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Enumeration of maps with self avoiding loops and the $\mathcal{O}(\mathfrak{n})$ model on random lattices of all topologies

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Abstract

We compute the generating functions of a $\mathcal{O}(\mathfrak{n})$ model (loop gas model) on a random lattice of any topology. On the disc and the cylinder, they were already known, and here we compute all the other topologies. We find that the generating functions (and the correlation functions of the lattice) obey the topological recursion, as usual in matrix models, i.e they are given by the symplectic invariants of their spectral curve.

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Introduction

The problem consists in counting random discrete surface, carrying random, self-avoiding, non intersecting loops, which can have $\mathfrak{n}$ possible colors. In statistical physics, this is called the $\mathcal{O}(\mathfrak{n})$ *model* (or *loop gas model*) on a random lattice, and it plays a very important role. The $\mathcal{O}(\mathfrak{n})$ model on a regular lattice is one of the exactly solvable models of Baxter [1]. Its limit $\mathfrak{n} \to 0$ (or more generally $\mathfrak{n} = \mathfrak{n}_0 + \delta\mathfrak{n}$ with $\delta\mathfrak{n} \to 0$) counts configurations of self-avoiding polymers in two dimensions [10]. On the random lattice, it is one of the basic toy models to understand the random geometry of discrete maps carrying structure.

Matrix models provide powerful techniques for the combinatorics of maps [12].The problem of counting random discrete surfaces without loops can be rephrased as a 1-matrix model [5, 11], and the 1-matrix model was solved in [13], by a "topological recursion" formula. This structure was later enhanced [17] to multi-matrix models, and it was used beyond matrix models to associate "symplectic invariants" to an arbitrary spectral curve.

Similarly, the $\mathcal{O}(\mathfrak{n})$ model was rewritten in 1989 as a matrix model [19, 25]. In [27], then [18], the phase diagram with respect to μ such that $\mathfrak{n} = -2\cos(\pi\mu)$ was established. Besides, the critical exponents for the geometry of large maps with the topology of a disc were found. In 1995, a closed set of loop equations for the generating function of the $\mathcal{O}(\mathfrak{n})$ maps was written [11, 15], and they were solved only for maps with the topology of a disc or a cylinder. Some sparse other cases were investigated before [24, 26]. So far, an efficient algorithm was lacking to compute the generating functions of the $\mathcal{O}(\mathfrak{n})$ model in higher topologies. It was not clear to which extent the method of [13] could be generalized. Indeed, the disc amplitude $\mathrm{W}_1^{(0)}(\mathrm{x})$, which ought to give the spectral curve, is not algebraic when μ is not rational, and the loop equations seem rather different from the 1-matrix model case.

In other words, do some symplectic invariants of [17] give the solution of the $\mathcal{O}(\mathfrak{n})$ model ? The answer is yes, with a slight deformation (of parameter $\mathfrak{n}$) of the notion of spectral curve. Thus, we obtain all correlation functions of the random lattice for any topology and any number of boundaries. We introduce the combinatorics of the $\mathcal{O}(\mathfrak{n})$ model and the matrix model representation in Section 1. Then, we give in Section 1.7 two derivations of the loop equations: one is straightforward from the matrix integral, the other is its combinatoric counterpart. We also gather here the main results of the article for the correlation functions. The solution of the loop equations is presented in Sections 2-4. Finally, we extend to the $\mathcal{O}(\mathfrak{n})$ model the properties coming along with the topological recursion of [17], in Section 5.

Besides, we recall the results concerning the limit of large maps (which were already known) in Section 6, and complete them for all topologies in the light of the double scaling limit and topological recursion.

1 The $\mathcal{O}(\mathfrak{n})$ model

1.1 Definition: gas of loops on a random surface

Roughly speaking, a random discrete surface[3] (also called "map" in combinatorics) is a graph drawn on an oriented connected surface, such that all faces are polygons. A random discrete surface is thus obtained by gluing polygons along their edges. In the $\mathcal{O}(\mathfrak{n})$ model, we glue the types of polygons shown below, i.e we have at our disposal empty polygons of size ≥ 3, and triangles carrying a piece of path of $\mathfrak{n}$ possible colors.

