

# Topological expansion of the Bethe ansatz, and quantum algebraic geometry

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**Abstract:** In this article, we solve the loop equations of the  $\beta$ -random matrix model, in a way similar to what was found for the case of hermitian matrices  $\beta = 1$ . For  $\beta = 1$ , the solution was expressed in terms of algebraic geometry properties of an algebraic spectral curve of equation  $y^2 = U(x)$ . For arbitrary  $\beta$ , the spectral curve is no longer algebraic, it is a Schrödinger equation  $((\hbar\partial)^2 - U(x)).\psi(x) = 0$  where  $\hbar \propto (\sqrt{\beta} - 1/\sqrt{\beta})$ . In this article, we find a solution of loop equations, which takes the same form as the topological recursion found for  $\beta = 1$ . This allows to define natural generalizations of all algebraic geometry properties, like the notions of genus, cycles, forms of 1st, 2nd and 3rd kind, Riemann bilinear identities, and spectral invariants  $F_g$ , for a quantum spectral curve, i.e. a D-module of the form  $y^2 - U(x)$ , where  $[y, x] = \hbar$ . Also, our method allows to enumerate non-oriented discrete surfaces.

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# 1 Introduction

## Spectral invariants and algebraic geometry

In [12, 16], was presented the definition of spectral invariants  $F_g$  for any algebraic plane curve, i.e. given by a polynomial equation

$$\mathcal{E}(x, y) = \sum_{i,j} \mathcal{E}_{i,j} x^i y^j = 0.$$

Those invariants  $F_g(\mathcal{E})$  are defined in terms of algebraic geometry quantities defined on the Riemann surface of equation  $\mathcal{E}(x, y) = 0$ . Their definition involves residues at branchpoints of some meromorphic forms. Their definition provides a natural basis of meromorphic forms of 1st, 2nd and 3rd kind, and a natural framework for all algebraic geometry notions.

Moreover, the invariants  $F_g$  of [16] have many nice properties, for instance their deformations under changes of the complex structure of  $\mathcal{E}$  is given by some "special geometry" relations, and provide a natural form-cycle duality. Also, they are invariants under changes of  $\mathcal{E}$  which conserve the symplectic form  $dx \wedge dy$  in  $\mathbb{C} \times \mathbb{C}$ , they have nice modular properties, and finally, they define the tau-function of some dispersionfull integrable system associated to  $\mathcal{E}$ .

Also, those invariants  $F_g$  have deep relationships with enumerative geometry, for instance they have been related to the Kodaira-Spencer theory [10], to combinatorics of discrete surfaces (maps), to intersection theory [16, 14], and they are conjectured to be equal to the Gromov-Witten invariants of some toric Calabi-Yau target 3-folds [3].

### Algebraic geometry on "quantum" curves

Here, our goal is to define those notions for a "**quantum curve**", where  $\mathcal{E}(x, y)$  is a non-commutative polynomial of  $x$  and  $y$ :

$$\mathcal{E}(x, y) = \sum_{i,j} \mathcal{E}_{i,j} x^i y^j \quad , \quad [y, x] = \hbar.$$

The notion of quantum curve has arised in many ways in the litterature [9], and is also called D-modules, i.e. a space of functions quotiented by  $\text{Ker } \mathcal{E}(x, y)$ , where  $y = \hbar \partial / \partial x$ .

In other words, one has to study functions  $\psi(x)$  such that:

$$\mathcal{E}(x, \hbar \partial_x) \cdot \psi(x) = 0.$$

In our attempt to define the spectral invariants analogous to those of [16] for such D-modules, we are naturally led to define all analogous properties of algebraic geometry. For instance we define the notions of **branch points**, **sheets**, **genus**, **cycles**, **forms**, **Bergman kernel**, and so on...

Because of non-commutativity, some notions like branch-points, cuts and sheets, become "blurred" or "non-localized", i.e. the branchpoint is no longer a point, but a "region" of the complex plane, and cuts are asymptotic accumulation lines of points.

But, otherwise, it is surprising to find that almost all relationships of classical algebraic geometry, remained unchanged when  $\hbar \neq 0$ , for instance the Riemann bilinear identity, the Rauch variational formula, and the topological recursion defining the spectral invariants.

Moreover, we shall find, that in order for our quantities to make sense, we must have a "vanishing monodromy" condition, which can be interpreted as a **Bethe ansatz**, and this gives a geometrical interpretation of the Bethe ansatz.

Let us also mention that in a previous article [15], we treated a special case, where the Schrödinger potential  $U(x)$  was quantized, and we shall see, under the light of this new work, that it was the case of a degenerate quantum surface, with no branchpoints.

### Hyperelliptical case

Here, for simplicity, we shall restrict ourselves to polynomials of degree 2 in  $y$  (called hyperelliptical in algebraic geometry), of the form:

$$\mathcal{E}(x, y) = y^2 - U(x) \quad , \quad [y, x] = \hbar$$

i.e. to the Schrödinger equation:

$$\hbar^2 \psi'' = U\psi.$$

We leave the higher degree case for a further work.

### Link with $\beta$ matrix models

The spectral invariants  $F_g$  were first introduced for the solution of loop equations arising in the 1-hermitian random matrix model [12, 6]. They were later generalized to other hermitian multi-matrix models [7, 17].

There exist other matrix models, which are defined with non hermitian matrices. In fact it is well known since Wigner [21] that depending on the symmetry of the problem, it is sometimes interesting to have matrices that are not hermitian. (For example, real-symmetric, unitary, orthogonal or quaternionic, ...). Therefore, it seems reasonable to extend the definition of the spectral invariants for those other models. Those other matrix models are often called  $\beta$ -ensembles, and they are classified by an exponent  $\beta$ . The 3 Wigner ensembles (see [21], and we changed  $\beta \rightarrow \beta/2$ ) correspond to  $\beta = 1$  (hermitian case),  $\beta = 1/2$  (real symmetric case),  $\beta = 2$  (real self-dual quaternion case), but it is easy to define a  $\beta$  one-matrix model for any other value of  $\beta$  (see section 9 for more details).

In [5], a first attempt to generalize the solution of [12] to other matrix models was proposed, but it was not as nice as the topological recursion of [12]. In [5], it was assumed that  $\beta = O(1)$  when the size  $N$  of the random matrix becomes large, and it was found that all spectral invariants were related to a double series expansion of the form:

$$\sum_{g,k} N^{2-2g-k} (\sqrt{\beta} - 1/\sqrt{\beta})^k F_{g,k}$$

The coefficients  $F_{g,k}$  were computed in [5]. Here, in this article, we shall work at fixed  $\hbar = (\sqrt{\beta} - 1/\sqrt{\beta})/N$ , instead of fixed  $\beta$ , i.e. we shall define the resummed  $F_g$ 's as:

$$F_g(\hbar) = \sum_k \hbar^k F_{g,k}.$$

The  $F_{g,k}$ 's of [5] can be recovered by computing the semi-classical small  $\hbar$  expansion of  $F_g(\hbar)$ . In this article we shall argue that  $F_g(\hbar)$  is the natural generalization of the symplectic invariants of [16] for a "quantum spectral curve"  $\mathcal{E}(x, y)$  with  $[y, x] = \hbar$ .

The tool which we use for studying the  $\beta$ -matrix model, is the loop equation method. Loop equations are related to the invariance of an integral under change of variable. They can be obtained by integrating by parts. Loop equations for the  $\beta$ -matrix model have been written many times [11, 13], and here we show how to solve them order by order in  $1/N$ , at fixed  $\hbar$ .

The  $\beta$ -matrix model and its loop equations are explained in section 9.

## 2 Schrödinger equation and Bethe ansatz

### 2.1 Schrödinger equation, generalities and notation

Let:

$$\hbar^2 \psi''(x) = U(x) \psi(x) \quad (2.1)$$

be a Schrödinger equation with  $U(x)$  a polynomial. Let  $U(x)$  be a polynomial of degree  $2d$ , and define the polynomial "potential"  $V(x)$  of degree  $d + 1$  by its derivative:

$$V'(x) = 2(\sqrt{U})_+ = \sum_{k=0}^d t_{k+1} x^k \quad (2.2)$$

where  $(\ )_+$  means the polynomial part of the Laurent series at  $x \rightarrow \infty$ . We also define:

$$P(x) = \frac{V'^2(x)}{4} - U(x) - \hbar \frac{V''(x)}{2} \quad (2.3)$$

so that  $P$  is a polynomial of degree  $d - 1$ .

Eventually, we define:

$$t_0 = \lim_{x \rightarrow \infty} \frac{xP(x)}{V'(x)} \quad (2.4)$$

**Remark 2.1** Just in order to give names to those parameters, let us say that in the language of integrable systems, the coefficients  $t_0, t_1, t_2, \dots, t_{d+1}$  are called the "Casimirs", and the remaining coefficients of  $P$  are the "conserved charges". They will play a special role later on in this article. In matrix model language (see section 9),  $t_1, \dots, t_{d+1}$  are called the times associated to the potential  $V(x)$ ,  $t_0$  is often called the temperature, and the remaining coefficients of  $P$  are called "filling fractions". In the language of algebraic geometry, parameters  $t_k$  with  $k \geq 1$  are coupled to 2nd kind meromorphic differential forms,  $t_0$  is coupled to 3rd kind, and the remaining coefficients of  $P$  are coupled to 1st kind holomorphic differentials, see section 6 about form-cycle duality.

### 2.1.1 Stokes Sectors

From the study of the Schrödinger equation we know that the function  $\psi(x)$  is subject to the Stokes phenomenon, i.e. although  $\psi(x)$  is an entire function, its asymptotics look discontinuous near  $\infty$ . We therefore need to introduce properly the Stokes sectors by defining the following quantities: Let

$$\theta_0 = \text{Arg}(t_{d+1})$$

be the argument of the leading coefficient of  $V(x)$ .

We define the Stokes lines going to  $\infty$  as:

$$L_k = \left\{ x / \text{Arg}(x) = -\frac{\theta_0}{d+1} + \pi \frac{k + \frac{1}{2}}{d+1} \right\} \quad (2.5)$$

Those are the lines where  $\text{Re}V(x)$  vanishes asymptotically.

We define the sectors:

$$S_k = \left\{ \text{Arg}(x) \in ] -\frac{\theta_0}{d+1} + \pi \frac{k - \frac{1}{2}}{d+1}, -\frac{\theta_0}{d+1} + \pi \frac{k + \frac{1}{2}}{d+1} [ \right\} \quad (2.6)$$

i.e.  $S_k$  is the sector between  $L_{k-1}$  and  $L_k$ .

Notice that in even sectors we have asymptotically  $\text{Re}V(x) > 0$  and in odd sectors we have  $\text{Re}V(x) < 0$ .

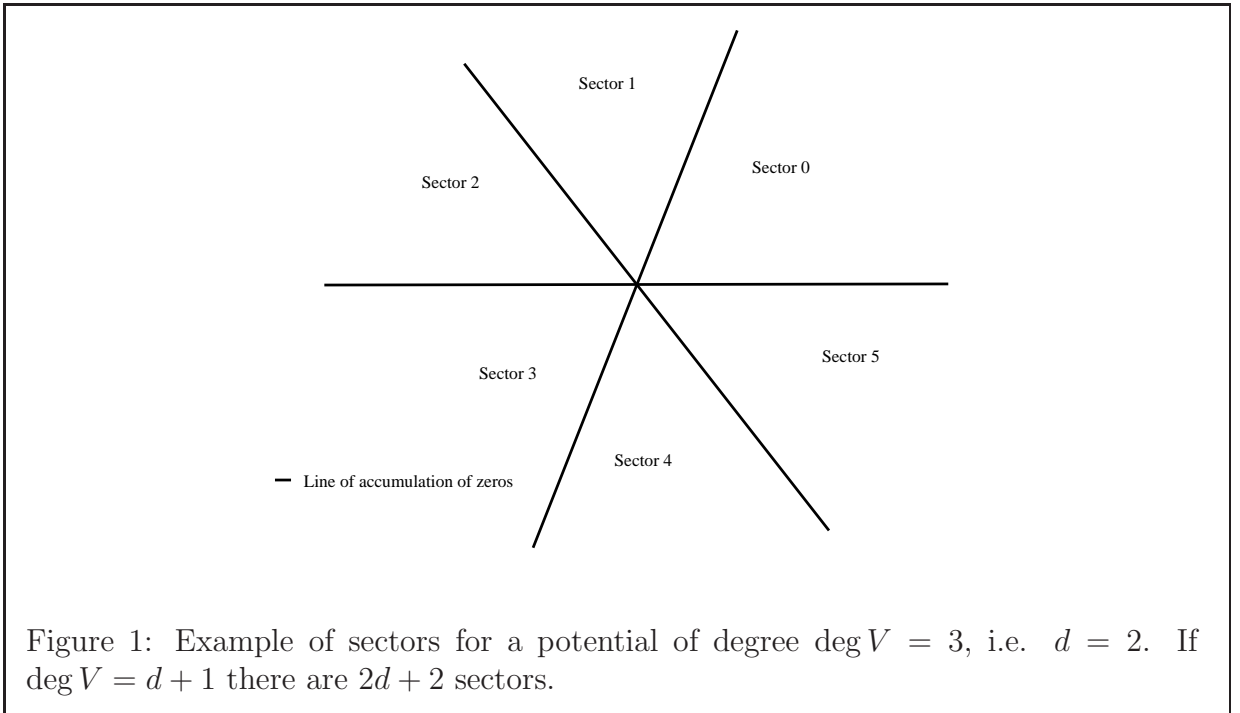


Figure 1: Example of sectors for a potential of degree  $\deg V = 3$ , i.e.  $d = 2$ . If  $\deg V = d + 1$  there are  $2d + 2$  sectors.

### 2.1.2 Stokes phenomenon

Any solution of a linear equation, is analytical where the coefficients of the equation are analytical, and it may possibly have essential singularities where the coefficients are singular. Here,  $U(x)$  is an entire function with a singularity (a pole), only at  $\infty$ , thus, any solution  $\psi$  is an entire function with a possible essential singularity at  $\infty$ . The asymptotics of  $\psi$  near  $\infty$  are subject to the Stokes phenomenon. This means that, although  $\psi$  is analytical in the whole complex plane, its asymptotics at infinity may change from sectors to sectors.

From the study of the Schrödinger equation it is known that in each sector  $S_k$ ,  $\psi(x)$  has a large  $x$  expansion:

$$\psi(x) \underset{S_k}{\sim} e^{\pm \frac{1}{2\hbar} V(x)} x^{C_k} \left( A_k + \frac{B_k}{x} + \dots \right)$$

and the sign  $\pm$ , may jump discontinuously from one sector to another as well as the numbers  $A_k, B_k, C_k, \dots$  (and in general, all the coefficients of the series in  $\frac{1}{x^j}$  at infinity).

## 2.2 Decreasing solution

Let us consider a specific solution  $\psi(x)$  of the Schrödinger equation which is exponentially decreasing in some even sector at infinity. For writing convenience, **we will choose  $\psi(z) = \psi_0(z)$  a decreasing solution in sector  $S_0$ . Without further indication,  $\psi(z)$  is now understood to be  $\psi_0(z)$  in the rest of the article.** Note that this choice is quite arbitrary at the moment, and one should wonder if the quantities we are about to compute depend on this choice, but we are presently not able to answer this question properly, and leave it for further study.

An important and useful result is the Stokes theorem which claims that if the asymptotics of  $\psi(x)$  is exponentially small in some sector, then the same asymptotics holds in the two adjacent sectors (and therefore  $\psi(x)$  is exponentially large in those two sectors).

In the general case, (i.e. a generic potential  $U(x)$ ) our solution  $\psi(x)$  is decreasing only in sector 0, and is exponentially large in all other sectors. But if the Schrödinger potential  $U(x)$  is non-generic (quantized), then there may exist several sectors in which  $\psi(x)$  is exponentially small. Due to Stokes theorem, if  $\psi$  is exponentially small in some sectors then it must be exponentially large in the adjacent sectors, this implies that there are at most  $d + 1$  sectors in which  $\psi$  is exponentially small.

The case studied in [15] was the most degenerate case, such that  $\psi$  is exponentially small in  $d + 1$  sectors.



### 2.2.1 Zeroes of $\psi$

The main difference with our previous article [15] is that we will not restrict ourselves to the case where  $\psi(x)$  is a quasi-polynomial which can only be obtained with very non-generic potential  $U(x)$ . Here  $\psi(x)$  is an entire function with an essential singularity at  $\infty$ , and with isolated zeroes labelled  $s_i$ :

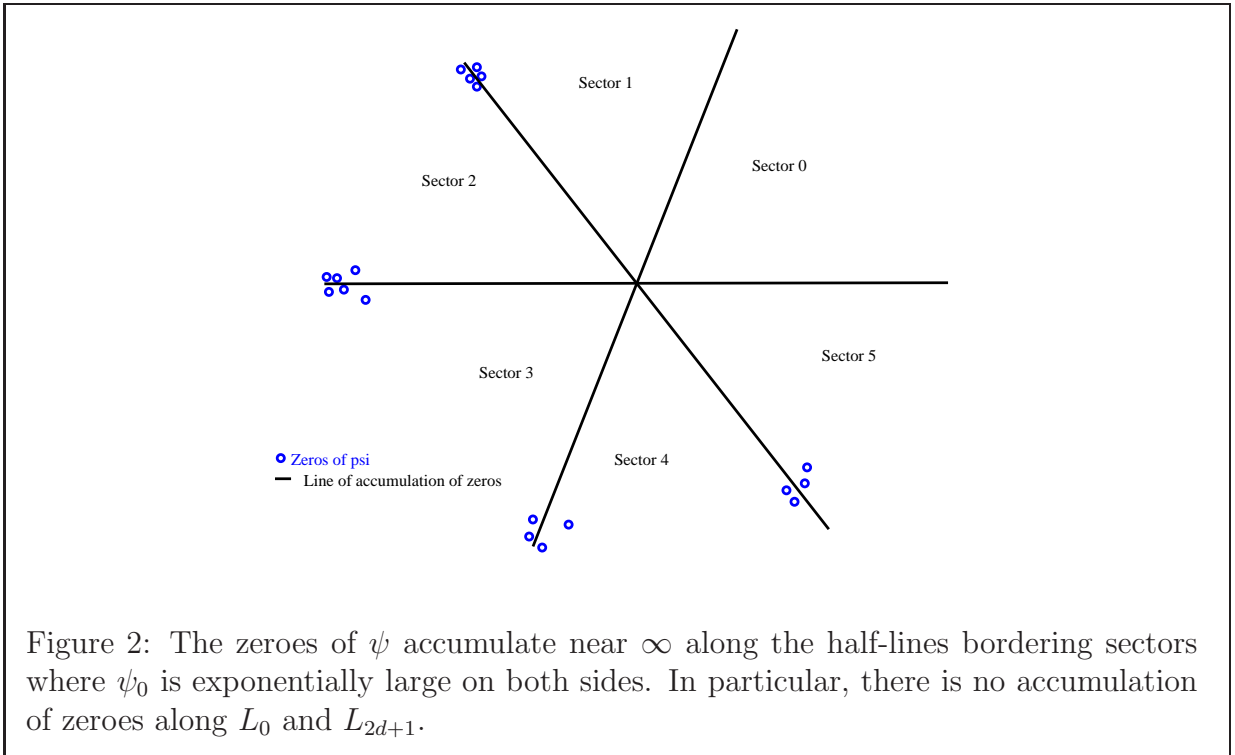
$$\psi(s_i) = 0 \tag{2.7}$$

**In particular, the number of zeroes of  $\psi$  may be finite or infinite.**

If  $\psi(x)$  has an infinite number of zeroes, it is known that the zeroes may only accumulate near  $\infty$ , and only along the Stokes half-lines  $L_j$ 's bordering the sectors (see fig.2). In fact, there is an accumulation of zeroes along the half-line  $L_j$  if and only if  $\psi$  is exponentially large on both sides of the half-line.

For example in the case where  $\psi(x) = \psi_k(x)$  is a solution that exponentially decreases in sector  $k$  then it implies that there is no accumulation of zeroes along the half-lines  $L_k$  and  $L_{k-1}$ .

If  $U(x)$  is generic, then  $\psi$  has an infinite number of zeroes, and the zeroes accumulate at  $\infty$  along all half-lines  $L_j$  with  $j \neq 1, 2d + 1$  (because remember that  $\psi$  is implicitly assumed to be  $\psi_0$  which decreases in sector 0), i.e. there are generically  $2d$  half-lines of zeroes. The situation is illustrated in fig 2.



If  $U(x)$  is non-generic (quantized), then there are additional sectors in which  $\psi$  is exponentially small, and thus there can be no zeroes accumulating along the two half-lines bordering these sectors. Remember that from Stokes theorem, each time we have a new sector in which  $\psi$  is decreasing, we have two half-lines less of zeroes. Therefore, the number of half-lines of zeroes is always even, and we call it:

**Definition 2.1** *The genus  $\mathbf{g}$  of the Schrödinger equation is defined by:*

$$2\mathbf{g} + 2 = \# \text{ half-lines of zeroes} \quad (2.8)$$

And if  $\psi$  has a finite number of zeroes (i.e. there is no half-line of zeroes), we define  $\mathbf{g} = -1$ . We have

$$-1 \leq \mathbf{g} \leq d - 1 \quad (2.9)$$

Note also that the definition of  $\mathbf{g}$  a priori depends on the choice of the solution  $\psi = \psi_0$  since two different solutions of the same Schrödinger equation may have different numbers of semi-lines of zero accumulation.

An exception, is in the special cases  $\mathbf{g} = -1$  where it is easy to see that every choice of  $\psi = \psi_{2k}$  would give the same value of  $\mathbf{g}$ .

Indeed, consider  $\mathbf{g}_{2k}$  and  $\mathbf{g}_{2k'}$  be the genus defined from the solutions  $\psi_{2k}$  exponentially small in sector  $S_{2k}$  and  $\psi_{2k'}$  exponentially small in sector  $S_{2k'}$ :

if  $\mathbf{g}_{2k} = -1$ , this means that  $\psi_{2k}$  is exponentially small in all even sectors, in particular in sector  $S_{2k'}$ , and therefore  $\psi_{2k} \propto \psi_{2k'}$ , and therefore  $\mathbf{g}_{2k'} = -1$ .

### 2.2.2 Case $\mathbf{g} = -1$

The case  $\mathbf{g} = -1$  was studied in [15]. This is the case where  $\psi$  has only a finite number of zeroes, it is a quasipolynomial:

$$\psi(x) e^{\frac{\hbar}{2} V(x)} = \text{polynomial}. \quad (2.10)$$

Notice that in order to diminish  $\mathbf{g}$  by 1, we need to quantize one parameter of  $U$ , and therefore to reach  $\mathbf{g}$ , we need to quantize  $d - 1 - \mathbf{g}$  parameters. In particular, to reach  $\mathbf{g} = -1$ , we need to quantize  $d$  parameters, i.e.  $P$  is completely fixed in terms of  $V'$ , and in particular,  $t_0$  is quantized.

In the applications to random matrices,  $t_0$  is usually a free parameter (called the temperature) and is never considered quantized, and therefore the case  $\mathbf{g} = -1$  is never obtained in random matrices.

Another way to say that, is that the case  $\mathbf{g} = -1$  has no  $\hbar \rightarrow 0$  classical limit, and therefore in classical geometry we always have  $\mathbf{g} \geq 0$ .

## 2.3 Resolvent

The first ingredient of our strategy is to define a resolvent similar to the one in matrix models.

**Definition 2.2** *We define the resolvent for a generic solution  $\psi$  by:*

$$\omega(x) = \hbar \frac{\psi'(x)}{\psi(x)} + \frac{V'(x)}{2} \quad (2.11)$$

It is clear that this function is analytical except at the zeros of  $\psi(x)$  where it has **simple** poles with residue  $\hbar$ :

$$\omega(x) \underset{x \rightarrow s_i}{\sim} \frac{\hbar}{x - s_i} + \text{reg.} \quad (2.12)$$

It also has a possible essential singularity at infinity with the same location of discontinuities as  $\psi(x)$ . Eventually, note again that the definition of  $\omega(x)$  depends on the choice of  $\psi(x)$ .

## 2.4 Sheets

In sector  $S_k$  we have the asymptotic:

$$\psi(x) \underset{S_k}{\sim} e^{\frac{\eta_k}{2\hbar} V(x)} x^{-\frac{\eta_k t_0}{\hbar} - d \frac{1+\eta_k}{2}} \left( A_k + \frac{B_k}{x} + \dots \right) \quad (2.13)$$

where  $\eta_k = \pm 1$ . That translates for the resolvent to:

$$\omega(x) \underset{x \rightarrow \infty_k}{\sim} \frac{1 + \eta_k}{2} \left( V'(x) - \hbar \frac{d}{x} \right) - \frac{\eta_k t_0}{x} + O(1/x^2), \quad (2.14)$$

Therefore it depends if the solution  $\psi$  is exponentially big or small in sector  $k$  (and of course on the parity of  $k$ ). For a generic  $\mathfrak{g} = d - 1$  solution which is exponentially big in every sector except  $S_0$  (and thus has an alternating sign in the exponential) then  $\eta_k = (-1)^k$  (except  $\eta_0 = -1$ ).

**Definition 2.3** *We call "physical sheet", the union of sectors where  $\eta_k = -1$ , in those sectors we have:*

$$\omega(x) \sim \frac{t_0}{x} + O(1/x^2) \quad (2.15)$$

*Notice that the sectors  $S_0, S_1$  and  $S_{2d+1}$  are always in the physical sheet.*

*And we call "second sheet", the union of sectors where  $\eta_k = +1$ , in those sectors we have:*

$$\omega(x) \sim V'(x) + O(1/x) \quad (2.16)$$

This definition comes from the analogy with the resolvent in matrix model (see section 9 for details).

For a generic potential  $U(x)$ , all odd sectors are in the physical sheet, and all even sectors except  $S_0$  are in the second sheet.

Notice that if  $\mathbf{g} = -1$ , there is only the physical sheet, i.e. there is no second sheet.

## 2.5 The Bethe ansatz

In the polynomial case studied before [15], a key ingredient for establishing results was the Bethe ansatz. This ansatz basically deals with the behaviour of  $\omega(x)$  around zeroes of  $\psi$ . The zeroes of  $\psi$  are called "Bethe roots".

The Bethe ansatz can be formulated in many ways. One way to formulate it, is to say that  $1/\psi^2$  has no residue at the  $s_i$ 's:

$$\text{Res}_{s_i} \frac{1}{\psi^2(x)} = 0 \quad (2.17)$$

in this way, it will play a key role in defining contour integrals, because all integrals of the type  $\int dx/\psi^2(x)$  are insensitive to the exact location integration path with respect to the  $s_i$ 's, i.e. such integrals will depend only on the homotopy classes of paths.

