} = -\frac{\partial\mathcal{H}}{\partial x^i}$; $\mathcal{H} = \frac{1}{2}g^{ij}p_i p_j$; $p_i = mu_i$

HJ eqn: $-\frac{\partial S}{\partial\lambda} = \frac{1}{2}g^{ij}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^j}$; write: $S = \frac{1}{2}\mu^2\tau - Et + L_z\phi + S_r(r) + S_\theta(\theta)$

and find $S_r(r) = \int\frac{\sqrt{R}\,dr}{\Delta}$; $S_\theta(\theta) = \int\sqrt{\Theta}\,d\theta$

then $\partial S/\partial K = 0$; $\partial S/\partial\mu^2 = 0$, $\partial S/\partial E = 0$; $\partial S/\partial L_z = 0$ yield;

$$\begin{cases} \int^r\frac{dr}{\sqrt{R}} = \int^\theta\frac{d\theta}{\sqrt{\Theta}}; \quad \tau = \int^r\frac{r^2 dr}{\sqrt{R}} + \int\frac{a^2\cos^2\theta}{\sqrt{\Theta}}d\theta \\ t = \int(r^2+a^2)\frac{\left[E(r^2+a^2) - L_z a\right]dr}{\Delta\sqrt{R}} - \int a\frac{(aE\sin^2\theta - L_z)}{\sqrt{\Theta}}d\theta \\ \phi = \int a\frac{\left[E(r^2+a^2) - L_z a\right]}{\Delta\sqrt{R}}dr - \int\frac{aE\sin^2\theta - L_z}{\sin^2\theta\sqrt{\Theta}}d\theta \end{cases}$$

From Chandra (83): "This was the first of the many properties which have endowed the Kerr metric with an aura of the miraculous"

which can be shown to be equivalent to the eqn above (Frolov-Novikov p 662)

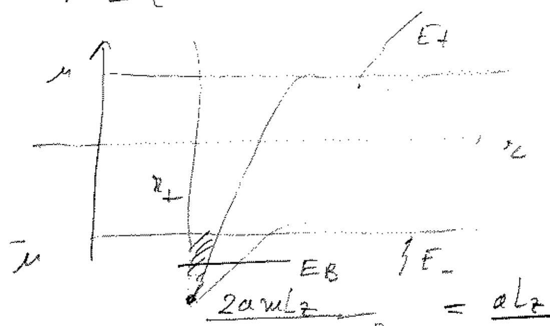
4. Penrose process (1969)

⊖ Consider for simplicity a geodesic in the "equatorial plane" $\theta = \pi/2$; $Q = 0$

then $\Sigma^2 = r^2$;

$$\begin{cases} r^2 \dot{t} = a(L_z - aE) + \frac{r^2+a^2}{\Delta}\left[E(r^2+a^2) - L_z a\right] \\ r^2 \dot{\phi} = L_z - aE + \frac{a}{\Delta}\left[E(r^2+a^2) - L_z a\right] \\ r^2 \dot{\theta} = 0 \\ r^2 \dot{r} = \pm\sqrt{E^2\left[(r^2+a^2)^2 - \Delta a^2\right] + 2E\left[-L_z a(r^2+a^2) + aL_z\Delta\right] + L_z^2 a^2 - \Delta \mu^2 r^2} \end{cases}$$

that is $\dot{r}^2 = (E - E_+)(E - E_-)$ with $E_\pm = \dfrac{2amL_z \pm \sqrt{\Delta}\,\left[r^2 L_z^2 + \mu^2 r\left[r(r^2+a^2) + 2a^2 m\right]\right]}{r(r^2+a^2) + 2a^2 m}$



$$\frac{2amL_z}{r_+(r_+^2+a^2) + 2a^2 m} = \frac{aL_z}{a^2+r_+^2} = \frac{aL_z}{2mr_+} = L_z \Omega_H \qquad (L_z < 0)$$

⊖ Send a particle A on a geodesic with $L_A, E_A > \mu$ from infinity down the hole. Suppose that close to the hole this particle A splits into 2 : $B\,(L_B, E_B)$ & $C\,(L_C, E_C)$ In the local freely falling frame special relativity holds and

$$E_A = E_B + E_C\,, \qquad L_A = L_B + L_C$$

Suppose now that $L_B \Omega_H < E_B < -\mu$; the B particle falls into the BH decreasing its angular momentum (ma) by $|L_B|$ and its mass by $|E_B|$. As for the particle C it will fly to ∞ carrying that energy. This "Penrose process" is a gedanken experiment to show how energy can be extracted from a rotating BH. Hence the name "ergoregion" given to the shaded part of the diagram (Christodoulou - Ruffini 1971).

⊖ The maximum energy which can be extracted corresponds to the case when $E \geq L_z \Omega_H$ (the particle turning point, $\dot{r} = 0$, is on the event horizon). Therefore, setting $E = \delta m$ (<0) & $L_z = \delta J$ we have $\delta m \geq \Omega_H \delta J$

Can this be rewritten as $\delta F \geq 0$? The answer is yes. Choose F to be a function of the geometry of the BH, for example the area of the event horizon $A = 4\pi(r_+^2 + a^2)$; then $\delta F = \frac{dF}{dA}\,\frac{16\pi m r_+}{\sqrt{m^2 - a^2}}\,(\delta m - \Omega_H \delta J)$

$= 8\pi m r_+$; $r_\pm = m \pm \sqrt{m^2 - a^2}$.

Hence $\delta m \geq \Omega_H \delta J \iff \delta F \geq 0$ with $F = F(A)$ (and $dF/dA > 0$)

⊖ "Princeton school" choice (Wheeler, Christodoulou, Ruffini ...) Choose $F(A)$ to have the dimension of a mass [they were interested in the "energetics" of BH (see Les Houches school book 1972) hence he chose $F = m_{irreducible}$: $m_{irr} = \sqrt{\frac{A}{16\pi}} = \sqrt{\frac{r_+^2+a^2}{4}}$

(the coefficient 16π is chosen so that $m_{irr} = m$ in the Schwarzschild case $a=0$)

hence the inequality $\boxed{\delta m_{irr} \geq 0}$

Recall its meaning: if we start with a BH of mass m, angular momentum $J=ma$ (and charge e) and devise processes near the horizon which produce particles in the "ergoregion", energy can be then extracted from the BH with the restriction $\delta m_{irr} \geq 0$. A "reversible" transformation corresponds to $\delta m_{irr} = 0$.

Hence the maximum energy which can be extracted from a rotating (and/or charged) BH is $m - m_{irr}$; writing $\boxed{m^2 = m_{irr}^2 + \frac{J^2}{4m_{irr}^2}}$

$m - m_{irr}$ is max if $a = m \Rightarrow r_+ = m \Rightarrow m_{irr} = \frac{m}{\sqrt{2}} \Rightarrow \frac{m - m_{irr}}{m} = 1 - \frac{1}{\sqrt{2}}$
$= 29\%$.

(charged case $\frac{m - m_{irr}}{m} = 50\%$).

Hence "Black holes are the largest storehouse of energy in the universe"

- NB: simultaneously Bekenstein, Bardeen, Carter, Hawking, Israel chose for F the area itself. $\delta m_{irr} \geq 0$ then reads $\boxed{\delta A \geq 0}$ (Hawking "area" theorem, 1971). [although by a completely, more general, approach of lecture 5]. Bekenstein conjectured that A should somehow measure the "entropy" of the BH .. For more: lecture 5.

I. Uniqueness theorems (or "a black hole has no hair", Wheeler 1971)

The question to be answered here is: consider a collapsing object and SUPPOSE it eventually settles down to a static or stationary final configuration which is a BH, that is, which possesses an event horizon; what will this BH look like? Answer: a Kerr-Newman solution (modulo a nb of definitions & hypotheses to be enunciated below).

1. Israel's uniqueness theorem for static BH

ref.: W. Israel, Phys. Rev 164 (1967) 1776.
V. Frolov & I. Novikov "Black hole Physics" (1998) p 225

- Consider a static 4D Space-Time; this means that $\exists$ coord. $(t\, \underbrace{r\, \theta\, \varphi}_{x^\alpha})$ s.t. the metric reads: $ds^2 = -V^2 dt^2 + h_{\alpha\beta}\, dx^\alpha dx^\beta$; $\begin{pmatrix} V = V(r\theta\varphi) \\ h_{\alpha\beta} = h_{\alpha\beta}(r\theta\varphi) \end{pmatrix}$

  Then $\begin{cases} \Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk}) \text{ decompose as:} \\ \Gamma^0_{0\alpha} = \partial_\alpha V / V \,; \quad \Gamma^\alpha_{00} = V \partial^\alpha V \,; \quad \Gamma^\nu_{\beta\gamma} = \bar{\Gamma}^\alpha_{\beta\gamma} \end{cases}$

  and $R_{ij} = R^k{}_{ikj}$; $R^i{}_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{km} \Gamma^m_{jl} - \Gamma^i_{lm} \Gamma^m_{jk}$

  reads: $\boxed{R_{00} = V \bar{\Delta} V}$; $\boxed{R_{\alpha 0} = 0}$; $R_{\alpha\beta} = \bar{R}_{\alpha\beta} - \frac{\bar{D}_{\alpha\beta} V}{V}$

  where barred quantities are built with $h_{\alpha\beta}$ (eg $\bar{\Delta} V = h^{\alpha\beta} \bar{D}_{\alpha\beta} V = h^{\alpha\beta}(\partial_{\alpha\beta} V - \bar{\Gamma}^\gamma_{\alpha\beta} \partial_\gamma V)$)

- $\boxed{\text{4D vacuum}}$ Einstein equations imply $\bar{\Delta} V = 0$: V is a "harmonic function".
  If ST is asymptotically flat $V \longrightarrow 1$ when $r \longrightarrow \infty$.
  properties of harmonic functions in 3D Riemannian space then imply that
  - if $V \neq 0$ everywhere than $V = 1$ everywhere; In this case $R_{ij} = 0$ and space time is flat [in other words: if a metric reads $ds^2 = -V^2(x^\alpha) dt^2 + h_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta$ and if $V \longrightarrow 1$ at ∞ty & does not vanish then ST is flat]

  - if $\exists$ a connected regular 2-surface such that $V(r, \theta, \varphi) = 0$ (that is called an "event horizon") then V varies monotonically from 0 to 1 (ie $\partial_\alpha V \neq 0$ on $t = $ ct hypersurfaces) if $\boxed{R_{ijkl} R^{ijkl}}$ is finite everywhere (shown by Robinson et al. 1973, 1977).

  We shall suppose that $\boxed{\text{ST possesses an event horizon}}$

- Since V is monotonic we can use it as a coordinate.

(that is $V(r\theta\varphi) = V \iff r = r(V, \underbrace{\theta,\varphi}_{x^A}))$ and we can rewrite the metric as: $ds^2 = -V^2 dt^2 + \rho^2 dV^2 + b_{AB} dx^A dx^B$ $\quad \begin{cases} \rho = \rho(V, x^A) \\ b_{AB} = b_{AB}(V, x^A) \end{cases}$

The idea is now to show that the hypersurfaces $t =$ cte are spherically symmetric.

That is that $\begin{cases} - \rho = \rho(V) \text{ only} \quad (\iff \partial_A \rho = 0) \\ - K_{AB} = \frac{1}{2} K\, b_{AB} \quad \text{where } K_{AB} \equiv \frac{1}{2\rho} \frac{\partial b_{AB}}{\partial V} \text{ is the} \\ \quad \text{extrinsic curvature of the 2 (closed) surfaces } V = \text{cte (which} \\ \quad \text{are hence 2-spheres)} \end{cases}$

If we can show that then, by Birkhoff theorem, ST is Schwarzschild ST.

- Outline of the proof.

(–) In the coordinates (t, V, θ, φ) $\bar{\Delta} V = 0 \iff \frac{1}{\sqrt{h}} \partial_\alpha (\sqrt{h}\, h^{\alpha\beta} \partial_\beta V) = 0$

$\iff \frac{1}{\rho\sqrt{b}} \partial_V \left(\frac{\rho\sqrt{b}}{\rho^2} \right) = 0 \iff \partial_V \left(\frac{\sqrt{b}}{\rho} \right) = 0 \iff \sqrt{b} = \rho\, c(\theta,\varphi).$

(!) Imposing that the event horizon $V = 0$ be a smooth regular surface implies that $\rho(V=0, \theta, \varphi) \neq 0$

(–) The other 4D vacuum equations $\left(\bar{R}_{\alpha\beta} - \frac{\bar{D}_{\alpha\beta} V}{V} = 0 \right)$ are then rewritten in terms of $K_{AB} \equiv \frac{1}{2\rho} \partial_V b_{AB}$ and the intrinsic curvature of the 2 surfaces $V =$ cte which (because they are 2-dimensional) is ${}^{(2)}R_{AB} = \frac{1}{2} b_{AB}\, {}^{(2)}R$. The curvature invariant $R_{ijkl} R^{ijkl}$ is then shown to read: $\frac{1}{8} R_{ijkl} R^{ijkl} = \frac{1}{(V\rho)^2} \left[2 K_{AB} K^{AB} + \frac{2 D_A \rho D^A \rho}{\rho^2} - \frac{2K}{\rho V} + {}^{(2)}R \right]$

(!) Imposing this quantity to be finite on the horizon $V = 0$ (where $\rho \neq 0$) implies that $K_{AB} = 0$; $\rho = \rho_0$; $\frac{K}{V} \to \frac{1}{2} \rho_0\, {}^{(2)}R$ when $V \to 0$

(–) let us now return to $\partial_V \left(\frac{\sqrt{b}}{\rho} \right) = 0 \Rightarrow \partial_V \int d\theta d\varphi \frac{\sqrt{b}}{\rho} = 0$

$\Rightarrow \int d\theta d\varphi \frac{\sqrt{b}}{\rho} \Big|_{V=1} = \int d\theta d\varphi \frac{\sqrt{b}}{\rho} \Big|_{V=0}$; let us call $\mathcal{A}_0 \equiv \int d\theta d\varphi \sqrt{b} \Big|_{V=0}$ the area of the event horizon

then $\int d\theta d\varphi \frac{\sqrt{b}}{\rho} \Big|_{V=0} = \frac{\mathcal{A}_0}{\rho_0}$ since $\rho(V=0, \theta\varphi) = \rho_0$.

now, asymptotic flatness at ∞^{ty} $(V \to 1)$ means that $V \sim 1 - \frac{m}{r}$ (4D again)

hence $dV \sim \frac{m}{r^2} dr$; hence $ds^2 = -V^2 dt^2 + \rho^2 dV^2 + b_{AB} dx^A dx^B$

$$\sim -\left(1 - \frac{2m}{r}\right) dt^2 + \rho^2 \frac{m^2}{r^4} dr^2 + b_{AB} dx^A dx^B$$

& asymptotic flatness hence implies that $\rho \sim \frac{r^2}{m}$ & $\sqrt{b} \sim r^2 \sin\theta$

therefore $\int d\theta d\varphi \frac{\sqrt{b}}{\rho}\Big|_{V\to 1} = \int d\theta\, d\varphi \frac{r^2 \sin\theta\, m}{r^2} = 4\pi m.$

We therefore have that $\underline{\mathcal{A}_0 = 4\pi m \rho_0}$

⊖ Manipulating the 2+1 split of $\bar{R}_{\nu\beta} - D_{\nu\beta}V/V = 0$ then yields

$$\begin{cases} \partial_V\left(\frac{\sqrt{b}\,K}{\sqrt{\rho}\,V}\right) = -\frac{2\sqrt{b}}{V}\left[{}^{(2)}\Delta\sqrt{\rho} + \frac{1}{\rho^{3/2}}\left(\frac{1}{2}\partial_A\rho\partial^A\rho + \Psi_{AB}\Psi^{AB}\right)\right] & (1) \\ \partial_V\left[\frac{\sqrt{b}}{\rho}\left(KV + \frac{4}{\rho}\right)\right] = -\sqrt{b}\,V\left[{}^{(2)}\Delta(\ln\rho) + \frac{1}{\rho^2}\left(\partial_A\rho\partial^A\rho + 2\Psi_{AB}\Psi^{AB}\right) + {}^{(2)}R\right] & (2) \end{cases}$$

where $\begin{cases} {}^{(2)}\Delta f \text{ is the 2-D laplacian built with } b_{AB} \\ \Psi_{AB} \equiv \left(K_{AB} - \frac{1}{2}K b_{AB}\right)\rho \end{cases}$

once again one integrates these 2 eqns over all space. Using the facts that

$$\begin{cases} \int_{V=cte} \left({}^{(2)}\Delta f\right)\sqrt{b}\, d\theta d\varphi = 0 & \text{for any smooth function } f \text{ (admettons!)} \\ \int_{V=cte} {}^{(2)}R\sqrt{b}\, d\theta d\varphi = 8\pi & \text{(Gauss-Bonnet theorem).} \end{cases}$$

and the boundary condition $\left(\begin{matrix} \rho \sim r^2/m\ ; \ \sqrt{b} \sim r^2\sin\theta \text{ etc at } \infty & (V=1) \\ K_{AB} \sim 0\ ; \ \rho \sim \rho_0\ :\ K/V \sim \frac{1}{2}\rho_0\, {}^{(2)}R & (V=0) \end{matrix}\right)$

(let us believe that!).

one arrives at $\begin{cases} \rho_0 \geqslant 4m \ \ \& \ \ \mathcal{A}_0 \geqslant \pi\rho_0^2 \\ \text{with } \rho_0 = 4m \ \ \& \ \ \mathcal{A}_0 = \pi\rho_0^2 \ \text{ iff } \ \underline{\Psi_{AB} = 0 \quad \partial_A\rho = 0} \end{cases}$

["my way". If $\partial_A\rho = 0$ & $\Psi_{AB} = 0$ then (1) reads. $\partial_V\left(\frac{\sqrt{b}\,K}{\sqrt{\rho}\,V}\right) = -\frac{2\sqrt{b}}{V}\,{}^{(2)}\Delta\sqrt{\rho}$

integrate over θ, φ, V: $\int \frac{\sqrt{b}\,K}{\sqrt{\rho}\,V} d\theta d\varphi = cte$; (since $\int d\theta d\varphi\, {}^{(2)}\Delta\sqrt{\rho} = 0$)

evaluate this constant at $V=0$ (event horizon) where $\rho = \rho_0$ & $K/V = \frac{1}{2}\rho_0\, {}^{(2)}R$

this yields $cte = \int_{V=0} d\theta d\varphi \sqrt{b}\sqrt{\rho_0}\,\frac{1}{2}\,{}^{(2)}R = 4\pi\sqrt{\rho_0}$ (because of Gauss-Bonnet)

evaluate now this const at $V=1$ (∞) where $\sqrt{b} \sim r^2\sin\theta$; $\rho \sim r^2/m$; $K \sim 2/r$

$$cte = \int_{V=1} d\theta d\varphi \frac{r^2 \sin\theta \sqrt{m}}{r}\frac{2}{r} = 8\pi\sqrt{m}.$$

hence $4\pi\sqrt{\rho_0} = 8\pi\sqrt{m} \iff \underline{\rho_0 = 4m}$

similarly eqn (2) must yield $\underline{\mathcal{A}_0 = \pi\rho_0^2}$]

- summary : Israel's theorem
  - if 4D spacetime is static
  - if it is asymptotically flat
  - if it has an event horizon
  - if $R_{ijkl}R^{ijkl}$ is regular on or outside event hor.

then this spacetime is spherically symmetric and coincides with Schwarzschild's spacetime. (NB: can be generalized in presence of electromagnetism; only sol is Reissner-Nordström)

(see Israel, 1987 (in 300 years of gravitation) for an account of his discovery and how he interpreted it as: "If it were permissible to consider the limiting external field of a gravitationally collapsing asymmetric (non rotating) body as static [then] only two alternatives would be open – either the body has to divest itself of all quadrupole & higher moments by some mechanism (perhaps gravitational radiation) or else an event horizon ceases to exist".) (NB: "modern proof" Masood-ul Alam, CQG (1992) 153 Lett.)

2. Uniqueness theorems for stationary BH (no proof here!)

- Stationary axisymmetric solutions of Einstein's equations for the vacuum, which have a smooth convex event-horizon, are asymptotically flat, and are non singular outside the horizon, are the Kerr solutions (specified by m & a). [Carter 1971; Robinson 1975].
- Extension to Einstein Maxwell eqns & Kerr-Newman solutions: Mazur (1982); Bunting (1983) (hence: the demo took 15 years).

- sketchy outline :

(1) the metric can be written as: $ds^2 = -V dt^2 + 2W d\phi\, dt + X d\phi^2 + U(\lambda,\mu)\left[\frac{d\lambda^2}{\lambda^2-c^2} + \frac{d\mu^2}{1-\mu^2}\right]$

(Carter 71) $\lambda = c$: event horizon

(2) Einstein-(Maxwell) eqns reduce to solving a system of two second order differential equations for two functions of λ & μ, (the "Ernst potentials") subject to some boundary conditions at ∞^{ty} and on the horizon (coming to the condition of asymptotic flatness & regularity on the horizon) (Carter 71).

Let $X(\lambda,\mu)$ & $Y(\lambda,\mu)$ be these 2 functions; and let $E(X,Y)=0$, $F(X,Y)=0$ be the 2 diff eqns they satisfy.

at $\lambda \to \infty$ $\quad Y \to 2J\mu(3-\mu^2)$

3) Consider two, a priori different, solution $(X_1 Y_1)$ $(X_2 Y_2)$ belonging to the same value of J. Robinson (75) (in the vacuum case) consider some integral $\left(R = \int_C^\infty d\lambda \int_{-1}^{+1} d\mu \left[\frac{X_1}{X_2} (Y_2 - Y_1) F(X_1, Y_1) + \text{etc} \right] \right)$ which is zero when the field eqn ($E = 0$, $F = 0$) are satisfied. By a "tour de force" he transforms this integral in such a way that the integrand is a sum of eight (!) positive definite expressions which, therefore, must vanish separately and imply that $X_2 = X_1$ $Y_2 = Y_1$ (see Chandra's book p 292 et seq.) Since the Kerr expressions for X_1 & Y_1 solve the field eqn, Kerr's solution is the only one.

(NB. Carter had showed the same when (X_2, Y_2) is ∞^{ly} close to (X_1, Y_1): $X_2 = X_1 + \delta X$; $Y_2 = Y_1 + \delta Y$)

→ The extension of the proof to the charged case by Mazur explained the "miraculous" Robinson identity (see eg Heusler Living Reviews 98
Carter hep-th/0411259)

→ remarks (a) There is NO proof that the Kerr-de Sitter metric (Carter, 1968) is the unique stationary, axisymmetric, asymptotically de Sitter solution of $G_{ij} = \Lambda g_{ij}$.

(b) The proofs specifically assume that the event horizon is non-degenerate (that is $m^2 \neq a^2$).

(I stop here)

Black hole perturbations & superradiance

Introductory remarks:

- For astrophysicists a Black Hole often reduces to a compact object with no surface for matter to hit on. The motion of stars, gas, accretion disks around the BH is described in Newtonian terms. The observed (electromagnetic) energy coming from this BH is usually supposed to originate from the gravitational binding energy of the material close to the hole $\left(W_{grav} = O\left(\frac{GM^2}{r}\right)\right)$.

slide
slide

- Here we shall rather try to see how one can extract energy from the BH itself. This should be possible since the mass of the BH can be decomposed as: $m^2 = m_{irr}^2 + \frac{J^2}{4 m_{irr}^2}$ where $J = ma$ (Kerr Black hole)

  and $(m - m_{irr})$ can in principle be extracted from the hole (Christodoulou Ruffini 1971) (see lecture 3).

  (if $a = m$ $\quad \frac{m - m_{irr}}{m} = 29\%$)

- We know also of a "gedanken" experiment to extract energy: the Penrose process (1969): a particle coming in from ∞^{ty} with a positive energy splits into 2 particles in the ergoregion of the BH, one with a negative energy (as measured from ∞^{ty}) falling into the BH, the other being expelled to infinity with an energy greater than the incident one

- We will study in (some) detail 2 ways of implementing the Penrose process: superradiant scattering & the Blandford-Znajek mechanism.

  (not efficient in astrophysical processes)

  (debatable efficiency)

(II) Black hole perturbations & their linear stability

(or "how do collapsing object loose their "hair" before becoming BH".)

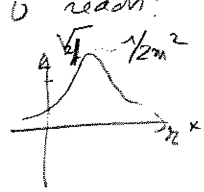
1. A toy model: scalar field in Schwarzschild geometry. (Price 1971, Thorne 1972)

(see MTW p 868 et sq ; Frolov-Novikov p 89).

- Consider a collapsing star with some "scalar charge" which produces a scalar field Φ; consider this scalar field as a perturbation, that is Φ solves $\Box\Phi = 0$ on the exterior, Schwarzschild background - $\left(ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-2m/r} + r^2 d\Omega_2^2\right)$.

- Decompose Φ as $\Phi = \sum_\ell \frac{1}{r}\Psi_\ell(t,r)\,Y_{\ell m}(\theta,\varphi)$ and find that $\Box\Phi = 0$ reads:

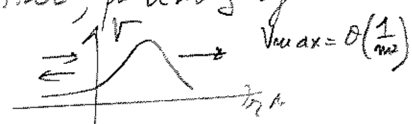
$$-\partial^2_{tt}\Psi_\ell + \partial^2_{r^*r^*}\Psi_\ell = V_{eff}\,\Psi \quad \text{where} \quad \begin{cases} r_* = r + 2m\ln(r/2m - 1) \\ V_{eff} = \left(1-\frac{2m}{r}\right)\left(\frac{2m}{r^3} + \frac{\ell(\ell+1)}{r^2}\right) \end{cases}$$



Fourier decomposition: $\Psi_\ell(t,r^*) = \int_{-\infty}^{+\infty} A(k)\,R^\ell_k(r^*)\,e^{-ikt}\,dk$; $\frac{d^2R^\ell_k}{dr_*^2} = \left(-k^2 + V_{eff}(r^*)\right)R^\ell_k(r^*)$

- (Boundary conditions (for example)): scalar waves emitted near collapsing star, partially reflected & partially transmitted; hence

$$\begin{cases} R^\ell_k = e^{ikr_*} + \Gamma_k e^{-ikr_*} & \text{as } r_* \longrightarrow -\infty \\ R^\ell_k = T_k e^{+ikr_*} & \text{as } r_* \longrightarrow +\infty \end{cases}$$



$|T|^2 + |R|^2 = 1$ (Wronskian)

- Now long wavelength modes (such that $km \ll 1$) are completely reflected by the potential barrier [as is intuitively clear ; short wavelength modes, on the other hand do not "feel" the barrier].