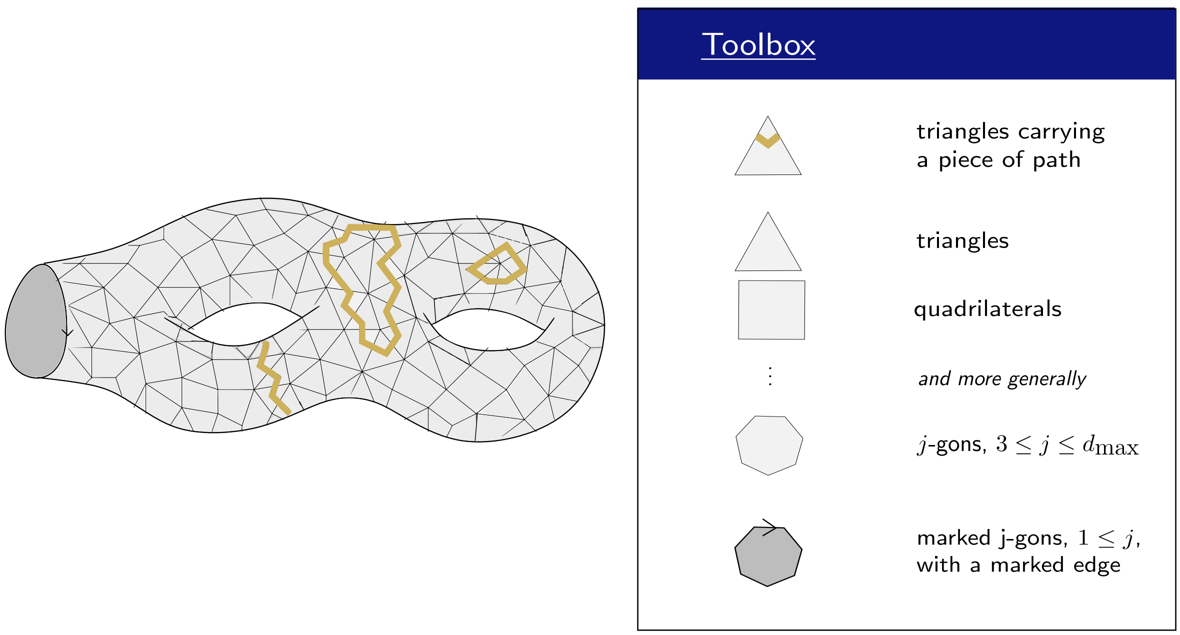


Figure 1: Example of a map in the $\mathcal{O}(\mathfrak{n})$ model: $\mathcal{M} \in \mathbb{M}_1^{(2)}$.

Each configuration is weighted by a "Boltzmann" weight:

$$t^v\, N^{2-2g}\, \frac{c^\ell\ \mathfrak{n}^{\#\text{loops}}}{\#\text{Aut}}\ \hat{t}_3^{n_3}\ \hat{t}_4^{n_4}\ \cdots\ \hat{t}_{d_{\max}}^{n_{d_{\max}}} \tag{1-1}$$

ℓ is the total length of all loops.

#loops is the number of closed paths, i.e. loops.

#Aut is the number of automorphisms.

[3]See [2] for a precise definition

n_j is the number of j-gons which do not carry a piece of path ($3 \leq j \leq d_{\max}$).

v is the number of vertices.

g is the genus of the surface on which the graph is embedded.

In addition, we may consider maps with k marked faces, with a marked edge on each marked face. We allow marked faces having more than one side. By abuse of language, we call those marked faces "boundaries" of the discrete surface. For $k=1$, one says that the maps are "rooted".

Definition 1.1 $\mathbb{M}_k^{(g)}(v)$ *is the set of connected oriented discrete surfaces of genus g, with v vertices, obtained by gluing unmarked j-gons of degree $3 \leq j \leq d_{max}$, k marked j-gons (of degree $1 \leq j$), and triangles carrying a piece of path, such that all the paths are loops (they are automatically self-avoiding).*

Proposition 1.1 $\mathbb{M}_k^{(g)}(v)$ *is a finite set.*

proof:

Consider a surface $\mathcal{M} \in \mathbb{M}_k^{(g)}(v)$, with n_j j-gons, ℓ triangles carrying a piece of path, and whose marked faces have lengths $l_1, \dots, l_k$. The total number of faces is:

$$\#\text{faces} = \sum_{j=3}^{d_{\max}} n_j + k + \ell \tag{1-2}$$

The total number of edges is half the number of half-edges i.e:

$$\#\text{edges} = \frac{1}{2}\Big(\sum_{j=3}^{d_{\max}} j n_j + \sum_{i=1}^{k} l_i + 3\ell\Big) \tag{1-3}$$

The Euler characteristics is:

$$\chi = 2 - 2g = v - \#\text{edges} + \#\text{faces} \tag{1-4}$$

and that gives:

$$2g - 2 + k + v = \frac{1}{2}\Big(\sum_{j=3}^{d_{\max}} (j-2) n_j + \sum_{i=1}^{k} l_i + \ell\Big) \tag{1-5}$$

Therefore for fixed g, k, v, we find that n_j, l_i and ℓ are bounded, and there can be only a finite number of such surfaces. $\square$

An interesting case to consider is the "fully packed" situation where all unmarked polygons are triangles carrying a piece of path, i.e $\hat{t}_j = 0$ for all $j \geq 3$.