Equation (2.17) can also be formulated, in a form very similar to the Bethe ansatz in the Gaudin model [20, 1] as follows:

**Theorem 2.1** *The roots  $s_i$  of  $\psi$  satisfy the Bethe ansatz:*

$$\forall i, \quad V'(s_i) = 2 \lim_{x \rightarrow s_i} \left( \omega(x) - \frac{\hbar}{x - s_i} \right). \quad (2.18)$$

It is a regularized version of the Bethe equation for Gaudin model:

$$\forall i, \quad V'(s_i) = 2\hbar \sum_{j \neq i} \frac{1}{s_i - s_j}$$

when the number of zeros is infinite and the sum is ill-defined.

**proof:**

This theorem is a classical result and is easy, it just consists in rewriting the Schrödinger equation as a Riccati equation. We proceed the same way as in [15] and compute:

$$\begin{aligned} & V'(x)\omega(x) - \omega^2(x) - \hbar\omega'(x) \\ = & V'(x)\left(\hbar\frac{\psi'(x)}{\psi(x)} + \frac{V'(x)}{2}\right) - \left(\frac{V'(x)^2}{4} + \hbar V'(x)\frac{\psi'(x)}{\psi(x)} + \hbar^2\frac{\psi'(x)^2}{\psi^2(x)}\right) \end{aligned}$$

$$\begin{aligned}
& -\hbar \left( \hbar \frac{\psi''(x)}{\psi(x)} - \hbar \frac{\psi'^2(x)}{\psi^2(x)} + \frac{V''(x)}{2} \right) \\
= & \frac{V'(x)^2}{4} - \hbar^2 \frac{\psi''(x)}{\psi(x)} - \hbar \frac{V''(x)}{2} \\
= & \frac{V'(x)^2}{4} - U(x) - \hbar \frac{V''(x)}{2} \\
= & P(x)
\end{aligned} \tag{2.19}$$

which is a polynomial in  $x$ , of degree  $d - 1$ .

From its definition, it is clear that  $\omega^2 + \hbar\omega'$  has no double pole at the  $s_i$ 's, but it could have simple poles. Consider now a zero  $s_i$  of  $\psi$ , and define:

$$\bar{\omega}_i(x) = \omega(x) - \frac{\hbar}{x - s_i}$$

Then,  $\bar{\omega}_i(x)$  is regular at  $x = s_i$ , and we may compute  $\bar{\omega}_i(s_i)$ . Compute:

$$\begin{aligned}
\operatorname{Res}_{x \rightarrow s_i} \omega^2(x) + \hbar\omega'(x) &= \operatorname{Res}_{x \rightarrow s_i} \bar{\omega}_i^2(x) + 2\hbar \frac{\bar{\omega}_i(x)}{x - s_i} + \frac{\hbar^2}{(x - s_i)^2} + \hbar\bar{\omega}_i'(x) - \frac{\hbar^2}{(x - s_i)^2} \\
&= \operatorname{Res}_{x \rightarrow s_i} 2\hbar \frac{\bar{\omega}_i(x)}{x - s_i} \\
&= 2\hbar \bar{\omega}_i(s_i)
\end{aligned} \tag{2.20}$$

On the other hand we have, from eq. (2.19) we have:

$$\begin{aligned}
\operatorname{Res}_{x \rightarrow s_i} \omega^2(x) + \hbar\omega'(x) &= \operatorname{Res}_{x \rightarrow s_i} V'(x)\omega(x) - P(x) \\
&= \operatorname{Res}_{x \rightarrow s_i} V'(x)\omega(x) \\
&= \hbar V'(s_i)
\end{aligned} \tag{2.21}$$

Therefore we find :

$$\forall i, \quad V'(s_i) = 2\hbar \bar{\omega}_i(s_i).$$

This equation is the Bethe equation for the roots  $s_i$ 's. Note that the potential  $V'(x)$  is completely determined by the data of the potential  $U(x)$  and does not depend on  $\psi$ . In particular, in the case where there are only a finite number of  $s_i$ 's we recognize the Bethe equation for Gaudin model [15]:

$$\forall i, \quad V'(s_i) = 2\hbar \sum_{j \neq i} \frac{1}{s_i - s_j}$$

which were completely defining the  $s_i$ 's.  $\square$

### 3 Towards a "Quantum Riemann Surface"

From the definition of our non-commutative spectral curve (i.e the Schrödinger equation), it is tempting to generalize the classical notions known in algebraic geometry

and Riemann surfaces to our "quantum" case ("quantum" is not to be understood as "quantized" but as "non-commutative"  $[y, x] = \hbar$ ). For a Riemann surface, the central notions are those of cuts, sheets, genus, cycles and meromorphic differential forms of 1st, 2nd and 3rd kind. In our context, the picture needs a proper adaptation in order to recover the terminology of Riemann surfaces and algebraic geometry.

In this section we will define the notions of genus,  $\mathcal{A}$ -cycles,  $\mathcal{B}$ -cycles and the first kind differentials dual to them. Here, let us assume that  $\mathfrak{g} \geq 0$ .

### 3.1 Cuts

First, we like to think of the 2 sheets, as the sectors which correspond to the 2 possible behaviors of the resolvent at  $\infty$ :  $\omega(x) \sim t_0/x$  (physical sheet) or  $\omega(x) \sim V'(x)$  (second sheet).

Then, we consider the cuts as sets of roots  $s_i$ 's. In some sense, each pair of half lines of accumulation of zeroes can be thought of as a cut.

**Definition 3.1** *We define cuts as pairs of half-lines of zeroes.*

*There is some arbitrariness in grouping the half-lines of zeroes by pairs.*

There is  $\mathfrak{g} + 1$  cuts, like in classical algebraic geometry, and notice that the case  $\mathfrak{g} = -1$  which has no classical counterpart, has no cuts.

Notice that, contrarily to classical geometry, where the endpoints of the cuts are zeroes of  $U(x)$ , here the endpoints are somehow blurred, we may move a finite number of  $s_i$ 's from one cut to another.

### 3.2 Cycles

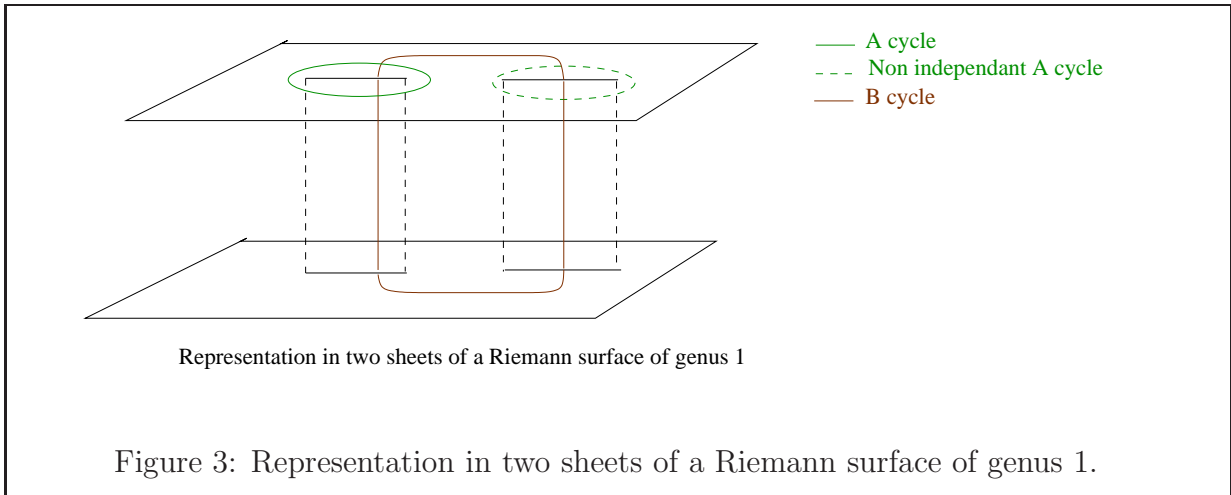
In standard algebraic geometry, the non-contractible  $\mathcal{A}$ -cycles are often thought of as surrounding cuts in the physical sheet, and their dual  $\mathcal{B}$ -cycles are going through the cuts, from one sheet to the other, see fig 3.

#### 3.2.1 A-Cycles

Consider the complex plane from which we remove the second sheet (sectors where  $\omega(x) \sim V'(x)$ ). It is clear that it contains  $\mathfrak{g} + 1$  sectors near  $\infty$ , and there are  $\mathfrak{g}$  homologically linearly independent contours which link them.

**Definition 3.2** *We define  $\mathcal{A}$ -cycles  $\mathcal{A}_1, \dots, \mathcal{A}_{\mathfrak{g}}$  as  $\mathfrak{g}$  linearly independent non-contractible contours going from  $\infty$  to  $\infty$  in the physical sheet.*

*A choice of  $\mathcal{A}$ -cycles is not unique.*



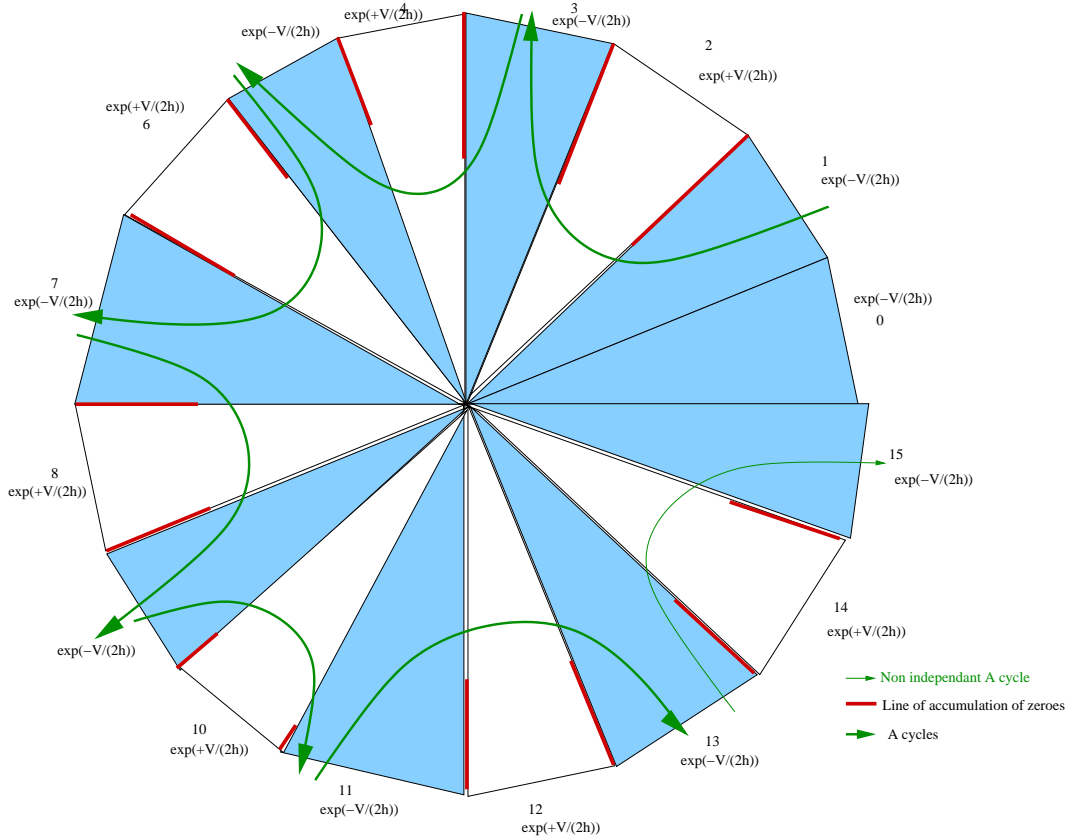
Remark that this notion really makes sense only for  $g \geq 1$ .

Notice that each time  $\psi(x) \sim e^{-V(x)/2\hbar}$  in an even sector, it means it is exponentially small and thus it also behaves like  $e^{-V(x)/2\hbar}$  in the neighboring odd sectors. That means that we can always choose  $\mathcal{A}$ -cycles going from odd sector to odd sector.

Since the first sheet and second sheet are separated by half-lines of accumulations of zeroes, every  $\mathcal{A}$ -cycle surrounds an even number of such half-lines of accumulations of zeroes, i.e. surrounds the cuts in the physical sheet. Like in standard algebraic geometry, the cuts are identified as pairs of half-lines of zeroes accumulations and the  $\mathcal{A}$  cycles are going enclosing these cuts.

### 3.2.2 Examples

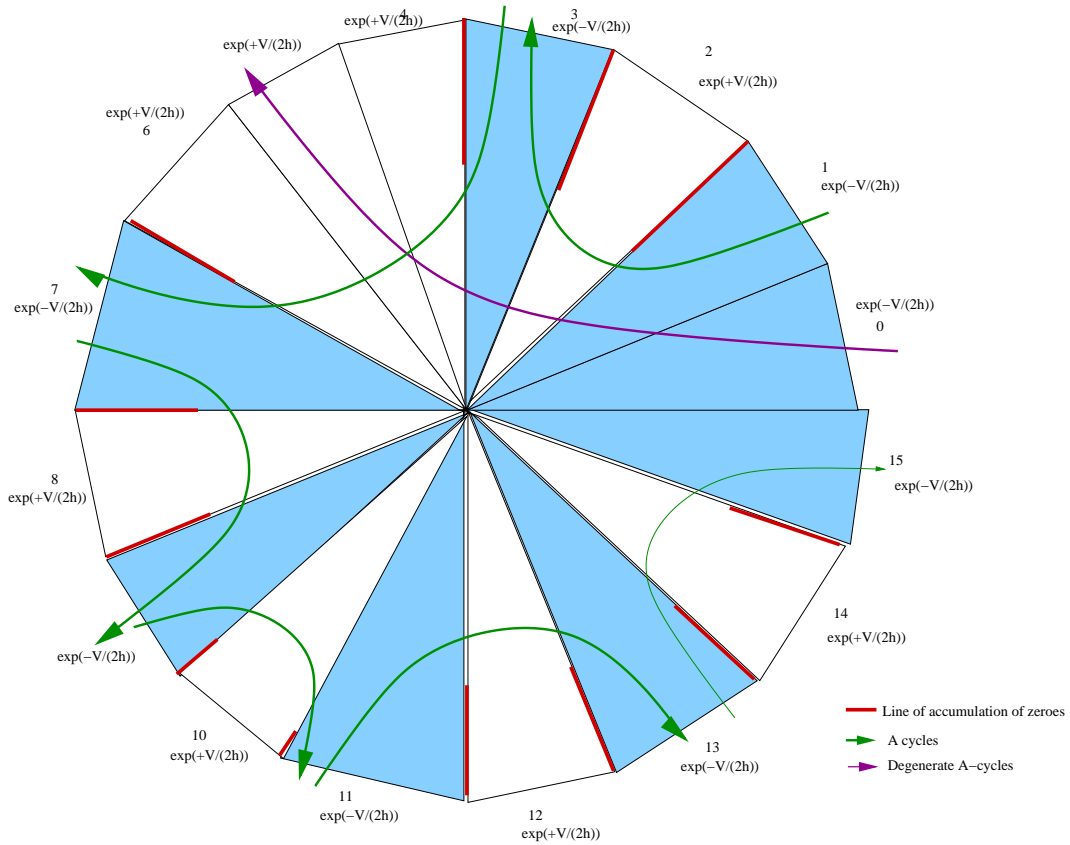
In the generic case  $g = d - 1$ , we can define  $d$   $\mathcal{A}$ -cycles but only  $d - 1$  are linearly independent. See picture where  $d = 7$ :



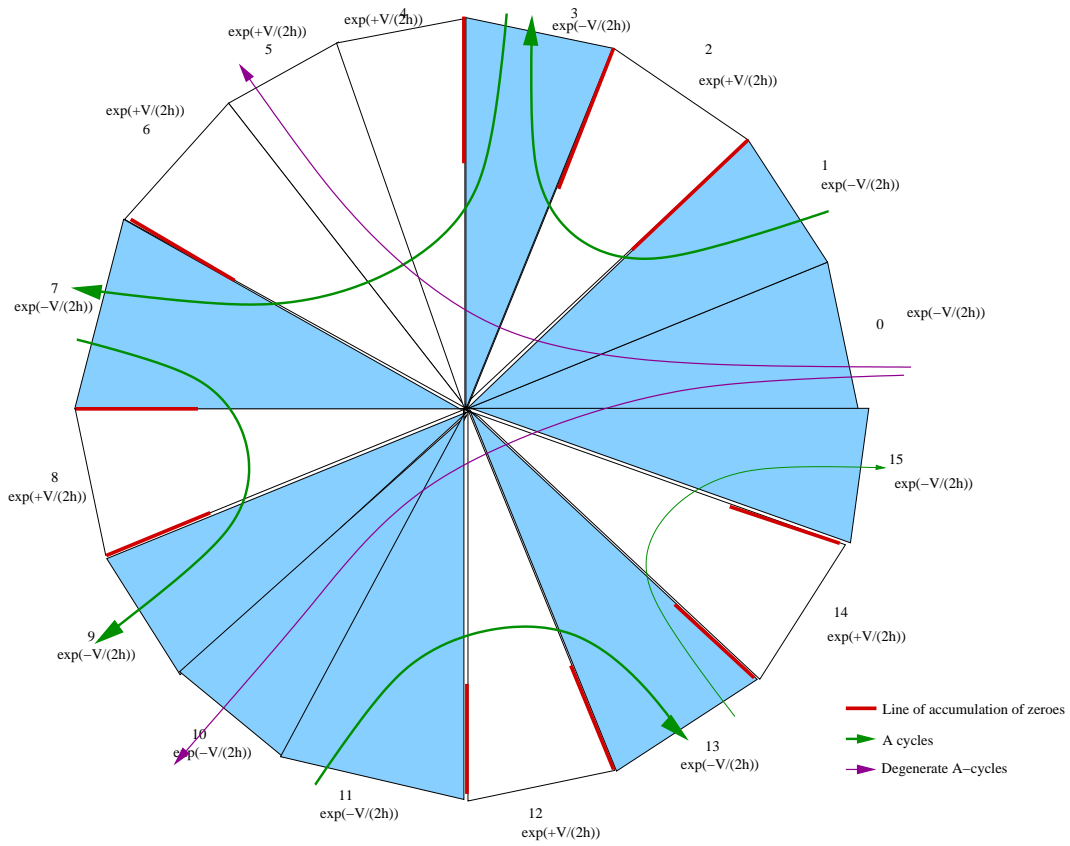
We clearly see that the dashed contour is not linearly independent with the others since the global sum of the contours (dashed included) is contractible in the physical sheet.

For a non-generic case, there are sectors at infinity where  $\psi$  is exponentially small. In these cases, the definition of the contours need some adaptations because these sectors correspond to "degenerate" cuts. Here are a few examples of how to deal with these cases. Basically, each time there are two sectors where  $\psi$  is small we can replace one of the standard  $\mathcal{A}$  cycle, by a  $\hat{\mathcal{A}}$  cycle (sometimes called also "degenerate"  $\mathcal{A}$  cycles) that connect them. Here are some examples of the contours in more and more peculiar situations for  $d = 7$ :





From then it is easy to generalize into more complicated frames:



It is then easy to generalize the method in more sophisticated situations.

In the extreme case where  $\psi$  is exponentially small in all even sectors, there are only  $d$  independant "degenerate"  $\hat{\mathcal{A}}$  cycles and no  $\mathcal{A}$  cycles, the genus is  $\mathfrak{g} = -1$ . This is the polynomial case studied in [15] where there are no  $\mathcal{A}$  cycles.

From the definitions, it is easy to see that the genus  $\mathfrak{g}$  defined above corresponds to the number of independant  $\mathcal{A}$  cycles (we exclude the  $\hat{\mathcal{A}}$  cycles). It is also obvious that the sum of independant  $\mathcal{A}$  and  $\hat{\mathcal{A}}$  cycles always equals  $d - 1$ .

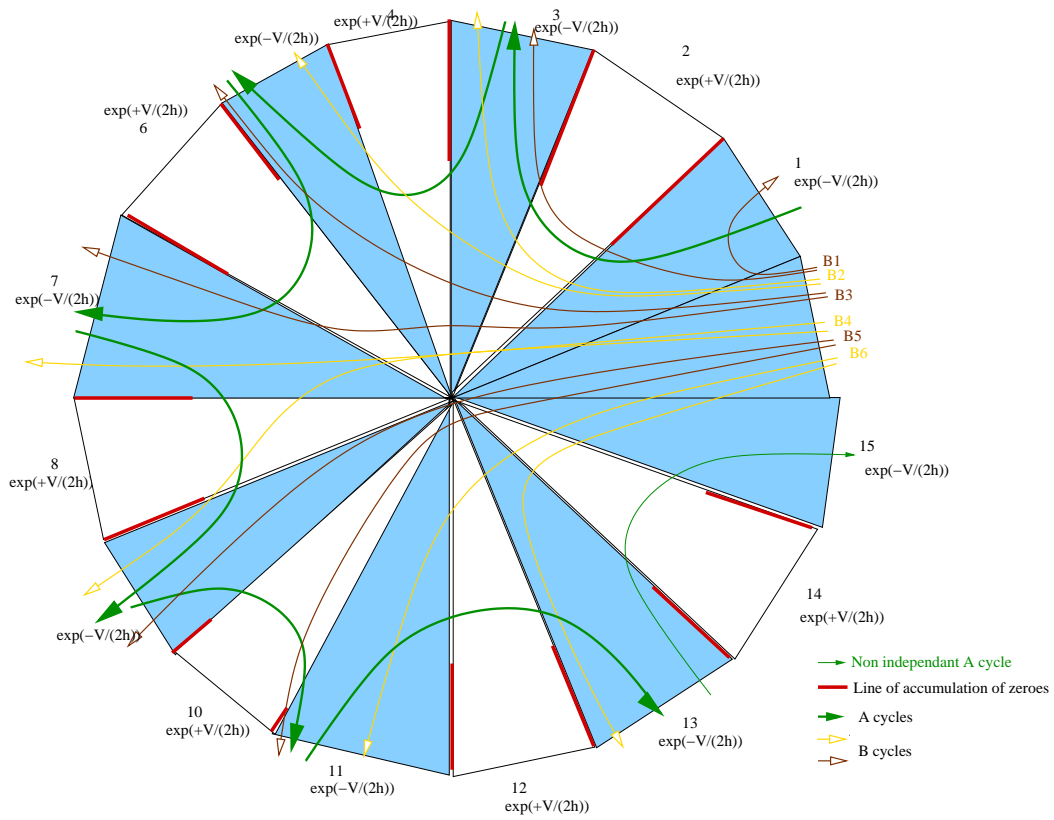
### 3.2.3 B-Cycles

As in classical algebraic geometry, it is standard to define the  $\mathcal{B}$  cycles with an origin lying in the non-independant cut. Moreover, although it would be possible to define  $\hat{\mathcal{B}}$  cycles attached to the  $\hat{\mathcal{A}}$  cycles, we prefer limiting ourselves to the definition of  $\mathcal{B}$  cycles attached only to the  $\mathcal{A}$  cycles. Basically, they start from the non-independant cut, goes through their corresponding  $\mathcal{A}$  cycle and end at infinity in the same sector as their corresponding  $\mathcal{A}$  cycle. As there are two sectors in which their corresponding  $\mathcal{A}$  cycle ends, we double them so that one goes into one sector and the other one in the second sector. We also choose the whole so that they intersect only with their corresponding  $\mathcal{A}$ -cycles:

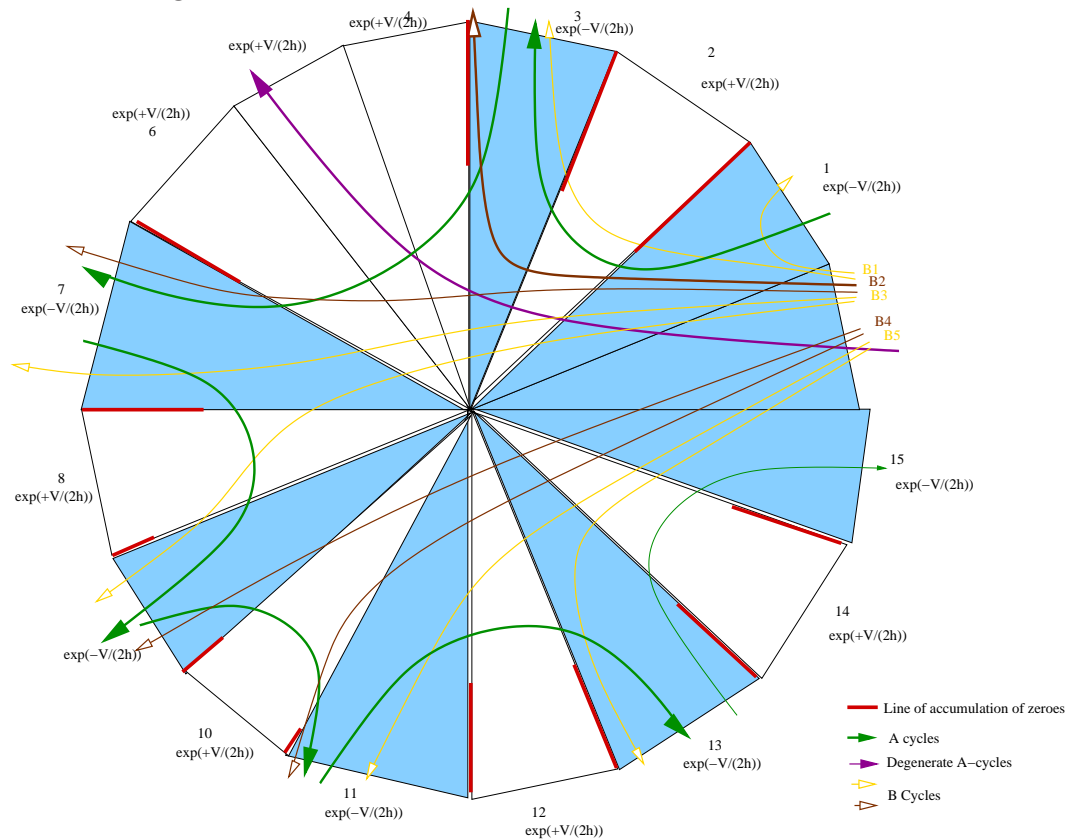
$$\mathcal{A}_\alpha \cap \mathcal{B}_\beta = 2\delta_{\alpha,\beta} \tag{3.1}$$

This definition is easier understandable with the following pictures:

Generic case:



And in a degenerate case:



### 3.3 First kind functions

After defining the cycles, another important step is to define the equivalent of the first, second and third kind differentials. In this section, we propose a definition of the first kind differentials.