- Argue now that as the star approaches its gravitational radius ($r = 2m$) the scalar waves emitted by the surface appear more & more redshifted (when measured in time t) so that they can be regarded as outgoing waves with infinite wavelength and are completely reflected by the barrier. All the multipoles of the scalar field hence "die out" at r^* finite as $t \to \infty$: the star has lost its "scalar hair".

2. gravitational perturbations of a Schwarzschild BH

- perturbed metric. $ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - \omega dt - q_2 dr - q_3 d\theta)^2 + e^{2\mu_2}dr^2 + e^{2\mu_3}d\theta^2$

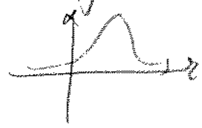
  Schwarzschild background: $e^{2\nu} = e^{-2\mu_2} = 1 - \frac{2m}{r}$; $e^{\mu_3} = r$; $e^{\psi} = r\sin\theta$; $q_1 = q_2 = q_3 = 0$

  "axial" perturbations: ω, q_2, q_3 (induce rotation) ("odd" parity) ; "polar" perturbations: $\delta\nu, \delta\psi, \delta\mu_2, \delta\mu_3$ (do not induce rotation) ("even" parity)

- (one can be shown it will: $10 - 4 = 6$).

- Writing down the perturbed Einstein equations it turns out (see Chandra 1983) that all axial perturb. satisfy the Regge-Wheeler (1957) equation:-

$$\left(\frac{d^2}{dr_*^2} + k^2\right) Z^- = V^- Z^- \quad ; \quad V^- = \left(1 - \frac{2m}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} - \frac{6m}{r^3}\right)$$



$$\begin{cases} \text{if } Q(t,r,\theta) \equiv r^2\left(1 - \frac{2m}{r}\right)(\partial_\theta q_2 - \partial_r q_3)\sin^3\theta & \text{(all time dependence: } e^{ikt}) \\ \qquad = e^{ikt} Q(r,\theta), \quad Q(r,\theta) = Q(r)\, C_{\ell+2}^{-3/2}(\theta) & \uparrow \text{Gegenbauer polynomial} \\ Q(r) = r Z^- \end{cases}$$

(in short: the function $Z^-(r)$ is related to the perturbations ω, q_2 & q_3).

As for the polar perturbation they satisfy the Zerilli (1970) eqn:

$$\left(\frac{d^2}{dr_*^2} + k^2\right) Z^+ = V^+ Z^+ \quad ; \quad \begin{cases} V^+ = 2\left(1 - \frac{2m}{r}\right) \dfrac{n^2(n+1)r^3 + 3n^2 m r^2 + 9nm^2 r + 9m^3}{r^3(nr + 3m)^2} \\ n = \frac{1}{2}(\ell-1)(\ell+2) \end{cases}$$

and Z^+ connected to $\delta\nu, \delta\psi, \delta\mu_2, \delta\mu_3$ (see Chandra's book for details).

These equations have remarkable properties (in particular the reflexion & transmission coefficients are related) (in fact they are equal).

- Stability of the Schwarzschild BH (original motivation of Regge-Wheeler)

  the eqns for the perturbations are of the type. $\frac{\partial^2 Z}{\partial t^2} = \frac{\partial^2 Z}{\partial r_*^2} - VZ$

  Multiply it by $\frac{\partial Z^*}{\partial t}$ and obtain, after integration by part:

$$\int_{-\infty}^{+\infty}\left(\frac{\partial Z^*}{\partial t}\frac{\partial^2 Z}{\partial t^2} + \frac{\partial Z}{\partial r_*}\frac{\partial^2 Z^*}{\partial t\,\partial r_*} + VZ\frac{\partial Z^*}{\partial t}\right)dr^* = 0$$

. Add the complex conjug.

(Detweiler-Ipser 1973)

$$\Rightarrow \int_{-\infty}^{+\infty}\left(\left|\frac{\partial Z}{\partial t}\right|^2 + \left|\frac{\partial Z}{\partial r_*}\right|^2 + V|Z|^2\right)dr^* = \text{Const}$$

<u>Now</u> V is <u>positive definite</u>, hence $|\partial Z/\partial t|^2$ is bounded $\Longrightarrow$ no catastrophic growth as t increases. More rigourous proof: Kay & Wald (1987).

3. Gravitational perturbations of Kerr BH

⊖ Chandra 1983: "The subject is one of considerable complexity; and, in spite of the length of this chapter (100p), the account, in large parts, is hardly more than an outline" [and: "In the event that some reader may wish to undertake a careful scrutiny of the entire development, the author's derivations (in some 600 legal-size pages and in 6 additional notebooks) have been deposited in the JR library of the U. of Chicago"].

⊖ pb. of the separability of the perturbation eqns – Carter (1968) had shown that the geodesic eqn could be separated (see above). Brill et al. 1972 manage to separate the scalar field eqn $\Box\Phi = 0$. For higher spin (that is, electromagnetic waves + metric perturbations) the breakthrough was due to Teukolsky 1973 who took a different approach based on the Newman-Penrose (1962) formalism

NP in a nutshell: consider Kerr metric in Boyer-Lindquist coordinates
Introduce the "Kinnersley tetrad" $\begin{cases} \ell^\mu = \frac{1}{\Delta}(r^2+a^2, \Delta, 0, a) ; \ n^\mu = \frac{1}{2\Sigma^2}(r^2+a^2, -\Delta, 0, a) \\ m^\mu = \frac{1}{\sqrt{2}(r+ia\cos\theta)}(ia\sin\theta, 0, 1, \frac{i}{\sin\theta}) \end{cases}$

and write Kerr metric as $g_{\mu\nu} = -2(\ell_{(\mu}n_{\nu)} - m_{(\mu}\bar{m}_{\nu)})$.
Then the Christoffel symbols are expressed in terms of 12 "spin coefficients"
$(\text{ex}: \ -\kappa \equiv \Gamma_{200} = \ell_{\mu;\nu} m^\mu \ell^\nu \ ; \ -\rho = \Gamma_{203} = \ell_{\mu;\nu} m^\mu \bar{m}^\nu \ \text{etc})$
Then one introduces the required nb of quantities related to the Ricci or Weyl tensor components $(\text{ex}: \ \Phi_{00} = \frac{1}{2}R_{00}; \ \Psi_2 \equiv C_{\alpha\beta\mu\nu}\ell^\alpha n^\beta(\ell^\mu n^\nu - m^\mu \bar{m}^\nu))$
Finally one relates these $\Phi_{00} \dots \Psi_2$ to derivatives of the spin coefficients.
$(\text{ex}: \ \underbrace{D}_{=\ell^\mu\nabla_\mu}\rho - \underbrace{\bar{\delta}}_{=\bar{m}^\mu\nabla_\mu}\kappa = \rho^2 + \text{terms quadratic in spin coeff} + \Phi_{00} \ \text{etc})$

Anyway Teukolsky managed the tour de force of obtaining a master equation for all perturbations:

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\phi^2} - \text{etc} = 0$$

(see eg Frolov-Novikov p 128)

where ψ stands for Φ if eqn is $\Box\Phi = 0$; $\psi = \Psi_0$ or Ψ_4 for grav. perturb.

⊖ This eqn can be separated: $\psi = R(r)\,S(\theta)\,e^{-ikt}\,e^{im\phi}$

- $S(\theta)$ satisfies an ordinary second order differential equation; $S(\theta)$ are the "spin weighted spheroidal harmonics": $\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + [f_s(k,m) + E]\,S = 0$
  separation constant found numerically
  $E = \ell(\ell+1) - \frac{2ma\omega s^2}{\ell(\ell+1)} + O(a\omega)^2$,

- radial equation: $\left[\frac{d^2}{dr_*^2} + \underbrace{G(r_*,E)}_{\text{complicated function of } r_*}\right]\chi = 0$; $\frac{d}{dr_*} = \frac{\Delta}{r^2+a^2}\frac{d}{dr}$, $\chi = \sqrt{r^2+a^2}\,\Delta^{s/2} R$

asymptotic behaviours: $\begin{cases} R \sim \frac{e^{i\omega r_*}}{r^{2s+1}} \text{ or } \frac{e^{-i\omega r_*}}{r} & \text{as } r \to \infty \\ R \sim e^{i\bar\omega r_*} \text{ or } \Delta^{-s} e^{-i\bar\omega r_*} & \text{as } r \to r_+ \end{cases}$ $\begin{cases} \bar\omega = \omega - m\Omega_H \\ \Omega_H = \frac{a}{2Mr_+} \end{cases}$

- finally χ, that is R, that is Ψ can be related to Φ, or metric perturbations (Wald 1978)

4. Superradiant scattering (Zel'dovich 1970, Misner 1972)

Consider a perturbation propagating on a Kerr background ($\Box\phi = 0$, electromagnetic waves or metric perturbations). Its evolution is given by Teukolsky's eqn above.

In the case of a scalar field, the radial eqn is $\left(\frac{d^2}{dr_*^2} + G\right)\chi = 0$ (see above)

Choose the asymptotic behaviours $\begin{cases} \chi \sim e^{-i\bar\omega r_*} & \text{as } r \to r_+ \\ \chi \sim A_{out} e^{i\omega r_*} + A_{in} e^{-i\omega r_*} & \text{as } r \to \infty \end{cases}$: (infalling wave in the locally dragged frame)

Wronskian theorem: $\left(1 - \frac{m\Omega_H}{\omega}\right)|T|^2 = 1 - |R|^2$ $\left(T = \frac{1}{A_{in}} ;\ R = \frac{A_{out}}{A_{in}}\right)$

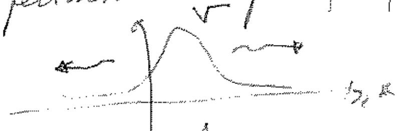
hence if $\omega < m\Omega_H = \frac{ma}{r_+^2+a^2}$ then $|R|^2 > 1$: superradiance

numerical integration (to compute $|T|^2$) yield an amplification of:
$\begin{cases} 0.3\% & \text{for scalar waves} \\ 4.4\% & \text{for electromagnetic waves} \\ 138\% & \text{for GW} \end{cases}$ (Press Teukolsky etc 1972 onwards)

FOR complete study (post Chandra) see Sasaki & Tagoshi Living Review 2003 and Sasaki IHP06 lecture Notes.

5. More on BH perturbations

- "quasi normal modes": solve the perturbation eqn for purely outgoing waves: $\left(\frac{d^2}{dr_*^2} + \omega^2 - V\right)\chi = 0$



ω must be complex ; finding the resonant modes is an ongoing enterprise
(Leaver 1988. to eg. Shahar Hod arXiv:0811.3806 [gr-qc]
and Living Review Kokkotas & Schmidt 1999 and Sasaki lect notes IHP06.

- stability of Kerr Black hole. NO definite proof as yet because of superradiant modes ; for scalar mode Teukolski eqn is:

$$\frac{\partial}{\partial r}\left[\Delta\frac{\partial\Phi}{\partial r}\right] + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\Phi}{\partial\theta}\right] + \underbrace{\left[\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right]}_{\text{not definite positive}}\frac{\partial^2\Phi}{\partial\varphi^2} + \dots = 0$$

hence one cannot straightforwardly obtain an "energy integral" as in the Schwarzschild case. However Whiting (1989) managed to exhibit one (in that case).

B l a c k H o l e s
in General Relativity

Nathalie Deruelle

Lecture Four

Astrophysics of Black Holes

CEA, January-February 2009

Outline of the Lecture

I. Observational evidence

1. Stellar mass black holes
2. Supermassive black holes

II. Extracting energy from black holes

1. BH perturbations and superradiance
2. The Blandford-Znajek mechanism
3. Remarks on the "membrane paradigm"

III. Black hole binary coalescence

1. Likehood and observability
2. Extreme mass ratio inspirals (EMRI)
3. Post-newtonian calculations
4. Quasi-normal modes
5. Effective one-body approach
6. Numerical breakthroughs

I.

Observational Evidence for the existence of Black Holes

Box 1. Neutron Star Maximum Mass

Ref : Eric Gourgoulhon Lecture Notes
Philippe Grandclément Lecture Notes. (LUTH website at obspm)

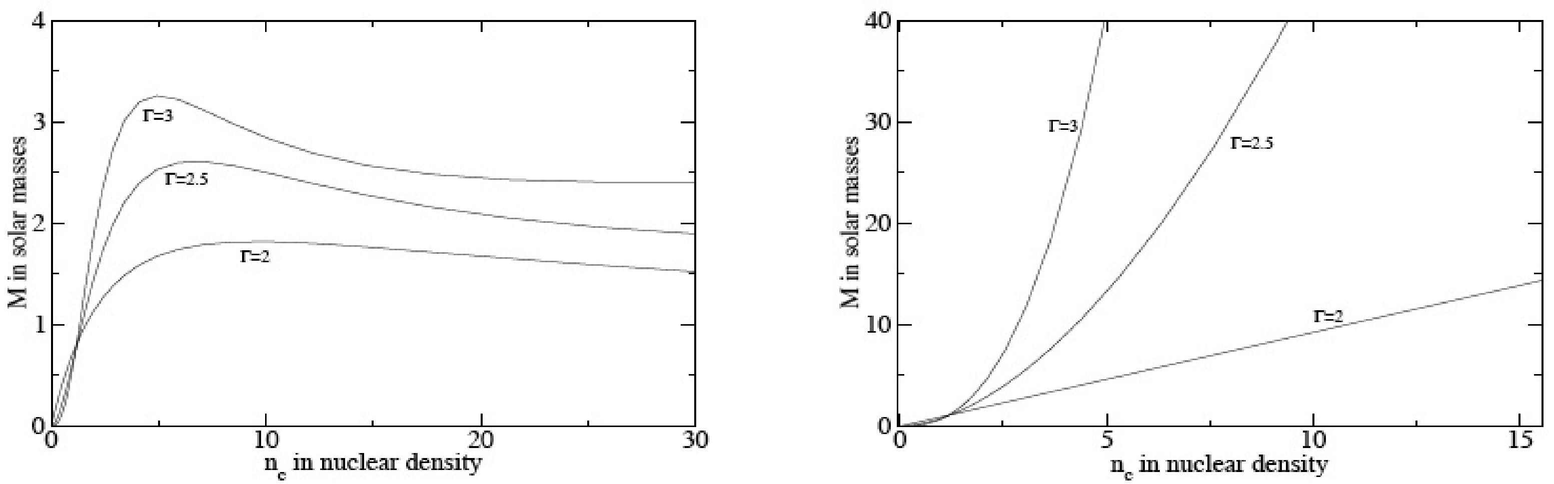


FIG. 5.1 – Masse gravitationnelle en fonction de la densité centrale, pour une équation polytropique. A gauche les configurations relativistes admettent un maximum, ce qui n'est pas le cas pour leur équivalent newtonien.

Box 1. Neutron Star Interior

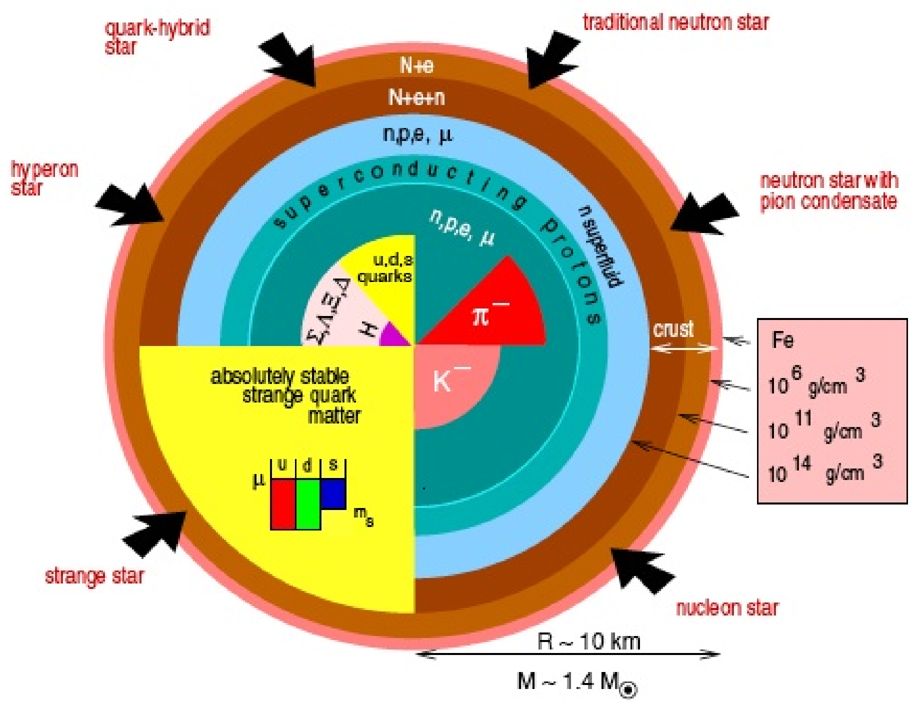


FIG. 5.4 – Différents modèles d'intérieurs d'étoiles à neutrons [d'après Weber (2001)].

Box 1. Neutron EOS

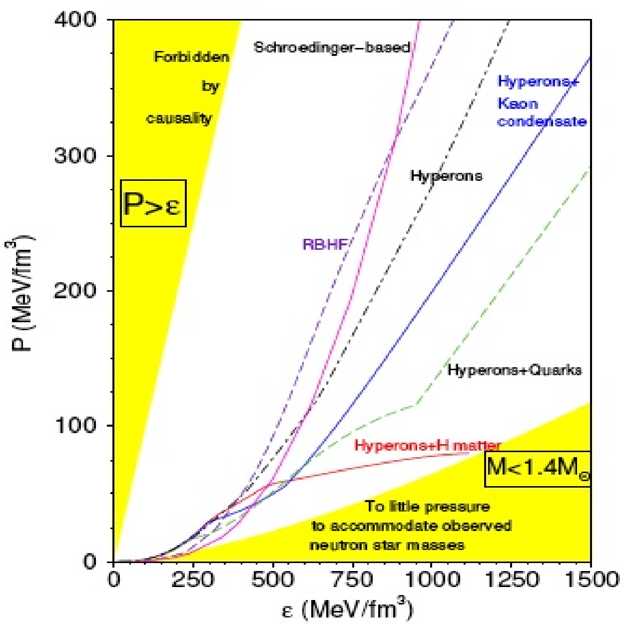


FIG. 5.5 – Différentes équations d'état $p = p(\varepsilon)$ de la matière à haute densité. ε représente la densité d'énergie totale, notée ρc^2 dans le texte. La densité nucléaire correspond à $\varepsilon \sim 140$ MeV fm^{-3} [d'après Weber (2001)].

Box 1. NS maximum mass : Influence of rotation

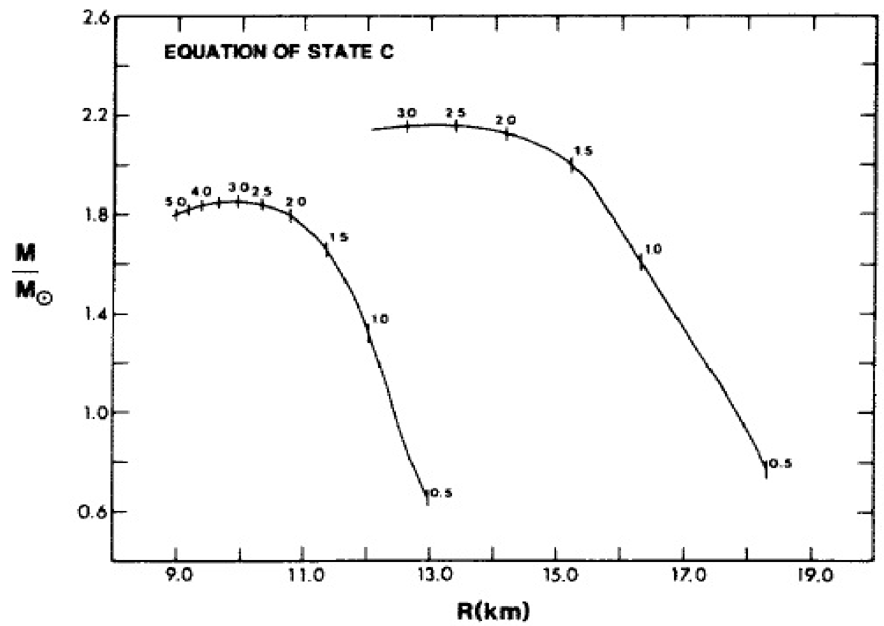


FIG. 5.9 – Relation entre la masse-énergie totale M et le rayon équatorial R pour l'équation d'état I de Bethe & Johnson (1974). La courbe la plus à gauche correspond au cas sans rotation (symétrie sphérique), celle la plus à droite au cas de la rotation maximale $\Omega = \Omega_K$. Sur chaque courbe, on a figuré la valeur de la densité centrale en unités de 10^{18} kg m^{-3}. [D'après Friedman et al. (1986)].

Box 2. Stellar Mass black holes

Riccardo Giacconi

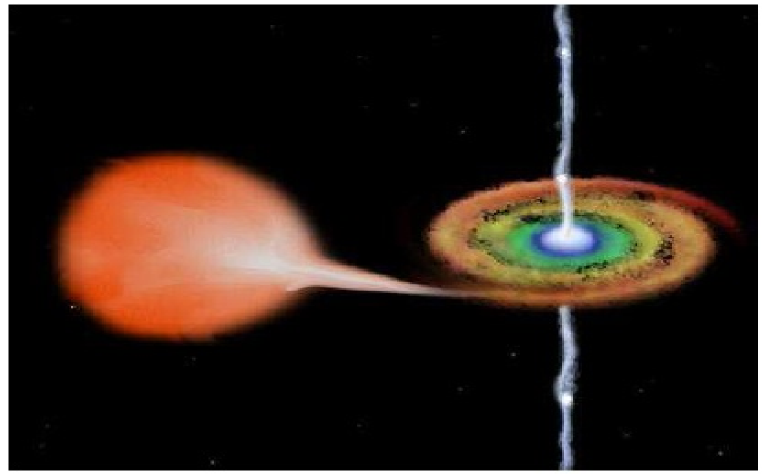


FIG. 6.4 – Vue d'artiste d'une binaire X de faible masse.

Box 2. Stellar black hole candidates

source X	autre nom	année publi.	P	type comp.	$f(M_1, M_2, i)$ $[M_\odot]$	M_2 $[M_\odot]$	M_1 $[M_\odot]$	Réf.
Cyg X-1		1972	5.6 j	O9I	0.25 ± 0.01	33	$7 - 20$	[1]
LMC X-3		1983	1.7 j	B3V	2.3 ± 0.3	6	$7 - 14$	[2]
LMC X-1		1987	4.2 j	O7III	0.14 ± 0.5	6	$4 - 10$	[2a]
A 0620-00	XN Mon 75	1986	7.8 h	K5V	2.91 ± 0.08	0.6	10 ± 5	[3]
GS 2023+338	V404 Cyg	1992	6.5 j	K0IV	6.08 ± 0.06	0.6	12 ± 2	[4]
GRS 1124-683	XN Mus 91	1992	10.4 h	K3/K5V	3.01 ± 0.15	0.8	6^{+5}_{-2}	[5]
GRO J1655-40	XN Sco 94	1995	2.6 j	F6IV	2.73 ± 0.09	$1.7 - 3.3$	$5.5 - 7.9$	[6a,b]
GS 2000+250	XN Vul 88	1995	8.3 h	K5V	5.01 ± 0.12	0.5	10 ± 0.4	[7a,b]
GRO J0422+32	XN Per 92	1995	5.1 h	M2V	1.21 ± 0.06	0.3	10 ± 5	[8]
H 1705-250	XN Oph 77	1996	12.5 h	K3/K7V	4.86 ± 0.13	0.3	6 ± 2	[9]
4U 1543-47	IL Lup	1998	27.0 h	A2V	0.22 ± 0.02	2.5	5.0 ± 2.5	[10]
GRS 1009-45	XN Vel 93	1999	6.8 h	K7/M0V	3.17 ± 0.12		$6 - 8$	[10a]
XTE J1859+226	V406 Vul	2001	9.2 h		7.4 ± 1.1		10 ± 3	[11]
XTE J1550-564	V381 Nor	2001	37.0 h	G8/K0IV	6.86 ± 0.71		> 7.4	[12]
SAX J1819.3-2525	V4641 Sgr	2001	2.8 j	B9III	2.74 ± 0.12		10 ± 1.5	[13]
XTE J1118+480	KV UMa	2001	4.1 h	K7/M0V	6.1 ± 0.3	$0.09 - 0.5$	$6.0 - 7.7$	[14]
GRS 1915+105	V1487 Aql	2001	33.5 j	K/MIII	9.5 ± 3.0	1.2 ± 0.2	14 ± 4	[15]
GX 339-4	V821 Ara	2003	42.1 h		5.8 ± 0.5			[16]

TAB. 7.1 – Trous noirs dans les binaires X. Les deux premières lignes correspondent à des HMXB, les suivantes à des LMXB. La colonne "année publi." donne l'année de la première publication qui annonce la détermination de M_1. Références : [1] Gies et al. 1982 ; [2] Cowley et al. 1983 ; [2a] Cowley et al. 1995 ; [3,4,5,9] Charles 1999 ; [6a] Bailyn et al. 1995 ; [6b] Shahbaz et al. 1999 ; [7a] Casares et al. 1995 ; [7b] Filippenko et al. 1995a ; [8] Filippenko et al. 1995b ; [10] Orosz et al. 1998 ; [10a] Filippenko et al. 1999 ; [14] Wagner et al. 2001., McClintock et al. 2003 ; [15] Greiner et al. 2001., Harlaftis & Grenier 2004 ; [16] Hynes et al. 2003

Box 2. Evidence for “horizons”

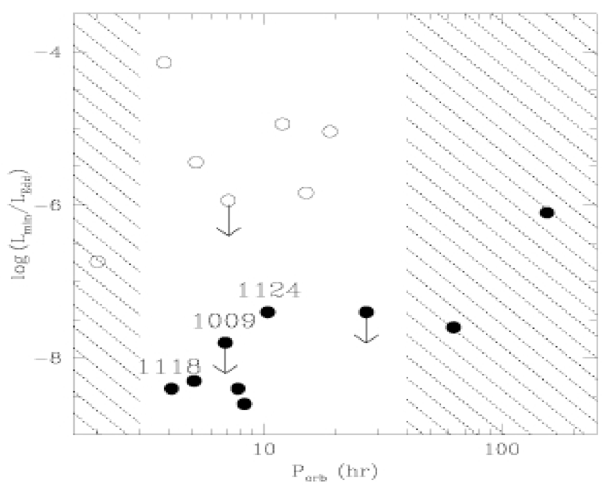


FIG. 7.6 – Luminosité (rapportée à la luminosité d'Eddington) dans la période de quiescence des novæ X, en fonction de leur période orbitale. Les cercles vides correspondent à des novæ X qui contiennent une étoile à neutrons et les pleins à celles qui contiennent un trou noir : de la gauche vers la droite, il s'agit respectivement de GRO J0422+32, XTE J1118+480, GRS 1009-45, A 0620+00, GS 2000+25, GRS 1124-683, 4U 1543-47, GRO J1655-40 et V404 Cyg (cf. Table 7.1). Les zones hachurées sont des domaines où la comparaison entre binaires à trou noir et à étoile à neutrons n'est pas possible, faute d'éléments [d'après McClintock et al. (2003)].