Let  $h_k$ ,  $k = 1, \dots, d-1$ , be a basis (arbitrary for the moment, but we will choose it orthonormal later on), of the complex vector space of polynomials of degree  $\leq d-2$ . To have more convenient notation, we will label the  $\hat{\mathcal{A}}$ -cycles as  $\mathcal{A}_\alpha$ ,  $\mathfrak{g}+1 \leq \alpha \leq d-1$  and the standard  $\mathcal{A}$  are labelled  $\mathcal{A}_\alpha$ ,  $1 \leq \alpha \leq \mathfrak{g}$ .

Consider the following functions:

$$v_k(x) = \frac{1}{\hbar \psi^2(x)} \int_{\infty_0}^x h_k(x') \psi^2(x') dx' \quad , \quad \deg h_k \leq d-2. \quad (3.2)$$

Notice that, thanks to the Bethe ansatz,  $v_k(x)$  has double poles with vanishing residues at the  $s_j$ 's (the zeroes of  $\psi$ ), and behaves like  $O(1/x^2)$  in sector  $S_0$  and in sectors where  $\psi$  is exponentially large. (because the polynomial is of degree less than  $d-2$ ). Therefore, the following integrals are well defined:

$$I_{k,\alpha} = \oint_{\mathcal{A}_\alpha} v_k(x) dx \quad , \quad \alpha = 1, \dots, \mathfrak{g}, k = 1, \dots, d-1. \quad (3.3)$$

For the degenerate contours  $\hat{\mathcal{A}}_\alpha$ , we cannot take the integral since it would not converge. We define instead:

$$I_{k,\alpha} = \int_{\hat{\mathcal{A}}_\alpha} h_k(x) \psi^2(x) dx \quad , \quad \alpha = \mathfrak{g}+1, \dots, d-1, k = 1, \dots, d-1. \quad (3.4)$$

The matrix  $I_{k,\alpha}$  with  $k, \alpha = 1, \dots, d-1$  is a square matrix, which gives a pairing between the set of paths  $\{ \mathcal{A}_\alpha, \hat{\mathcal{A}}_\alpha \}$  and the space of polynomials of degree at most  $d-2$ . Let us choose a basis  $h_k$ , dual to the  $\mathcal{A}$ -cycles, i.e.:

$$I_{k,\alpha} = \delta_{k,\alpha}. \quad (3.5)$$

Choosing this set of polynomials gives then the following relations:

$$\forall i = 1, \dots, \mathfrak{g}, j = 1, \dots, d-1, \quad \oint_{\mathcal{A}_i} v_j(x) dx = \delta_{i,j} \quad (3.6)$$

$$\forall i = \mathfrak{g}+1, \dots, d-1, j = 1, \dots, d-1 : \int_{\hat{\mathcal{A}}_i} h_j(x) \psi^2(x) dx = \delta_{j,i} \quad (3.7)$$

Moreover, from the definitions, we get an asymptotic expression of  $v_k(x)$  at infinity:

**Theorem 3.1** *The functions  $v_k(x)$  with  $k \leq \mathfrak{g}$  are such that:*

$$k = 1, \dots, \mathfrak{g}, \quad v_k(x) = O(x^{-2}) \quad (3.8)$$

*in all sectors at infinity.*

*And the functions  $v_k(x)$  with  $\mathfrak{g} + 1 \leq k \leq d - 1$  are such that:*

$$k = \mathfrak{g} + 1, \dots, d - 1, \quad v_k(x) = O(x^{-2}) \quad (3.9)$$

*in all sectors except in the sector where  $\hat{\mathcal{A}}_k$  ends, where we have:*

$$v_k(x) = \frac{1}{\hbar\psi(x)^2} + O(1/x^2). \quad (3.10)$$

**proof:**

In sector  $\infty_0$ , we clearly have  $v_k(x) \sim O(x^{\deg h_k - d}) = O(x^{-2})$ . And in a sector  $S_i$  where  $\psi$  is exponentially small we have:

$$v_k(x) = \frac{1}{\hbar\psi^2(x)} \left[ \int_{\infty_i}^x h_k(x') \psi^2(x') dx' + \int_{\infty_0}^{\infty_i} h_k(x') \psi^2(x') dx' \right], \quad (3.11)$$

and due to our choice of basis eq. (3.5), we have

$$v_k(x) = \frac{\delta_{i,i_k}}{\hbar\psi(x)^2} + \frac{1}{\hbar\psi^2(x)} \int_{\infty_i}^x h_k(x') \psi^2(x') dx' = \frac{\delta_{i,i_k}}{\hbar\psi(x)^2} + O(1/x^2), \quad (3.12)$$

in sector  $S_i$ .  $\square$

We claim that the function  $v_k(x)$   $k = 1, \dots, \mathfrak{g}$  are the generalization of holomorphic forms (1st kind differentials).

**Remark 3.1** Classical limit.

The small  $\hbar$  BKW expansion  $\psi \sim e^{\pm \frac{1}{2\hbar} \int \sqrt{U}}$  gives:

$$v_k(x) \sim \frac{\pm h_k(x)}{\sqrt{U(x)}} \quad (3.13)$$

and  $v_k(x)dx$  are indeed the holomorphic forms on the algebraic curve  $y^2 = U(x)$ .

### 3.4 Riemann matrix of periods

An interesting quantity in standard algebraic geometry is the Riemann matrix of periods which is the integrations of the holomorphic differentials over  $\mathcal{B}$ -cycles. Now that we have defined properly the cycles, we can define a similar ‘‘quantum’’ Riemann period matrix  $\tau_{i,j}$ ,  $i, j = 1, \dots, \mathfrak{g}$  by:

$$\tau_{i,j} \stackrel{\text{def}}{=} \oint_{\mathcal{B}_i} v_j(x) dx. \quad (3.14)$$

Note that this definition makes sense since  $v_j(x)$  ( $j = 1, \dots, \mathfrak{g}$ ) behaves as  $O(1/x^2)$  in the sectors where the  $\mathcal{B}$ -cycles go. Also, thanks to the Bethe ansatz,  $v_j$  has no residue at the roots  $s_i$ 's, therefore those integrals depend only on the homology class of  $\mathcal{B}$ -cycles, and not on a representent.

Like for the classical Riemann matrix of periods we have the following property:

**Theorem 3.2** *The period matrix  $\tau$  is symmetric:  $\tau_{i,j} = \tau_{j,i}$ .*

**proof:**

We anticipate on results which shall be proved later, but which don't depend on this theorem. The proof comes directly from theorem 4.10 below, since:

$$\oint_{\mathcal{B}_\beta} dx \oint_{\mathcal{B}_\alpha} B(x, z) dz = 2i\pi \oint_{\mathcal{B}_\beta} dx v_\alpha(x) = 2i\pi \tau_{\beta, \alpha}$$

and from the symmetry theorem 4.11 for the Bergman kernel  $B(x, z) = B(z, x)$ :

$$\oint_{\mathcal{B}_\beta} dx \oint_{\mathcal{B}_\alpha} B(x, z) dz = \oint_{\mathcal{B}_\alpha} dz \oint_{\mathcal{B}_\beta} dx B(x, z) = 2i\pi \oint_{\mathcal{B}_\alpha} dz v_\beta(z) = 2i\pi \tau_{\alpha, \beta}.$$

□

### 3.5 Filling fractions

In random matrices, the notion of filling fractions, is just the  $\mathcal{A}$ -cycle integrals of the resolvent. Here, we easily generalize it by the definition:

**Definition 3.3** *The filling fractions  $\epsilon_1, \dots, \epsilon_d$  are defined as follows:*

$$\alpha = 1, \dots, \mathfrak{g}, \quad \epsilon_\alpha = \frac{1}{2i\pi} \oint_{\mathcal{A}_\alpha} \left( \omega(x) - \frac{t_0}{x} \right) + \frac{t_0 n_\alpha}{(d+1)} \quad (3.15)$$

where the integer  $n_\alpha$  is half the number of Stokes half-lines surrounded by the cycle  $\mathcal{A}_\alpha$ . In other words,  $\frac{2n_\alpha}{2d+2}$  corresponds to the angular fraction of the complex plane defined by the cycle  $\mathcal{A}_\alpha$ .

For  $\alpha = \mathfrak{g} + 1, \dots, d - 1$  we define

$$\alpha = \mathfrak{g} + 1, \dots, d - 1, \quad \epsilon_\alpha = 0 \quad (3.16)$$

And for  $\alpha = d$ , we choose a non-independent  $\mathcal{A}$ -cycle  $\mathcal{A}_d$ , which surrounds all the  $s_i$ 's which are not surrounded by  $\mathcal{A}_1, \dots, \mathcal{A}_\mathfrak{g}$ , and define:

$$\epsilon_d = \frac{1}{2i\pi} \oint_{\mathcal{A}_d} \left( \omega(x) - \frac{t_0}{x} \right) + \frac{t_0 n_d}{(d+1)} \quad (3.17)$$

Note that this definition makes sense because all the cycles  $\mathcal{A}_\alpha$  go from an infinity where  $\omega(x) - \frac{t_0}{x} \sim O\left(\frac{1}{x^2}\right)$ . Note also that this definition depends on the exact locus of the contour  $\mathcal{A}_\alpha$  and not only on its homotopy class, since  $\omega(x)$  has simple poles at the  $s_i$ 's with residue  $\hbar$ . If we deform the contour  $\mathcal{A}_\alpha$ , the filling fractions can change by some integer times  $\hbar$ .

In other words, the filling fractions are "blurred" when  $\hbar \neq 0$ , they are defined modulo an integer times  $\hbar$ . In the classical limit  $\hbar \rightarrow 0$ , they become deterministic.

We have:

**Theorem 3.3**

$$\sum_{\alpha=1}^d \epsilon_\alpha = t_0 \tag{3.18}$$

**proof:**

When we perform the sum over the contours  $\mathcal{A}_\alpha$ , the contour  $\mathcal{A}_d$  was defined as the "complementary" of the others, i.e. so that the sum is contractible. Since the function  $x \rightarrow \omega(x) - t_0/x$  is integrable at infinity, we find that its global integral is null. With the same argument, it is easy to see that  $\sum_{\alpha=1}^d n_\alpha = (d+1)$  because we take all Stokes lines once and only once. Therefore we get:

$$\sum_{\alpha=1}^d \epsilon_\alpha = 0 + \frac{t_0}{d+1} \sum_{\alpha=1}^d n_\alpha = t_0.$$

Note that it also tells us that only  $d-1$  of the epsilon's are independent.  $\square$

**Remark 3.2** In the case  $\mathfrak{g} = -1$ , the only filling fraction is  $\epsilon_d = t_0$ , and it is also the sum of residues of  $\omega$  at the  $s_i$ 's:

$$\epsilon_d = t_0 = \sum_i \operatorname{Res}_{s_i} \omega = \hbar \#\{s_i\}$$

This shows again, that  $\mathfrak{g} = -1$  corresponds to a case where  $t_0$  is quantized, namely  $t_0$  is an integer times  $\hbar$ :

$$t_0/\hbar \in \mathbb{N}.$$

## 4 Kernels

One of the key geometric objects in [15] and in [16], is the "recursion kernel"  $K(x, z)$ . It was used in the context of matrix models, to find a solution of loop equations. Here, it will also allow us to define the 3rd and 2nd kind differentials.

## 4.1 The recursion kernel $K$

First we define:

$$\hat{K}(x, z) = \frac{1}{\hbar} \frac{1}{\psi^2(x)} \int_{\infty_0}^x \psi^2(x') \frac{dx'}{x' - z} \quad (4.1)$$

and for each  $\alpha = 1, \dots, \mathfrak{g}$ , we choose a point  $P_\alpha \in \mathcal{A}_\alpha$  and we define:

$$\hbar C_\alpha(z) = \oint_{\mathcal{A}_\alpha} \frac{dx''}{\psi^2(x'')} \int_{\infty_0}^{P_\alpha} \psi^2(x') \frac{dx'}{x' - z} + \oint_{\mathcal{A}_\alpha} \frac{dx''}{\psi^2(x'')} \int_{P_\alpha}^{x''} \psi^2(x') \frac{dx'}{x' - z} \quad (4.2)$$

where in the last integral, the integration contour between  $P_\alpha$  and  $x''$ , is along  $\mathcal{A}_\alpha$ . This is described in fig.4.

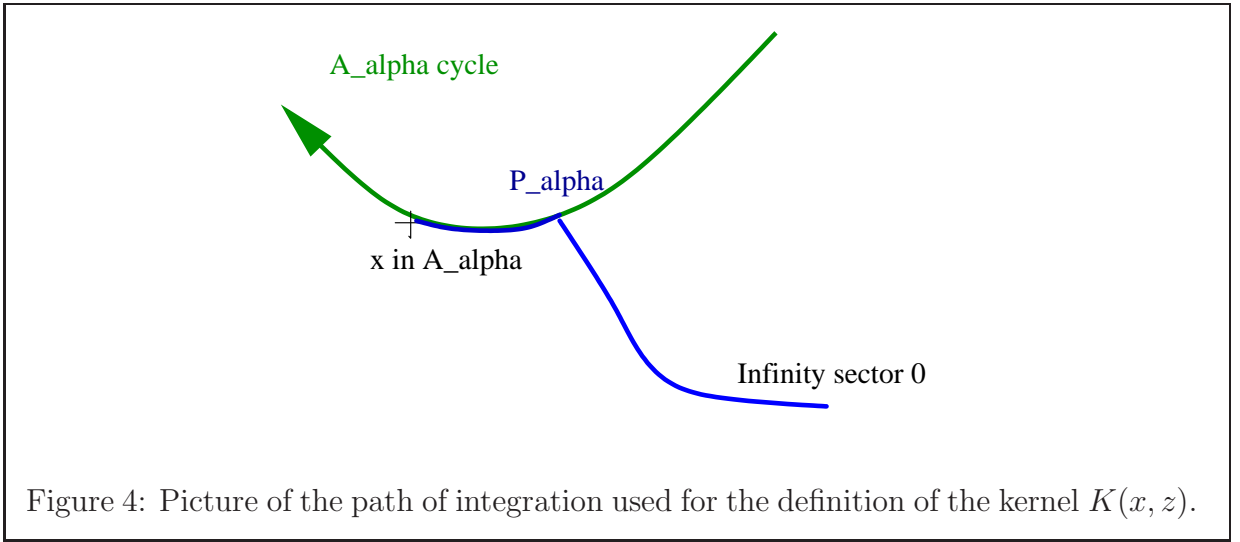


Figure 4: Picture of the path of integration used for the definition of the kernel  $K(x, z)$ .

For each  $\alpha = \mathfrak{g} + 1, \dots, d - 1$ , we define:

$$C_\alpha(z) = \int_{\hat{\mathcal{A}}_\alpha} \psi^2(x') \frac{dx'}{x' - z}. \quad (4.3)$$

We now need to describe the domain of definition of these functions.

First, one can see that for a fixed  $x$ , these functions are defined for  $z$  outside of some "cuts" (see figure 4)

- Choose a path between  $\infty_0$  and  $x$ , then  $\hat{K}(x, z)$  is defined for  $z$  outside of this path. Across the path  $]\infty_0, x]$ ,  $\hat{K}(x, z)$  has a discontinuity:

$$\delta \hat{K}(x, z) = \frac{2i\pi}{\hbar} \frac{\psi^2(z)}{\psi^2(x)} \quad (4.4)$$

- For each  $\alpha = 1, \dots, \mathfrak{g}$ , choose a path between  $\infty_0$  and  $P_\alpha$ , then  $C_\alpha(z)$  is defined for  $z$  outside of this path, and outside  $\mathcal{A}_\alpha$ . Across the path  $]\infty_0, P_\alpha]$ ,  $C_\alpha(z)$  has a discontinuity:

$$\delta C_\alpha(z) = \frac{2i\pi}{\hbar} \psi^2(z) \oint_{\mathcal{A}_\alpha} \frac{dx''}{\psi^2(x'')} \quad (4.5)$$



and across the path  $\mathcal{A}_\alpha$ ,  $C_\alpha(z)$  has a discontinuity:

$$\delta C_\alpha(z) = \frac{2i\pi \psi^2(z)}{\hbar} \int_{P_\alpha}^z \frac{dx''}{\psi^2(x'')} \quad (4.6)$$

• For each  $\alpha = \mathbf{g} + 1, \dots, d - 1$ ,  $C_\alpha(z)$  is defined for  $z$  outside of the path  $\hat{\mathcal{A}}_\alpha$ . Across the path  $\hat{\mathcal{A}}_\alpha$ ,  $C_\alpha(z)$  has a discontinuity:

$$\delta C_\alpha(z) = 2i\pi \psi^2(z) \quad (4.7)$$

From these remarks, we now define the recursion kernel  $K(x, z)$  by:

**Definition 4.1** *Definition of the recursion kernel:*

$$K(x, z) = \hat{K}(x, z) - \sum_{\alpha=1}^{d-1} v_\alpha(x) C_\alpha(z) \quad (4.8)$$

*it is defined for  $z$  outside the cuts mentionned above.*

For a fixed  $z$ , the analytical properties in  $x$  of  $K(x, z)$  are the same as those of  $\hat{K}(x, z)$  since all  $v_\alpha(x)$  are analytic. For a fixed  $z$ , the primitive of  $\psi^2(x') \frac{dx'}{x'-z}$  can be defined locally but not globally on the complex plane. In fact there is a logarithmic cut to be arbitrarily chosen on  $]\infty_0, z]$ . Anywhere out of this cut the function  $x \rightarrow K(x, z)$  is analytic.

#### 4.1.1 Properties of kernel $K$

The definition of the kernel  $K(x, z)$  might seem arbitrary at first glance. But in fact, the main reason for the introduction of such kernel is that it has many interesting properties:

It is clear from our definitions that:

**Theorem 4.1** *For a given  $z$ , the kernel  $K$  behaves like:*

$$K(x, z) \sim O(x^{-2}) \quad (4.9)$$

*when  $x \rightarrow \infty$  in all sectors.*

**proof:**

The result is obvious for sector  $S_0$  and for sectors where  $\psi$  is exponentially big. When it is not, the fact that we subtract  $C_\alpha, \alpha = \mathbf{g} + 1, \dots, d - 1$  gives the result.  $\square$

**Theorem 4.2** *We have in all sectors at infinity :*

$$K(x, z) \underset{z \rightarrow \infty}{\sim} O(z^{-d}). \quad (4.10)$$

*More precisely we have:*

$$K(x, z) \sim - \sum_{k=d-1}^{\infty} \frac{K_k(x)}{z^{k+1}} \quad (4.11)$$

*with*

$$\hat{K}_k(x) = \frac{1}{\hbar \psi^2(x)} \int_{\infty_0}^x x'^k \psi^2(x') dx', \quad (4.12)$$

*and*

$$K_k(x) = \hat{K}_k(x) - \sum_{\alpha=1}^{\mathfrak{g}} v_{\alpha}(x) \oint_{\mathcal{A}_{\alpha}} \hat{K}_k(x') dx' - \sum_{\alpha=\mathfrak{g}+1}^{d-1} v_{\alpha}(x) \oint_{\mathcal{A}_{\alpha}} \psi^2(x') x'^k dx'. \quad (4.13)$$

**proof:**

It is clear that

$$\hat{K}(x, z) \sim - \sum_{k=0}^{\infty} \frac{\hat{K}_k(x)}{z^{k+1}} \quad (4.14)$$

where

$$\hat{K}_k(x) = \frac{1}{\hbar \psi^2(x)} \int_{\infty_0}^x x'^k \psi^2(x') dx', \quad (4.15)$$

and therefore

$$K_k(x) = \hat{K}_k(x) - \sum_{\alpha=1}^{\mathfrak{g}} v_{\alpha}(x) \oint_{\mathcal{A}_{\alpha}} \hat{K}_k(x') dx' - \sum_{\alpha=\mathfrak{g}+1}^{d-1} v_{\alpha}(x) \oint_{\mathcal{A}_{\alpha}} \psi^2(x') x'^k dx' \quad (4.16)$$

Now, if  $k \leq d - 2$ , notice that  $x'^k$  is a polynomial of degree  $\leq d - 2$ , and it is thus a linear combinations of  $h_{\alpha}(x)$ 's:

$$x'^k = \sum_{\beta=1}^{d-1} b_{k,\beta} h_{\beta}(x') \quad (4.17)$$

This implies:

$$\hat{K}_k(x) = \sum_{\beta=1}^{d-1} b_{k,\beta} v_{\beta}(x) \quad (4.18)$$

Taking now the integral over an  $\mathcal{A}$  cycle and using the normalization choice of  $h_k(x)$  gives: If  $\alpha \leq \mathfrak{g}$

$$\oint_{\mathcal{A}_{\alpha}} \hat{K}_k(x') dx' = b_{k,\alpha} \quad (4.19)$$

and if  $\alpha > \mathfrak{g}$

$$\oint_{\mathcal{A}_{\alpha}} \psi^2(x') x'^k dx' = b_{k,\alpha} \quad (4.20)$$

This implies that  $K_k(x) = 0$  if  $k \leq d - 2$ , and therefore

$$K(x, z) = O(z^{-d}). \quad (4.21)$$

□

**Theorem 4.3** *Let  $\alpha = 1, \dots, \mathfrak{g}$ , and  $z$  on the side of  $\mathcal{A}_\alpha$  which does not contain  $\infty_0$ , then:*

$$\oint_{\mathcal{A}_\alpha} K(x, z) dx = 0 \quad (4.22)$$

**proof:**

Notice that if  $z$  is on that side of  $\mathcal{A}_\alpha$ , we have  $C_\alpha(z) = \oint_{\mathcal{A}_\alpha} \hat{K}(x, z) dx$ , and therefore  $\oint_{\mathcal{A}_\alpha} K(x, z) dx = 0$ . In fact one can see that the addition of the part with the  $C_\alpha(z)$  was just put there to cancel out the  $\mathcal{A}$ -cycle integrals. □

## 4.2 Third kind differential: kernel $G(x, z)$

The second important kernel to define is the equivalent of the third kind differential. In [15] this kernel was computed from  $K$  by derivation, and we use the same definition.

**Definition 4.2** *We define the kernel  $G(x, z)$  by:*

$$G(x, z) = -\hbar \psi^2(z) \partial_z \frac{K(x, z)}{\psi^2(z)} = 2\hbar \frac{\psi'(z)}{\psi(z)} K(x, z) - \hbar \partial_z K(x, z) \quad (4.23)$$

From an easy integration by parts we find:

$$\begin{aligned} G(x, z) &= -\frac{1}{x-z} + \frac{2}{\psi^2(x)} \int_{\infty_0}^x \frac{dx'}{x'-z} \psi^2(x') \left( \frac{\psi'(x')}{\psi(x')} - \frac{\psi'(z)}{\psi(z)} \right) \\ &\quad - \hbar \sum_{\alpha} v_{\alpha}(x) \psi^2(z) \partial_z \frac{C_{\alpha}(z)}{\psi^2(z)} \end{aligned} \quad (4.24)$$

which shows that near  $x = z$  we have  $G(x, z) \sim \frac{1}{z-x}$ , i.e. there is a simple pole of residue 1 at  $z = x$ . Note in particular that  $\frac{1}{x'-z} \left( \frac{\psi'(x')}{\psi(x')} - \frac{\psi'(z)}{\psi(z)} \right)$  has no singularity at  $x' = z$  and therefore for a fixed  $z$ , there is no more any logarithmic cut  $]\infty, z]$  as we had for  $K(x, z)$ .

Note again that a priori, this function of  $z$  has the same lines of discontinuity as the kernel  $K(x, z)$ . But notice that the definition of  $G$  ensures that all discontinuities of  $K$  which are proportional to  $\psi^2(z)$  cancel.

**Theorem 4.4**  $G(x, z)$  is an analytical function of  $x$ , with a simple pole at  $x = z$  with residue  $-1$ , and double poles at the  $s_j$ 's (zeros of  $\psi(x)$ ) with vanishing residue, and possibly an essential singularity around  $\infty$ .

$G(x, z)$  is an analytical function of  $z$ , with a simple pole at  $z = x$  with residue  $+1$ , simple poles at  $z = s_j$ , and with a discontinuity across  $\mathcal{A}_\alpha$ -cycles with  $\alpha = 1, \dots, \mathfrak{g}$  (and thus no discontinuity on  $\hat{\mathcal{A}}_\alpha$ ):

$$\delta G(x, z) = -2i\pi v_\alpha(x) \quad (4.25)$$

**proof:**

$K(x, z)$  is discontinuous when  $z$  crosses either  $]\infty_0, x]$ ,  $]\infty_0, P_\alpha]$  or  $\mathcal{A}_\alpha$ . However, the discontinuity of  $K(x, z)$  across  $]\infty_0, x]$ ,  $]\infty_0, P_\alpha]$ , and  $\hat{\mathcal{A}}_\alpha$  is proportional to  $\psi^2(z)$ , and this means by derivation that  $G(x, z)$  is not discontinuous there. Across  $\mathcal{A}_\alpha$  with  $\alpha \leq \mathfrak{g}$ , the discontinuity of  $K(x, z)$  is given by eq. (4.7), and thus, the discontinuity of  $G(x, z)$  is  $\delta G(x, z) = -2i\pi v_\alpha(x)$ .

Since  $K(x, z)$  is regular when  $z = s_j$ , then it is clear that  $G(x, z)$  has simple poles at  $z = s_j$ , with residue  $-2\hbar K(x, s_j)$ .

In the variable  $x$ , it is clear from the definition and from the Bethe ansatz 2.18, that  $K(x, z)$  has double poles at  $x = s_j$  without residue, and this properties follows for  $G(x, z)$ .

□

**Theorem 4.5**

$$G(x, z) = O(1/x^2) \quad (4.26)$$

when  $x \rightarrow \infty$  in all sectors.

And at large  $z$  in sector  $S_k$ :

$$\lim_{z \rightarrow \infty_k} G(x, z) = G(x, \infty_k) = \eta_k t_{d+1} K_{d-1}(x) \quad (4.27)$$

where  $\eta_k = \pm 1$  is such that  $\psi \sim e^{\eta_k V/2\hbar}$  in sector  $S_k$ .

**proof:**

The large  $x$  behavior follows from theorem 4.1. The large  $z$  behavior is given by theorem 4.2, i.e.  $G(x, z) \sim \eta_k V'(z) K(x, z) \sim \eta_k t_{d+1} K_{d-1}(x)$ . The sign depends on the behavior of the solution in this sector. □.

**Theorem 4.6** Let  $\alpha = 1, \dots, \mathfrak{g}$ , and  $z$  on the side of  $\mathcal{A}_\alpha$  which does not contain  $\infty_0$ , then:

$$\oint_{\mathcal{A}_\alpha} G(x, z) dx = 0 \quad (4.28)$$

**proof:**

Immediate from theorem 4.3 □

### 4.2.1 Semi-classical limit

We claim that this kernel is the quantum version of the third kind differential. Indeed, in classical algebraic geometry a third kind differential is characterized by analyticity except a simple pole with non vanishing residue and a proper normalization on  $\mathcal{A}$ -cycles. Here, apart from the discontinuity along the  $\mathcal{A}$ -cycles which is expected since these contours represent the "quantum cuts", we have analyticity (apart from the  $s_i$ 's which also define the cuts), a simple pole with residue and a good normalization on  $\mathcal{A}$ -cycles.

In the BKW semiclassical expansion we have  $\psi \sim e^{\frac{\pm 1}{\hbar} \int \sqrt{U}}$  and thus

$$\hat{K}(x, z) \sim \frac{2}{x-z} \frac{1}{\sqrt{U(x)}} \quad (4.29)$$

and

$$K(x, z) \sim \frac{1}{x-z} \frac{1}{2\sqrt{U(x)}} - \sum_{\alpha} v_{\alpha}(x) C_{\alpha}(z) \quad (4.30)$$

and

$$G(x, z) \sim 2\sqrt{U(z)}K(x, z) \sim \frac{1}{x-z} \frac{\sqrt{U(z)}}{\sqrt{U(x)}} - 2 \sum_{\alpha} v_{\alpha}(x) C_{\alpha}(z) \sqrt{U(z)} \quad (4.31)$$

The form  $G(x, z)dx$  has thus a simple pole at  $x = z$ , in the physical sheet with residue  $+1$  and in the other sheet with residue  $-1$ , and it is normalized on  $\mathcal{A}$ -cycles  $\oint_{\mathcal{A}_i} G(x, z)dx = 0$ . This is indeed the usual 3rd kind differential in classical algebraic geometry.

### 4.3 The Bergman kernel $B(x, z)$

In classical algebraic geometry, the Bergman kernel is the fundamental second kind differential, it is the derivative of the 3rd kind differential, and it is another major tool in classical algebraic geometry. Following the same definition as in [15], we define:

$$B(x, z) = -\frac{1}{2} \partial_z G(x, z). \quad (4.32)$$

The kernel  $B$  is going to be called the "quantum" Bergman kernel.

**Theorem 4.7**  *$B(x, z)$  is an analytical function of  $x$ , with a double pole at  $x = z$  with no residue, and double poles at the  $s_j$ 's with vanishing residues, and possibly an essential singularity around  $\infty$ .*

*$B(x, z)$  is an analytical function of  $z$ , with a double pole at  $z = x$  with no residue, and double poles at the  $s_j$ 's with vanishing residues, and possibly an essential singularity*

around  $\infty$ . *In particular it has no discontinuity along the  $\mathcal{A}$  cycles, it is defined analytically in the whole complex plane except at those double poles.*

**proof:**

Those properties follow easily from those of  $G(x, z)$  of theorem 4.4. In particular, it is important to notice that the only discontinuity of  $G(x, z)$  is along the  $\mathcal{A}$ -cycles, and is independent of  $z$ , therefore  $B(x, z)$  has no discontinuity there.  $\square$

### 4.3.1 Properties of the Bergman kernel

**Theorem 4.8**

$$B(x, z) = O(1/x^2) \quad (4.33)$$

when  $x \rightarrow \infty$  in all sectors.

And

$$B(x, z) = O(1/z^2) \quad (4.34)$$

when  $z \rightarrow \infty$  in all sectors.

**proof:**

Follows from the large  $x$  and  $z$  behaviors of  $G(x, z)$ .  $\square$

**Theorem 4.9** *B satisfies the loop equations:*

$$\left(2\frac{\psi'(x)}{\psi(x)} + \partial_x\right) \left(B(x, z) - \frac{1}{2(x-z)^2}\right) + \partial_z \frac{\frac{\psi'(x)}{\psi(x)} - \frac{\psi'(z)}{\psi(z)}}{x-z} = P_2^{(0)}(x, z) \quad (4.35)$$

where  $P_2^{(0)}(x, z)$  is a polynomial in  $x$  of degree at most  $d-2$ . And

$$\left(2\frac{\psi'(z)}{\psi(z)} + \partial_z\right) \left(B(x, z) - \frac{1}{2(x-z)^2}\right) + \partial_x \frac{\frac{\psi'(x)}{\psi(x)} - \frac{\psi'(z)}{\psi(z)}}{x-z} = \tilde{P}_2^{(0)}(z, x) \quad (4.36)$$

where  $\tilde{P}_2^{(0)}(z, x)$  is a polynomial in  $z$  of degree at most  $d-2$ .

**proof:**

This theorem is crucial for all what follows, and its proof is rather non-trivial. Since it is very long and technical, we present the proof in appendix A. Those equations are indeed the loop equations for the 2-point function in the  $\beta$  matrix model, see section 9.  $\square$

**Theorem 4.10** *We have for every  $\alpha = 1, \dots, \mathfrak{g}$ :*

$$\oint_{\mathcal{A}_\alpha} B(x, z) dx = 0 \quad , \quad \oint_{\mathcal{A}_\alpha} B(x, z) dz = 0 \quad (4.37)$$

and

$$\oint_{\mathcal{B}_\alpha} B(x, z) dz = 2i\pi v_\alpha(x) \quad (4.38)$$

**proof:**

The vanishing of  $\mathcal{A}$ -cycle integrals in the  $x$  variable is by construction and can be seen as the consequence of the same result known for  $G(x, z)$  on one side of  $\mathcal{A}$  and the fact that  $B(x, z)$  has no discontinuity along the  $\mathcal{A}$ -cycles. (Therefore, the nullity extend on both sides which no longer need to be treated separately).

For the  $z$  variable, notice that if  $\mathcal{A}_\alpha = ]\infty_i, \infty_j[$  goes from  $\infty_i$  to  $\infty_j$ , where both  $\infty_i$  and  $\infty_j$  are in the physical sheet, we have:

$$\oint_{\mathcal{A}_\alpha} B(x, z) dz = \int_{\infty_i}^{\infty_j} B(x, z) dz = -\frac{1}{2} (G(x, \infty_j) - G(x, \infty_i)) \quad (4.39)$$

and from theorem 4.5  $G(x, \infty_i) = \eta_i t_{d+1} K_{d-1}(x)$ , we get:

$$\oint_{\mathcal{A}_\alpha} B(x, z) dz = \int_{\infty_i}^{\infty_j} B(x, z) dz = \frac{\eta_i - \eta_j}{2} t_{d+1} K_{d-1}(x) \quad (4.40)$$

and since  $\infty_i$  and  $\infty_j$  are both in the physical sheet we have  $\eta_i = \eta_j = -1$ , and therefore

$$\oint_{\mathcal{A}_\alpha} B(x, z) dz = 0. \quad (4.41)$$

And similarly, when performing the integral over  $\mathcal{B}_\alpha$ , the contribution from infinities cancels out since the contour goes in the same sheet. But since  $\mathcal{B}_\alpha$  intersects its corresponding  $\mathcal{A}_\alpha$  (and only this one) where the primitive  $-\frac{1}{2}G(x, z)$  is discontinuous, the result is the jump of  $G(x, z)$  along this  $\mathcal{A}_\alpha$ , that is to say  $i\pi v_\alpha(x)$ . Eventually, since  $\mathcal{B}_\alpha$  and  $\mathcal{A}_\alpha$  intersect twice, we find eq. (4.10).  $\square$

One of our key theorems is:

**Theorem 4.11**  *$B(x, z)$  is symmetric*

$$B(x, z) = B(z, x) \quad (4.42)$$

**proof:**

The proof relies essentially on the fact that  $B(x, z)$  satisfies the loop equation in the two variables. We have:

$$\left(2 \frac{\psi'(z)}{\psi(z)} + \partial_z\right) \left(2 \frac{\psi'(x)}{\psi(x)} + \partial_x\right) \left(B(x, z) - \frac{1}{2(x-z)^2}\right)$$

$$\begin{aligned}
&= \left(2\frac{\psi'(z)}{\psi(z)} + \partial_z\right) \left(P_2^{(0)}(x, z) - \partial_z \frac{\frac{\psi'(x)}{\psi(x)} - \frac{\psi'(z)}{\psi(z)}}{x-z}\right) \\
&= \left(2\frac{\psi'(x)}{\psi(x)} + \partial_x\right) \left(\tilde{P}_2^{(0)}(z, x) - \partial_x \frac{\frac{\psi'(x)}{\psi(x)} - \frac{\psi'(z)}{\psi(z)}}{x-z}\right)
\end{aligned} \tag{4.43}$$

This implies:

$$\begin{aligned}
&\left(2\frac{\psi'(z)}{\psi(z)} + \partial_z\right) P_2^{(0)}(x, z) - \left(2\frac{\psi'(x)}{\psi(x)} + \partial_x\right) \tilde{P}_2^{(0)}(z, x) \\
&= \left(2\frac{\psi'(z)}{\psi(z)} + \partial_z\right) \partial_z \frac{\frac{\psi'(x)}{\psi(x)} - \frac{\psi'(z)}{\psi(z)}}{x-z} \\
&\quad - \left(2\frac{\psi'(x)}{\psi(x)} + \partial_x\right) \partial_x \frac{\frac{\psi'(x)}{\psi(x)} - \frac{\psi'(z)}{\psi(z)}}{x-z} \\
&= 2\frac{U(x) - U(z)}{(x-z)^2} - \frac{U'(x) + U'(z)}{x-z}
\end{aligned} \tag{4.44}$$

and therefore:

$$\begin{aligned}
&(x-z)^2 \left(2\frac{\psi'(z)}{\psi(z)} + \partial_z\right) P_2^{(0)}(x, z) + 2U(z) + (x-z)U'(z) \\
&= (x-z)^2 \left(2\frac{\psi'(x)}{\psi(x)} + \partial_x\right) \tilde{P}_2^{(0)}(z, x) + 2U(x) + (z-x)U'(x) \\
&\stackrel{\text{def}}{=} R(x, z)
\end{aligned} \tag{4.45}$$

Here, the first line is a polynomial in  $x$ , whereas the second line is also a polynomial in  $z$ . Therefore,  $R(x, z)$  is a polynomial in both variables, of degree at most  $d$  in each variable. Moreover, we must have:

$$R(x, x) = 2U(x) \tag{4.46}$$

Therefore we must have:

$$R(x, z) = \frac{1}{\hbar^2} \left( \frac{1}{2} V'(x)V'(z) - \hbar \frac{V'(x) - V'(z)}{x-z} - P(x) - P(z) \right) + (x-z)^2 \tilde{R}(x, z) \tag{4.47}$$

where  $\tilde{R}(x, z)$  is a polynomial of both variables of degree at most  $d-2$  in each variable.

Putting this back into 4.45 and using the symmetry  $x \leftrightarrow z$  it implies that:

$$\left(2\frac{\psi'(z)}{\psi(z)} + \partial_z\right) (P_2^{(0)}(x, z) - \tilde{P}_2^{(0)}(x, z)) = \tilde{R}(x, z) - \tilde{R}(z, x) \tag{4.48}$$

Then, we can decompose the r.h.s into the basis  $h_\alpha(x)h_\beta(z)$  introduced in 3.6:

$$\tilde{R}(x, z) - \tilde{R}(z, x) = \sum_{\alpha, \beta=1}^{d-1} (\tilde{R}_{\alpha, \beta} - \tilde{R}_{\beta, \alpha}) h_\alpha(x) h_\beta(z) \tag{4.49}$$



Integrating the differential equation eq. (4.48) then gives:

$$P_2^{(0)}(x, z) - \tilde{P}_2^{(0)}(x, z) = \sum_{\alpha, \beta=1}^{d-1} (\tilde{R}_{\alpha, \beta} - \tilde{R}_{\beta, \alpha}) h_\alpha(x) v_\beta(z) + A_1(x) \quad (4.50)$$

where  $A_1(x)$  is some integration constant.

Then using the loop equations 4.9 we find by subtraction that:

$$\left(2 \frac{\psi'(y)}{\psi(y)} + \partial_y\right) (B(y, z) - B(z, y)) = P_2^{(0)}(y, z) - \tilde{P}_2^{(0)}(y, z) \quad (4.51)$$

and again, integrating this differential equation we find:

$$B(x, z) - B(z, x) = \sum_{\alpha, \beta=1}^{d-1} (\tilde{R}_{\alpha, \beta} - \tilde{R}_{\beta, \alpha}) v_\alpha(x) v_\beta(z) + A(x) + \tilde{A}(z) \quad (4.52)$$

where  $(2\psi'/\psi + \partial)A = A_1$ , and  $\tilde{A}(z)$  is some other integration constant.

The large  $x$  and large  $z$  behavior of  $B$  imply that  $A(x) = \tilde{A}(z) = 0$ . We thus get:

$$B(x, z) - B(z, x) = \sum_{\alpha, \beta} (\tilde{R}_{\alpha, \beta} - \tilde{R}_{\beta, \alpha}) \tilde{v}_\alpha(x) \tilde{v}_\beta(z) \quad (4.53)$$

Then, using theorem 4.10

$$\oint_{\mathcal{A}_\alpha} B(x, z) dx = 0 = \oint_{\mathcal{A}_\beta} B(x, z) dz \quad (4.54)$$

We find:

$$\forall \alpha, \beta, \quad \tilde{R}_{\alpha, \beta} = \tilde{R}_{\beta, \alpha} \quad (4.55)$$

that is to say by 4.53 that the Bergman kernel is symmetric.  $\square$

We claim that all these properties are essential to name this function a "quantum Bergman kernel". Indeed, the symmetry is absolutely necessary and is completely non-trivial. The fact that  $B(x, z)$  has no discontinuity is also essential since in standard algebraic geometry, it is defined everywhere on the Riemann surface. Using all these kernels and their properties, we can then generalize easily the recursion of [12, 15] defining the correlation functions.

## 4.4 Meromorphic forms and properties

### 4.4.1 Definition of meromorphic forms

**Definition 4.3** *A meromorphic form  $\mathcal{R}(x)$  is defined as:*

$$\mathcal{R}(x) = \frac{1}{\hbar \psi^2(x)} \int_{\infty_0}^x r(x') \psi^2(x') dx' \quad (4.56)$$

where  $r(x)$  is a rational function of  $x$ , which behaves at most like  $O(x^{d-2})$  at large  $x$ , and whose poles  $r_i$  are such that:

$$\operatorname{Res}_{x \rightarrow r_i} \psi^2(x) r(x) = 0 \quad (4.57)$$

and for all degenerate  $\hat{\mathcal{A}}_\alpha$  cycles

$$\int_{\hat{\mathcal{A}}_\alpha} \psi^2(x') r(x') dx' = 0. \quad (4.58)$$

It is easy to see, that with this definition, the holomorphic forms  $v_\alpha(x)$ , the kernels  $G(x, z)$  and  $B(x, z)$  are meromorphic forms of  $x$ .

#### 4.4.2 Analiticity properties

A meromorphic forms  $\mathcal{R}(x)$ , has poles at  $x = r_i$  the poles of  $r(x)$ , with degree 1 less than that of  $r$ , it behaves like  $O(x^{-2})$  in all sectors of the physical sheet. From the Bethe ansatz, it has double poles at the  $s_i$ 's, with vanishing residues.

In particular, it has an accumulation of poles along the half-lines  $L_i$  of accumulations of zeroes of  $\psi$ .

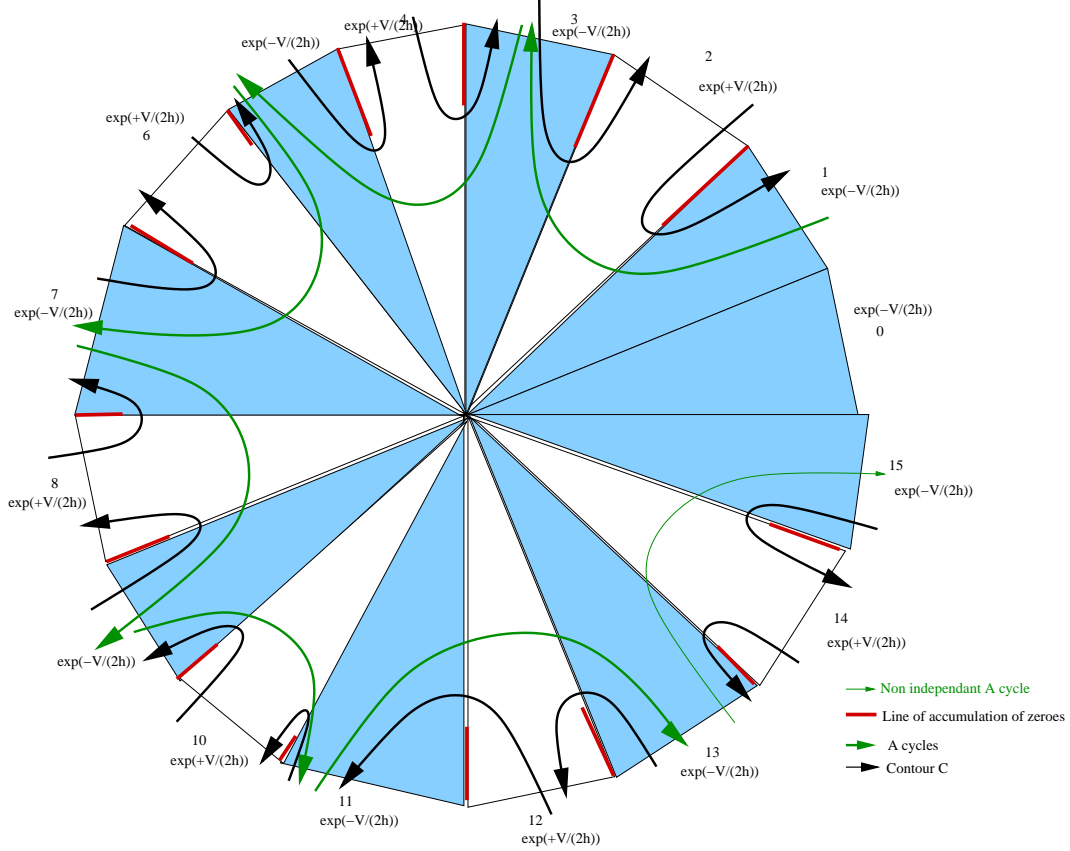
Also, notice that the following integrals are well defined, and independent of homotopic deformations of  $\mathcal{A}_\alpha$  (in particular independent of where are the  $s_i$ 's):

$$\oint_{\mathcal{A}_\alpha} \mathcal{R}(x) dx. \quad (4.59)$$

#### 4.4.3 The integration contours around branch-points

Let us choose some contour  $\mathcal{C}_i$ , such that each  $\mathcal{C}_i$  surrounds (in the trigonometric direction) a half-line  $L_i$  of accumulation of zeroes. In other words it surrounds a "branch point". Let us also assume that  $\sum_i \mathcal{C}_i$  surrounds all roots of  $\psi$ , i.e. each root of  $\psi$  is enclosed in one  $\mathcal{C}_i$ . We also assume that contours  $\mathcal{C}_i$  and  $\mathcal{A}_\alpha$  do not intersect (they have vanishing intersection numbers):

$$\forall i = 1, \dots, 2\mathbf{g} + 2, \quad \forall \alpha = 1, \dots, d - 1, \quad \mathcal{C}_i \cap \mathcal{A}_\alpha = 0 \quad (4.60)$$



#### 4.4.4 Riemann bilinear identity

For the Riemann bilinear identity, we need the following useful lemma, which we shall use very often in this article:

**Lemma 4.1** *For every analytical function  $f(x)$  which behaves at infinity at most like  $f(x) = O(x^{d-2})$  in all directions, and such that it has no singularities inside every contour  $\mathcal{C}_i$  (and thus must be regular at the root  $s_j$ 's) we have, for  $x_0$  outside of all  $\mathcal{A}$ -cycles (i.e. on the same side as  $\infty_0$ ) :*

$$\forall i, \quad \frac{1}{2i\pi} \oint_{\mathcal{C}_i} dx K(x_0, x) f(x) = 0$$

**proof:**

Clearly, the contours  $\mathcal{C}_i$  enclose no singularity of  $K(x_0, x)f(x)$  and can be contracted to 0.  $\square$

Then we can write the bilinear Riemann identity:

#### Theorem 4.12 Riemann bilinear identity

*Consider a meromorphic form  $\mathcal{R}(x)$ , with poles  $r_i$ .*

Then we have for  $x$  outside of all  $\mathcal{A}$ -cycles (i.e. on the same side as  $\infty_0$ ):

$$\mathcal{R}(x) = - \sum_i \operatorname{Res}_{r_i} G(x, z) \mathcal{R}(z) dz + \sum_{\alpha=1}^g v_\alpha(x) \oint_{\mathcal{A}_\alpha} \mathcal{R}(z) dz. \quad (4.61)$$

**proof:**

Since  $G(x, z) = 1/(z - x) + \dots$ , we write Cauchy formula:

$$\mathcal{R}(x) = \operatorname{Res}_{z \rightarrow x} G(x, z) \mathcal{R}(z) dz \quad (4.62)$$

and we deform the contour of integration from a small circle around  $x$ , to contours enclosing all other singularities, i.e. the  $r_i$ 's and the  $s_i$ 's. By doing so,  $G(x, z)$  has to cross the  $\mathcal{A}$ -cycles, and picks a discontinuity equal to  $2i\pi v_\alpha(x)$  i.e. independent of  $z$ , so the contour integral of the product factorizes for each  $\mathcal{A}_\alpha$ . We thus arrive to:

$$\begin{aligned} \mathcal{R}(x) &= - \sum_i \operatorname{Res}_{r_i} G(x, z) \mathcal{R}(z) dz - \sum_i \frac{1}{2i\pi} \oint_{\mathcal{C}_i} G(x, z) \mathcal{R}(z) dz \\ &\quad + \sum_{\alpha=1}^g v_\alpha(x) \oint_{\mathcal{A}_\alpha} \mathcal{R}(z) dz. \end{aligned} \quad (4.63)$$

Then, we need to compute

$$\oint_{\mathcal{C}_i} G(x, z) \mathcal{R}(z) dz.$$

Write that  $G(x, z) = \psi^2(z) \partial_z K(x, z) / \psi^2(z)$ , and integrate by parts:

$$\oint_{\mathcal{C}_i} G(x, z) \mathcal{R}(z) dz = - \oint_{\mathcal{C}_i} K(x, z) r(z) dz$$

and using lemma 4.1, we see that this vanishes.

□

## 5 Definition of correlators and free energies

In this section, we define the quantum deformations of the correlation functions introduced in [12, 16]. Although the following definitions are inspired from (non hermitian) matrix models (see section 9), they are valid in the present framework of an arbitrary Schrödinger equation, not necessarily linked to a matrix model. The special case of their application to matrix models will be discussed in section 9.

### 5.1 Definition of correlators

**Definition 5.1** We define the following functions  $W_n^{(g)}(x_1, \dots, x_n)$  called  **$n$ -point correlation function of "genus"  $g$**  by the recursion<sup>1</sup>:

$$W_1^{(0)}(x) = \omega(x) \quad , \quad W_2^{(0)}(x_1, x_2) = B(x_1, x_2) \quad (5.1)$$

---

<sup>1</sup>here  $g$  is any given integer, it has nothing to do with the genus  $g$  of the spectral curve.

$$\begin{aligned}
W_{n+1}^{(g)}(x_0, J) &= \frac{1}{2i\pi} \sum_{i=1}^{2g+2} \oint_{\mathcal{C}_i} dx K(x_0, x) \left( \overline{W}_{n+2}^{(g-1)}(x, x, J) \right. \\
&\quad \left. + \sum_{h=0}^g \sum_{I \subset J} W_{|I|+1}^{(h)}(x, x_I) W_{n-|I|+1}^{(g-h)}(x, J/I) \right) \\
(5.2)
\end{aligned}$$

where  $J$  is a collective notation for the variables  $J = \{x_1, \dots, x_n\}$ , and where  $\sum \sum'$  means that we exclude the terms  $(h = 0, I = \emptyset)$  and  $(h = g, I = J)$ , and where:

$$\overline{W}_n^{(g)}(x_1, \dots, x_n) = W_n^{(g)}(x_1, \dots, x_n) - \frac{\delta_{n,2} \delta_{g,0}}{2} \frac{1}{(x_1 - x_2)^2} \quad (5.3)$$

Here  $x_0$  and all the  $x_i$ 's are outside of the  $\mathcal{A}$ -cycles, i.e. on the same side as  $\infty_0$ . The contour  $\mathcal{C}_i$  (defined in section 4.4.3) is a contour which surrounds the branchpoint  $L_i$ , i.e. a half-line of accumulation of zeroes, and chosen such that every  $s_j$  is surrounded by exactly one  $\mathcal{C}_i$ , and such that  $\mathcal{C}_i$  doesn't intersect any  $\mathcal{A}$ -cycle. Very often we shall write

$$\mathcal{C} = \sum_{i=1}^{2g+2} \mathcal{C}_i. \quad (5.4)$$

Appart from the precise definition of the kernel  $K$ , this definition is exactly the same topological recursion as in [16], a sum of residues around all branchpoints of the same expression. In other words, the topological recursion is independent of  $\hbar$ .