(Taken from Eric Gourgoulhon's Lecture Notes on Compact Objects)

Box 2. Gamma Ray Bursts Fireball Model

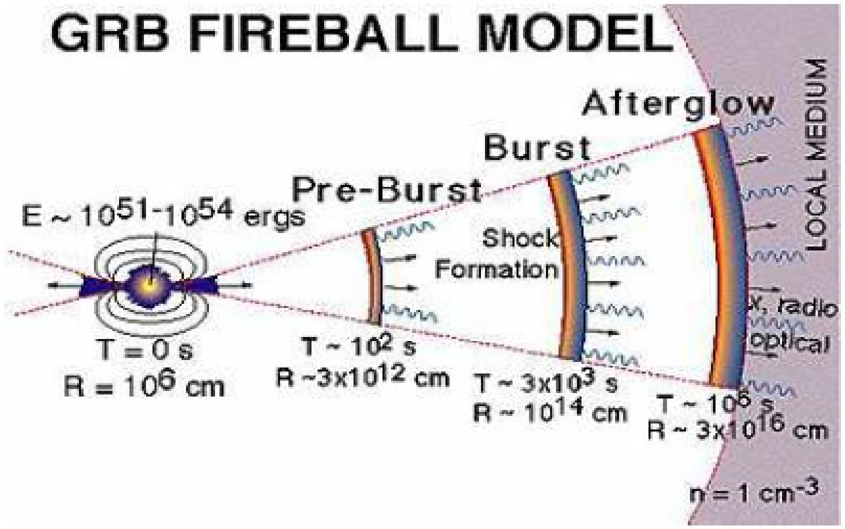


FIG. 4.26 – Différentes étapes d'un sursaut γ, dans le modèle standard de la boule de feu.

(see e.g. Tsvi Piran ArXiv: 0804.2074 [astro-ph]; and P. Grandclément Lecture Notes on compact objects)

Box 2. Gamma Ray Bursts “Inner Engine”

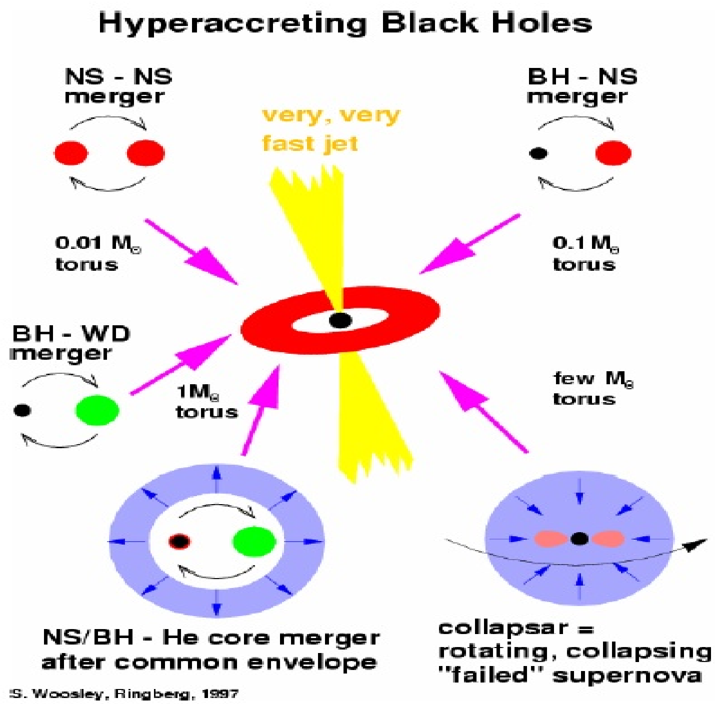
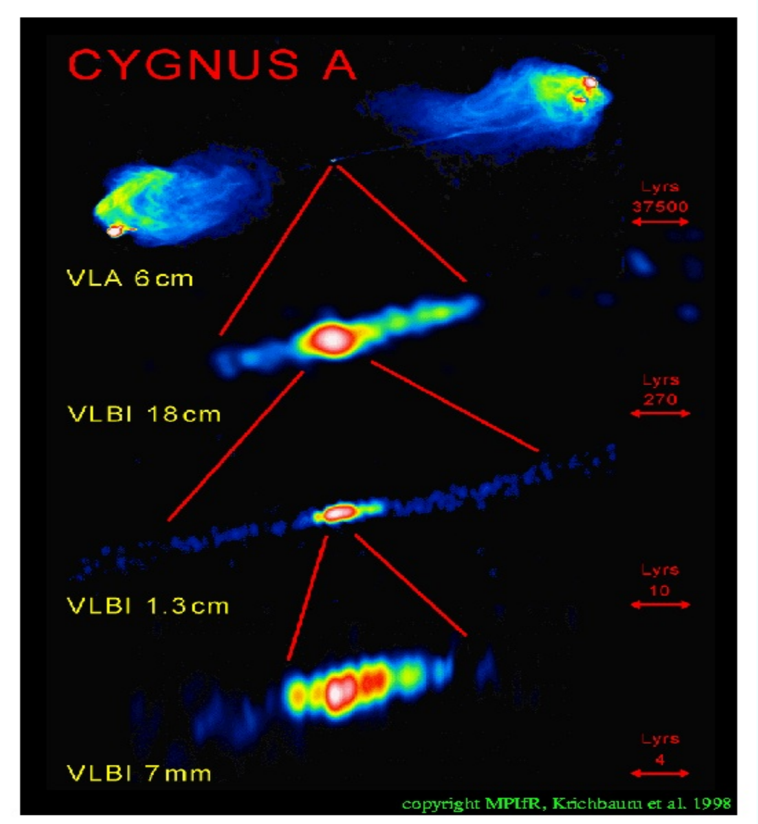


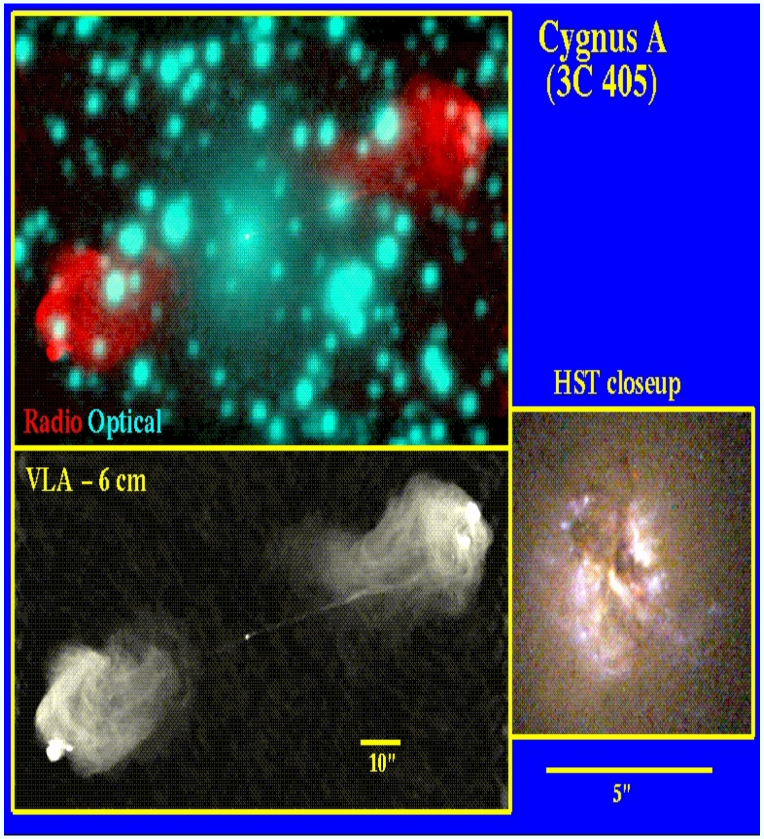
FIG. 4.37 – Différents scenarii permettant la création d’un disque d’accrétion autour d’un trou noir stellaire.

(for “alternative” model see Remo Ruffini et al., arXiv:0706.2572[astro-ph])

Box 3. Super Massive Black Holes

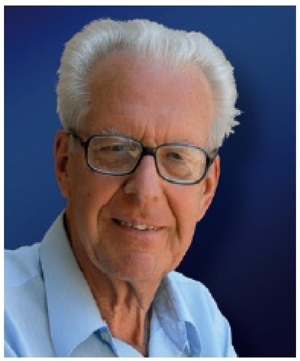


The example of Cygnus A



Box 3. Two “Fathers” of SMBH

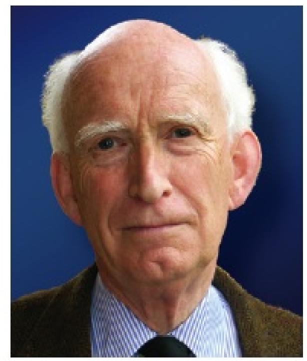
THE KAVLI PRIZE IN ASTROPHYSICS
is awarded to Maarten Schmidt



Maarten Schmidt gained his PhD under the famous Dutch radio astronomy pioneer Jan H. Oort. He emigrated to the US and joined the California Institute of Technology in 1959. Initially he continued earlier work on the mass distribution and dynamics of the galaxy, before taking over a project obtaining spectra of objects found to be radio wave emitters.

Schmidt has worked on quasars ever since demonstrating in 1963 that the peculiarities of the visible light spectrum of particularly bright quasar 3C273 were caused by a massive red shift. Since then he has studied the evolution and distribution of quasars, discovering they were more abundant in the early Universe. This finding contributed to the decline of the Steady State theory, which previously competed with the Big Bang theory as a model for the origin of the Universe.

THE KAVLI PRIZE IN ASTROPHYSICS
is awarded to Donald Lynden-Bell

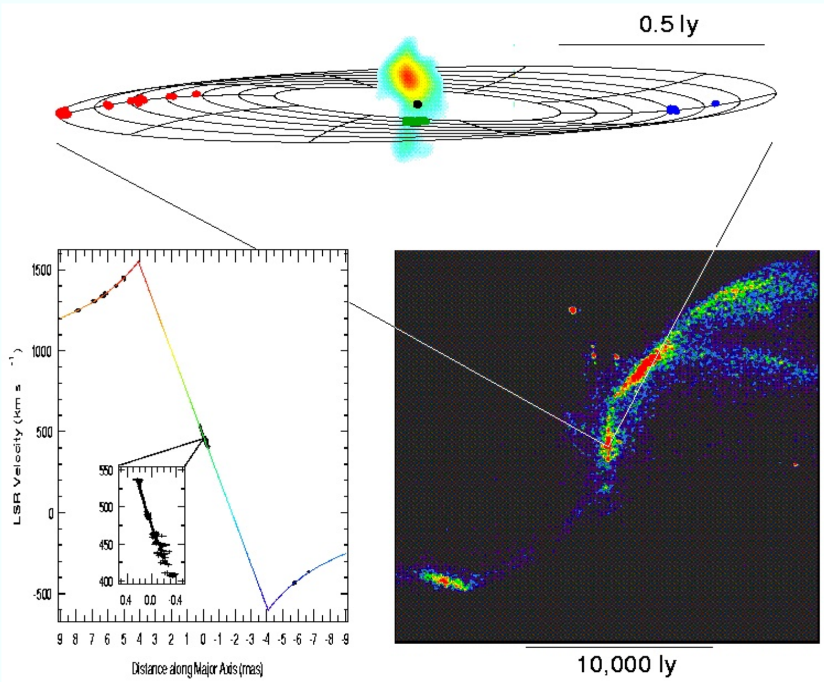


Donald Lynden-Bell studied astronomy at the University of Cambridge in the UK. After periods at the California Institute of Technology and the Royal Greenwich Observatory, he returned in 1972 to become Professor of Astrophysics and the first Director of the Institute of Astronomy.

He is best known in the field for work on the motion of stars, the formation of the galaxy, spiral structures and chemical evolution of galaxies, and the distributions and motions of galaxies and quasars. His 1962 paper, published with Olin Eggen and Allan Sandage, argued that our galaxy originated from the collapse of a single large gas cloud. It stimulated huge interest and further research in the area. In 1969 Lynden-Bell proposed that quasars are able to generate the vast quantities of energy that make them visible thousands of millions of light years away thanks to the presence of black holes at their centres. He argued their extreme luminosity arose from frictional heating in rapidly rotating gaseous disks rotating around black holes.

Maarten Schmidt and Donald Lynden-Bell, Kavli prize 2008

Box 3. Observational evidence for SMBH



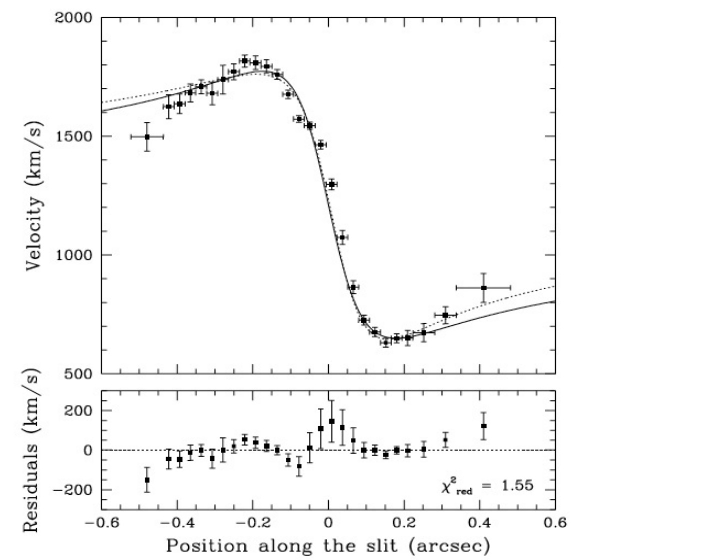
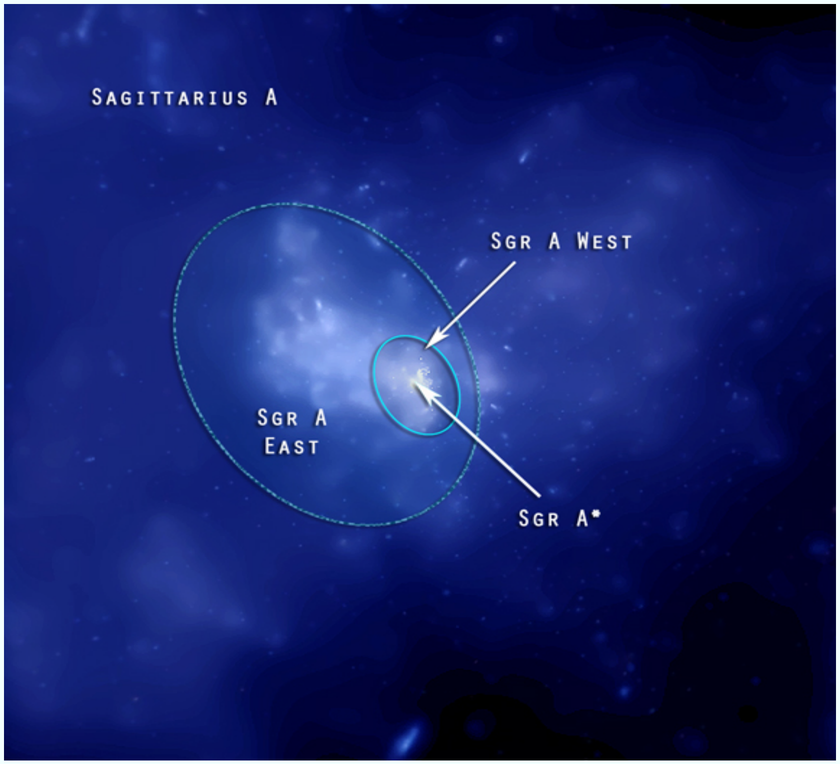
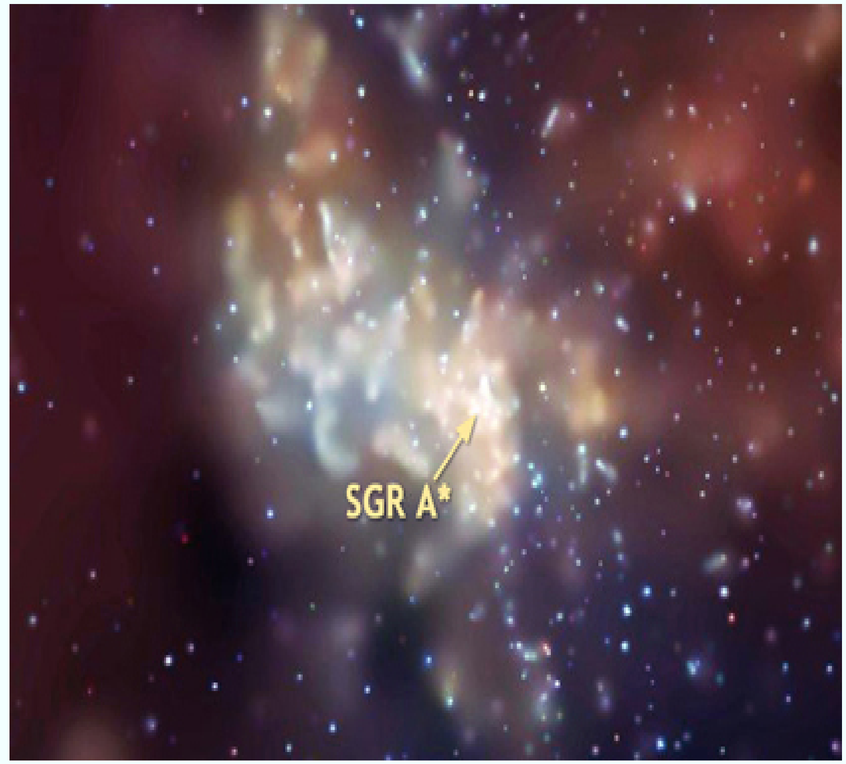


FIG. 6.14 – Mesure de la courbe de rotation du disque de M87, par le HST. La courbe du bas montre les résidus entre les données et le meilleur modèle qui indique une masse de $M = 3\,10^9\,M_\odot$ pour le candidat trou noir.

Water Maser (VLBI) in NGC4258 versus disk motion (HST) in M87

Box 3. A SMBH in the Milky Way





Sagittarius A* (at 26 000 light-years from us) $M = 4 \times 10^6 M\odot$

Box 3. Stars orbiting around SgrA*

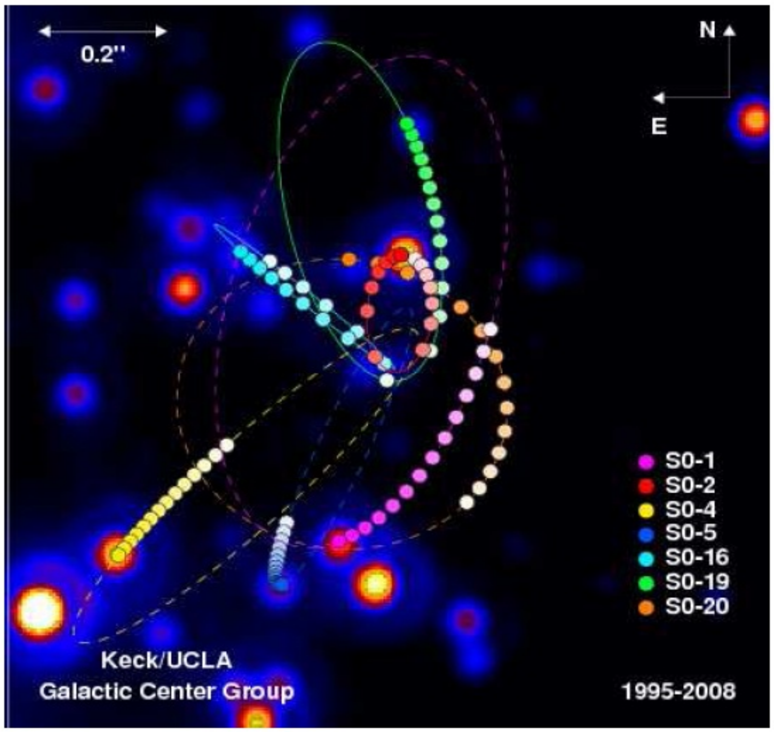


Fig. 7. Stars within the 0.02 parsecs of the Galactic center orbiting an unseen mass. Yearly positions of seven stars are indicated with filled colored circles. Both curved paths and accelerations (note the non-uniform spacings between yearly points) are evident. Partial and complete elliptical orbital fits for these stars are indicated with lines. All orbital fits require the same central mass of $\approx 4 \times 10^6$ $M_\odot$ and a common focus at the center of the image, the position of the radio source Sgr A*. (Image courtesy A. Ghez.)

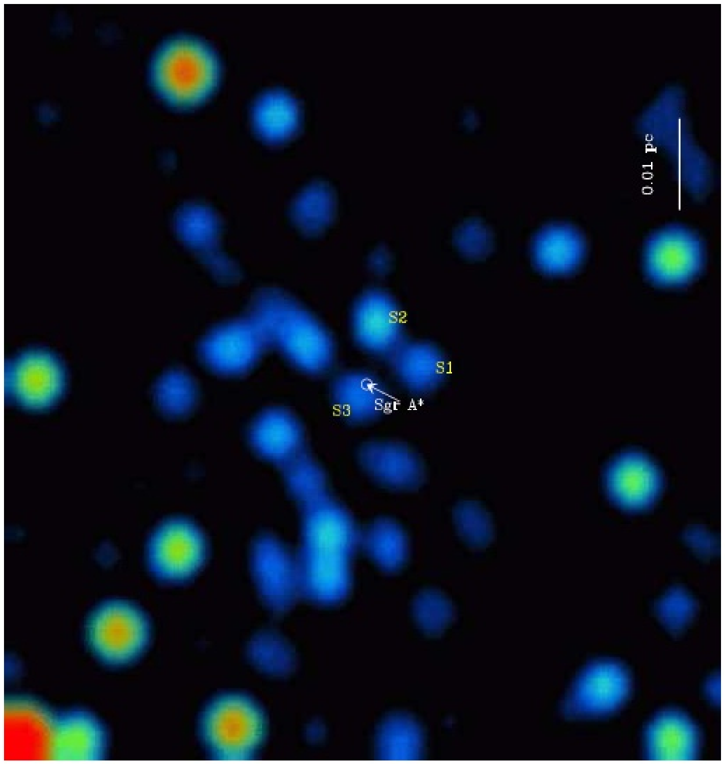


Fig. 9. An infrared frame from July 1995 covering the inner ±1 arcsecond from Reid et al. (2003). Three of the stars that orbit the Galactic center, S1, S2, and S3, are labeled, as is the position of Sgr A*. The image of star S3 overlaps slightly with Sgr A*, whose emission is much weaker than a single star. (Reproduced by permission of AAS.)

Box 3. An “overwhelming” evidence

The major observational results that provide overwhelming evidence that Sgr A* is a SMBH are as follows:

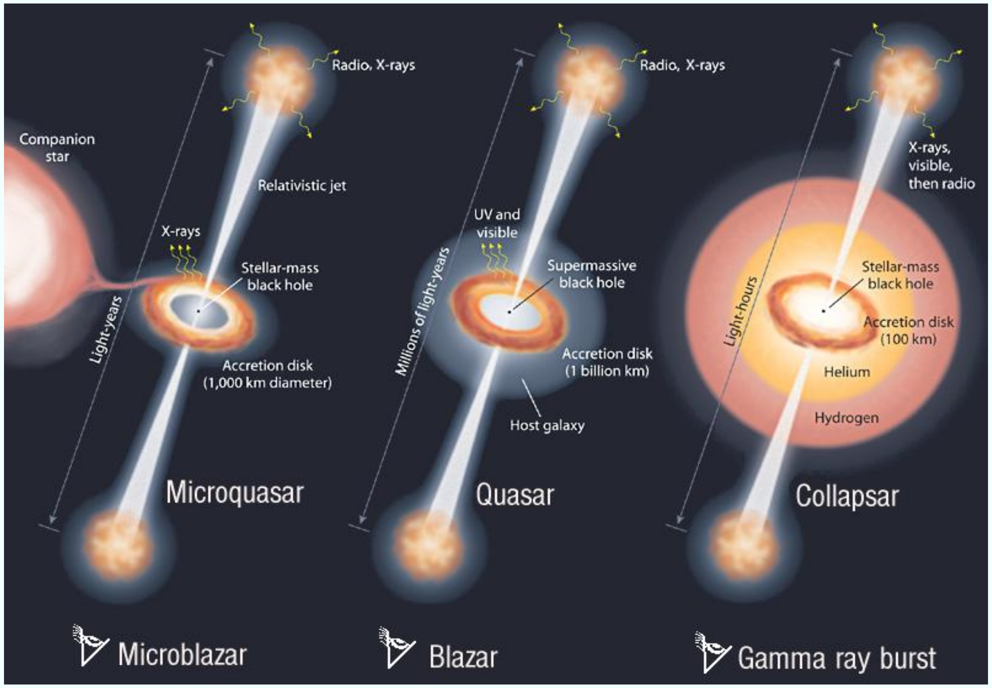
- Stars near Sgr A* move on elliptical orbits with a common focal position.
- The required central mass is 4×10^6 $M_\odot$ within a radius of 100 AU.
- The position of Sgr A* agrees with the orbital focus to within measurement uncertainty of ± 80 AU.
- The infrared emission from Sgr A* is far less luminous than a single star.
- The intrinsic size of Sgr A* at mm-wavelengths is $< 6R_{\rm Sch}$.
- Sgr A* is intrinsically motionless at the km s^{-1} level at the dynamical center of the Galaxy.