To shorten equation we will introduce the notation:

$$\begin{aligned}
U_n^{(g)}(x, J) &= \overline{W}_{n+2}^{(g-1)}(x, x, J) + \sum_{I \subset J} \overline{W}_{|I|+1}^{(h)}(x, x_I) \overline{W}_{n-|I|+1}^{(g-h)}(x, J/I) \\
&\quad + \sum_j \partial_{x_j} \left( \frac{\overline{W}_n^{(g)}(x, J/\{j\}) - \overline{W}_n^{(g)}(x_j, J/\{j\})}{(x - x_j)} \right) \\
(5.5)
\end{aligned}$$

To get:

**Theorem 5.1**

$$W_{n+1}^{(g)}(x_0, J) = \frac{1}{2i\pi} \oint_{\mathcal{C}} dx K(x_0, x) U_n^{(g)}(x, J) \quad (5.6)$$

**proof:**

The only difference with the definition, is when we face a term like  $B(x, x_j) W_n^{(g)}(x, J/\{j\})$ . (note that there are twice this term). It can be split into two terms:  $\overline{B}(x, x_j) W_n^{(g)}(x, J/\{j\})$  and  $\frac{1}{(x-x_j)^2} W_n^{(g)}(x, J/\{j\})$ . The second term compensate exactly the  $\partial_{x_j} \frac{\overline{W}_n^{(g)}(x, J/\{j\})}{(x-x_j)}$ . Thus, the only difference between the two definitions is the term:  $\frac{1}{2i\pi} \oint_{\mathcal{C}} dx K(x_0, x) \sum_j \partial_{x_j} \frac{\overline{W}_n^{(g)}(x, J/\{j\})}{(x-x_j)}$ . Therefore the definitions are only the same if these terms are null. This is the case because of Lemma 4.1.  $\square$

## 5.2 Properties of correlators

The main reason of definition. 5.1, is because the  $W_n^{(g)}$ 's have many beautiful properties, which generalize those of [16], and in particular they provide a solution of loop equations. We shall prove the following properties:

**Theorem 5.2** *Each  $W_n^{(g)}(x_1, \dots, x_n)$  with  $2 - 2g - n < 0$ , is an analytical functions of all its arguments, with poles only when  $x_i \rightarrow s_j$ . Moreover, it vanishes at least as  $O(1/x_i^2)$  when  $x_i \rightarrow \infty$  in all sectors. It has no discontinuity across  $\mathcal{A}$ -cycles.*

**proof:**

in appendix B  $\square$

**Theorem 5.3** *For all  $(n, g) \neq (0, 0)$  we have*

$$\forall \alpha \leq \mathfrak{g} : \oint_{\mathcal{A}_\alpha} W_{n+1}^{(g)}(x_0, x_1, \dots, x_n) dx_1 = 0 \quad (5.7)$$

$$\forall \alpha \leq \mathfrak{g} : \oint_{\mathcal{A}_\alpha} W_{n+1}^{(g)}(x_0, x_1, \dots, x_n) dx_0 = 0 \quad (5.8)$$

**proof:**

We clearly have these properties for  $W_2^{(0)}(x_0, x_1)$ . By an easy recursion, the first property holds for  $x_1, \dots, x_n$ . The case of the variable  $x_0$  is special and requires explanation. Indeed for fixed values of  $x_1, \dots, x_n$ , the dependance in  $x_0$  comes from  $K(x_0, x)$ . The theorem then comes from a permutation of integrals. Indeed, since the contour  $\mathcal{C}$  never crosses any  $\mathcal{A}$ -cycles by prescription then we can permute the integrals in  $x$  and  $x_0$ . The nullity of the integral for  $K(x_0, x)$  in 4.3 then gives the result.  $\square$

**Theorem 5.4** *For  $2 - 2g - n < 0$ , the  $W_n^{(g)}$ 's satisfy the loop equation, i.e. Virasoro-like constraints. This means that the quantity:*

$$\begin{aligned} P_{n+1}^{(g)}(x; x_1, \dots, x_n) &= 2\hbar \frac{\psi'(x)}{\psi(x)} \overline{W}_{n+1}^{(g)}(x, x_1, \dots, x_n) + \hbar \partial_x \overline{W}_{n+1}^{(g)}(x, x_1, \dots, x_n) \\ &+ \sum_{I \subset J} \overline{W}_{|I|+1}^{(h)}(x, x_I) \overline{W}_{n-|I|+1}^{(g-h)}(x, J/I) + \overline{W}_{n+2}^{(g-1)}(x, x, J) \\ &+ \sum_j \partial_{x_j} \left( \frac{\overline{W}_n^{(g)}(x, J/\{j\}) - \overline{W}_n^{(g)}(x_j, J/\{j\})}{(x - x_j)} \right) \end{aligned} \quad (5.9)$$

*is a polynomial in the variable  $x$ , of degree at most  $d - 2$ .*

**proof:**

in appendix C  $\square$

**Theorem 5.5** Each  $W_n^{(g)}$  is a symmetric function of all its arguments.

**proof:**

in appendix E, with the special case of  $W_3^{(0)}$  in appendix D.  $\square$

**Theorem 5.6** The 3 point function  $W_3^{(0)}$  can also be written:

$$W_3^{(0)}(x_1, x_2, x_3) = \frac{4}{2i\pi} \sum_i \oint_{\mathcal{C}_i} \frac{B(x, x_1)B(x, x_2)B(x, x_3)}{Y'(x)} dx \quad (5.10)$$

(this can be seen as a quantum version of Rauch variational formula)

**proof:**

in appendix D  $\square$

**Theorem 5.7** For  $2-2g-n < 0$ ,  $W_n^{(g)}(x_1, \dots, x_n)$  is homogeneous of degree  $2-2g-n$ :

$$\left( \hbar \frac{\partial}{\partial \hbar} + \sum_{j=0}^{d+1} t_j \frac{\partial}{\partial t_j} + \sum_{i=1}^g \epsilon_i \frac{\partial}{\partial \epsilon_i} \right) W_n^{(g)}(x_1, \dots, x_n) = (2-2g-n) W_n^{(g)}(x_1, \dots, x_n) \quad (5.11)$$

**proof:**

Under a change  $t_k \rightarrow \lambda t_k$ ,  $\hbar \rightarrow \lambda \hbar$ ,  $\epsilon_i \rightarrow \lambda \epsilon_i$ , the Schrödinger equation remains unchanged, and thus  $\psi$  is unchanged. The kernel  $K$  is changed to  $K/\lambda$  and nothing else is changed. By recursion,  $W_n^{(g)}$  is changed by  $\lambda^{2-2g-n}$ .  $\square$

## 6 Deformations

In this section, we will consider the variations of correlators  $W_n^{(g)}$  under infinitesimal variations of the Schrödinger potential  $U(x)$  or  $\hbar$ . Infinitesimal variations of the resolvent  $\omega(x)$  can be decomposed on the basis of "meromorphic forms", and forms can be put in duality with cycles. The duality kernel pairing is the Bergman kernel. We will find in this section, that the classical  $\hbar = 0$  formulae remain valid for  $\hbar \neq 0$ , and generalize the corresponding form-cycle duality in special geometry.

### 6.1 Variation of the resolvent

Let's consider an infinitesimal polynomial variation:

$$U \rightarrow U + \delta U \quad , \quad \hbar \rightarrow \hbar + \delta \hbar$$

where  $\delta U$  is a polynomial of degree:  $\deg \delta U \leq 2d$ . Since we have written  $U = V'^2/4 - \hbar V''/2 - P$ , we have:

$$\delta U = \frac{V'}{2} \delta V' - \frac{\hbar}{2} \delta V'' - \frac{\delta \hbar}{2} V'' - \delta P \quad (6.1)$$

with

$$\delta V'(x) = \sum_{k=1}^{d+1} \delta t_k x^{k-1}, \quad (6.2)$$

and  $\delta P$  is of degree at most  $d - 1$ :

$$\delta P = (t_{d+1} \delta t_0 + t_0 \delta t_{d+1}) x^{d-1} + \text{lower degree}. \quad (6.3)$$

Let us compute  $\delta \psi$ , or more precisely  $f = \delta \ln \psi = \delta \psi / \psi$ , let us write it:

$$\delta \psi(x) = f(x) \psi(x) \quad (6.4)$$

The Schrödinger equation  $\hbar^2 \psi'' = U \psi$  implies:

$$\hbar^2 (f \psi)'' - U f \psi = \delta U \psi - \delta \hbar^2 \psi'' \quad (6.5)$$

i.e.

$$\hbar^2 (f'' \psi + 2f' \psi') = (\delta U - 2 \frac{\delta \hbar}{\hbar} U) \psi \quad (6.6)$$

Multiplying by  $\psi$  we get:

$$\hbar^2 (f' \psi^2)' = (\delta U - 2 \frac{\delta \hbar}{\hbar} U) \psi^2 \quad (6.7)$$

i.e.:

$$\delta(\psi'/\psi) = f'(x) = \frac{1}{\hbar^2 \psi^2(x)} \int_{\infty_0}^x \psi^2(x') (\delta U(x') - 2 \frac{\delta \hbar}{\hbar} U(x')) dx'. \quad (6.8)$$

therefore, since  $\omega = V'/2 + \hbar \psi'/\psi$ :

$$\delta \omega(x) = \frac{\delta V'(x)}{2} + \delta \hbar \frac{\psi'(x)}{\psi(x)} + \frac{1}{\hbar \psi^2(x)} \int_{\infty_0}^x \psi^2(x') (\delta U(x') - 2 \frac{\delta \hbar}{\hbar} U(x')) dx'.$$

(6.9)

If we write:

$$\delta U = \frac{V'}{2} \delta V' - \frac{\hbar}{2} \delta V'' - \frac{\delta \hbar}{2} V'' - \delta P \quad (6.10)$$

where  $\delta P$  is of degree at most  $d - 1$ , and  $V'/2 = \omega - \hbar \psi'/\psi$ , we have by integration by parts:

$$\begin{aligned} \delta \omega(x) = & \frac{1}{\hbar \psi^2(x)} \int_{\infty_0}^x \psi^2(x') \left( \omega(x') \delta V'(x') - \delta P(x') \right. \\ & \left. - \delta \hbar (\omega'(x') - \frac{1}{2} V''(x')) \right) dx'. \end{aligned} \quad (6.11)$$



## 6.2 Decomposition of variations

$U(x)$  is a polynomial of degree  $2d$ , it has  $2d + 1$  independent coefficients. If we assume that we have a solution of genus  $\mathfrak{g} < d - 1$ , this means that  $U$  is non generic, and satisfies  $d - 1 - \mathfrak{g}$  constraints. In the space of all possible  $U$ 's, we shall consider the submanifold corresponding to  $U$  of genus  $\mathfrak{g}$ , which is a submanifold of dimension

$$\dim = d + 2 + \mathfrak{g} \quad (6.12)$$

and we shall consider variations of  $U$  within that submanifold. Variations transverse to the genus  $\mathfrak{g}$  submanifold, are variations of higher genus and should be computed within a higher genus submanifold.

Instead of the  $d + 2 + \mathfrak{g}$  independent coefficients of the polynomial  $U$ , it is more convenient to choose a system of "flat" coordinates in our genus  $\mathfrak{g}$  submanifold, given by:

$$t_0, t_1, \dots, t_{d+1}, \epsilon_1, \dots, \epsilon_{\mathfrak{g}}. \quad (6.13)$$

We have indeed  $d + 2 + \mathfrak{g}$  coordinates.

Let us write the variations as:

$$\delta U = \sum_{k=0}^{d+1} U_{t_k} \delta t_k + \sum_{i=1}^{\mathfrak{g}} U_{\epsilon_i} \delta \epsilon_i + U_{\hbar} \delta \hbar. \quad (6.14)$$

### 6.2.1 Variations relatively to the filling fractions

For the filling fraction  $\delta \epsilon_{\alpha}$  we have  $\delta V' = 0$  and thus:

$$\delta U(x) = -\delta P(x) \quad (6.15)$$

where  $\deg \delta P \leq d - 2$ , so we decompose it on the basis of  $h_{\alpha}$ 's:

$$\delta P(x) = \sum_{\alpha'} c_{\alpha'} h_{\alpha'}. \quad (6.16)$$

and therefore, from eq. (6.9):

$$\delta \omega(x) = - \sum_{\alpha'} c_{\alpha'} v_{\alpha'}(x). \quad (6.17)$$

Since  $2i\pi \epsilon_{\alpha'} = \oint_{\mathcal{A}_{\alpha'}} \omega$ , we have:

$$2i\pi \delta_{\alpha, \alpha'} = \oint_{\mathcal{A}_{\alpha'}} \delta \omega = - \sum_{\alpha''} \oint_{\mathcal{A}_{\alpha'}} c_{\alpha''} v_{\alpha''} = -c_{\alpha'} \quad (6.18)$$

This implies:

$$U_{\epsilon_\alpha}(x) = 2i\pi h_\alpha(x) \quad (6.19)$$

and

$$\delta_{\epsilon_i}\omega(x) = 2i\pi v_\alpha(x) = \oint_{\mathcal{B}_\alpha} B(x, z) dz. \quad (6.20)$$

We shall say that the flat coordinate  $\epsilon_\alpha$  is dual to the holomorphic form  $v_\alpha$ , which is itself dual to the cycle  $\mathcal{B}_\alpha$ :

$$\epsilon_\alpha = \frac{1}{2i\pi} \oint_{\mathcal{A}_\alpha} \omega, \quad \delta_{\epsilon_\alpha}\omega = 2i\pi v_\alpha = \oint_{\mathcal{B}_\alpha} B. \quad (6.21)$$

### 6.2.2 Variations relatively to $t_0$

We have:

$$\delta U(x) = -\delta P(x) = -t_{d+1} x^{d-1} + Q(x) \quad (6.22)$$

where  $\deg Q \leq d-2$ . Using eq. (6.9) we get:

$$\delta\omega(x) = \frac{1}{\psi^2(x)} \int_{\infty_0}^x (-t_{d+1} x'^{d-1} + Q(x')) \psi^2(x') dx' \quad (6.23)$$

and the polynomial  $Q$  is chosen such that  $\oint_{\mathcal{A}_i} \delta\omega = 0$  so that when decomposing  $Q(x)$  on the basis  $v_\alpha(x)$  and performing integrals over  $\mathcal{A}$ -cycles one finds the coefficients of the decomposition as integrals. Therefore we have:

$$\begin{aligned} \delta\omega(x) &= -t_{d+1} K_{d-1}(x) \\ &= -t_{d+1} \left( \hat{K}_{d-1}(x) - \sum_{\alpha=1}^g v_\alpha(x) \oint_{\mathcal{A}_\alpha} \hat{K}_{d-1}(x') dx' - \sum_{\alpha=g+1}^{d-1} v_\alpha(x) \oint_{\mathcal{A}_\alpha} \psi^2(x') x'^{d-1} dx' \right) \end{aligned} \quad (6.24)$$

where

$$\hat{K}_k(x) = \frac{1}{\psi^2(x)} \int_{\infty_0}^x x'^k \psi^2(x') dx', \quad (6.25)$$

and  $K_k(x)$  is the  $k^{\text{th}}$  term in the large  $z$  expansion of  $K(x, z) = -\sum_{k=0}^{\infty} \frac{K_k(x, z)}{z^{k+1}}$  computed in theorem 4.2. From theorem 4.5 we have  $G(x, \infty_i) = \eta_i t_{d+1} K_{d-1}(x)$ . This shows that

$$\delta_{t_0}\omega(x) = G(x, \infty_0) = \frac{1}{2} (G(x, \infty_0) - G(x, \infty_-)) = \int_{\infty_0}^{\infty_-} B(x, z) dz \quad (6.26)$$

where  $\infty_0$  is in the physical sheet, and  $\infty_-$  is any infinity chosen in the second sheet.

We shall say that the flat coordinate  $t_0$  is dual to the 3rd kind meromorphic form  $-2G(x, \infty_0)$ , which is itself dual to the chain  $[\infty_0, \infty_-]$ :

$$\boxed{t_0 = \operatorname{Res}_{\infty_0} \omega \quad , \quad \delta_{t_0} \omega = -2G(x, \infty_0) = \int_{\infty_0}^{\infty_-} B(x, z) dz} \quad (6.27)$$

where Res means the coefficient of  $1/z$  in the given sector.

### 6.2.3 Variation relatively to $t_k, k = 1 \dots d$

For  $k = 1, \dots, d$  we have:

$$U_{t_k}(x) = \frac{V'(x)}{2} x^{k-1} - Q(x) \quad , \quad \deg Q \leq d-2 \quad (6.28)$$

and  $Q$  is chosen such that  $\oint_{\mathcal{A}_i} \delta \omega = 0$ . Using eq. (6.9) we write:

$$\delta \omega(x) = \delta \hat{\omega}(x) - \sum_{\alpha} v_{\alpha}(x) \oint_{\mathcal{A}_{\alpha}} \delta \hat{\omega}(x') dx' \quad (6.29)$$

where

$$\delta \hat{\omega}(x) = \frac{1}{\psi^2(x)} \int_{\infty_0}^x \frac{V'(x')}{2} x'^{k-1} \psi^2(x') dx' \quad (6.30)$$

Since  $V'(x') = \sum_j t_{j+1} x'^j$ , we have:

$$2\delta \omega(x) = \sum_{j=0}^d t_{j+1} K_{k+j-1} \quad (6.31)$$

Let us compare it with the large  $z$  behaviour of  $G(x, z)$  in the physical sheet. We have:

$$G(x, z) = V'(z) K(x, z) + O(z^{-d-1}) \quad (6.32)$$

which means that the large  $z$  expansion of  $G(x, z) = \sum_k G_k(x) z^{-k}$  is given for  $k = 1, \dots, d$  by:

$$G_k(x) = - \sum_{j=0}^d t_{j+1} K_{k+j-1} \quad (6.33)$$

and therefore

$$\delta \omega(x) = -\frac{1}{2} G_k(x) \quad (6.34)$$

If we write the large  $z$  expansion of  $B(x, z)$  in the physical sheet, we have

$$B(x, z) = \sum_k B_k(x) z^{-k-1} = -\frac{1}{2} \sum_k k G_k(x, z) z^{-k-1} \quad (6.35)$$

and thus

$$\boxed{\delta_{t_k} \omega(x) = \frac{1}{k} B_k(x) = \operatorname{Res}_{\infty_0} \frac{z^k}{k} B(x, z) dz} \quad (6.36)$$

We shall say that the flat coordinate  $t_k$  is dual to the 2nd kind meromorphic form  $\frac{1}{k} B_k(x)$ , which is itself dual to a residue of  $B$ .

#### 6.2.4 Variations relatively to $t_{d+1}$

When  $k = d + 1$ , we have a few additional terms of degree  $> d - 2$ :

$$U_{t_{d+1}}(x) = \frac{V'(x)}{2} x^d - \frac{d\hbar}{2} x^{d-1} - t_0 x^{d-1} - Q(x) \quad , \quad \deg Q \leq d - 2 \quad (6.37)$$

and  $Q$  is chosen such that  $\oint_{\mathcal{A}_i} \delta\omega = 0$ . Using eq. (6.9) we write:

$$\delta\omega(x) = \delta\hat{\omega}(x) - \sum_{\alpha} v_{\alpha}(x) \oint_{\mathcal{A}_{\alpha}} \delta\hat{\omega}(x') dx' \quad (6.38)$$

where

$$\delta\hat{\omega}(x) = \frac{1}{\psi^2(x)} \int_{\infty_0}^x \left( \frac{V'(x')}{2} x'^d - \frac{d\hbar}{2} x'^{d-1} - t_0 x'^{d-1} \right) \psi^2(x') dx' \quad (6.39)$$

In other words we have:

$$2\delta\omega(x) = \sum_{j=0}^d t_{j+1} K_{d+j} - d\hbar K_{d-1} - 2t_0 K_{d-1} \quad (6.40)$$

Let us compare it with the large  $z$  behaviour of  $G(x, z)$ . We have:

$$G(x, z) = \left( V'(z) - \frac{2t_0}{z} \right) K(x, z) - \hbar \partial_z K(x, z) + O(z^{-d-2}) \quad (6.41)$$

which means that the large  $z$  expansion of  $G(x, z) = \sum_k G_k(x) z^{-k}$  is given for  $k = d+1$  by:

$$G_{d+1}(x) = - \sum_{j=0}^d t_{j+1} K_{d+j} + \hbar d K_{d-1} + 2t_0 K_{d-1} \quad (6.42)$$

and therefore

$$\delta\omega(x) = -\frac{1}{2} G_{d+1}(x) \quad (6.43)$$

If we write the large  $z$  expansion of  $B(x, z)$ , we have

$$B(x, z) = \sum_k B_k(x) z^{-k-1} = -\frac{k}{2} \sum_k G_k(x, z) z^{-k-1} \quad (6.44)$$

and thus

$$\delta_{t_{d+1}}\omega(x) = \frac{1}{d+1} B_{d+1}(x) = \operatorname{Res}_{\infty_0} \frac{z^d}{d} B(x, z) dz \quad (6.45)$$

We shall say that the flat coordinate  $t_{d+1}$  is dual to the 2nd kind meromorphic form  $\frac{1}{d+1} B_{d+1}(x)$ , which is itself dual to a residue of  $B$ .

### 6.3 Variation relatively to $\hbar$

We have:

$$\delta_{\hbar}\omega(x) = -\frac{1}{\hbar\psi^2(x)} \int_{\infty_0}^x \psi^2(x') \left( \omega'(x') - \frac{1}{2} V''(x') - \delta_{\hbar}P(x') \right) dx' \quad (6.46)$$

where  $\delta_{\hbar}P$  is a polynomial of degree  $\leq d-2$  chosen so that  $\oint_{\mathcal{A}_i} \delta\omega = 0$ . For the moment, we have not found a good way of writing this expression as an integral with  $B$ , and we leave that question for a future work.

### 6.4 Form-cycle duality

Notice that in all cases, except  $\delta_{\hbar}$ , there exist a cycle  $\delta\omega^*$  and a function  $\Lambda_{\delta\omega}^*$  such that:

$$\delta\omega(x) = \int_{\delta\omega^*} B(x, z) \Lambda_{\delta\omega}^*(z) dz. \quad (6.47)$$

We will use this generic notation later on in order to avoid specifying the 3 different cases.

Under a suitable reparametrization  $z \rightarrow z'$  such that  $dz' = \Lambda_{\delta\omega}^*(z) dz$ , we say that  $\delta\omega^*$  in the variable  $z'$  is the cycle dual to the "meromorphic form"  $\delta\omega$ .

### 6.5 Variation of higher correlators

The following theorem allows to compute the infinitesimal variation of any  $W_n^{(g)}$  under a variation of the Schrödinger equation. It tells about the "complex structure deformation" of our quantum Riemann surface. It can be regarded as special geometry relations.

**Theorem 6.1** *Under an infinitesimal deformation  $U \rightarrow U + \delta U$ , we have:*

$$\delta W_n^{(g)}(x_1, \dots, x_n) = \int_{\delta\omega^*} W_{n+1}^{(g)}(x_1, \dots, x_n, x') \Lambda^*(x') dx' \quad (6.48)$$

where  $(\delta\omega^*, \Lambda_{\delta\omega}^*)$  is the dual cycle to the deformation of the resolvent  $\omega \rightarrow \omega + \delta\omega$ .

**proof:**

The loop equation for  $W_n^{(g)}(x, J)$  is:

$$(2\omega(x) - V'(x) + \hbar\partial_x) W_n^{(g)}(x, J) + U_n^{(g)}(x, x; J) = P_n^{(g)}(x, J) \quad (6.49)$$

taking a variation  $\delta$  we have:

$$\begin{aligned} & (2\omega(x) - V'(x) + \hbar\partial_x) \delta W_n^{(g)}(x, J) + (2\delta\omega(x) - \delta V'(x)) W_n^{(g)}(x, J) + \delta U_n^{(g)}(x, x; J) \\ & = \delta P_n^{(g)}(x, J) \end{aligned} \quad (6.50)$$

notice that  $\delta P_n^{(g)}(x, J)$  is a polynomial in  $x$ , of degree at most  $d - 2$ .

On the other hand, consider the loop equation for  $W_{n+1}^{(g)}(x, J, x')$  and multiply it by  $\Lambda^*(x')$  and integrate  $x'$  along  $\omega^*$ , one gets:

$$\begin{aligned} & (2\omega(x) - V'(x) + \hbar\partial_x) \int_{\omega^*} W_{n+1}^{(g)}(x, J, x') \Lambda^*(x') dx' + \int_{\omega^*} \delta U_{n+1}^{(g)}(x, x; J, x') \Lambda^*(x') dx' \\ & = \int_{\omega^*} P_{n+1}^{(g)}(x, J, x') \Lambda^*(x') dx' \end{aligned} \quad (6.51)$$

That gives by recursion hypothesis for the computation of  $\int_{\omega^*} \delta U_{n+1}^{(g)}(x, x; J, x') \Lambda^*(x') dx'$  and using 6.9:

$$\begin{aligned} & (2\omega(x) - V'(x) + \hbar\partial_x) \left( \int_{\omega^*} W_{n+1}^{(g)}(x, J, x') \Lambda^*(x') dx' - \delta W_n^{(g)}(x, J) \right) \\ & = \delta P_n^{(g)}(x, J) - \int_{\omega^*} P_{n+1}^{(g)}(x, J, x') \Lambda^*(x') dx' \\ & = \sum_i \alpha_i(J) h_i(x) \end{aligned} \quad (6.52)$$

where the right hand side is a polynomial of degree at most  $d - 2$  in  $x$ , which can be decomposed on the basis  $h_i(x)$ .

Solving the differential equation gives:

$$\int_{\omega^*} W_{n+1}^{(g)}(x, J, x') \Lambda^*(x') dx' - \delta W_n^{(g)}(x, J) = \sum_i \alpha_i(J) v_i(x) \quad (6.53)$$

but since  $W_n^{(g)}$  and  $W_{n+1}^{(g)}$  are normalized on  $\mathcal{A}$ -cycles, this implies  $\alpha_i = 0$ , i.e.:

$$\int_{\omega^*} W_{n+1}^{(g)}(x, J, x') \Lambda^*(x') dx' = \delta W_n^{(g)}(x, J) \quad (6.54)$$

□

## 7 Free energies

We use the variations and theorem 5.7 to define the  $F_g$ 's.

Theorem 5.7 gives:

$$(2 - 2g - n - \hbar \partial_{\hbar}) W_n^{(g)} = \left( t_0 \partial_{t_0} + \sum_{k=1}^{d+1} t_k \partial_{t_k} + \sum_{i=1}^{\mathfrak{g}} \epsilon_i \partial_{\epsilon_i} \right) W_n^{(g)} \quad (7.1)$$

And in the previous section, we have seen how to write the derivatives of  $W_n^{(g)}$  as integrals of  $W_{n+1}^{(g)}$ , that gives:

$$(2 - 2g - n - \hbar \partial_{\hbar}) W_n^{(g)} = \hat{H}. W_{n+1}^{(g)} \quad (7.2)$$

where  $\hat{H}$  is the linear operator acting as follows:

$$\hat{H}.f(x) = t_0 \int_{\infty_0}^{\infty^-} f + \sum_{j=1}^{d+1} \text{Res}_{\infty_0} \frac{t_j x^j}{j} f + \sum_{i=1}^{\mathfrak{g}} \epsilon_i \oint_{\mathcal{B}_i} f. \quad (7.3)$$

Those equations allow to define  $W_0^{(g)} = F_g$  for  $n = 0$  and  $g \geq 2$  as:

**Definition 7.1** *We define  $F_g$  for  $g \geq 2$  such that:*

$$(2 - 2g - \hbar \partial_{\hbar}) F_g = \hat{H}. W_1^{(g)} \quad (7.4)$$

It would remain to find the correct definitions of  $F_0$  (called the prepotential) and  $F_1$ .  $F_0$  and  $F_1$  should be such that under every deformation  $\delta = \partial_{t_k}, \partial_{t_0}, \partial_{\epsilon_i}$  we should have

$$\delta F_g = H_{\delta} W_1^{(g)}. \quad (7.5)$$

For example  $\partial F_0 / \partial t_k = \text{Res } x^k \omega(x) / k$  i.e. the coefficient of the term  $1/x^{k-1}$  in the expansion of  $\omega(x)$  near  $\infty_0$ .

We leave the definitions of  $F_0$  and  $F_1$  for a future work.

## 8 Classical and quantum geometry: summary

Let us summarize the comparison between classical algebraic geometry, and its quantum counterpart introduced here.

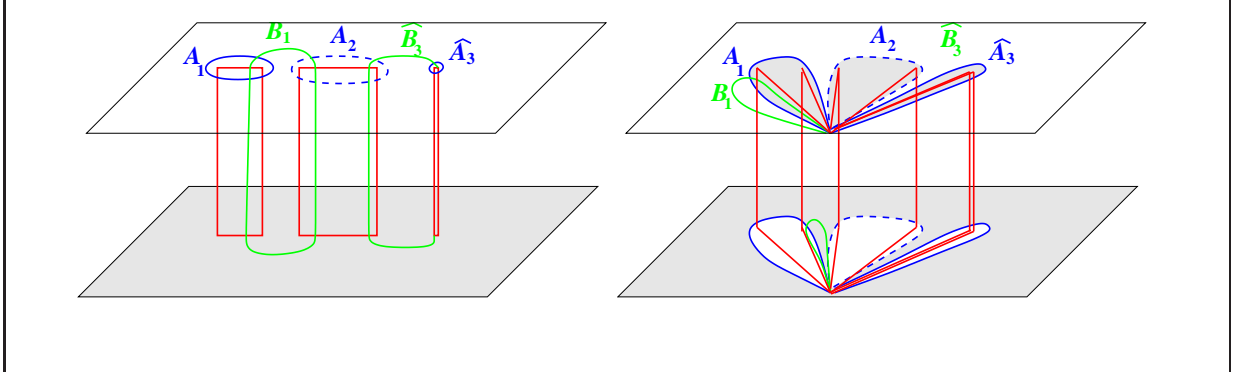


Figure 5: Classical case  $\hbar = 0$  of a two sheeted Riemann surface. The branchpoints are paired (in an arbitrary way) to form cuts, and the two sheets are glued along the cuts. Another possibility, is to draw a cut from each branchpoint to  $\infty$ . The  $\mathcal{A}$ -cycles surround pairs of branchpoints in the physical sheet. There are also some degenerate branchpoints, which correspond to cuts of vanishing length.

<b>Summary</b>		
	<b>classical</b> $\hbar = 0$	<b>quantum</b>
plane curve:	$E(x, y) = \sum_{i,j} E_{i,j} x^i y^j$ $E(x, y) = 0$	$E(x, y) = \sum_{i,j} E_{i,j} x^i y^j$ , $[y, x] = \hbar$ $E(x, \hbar \partial_x) \psi = 0$
hyperelliptical plane curve:	$y^2 = U(x)$ $\deg U = 2d$	$y^2 - U(x)$ , $[y, x] = \hbar$ , $\hbar^2 \psi'' = U \psi$
Potential:		$V'(x) = 2(\sqrt{U(x)})_+$
2 sheets:	$y \sim_{\infty} \pm \frac{1}{2} V'(x)$	$\hbar \psi' / \psi \sim_{\infty} \pm \frac{1}{2} V'(x) \sim_{\infty} \frac{\eta_k}{2} V'(x)$ choice $\psi = \psi_0 \searrow$ in sector $\infty_0$ , $\eta_0 = -1$
resolvent:	$\omega(x) = V'(x)/2 + y$ .	$\omega(x) = V'(x)/2 + \hbar \frac{\psi'}{\psi}$ .
physical sheet:	$y \sim_{\infty} -\frac{1}{2} V'(x)$ , $\omega \sim t_0/x$	$\hbar \psi' / \psi \sim_{\infty} -\frac{1}{2} V'(x)$ , $\omega \sim t_0/x$ sectors where $\psi \sim e^{-\frac{V}{2\hbar}}$
branchpoints:	simple zeroes of $U(x)$ $U(a_i) = 0$ , $U'(a_i) \neq 0$ $i = 1, \dots, 2\mathbf{g} + 2$	half-lines of accumulations of zeroes of $\psi$ $L_i$ , $i = 1, \dots, 2\mathbf{g} + 2$
genus $\mathbf{g}$ :		$2\mathbf{g} + 2 = \#$ branch points $-1 \leq \mathbf{g} \leq d - 1$
double points:	double zeroes of $U(x)$ $U(\hat{a}_i) = 0$ , $U'(\hat{a}_i) = 0$	half-lines without accumulations of zeroes of $\psi$
genus $\mathbf{g} = -1$	degenerate surface	$\psi e^{V/2\hbar} = \text{polynomial}$
$\mathcal{A}_\alpha$ -cycles $\alpha = 1, \dots, \mathbf{g}$	surround pairs of branchpoints	surround pairs of half-lines of accumulating zeroes
false $\hat{\mathcal{A}}_\alpha$ -cycles $\mathbf{g} < \alpha < d$	surround double points	links 2 sectors where $\psi \searrow$
extra $\mathcal{A}_d$ -cycle $\alpha = d$	surrounds last pair of branchpoints	surrounds last pair of half-lines of accumulating zeroes
$\mathcal{B}$ -cycles		$\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}$
Holomorphic forms, 1st kind differentials	$v_i(x) = \frac{-h_i(x)}{2\sqrt{U(x)}}$ $h_i(\hat{a}_\alpha) = -\frac{1}{2} \delta_{\alpha,i} \sqrt{U''(\hat{a}_\alpha)}$	$v_i(x) = \frac{1}{\hbar \psi^2(x)} \int_{\infty_0}^x \psi^2(x') h_i(x') dx'$ $h_i = \text{polynomials, } \deg h_i \leq d - 2$ normalized: $\oint_{\mathcal{A}_\alpha} v_i(x) dx = \delta_{\alpha,i}$ , $\alpha = 1, \dots, \mathbf{g}$ $\oint_{\hat{\mathcal{A}}_\alpha} \psi^2 h_i = \delta_{\alpha,i}$ , $\alpha = \mathbf{g} + 1, \dots, d - 1$
Period matrix		$\tau_{i,j} = \oint_{\mathcal{B}_j} v_i$ , $i, j = 1, \dots, \mathbf{g}$ , $\tau_{j,i} = \tau_{j,i}$



<b>Summary</b>	
	classical $\hbar = 0$   quantum
3rd kind form	$G(x, z) \sim_{x \rightarrow z} 1/(z - x)$ $G(x, z) = (2\omega(z) - V'(z) - \hbar\partial_z)K(x, z)$
Recursion kernel	$K(x, z)$ $K(x, z) = \hat{K}(x, z) - \sum_{\alpha} v_{\alpha}(x) C_{\alpha}(z)$ $C_{\alpha}(z) = \oint_{\mathcal{A}_{\alpha}} \hat{K}(x', z) dx'$ $\hat{K}(x, z) = \frac{1}{z-x} \frac{1}{2\sqrt{U(x)}} \quad   \quad \hat{K}(x, z) = \frac{1}{\hbar\psi^2(x)} \int_{\infty_0}^x \frac{\psi^2(x') dx'}{x' - z}$
Bergman kernel 2nd kind	$B(x, z) = -\frac{1}{2} \partial_z G(x, z)$ $B(x, z) \sim 1/2(x - z)^2$
Symmetry:	$B(x, z) = B(z, x)$
	$\oint_{\mathcal{A}_{\alpha}} B(x, z) dx = 0$ $\oint_{\mathcal{B}_{\alpha}} B(x, z) dx = 2i\pi v_{\alpha}(z)$
Meromorphic forms	$\mathcal{R}(x) dx = \frac{r(x) dx}{2\sqrt{U(x)}} \quad   \quad \mathcal{R}(x) = \frac{1}{\hbar\psi^2(x)} \int_{\infty_0}^x r(x') \psi^2(x') dx'$ $r(x) = \text{rational with poles } r_i, r(x) = O(x^{d-2})$ $  \quad \text{Res}_{r_i} r(x') \psi^2(x') = 0$
Higher correlators	$W_{n+1}^{(g)}(x, J) = \sum_i \frac{1}{2i\pi} \oint_{\mathcal{C}_i} K(x, z) dz \left( W_{n+2}^{(g-1)}(z, z, J) \right.$ $\left. + \sum'_{h+h'=g, I \sqcup I' = J} W_{1+ I }^{(h)}(z, I) W_{1+ I' }^{(h')}(z, I') \right)$ where $\mathcal{C}_i$ surrounds the branchpoint $L_i$
Symmetry	$W_n^{(g)}(x_1, x_2, \dots, x_n) = W_n^{(g)}(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}), \quad \sigma \in S_n$
Variations and dual cycle	$U(x) \rightarrow U(x) + \delta U(x)$ $\delta U^*: \delta\omega(x) = \int_{\delta U^*} B(x, x') \Lambda_{\delta U}(x') dx'$
$\delta V' = \sum \delta t_k x^{k-1}$	$\delta_{t_k} \omega(x) = \text{Res}_{\infty} B(x, x') \frac{x'^k}{k} dx'$
variation $\delta t_0$	$\delta_{t_0} \omega(x) = \int_{\infty_0}^{\infty^-} B(x, x') dx'$
variation $\delta \epsilon_i$	$\delta_{\epsilon_i} \omega(x) = \oint_{\mathcal{B}_i} B(x, x') dx'$
Variations of higher correlators	$\delta W_n^{(g)}(x_1, \dots, x_n) = \int_{\delta U^*} W_{n+1}^{(g)}(x_1, \dots, x_n, x') \Lambda_{\delta U}(x') dx'$
Rauch formula	$W_3(x_1, x_2, x_3) = \oint_{\mathcal{C}} \frac{B(x_1, z) B(x_2, z) B(x_3, z)}{4Y'(z)} dz$

## 9 Application: Matrix models

The reason why we introduced those  $W_n^{(g)}$ 's is because they satisfy the loop equations for  $\beta$ -random matrix ensembles.

Consider a (possibly formal) matrix integral:

$$Z = \int_{E_{N,\beta}} dM e^{-\frac{N\sqrt{\beta}}{t_0} \text{tr} V(M)} \quad (9.1)$$

where  $V(x)$  is some polynomial, and where  $E_{N,1} = H_N$  is the set of hermitian matrices

of size  $N$ ,  $E_{N,1/2}$  is the set of real symmetric matrices of size  $N$  and  $E_{N,2}$  is the set of quaternion self dual matrices of size  $N$  (see [21]).

Alternatively, we can integrate over the angular part and get an integral over eigenvalues only [21]:

$$Z = \int d\lambda_1 \dots d\lambda_N \Delta(\lambda)^{2\beta} \prod_{i=1}^N e^{-\frac{N\sqrt{\beta}}{t_0} V(\lambda_i)} \quad (9.2)$$

where  $\Delta(\lambda) = \prod_{i<j}(\lambda_j - \lambda_i)$  is the Vandermonde determinant.

This allows to generalize the matrix model to arbitrary values of  $\beta$ . In particular, we shall choose  $\beta$  of the form:

$$\sqrt{\beta} = \frac{\hbar N}{2t_0} \left( 1 \pm \sqrt{1 + \frac{4t_0^2}{\hbar^2 N^2}} \right) \quad (9.3)$$

i.e.

$$\hbar = \frac{t_0}{N} \left( \sqrt{\beta} - \frac{1}{\sqrt{\beta}} \right). \quad (9.4)$$

Notice that  $\hbar = 0$  correspond to the hermitian case  $\beta = 1$ , and  $\hbar \rightarrow -\hbar$  corresponds to  $\beta \rightarrow 1/\beta$ .

## 9.1 Correlators and loop equations

Then we define the correlators:

$$W_k(x_1, \dots, x_k) = \beta^{k/2} \left\langle \sum_{i_1, \dots, i_k} \frac{1}{x_1 - \lambda_{i_1}} \dots \frac{1}{x_k - \lambda_{i_k}} \right\rangle_c \quad (9.5)$$

and

$$W_0 = F = \ln Z. \quad (9.6)$$

And we assume (this is automatically true if we are considering formal matrix integrals), that there is a large  $N$  expansion of the type (where we assume  $\hbar = O(1)$ ):

$$W_k(x_1, \dots, x_k) = \sum_{g=0}^{\infty} (N/t_0)^{2-2g-k} W_k^{(g)}(x_1, \dots, x_k) \quad (9.7)$$

$$W_0 = F = \sum_g (N/t_0)^{2-2g} W_0^{(g)} = \sum_g (N/t_0)^{2-2g} F_g. \quad (9.8)$$

The loop equations are obtained by integration by parts, for example:

$$0 = \sum_i \int d\lambda_1 \dots d\lambda_N \frac{\partial}{\partial \lambda_i} \left( \frac{1}{x - \lambda_i} \Delta(\lambda)^{2\beta} \prod_j e^{-\frac{N\sqrt{\beta}}{t_0} V(\lambda_j)} \right) \quad (9.9)$$

gives:

$$\begin{aligned}
0 &= \sum_i \left\langle \frac{1}{(x - \lambda_i)^2} + 2\beta \sum_{j \neq i} \frac{1}{x - \lambda_i} \frac{1}{\lambda_i - \lambda_j} - \frac{N\sqrt{\beta}}{t_0} \frac{V'(\lambda_i)}{x - \lambda_i} \right\rangle \\
&= \sum_i \left\langle \frac{1}{(x - \lambda_i)^2} + \beta \sum_{j \neq i} \frac{1}{x - \lambda_i} \frac{1}{x - \lambda_j} - \frac{N\sqrt{\beta}}{t_0} \frac{V'(\lambda_i)}{x - \lambda_i} \right\rangle \\
&= \sum_i \left\langle \frac{1 - \beta}{(x - \lambda_i)^2} + \beta \sum_j \frac{1}{x - \lambda_i} \frac{1}{x - \lambda_j} - \frac{N\sqrt{\beta}}{t_0} \frac{V'(\lambda_i)}{x - \lambda_i} \right\rangle \\
&= (\beta - 1) \frac{1}{\sqrt{\beta}} W_1'(x) + \beta \left( \frac{1}{\beta} W_1^2(x) + \frac{1}{\beta} W_2(x, x) \right) \\
&\quad - \frac{N\sqrt{\beta}}{t_0} \left( \frac{1}{\sqrt{\beta}} V'(x) W_1(x) - \sum_i \left\langle \frac{V'(x) - V'(\lambda_i)}{x - \lambda_i} \right\rangle \right)
\end{aligned} \tag{9.10}$$

We define the polynomial

$$P_1(x) = \sqrt{\beta} \sum_i \left\langle \frac{V'(x) - V'(\lambda_i)}{x - \lambda_i} \right\rangle = (V' W_1)_+. \tag{9.11}$$

We thus have:

$$W_1^2(x) + \hbar \frac{N}{t_0} W_1'(x) + W_2(x, x) = \frac{N}{t_0} (V'(x) W_1(x) - P_1(x)) \tag{9.12}$$

Using the expansion eq. (9.7), that gives the Ricatti equation

$$W_1^{(0)}(x)^2 + \hbar \partial_x W_1^{(0)}(x) = V'(x) W_1^{(0)}(x) - P_1^{(0)}(x) \tag{9.13}$$

which is satisfied by  $\omega(x)$ :

$$W_1^{(0)}(x) = \omega(x). \tag{9.14}$$

generalizing to the integration by parts of

$$0 = \sum_i \int d\lambda_1 \dots d\lambda_N \frac{\partial}{\partial \lambda_i} \left( \frac{1}{x - \lambda_i} \sum_{i_1, \dots, i_k} \prod_{j=1}^k \frac{1}{x_j - \lambda_{i_j}} \Delta(\lambda)^{2\beta} \prod_j e^{-\frac{N\sqrt{\beta}}{t_0} V(\lambda_j)} \right) \tag{9.15}$$

and using the expansion eq. (9.7) to higher orders in  $t_0/N$ , one gets the loop equations of theorem 5.4, where

$$\begin{aligned}
P_{k+1}(x; x_1, \dots, x_k) &= \sum_g (N/t_0)^{2-2g-k} P_{k+1}^{(g)}(x; x_1, \dots, x_k) \\
&= \beta^{k/2} \left\langle \sum_i \frac{V'(x) - V'(\lambda_i)}{x - \lambda_i} \sum_{i_1, \dots, i_k} \prod_{j=1}^k \frac{1}{x_j - \lambda_{i_j}} \right\rangle_c.
\end{aligned} \tag{9.16}$$

In other words, **the correlation functions of  $\beta$  matrix models, obey the topological recursion of def. 5.1.**

**Remark:**

In [15], a solution of loop equations for the  $\beta$ -matrix ensemble was proposed, but that solution was such that  $U(x)$  was non-generic, corresponding to  $\mathfrak{g} = -1$ , and that  $\psi(x)$  had only a finite number of zeroes. This case implied that  $t_0$  was quantized. Generic matrix models cannot correspond to that situation.

That solution was thus not very useful for actual matrix models. Here instead, we have the solution for every  $U(x)$ , i.e. every contour of integration for the  $\lambda_i$ 's, and therefore we have the solution of loop equations for the actual matrix model.

## 9.2 Example: real eigenvalues

Very often, we are interested in a matrix model with real potential  $V(x)$  of even degree (i.e.  $d$  is odd) and such that the eigenvalues are integrated along the real axis. The resolvent  $\omega(x)$  is the Stieljes transform of the density of eigenvalues:

$$\omega(x) = \frac{t_0 \sqrt{\beta}}{N} \int_{\mathbb{R}} \frac{\rho(x') dx'}{x - x'} \quad , \quad \rho(x) = \left\langle \sum_i \delta(x - \lambda_i) \right\rangle \quad (9.17)$$

Let us consider that it is defined by this integral in the upper half-plane for  $x \in \mathbb{H}_+$ , and it is extended to the lower half-plane by analytical continuation.

By definition,  $\omega(x)$  is regular in the upper half-plane, therefore we look for a  $\psi(x)$  which has no zero in the upper half-plane, i.e. no zero on the half-lines  $L_0, L_1, \dots, L_{d-1}$ . I.e. it has at most  $d + 1$  half-lines of zeroes, and thus:

$$\mathfrak{g} \leq \frac{d - 1}{2}.$$

## 10 Non-oriented Ribbon graphs

Consider the set of all closed connected ribbon graphs obtained by gluing the pieces represented in fig. 6. Closed means every half-edge is glued to another half-edge. Connected means every vertex is connected to any other vertex. See for example fig. 7.

We define the genus of such a ribbon graph  $\mathcal{G}$  as follows. We replace every twisted edge of  $\mathcal{G}$  by a non-twisted one, we thus obtain another ribbon graph  $\mathcal{G}'$ , which is oriented. We define the genus of  $\mathcal{G}$  equal to that of  $\mathcal{G}'$ :

$$g(\mathcal{G}) = g(\mathcal{G}').$$

The genus of  $\mathcal{G}'$  is computed as usual for oriented ribbon graphs, from the Euler characteristics of  $\mathcal{G}'$ :

$$\chi(\mathcal{G}') = 2 - 2g = \#\{\text{vertices}(\mathcal{G})\} - \#\{\text{edges}(\mathcal{G})\} + \#\{\text{single lines}(\mathcal{G}')\}$$

where single lines are the lines bordering each side of the ribbon edges. One should follow single lines and see how many connected single lines a graph contains. Obviously  $\mathcal{G}$  and  $\mathcal{G}'$  have the same number of fat vertices and fat edges (each edge containing two single lines), but they may have different number of single lines.

This defines what we call the genus  $g$  of a ribbon graph.

For a given Ribbon graph  $\mathcal{G}$  we call:

- $n_i(\mathcal{G}) = \#\text{unmarked vertices of degree } i, \text{ for } 3 \leq i \leq d + 1,$
- $l_i(\mathcal{G}) = \text{size of the } i^{\text{th}} \text{ marked vertex, we have } l_i(\mathcal{G}) \geq 1.$
- $e(\mathcal{G}) = \#\text{edges},$
- $q(\mathcal{G}) = \#\text{twisted edges},$
- $v(\mathcal{G}) = \#\text{connected single lines},$
- $g(\mathcal{G}) = \text{genus},$
- $\#Aut(\mathcal{G}) = \text{symmetry factor of } \mathcal{G}.$

**Definition 10.1** Let  $\mathbb{M}_k^{(g)}(v')$ , be the set of ribbon graphs  $\mathcal{G}$  with  $k$  marked vertices,  $q$  twisted edges, and of genus  $g$ , and such that  $\mathcal{G}'$  has  $v(\mathcal{G}') = v'$  connected single-lines, .

**Proposition 10.1**  $\mathbb{M}_k^{(g)}(v')$  is a finite set.

**proof:**

The number of vertices of  $\mathcal{G}'$  is:

$$\#\{\text{vertices}\} = k + \sum_{i \geq 3} n_i$$

The number of edges is twice the number of half-edges, i.e.

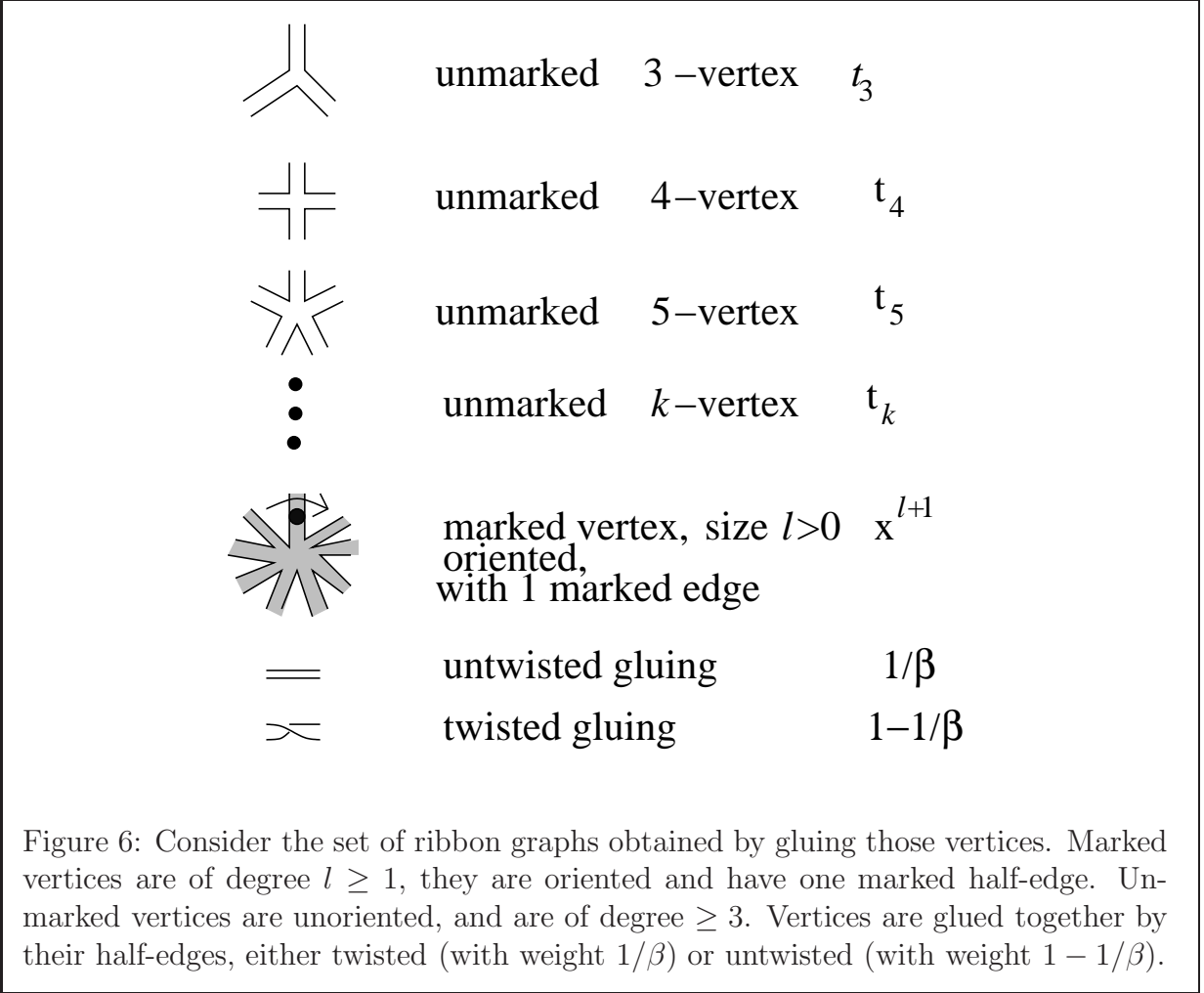
$$2 \#\{\text{edges}\} = \sum_{i \geq 3} i n_i + \sum_{i=1}^k l_i$$

That gives:

$$2 - 2g = \#\{\text{vertices}\} - \#\{\text{edges}\} + v' = k - \frac{1}{2} \sum_{i \geq 3} (i - 2)n_i - \frac{1}{2} \sum_{i=1}^k l_i + v$$

i.e.

$$k + v' + 2g - 2 = \frac{1}{2} \sum_{i \geq 3} (i - 2)n_i + \frac{1}{2} \sum_{i=1}^k l_i$$



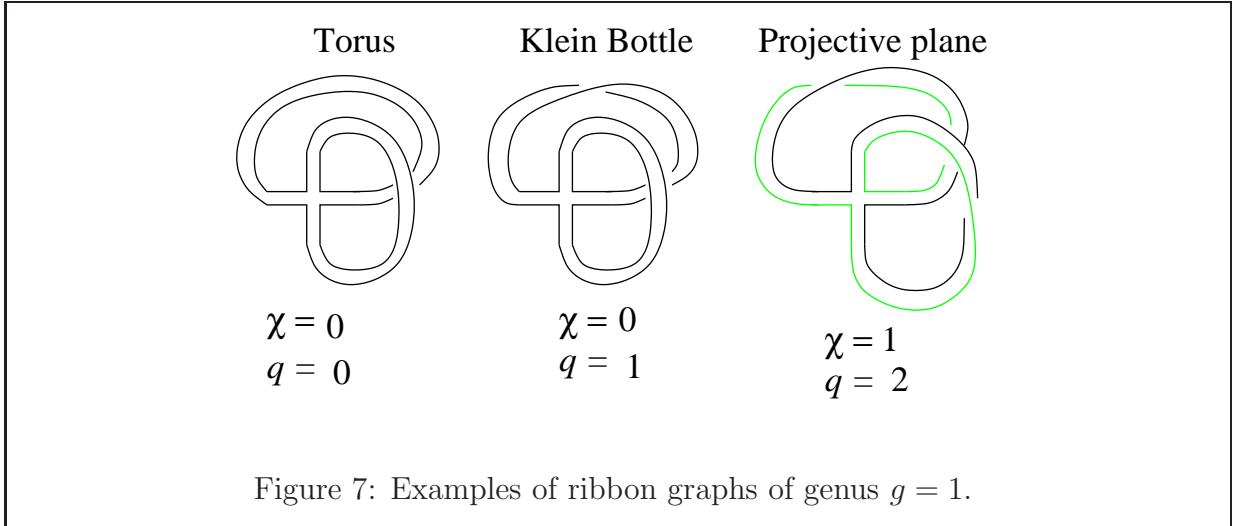
Since the left hand side is fixed, we see that the number and size of vertices are bounded, so that there is only a finite number of possible oriented ribbon graphs  $\mathcal{G}'$ . Since  $\mathcal{G}'$  has a bounded number of edges, there is only a finite number of possibilities of twisting them, i.e. there are also only a finite number of graphs  $\mathcal{G}$ .  $\square$

## 10.1 Generating functions

In order to enumerate the sets  $\mathbb{M}_k^{(g)}(v')$ , we define the following generating functions:

**Definition 10.2** *We define:*

$$\begin{aligned}
& W_k^{(g)}(x_1, \dots, x_k; t_3, \dots, t_{d+1}, \beta; t_0) \\
= & \beta^{-k/2} \sum_{v' \geq 1} t_0^{v'} \sum_{\mathcal{G} \in \mathbb{M}_k^{(g)}(v')} \frac{1}{\#\text{Aut}(\mathcal{G})} \frac{t_3^{n_3(\mathcal{G})} t_4^{n_4(\mathcal{G})} \dots t_{d+1}^{n_{d+1}(\mathcal{G})}}{x_1^{l_1(\mathcal{G})} x_2^{l_2(\mathcal{G})} \dots x_k^{l_k(\mathcal{G})}} \beta^{-e(\mathcal{G})} (\beta - 1)^{q(\mathcal{G})} \\
& + \delta_{k,1} \delta_{g,0} \delta_{q,0} \frac{t_0}{x_1} + \delta_{k,2} \delta_{g,0} \delta_{q,0} \frac{1}{2(x_1 - x_2)^2}. \tag{10.1}
\end{aligned}$$



It is a formal series in powers of  $t_0$ .