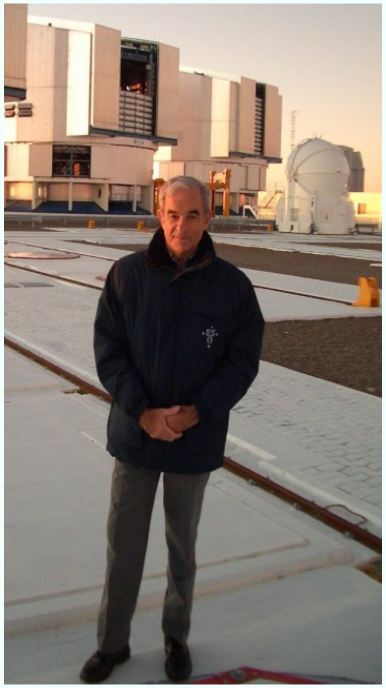
(from Mark J. Reid, ArXiv:0808.2624[astro-ph])

And also : “Milky Way’s Giant Black Hole Awoke from Slumber 300 Years Ago” , Koyama, Inui et al., arXiv:0711.2853 [astro-ph]

II.
Extracting Energy from Black Holes

(...as opposed to converting gravitational binding energy)





(taken from Felix Mirabel, CEA and ESO, Chile)

1. BH perturbations and superradiance

See Notes from Lecture 3; and : “Black Hole Physics” Frolov-Novikov (1998); Misao Sasaki IHP06 Lecture Notes and Living Review (2003)

2. The Blandford-Znajek mechanism

See e.g. Komissarov, ArXiv:0804.1912[astro-ph]

3. Remarks on the “membrane paradigm”

See Kip Thorne et al.’s book, Yale University Press, 1986

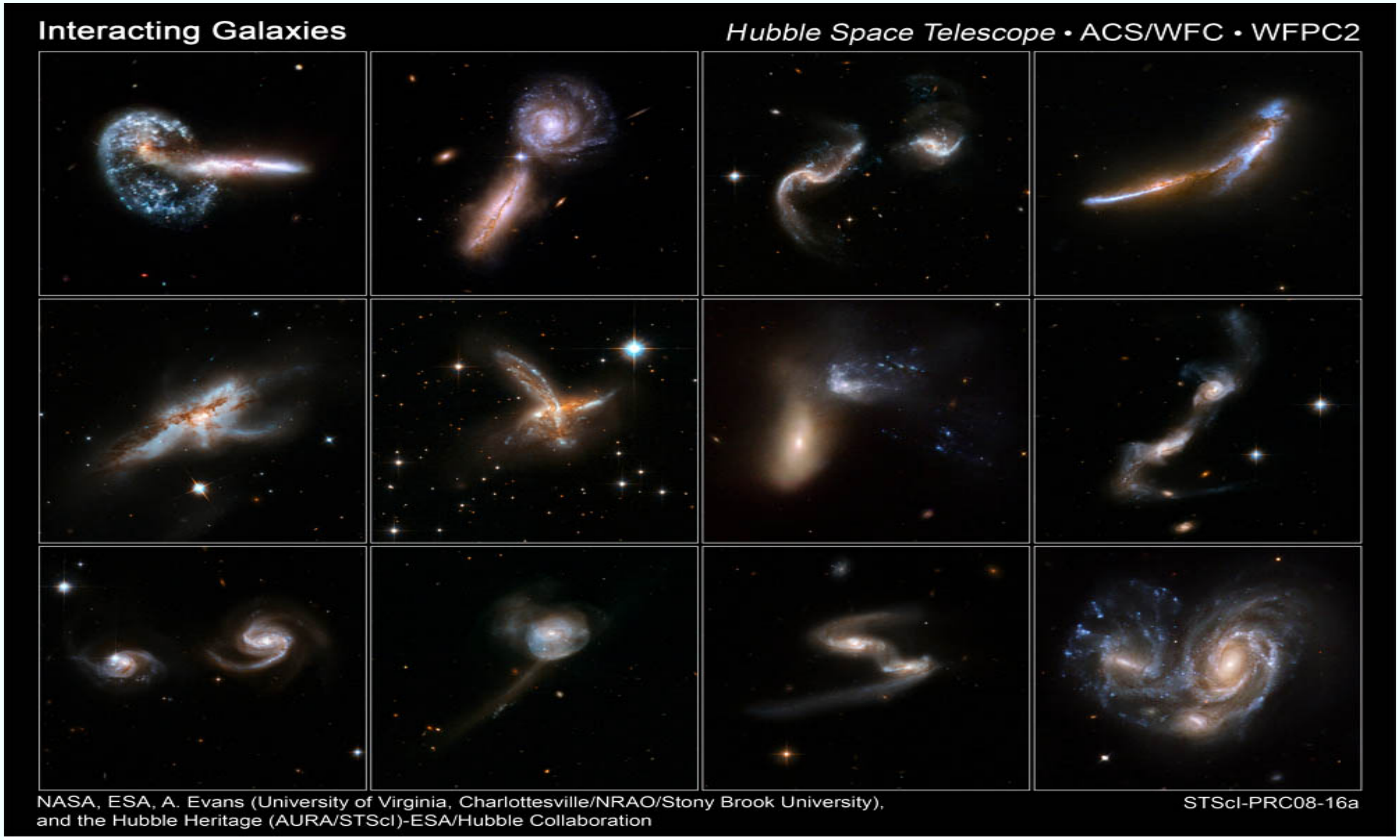
III.

Black Hole binary coalescence

1. Likehood and observability
2. Extreme mass ratio inspirals (EMRI)
3. Post-newtonian calculations
4. Quasi-normal modes
5. Numerical breakthroughs
6. Effective one-body approach

(see, e.g. T. Damour et al., ArXiv: 0803.3162[gr-qc])

Box 4. Likehood and Observability



Box 4. Binary BH merging Observability

If you ask a learned astronomer to name the most powerful class of events to occur in our Universe since the Big Bang, he or she will probably reply with either "supernovae," the cataclysmic explosions of massive stars, or "gamma-ray bursts," intense outpourings of high-energy radiation coming from the distant Universe.

Both responses are well informed, but they are off by several orders of magnitude. The most stupendous events that have occurred in our Universe since it came into being are the mergers of supermassive black holes. These beasts contain millions of times the mass of our Sun, and their titanic collisions rock the surrounding spacetime, sending out ripples of energy called gravitational waves, first predicted by Albert Einstein in his general theory of relativity. The energy in these waves exceeds the energy emitted by the trillions upon trillions of stars in the visible Universe over the course of minutes to hours.

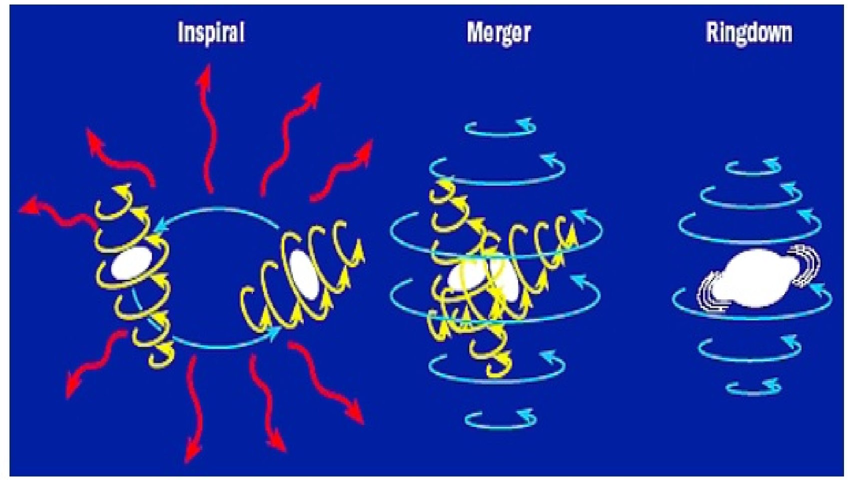


Figure 4. Schematic illustration of the three stages in the coalescence of a black hole binary. The gravitational waveforms from the inspiral and ringdown phases can be calculated approximately with pen and paper; the merger waveforms must be obtained from large-scale numerical simulations.

Joan Centrella et al (Goddard Space Flight Center)

($10^{23} L_{\odot}$... in weakly interating GW)

Box 4. SMBH coalescence Observability by LISA

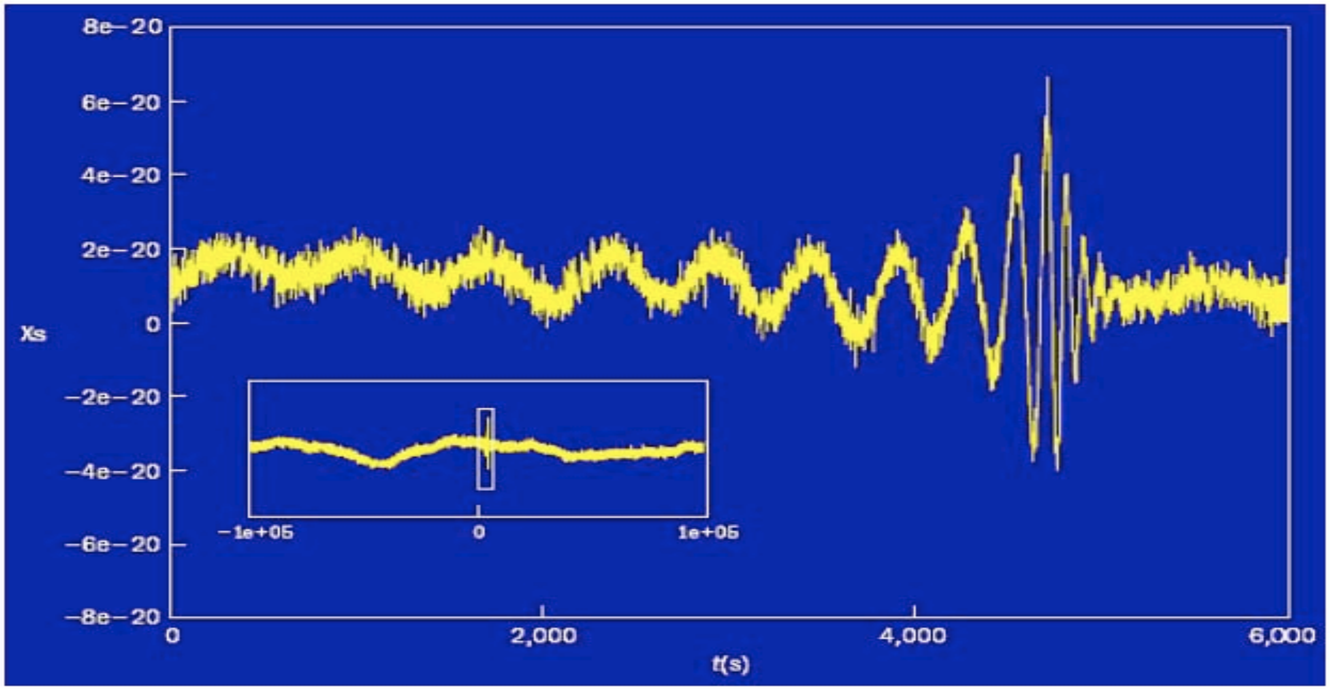
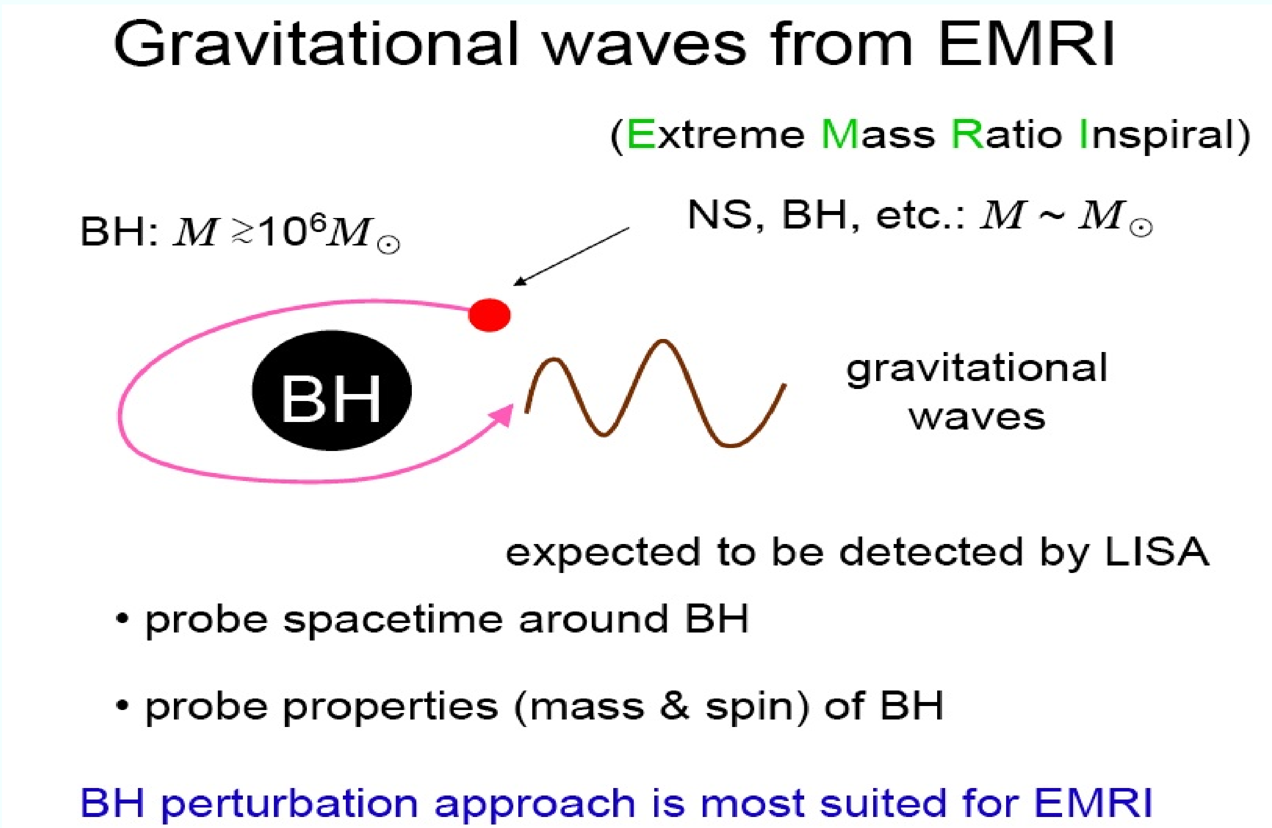


Figure 10. Simulated gravitational-wave signal from two merging equal-mass black holes, deliberately contaminated by simulated instrumental and astrophysical noise for the LISA detector.

Joan Centrella et al (Goddard Space Flight Center)

Box 5. Inspiral Phase



(Taken from Misao Sasaki, IHP06 Lecture Notes)

Box 5. Extreme Mass Ratio Inspirals : Energy Loss

Energy loss rate for circular orbit on the equatorial plane
in Kerr background to 4PN order

$$\left\langle\frac{dE}{dt}\right\rangle = \frac{32}{5}\left(\frac{\mu}{M}\right)v'^{10}\left[1-\frac{1247v'^2}{336}+\left(4\pi-\frac{11q}{4}\right)v'^3+\left(-\frac{44711}{9072}+\frac{33q^2}{16}\right)v'^4+\left(\frac{-8191\pi}{672}-\frac{59q}{16}\right)v'^5\right.$$
$$+\left(\frac{6643739519}{69854400}-\frac{1712\gamma}{105}+\frac{16\pi^2}{3}-\frac{65\pi q}{6}+\frac{611q^2}{504}-\frac{3424\ln 2}{105}-\frac{1712\ln v'}{105}\right)v'^6$$
$$+\left(\frac{-16285\pi}{504}+\frac{162035q}{3888}+\frac{65\pi q^2}{8}-\frac{71q^3}{24}\right)v'^7+\left(-\frac{323105549467}{3178375200}+\frac{232597\gamma}{4410}-\frac{1369\pi^2}{126}-\frac{359\pi q}{14}\right.$$
$$\left.\left.+\frac{22667q^2}{4536}+\frac{17q^4}{16}+\frac{39931\ln 2}{294}-\frac{47385\ln 3}{1568}+\frac{232597\ln v'}{4410}\right)v'^8\right].$$

$$q = a/M \qquad v' = (M\underline{\Omega})^{1/3}$$

Orbital freq. as measured
by observer at infinity

(Taken from Misao Sasaki, IHP06 Lecture Notes)
For the thread of the argument yielding the formula above, see Notes

Box 5. Post-Newtonian calculations : Energy Loss

of motion. At long last we obtain our end result:

$$\mathcal{L} = \frac{32c^5}{5G}\nu^2 x^5 \left\{ 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right) x + 4\pi x^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 \right.$$

[39] All formulas incorporate the changes in some equations following the published Errata (2005) to the works [16, 19, 45, 40, 4].

$$\begin{aligned}
&+ \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} \\
&+ \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}C - \frac{856}{105}\ln(16\,x) \right. \\
&\quad \left. + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right] x^3 \\
&\left. + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right)\right\}. \qquad (231)
\end{aligned}$$

In the test-mass limit $\nu \to 0$ for one of the bodies, we recover exactly the result following from linear black-hole perturbations obtained by Tagoshi and Sasaki [205]. In particular, the rational fraction 6643739519/69854400 comes out exactly the same as in black-hole perturbations. On the other hand, the ambiguity parameters λ and θ are part of the rational fraction $-134543/7776$, belonging to the coefficient of the term at 3PN order proportional to ν (hence this coefficient cannot be computed by linear black hole perturbations)[40].

(Taken from Luc Blanchet Living Review, 2006)

Box 4. Numerical Breakthough

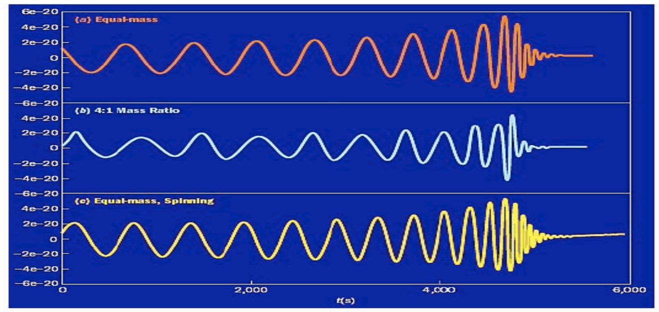


Figure 8. Waveforms as viewed by an observer on the equatorial plane of a black hole binary, for three different physical cases: (a) equal mass; (b) 4:1 mass ratio; (c) equal mass with spinning holes. In each case the total system mass is 10^5 solar masses, at a redshift of 15, or around 550 billion light-years away.

Bern Bruegmann et al (2003) (now at Iena)
Frans Pretorius (2005) (now at Princeton)
Manuella Campanelli et al (2005) (now at Rochester)
Joan Centrella et al (2005) (Goddard Space Flight Center)

(See eg Pretorius ArXiV: 0710.1338[gr-qc])

(NINJA Collaboration, ArXiV: 0901.4399[gr-qc];
SAMURAI Collaboration, ArXiV: 0901.2437[gr-qc])

BH in GR
Lecture 5

IPhT-CEA-09
N. Deruelle

Hawking's radiation and Black hole Thermodynamics

1. BH Quantum superradiance (recap)
2. Bekenstein entropy
3. Hawking radiation
4. radiation from "dynamical" BH

I. BH Quantum superradiance

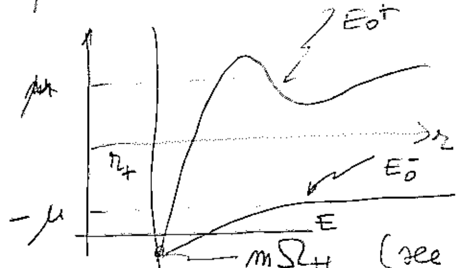
- Consider a rotating Kerr BH & describe its geometry using Boyer-Lindquist coordinates : $ds^2 = -\frac{\Delta}{\Sigma^2}(dt - a\sin^2\theta\, d\varphi)^2 + \frac{\sin^2\theta}{\Sigma^2}[(r^2+a^2)d\varphi - a\,dt]^2 + \frac{\Sigma^2}{\Delta}dr^2 + \Sigma^2 d\theta^2$, $\left(\begin{array}{l}\Sigma^2 = r^2 + a^2\cos^2\theta \\ \Delta = r^2 - 2mr + a^2\end{array}\right)$

- Consider a scalar field propagating on this background : $-\Box\hat{\varphi} + \mu^2\hat{\varphi} = 0$

Decompose $\hat{\varphi}$ into modes (see lecture 3) : $\hat{\varphi} = \int_{E>0} dE \sum_{\ell m} e^{-iEt}\, e^{im\varphi} K_{m\ell}(\theta) R(r)\hat{a} + \text{h.c.}$

the radial function satisfies a Teukolsky equation : $\left\{\begin{array}{l}\frac{d^2R}{dr_*^2} + (E - E_0^+)(E - E_0^-)R = 0 \\ \text{with } \frac{dr^*}{dr} = \frac{1}{\Delta}\end{array}\right.$

and :



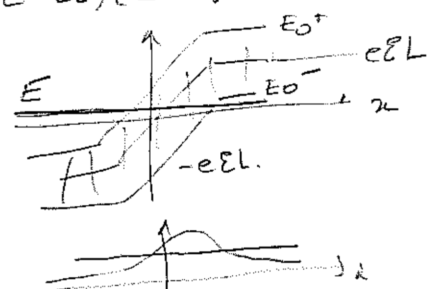
(see lecture 3) where $\Omega_H = \frac{a}{r_+^2 + a^2}$ is the angular velocity of the hole.

if $m\Omega_H < E < -\mu$: superradiant mode.

- Compare now with the problem of quantizing ϕ in M_4 in the presence of an constant electric field in the region $x = -0$, $x = +L$. The equation of motion is $(\partial_\alpha + ieA_\alpha)(\partial^\alpha + ieA^\alpha)\phi + \mu^2\phi = 0$; $\phi = e^{-iEt}e^{ip_y y}e^{ip_z z}\psi(x)$ with :

$\frac{d^2\psi}{dx^2} + (E - E_0^+)(E - E_0^-)\psi = 0$ with :

(Schwinger 1951)



If there is "level crossing" then (in the Dirac interpretation) the "holes" in the Dirac "sea" can cross the potential barrier and there is creation of a pair of particle/antiparticle ("Klein paradox" 1926)

In quantum field theory : the definition of positive frequency is not the same on the left & on the right of the region where there is an electric field

Hence two decomposition of $\hat{\varphi}$ with respect to the L/R positive frequencies

Since the two sets of modes form two $\neq$ bases for $\hat{\varphi}$ one can express the annihilation operator a_j^R as $a_j^R = \sum_i (\alpha_{ji} a_i^L + \beta_{ji} a_i^{+L})$

so that the expectation value of the particle operator $N_i^R = a_R^{+i} a_R^i$ when evaluated on the "left" vacuum $|0_L\rangle$ (such that $a_L|0\rangle = 0$) is

$\langle 0_L | N_i^R | 0_L\rangle = \sum_j |\beta_{ij}|^2$ with the normalisation $|\alpha|^2 + |\beta|^2 = 1$.

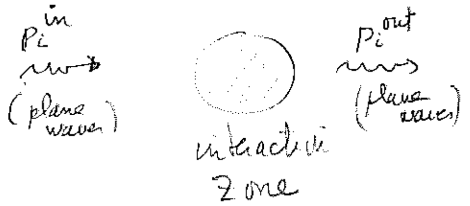
- The same occurs in the field of a Kerr Black hole (Zeldovich, Misner, Unruh, ND + Ruffini)

– Particle creation by a background field (recap):

$$\hat{\varphi}(x) = \sum_i \left(\hat{a}_i^{\,in}\, p_i^{in}(x) + \hat{a}_i^{\dagger\, in}\, n_i^{in}(x) \right) \qquad n_i^{in} = p_i^{*\,in}(x).$$

↳ complete set of modes with incoming positive frequencies ($p_i^{in}(x) = e^{-iEt} \times \ldots,\ E > 0$)

describing free incoming particles



$$= \sum_i \left(\hat{a}_i^{\,out}\, p_i^{out}(x) + \hat{a}_i^{\dagger\, out}\, n_i^{out}(x) \right)$$

↳ complete set of modes with outgoing positive frequencies

describing free outgoing particles

$[\hat{a}_i, \hat{a}_j^{\dagger}]^{in}_{out} = \delta_{ij}$; modes normalized as $\begin{cases} (p_i, p_j)^{in}_{out} = \delta_{ij} ; \quad (p_i, n_j)^{in}_{out} = 0 \\ (n_i, n_j)^{in}_{out} = -\delta_{ij} \end{cases}$

where $(\,,\,)$ mean the KG scalar product.

The $|in\rangle$ vacuum is such that $a_{in}|in\rangle = 0$.

Now the in-modes can be decomposed on the out-basis (and vice-versa)

$$p_i^{in} = \sum_j \left(\alpha_{ij}\, p_j^{out} + \beta_{ij}\, n_j^{out} \right) \quad \text{where } \beta_{ij} = (p_i^{out}, n_j^{in}) \text{ is a}$$

"transition amplitude" (or Bogolubov coefficient).

Therefore the mean number of i-type "out" particles present in the in-vacuum is:

$$\langle N_i \rangle \equiv \langle in | (a_i^{out})^\dagger a_i^{out} | in \rangle = \sum_j |\beta_{ij}|^2$$

– <u>remark</u> all this is standard QFT where it is assumed that the observations are made in an INERTIAL frame –

and that the interaction zone is local (in that it does not extend to infinity.

Typical example: an electric field between to plates.

(Schwinger effect).

(II) Bekenstein's BH entropy

(see eg T. Damour hep-th/0401160).