Most often, for readability, we shall write only the dependence in the  $x_i$ 's:

$$W_k^{(g)}(x_1, \dots, x_k; t_3, \dots, t_{d+1}, \beta; t_0) \equiv W_k^{(g)}(x_1, \dots, x_k).$$

Also, for  $k = 0$  we write

$$W_0^{(g)} = F_g.$$

## 10.2 Tutte's recursive equations

Tutte's equation is a recursion on the number of edges to construct the ribbon graphs. It consists in finding a bijection between ribbon graphs of various ensembles, by recursion on the number of edges. Let  $\mathbb{M}_{l_1, \dots, l_k}^{(g)}$  be the set of ribbon graphs of genus  $g$ , and with  $k$  marked vertices of size  $l_1, \dots, l_k$ .

Consider a ribbon graph  $\mathcal{G} \in \mathbb{M}_{l_0+1, L}^{(g)}$  where  $L = \{l_1, \dots, l_k\}$ , with marked vertices of degrees  $l_0 + 1, L$ .

Consider the marked edge of marked face 0. It is either twisted or untwisted. Several mutually exclusive situations may occur (see fig 8):

- on the other side of the marked edge, there is an unmarked vertex of size  $j + 1$  with  $j \geq 2$ . We then shrink the marked edge to concatenate the two vertices into one marked vertex of degree  $l_0 + j$ . The orientation is inherited from the initial marked vertex, and the marked edge is chosen as the first edge to the left of the shrunked edge. It is clear that we don't change the number of single lines in  $\mathcal{G}$  or  $\mathcal{G}'$ . We decrease the number of vertices and edges by 1, so we don't change the genus. We thus get a ribbon graph in  $M_{l_0+j, L}^{(g)}$ , and this is weighted with weight  $t_{j+1}(1/\beta + (1 - 1/\beta)) = t_{j+1}$ .

- on the other side of the marked edge, there is the marked vertex  $i \neq 0$ , of size  $l_i \geq 1$ . We then shrink the marked edge to concatenate the two vertices into one marked vertex of degree  $l_0 + l_i - 1$ . The orientation is inherited from the initial marked vertex, and the marked edge is chosen as the first edge to the left of the shrunked edge. It is also clear that we don't change the genus. Since we forget the marking of the other face, we shall get a symmetry factor  $l_i$ , corresponding to the  $l_i$  places where we glue to the  $i^{\text{th}}$  marked vertex. We thus get a ribbon graph in  $M_{l_0+l_i-1, L/\{l_i\}}^{(g)}$ , and this is weighted with weight  $l_i$ .

- on the other side of the marked edge, there is the same marked vertex 0. Again we shall shrink the marked edge, i.e. shrink the 2 single lines. Several sub-situations may occur:

- \* if the edge is untwisted, shrinking the 2 single lines splits the marked vertex of size  $l_0 + 1$  into two vertices of size  $l'$  and  $l_0 - l' - 1$ . They inherit their orientation and marked edge from the initial marked vertex. We have increased the number of marked vertices by 1. The two new vertices are either connected together, or not.

- \*\* If they are not connected, this means that the number of other marked vertices and the genus simply add up. We thus get two ribbon graphs in  $M_{l', l'}^{(g')} \times M_{l_0-l'-1, L/L'}^{(g-g')}$ , and this is weighted with weight  $1/\beta$ .

- \*\* If they are connected, we see that we get a new ribbon graph, with one more vertex, 1 less edge, and we have not changed the connectivity of single lines. The genus has thus decreased by 1. We thus get a ribbon graph in  $M_{l', l_0-l'-1, L}^{(g-1)}$ , and this is weighted with weight  $1/\beta$ .

- \* if the edge is twisted, shrinking the 2 single lines doesn't split the marked vertex. Instead we get a new vertex of size  $l_0 - 1$ . We assign to it the orientation of the half-vertex situated left of the marked edge, and we mark the edge left of the initial one. We have decreased  $q$  by 1, and the genus is unchanged. We thus get a ribbon graph in  $M_{l_0-1, L}^{(g)}$ , and this is weighted with weight  $(1 - 1/\beta) l_0$  (indeed, there are  $l_0$  places where we can glue the marked edge).

For the generating function, those bijections read:

$$\begin{aligned}
x W_{k+1}^{(g)}(x, X_L) &= \sum_{j=2}^d t_{j+1} x^j W_{k+1}^{(g)}(x, X_L) \\
&+ \frac{1}{\sqrt{\beta}} \sum_{i=1}^k \partial_{x_i} \frac{W_k^{(g)}(x, X_L/\{x_i\}) - W_k^{(g)}(x_i, X_L/\{x_i\})}{x - x_i} \\
&+ \frac{1}{\sqrt{\beta}} \sum_{g', L' \subset L} W_{1+\#L'}^{(g')} (x, X_{L'}) W_{1+k-\#L'}^{(g-g')} (x, X_{L/L'})
\end{aligned}$$



$$\begin{aligned}
& + \frac{1}{\sqrt{\beta}} W_{k+2}^{(g-1)}(x, x, X_L) \\
& + \left(1 - \frac{1}{\beta}\right) \partial_x W_{k+1}^{(g)}(x, X_L) \\
& + \frac{1}{\sqrt{\beta}} P_{k+1}^{(g)}(x, X_L)
\end{aligned} \tag{10.2}$$

we define

$$V'(x) = \sqrt{\beta} \left( x - \sum_{j=2}^d t_{j+1} x^j \right) \tag{10.3}$$

and the last term  $P_{k+1}^{(g)}(x; X_L)$  accounts for all the boundary terms, and it is necessarily equal to:

$$P_{k+1}^{(g)}(x; X_L) = \left( V'(x) W_{k+1}^{(g)}(x; X_L) \right)_+ . \tag{10.4}$$

This can be rewritten:

$$\begin{aligned}
V'(x) W_{k+1}^{(g)}(x, X_L) &= \sum_{i=1}^k \partial_{x_i} \frac{W_k^{(g)}(x, X_{L/\{x_i\}}) - W_k^{(g)}(x_i, X_{L/\{x_i\}})}{x - x_i} \\
&+ \sum_{g', L'} W_{1+\#L'}^{(g')}(x, X_{L'}) W_{1+k-\#L'}^{(g-g')}(x, X_{L/L'}) \\
&+ W_{k+2}^{(g-1)}(x, x, X_L) \\
&+ \hbar \partial_x W_{k+1}^{(g)}(x, X_L) \\
&+ P_{k+1}^{(g)}(x, X_L)
\end{aligned} \tag{10.5}$$

where

$$\hbar = \sqrt{\beta} - 1/\sqrt{\beta}.$$

In other words, the  $W_n^{(g)}$ 's defined in section 5 provide a solution to Tutte's equations. They are the generating functions counting our non-oriented ribbon graphs. One just needs to find the polynomial  $P_1^{(0)}(x)$ , i.e.  $U(x)$ , and the choice of  $\psi$  which is such that  $W_1^{(0)}$  is a formal power series in  $t_0$ .

## 11 Conclusion

In this article, we have defined some "quantum" versions of quantities known in algebraic geometry and applied them to the resolution of the loop equations in the arbitrary  $\beta$ -random matrix model case, and in particular the enumeration of some non-orientable ribbon graphs.

Our formalism recovers standard algebraic geometry and the invariants of [15] in the classical limit  $\hbar \rightarrow 0$ .

Instead of an algebraic equation, we have to deal with a differential equation, which we interpreted as a "quantum spectral curve", and we were able to generalize the

basic notions arising in classical algebraic geometry, like genus, sheets, branchpoints, meromorphic forms, of 1st kind, 2nd kind, 3rd kind, matrix of periods,...

It is surprising to see that the notion of branchpoints become "blurred", a branchpoint is no longer a point, but an asymptotic accumulation line. Also, there are two sheets, corresponding of the two possible large  $x$  asymptotic behaviors of  $\psi(x) \sim \exp \pm V/2\hbar$ , but in fact any solution is a linear combination of these two, so that we could say that we are always in a "linear superposition" of two states like in quantum mechanics.

Another surprising thing, is that, in order for any cohomology theory to make sense, we need the cycle integrals of any forms to depend only on the homology class of the cycles, i.e. we need all forms to have vanishing residues at the  $s_i$ 's. This "no-monodromy" condition is equivalent to a Bethe ansatz satisfied by the  $s_i$ 's, like in the Gaudin model [2]. This provides a geometric interpretation of the Bethe ansatz, as the condition for cohomology to make sense.

However, we still lack of a complete understanding of the situation, since most of our results explicitly depend on an initial sector  $S_0$  which we choose, whereas in algebraic geometry most of them only depend on the spectral curve and not on its parametrization. For instance the genus itself depends on a choice of sector. In some sense, the genus is no longer deterministic.

Moreover, we still lack the proper definition of the spectral invariants  $F_g$ , indeed we have defined the  $F_g$ 's only through solving a differential equation with respect to  $\hbar$ , which is not as explicit as [16] or [5]. Out of the  $F_g$ 's, we could expect the possibility to make the link with integrable systems and define a "quantum Tau-function", like in [16].

Also, we restricted ourselves to the case of hyperelliptic curves, i.e. second order differential equations, or also a 1-matrix model. In a forthcoming paper, we shall generalize all this construction to arbitrary linear differential equations of any order, and generalize to a 2-matrix model. This work is underway, almost finished and the article is being written at this time. As for the hyperelliptical case, the notions of genus, sheets, branchpoints, forms,  $W_n^{(g)}$ 's ... can be defined. Again there is a Bethe ansatz ensuring a no-monodromy condition so that all cycle integrals depend only on the homology class of cycles. So, there is no qualitative change, the difference is only technical, because the hyperelliptical case has big simplifications due to the involutive symmetry. The difference between the hyperelliptical case and the general case is comparable to the difference between [12] and [7], i.e. the definition of the kernel  $K$  is really more complicated, and there is a rather "big" technical step.

Then it would be interesting to see if the  $F_g$ 's have some sort of symplectic invariance, or more precisely some "canonical invariance", i.e. are unchanged under any change  $(x, y) \rightarrow (\tilde{x}, \tilde{y})$  such that  $[\tilde{y}, \tilde{x}] = [y, x] = \hbar$ .

Finally, let us mention that we have developed a new geometrical approach to the study of D-modules, and it would be interesting to see how to relate it to more standard approaches, and also to the resurgence theory for studying the Schrödinger equation.

## Acknowledgments

We would like to thank O. Babelon, M. Bergère, G. Borrot, P. Di Francesco, S. Guillermou, V. Pasquier, A. Prats-Ferrer, A. Voros for useful and fruitful discussions on this subject. The work of B.E. and O.M. is partly supported by the Enigma European network MRT-CT-2004-5652, ANR project GranMa "Grandes Matrices Aléatoires" ANR-08-BLAN-0311-01, by the European Science Foundation through the Misgam program, by the Quebec government with the FQRNT. O. Marchal would like to thank the CRM (Centre de recherche mathématiques de Montréal, QC, Canada) for its hospitality.

## A Appendix: Proof of the loop equation for $B(x, z)$

Let's first prove the first loop equation for  $B(x, z)$ : Let's define:

$$\hat{B}(x, z) = \frac{1}{2} \partial_z \left( 2 \frac{\psi'(z)}{\psi(z)} - \partial_z \right) \hat{K}(x, z) \quad (\text{A.1})$$

i.e. we have:

$$B(x, z) = \hat{B}(x, z) - \sum_{\alpha=1}^{d-1} v_\alpha(x) \oint_{\mathcal{A}_\alpha} \hat{B}(x'', z) dx'' \quad (\text{A.2})$$

Since  $(2 \frac{\psi'(x)}{\psi(x)} + \partial_x) v_\alpha(x) = h_\alpha(x)$  is a polynomial of degree  $\leq d-2$ , it suffices to prove eq. (4.35) for  $\hat{B}(x, z)$ .

Let us compute:

$$\begin{aligned} \left( 2 \frac{\psi'(x)}{\psi(x)} + \partial_x \right) \hat{B}(x, z) &= \frac{1}{2} \partial_z \left( 2 \frac{\psi'(z)}{\psi(z)} - \partial_z \right) \frac{1}{x-z} \\ &= \frac{1}{2} \partial_z \left( 2 \frac{\psi'(z)}{\psi(z)(x-z)} - \frac{1}{(x-z)^2} \right) \\ &= -\frac{1}{(x-z)^3} + \partial_z \frac{\psi'(z)}{\psi(z)(x-z)} \end{aligned} \quad (\text{A.3})$$

and therefore:

$$\left(2\frac{\psi'(x)}{\psi(x)} + \partial_x\right) \left(\hat{B}(x, z) - \frac{1}{2(x-z)^2}\right) + \partial_z \frac{\frac{\psi'(x)}{\psi(x)} - \frac{\psi'(z)}{\psi(z)}}{x-z} = 0 \quad (\text{A.4})$$

This proves eq. (4.35), with:

$$P_2^{(0)}(x, z) = -\sum_{\alpha=1}^g h_\alpha(x) \oint_{\mathcal{A}_\alpha} \hat{B}(x'', z) dx'' - \sum_{\alpha=g+1}^{d-1} h_\alpha(x) \oint_{\mathcal{A}_\alpha} dx'' \partial_{x''} \psi^2(x'') \hat{B}(x'', z). \quad (\text{A.5})$$

Let's now prove the second loop equation for  $B(x, z)$ : Similarly, let us compute  $(2\frac{\psi'(z)}{\psi(z)} + \partial_z) \hat{B}(x, z)$ :

$$\left(2\frac{\psi'(z)}{\psi(z)} + \partial_z\right) \hat{B}(x, z) = \frac{1}{2} \left(2\frac{\psi'(z)}{\psi(z)} + \partial_z\right) \partial_z \left(2\frac{\psi'(z)}{\psi(z)} - \partial_z\right) \hat{K}(x, z) \quad (\text{A.6})$$

Notice that the operator  $\hat{U}(z) = \frac{1}{2} \left(2\frac{\psi'(z)}{\psi(z)} + \partial_z\right) \partial_z \left(2\frac{\psi'(z)}{\psi(z)} - \partial_z\right)$ , is equal to:

$$\hat{U}(z) = -\frac{1}{2} \partial_z^3 + 2U(z) \partial_z + U'(z) \quad (\text{A.7})$$

which is also known in the litterature as the Gelfand-Dikii operator [8] (The Gelfand-Dikii differential polynomials  $R_k(U)$  are computed recursively by  $R_0 = 1$  and  $\partial_z R_{k+1} = \hat{U} \cdot R_k$ ), which plays a key role in the KdV hierarchy.

However, independently of any relationship with KdV, we get:

$$\begin{aligned} & \left(2\frac{\psi'(z)}{\psi(z)} + \partial_z\right) \hat{B}(x, z) \\ &= \frac{1}{\psi^2(x)} \int_{\infty_0}^x \psi^2(x') dx' \hat{U}(x') \cdot \frac{1}{x' - z} \\ &= \frac{1}{\psi^2(x)} \int_{\infty_0}^x \psi^2(x') dx' \left( -\frac{3}{(x' - z)^4} + \frac{2U(z)}{(x' - z)^2} + \frac{U'(z)}{x' - z} \right) \end{aligned} \quad (\text{A.8})$$

We integrate the first term by parts three times, and we write  $Y = \psi'/\psi$  (we have  $Y' + Y^2 = U$ ):

$$\begin{aligned} & \left(2\frac{\psi'(z)}{\psi(z)} + \partial_z\right) \hat{B}(x, z) \\ &= \frac{1}{(x-z)^3} - \frac{2}{\psi^2(x)} \int_{\infty_0}^x \psi^2(x') dx' \frac{Y(x')}{(x' - z)^3} \\ & \quad + \frac{1}{\psi^2(x)} \int_{\infty_0}^x \psi^2(x') dx' \left( \frac{2U(z)}{(x' - z)^2} + \frac{U'(z)}{x' - z} \right) \\ &= \frac{1}{(x-z)^3} + \frac{Y(x)}{(x-z)^2} - \frac{1}{\psi^2(x)} \int_{\infty_0}^x \psi^2(x') dx' \frac{Y'(x') + 2Y^2(x')}{(x' - z)^2} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\psi^2(x)} \int_{\infty_0}^x \psi^2(x') dx' \left( \frac{2U(z)}{(x'-z)^2} + \frac{U'(z)}{x'-z} \right) \\
= & \frac{1}{(x-z)^3} + \frac{Y(x)}{(x-z)^2} + \frac{1}{\psi^2(x)} \int_{\infty_0}^x \psi^2(x') dx' \frac{Y'(x')}{(x'-z)^2} \\
& + \frac{1}{\psi^2(x)} \int_{\infty_0}^x \psi^2(x') dx' \left( \frac{2(U(z) - U(x'))}{(x'-z)^2} + \frac{U'(z)}{x'-z} \right) \\
= & \frac{1}{(x-z)^3} + \frac{Y(x)}{(x-z)^2} - \frac{Y'(x)}{x-z} + \frac{1}{\psi^2(x)} \int_{\infty_0}^x \psi^2(x') dx' \frac{Y''(x') + 2Y(x')Y'(x')}{(x'-z)^2} \\
& + \frac{1}{\psi^2(x)} \int_{\infty_0}^x \psi^2(x') dx' \left( \frac{2(U(z) - U(x'))}{(x'-z)^2} + \frac{U'(z)}{x'-z} \right) \\
= & \frac{1}{(x-z)^3} - \frac{\partial}{\partial x} \frac{Y(x)}{x-z} + \frac{1}{\psi^2(x)} \int_{\infty_0}^x \psi^2(x') dx' \left( \frac{2(U(z) - U(x'))}{(x'-z)^2} + \frac{U'(z) + U(x')}{x'-z} \right)
\end{aligned} \tag{A.9}$$

This implies that:

$$\begin{aligned}
& \left( 2 \frac{\psi'(z)}{\psi(z)} + \partial_z \right) \left( \hat{B}(x, z) - \frac{1}{2(x-z)^2} \right) + \frac{\partial}{\partial x} \frac{Y(x) - Y(z)}{x-z} \\
= & \frac{1}{\psi^2(x)} \int_{\infty_0}^x \psi^2(x') dx' \left( \frac{2(U(z) - U(x'))}{(x'-z)^2} + \frac{U'(z) + U(x')}{x'-z} \right) \\
= & Q(z, x)
\end{aligned} \tag{A.10}$$

which is clearly a polynomial in  $z$ . Taking integrals over  $x$  along  $\mathcal{A}_\alpha$  does not change its structure in  $z$ , and therefore:

$$\begin{aligned}
& \left( 2 \frac{\psi'(z)}{\psi(z)} + \partial_z \right) \left( B(x, z) - \frac{1}{2(x-z)^2} \right) + \frac{\partial}{\partial x} \frac{Y(x) - Y(z)}{x-z} \\
= & \frac{1}{\psi^2(x)} \int_{\infty_0}^x \psi^2(x') dx' \left( \frac{2(U(z) - U(x'))}{(x'-z)^2} + \frac{U'(z) + U(x')}{x'-z} \right) \\
= & \tilde{P}_2^{(0)}(z, x)
\end{aligned} \tag{A.11}$$

is of the required form.

By looking at the behavior of the various terms in the LHS of eq. (4.36) when  $z \rightarrow \infty$ , we find that  $\tilde{P}_2^{(0)}(z, x)$  is a polynomial of degree at most  $d-2$  in  $z$ .

## B Appendix: Proof of theorem 5.2

**Theorem 5.2** *Each  $W_n^{(g)}(x_1, \dots, x_n)$  with  $2 - 2g - n < 0$ , is an analytical functions of all its arguments, with poles only when  $x_i \rightarrow s_j$ . Moreover, it vanishes at least as  $O(1/x_i^2)$  when  $x_i \rightarrow \infty$  in all sectors. It has no discontinuity across  $\mathcal{A}$ -cycles.*

**proof:**

We proceed by recursion on  $2g+n$ . The theorem is true for  $W_2^{(0)}$ . Assume it is true up to  $2g+n$ , we shall prove it for  $W_{n+1}^{(g)}(x_0, x_1, \dots, x_n)$ .

The integrand  $U_n^{(g)}$  of theorem 5.1 is singular only at  $x = s_j$ 's. As long as  $x_0$  is away from the  $s_j$ 's, we can continuously deform the  $\mathcal{A}$ -cycles and the contour  $\mathcal{C}$  in order to have  $x_0$  outside of the  $\mathcal{A}$ -cycles, and the integral can be evaluated and is analytical in  $x_0$ . When  $x_0$  approaches  $s_i$ , we define  $\hat{\mathcal{C}}_i$ , a contour which surrounds all roots except  $s_i$ , i.e:

$$\oint_{\mathcal{C}} = \oint_{\hat{\mathcal{C}}_i} + 2i\pi \operatorname{Res}_{s_i} \quad (\text{B.1})$$

The integral over  $\hat{\mathcal{C}}_i$  can be evaluated and is convergent, thus it is analytical in  $x_0$ .

From the recursion hypothesis, all terms in the integrand are meromorphic in the vicinity of  $s_i$ , and thus the residue at  $s_i$  can be computed by taking a finite Taylor expansion of  $K(x_0, x) = \sum_k (x - s_i)^k K_{i,k}(x_0)$  in the vicinity of  $x \rightarrow s_i$ . The result is a finite sum of terms of the type  $K_{i,k}(x_0)$ . It is easy to see from the definition of  $K$ , that each  $K_{i,k}(x_0)$  has only poles at  $x_0 = s_i$ . Thus we have proved that  $W_{n+1}^{(g)}$  has poles at the  $s_i$ 's in its first variable.

In the other variables, the result comes from an obvious recursion.

□

## C Appendix: Proof of theorem 5.4

In this subsection we prove theorem 5.4, that all  $W_n^{(g)}$ 's satisfy the loop equation.

$$\begin{aligned} P_{n+1}^{(g)}(x, x_1, \dots, x_n) &= 2\hbar \frac{\psi'(x)}{\psi(x)} \overline{W}_{n+1}^{(g)}(x, x_1, \dots, x_n) + \hbar \partial_x \overline{W}_{n+1}^{(g)}(x, x_1, \dots, x_n) \\ &+ \sum_{I \subset J} \overline{W}_{|I|+1}^{(h)}(x, x_I) \overline{W}_{n-|I|+1}^{(g-h)}(x, J/I) + \overline{W}_{n+2}^{(g-1)}(x, x, J) \\ &+ \sum_j \partial_{x_j} \left( \frac{\overline{W}_n^{(g)}(x, J/\{j\}) - \overline{W}_n^{(g)}(x_j, J/\{j\})}{(x - x_j)} \right) \end{aligned} \quad (\text{C.1})$$

is a polynomial in  $x$  of degree at most  $d - 2$ .

**proof:**

From the definition we have:

$$\begin{aligned} W_{n+1}^{(g)}(x, J) &= \frac{1}{2i\pi} \oint_{\mathcal{C}} dz K(x, z) U_n^{(g)}(z, J) \\ &= \frac{1}{2i\pi} \oint_{\mathcal{C}} dz \hat{K}(x, z) U_n^{(g)}(z, J) \\ &\quad - \sum_{\alpha} \frac{v_{\alpha}(x)}{2i\pi} \oint_{\mathcal{C}} dz C_{\alpha}(z) U_n^{(g)}(z, J) \end{aligned} \quad (\text{C.2})$$

Then, notice that  $\hat{K}(x, z)$  has a logarithmic cut along  $]\infty_0, x]$ , and the discontinuity across that cut is:

$$\delta\hat{K}(x, z) = \frac{2i\pi}{\hbar} \frac{\psi^2(z)}{\psi^2(x)} \quad (\text{C.3})$$

$U_n^{(g)}$  has no singularity outside of  $\mathcal{C}$ , and thus we can deform the contour into a contour enclosing only the logarithmic cut of  $\hat{K}(x, z)$ , and therefore:

$$\frac{1}{2i\pi} \oint_{\mathcal{C}} dz \hat{K}(x, z) U_n^{(g)}(z, J) = -\frac{1}{\hbar} \int_{\infty_0}^x dz \frac{\psi^2(z)}{\psi^2(x)} U_n^{(g)}(z, J) \quad (\text{C.4})$$

We then apply the operator:  $2\frac{\psi'(x)}{\psi(x)} + \partial_x$ , that gives:

$$(2\hbar \frac{\psi'(x)}{\psi(x)} + \hbar\partial_x) \frac{1}{2i\pi} \oint_{\mathcal{C}} dz \hat{K}(x, z) U_n^{(g)}(z, J) = -U_n^{(g)}(x, J) \quad (\text{C.5})$$

and therefore:

$$\begin{aligned} P_{n+1}^{(g)}(x, J) &= (2\hbar \frac{\psi'(x)}{\psi(x)} + \hbar\partial_x) W_{n+1}^{(g)}(x, J) + U_n^{(g)}(x, J) \\ &= -(2\hbar \frac{\psi'(x)}{\psi(x)} + \hbar\partial_x) \sum_{\alpha} \frac{v_{\alpha}(x)}{2i\pi} \oint_{\mathcal{C}} dz C_{\alpha}(z) U_n^{(g)}(z, J) \\ &= -\sum_{\alpha} h_{\alpha}(x) \oint_{\mathcal{C}} dz C_{\alpha}(z) U_n^{(g)}(z, J) \end{aligned} \quad (\text{C.6})$$

which is indeed a polynomial of  $x$  of degree at most  $d - 2$ .