1. Christodoulou - Ruffini irreducible mass (1971). (recap). (see lect 3).

- Consider a Kerr-Newman metric in Boyer-Lindquist coordinates describing the gravitational & electric field of a rotating BH:

$$\begin{cases} ds^2 = -\dfrac{\Delta}{\Sigma^2}\omega_t^2 + \dfrac{\Sigma^2}{\Delta}dr^2 + \Sigma^2 d\theta^2 + \dfrac{\sin^2\theta}{\Sigma^2}\omega_\varphi^2 \ ; \quad A_i dx^i = -\dfrac{Qr}{\Sigma^2}\omega_t \\ a = J/M \ ; \quad \Delta = r^2 - 2Mr + a^2 + Q^2 \ ; \quad \Sigma^2 = r^2 + a^2\cos^2\theta \ ; \quad \omega_t = dt - a\sin^2\theta\, d\varphi \ ; \quad \omega_\varphi = (r^2+a^2)d\varphi - a\,dt \\ a^2 + Q^2 \leq M^2 \ ; \quad r_+ = M + \sqrt{M^2 - a^2 - Q^2} \end{cases}$$

- The equation of motion of a charged particle (e,μ) is $\mu \dfrac{Du^\mu}{d\tau} = e F^\mu{}_\nu u^\nu$; $u^\mu = \dfrac{dx^\mu}{d\tau}$

ST being stationary & axisymmetric we have that $\begin{cases} p_\varphi \equiv \mu u_\varphi = \text{cte} & \leftarrow \xi^{(\varphi)}\cdot u = \text{cte} \\ p_t \equiv \mu u_t = \text{cte} & \leftarrow \xi^{(t)}\cdot u = \text{cte} \end{cases}$

Moreover the $\exists$ of a Killing tensor st $\xi_{ij}u^i u^j = \underset{\uparrow \text{ Carter's constant}}{K}$ yields a third first integral;

Finally the "mass-shell constraint" (that is the normalisation of u^μ: $u^\mu u_\mu = -1$) yields: $g^{ij}(p_i - eA_i)(p_j - eA_j) = -\mu^2$ and gives p_r as a function of p_t, p_φ, K.

If the particle falls in the BH we have $\begin{cases} \delta M = E \equiv -p_t \\ \delta J = p_\varphi \ ; \quad \delta Q = e \end{cases}$

- As we saw in lecture 3 we have that (generalised here to the charged case).

$$(*) \qquad \delta M - \frac{a\delta J + r_+ Q\delta Q}{r_+^2 + a^2} \geq 0 \qquad (=0 \text{ if the particle has zero radial impulsion } p_r \text{ on the horizon})$$

we also saw that this can be rewritten as $\delta M_{irr} \geq 0$ with

$$M_{irr} = \frac{1}{2}\sqrt{r_+^2 + a^2} \quad \text{so that} \quad M^2 = \left(M_{irr} + \frac{Q^2}{4M_{irr}}\right)^2 + \frac{J^2}{4M_{irr}^2}$$

so that the "free energy" of a BH that is the maximal energy which can be extracted by depleting the BH from its angular momentum J & charge Q is $M - M_{irr}$. This free energy has a Coulomb & rotational contribution.

- Recall too that we found M_{irr} by looking for a function of the area of the BH $[(r_+^2 + a^2)]$ and showed that the inequality (*) could be written as $\delta F(A) \geq 0$ (if $dF/dA > 0$). Therefore $\delta M_{irr} \geq 0$ is equivalent to

$$\delta A \geq 0$$: the surface area of a KN Black hole must increase when "swallowing" outside matter.

$$(A = 4\pi(r_+^2 + a^2) = 16\pi M_{irr}^2)$$

2. Hawking's area theorem (1972)

At the same time Hawking also found this theorem: $\delta A \geq 0$ using however more general concepts than Christodoulou & Ruffini.

To prove it he needed to give a precise definition of a horizon (and introduced the concept of "absolute" or "event" horizon, see below) and used the fact that these horizons are null surfaces (they are generated by light rays).

He could therefore prove that if one considers a system made of several BH (that is a ST possessing several event horizons, the metric no longer being that of KN), then the sum of the areas of the horizons cannot decrease: $\delta \sum A \geq 0$.

3. Bekenstein entropy

⊙ John Wheeler who had conjectured that "black hole have no hair" (ie the unicity of the Kerr-Newman BH, showed by Israel, Carter, Robinson, Bunting, Mazur ... see Part 3) was concerned about the fact that when one throws an object in a BH one irretrievably looses all information about this object (apart from its mass, angular momentum and charge).

Hence the idea of Bekenstein to try and attribute an entropy to BH (JAW is said to have told him: "BH thermodynamics is crazy, perhaps crazy enough to work" See eg Israel 1987 300 years of grav.)

⊙ Since $\delta A \geq 0$ looks like the 2nd law of thermodynamics: $\delta S \geq 0$ Bekenstein (1972-1973) conjectured that one could endow a BH with an entropy: $\boxed{S_{BH} = \alpha A}$ where $[\alpha] = 1/L^2$. By varying the mass formula ($M^2 = (M_{ir} + Q^2/4M_{ir})^2 + J^2/4M_{ir}^2$) one arrives then to the "1st law of BH thermodynamics":

$$dM = \Omega\, dJ + \Phi\, dQ + T_{BH}\, dS_{BH}$$

where $\Omega \overset{KN}{=} \frac{a}{r_+^2+a^2}$ is the "angular velocity" of the hole (the angular velocity of photons orbiting the horizon as measured with time t (proper time at ∞!))

and where $\Phi \overset{KN}{=} \frac{Q r_+}{r_+^2+a^2}$ is the "Coulomb potential" of the horizon

and where $T_{BH} = \frac{1}{\alpha}\frac{\partial M}{\partial A} = \frac{\kappa}{8\pi\alpha}$

6

where $\varkappa = \frac{1}{2}\frac{r_+ - r_-}{r_+^2 + a^2}$ (for a KN BH) is the "surface gravity" of the BH, that is the acceleration of a particle on the horizon (which is oste when measured in terms of the proper time of the particle) CORRECTED by the redshift factor (which converts local proper time to time t at infinity). In more geometrical terms, if ℓ^i is the (null) vector normal to the horizon [$\ell^i \partial_i = \partial_t + \Omega \partial_\varphi$] then:
$\ell^i \nabla_i \ell^j = \varkappa \ell^j$.

⊛ The first law was obtained here in the particular case of an isolated BH (Kerr Newman geometry). Bardeen, Carter and Hawking using the geometrical methods introduced by Penrose and Hawking's concept of event horizon generalized it to any (asymptotically flat) ST containing event horizons characterized by their angular velocities, Coulomb potentials & surface gravities [that Carter showed to be constants in the stationary case ("rigidity theorem")]

The purpose of this lecture is to show that by studying quantum fields in BH space times Hawking found that they radiate with a temperature $T_{BH} = \frac{1}{2\pi}\varkappa \overset{\text{Schw}}{=} \frac{1}{8\pi M}$ (thus $\alpha = \frac{1}{4}$)

(Considerations about the meaning of the Bekenstein Hawking entropy $S_{BH} = \frac{A}{4}$ is left to lecture 6)

(III) Hawking radiation

1. The Unruh effect

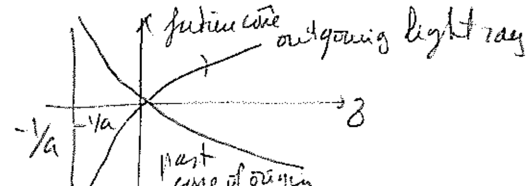
- NB: discovered after Hawking (1974) discovered that BH radiate
  (Unruh (1976); Davies 1975); for a review Takagi (1986), (Progr. of Theor. Phys 88, 1)
  I shall adapt Damour-Ruffini's method to the case at hand (ND, 1977).
  for a recent, simple, derivation see Alsing & Milonni arXiv: quant-phys/0401170

- Consider flat M_4 in the coordinates (t, z, x, y) such that the metric reads:

$$ds^2 = -(1+az)\,dt^2 + \frac{dz^2}{4(1+az)} + dx^2 + dy^2$$

(this reduces to the Rindler metric by the change of coord $z \to \zeta$: $1+az = (1+a\zeta)^2$

The metric is well behaved for $z > -1/a$; $z = -1/a$ is a horizon -



- as in the Schwarzschild/Rindler case the manifold can be extended beyond $z = -1/a$

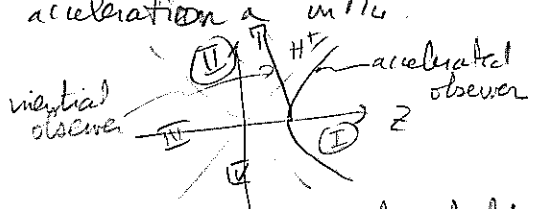
In the "Kruskal" coordinates:

$$\begin{cases} aT = \sqrt{1+az}\,\mathrm{sh}\,at \; ; \; aZ = \sqrt{1+az}\,\mathrm{ch}\,at & (z \geq -1/a) \\ aT = -\sqrt{-(1+az)}\,\mathrm{ch}\,at \; ; \; aZ = -\sqrt{-(1+az)}\,\mathrm{sh}\,at & (z \leq -1/a) \end{cases}$$

the metric reduces to $ds^2 = -dT^2 + dZ^2 + dx^2 + dy^2$ ie M_4 in minkowskian coord.

since $Z^2 - T^2 = 1+az$; $T/Z = \mathrm{th}\,at$ $(z \geq -1/a)$; $T/Z = \coth at$ $(z \leq -1/a)$

the hyperbola $z = \text{cte}$ is the worldline of an object having a constant acceleration a in M_4.



- consider a massless scalar field propagating in M_4 and decompose it in the following modes: $\phi = e^{-iEt}\, e^{ip_x x}\, e^{ip_y y}\, \Psi(z)$ where $(t\,x\,y\,z)$ are the Rindler-like coordinates. $\Box\phi = 0$ then reduces to:

$$\frac{4}{1+az}\frac{d}{dz}(1+az)\frac{d\Psi}{dz} = -\frac{1}{(1+az)^2}(E - E_0^+)(E - E_0^-) \quad \text{with } E_0^\pm = \pm\lambda\sqrt{1+az}$$

$$\& \; \lambda = \sqrt{p_x^2 + p_y^2}$$

If we restrict the manifold to the region I of M_4 (which is the equivalent of restricting the support of quantum fields to the exterior of a Schw BH) there is NO pair creation since there is no "level crossing".

- If we decide to extend the manifold to regions (I + II) appropriate to describing the physical situation of the description of a field ϕ in the frame of an accelerated observer whose acceleration started at some time t_0 in the distant past (the analogue of the collapsing star situation) then we must extend the mode functions beyond the future horizon H^+.
- Now, near the horizon: $\phi \sim \alpha e^{-iE(t-z^*)} + \beta e^{-iE(t+z^*)}$; $z^* = \frac{1}{2a}\ln(1+az)$

$$= \alpha e^{-iEv}(1+az)^{iE/a} + \beta e^{-iEv} \quad ; v = t+z^* \quad (E>0).$$

(in this advanced Eddington-Finkelstein coordinates the metric reads:

$$ds^2 = (1+az)\,dv^2 - dv\,dz - dx^2 - dy^2$$ and is well behaved on the horizon

($z=-1/a$) where $(v\,x\,y)$ remain finite (but $t \to +\infty$))

- To continue the wave $\alpha e^{-iEv}(1+az)^{iE/a}$ we complexify z. Since $z=-1/a$ is a branch point the function is multivalued and we have:

$$(*) \quad \alpha e^{-iEv}(1+az)^{iE/a} \xrightarrow[\text{continuation}]{\text{analytical}} \alpha e^{-iEv}\left[\Theta(1+az)\times(1+az)^{iE/a} + \Theta(-1-az)\times(-1-az)^{iE/a}\exp(\pm\pi E/a)\right]$$

+ sign for $\mathrm{Im}\,z<0$; − sign for $\mathrm{Im}\,z>0$. ("Unruh" vacuum)

- In order to choose a sign we translate to the present situation the well known result in M_4: wave functions can be analytically continued to complex coordinates $(T\,Z)$ and describe a particle (resp. antiparticle) state if $\mathrm{Im}\,Z$ belongs to the past (future cone). Since $\frac{\partial}{\partial v}$ is future directed the prescription $\mathrm{Im}\,z<0$ (+ sign) will describe an antiparticle.
- We can then interpret $(*)$: The wave outside the horizon describes a flux of $|\alpha|^2$ particles going towards infinity; behind the horizon the Killing vector $\frac{\partial}{\partial t}$ is spacelike and the wave represents a flux of $\alpha^2 e^{2\pi E/a}$ of ANTI-particles going towards $z=-\infty$. Finally since the continued wave represents an antiparticle its flux must be normalized to -1:

$$\alpha^2 - \alpha^2 e^{2\pi E/a} = -1 \implies \alpha^2 = \frac{1}{e^{\frac{2\pi E}{a}}-1}$$

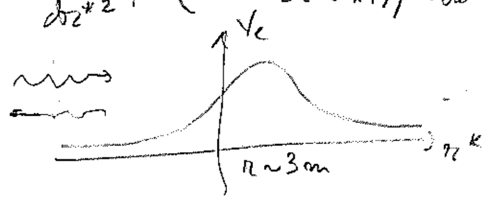
α^2 is the transmission coefficient between the particle states outside the horizon & the antiparticle state inside the horizon and is the rate of particle creation from the vacuum in the state E, p_x, p_y

- The spectrum is Planckian & the accelerated observer will detect a flux of α^2 particles coming from the horizon at temperature $T = a/2\pi$ (Unruh temperature). in numbers: $T \sim 1°K$ for $a \sim 1.5\times10^{19}\,g$ (!)
Earth gravity

2. Hawking radiation (see eg Damour arXiv:0802.4169, arXiv: hep-th/0401160)
(based on Damour-Ruffini 1976)

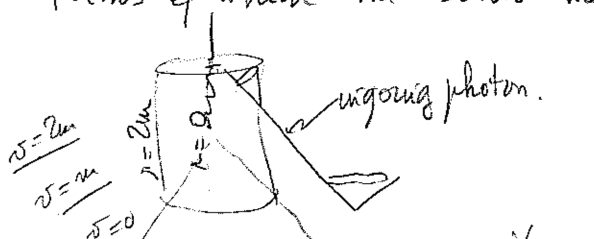
⊙ Consider a Schwarzschild BH and (for simplicity) a massless scalar field ϕ propagating in that background. If one describes the geometry in Schwarzschild $(t\, r\, \theta\, \varphi)$ coordinates [such that $ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-2m/r} + r^2 d\Omega^2$] the field $\phi(t\, r\, \theta\, \varphi)$ can be decomposed into modes: $\phi_{\omega\ell m}(t r \theta \varphi) = \frac{e^{-i\omega t}}{\sqrt{2\pi|\omega|}} \frac{u_{\ell m}(r)}{r} Y_{\ell m}(\theta\varphi)$
which satisfy: $\Box \phi_{\ell m \omega} = \frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}\, g^{ij}\partial_j \phi_{\omega\ell m}\right) = 0$, that is:

$$\frac{d^2 u_{\ell\omega}}{dr^{*2}} + \left(\omega^2 - V_\ell(r(r_*))\right)u_{\ell\omega} = 0 \quad \text{with} \quad \begin{cases} r_* = r + 2m \ln \frac{r-2m}{2m} \\ V_\ell = \left(1-\frac{2m}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2m}{r^3}\right) \end{cases}$$



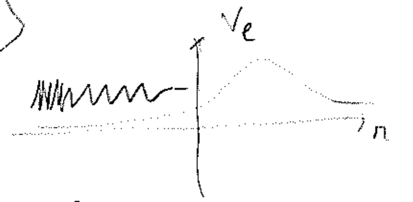
near the horizon: $\phi \propto e^{-i\omega(t\pm r_*)}$

⊙ to eliminate the coordinate singularity at $r=2m$ one introduces the ingoing Eddington-Finkelstein coordinate (see lect 2): $v = t + r_*$ in terms of which the Schw metric reads: $ds^2 = -\left(1-\frac{2m}{r}\right)dv^2 + 2dv\,dr + r^2 d\Omega^2$



at $r=2m$, φ, θ, v remain finite (but $t\to+\infty$)

in these coordinates an OUTgoing wave near $r>2m$ $(\omega>0)$: $\phi^{out}_\omega \propto e^{-i\omega(t-r_*)} = e^{-i\omega v}\, e^{2i\omega r_*}$



$$= e^{-i\omega v}\, e^{2i\omega\left[r + 2m\ln\frac{r-2m}{2m}\right]}$$
$$= e^{-i\omega v}\, e^{2i\omega r}\left(\frac{r-2m}{2m}\right)^{4i\omega m}$$

This mode piles up an ∞ nb of oscillations of higher and higher frequencies and it is, a priori, not defined inside the horizon

⊙ let us now extend this mode inside the horizon; to do so we have to give sense to $\left[-\frac{(2m-r)}{2m}\right]^{4i\omega m}$ that is to $(-)^{4i\omega m}$. The standard way is to go to the complex plane and write $(-1) = e^{\pm i\pi}$ the + sign meaning that we took a path in the upper complex plane; the − sign in the lower part.

Now, in keeping with what is done in flat ST analyticity in the lower part of the complex plane means that we are describing a negative frequency wave packet so that the mode:
$M_\omega(v,r) = N_\omega\left[\Theta(r-2m)\,\phi^{out}_\omega(r-2m) + e^{+4\pi m\omega}\,\Theta(2m-r)\,\phi^{out}_\omega(2m-r)\right]$ describes a negative frequency state. (ie an antiparticle state)

- let us interpret the result (considering that our plane waves represent "wave packets"): The wave outside the horizon represents a flux of $|N_\omega|^2$ particles ($\omega > 0$) going towards infinity ($\mathcal{J}^+$); behind the horizon the Killing vector $\frac{\partial}{\partial t}$ is spacelike: the wave therefore represents a flux of $N_\omega|^2 e^{8\pi m\omega}$ particles going towards the horizon, that is backward in time; according to the Stueckelberg-Feynman interpretation it thus represents a flux of $|N_\omega|^2 e^{8\pi m\omega}$ antiparticles following the arrow of time i.e. going to the singularity. Finally, since the continued wave n_ω describes an antiparticle it must be normalized to -1. Hence:

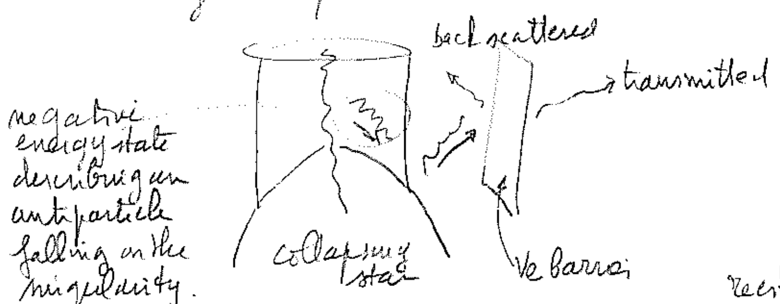
$$|N_\omega|^2 - |N_\omega|^2 e^{8\pi m\omega} = -1 \iff |N_\omega|^2 = \frac{1}{e^{8\pi m\omega} - 1}$$

which is the number of particles created in the mode ω per unit time.

- Finally the rate of particle creation by the black hole is:

$$\frac{d\langle N\rangle}{dt} = \sum_{\ell m} \int \frac{d\omega}{2\pi} \frac{\Gamma_\ell(\omega)}{e^{8\pi m\omega} - 1} = \sum_{\ell m} \int d\omega \frac{\Gamma_\ell(\omega)}{e^{\frac{\omega}{T}} - 1} \qquad T \equiv \frac{1}{8\pi m}$$

where the "grey body" factor $\Gamma_\ell(\omega)$ accounts for the transmission coefficient through the potential barrier $V_\ell(r)$



If $\Gamma_\ell(\omega)$ was absent an observer at infinity would detect a steady flow of black-body radiation at the temperature

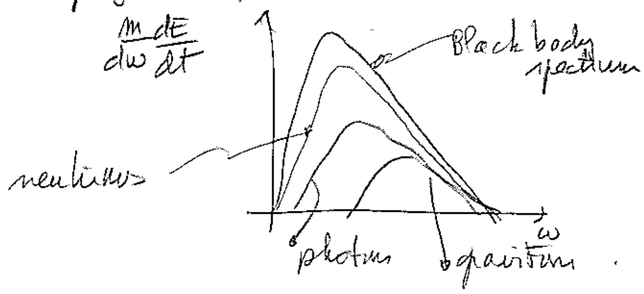
$$\boxed{T = \frac{1}{8\pi m}}$$

reestablishing units: $T = 6 \times 10^{-8} \left(\frac{M_\odot}{M}\right)$ Kelvin

- Remarks: – the chosen "in vacuum" is defined via the waves $n_\omega(v, r)$ and is a "Unruh vacuum" (cf previous section).
  – the only essential ingredient is the "piling up" of the modes near the horizon (see below M. Visser comments)
  – Note that because of the "piling up" of the modes near the horizon a mode with a given ω at finite distance is more and more blueshifted near the horizon ("transplanckian frequencies"). Introducing cut-offs at high frequencies did not change the result (see eg B. Wald Living Review p13).
  – There are MANY ways to deduce the Hawking temperature see eg Carlip arXiV:0807.4520 who describes 6 different routes.

⊙ More detailed calculations: instead of computing the rate of creation of massless spin 0 particles (obeying the KG eqn) one can study neutrinos, photons, gravitons ... The energy flux is $\frac{d\langle E\rangle}{dt} = \sum_{\ell m} \int_{\omega \geq 0} \frac{d\omega}{2\pi} \frac{\Gamma_\ell(\omega)\,\omega}{e^{8\pi m\omega}-1}$ with various $\Gamma_\ell(\omega)$ grey body factors. Studied in the 70s (eg by Don Page: 1976)

(see Frolov. Novikov p 401)



⊙ Temperature measured by an infalling observer.

- preliminaries: The distribution function (ie number density in phase space) $\mathcal{N}$ of a ensemble of particles is conserved along a particle trajectory (collisionless Boltzmann equation). One can also show that $\mathcal{N} = I_\nu/\nu^3$ when one considers photons with frequency ν and intensity I_ν ($I_\nu = \frac{d\,\text{Energy}}{dt\, d(\text{area})\, d\nu\, d(\text{solid angle})}$). For a black body: $I_\nu = \frac{2\nu^3}{e^{\nu/kT}-1}$; hence if $\frac{I_\nu}{\nu^3} = \text{ctt}$ then $\frac{\nu}{T} = \text{ctt}$. (see eg MTW p 588)

  Now in Schw geometry: $ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-2m/r} + r^2 d\Omega^2$

  proper time of an observer at rest: $d\tau = \sqrt{1-\frac{2m}{r}}\, dt$; $dt \propto \frac{1}{\nu_\infty}$ $\Rightarrow \nu_{em} = \frac{\nu_\infty}{\sqrt{1-2m/r}}$

  $d\tau \to$ period of emitted wave $\propto \frac{1}{\nu_{em}}$

  hence $\frac{\nu_\infty}{T_\infty} = \frac{\nu_r}{T_r} \Longrightarrow \boxed{T_r = T_\infty \frac{\nu_r}{\nu_\infty} = \frac{T_\infty}{\sqrt{1-2m/r}}}$ (Tolman relation 1934)

- hence an observer at rest at r in the Schwarzschild gravitational field (and hence NOT in free fall) observes a temperature $T_r = \frac{T_\infty}{\sqrt{1-2m/r}}$ with $T_\infty = \frac{1}{8\pi m}$

  this temperature diverges when approaching the horizon (see however below the smoothing effect of back reaction.

- Infalling observer: the issue is not clear yet (at least to me!).
  * B Wald (p15 of his Living Review) states (with no ref) that a freely falling observer will not notice any important effect as they approach the horizon.
  * Greenwood & Stojkovic (arXiV: 0806.0628 [gr-qc]) find that the spectrum is not thermal but that the temp. diverge. However their model implies that their infalling observers "have to stop each time they are taking the temperature"...
  * Brynjolfsson & Thorlacius (arXiV: 0805.1876) using a higher-dimensional embedding technique due to Deser & Levin find that the temperature remains finite (but with a non planckian spectrum) (in agreement with Unruh 77)

3. back reaction (ref: Birrell & Davies 1982; Frolov-Novikov 1998)

⊙ BH evaporation time: order of magnitude estimate.

Black body radiation; temperature $T_H = \frac{1}{8\pi m}$ & surface area $4\pi(2m)^2 \equiv \mathcal{A}$
energy σT_H^4 (Stefan law); hence $\dot m \sim -\sigma T_H^4 \mathcal{A} \propto \frac{1}{m^2}$

Putting in numbers: $\begin{cases} -\dot m \sim b\left(\frac{m_{pl}}{m}\right)^2\left(\frac{m_p}{t_p}\right) N & \text{nb of states \& species radiated} \\ b \sim 2.6\times 10^{-6} \end{cases}$

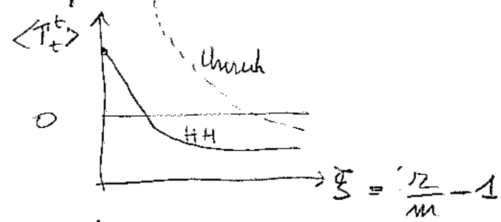
hence $t_{ev} \sim \frac{1}{3b} t_{pl}\left(\frac{m}{m_{pl}}\right)^3 \sim 2.4\times 10^{-24}\,s\left(\frac{m}{1g}\right)^3 N^{-1} \sim$ Hubble time for $m \sim 5\times 10^{14}g$.

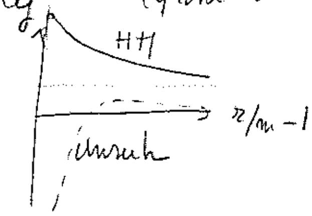
⊙ To do better: $G_{ij} = 8\pi \langle \hat T_{ij} \rangle$ vacuum expectation value of the stress energy tensor.

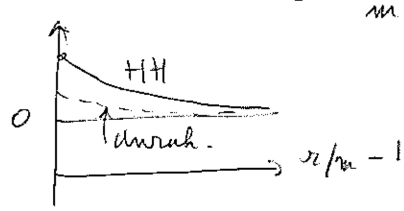
- $\langle \hat T_{ij} \rangle$ diverges (as in flat spacetime). Hence renormalisation procedure is required.
- However Wald showed (1977-78) that all $\langle \hat T^i_j \rangle^{ren}$ which are conserved ($D_i \langle T^i_j \rangle^{ren} = 0$) (plus some other conditions see eg Frolov-Novikov p405) differ from one another by conserved tensors built with the Riemann tensor and its derivatives.
- "conformal trace anomaly": it is also possible to show that whatever the state $\hat T^i_j$ is averaged on, its trace (for a conformal field, eg electromagnetism) does NOT vanish and is of the form: (see Birrell & Davies 82; Fulling 89)
$$\langle T^i_i \rangle^{ren} = \alpha\left(Weyl^2 + \frac{2}{3}\Box R\right) + \beta\, GB + \gamma \Box R \quad = \alpha\, Weyl^2 + \beta\, GB = (\alpha+\beta)\, Weyl^2 \text{ if } R_{ij} = 0.$$
($Weyl^2 = Riem^2 - 2Ric^2 + \frac{1}{3}R^2$; $GB = Riem^2 - 4Ric^2 + R^2$)
The coefficients α & β have been computed. (hence calculation of $\langle T^i_j \rangle^{ren}$ is easier in 2D)
- choice of vacuum: for example: the "Unruh vacuum" adapted to the description of a collapsing star (quantum field being originally in ground state, see above); "Hartle-Hawking vacuum" adapted to the description of a BH in a cavity (see section on Unruh effect). + "Boulware vacuum" ...
- in Schwarzschild spacetime, Christensen-Fulling (1977) found that (massless)
$$\langle \hat T^\mu_\nu \rangle = \sum_{i=1}^4 t_{(i)}{}^\mu{}_\nu \quad \text{where } t_{(i)}{}^\mu{}_\nu \text{ are separately conserved and depend}$$
on two functions: $T \equiv \langle T^i_i \rangle$ and $\Theta = \langle T^\theta_\theta \rangle - \frac{1}{4}T$ as well as two constants: W & N. $W = -dM/dt$ is the intensity of the radiation at ∞ (hence is zero in the Hartle-Hawking vacuum); N vanishes for the Unruh & HH vacua.
- asymptotic expressions for $\langle T^i_j \rangle^{ren}_{U,HH}$ can be obtained analytically
(eg $\langle T^i_j \rangle^{HH} \underset{r\to\infty}{\sim} \frac{\pi^2 T^2}{90}\, \mathrm{diag}(-3,1,1,1)$ where $T = \frac{1}{8\pi m}$
and $\langle T^i_j \rangle^{HH} \sim (A,A,B,B)$ for $r \to 2m$ with A & B finite)

– otherwise one must integrate over the modes numerically (from Frolov. Novikov p424)

example of result







⊙ Thermal atmosphere of a BH

– as we saw above the temperature measured by an observer at rest at r is

$$T(r) = \frac{1}{8\pi m} \frac{1}{\sqrt{1-2m/r}} \xrightarrow{r \to 2m} \infty$$ <u>without</u> taking into account the back reaction

– now the energy density of this radiation is $-\langle T^t_t \rangle$ which is finite at the horizon (in the HH vacuum).

Interpretation: the contribution of the vacuum polarisation caused by the gravitational field near the horizon compensates for the divergence that would take place if the law σT^4 were not violated.

(IV) Radiation from dynamical black holes

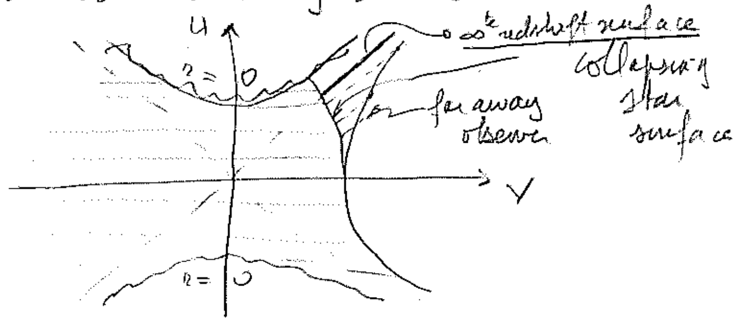
1. Introductory remarks

Up to now we have almost exclusively only considered isolated, stationary, that is, Kerr, BH; moreover we have loosely defined the "horizon" of these BH (we mainly defined them as surfaces of ∞ redshift).

A question is: how does one define "dynamical" BH (eg binary BH: important in numerical relativity!); and what are their thermodynamical properties

2. BH "event" horizon

⊙ For Oppenheimer & Snyder (as well as $\simeq$ anybody else up to the 60's) a "BH" was the end point of gravitational collapse, a gravitationally bound object delimited by an immaterial boundary of infinite redshift called "horizon" or, rather, an "apparent horizon" (in Hawking's terminology).

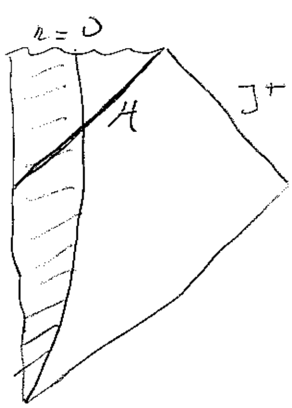


- created at full size when star crosses its Schwarzschild radius
- observer dependent (defined here wrt static observers)

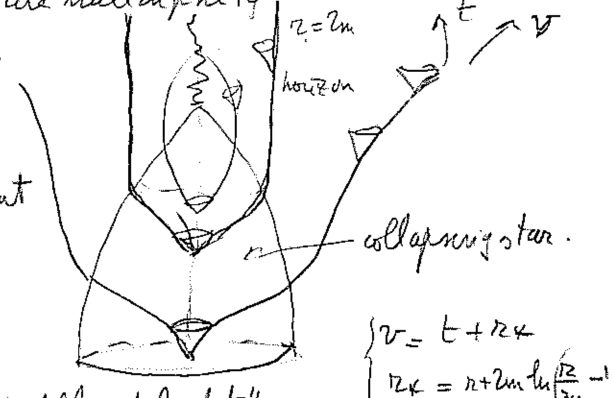
⊙ In 1970 Stephen Hawking introduced the concept of "event horizon": the "absolute" or "event" horizon $\mathcal{H}$ is $\dot{\mathcal{B}}$ = the boundary of $\mathcal{B}$, $\mathcal{B}$ being the def of a BH: $\mathcal{B} \equiv \mathcal{M} - I^-(\mathcal{J}^+)$

future null infinity

causal past



This definition is observer independent

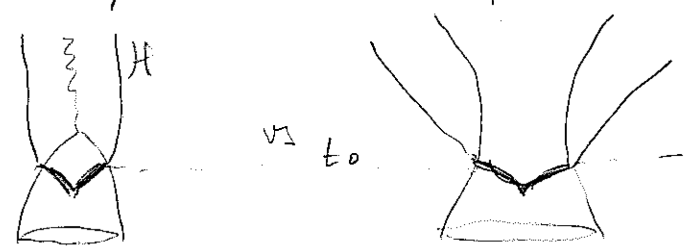


$$\begin{cases} v = t + r_* \\ r_* = r + 2m \ln\left(\frac{r}{2m} - 1\right) \end{cases}$$

The event horizon is the time-development of a "bubble of light" emitted from the center of the star which stabilizes itself under the pull of gravity (see Hawking-Ellis 73; and MTW p 933 et seq.)

It is with this definition of a horizon that Hawking could prove his "area theorem": If matter satisfies the null energy condition ($T_{ij}k^ik^j \geq 0$ for all null k^i); and if ST is "strongly asymptotically predictible" (ie $\exists$ a globally hyperbolic region containing $I^-(\mathcal{J}^+) \cup \mathcal{H}$) then the surface area of the event horizon of a BH cannot decrease.

• drawback: the event horizon is "teleological", that is: it is formed BEFORE the star has reached its Schwarzschild radius. Hence its position at a given time depends on the entire future history of ST. For example:



Two stars have the same history until t_0; one forms a BH, the other explodes; in the 1st case the underlined null geodesics $\in \mathcal{H}$; in the 2nd case there is no event hori

• aside: In the "membrane paradigm" approach to BH (Znajek 1978; Damour 1978; Thorne et al 1986), when one wants to describe electromagnetic fields or, more generally, matter fields, interacting with the gravitational field of the hole, one ignores the "inside" of the hole by introducing suitable quantities defined — on the event horizon of the hole [ie on a "fictitious brane" stabilized by the teleological condition that $\exists$ equil state].

[eg: $\nabla_i F^{ij} = 4\pi J^j$; introduce $T_{ij}\,\Theta_H$ where Θ_H is a Heaviside function which is 1 outside the BH & 0 inside; rewrite Maxwell eqn as:

$$\nabla_i (\Theta_H F^{ij}) = 4\pi (J^j \Theta + j^j_H) \quad \text{with } j^i_H = \frac{1}{4\pi} F^{ij} \underbrace{\partial_j \Theta_H}_{\text{Dirac}} \quad \text{BH "current"}]$$

(surface electric resistivity $\rho = 4\pi = 377\,\Omega$)

[similarly one can write a Navier-Stokes like eqn for viscous fluid brane: the shear viscosity $\eta = +\frac{1}{16\pi}$; the surface viscosity is $\zeta = -\frac{1}{16\pi}$: (for an ordinary fluid, would signal instability wrt contraction/dil.) surface pressure $p = \frac{g}{8\pi}$; exterior force $f_A = -\ell^i T_{iA}$.]

For a pedagogical introduction see Damour arXiv: 0802.4169 [hep-th]

NB: the "stretched horizon" of Thorne et al (1982) is a timelike surface "just outside" $\mathcal{H}$.

3. "Quasi-local horizons"

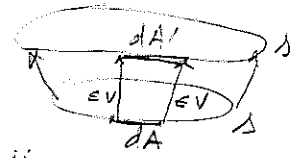
• The teleological nature of event horizons makes them very difficult to localize beyond the stationary regime (ie beyond Kerr ST); hence it is NOT a very useful concept in numerical relativity (as eg the study of BH coalescence) or in quantum gravity (in eg the study of BH evaporation). (For a review of the "Event Horizon finders" in numerical relativity, see eg Cohen et al, ArXiv: 0809.2628 [gr-qc])

• Hence the need to introduce a local def of the horizon of a BH (or rather "quasi" local as it will involve small regions of ST rather than points). Various definitions have been introduced in the last decade or so:

- trapping horizons (Hayward 1994)
- isolated horizons (Ashtekar et al 1999)
- dynamical horizons (Ashtekar & Krishnan 2002)
- slowly evolving horizons (Booth & Fairhurst 2004)

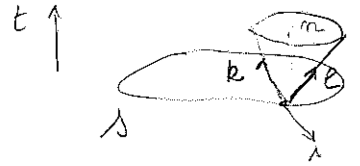
(see Gourgoulhon - Jaramillo arXiv: 0803.2944 [gr-qc])

⊙ trapped surfaces (Penrose 1965)   (from Gourgoulhon - Jaramillo)



- consider a 2-D closed spacelike surface S
  define a vector V on S; $\perp$ to S everywhere
  displace each point of S by ϵV; this defines S' (Lie dragging)
  the "expansion" of S is defined by $\theta^{(V)} = \lim_{\epsilon \to 0} \frac{\delta A' - \delta A}{\epsilon\, \delta A} = \mathcal{L}_V \ln\sqrt{q} = q^{ij} D_i V_j$
  where q_{ij} is the induced metric on S.

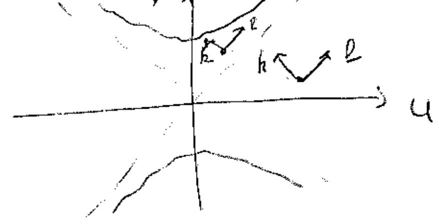
- there are 2 future-null null directions orthogonal to S: ℓ^i: outgoing null normal; k^i: ingoing null normal



in flat spacetime the expansion of S along ℓ is always positive: $\theta^{(\ell)} > 0$ (and $\theta^{(k)} < 0$) (in plain language outgoing light-rays propagate outwards).

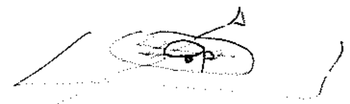
a "trapped surface" is such that $\theta^{(k)}$ AND $\theta^{(\ell)}$ are negative.
a "marginally trapped surface" is such that $\theta^{(k)} < 0$ & $\theta^{(\ell)} = 0$

example



inside the horizon of a Schwarzschild BH all 2 spheres (represented by a point in the Kruskal diagram) are trapped surfaces.

- if ST is asymptotically flat the "apparent horizon" (defined above) is a (connected) boundary of an trapped region (then called "outer" trapped region).



Σ (spacelike hypersurface, ie V_3 at time t)

Ω: trapped region ie $\{p\}$ through which there is at least one outer trapped surface

- Penrose used this concept to prove the singularity theorem: provided the weak energy condition holds, if $\exists$ a trapped surface S then $\exists$ a singularity in $(\mathcal{M}, g)$ (in the form of future non-extendible null geodesics).
  Also: if cosmic censorship conjecture holds (ie no singularities outside the trapped surface) then, if $\exists$ a trapped surface S, then $\exists$ a black hole B and $S \subset B$ [in plain language: the apparent horizon is inside the event horizon].

⊙ local definitions of horizons: world-tubes of apparent horizons

A hypersurface $\mathcal{H}$ is said to be

- a FOTH (future outer trapping horizon) if $\mathcal{H}$ is foliated by marginally outer trapped 2 surfaces [$\theta^{(k)} < 0$, $\theta^{(\ell)} = 0$; $\mathcal{L}_k \theta^{(\ell)} < 0$] (distinguishes outer/inner horizons) $\{k^i \nabla_i \theta^{(\ell)} < 0\}$ ("non expanding" if, moreover, $\mathcal{H}$ is null ($\mathcal{L}_\ell q = 0$))
- a DH (dynamical horizon): $\mathcal{H}$ is foliated by marginally trapped surfaces and is spacelike (Ashtekar-Krishnan 2002)

in dynamical situation the 2 def. can be shown to be equivalent; in stationary situations (eg Kerr) a FOTH becomes a null surface whereas DH do not exist

- There $\exists$ for these horizons $\exists$ & uniqueness theorems etc plus a generalisation of the "membrane paradigm" approach; Gourgoulhon (2005) in particular generalized the Damour-Navier Stokes eqn (where, recall, one ignores the "inside" of a BH and introduces suitable quantities defined on the (event) horizon). An important ~~di~~ difference is that the surface viscosity of the "brane" becomes $\zeta = +\frac{1}{16\pi} > 0$; hence the FOTH & DH behave like ordinary fluids

4. Example of trapping horizon (see e.g. Nielsen ArXiV:0809.3850[hep-th]).

in Painlevé-Gullstrand coordinates (see lectures 1-2) the metric of a spherically symmetric ST can be written as:

$$ds^2 = -e^{-2\Phi(\tau,r)}\left(1-\frac{2m(\tau,r)}{r}\right)d\tau^2 + 2e^{-\Phi(\tau,r)}\sqrt{\frac{2m(\tau,r)}{r}}\,d\tau\,dr + dr^2 + r^2 d\Omega^2$$

radial null geodesics: $\frac{dr}{d\tau} = e^{-\Phi(\tau,r)}\left(\pm 1 + \sqrt{\frac{2m(\tau,r)}{r}}\right)$ [from $ds^2 = 0$

tangent vectors: $l^i = \left(e^{\Phi}, 1-\sqrt{\frac{2m(\tau,r)}{r}}, 0, 0\right)$; $k^i = \frac{1}{2}\left(e^{\Phi}, -1-\sqrt{\frac{2m}{r}}, 0, 0\right)$

[from $g_{ij}l^i l^j = 0$ which yields $l^r = l^\tau\, dr/d\tau$; $l^\tau = e^{\Phi}$ (choice)
factor $\frac{1}{2}$ so that $k^i l_i = -1$]

hence
$$\begin{cases} \theta^{(l)} = q^{ij}\nabla_i l_j = g^{ij}\nabla_i l_j + k^i l^j \nabla_i l_j + l^i k^j \nabla_i l_j = \frac{2}{r}\left(1-\sqrt{\frac{2m}{r}}\right) \\ \theta^{(k)} = -\frac{1}{r}\left(1+\sqrt{\frac{2m}{r}}\right) \quad : \text{at } r = 2m \;\; \theta^{(l)} = 0 \quad \boxed{H : r = 2m} \\ k^i \nabla_i l_j \Big|_H = -\frac{(1-2m_H')}{r_H^2}\left(1 + \frac{\dot{r}_H}{2e^{-\Phi_H}}\right) \quad (r_H = 2m)\text{; impose } m_H' < 1 \end{cases}$$

normal to $r = 2m$: n^i such that $n^i n_i = -4\dot{m}e^{2\Phi} - 4\dot{m}e^{\Phi}(1-2m')$
$= 0$ if $\dot{m} = 0$

5. Hawking radiation for trapping horizons (same ref as above)(see also 7.)

Write $\Box\phi = 0$ for a spherically symmetric ST in Painlevé coordinates and restrict one's attention to s-waves (that is, $\phi = \phi(\tau, r)$).
Write $\phi = e^{iS(\tau,r)}$. In the WKB approximation (ie geometrical optics, ie classical, limit)

S satisfies the hamilton-Jacobi equation: $g^{ij}\partial_i S\,\partial_j S = 0$
inverse of Painlevé metric

separate the variable as $S(\tau, r) = \omega\tau - \int p(r)\,dr$, we get:

$$\omega^2 + 2e^{-\Phi}\sqrt{\frac{2m}{r}}\,\omega p - e^{-2\Phi}\left(1-\frac{2m}{r}\right)p^2 = 0 \qquad (\text{NB}: p \equiv u_r)$$

$$\Leftrightarrow \quad p = \pm\frac{\omega e^{\Phi}}{1 \mp \sqrt{\frac{2m}{r}}}$$
+ sign: outgoing modes
− sign: ingoing modes

Examine now the contribution to the phase S of the outgoing mode by expanding around the horizon: $(r_H = 2m)$

$$S \sim \omega t + \frac{2r_H\, \omega e^{\phi_H}}{1-2m'_H} \int \frac{dr}{r-r_H}$$

the integral is performed "à la Damour-Ruffini" by deforming the contour into the lower half of the complex plane which yields: (see also Parikh & Wilczek)

$$\mathrm{Im}\, S = \frac{4\pi\, r_H\, \omega e^{\phi_H}}{1-2m'_H}$$

Therefore the tunnelling probability $\Gamma \sim \phi\phi^* \sim e^{-2\,\mathrm{Im}\,S} \sim e^{-\frac{\omega}{T_H}}$ with

$$T_H = \frac{1}{2\pi}\,\frac{e^{-\phi_H}}{2r_H}\,(1-2m'_H)$$

(factor 2 pb in this derivation)
(Hayward et al arrive at the same temperature ArXiV: 0812.2534 [gr-qc])
(similar method in Eddington Finkelstein coordinates where $e^{-\phi_H} = 1$)

clearly this is still work in progress! (first paper by Di Criscienzo et al: arXiV: 0707:4425)
(see 7 for a more detailed derivation)

6. Conclusion: Essential vs inessential features of Hawking radiation (M. Visser arXiV: hep-th/0106111)

"All is really necessary is quantum physics plus a slowly evolving future apparent horizon (NOT an event horizon). In particular neither the Einstein equation nor Bekenstein entropy are necessary (nor even useful) in deriving Hawking radiation"

For historical reasons [– Hawking rad was thought to have sth to do with gravity
– and was thought to imply Bekenstein's entropy

however it was soon realized that Hawking rad also arises in eg condensed matter systems [Unruh 1976; Visser "Acoustic BH" 1997] "Analog models of GR".

Indeed, as is clear from Damour-Ruffini's derivation of Hawking's radiation what is required to obtain it is the ∃ of a horizon. See also Brout, Parentani et al who emphasize that the ingredient that matters is an exponential stretching of frequencies associated with an horizon.

(In contrast Bekenstein's entropy is intimately linked to the validity of Einstein's equations: we need to define eg the mass of the BH, which necessitates a theory of gravity).

In conclusion the input to derive Hawking's radiation is basic quantum field theory plus the existence of a Lorentzian metric with an apparent horizon; non zero surface gravity & slow-evolution (to be able to expand fields in modes).

7. Appendix: M. Visser's derivation of Hawking radiation [arXiv: hep-th/0106111]

- general sph. sym. metric in Painlevé coordinates:

$$ds^2 = -[c^2 - v^2]dt^2 - 2v\,dr\,dt + dr^2 + r^2 d\Omega^2 \qquad c = c(t,r);\ v = v(t,r)$$

- apparent horizon (or FOTH) located at $c^2 = v^2$; $\det g = -c^2$ regular at hor.

- define $g_H = \frac{1}{2}(c^2 - v^2)'\Big|_H = c_H(c-v)'_H$; (if geom. is static $\varkappa = g_H/c_H$ is the "surface gravity")

- $\Box\phi = 0$; s-wave: $\phi = \phi(t,r)$, eikonal approximation: $\phi = \phi_0(r,t)e^{\mp i\varphi(r,t)}$

$\varphi(r,t)$ is the action/radial momentum satisfying the Hamilton-Jacobi eqn:

$$g^{ij}\partial_i\varphi\partial_j\varphi + i\epsilon = 0$$

$i\epsilon$: "Feynman's prescription" see below. (in order to enforce analyticity of the fields)

Separate variables $\varphi = \mp i(\omega t - \int k\,dr)$: $\omega^2 - 2v\omega k - (c^2 - v^2)k^2 + i\epsilon = 0$

hence $k = \dfrac{\pm\omega}{(1+i\epsilon)c \pm v}$; current $J_i = |\phi_0|^2(\omega, k, 0, 0)$

$$\nabla_i J^i = 0 \Rightarrow |\phi_0|^2 \propto \frac{1}{r}$$

hence: $\phi(r,t) \sim \dfrac{N}{\sqrt{2\omega}\,r}\exp\left[\mp i\left(\omega t - \int k\,dr\right)\right]$ (N: norm. factor).

- ingoing modes: $k_{in} = -\dfrac{\omega}{(1+i\epsilon)c - v} \overset{r\sim r_H}{\sim} \dfrac{\omega}{(v-c)} - \dfrac{\omega}{2c_H} \Rightarrow \phi_{in} \sim \dfrac{N_{in}}{\sqrt{2\omega}\,r_H}\exp\left[\mp i\omega\left(t + \dfrac{r - r_H}{2c_H}\right)\right]$

(they cross the horizon at velocity $2c_H$)

- outgoing modes: $k_{out} = \dfrac{\omega}{(1+i\epsilon)c + v} \overset{r\sim r_H}{\sim} \dfrac{\omega}{v - c}\ \dfrac{\omega}{(g_H/c_H)(r - r_H) + i\epsilon c_H} \overset{r>r_H}{\sim} \dfrac{\omega c_H}{g_H(r - r_H)}$

hence for $r > r_H$ $\phi_{out} \sim \dfrac{N_{out}}{\sqrt{2\omega}\,r_H}e^{\mp i\omega t}(r - r_H)^{\pm i\omega c_H/g_H}$; mode oscillates with exponentially growing period when $r \to r_H$

- let us now look at outgoing modes that straddle the horizon.

then: $\displaystyle\int_{r_-}^{r_+} k_{out} = \int_{r_-}^{r_+}\frac{\omega}{(g_H/c_H)(r - r_H) + i\epsilon c_H} \overset{\text{contour integral}}{=} \int_{r_-}^{r_+}\frac{c_H\omega}{g_H}\left\{\mathcal{P}\left(\frac{1}{r - r_H}\right) - i\pi\delta(r - r_H)\right\}$

so that
$$\begin{cases} \phi^{\text{straddle}}_{r<r_H} \sim N_{\text{straddle}}\dfrac{|r - r_H|^{\pm i\frac{\omega c_H}{g_H}}}{\sqrt{2\omega}\,r_H}\left(\exp\dfrac{\pi\omega c_H}{g_H}\right)e^{\mp i\omega t} \\[2ex] \phi^{\text{straddle}}_{r>r_H} \sim N_{\text{straddle}}\dfrac{(r - r_H)^{\pm i\frac{\omega c_H}{g_H}}}{\sqrt{2\omega}\,r_H}\exp\mp i\omega t \end{cases}$$

'Boltzmann factor' (Hartle-Hawking Damour-Ruffini Parikh-Wilczek)

full $\phi^{\text{straddle}} = \Theta(r - r_H)\,\phi_{r>r_H} + \Theta(r - r_H)\,\phi_{r<r_H}$

Wronskian property $\Rightarrow$ $|N_{\text{straddle}}|^2\left[\exp\left(\dfrac{2\pi\omega c_H}{g_H}\right) - 1\right] = |N_{\text{escape}}|^2$

$$\Longrightarrow \quad \frac{|N_{straddle}|^2}{|W_{escape}|^2} = \frac{1}{\left(\exp\frac{2\pi\omega c_H}{g_H} - 1\right)}$$

: Planckian spectrum; quantum flux of out. part (assuming that the quantum vacuum state corresponds to $\Phi_{straddle}$; that is freely falling observers do not see anything particular when crossing the horizon (Unruh vacuum),

hence a Temperature $\boxed{T_H = \frac{\hbar}{2\pi}\frac{g_H}{c_H}\frac{1}{k}}$

k: Boltzmann const.

- looking at non s-waves indicates that
  the near-horizon behaviour of the modes is independent of $L_\perp$
  the phase pile up too; as well as the analytic continuation of the modes through the horizon

Note that this derivation is very similar to Damour-Ruffini's.
Note also that the "new" derivation of the Hawking temperature by Parikh & Wilczek hep-th/9907001 is also in the same vein.

BH in GR
lecture 6

IPh-T-6A-09
N. Deruelle

Thermodynamics of Black holes

1. Bekenstein-Hawking entropy

- recap: • Recall Hawking's definition of an "event horizon" (the time development of a bubble of light emitted from the centre of a collapsing object which stabilizes itself under the pull of gravity)

  • Recall Hawking's area theorem : $\delta(\Sigma A) \geqslant 0$ where A is the area of an event horizon (measured at same time t of a distant, inertial, observer). In words: whatever interactions BH may have the sum of the areas of their event horizon always increases (as measured by a distant static observer).

  • Recall Bekenstein's proposal : attribute an entropy to a BH proportional to its area : $S = \alpha \frac{A}{4}$; use Christodoulou-Ruffini's mass formula : $M^2 = \left(M_{irr} + \frac{Q^2}{4M_{irr}}\right)^2 + \frac{J^2}{4M_{irr}}$ with $M_{irr}^2 = \frac{A}{16\pi} = \frac{r_+^2+a^2}{4}$

and differentiate $M = M(J, Q, A)$ to obtain the 1st law of BH thermodynamics : $\boxed{dM = \Omega\, dJ + \Phi\, dQ + T dS}$

where
$$\begin{cases} \Omega \overset{KN}{=} \dfrac{a}{r_+^2+a^2} & \text{is the "angular velocity" of the hole} \\ \Phi \overset{KN}{=} \dfrac{Q r_+}{r_+^2+a^2} & \text{is the "Coulomb potential" on the horizon} \\ T \equiv \dfrac{\kappa}{2\pi}\dfrac{1}{\alpha} & \text{where } \kappa \overset{KN}{=} \dfrac{1}{2}\dfrac{r_+-r_-}{r_+^2+a^2} \text{ is the "surface gravity" of the BH} \end{cases}$$

T is the BH "temperature"

[zeroth law: the temperature T (that is, the surface gravity) of a stationary BH is a constant (Carter)]

[2nd law: $\delta S \geqslant 0$: Hawking's area theorem, which supposes that matter interacting with the BH satisfies the "weak energy condition" (that is $T_{ij}u^i u^j \geqslant 0$)]

meaning of 1st law: a BH is a kind of oven ; when interacting with its surroundings, a variation of δJ & δQ of its angular momentum and charge will induce a variation of its mass ($\equiv$ energy) which will be greater than $\Omega dJ + \Phi dQ$ ($\equiv dW$: the work done by exterior agents)

(2)

• Recall Hawking's discovery that BH radiate like BH with a temperature $T = \frac{\kappa}{2\pi}$ (hence fixing Bekenstein's parameter $\alpha = 1$)
(T is the temperature – as deduced from the Black body spectrum of the radiation – measured by a static observer far from the hole)

• Finally recall that while the definitions of M, J, Q use the field (Einstein) eqns, the obtention of the temperature only requires QM & the ∃ of an horizon.