□

## D Appendix: Proof of theorem 5.6

**Theorem 5.6** *The 3 point function  $W_3^{(0)}$  is symmetric and we have:*

$$W_3^{(0)}(x_1, x_2, x_3) = \frac{4}{2i\pi} \oint_{\mathcal{C}} dx \frac{B(x, x_1)B(x, x_2)B(x, x_3)}{Y'(x)} \quad (\text{D.1})$$

where  $Y(x) = -2\hbar \frac{\psi'(x)}{\psi(x)}$

**proof:**

The definition of  $W_3^{(0)}$  is:

$$\begin{aligned} &W_3^{(0)}(x_0, x_1, x_2) \\ &= \frac{1}{i\pi} \oint_{\mathcal{C}} dx K(x_0, x) B(x, x_1) B(x, x_2) \\ &= \frac{1}{4i\pi} \oint_{\mathcal{C}} dx K_0 G'_1 G'_2 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4i\pi} \oint_{\mathcal{C}} dx K_0 ((\hbar K_1'' + Y K_1' + Y' K_1)(\hbar K_2'' + Y K_2' + Y' K_2)) \\
&= \frac{1}{4i\pi} \oint_{\mathcal{C}} dx K_0 (\hbar^2 K_1'' K_2'' + \hbar Y (K_1' K_2'' + K_1'' K_2') + \hbar Y' (K_1'' K_2 + K_2'' K_1) \\
&\quad + Y^2 K_1' K_2' + Y Y' (K_1 K_2' + K_1' K_2) + Y'^2 K_1 K_2) \\
&\tag{D.2}
\end{aligned}$$

where we have written for short  $K_p = K(x_p, x)$ ,  $G_p = G(x_p, x)$ , and derivative are w.r.t.  $x$ . Note also that introducing  $K_1$  and  $K_2$  makes appear some additional and arbitrary logarithmic cuts from  $x_1$  to  $\infty_0$  and from  $x_2$  to  $\infty_0$ . But these cuts can be chosen arbitrarily since from the definition of  $W_3^{(0)}(x_0, x_1, x_2)$  it should not depend on that. Remember also that to use this definition of  $W_3^{(0)}$  we need to assume that  $x_1$  and  $x_2$  are not circled by the contour  $\mathcal{C}$ . Therefore we can choose the logarithmic cut of  $K_1$  and  $K_2$  inside the contour  $\mathcal{C}$  like we have done it for  $x_0$ . We now see that for example  $K_0 K_1 K_2$  has no singularity outside  $\mathcal{C}$  and thus will not contribute because of theorem 4.1. Many other manipulations involving globally defined functions with no singularities outside  $\mathcal{C}$  can be done.

For example, using the Ricatti equation  $Y_i'^2 = 2\hbar Y_i' + 4U$ , we may replace  $Y_i'^2$  by  $2\hbar Y_i'$  and  $Y_i Y_i'$  by  $\hbar Y_i''$ .

$$\begin{aligned}
&W_3^{(0)}(x_0, x_1, x_2) \\
&= \frac{1}{4i\pi} \oint_{\mathcal{C}} dx K_0 (\hbar Y (K_1' K_2'' + K_1'' K_2') + \hbar Y' (K_1'' K_2 + K_2'' K_1) \\
&\quad + 2\hbar Y' K_1' K_2' + \hbar Y'' (K_1 K_2' + K_1' K_2) + Y'^2 K_1 K_2) \\
&= \frac{1}{4i\pi} \oint_{\mathcal{C}} dx K_0 (\hbar Y (K_1' K_2')' + \hbar Y' (K_1 K_2)'' + \hbar Y'' (K_1 K_2)' + Y'^2 K_1 K_2) \\
&= \frac{1}{4i\pi} \oint_{\mathcal{C}} dx Y'^2 K_0 K_1 K_2 + \hbar (Y'' K_0 (K_1 K_2)' - (Y K_0)' K_1' K_2' - (Y' K_0)' (K_1 K_2)') \\
&= \frac{1}{2} \frac{1}{4i\pi} \oint_{\mathcal{C}} dx Y'^2 K_0 K_1 K_2 - \hbar ((Y K_0)' K_1' K_2' + Y' K_0' (K_1 K_2)') \\
&= \frac{1}{4i\pi} \oint_{\mathcal{C}} dx Y'^2 K_0 K_1 K_2 - \hbar Y K_0' K_1' K_2' - \hbar Y' (K_0 K_1' K_2' + K_0' K_1 K_2' + K_0' K_1' K_2) \\
&\tag{D.3}
\end{aligned}$$

This expression is clearly symmetric in  $x_0, x_1, x_2$  as claimed in theorem 5.5.

Let us give an alternative expression, in the form of the Verlinde or Krichever formula.

$$W_3^{(0)}(x_0, x_1, x_2) = \frac{2}{i\pi} \oint_{\mathcal{C}} dx \frac{B(x, x_1) B(x, x_2) B(x, x_3)}{Y'(x)} \tag{D.4}$$

**proof:**

In order to prove formula D.4, compute:

$$B(x, x_i) = -\frac{1}{2} G'(x, x_i) = -\frac{1}{2} G_i' = \frac{1}{2} (\hbar K_i'' + Y K_i' + Y' K_i) \tag{D.5}$$



thus:

$$\begin{aligned}
& \frac{1}{2i\pi} \oint_{\mathcal{C}} dx \frac{B(x, x_1)B(x, x_2)B(x, x_3)}{Y'(x)} \\
= & \frac{1}{16i\pi} \oint_{\mathcal{C}} dx \frac{1}{Y'(x)} (\hbar K_0'' + YK_0' + Y'K_0)(\hbar K_1'' + YK_1' + Y'K_1)(\hbar K_2'' + YK_2' + Y'K_2) \\
= & \frac{1}{16i\pi} \oint_{\mathcal{C}} dx \frac{\hbar^3}{Y'} K_0'' K_1'' K_2'' + \hbar^2 \frac{Y}{Y'} (K_0' K_1'' K_2'' + K_0'' K_1' K_2'' + K_0'' K_1'' K_2') \\
& + \hbar^2 (K_0 K_1'' K_2'' + K_0'' K_1 K_2'' + K_0'' K_1'' K_2) + \hbar \frac{Y^2}{Y'} (K_0'' K_1' K_2' + K_0' K_1'' K_2' + K_0' K_1' K_2'') \\
& + \hbar Y (K_0 K_1' K_2'' + K_0 K_1'' K_2' + K_0' K_1 K_2'' + K_0' K_1'' K_2 + K_0'' K_1 K_2' + K_0'' K_1' K_2) \\
& + \hbar Y' (K_0'' K_1 K_2 + K_0 K_1'' K_2 + K_0 K_1 K_2'') + \frac{Y^3}{Y'} K_0' K_1' K_2' \\
& + Y^2 (K_0 K_1' K_2' + K_0' K_1 K_2' + K_0' K_1' K_2) + Y Y' (K_0' K_1 K_2 + K_0 K_1' K_2 + K_0 K_1 K_2') \\
& + Y'^2 K_0 K_1 K_2
\end{aligned} \tag{D.6}$$

$$\begin{aligned}
& \frac{1}{2i\pi} \oint_{\mathcal{C}} dx \frac{B(x, x_1)B(x, x_2)B(x, x_3)}{Y'(x)} \\
= & \frac{1}{16i\pi} \oint_{\mathcal{C}} dx \hbar Y (K_0 K_1' K_2'' + K_0 K_1'' K_2' + K_0' K_1 K_2'' + K_0' K_1'' K_2 + K_0'' K_1 K_2' + K_0'' K_1' K_2) \\
& + \hbar Y' (K_0'' K_1 K_2 + K_0 K_1'' K_2 + K_0 K_1 K_2'') + \frac{Y^3}{Y'} K_0' K_1' K_2' \\
& + Y^2 (K_0 K_1' K_2' + K_0' K_1 K_2' + K_0' K_1' K_2) + Y Y' (K_0' K_1 K_2 + K_0 K_1' K_2 + K_0 K_1 K_2') \\
& + Y'^2 K_0 K_1 K_2
\end{aligned} \tag{D.7}$$

Notice that  $Y^2 = 2\hbar Y' + 4U$ , thus we may replace  $Y^3/Y'$  by  $2\hbar Y$ , and  $Y^2$  by  $2\hbar Y'$  and  $Y Y'$  by  $\hbar Y''$ , for the same reasons as before. Thus:

$$\begin{aligned}
& \frac{1}{2i\pi} \oint_{\mathcal{C}} dx \frac{B(x, x_1)B(x, x_2)B(x, x_3)}{Y'(x)} \\
= & \frac{1}{16i\pi} \oint_{\mathcal{C}} dx \hbar Y (K_0 K_1' K_2'' + K_0 K_1'' K_2' + K_0' K_1 K_2'' + K_0' K_1'' K_2 + K_0'' K_1 K_2' + K_0'' K_1' K_2) \\
& + \hbar Y' (K_0'' K_1 K_2 + K_0 K_1'' K_2 + K_0 K_1 K_2'') + 2\hbar Y K_0' K_1' K_2' \\
& + 2\hbar Y' (K_0 K_1' K_2' + K_0' K_1 K_2' + K_0' K_1' K_2) + \hbar Y'' (K_0' K_1 K_2 + K_0 K_1' K_2 + K_0 K_1 K_2') \\
& + Y'^2 K_0 K_1 K_2 \\
= & \frac{1}{16i\pi} \oint_{\mathcal{C}} dx \hbar Y (K_0 (K_1' K_2')' + K_1 (K_0' K_2')' + K_2 (K_0' K_1')') \\
& + 2\hbar Y K_0' K_1' K_2' + Y'^2 K_0 K_1 K_2 + \hbar (Y' (K_0' K_1 K_2 + K_0 K_1' K_2 + K_0 K_1 K_2'))' \\
= & \frac{1}{16i\pi} \oint_{\mathcal{C}} dx \hbar Y (K_0 (K_1' K_2')' + K_1 (K_0' K_2')' + K_2 (K_0' K_1')') \\
& + 2\hbar Y K_0' K_1' K_2' + Y'^2 K_0 K_1 K_2 \\
= & -\frac{1}{16i\pi} \oint_{\mathcal{C}} dx 3\hbar Y K_0' K_1' K_2' + \hbar Y' (K_0 K_1' K_2' + K_0' K_1 K_2' + K_0' K_1' K_2)
\end{aligned}$$

$$\begin{aligned}
& -2\hbar Y K'_0 K'_1 K'_2 - Y'^2 K_0 K_1 K_2 \\
= & \frac{1}{8i\pi} \oint_{\mathcal{C}} dx W_3^{(0)}(x_0, x_1, x_2)
\end{aligned} \tag{D.8}$$

□

## E Appendix: Proof of theorem 5.5

**Theorem 5.5** *Each  $W_n^{(g)}$  is a symmetric function of all its arguments.*

**proof:**

The special case of  $W_3^{(0)}$  is proved in appendix D.1 above.

It is obvious from the definition that  $W_{n+1}^{(g)}(x_0, x_1, \dots, x_n)$  is symmetric in  $x_1, x_2, \dots, x_n$ , and therefore we need to show that (for  $n \geq 1$ ):

$$W_{n+1}^{(g)}(x_0, x_1, J) - W_{n+1}^{(g)}(x_1, x_0, J) = 0 \tag{E.1}$$

where  $J = \{x_2, \dots, x_n\}$ . We prove it by recursion on  $-\chi = 2g - 2 + n$ .

Assume that every  $W_k^{(h)}$  with  $2h + k - 2 \leq 2g + n$  is symmetric. We have:

$$\begin{aligned}
& W_{n+1}^{(g)}(x_0, x_1, J) \\
= & \frac{1}{2\pi i} \oint_{\mathcal{C}} dx K(x_0, x) \left( W_{n+2}^{(g-1)}(x, x, x_1, J) + 2 B(x, x_1) W_n^{(g)}(x, J) \right. \\
& \left. + 2 \sum_{h=0}^g \sum'_{I \in J} W_{2+|I|}^{(h)}(x, x_1, I) W_{n-|I|}^{(g-h)}(x, J/I) \right)
\end{aligned} \tag{E.2}$$

where  $\sum'$  means that we exclude the terms ( $I = \emptyset, h = 0$ ) and ( $I = J, h = g$ ). Notice also that  $\overline{W}_{n+2}^{(g-1)} = W_{n+2}^{(g-1)}$  because  $n \geq 1$ . Then, using the recursion hypothesis, we have:

$$\begin{aligned}
& W_{n+1}^{(g)}(x_0, x_1, J) \\
= & 2 \oint_{\mathcal{C}} dx K(x_0, x) B(x, x_1) W_n^{(g)}(x, J) \\
& + \oint_{\mathcal{C}} dx \oint'_{\mathcal{C}} dx' K(x_0, x) K(x_1, x') \left( W_{n+3}^{(g-2)}(x, x, x', x', J) \right. \\
& + 2 \sum_h \sum'_{I} W_{2+|I|}^{(h)}(x', x, I) W_{1+n-|I|}^{(g-1-h)}(x', x, J/I) \\
& + 2 \sum_h \sum'_{I} W_{3+|I|}^{(h)}(x', x, x, I) W_{n-|I|}^{(g-1-h)}(x', J/I) \\
& \left. + 2 \sum_h \sum'_{I \in J} W_{n-|I|}^{(g-h)}(x, J/I) \left[ W_{3+|I|}^{(h-1)}(x, x', x', I) \right] \right)
\end{aligned}$$

$$(E.3) \quad +2 \sum_{h'} \sum_{I' \subset I} W_{2+|I'|}^{(h')} (x', x, I') W_{1+|I|-|I'|}^{(h-h')} (x', I/I') \Big] )$$

Now, if we compute  $W_{n+1}^{(g)}(x_1, x_0, J)$ , we get the same expression, with the order of integrations exchanged, i.e. we have to integrate  $x'$  before integrating  $x$ . Notice, by moving the integration contours, that:

$$(E.4) \quad \oint_{\mathcal{C}} dx \oint_{\mathcal{C}} dx' - \oint_{\mathcal{C}} dx' \oint_{\mathcal{C}} dx = - \oint_{\mathcal{C}} dx \frac{1}{2\pi i} \text{Res}_{x' \rightarrow x}$$

Moreover, the only terms which have a pole at  $x = x'$  are those containing  $B(x, x')$ . Therefore:

$$(E.5) \quad \begin{aligned} & W_{n+1}^{(g)}(x_0, x_1, J) - W_{n+1}^{(g)}(x_1, x_0, J) \\ = & 2 \oint_{\mathcal{C}} dx (K(x_0, x) B(x, x_1) - K(x_1, x) B(x, x_0)) W_n^{(g)}(x, J) \\ & - 2 \oint_{\mathcal{C}} dx \frac{1}{2i\pi} \text{Res}_{x' \rightarrow x} K(x_0, x) K(x_1, x') B(x, x') \left( \right. \\ & \left. 2W_{1+n}^{(g-1)}(x', x, J) + 2 \sum_h \sum_{I \in J} W_{n-|I|}^{(g-h)}(x, J/I) W_{1+|I|}^{(h)}(x', I) \right) \end{aligned}$$

The residue  $\text{Res}_{x' \rightarrow x}$  can be computed:

$$\begin{aligned} & W_{n+1}^{(g)}(x_0, x_1, J) - W_{n+1}^{(g)}(x_1, x_0, J) \\ = & 2 \oint_{\mathcal{C}} dx (K(x_0, x) B(x, x_1) - K(x_1, x) B(x, x_0)) W_n^{(g)}(x, J) \\ & - \oint_{\mathcal{C}} dx K(x_0, x) \frac{\partial}{\partial x'} \left( K(x_1, x') \left( \right. \right. \\ & \left. \left. 2W_{1+n}^{(g-1)}(x', x, J) + 2 \sum_h \sum_{I \in J} W_{n-|I|}^{(g-h)}(x, J/I) W_{1+|I|}^{(h)}(x', I) \right) \right)_{x'=x} \\ = & 2 \oint_{\mathcal{C}} dx (K(x_0, x) B(x, x_1) - K(x_1, x) B(x, x_0)) W_n^{(g)}(x, J) \\ & - \oint_{\mathcal{C}} dx K(x_0, x) K'(x_1, x) \left( \right. \\ & \left. 2W_{1+n}^{(g-1)}(x, x, J) + 2 \sum_h \sum_{I \in J} W_{n-|I|}^{(g-h)}(x, J/I) W_{1+|I|}^{(h)}(x, I) \right) \\ & - \oint_{\mathcal{C}} dx K(x_0, x) K(x_1, x') \frac{\partial}{\partial x'} \left( \right. \\ & \left. 2W_{1+n}^{(g-1)}(x', x, J) + 2 \sum_h \sum_{I \in J} W_{n-|I|}^{(g-h)}(x, J/I) W_{1+|I|}^{(h)}(x', I) \right)_{x'=x} \\ = & 2 \oint_{\mathcal{C}} dx (K(x_0, x) B(x, x_1) - K(x_1, x) B(x, x_0)) W_n^{(g)}(x, J) \end{aligned}$$

$$\begin{aligned}
& - \oint_{\mathcal{C}} dx K(x_0, x) K'(x_1, x) \left( \right. \\
& 2W_{1+n}^{(g-1)}(x, x, J) + 2 \sum_h \sum_{I \in J} W_{n-|I|}^{(g-h)}(x, J/I) W_{1+|I|}^{(h)}(x, I) \left. \right) \\
& - \frac{1}{2} \oint_{\mathcal{C}} dx K(x_0, x) K(x_1, x) \frac{\partial}{\partial x} \left( \right. \\
& 2W_{1+n}^{(g-1)}(x, x, J) + 2 \sum_h \sum_{I \in J} W_{n-|I|}^{(g-h)}(x, J/I) W_{1+|I|}^{(h)}(x, I) \left. \right) \\
\end{aligned} \tag{E.6}$$

The last term can be integrated by parts, and we get:

$$\begin{aligned}
& W_{n+1}^{(g)}(x_0, x_1, J) - W_{n+1}^{(g)}(x_1, x_0, J) \\
= & 2 \oint_{\mathcal{C}} dx (K(x_0, x) B(x, x_1) - K(x_1, x) B(x, x_0)) W_n^{(g)}(x, J) \\
& + \frac{1}{2} \oint_{\mathcal{C}} dx \left( K'(x_0, x) K(x_1, x) - K(x_0, x) K'(x_1, x) \right) \left( \right. \\
& 2W_{1+n}^{(g-1)}(x, x, J) + 2 \sum_h \sum_{I \in J} W_{n-|I|}^{(g-h)}(x, J/I) W_{1+|I|}^{(h)}(x, I) \left. \right) \\
\end{aligned} \tag{E.7}$$

Then we use theorem 5.4:

$$\begin{aligned}
& W_{n+1}^{(g)}(x_0, x_1, J) - W_{n+1}^{(g)}(x_1, x_0, J) \\
= & 2 \oint_{\mathcal{C}} dx (K(x_0, x) B(x, x_1) - K(x_1, x) B(x, x_0)) W_n^{(g)}(x, J) \\
& + \oint_{\mathcal{C}} dx \left( K'(x_0, x) K(x_1, x) - K(x_0, x) K'(x_1, x) \right) \left( P_n^{(g)}(x, J) \right. \\
& \left. + (Y(x) - \hbar \partial_x) W_n^{(g)}(x, J) + \sum_j \partial_{x_j} \left( \frac{W_{n-1}^{(g)}(x_j, J/\{x_j\})}{x - x_j} \right) \right) \\
\end{aligned} \tag{E.8}$$

Since  $P_n^{(g)}(x, J)$  and  $W_{n-1}^{(g)}(x_j, J/\{x_j\})$  are entire functions of  $x$ , we can use the usual theorem 4.1 to say that they do not contribute. (Note again that we choose the logarithmic cut of  $K_1$  inside the contour  $\mathcal{C}$ , and that we can do that because the contour  $\mathcal{C}$  contains  $x_1$ .)

$$\begin{aligned}
& W_{n+1}^{(g)}(x_0, x_1, J) - W_{n+1}^{(g)}(x_1, x_0, J) \\
= & 2 \oint_{\mathcal{C}} dx (K(x_0, x) B(x, x_1) - K(x_1, x) B(x, x_0)) W_n^{(g)}(x, J) \\
& + 2 \oint_{\mathcal{C}} dx \left( K'(x_0, x) K(x_1, x) - K(x_0, x) K'(x_1, x) \right) \\
& (Y(x) - \hbar \partial_x) W_n^{(g)}(x, J)
\end{aligned}$$

(E.9)

Notice that:

$$K'_0 K_1 - K_0 K'_1 = -\frac{1}{\hbar} (G_0 K_1 - K_0 G_1) \quad (\text{E.10})$$

and  $B = -\frac{1}{2} G'$ , therefore:

$$\begin{aligned} & W_{n+1}^{(g)}(x_0, x_1, J) - W_{n+1}^{(g)}(x_1, x_0, J) \\ = & -2 \oint_{\mathcal{C}} dx (K_0 G'_1 - K_1 G'_0) W_n^{(g)}(x, J) \\ & - \frac{1}{\hbar} 2 \oint_{\mathcal{C}} dx (G_0 K_1 - K_0 G_1) (Y(x) - \hbar \partial_x) W_n^{(g)}(x, J) \end{aligned} \quad (\text{E.11})$$

we integrate the first line by parts:

$$\begin{aligned} & W_{n+1}^{(g)}(x_0, x_1, J) - W_{n+1}^{(g)}(x_1, x_0, J) \\ = & \oint_{\mathcal{C}} dx (K'_0 G_1 - K'_1 G_0) W_n^{(g)}(x, J) \\ & + \oint_{\mathcal{C}} dx (K_0 G_1 - K_1 G_0) W_n^{(g)}(x, J)' \\ & - \frac{1}{\hbar} \oint_{\mathcal{C}} dx (G_0 K_1 - K_0 G_1) (Y(x) - \hbar \partial_x) W_n^{(g)}(x, J) \end{aligned} \quad (\text{E.12})$$

Notice that:

$$K'_0 G_1 - G_0 K'_1 = -\frac{Y}{\hbar} (K_0 G_1 - G_0 K_1) \quad (\text{E.13})$$

So we find

$$W_{n+1}^{(g)}(x_0, x_1, J) - W_{n+1}^{(g)}(x_1, x_0, J) = 0 \quad (\text{E.14})$$

## References

- [1] O. Babelon, D. Bernard, M. Talon, Introduction to Classical Integrable Systems, Cambridge University Press, 1999.
- [2] O. Babelon, D. Talalaev, On the Bethe Ansatz for the Jaynes-Cummings-Gaudin model, hep-th/0703124.
- [3] V.Bouchard, A.Klemm, M.Mariño, S.Pasquetti, “Remodeling the B-model”, arXiv:0709.1453.
- [4] W. Bryc, V. Pierce, Duality of real and quaternionic random matrices, arxiv:0806.3695.
- [5] L. Chekhov, B. Eynard, Matrix eigenvalue model: Feynman graph technique for all genera, JHEP 0612 (2006) 026, math-ph/0604014.
- [6] L. Chekhov, B. Eynard, Hermitean matrix model free energy: Feynman graph technique for all genera, JHEP 009P 0206/5, hep-th/0504116.
- [7] L. Chekhov, B. E., N. Orantin, Free energy topological expansion for the 2-matrix model, JHEP 0612 (2006) 053, math-ph/0603003.
- [8] P. Di Francesco, P. Ginsparg, J. Zinn-Justin, “2D Gravity and Random Matrices”, *Phys. Rep.* **254**, 1 (1995).
- [9] R. Dijkgraaf, L. Hollands, P. Sułkowski, Quantum Curves and D-Modules, arXiv:0810.4157.
- [10] R.Dijkgraaf and c.Vafa, “Two Dimensional Kodaira-Spencer Theory and Three Dimensional Chern-Simons Gravity”, arXiv:0711.1932.
- [11] I. Dumitriu, A. Edelman, Matrix models for beta ensembles, 2002.
- [12] B. Eynard, Topological expansion for the 1-hermitian matrix model correlation functions, JHEP/024A/0904, hep-th/0407261.
- [13] B. Eynard, Asymptotics of skew orthogonal polynomials, J. Phys A. 34 (2001) 7591, cond-mat/0012046.
- [14] B. Eynard, Recursion between Mumford volumes of moduli spaces, math-ph: arXiv:0706.4403.

- [15] B. Eynard, O. Marchal, Topological expansion of the Bethe ansatz, and non-commutative algebraic geometry , arXiv:0809.3367, JHEP03(2009)094
- [16] B. Eynard, N. Orantin, Invariants of algebraic curves and topological expansion, Communications in Number Theory and Physics, Vol 1, Number 2, p347-452, math-ph/0702045.
- [17] B. Eynard, A. Prats Ferrer, Topological expansion of the chain of matrices, math-ph: arxiv.0805.1368.
- [18] H.M. Farkas, I. Kra, "Riemann surfaces" 2nd edition, Springer Verlag, 1992.
- [19] J.D. Fay, "Theta functions on Riemann surfaces", Springer Verlag, 1973.
- [20] M. Gaudin, "La fonction d'onde de Bethe", ed. Masson, 1997.
- [21] M. L. Mehta, Random matrices (3e edition), Pure and Applied Mathematics Series 142, Elsevier (London - 2004), 688 pp. ISBN 0120884097.
- [22] M. Mulase, A.Waldron, Duality of Orthogonal and Symplectic Matrix Integrals and Quaternionic Feynman Graphs , arXiv:math-ph/0206011, Communications in mathematical physics Phys. 240, 553586 (2003).
- [23] H. Hezari, Complex zeros of eigenfunctions of 1D Schödinger operators, arxiv:math-ph/0703028.
- [24] A. Eremenko, A. Gabrielov, B. Shapiro, zeros of eigenfunctions of some anharmonic oscillators, Annales de l'Institut Fourier, 2008.
- [25] A. Eremenko, A. Gabrielov, B. Shapiro, high energy eigenfunctions of one-dimensional Schrödinger operators with polynomial potentials, arxiv:math-ph/0703049.