— Miscellaneous remarks (see eg Scholarpedia article by Bekenstein)

• $S = \frac{A}{4\hbar} = \frac{c^3 A}{4G\hbar} = \frac{A}{4\ell_p^2}$; for a BH of $1 M_\odot$: $S \sim 4\times 10^{77}$
$(A = 4\pi(r_+^2+a^2))$

which is 20 orders of magnitude larger than the thermodynamic entropy of the Sun
Therefore the entropy of a BH is NOT the entropy of the material which formed the BH or fell into it

• an apparent paradox: the energy density of the Hawking black body radiation is σT^4 (Stefan's law); this energy is lost by the BH whose mass, hence, decreases $\left[\dot{m} \sim -\sigma T^4 \times (\text{BH area})\right.$; $T \propto \frac{1}{m}$; $A \propto m^2$
$\left.\Rightarrow \dot{m} \propto -\frac{1}{m^2}\right]$.
now if $m \downarrow$ then $A \downarrow$ too, in apparent contradiction with the area theorem!
reason: the area theorem is violated because the particle falling in the hole carries negative energy so that the weak energy condition is not fulfilled.

• Therefore the 2nd law must be generalized into (Bekenstein 1972):

$\Delta S_0 + \Delta S_{BH} \geq 0$ (where ΔS_0 is the entropy of matter OUTSIDE the holes)

that is: (1) when ordinary matter flows into a BH the increase of the BH entropy (that is, its area) will more than compensate for the disappearance of ordinary entropy from sight

(2) when taking into account Hawking's radiation then the entropy of the emergent radiation more than compensates the drop of black hole entropy due to the emission of this radiation.

This generalized 2nd law has been verified in a number of examples, as well as having been given various theoretical supports but is still a conjecture

• Third law: "The entropy of a system at $T=0$ vanishes". This (Nernst) version of the 3rd law fails for BH. Indeed an extreme BH ($r_+ = r_-$, $M^2 = a^2 + Q^2$) has zero surface gravity, hence zero Temp, but $A \neq 0$.

(see la Recherche Feb 2009 issue!)

— The "information loss" paradox. (see eg Hossenfelder & Smolin 0901.3156 [gr-qc])

- Consider a cold piece of coal (in its ground state) illuminated by a laser beam [ex due to Coleman, see Bekenstein arXiv: quant-phys/0311049; see also Brout et al: arXiv: 0710.4345 [gr-qc]]. The coal will heat up & radiate. The beam is interrupted. The coal radiates thermally and eventually disappears so that we are left with thermal radiation. (Hence we went from a pure to a mixed state). There is an entropy increase, due to "coarse graining", that is, to the fact that information about the original state is "traced out": the information, in fact, is still there, present in subtle correlations between the early & late radiation. (Another example is the evolution of the vacuum quantum fluctuations present in the early universe which are observed today as classical, statistical CMB anisotropies. Here too there is no loss of information, at least in principle, information being "hidden" in the decaying modes which are ignored.).
- The situation is (may be?) different in the case of BH radiation as information about the partner of the emitted Hawking photon is inextricably lost into the BH. The final mixed state is dictated by physics (ie the ∃ of an horizon) and not as a result of coarse graining. Hawking was thus led to state that gravity violates the unitarity principle of QM. (1976)
- Another way to say the same thing! If you burn down your TV set you can, in principle, tell from the remaining ashes everything about your TV set (although you usually choose not too: coarse graining). If you throw your TV set in a BH, Hawking's radiation will not tell you anything about it after it has crossed the horizon and you will not get the info back either after the hole has completely evaporated. Quantum mechanically: the correlations of the states are lost; a pure state has turned into a mixed one; unitarity is lost
- Hawking however changed his mind in 2004 (hep-th/0507171). He looked at BH formation & evaporation as a scattering process; the fact that one cannot be certain that a BH has formed preserves unitarity.

– The "holographic bound". (Susskind 1995)

* consider a neutral non rotating object containing entropy S within an area A suppose it collapses to a Schwarzschild BH; the area of the BH is $< A$. Now the GSL says that $\boxed{S < \frac{A}{4}}$ (initial entropy $(=S)$ < final entropy $(=S_{BH})$)

The equality is the "holographic bound". Hence information (that is, S) which scales like the volume is bounded by the surface of the medium where information is stored

* of course link to "holographic principle" ('t Hooft) which asserts that processes in a D-dim universe as described by say string theory of field theory are in one-to-one correspondence with processes taking place on its (D-1)-dim boundary described by a different theory (AdS/CFT). ["just like" the hologramme we see in 3D is built from information coded in a 2D film]

* Thus if processes in the bulk can be understood by correspondence with processes on its boundary then the information about the bulk is not so large that it cannot be bounded in terms of the surface of the boundary.

(see bekenstein quant-ph/0311049)

— *attempts to give a statistical meaning to S*:

(ie to compute S as $S = \ln W$ where W is the number of equally probable microstates of a particular macrostate).

- quantize the area $\frac{1}{4}A = \ell_p^2 (n + \text{cte})$; then each level A_n has a huge degeneracy $g_n = \exp \frac{1}{4}\frac{A_n}{\ell_p^2}$ (see Bekenstein, Mukhanov eg Bekenstein gr-qc/9808028)
- Loop quantum gravity (see eg arXiv: 0901.1302 (Corichi))
- possible link with quasi-normal modes (S. Hod, 1998)
- breakthrough: string theory allows a counting of the microstates of (extremal) black holes and yields $S = \frac{1}{4}A$ (with the factor 1/4). see A. Strominger & C. Vafa (1996); C. Callan & J. Maldacena (1997).

2. The first law of BH thermodynamics as a tool to study BH ST.

- suppose we have found or we are given a (stationary and axisymmetric) ST describing a BH. In practice this means that we are given a metric which, in coordinates adapted to the symmetries is of the form $g_{ij} = g_{ij}(r, \theta, \mu, a_i)$ where μ & a_i are integration constants
- a first task is to ensure we are dealing with a BH and find the location of the horizon; various ways; one is to compute the determinant of the metric induced on the 3-surfaces $r = \text{cte}$; if it vanishes for $r = r_+$ then $r = r_+$ is a null surface and a good candidate. Since $r_+ = r_+(\mu, a_i)$ we can trade μ for r_+ by inversion: $\mu = \mu(r_+, a_i)$; One also knows the area of the BH: $A = A(r_+ a_i)$.
- having located the horizon one can define the angular velocity of the hole [angular velocity of a photon orbiting the horizon, as measured by an observer at infinity (flat) who is static]: $\Omega = \Omega(r_+, a_i)$. Similarly one can define its surface gravity [if the BH does not rotate it is the force per unit mass an observer at ∞ must exert on a string holding the particle close to the horizon]. In the more general case

the def of surface gravity is the following (Carter 1968, see R. Wald p 331):
If $\xi^i_{(t)} = (1000)$ and $\xi^i_\varphi = (0001)$ are the Killing vectors for stationarity & axisymmetry then $\chi^i = \xi^i_{(t)} + \Omega \xi^i_{(\varphi)}$ is also a Killing vector which is null & $\perp$ to $\xi_{(t)}$ & $\xi_{(\varphi)}$ on the horizon ; moreover one has:

$$D^i(\chi_j \chi^j) = -2\kappa \chi^i \quad \text{where } \kappa \text{ is a constt : the surface gravity.}$$

Hence $\kappa = \kappa(r_+, a_i)$ is known.

- let us now turn to the definition of the "global charges" associated with the BH, that is masse M & angular momentum (a) J (vacuum case)
Up to now we have defined them in terms of the integration constants ($M = M(\mu, a_i)$; $J = J(\mu, a_i)$) by looking at the asymptotic form of the metric and identifying it with the PN metric describing the asymptotic grav. field of a material body (not BH) : [see eg lect 1 on the gravit. field created by a star of ctt density (Schwarzschild's interior solution); we found that outside the star $g_{00} = -\left(1 - \frac{2\mu}{r}\right)$ where $\frac{\mu}{r}$ had to be identified with the Newtonian potential, hence the integration constant μ must be identified to $M_{grav.}$; we also linked μ to the stress energy tensor of the star : $\mu = \frac{4}{3}\pi \rho R^3$]. (NB) M & J depend on the gravity theory used.

⚠ When dealing with "exotic" BH (higher dimension, non asymptotically flat ST, gravity theories with no newtonian limit...) The identification of these global charges will have to be given a more precise meaning.

- Now that we have in hand : $M(r_+, a_i)$; $J(r_+, a_i)$; $\Omega(r_+, a_i)$; $\kappa(r_+, a_i)$ we can introduce the temperature of the BH. As we saw in lect 5 this is a pretty universal concept (which requires an horizon only; not even field equations). And this temperature is given by
$T = \frac{\kappa}{2\pi}$ where κ is the surface gravity of the hole.

- At this stage the 1st law of BH thermodynamics can be a useful tool to check & compare results, or to DEFINE the entropy!

Indeed $dS = \frac{1}{T}(dM - \Omega dJ) = \underbrace{\frac{1}{T}\left(\frac{\partial M}{\partial r_+} - \Omega\frac{\partial J}{\partial r_+}\right)}_{\equiv \frac{\partial S}{\partial r_+}} dr_+ + \underbrace{\frac{1}{T}\left(\frac{\partial M}{\partial a_i} - \Omega\frac{\partial J}{\partial a_i}\right)}_{\equiv \frac{\partial S}{\partial a_i}} da_i$

— a first check (of the definitions of M & J) is to check that $\frac{\partial^2 S}{\partial a_i \partial r_+} = \frac{\partial^2 S}{\partial r_+ \partial a_i}$

— Then integration gives $S = S(r_+, a_i)$. If the BH is a solution of Einstein's equation then one <u>must</u> find $S = \frac{1}{4}A$

— trivial ex: Schwarz ("top to bottom"): $T = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}$; $dS = \frac{dM}{T} = 8\pi M dM$

$\Rightarrow S = 4\pi M^2 = \frac{1}{4}A$: fine!

— next to trivial ex: Kerr: $\begin{cases} r_+ = M - \sqrt{M^2 - a^2} \Leftrightarrow M = \frac{r_+^2 + a^2}{2r_+} ; \; J \equiv Ma = \frac{a(r_+^2 + a^2)}{2r_+} \\ \Omega = \frac{a}{r_+^2 + a^2} ; \; T = \frac{1}{4\pi}\frac{r_+^2 - a^2}{r_+^2 + a^2} \end{cases}$

it is a simple (Mathematica) exercise to check that

$$(1): \frac{\partial}{\partial a}\left[\frac{1}{T}\left(\frac{\partial M}{\partial r_+} - \Omega\frac{\partial J}{\partial r_+}\right)\right] = \frac{\partial}{\partial r_+}\left[\frac{1}{T}\left(\frac{\partial M}{\partial a} - \Omega\frac{\partial J}{\partial a}\right)\right]$$

and to see that $S = \frac{1}{4}A$ with $A = 4\pi(r_+^2 + a^2)$.

3. <u>Euclidean path integral approach & the 1st law</u> (Gibbons & Hawking 1977, Gibbons & Perry 1977)

[partition function $Z(\beta) = \int d[g]\, e^{-I[g]} \to \ln Z = -I_{\text{class on shell}}$, $Z = e^{-\beta F} = e^{-\beta E + S}$]

<u>The recipe</u>: the ex of the metric: $\begin{cases} ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{n-1}^2 \\ f(r) = 1 - \frac{\mu}{r^{n-2}} \end{cases}$ ($D = n+1$) (D-dim Schw. solution)

— horizon: $f(r_+) = 0 \Rightarrow \mu = r_+^{n-2}$; Area: $A = r_+^{n-1}\,\Omega_{n-1}$ ($n=3$: $A = 4\pi r_+^2$).

— to find the temperature: go to Euclidean signature: $ds_E^2 = +f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{n-1}^2$ and find that the metric is well-behaved at $r = r_+$ if t is periodic with period: $t = \beta = \frac{4\pi}{f'_+}$; and deduce T as $T = \frac{1}{\beta} = \frac{f'_+}{4\pi}$ ($n=3$ $T = \frac{1}{8\pi M}$, ok) ($\mu = 2M$).

(for justification see courses on thermal Green functions + Gibbons et al.)

[indeed: $d\sigma^2 = f(r)dt^2 + \frac{dr^2}{f(r)}$; near horizon $f(r) = f_+ + f'_+(r - r_+) + \ldots = f'_+(r - r_+) + \ldots$

introduce x: $\frac{dr^2}{f} = dx^2 \Leftrightarrow \frac{dr}{\sqrt{f'_+}\sqrt{r - r_+}} = dx \Leftrightarrow x = \frac{2\sqrt{r - r_+}}{\sqrt{f'_+}} \Leftrightarrow r - r_+ = \frac{x^2 f'_+}{4}$

$\Rightarrow d\sigma^2 = f'_+(r - r_+)dt^2 + dx^2 = \frac{f_+'^2}{4}x^2 dt^2 + dx^2$; introduce $\bar{T} = \frac{f'_+}{2}t$; $d\sigma^2 = x^2 d\bar{T}^2 + dx^2$

flat S,T if $\bar{T}$ has a period 2π, that is, if t has a period $4\pi/f'_+$]

– compute now the Euclidean action on shell. In Einstein gravity:

$$I = -\frac{1}{16\pi}\int_{\mathcal{V}} d^{n+1}x \,\underbrace{\sqrt{g}\,R}_{\text{Hilbert Lagrangian}} + I_b \; ; \quad I_b = -\frac{1}{8\pi}\int_{\partial\mathcal{V}} \underbrace{\sqrt{h}}_{\text{induced metric in } \partial\mathcal{V}}\, \underbrace{K}_{\text{extrinsic curvature of } \partial\mathcal{V}}$$

I_b (the Gibbons-Hawking surface term) is such that when varying the action with respect to the metric one gets Einstein's equations by keeping g_{ij} only fixed at the boundary (and not g_{ij} plus its normal derivatives).

On shell $R = 0$ & I reduces to the boundary term. $\left(I_b = -\frac{1}{8\pi}\beta\,\sigma^{n-1}\, \underbrace{\sqrt{f}\,\frac{\partial}{\partial r}\left(r^{n-1}\sqrt{f}\right)}_{\frac{\partial}{\partial n}\sqrt{h}}\Big|_{r_0}\right)$

I_b diverges $\left(I_b = -\frac{\sigma_{n-1}\, r_+}{2(n-2)}\left[(n-1)\, r_0^{n-2} - \frac{n}{2}\mu\right] \quad r_0 \to \infty\right)$ and must be regularized (by subtracting the flat space value computed on $r = r_0$ with the same induced geometry so that $\beta_0 \sqrt{f}(r_0, \mu = 0) = \beta\sqrt{f}(r_0)$). Hence the final result:

$$I = \lim_{r_0\to\infty} (I_b - I_0) = \frac{\sigma_{n-1}\, r_+^{n-1}}{4(n-2)}$$

– It is then known (to the expert) that the energy E & entropy S are obtained as:

$$E = \frac{\partial I}{\partial\beta} = \frac{(n-1)\,\sigma_{n-1}\, r_+^{n-2}}{16\pi} = \frac{(n-1)\,\sigma_{n-1}}{16\pi}\mu$$

which must be identified to the Energy $= M$ of the BH (that one gets the exact mass is a "miracle")

and: $$S \equiv \beta E - I = \frac{r_+^{n-1}\,\sigma_{n-1}}{4} = \frac{A}{4}$$

– resumé: we have a series of recipes which, given a BH spacetime, yield

(1) the temperature of the hole $T = f'_+/4\pi$

(2) the mass & entropy of the hole once the (regularized) euclidean action has been computed

(NB: in the case of AdS Schw. BH $\left(f = 1 - \frac{\mu}{r^{n-2}} + r^2/\ell^2\right)$ the bulk action on shell $\left(I_{bulk} = -\frac{1}{16\pi}\int d^{n+1}x\sqrt{g}\,R\right)$ is no longer zero and the (regularized) boundary term does not contribute. The regularization procedure of I_{bulk} proceeds along the same line & yields:

$$I = \sigma^{n-1}\frac{\left(\ell^2 r_+^{n-1} - r_+^{n+1}\right)}{4\left(n r_+^2 + (n-2)\ell^2\right)}$$

so that

$$\begin{cases} E \equiv M = \dfrac{\partial I}{\partial\beta} = \dfrac{(n-1)\,\sigma^{n-1}}{16\pi} r_+^{n-2}\left(r_+^2\ell^{-2} + 1\right) = \dfrac{(n-1)\,\sigma^{n-1}}{16\pi} r_+^{n-2}\mu \\[2ex] S = \beta E - I = \dfrac{r_+^{n-1}\sigma^{n-1}}{4} = \dfrac{A}{4} \end{cases}$$

– The entropy & mass thus obtained must coincide with the results deduced from the 1st law AND the definition of the mass as a global charge.

Conservation laws & definition of "global charges"

1. Introduction

- As already mentioned a number of times the definition of the mass & angular momentum (more generally the "global charges") of a ST is not a trivial matter (when one leaves the firm grounds of 4D asymptotically flat ST).
- It is important to have reliable definitions when one want to study the thermodynamics of (exotic) BH (see above).

- The problem is old (Einstein 1916). (*) In SR the conservation law of the stress-energy tensor of matter, $\partial_i T^{ij} = 0$, in an inertial frame can be integrated to define the inertial mass of the system: $\partial_0 T^{00} = -\partial_\alpha T^{0\alpha}$

$$\Rightarrow \quad \partial_0 \int_{V_3} d^3X\, T^{00} = -\int_{V_3} \partial_\alpha T^{0\alpha} d^3x = \int_{S_{2\infty}} T^{0\alpha} n_\alpha d^2x = 0 \text{ if isolated system}$$

hence $M \equiv \int_{V_3} d^3x\, T^{00}$ is a constant: the inertial mass of the system.

(*) in GR: $D_i T^{ij} = 0 \iff \partial_i T^{ij} + \underbrace{\Gamma^i_{ik} T^{kj} + \dots}_{\neq 0} = 0$

the extra terms "encode" the gravitational contribution to the mass-energy of the system. If one can extract some global conserved quantity from this it will be the INERTIAL mass of the system ($M = \Sigma m + \frac{W}{c^2}$ where W is the binding energy of all interactions, including gravity). This mass will be equal to the gravitational mass (which appear in the asymptotic form of the metric: $g_{00} \sim -\left(1 - \frac{2M_{gr}}{r}\right)$) (IF) the strong equivalence principle holds [Whether the principle holds or not depends on the gravity theory used; OK for GR (& 1913 Nordström's theory); not OK for scalar-tensor theories; status of other theories unclear].

(*) in the 1st law of thermodynamics we have to decide which M appears. Choice: $M_{inertial}$, that is, the global "charge".

- what one has then to do in practice: convert $D_i T^{ij} = 0$ into $\partial_i \Theta^{ij} = 0$ (Einstein 1916, Landau-Lifchitz 1946; $\Theta_i^j = T_i^j + \Gamma\Gamma$ is NOT a tensor. Various ways to cure that: Komar (59); ADM; Ashtekar; York etc and BKL.

2. The Katz - Bičák - Lynden Bell proposal (1985, 1997)

- Noether identities, conserved current, superpotential, charge.
  - Consider a gravitational lagrangian $\hat{\mathcal{L}} \equiv \underbrace{\hat{L}(g, \partial g)}_{\text{eg } \sqrt{g}R} + \underbrace{D_i \hat{k}^i}_{\text{surface term}}$ ($\hat{} = \sqrt{g} \times$)

    vector (see below)

    ($\neq$ Gibbons-Hawking)
  - vary $\hat{\mathcal{L}}$ wrt metric: $\delta\hat{\mathcal{L}} = -\underbrace{\hat{G}^{ij}}_{\text{Einstein tensor}} \delta g_{ij} + \partial_i(\underbrace{\hat{V}^i + \delta\hat{k}^i}_{\text{boundary term}})$ (1) ($\hat{V}^i = \otimes\delta g + \otimes\delta\Gamma$)
  - suppose the variation of the metric is due to a mere coordinate transformation:

    $x^i \to x^i + \xi^i$ [$\Rightarrow \delta\hat{\mathcal{L}} = \partial_i(\hat{\mathcal{L}}\xi^i)$; $\delta g_{ij} = 2D_{(i}\xi_{j)}$]

    and rewrite (1) as: $\partial_i \hat{j}^i = 0$ with $\hat{j}^i \equiv (2\hat{G}^{ij} + g^{ij}\hat{L})\xi_j - \hat{V}^i + 2D_j(\xi^{[i}\hat{k}^{j]})$
  - $\hat{j}^i$ is a "conserved current"; hence $\exists$ a "superpotential" $\hat{j}^{[ij]}$ such that $\partial_j \hat{j}^{[ij]} \equiv \hat{j}^i$; ansatz: $\hat{j}^{[ij]} = A\xi + B(D\xi) + C\xi k$

    Compute $\partial_j \hat{j}^{[ij]}$; identify with $\hat{j}^i$ and get A B & C.
  - since $\partial_i \hat{j}^i = 0 \Rightarrow$ by integration on V_3 & use of Stokes theorem we have that $q \equiv \int_{S_{2,\infty}} d^{D-2}x \; \hat{j}^{[01]}$ is a constt in time (surface integral at ∞)

- choice for the vector ξ^i: $\begin{cases} \text{time-translation Killing vector} & \to \text{mass} = M \\ \text{rotation} & \to J \end{cases}$

- covariantisation / regularisation: subtract to q an asymptotic background value (same thing is done in Euclidean path integral calcul.).

  and define $Q \equiv \int_{S_{\infty 2}} d^{D-2}x \; \hat{J}^{[01]}$ $\hat{J}^{[ij]} = \hat{j}^{ij} - \overline{\hat{j}^{ij}}$

- choice for the vector k^i:

  on shell $\delta I \equiv \int_{\mathcal{V}} d^D x \, \delta\hat{\mathcal{L}} = \int_{\partial\mathcal{V}} d^{D-1}x \, n_i(\hat{V}^i + \delta\hat{k}^i)$ (see (1))

  $\otimes\delta g + \otimes\delta\Gamma$

  choose k^i so as to cancel as many $\delta\Gamma$'s as possible [this will depend on the theory that is on the choice for $\mathcal{L}$].

3. KBL charge in Einstein theory

$$L = -2\Lambda + R \quad ; \quad V^i = -\left(g^{ij}\,\delta\Gamma^k_{jk} - g^{jk}\,\delta\Gamma^i_{jk}\right)$$

$$\hat{j}^{[ij]} = 2\left(D^{[i}\hat{\xi}^{j]} + \xi^{[i}\hat{k}^{j]}\right) \quad ; \text{ choose } k^i = \left(g^{ij}\Delta^k_{jk} - g^{jk}\Delta^i_{jk}\right)$$

where $\Delta^i_{jk} = \Gamma^i_{jk} - \bar{\Gamma}^i_{jk}$

$$\Rightarrow \quad Q = -\frac{1}{8\pi}\int_{S_\infty} d^{D-2}x \left(D^{[0}\hat{\xi}^{1]} - \overline{D^{[0}\hat{\xi}^{1]}} + \xi^{[0}\hat{k}^{1]}\right)$$

$\xi^i = (1000) \Rightarrow Q \equiv M$ (of course shown to work for Kerr!)

$\xi^i = (0001) \Rightarrow Q \equiv J$ (........................)

(The proposal was extended to various other theories).

Stationary black holes beyond GR
a few examples

1. Introduction

- No hair theorem:
  All stationary, asymptotically flat, 4D black hole equilibrium solutions of the Einstein equations in vacuum or with an electromagnetic field are characterized by their mass, angular momentum and (electric or magnetic) charge.
  Hence such BH are Kerr-Newmann BH

If one or several assumptions are relaxed then ... there is room for other sol.!

- Class (1) if a negative cosmological constant is added ($G_{ij} = \Lambda g_{ij}$) so that ST is asymptotically anti-de Sitter at ∞ ($R_{ijkl} = -\frac{1}{\ell^2}(g_{ik}g_{jl} - g_{il}g_{jk})$) then the event horizon of the BH is not necessarily spherical, giving rise to "topological" black holes. (in 4D). (see e.g. Vanzo PRD (1997))

- Class (2) keep Einstein vacuum equ but change dimensions
  eg | $D = 3$: BTZ BH (see below)
  | $D = 5$ & higher : * Myers-Perry generalisation of Kerr [illegible]
  * black rings (Emparan & Reall 2002)

- Class (3) 4D but f(Riemann) action instead of the Hilbert action R. example $f = f(R)$; the field equations are 4-th order in the metric g_{ij} and are equivalent to the Einstein equations for a (conformally) related metric g^*_{ij} minimally coupled to a scalar field ($G^*_{ij} = \partial_i\varphi\partial_j\varphi - g_{ij}(\frac{1}{2}(\partial^*\varphi)^2 + V(\varphi))$ where the shape of $V(\varphi)$ is related to $f(R)$ — and φ is related to R).
  In this case it is possible to extend the no-hair theorem: if $f(R) = R + a_2 R^2 + a_3 R^3 + \dots$ with $a_2 > 0$ then the only solution is Schwarzschild's (Mignemi-Wiltshire 1992) (if asymptotically flat)
  (Beware however that there is no Birkhoff theorem so that the gravitational field outside a STAR is NOT Schwarzschild's).

- Class (4) go to higher dimension with a f(Riemann) action - a particularly well studied case is when f(Riemann) is such that the field equations remain 2nd order:

$$f = Riem^2 - 4Ric^2 + R^2$$ ("Gauss-Bonnet" Lagrangian).

(and its generalisations by Lovelock). (see below)

- Class (5) hairy black holes in 4D → scalar field + electromagnetic field (Gibbons 1982, first example of "hairy BH")

→ Yang-Mills fields; first (Bartnik & McKinnon 1988)

(see the review by Volkov & Gal'tsov Phys. Report 1999)
see also E. Winstanley arXiv: 0801.0527 [gr-qc]

- where not only are the solutions characterized by more than the "standard global charges" M, J & Q but where the geometry outside the horizon is not uniquely determined by the global charges
- all known sol are unstable however (in pure YM & asympt flat)
- however the situation changes if ST is asymptotically AdS where ∃ stable SU(2) EYM BH.

- Class (6) mixture of all the above!
[as an example Gemonini et al 0812.3572 [hep-th]]

2. Two examples from "class" (5) (hairy BH)

- Gibbons solution (1982) : the first "flawless" BH sol with hair
- Consider the action

$$W = \frac{1}{16\pi}\int d^4x\,\sqrt{-g}\;\underset{\uparrow\ \text{"dilaton"}}{e^{-2\varphi}}\,[\underset{\uparrow\ \text{Hilbert-term (GR)}}{R} - 4(\nabla\varphi)^2 - \underset{\uparrow\ \text{Maxwell-term}}{F_{ij}F^{ij}}]$$

(This is the "string" frame action; by making the conformal transformation: $g^{(E)}_{ij} = e^{-2\varphi} g_{ij}$ one can rewrite W in the "Einstein frame" as:

$$W = \frac{1}{16\pi}\int d^4x\,\sqrt{-g_E}\,[R_E - 2(\nabla\varphi)^2 - e^{-2\varphi}F_{ij}F^{ij}]$$

- write the eom ($\delta W/\delta g_E^{ij} = 0$; $\partial W/\delta\varphi = 0$; $\delta W/\delta A_i = 0$); look for a static, spherically symm sol ie

$$\begin{cases} ds_E^2 = -f(r)dt^2 + g(r)dr^2 + h(r)(d\theta^2 + \sin^2\theta\, d\varphi^2) \\ \varphi = \varphi(r); \quad A^0 = A^0(r), \quad A^\alpha = 0 \end{cases}$$

Choose $f = 1/g$ (choice of r coordinate) & find (Gibbons):

$f = 1 - \frac{2m}{r}$; $h = r - \frac{Q^2}{m}$; $F_{rt} = Q/r^2$; $\exp 2\varphi = 1 - Q^2/mr$.
m: gravit. mass; Q: charge.

which translates in the "string" frame as:

$$ds^2 = -\frac{(1-2m/r)}{(1-Q^2/mr)}dt^2 + \frac{dr^2}{(1-2m/r)(1-Q^2/mr)} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

(singularity at $r = Q^2/m$)

- EYM BH (4D) $\quad W_{EYM} = \frac{1}{2}\int d^4x\,\sqrt{-g}\,[R - 2\Lambda - \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}]$

eom:

$$\begin{cases} T_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} \\ 0 = \nabla_i F^i_j + [A_i, F_j{}^i] \end{cases} ; \quad T_{ij} = \mathrm{Tr}\, F_{ik}F_j{}^k - \frac{1}{4}g_{ij}\,\mathrm{Tr}\, F_{kl}F^{kl}$$

ansatz: $ds^2 = -\mu S^2 dt^2 + \frac{dr^2}{\mu} + r^2 d\omega^2$ $\quad \mu = 1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3}$; $S = S(r)$

$$A = \mathcal{A}\,dt + \mathcal{B}\,dr + \frac{1}{2}(C - C^H)d\theta - \frac{i}{2}[(C + C^H)\sin\theta + D\cos\theta]\,d\phi$$

SU(N):
- $\mathcal{A}$, $\mathcal{B}$, C, D are $N\times N$ matrices & C^H is the hermitian conj of C
- $\mathcal{A}$ $\mathcal{B}$ are purely imaginary, diagonal, traceless
- C is upper-triangular : $C_{j,j+1} = \omega_j(r)\, e^{-i\gamma_j(r)}$
- D is a const matrix

$\mathcal{A} = 0$ if purely magnetic sol; $\quad \mathcal{B} = 0$: choice of gauge

& YM eqns give $\gamma_i = 0$

hence: $A = \frac{1}{2}(C - C^H)d\theta - \frac{i}{2}[(C + C^H)\sin\theta + D\cos\theta]\,d\phi$

write E's eqns ... solve! (see E. Winstanley 0801.0527 [gr-qc])

3. Two examples from "Class 2" (change D and allow for Λ)

• **BTZ solution** (Banados, Teitelboim, Zanelli, 1992). 3D

$$I = \frac{1}{2\pi}\int \sqrt{-g}\,[R + 2\ell^{-2}]\,d^2x\,dt \qquad (\Lambda = -\ell^{-2})$$

$\exists$ the solution:
$$\begin{cases} ds^2 = -N^2dt^2 + \dfrac{dr^2}{N^2} + r^2(N^\phi dt + d\varphi)^2 & (0 \le \varphi \le 2\pi) \\ N^2 = -M + \dfrac{r^2}{\ell^2} + \dfrac{J^2}{4r^2}\ ; \quad N^\phi = -\dfrac{J}{2r^2} \end{cases}$$

$\exists$ 2 horizons ($N=0$ pour $r_\pm = \ell\sqrt{\frac{M}{2}\left(1 \pm \sqrt{1 - \left(\frac{J}{M\ell}\right)^2}\right)}$ $(\Rightarrow M>0,\ |J|<M\ell)$ — rotating solution ("vacuum": when $r_+ \to 0$ ie M AND $J \to 0$ (vacuum $\neq$ AdS))

• **5D Kerr AdS BH** (Myers & Perry 1986 for $\Lambda \neq 0$; Gibbons et al 2004 when $\Lambda \neq 0$) (certainly NOT unique)

Recap on Kerr-Schild metrics:
$$\begin{cases} g_{ij} = \bar g_{ij} + l_{ij}\ ; \quad l_{ij} = f(x^k)\, l_i l_j \\ \bar g_{ij} l^i l^j = 0\ ; \quad l^i \bar D_i l^j = 0 \quad (l^i \text{ is null \& geodesic}) \end{cases}$$

Then $R^i_j = \bar R^i_j - l^{ik}\bar R_{kj} + \bar D_k(g^{il}\Delta^k_{jl})$; $\left(\Delta^i_{jk} = \frac{1}{2}(\bar D_j l^i_k + \bar D_k l^i_j - \bar D^i l_{jk})\right)$
is LINEAR in the "perturbation" f $\quad [R = \bar R - l^{ij}\bar R_{ij} + \frac{1}{\sqrt{-g}}\partial_k(l^k \partial_l(\sqrt{-g} f l^l))]$

choose the background to be AdS: $\bar R_{ijkl} = -\frac{1}{\ell^2}(\bar g_{ik}\bar g_{jl} - \bar g_{il}\bar g_{jk})$ (*)

5D in spheroidal coordinates $(t, r, \theta, \varphi, \psi)$

$$\begin{cases} d\bar s^2 = -\dfrac{(1 + r^2/\ell^2)\Delta_\theta}{\Xi_a \Xi_b}dt^2 + \dfrac{r^2\rho^2}{(1+r^2/\ell^2)(r^2+a^2)(r^2+b^2)}dr^2 + \dfrac{\rho^2}{\Delta_\theta}d\theta^2 + \\ \quad + \dfrac{r^2+a^2}{\Xi_a}\sin^2\theta\, d\varphi^2 + \dfrac{r^2+b^2}{\Xi_b}\cos^2\theta\, d\psi^2 \\ \text{where } \Delta_\theta = \Xi_a\cos^2\theta + \Xi_b\sin^2\theta\ ; \quad \rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta\ ; \begin{cases}\Xi_a = 1 - a^2/\ell^2 \\ \Xi_b = 1 - b^2/\ell^2\end{cases} \end{cases}$$

(such that (*) is verified — Mathematica helps!)

Then
$$\begin{cases} l_i dx^i = \dfrac{\Delta_\theta}{\Xi_a\Xi_b}dt + \dfrac{r^2\rho^2}{(1+r^2/\ell^2)(r^2+a^2)(r^2+b^2)}dr + \dfrac{a\sin^2\theta}{\Xi_a}d\varphi + \dfrac{b\cos^2\theta}{\Xi_b}d\psi \\ \text{is null \& geodesic (again, Mathematica helps).} \end{cases}$$

Impose Einstein's equations: $\Lambda\delta^i_j + G^i_j = 0$; if $\Lambda = -\frac{6}{\ell^2}$ then the AdS background satisfies the EOM.

Thanks to the linearity property of G^i_j the trace equation $(5\Lambda - R = 0)$ reads: $(r\rho^2 f)'' = 0 \iff \rho^2 f = 2m(\theta) + \frac{d(\theta)}{r}$

Plugging back this expression for f into the remaining field equations shows that indeed f is a solution if $d(\theta) = 0$ and $m(\theta) = m$ ie if $\boxed{f = \frac{2m}{\rho^2}}$

= Just like the Kerr-Schild form of the Kerr metric (see lect 2) the geometry thus obtained is not easily interpreted. To get rid of the $dt\,dr$ terms one goes to Boyer-Lindquist type coordinates:

$$dt = d\tau + \frac{2m\,dr}{(1+r^2/L^2)(V-2m)} \quad ; \quad \begin{cases} d\varphi = d\bar{\varphi} + \frac{a\,d\tau}{L^2} + \frac{2ma\,dr}{(r^2+a^2)(V-2m)} \\ d\psi = d\bar{\psi} + \frac{b\,d\tau}{L^2} + \frac{2mb\,dr}{(r^2+b^2)(V-2m)} \end{cases}$$

where $V = \frac{1}{r^2}\left(1+\frac{r^2}{L^2}\right)(r^2+a^2)(r^2+b^2)$

so that the metric becomes:

$$ds^2 = -\frac{\Delta_\theta}{\Xi_a\Xi_b}(1+r^2/L^2)d\tau^2 + \frac{r^2\rho^2 dr^2}{(1+r^2/L^2)(r^2+a^2)(r^2+b^2)-2mr^2} + \frac{2m}{\rho^2}\left(d\tau - \frac{a\sin^2\theta\,d\bar\varphi}{\Xi_a} - \frac{b\cos^2\theta\,d\bar\psi}{\Xi_b}\right)^2$$
$$+\frac{r^2+a^2}{\Xi_a}\left[\cos^2\theta\,d\theta^2 + \sin^2\theta\left(d\bar\varphi + \frac{a}{L^2}d\tau\right)^2\right] + \frac{r^2+b^2}{\Xi_b}\left[\sin^2\theta\,d\theta^2 + \cos^2\theta\left(d\bar\psi + \frac{b}{L^2}d\tau\right)^2\right] -$$
$$-\frac{\sin^2\theta\cos^2\theta}{L^2\Delta_\theta}(a^2-b^2)^2\frac{(1+r^2/L^2)}{\Xi_a\Xi_b}d\theta^2. \qquad (!)$$

(In the case when $L^2\to\infty$ (no cosmological constant, flat background) and when $b=0$, the metric simplifies into:

$$ds^2 = -d\tau^2\left(1-\frac{2m}{\rho^2}\right) - 4\frac{ma}{\rho}\sin^2\theta\,d\tau\,d\bar\varphi + \sin^2\theta\,d\bar\varphi^2\left[r^2+a^2+\frac{2ma^2}{\rho^2}\sin^2\theta\right] + r^2\cos^2\theta\,d\bar\psi^2$$
$$+\frac{\rho^2dr^2}{r^2+a^2-2m} + \rho^2 d\theta^2$$

(resembles 4D Kerr). ($\rho^2 = r^2+a^2\cos^2\theta$)

(r = cte surfaces: det of induced metric: $g^{(3)} \propto V-2m$)

= horizon: where the denominator of dr^2 vanishes: $(1+r^2/L^2)(r^2+a^2)(r^2+b^2) = 2mr^2$ (where also $g^{(3)}=0$; null surface).

angular velocities of horizon are found to be $\Omega^1_H = \frac{a\,\Xi_a}{r_H^2+a^2}$; $\Omega^2_H = \frac{b\,\Xi_b}{r_H^2+b^2}$

area of horizon is found to be $A = \frac{V_3}{r_H}\frac{(r_H^2+a^2)(r_H^2+b^2)}{\Xi_a\Xi_b}$ $\quad (V_3 = 2\pi^2)$

surface gravity: $\kappa = \frac{1}{2}\left(1+r_H^2/L^2\right)\left.\frac{V'}{V}\right|_H$ (V given above).

(see Gibbons et al hep-th/0404008 for details).

- by analogy with 4D-Kerr (we shall do better below) one can define the mass of the black hole as m (coefficient of $d\tau^2$); and define its 2 angular momenta as $J_a = ma$ & $J_b = mb$.

5D Kerr AdS and the first law

(1) it was shown that the mass m & angular momenta J_a are indeed those obtained using various definitions of global charges (eg KBL)

(2) Then the 1st law of Thermody $T\,dS = dM - \Omega\,dJ$ could be explicitely checked

(3) The mass & entropy could also be obtained via path integral method to yield consistent results.

Conclusion

In this set of lectures I tried to put the development of the BH concept in a historical perspective:

1915-1925: early years of discoveries & debates (Schwarzschild & R.N. solutions found, the "magic circle" $r = 2m$ explored & rejected).

1925-1939: the input of cosmology (Lemaître) & quantum physics (Chandrasekhar, Landau) to understand that $r = 2m$ is not a singular surface that can be crossed by a collapsing star. 1939: Oppenheimer-Snyder paper.

1940-1955. "Low water mark" (WW2)

1955-1963: Renaissance: The role of J. Wheeler, Y. Zel'dovich, D. Sciama and R. Penrose. Maximal extension of Schwarzschild ST (Kruskal); discovery of quasars (1963), Kerr solution (63)

1963-1975: "Golden age". geometry of BH understood; singularity & uniqueness theorems; birth of "relativistic astrophysics". Launch of UHURU satellite. Les Houches 1972 summer school. Hawking's discovery of BH radiation.

1976-1993: "Consolidation" (trying to understand BH entropy; membrane paradigm; "mathematical theory of BH"; growing evidence of ∃ of stellar & supermassive BH; EOM within PN approximation). 1993: Hulse & Taylor Nobel prize (the 1st in GR)

1993-present: BH entropy computed within string theory (1996) Development of numerical relativity and the 2005 breakthroughs (binary BH coalescence).

a new golden age?

– wealth of new solutions (in higher dimensions, with "hair") which may help strengthen the link between Quantum gravity & GR.

– gravitational wave astronomy and the need to describe (numerically or within approximation schemes) binary coalescence

Six Lectures on Black Holes in General Relativity
IPhT-CEA, Saclay, January-February 2009

Nathalie Deruelle

Selected bibliography

1. The development of a concept

1. Jean Eisenstaedt, "Avant Einstein", Seuil 2005
2. Jean Eisenstaedt, "Einstein et la relativité générale", Cnrs-Eds, 2002; English translation : Oxford and Princeton U-Press, 2006 (see also his technical papers quoted therein)
3. I. Todorov, "Einstein and Hilbert: the creation of General Relativity", arXiv:physics/0504179
4. W. Israel, "Dark stars : the evolution of an idea" in 300 years of gravitation, edts S.W. Hawking and W. Israel, Cambridge University Press, 1987
5. Kip Thorne, "Black holes and time warps", Norton Publ., 1994
6. B. Carter, "Half a century of black hole theory: from physicists' purgatory to mathematicians' paradise", arXiv:gr-qc/0604064
7. J.R. Oppenheimer and H. Snyder : "On continued gravitational contraction", Phys. Rev. 56 (1939) 455

2. The geometry of black holes

1. C.W. Misner, K.S. Thorne and J.A.Wheeler, "Gravitation", Freeman, San Fransico, 1973
2. S.W. Hawking and G.F.R. Ellis, "The large scale structure of space-time", Cambridge Univ. Press, 1973
3. R. Wald, "General relativity", Chicago University Press, 1984
4. R. Kerr, "Discovering the Kerr and Kerr-Schild metrics", arXiv:0706.1109 [gr-qc]
5. G. Dautcourt, "Race for the Kerr field", arXiV:0807.3473 [physics]
6. M. Visser, "The Kerr spacetime: a brief introduction", arXiv:0706.0622 [gr-qc]
7. Black Holes (1972 Les Houches Lectures), eds. B.S. DeWitt and C. DeWitt (Gordon and Breach, New York, 1973). In particular: B. Carter, "Black hole equilibrium states"
8. B. Carter, "Mechanics and equilibrium properties of black holes, membranes and strings", arXiv: hep-th/0411259
9. M.D. Kruskal, "Maximal extension of the Schwarzschild metric" Phys. Rev. 119 (1960) 1743
10. R. Kerr, "Gravitational field of a spinning mass as an example of algebraically special metrics", Phys. Rev. Lett. 11 (1963) 237-238

3. Stability and uniqueness of black holes

1. V. Frolov and I. Novikov, "Black hole physics", Kluwer academic Publisher, 1998
2. S. Chandrasekhar, "The mathematical theory of black holes", Clarendon Press, Oxford, 1983
3. K. Kokkotas and B. Schmidt, "Quasi-Normal modes of stars and black holes", Living Reviews, Relativity 2 (1999)

4. S. Hod, "Slow relaxation of rapidly rotating black holes", arXiv: 0811.3806 [gr-qc]
5. E. Poisson, W. Israel, "Internal structure of black holes", Phys. Rev. D41 (1990) 1796-1809
6. A. Ori, "Structure of the singularity inside a realistic rotating black hole" Phys. Rev. Lett 68 (1992) 2117-2120
7. B. Carter, "Mechanics and equilibrium properties of black holes, membranes and strings", arXiv: hep-th/0411259
8. M. Heusler, "Stationary black holes: uniqueness and beyond", Living Review, Relativity 1 (1998)
9. P. Crusciel and J. Lopes Costa, "On uniqueness of stationary vacuum black holes", arXiv: 0806.0016
10. T. Regge and J.A. Wheeler, Phys. Rev. 108 (1957) 1063
11. S. Teukolsky, Phys. Rev.Lett 29 (1972) 1114
12. W. Israel, "Event horizons in static vacuum space-times", Phys. Rev. 164 (1967) 1776
13. R. Penrose, "Gravitational collapse and spacetime singularities", Phys. Rev. Lett. 14 (1965) 57-59

4. The astrophysics and energetics of black holes

1. E. Gourgoulhon, "Objets compacts" notes de cours, sur le site du LUTH (http://www.luth.obspm.fr/luthier/gourgoulhon/index.html)
2. P. Grandclément, "Objets compacts", notes de cours, sur le site du LUTH (http://luth.obspm.fr/fichiers/enseignement/objets-compacts-F1.pdf)
3. T. Piran, "Bohdan's impact on our understanding of Gamma Ray Bursts", arXiv: 0804.2074 [astro-ph]
4. A. Müller, "Experimental evidence for black holes", arXiv: astro-ph/0701228
5. M. Reid, "Is there a super massive black hole at the center of the Milky Way?", arXiv: 0808.2624 [astro-ph]
6. Black Holes (1972 Les Houches Lectures), eds. B.S. DeWitt and C. DeWitt (Gordon and Breach, New York, 1973). See in particular : R. Ruffini, "On the energetics of black holes"
7. S. Komissarov, "Blandord-Znajek mechanism versus Penrose process", arXiv:0804.1921 [astro-ph]
8. "Black holes, the membrane paradigm", Eds K.S. Thorne, R.H. Price and A. MacDonald, Yale Univ. Press, New Haven, 1986
9. T. Damour, "Classical black holes as dissipative branes" in "String theory, gravity and experiment", arXiv:0802.416 [hep-th]
10. M. Sasaki and H. Tagoshi, "Analytic black hole perturbation approach to gravitational radiation", Living Review, Relativity 6 (2003)
11. L. Blanchet, "Gravitational radiation from post-newtonian sources and inspiralling compact binaries", Living Review, Relativity 9 (2006)
12. T. Damour and A. Buonanno, "Effective one-body approach to general relativistic two-body dynamics", arXiv: gr-qc/9811091; see also T. Damour and A. Nagar, "An improved analytical description of inspiralling and coalescing black hole binaries", arXiv: 0902.0136 [gr-qc]
13. E. Gourgoulhon and J. Jaramillo, "New theoretical approaches to black holes", arXiv: 0803.2944 [gr-qc]
14. F. Pretorius, "Binary black hole coalescence", arXiv: 0710.1338 [gr-qc]
15. M. Hannam, "Status of black hole binary simulations for gravitational wave detection", arXiv: 0901.2931 [gr-qc]
16. D. Lynden-Bell, "Galactic nuclei as collapsed old quasars", Nature 223 (1969) 690
17. D. Christodoulou, "Reversible and irreversible transformations in black hole physics", Phys. Rev. Lett. 25 (1970) 1596-1597
18. R. Penrose and R.M. Floyd, Nature Phys. Sci. 229 (1971) 177

5. Hawking radiation and black hole thermodynamics

1. M. Visser, "Essential and inessential features of Hawking radiation", arXiv: hep-th/0106111
2. T. Damour, "The entropy of black holes: a primer", arXiv: hep-th/0401160

3. R. Brout, S. Massar, R. Parentani and Ph. Spindel, “A primer for black hole quantum physics”, arXiv: 0710.4345
4. R. Wald, “The thermodynamcis of black holes”, Living Reviews, Relativity 4 (2001)
5. S. Carlip, “Black hole thermodynamics and statistical physics”, arXiv: 0807.4520 [gr-qc]
6. A. Nielsen, “Black holes and black hole thermodynamics without event horizons”, arXiv: 0809.3850 [hep-th]
7. J. Bekenstein, “Bekenstein-Hawking entropy”, in Scholarpedia
8. J. Bekenstein, “Black holes and information theory”, arXiv: quant-phys/0311049
9. J. Bekenstein, Phys Rev D7 (1973) 2333
10. S.W. Hawking, Nature 248 (1974) 30; and Comm. Math. Phys. 43 (1975) 199
11. T. Damour and R. Ruffini, Phys. Rev. D14 (1976) 332
12. S.W. Hawking, “Information loss in black holes”, arXiv: hep-th/0507171

6. Hairy and higher dimensional black holes

1. E. Winstanley, “Classical Yang-Mills black hole hair in anti-de Sitter Space”, arXiv: 0801.0527 [gr-qc]
2. M. Banados, C. Teiltelboim and J. Zanelli, “The black hole in three dimensional spacetime”, hep-th/9204099
3. G. Gibbons et al., “The general Kerr-de Sitter metrics in all dimensions”, hep-th/0404008
4. G. Gibbons, M. Perry and C. Pope, “The first law of thermodynamics for Kerr-anti-de-Sitter black holes”, arXiv: hep-th/0408217
5. B. Kleihaus, J. Kunz, F. Navarrro-Lerida, “Rotating black holes in higher dimensions”. arXiv: 0710.2291 [hep-th]
6. G. Horowitz, “Higher dimensional generalizations of the Kerr black hole”, arXiv: gr-qc/0507080
7. R. Emparan, “Black holes in higher dimensions”, arXiv: 0801.3471
8. S. Mignemi and D. Wiltshire, “Black holes in higher derivative gravity theories”, hep-th/9202031
9. T. Jacobson, Gungwon Kang and R. Myers, “Black hole entropy in higher curvature gravity”, arXiv: gr-qc/9502009
10. R. Myers, “Higher curvature gravity”, gr-qc/9811042
11. C. Garraffo and G. Giribet, “The Lovelock black holes”, srXiv:0805.3575
12. C. Charmousis, “Higher order gravity theories and their black hole solutions”, arXiv: 0805.0568
13. A. Komar, “Covariant conservation laws in general relativity”, Phys. Rev 113 (1959) 934
14. R. Arnowitt, S. Deser and C. Misner, “The dynamics of general relativity”, in “Gravitation, an introduction to current research”, L. Witten Edt, Wiley, New-York, 1962; also in arXiv: gr-qc/0405109
15. J. Katz, “A note on Komar’s anomalous factor”, Class and Quant Grav 2 (1985) 423

Further readings

1. E. Gallo and D. Marolf, “Resource Letter BH-2: black holes”, arXiv: 0806.2316 [astro-ph]
2. M. Visser, “Black holes in general relativity”, arXiv: 0901.4365 [gr-qc]
3. F. Lobo, “Exotic solutions in GR : traversable wormholes and “warp drive” spacetimes”, arXiv: 0710. 4474 [gr-qc]
4. S. Hod and T. Piran, “Mass-Inflation in dynamical collapse of a charged scalar field”, arXiv: gr-qc/9803004
5. R. Ruffini and She-Sheng Xue, “Dyadosphere formed in gravitational collapse”, arXiv: 0810.1438 [astro-ph]
6. B. Koch et al., “Exclusion of black hole disaster scenarios at the LHC”, arXiv: 0807.3349
7. M. Dafermos and I. Rodnianski, “Lectures on black holes and linear waves”, arXiv: 0811.0354 [gr-qc]
8. D. Christodoulou, “The formation of black holes in general relativity”, arXiv: 0805.3880 [gr-qc]
9. C. Gundlach, “Critical phenomena in gravitational collapse”, Living Reviews, Relativity 2 (1999) 4

10. L. Barack and N. Sago, "Gravitational self-force correction to the innermost stable circular orbit of a Schwzarzschild black hole", arXiv: 0902.0573 [gr-qc]
11. C. Barcelo, S. Liberati and M. Visser, "Analogue gravity", Living Review, Relativity, 8 (2006) 12
12. J. Bekenstein, "Black holes: classical properties, thermodynamics and heuristic quantization", arXiv: gr-qc/9808028
13. S. Hod, "Bohr's correspondence principle and the area spectrum of quantum black holes", arXiv: gr-qc/9812002
14. S. Hossenfelder and L. Smolin, "Conservative solutions to the black hole information problem", arXiv: 0901.3156
15. A. Ashtekar and B. Krishnan, "Isolated and dynamical horizons and their applications", Living Reviews, Relativity 7 (2004) 10
16. S. Hayward, et al., "Local Hawking temperature for dynamical black holes", arXiv: 0806.0014
17. E. Gourgoulhon, "From geometry to numerics: interdisciplinary aspects in mathematical and numerical relativity", arXiv: 0712.2332 [gr-qc]
18. J. Maldacena, "Black holes in string theory", PhD Thesis, arXiv: hep-th/0607235
19. G. Horowitz, "Black holes, entropy and information", arXiv: 0708.3680 [gr-qc]
20. A. Corichi, "Black holes and entropy in loop quantum gravity: an overview", arXiv: 0901.1302
21. P. Nicolini, "Noncommutative black holes, the final appeal to quantum gravity: a review", arXiv: 0807.1939