1.2 Generating functions

Definition 1.2 *We define a formal generating function* $\mathrm{W}_k^{(g)}$ *which counts the elements of* $\mathbb{M}_k^{(g)}$*, as a power series in a formal variable* t*:*

$$\mathrm{W}_k^{(g)} = \sum_{v=1}^{\infty} t^v \sum_{\mathcal{M}\in\mathbb{M}_k^{(g)}(v)} \frac{(-c)^{\ell(\mathcal{M})}\ \mathfrak{n}^{\#\mathrm{loops}(\mathcal{M})}}{\#\mathrm{Aut}(\mathcal{M})} \frac{\hat{t}_3^{n_3(\mathcal{M})}\,\hat{t}_4^{n_4(\mathcal{M})}\dots\hat{t}_{d_{max}}^{n_{d_{max}}(\mathcal{M})}}{(\mathrm{x}_1-\frac{c}{2})^{l_1(\mathcal{M})+1}\dots(\mathrm{x}_k-\frac{c}{2})^{l_k(\mathcal{M})+1}} \tag{1-6}$$

where the i*-th marked face is a* $l_i(\mathcal{M})$*-gon.*

The coefficient of t^v *is a finite sum, and thus it is a polynomial in* $\mathfrak{n}$*, in the* $\hat{t}_j$*'s, and a rational fraction of the* x_i*'s with poles only at* $\mathrm{x}_i = \frac{c}{2}$*.*

We define by convention $\mathbb{M}_1^{(0)}(1)$: it contains only one element, which is a map having $n_j = 0$ $(3 \leq j \leq d_{\max})$, $l_1 = 0$, $\ell = 0$. This amounts to add to the generating function of rooted planar maps $(g = 0, k = 1)$ the linear term $\frac{t}{\mathrm{x}_1-c/2}$.

$$\mathrm{W}_1^{(0)} = \frac{t}{\mathrm{x}_1-\frac{c}{2}} + \sum_{v=2}^{\infty} t^v \sum_{\mathcal{M}\in\mathbb{M}_1^{(0)}(v)} (-c)^{\ell(\mathcal{M})}\ \mathfrak{n}^{\#\mathrm{loops}(\mathcal{M})} \frac{\hat{t}_3^{n_3(\mathcal{M})}\,\hat{t}_4^{n_4(\mathcal{M})}\dots\hat{t}_{d_{\max}}^{n_{d_{\max}}(\mathcal{M})}}{(\mathrm{x}_1-\frac{c}{2})^{l_1(\mathcal{M})+1}} \tag{1-7}$$

(Notice that $\#\mathrm{Aut}(\mathcal{M}) = 1$ and $v \geq 2$ when there is only one marked face with one marked edge).

The partition function without marked faces $(k = 0)$ is denoted F_g:

$$\mathrm{F}_g = \mathrm{W}_0^{(g)} = \sum_{v=1}^{\infty} t^v \sum_{\mathcal{M}\in\mathbb{M}_0^{(g)}(v)} \frac{(-c)^{\ell(\mathcal{M})}\ \mathfrak{n}^{\#\mathrm{loops}(\mathcal{M})}}{\#\mathrm{Aut}(\mathcal{M})}\ \hat{t}_3^{n_3(\mathcal{M})}\,\hat{t}_4^{n_4(\mathcal{M})}\dots\hat{t}_{d_{\max}}^{n_{d_{\max}}(\mathcal{M})} \tag{1-8}$$

Eventually we define:

$$\ln \mathrm{Z} = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g} \mathrm{F}_g \tag{1-9}$$

$$\mathrm{W}_k^{(g)} = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g-k} \mathrm{W}_k^{(g)} \tag{1-10}$$

Notice that, according to Eqn. 1-5, $2g-2+k+v \geq 0$. These equalities, like in [5], are to be understood as equalities between formal power series of t. To each power t^v, the sum over g in the right hand side ranges over a finite number of surfaces, with a maximal genus $g_{\max}(v)$. There is no problem of exchange of limits. Notice that, according to Eqn. 1-5, $2g-2+k+v \geq 0$, so that only nonnegative powers of t appear in $\mathrm{W}_k^{(g)}$. Z is by construction the generating function of $\mathcal{O}(\mathfrak{n})$ maps (possibly not connected) with the weight 1-1.

1.3 Matrix model

These partition functions can be written as a formal matrix integral like in [5]. The $\mathcal{O}(\mathfrak{n})$ matrix model was first introduced by I. Kostov [25]:

$$Z = \int_{\text{formal}} d\hat{M}\; dA_1 \cdots dA_{\mathfrak{n}}\; e^{-\frac{N}{t}\,\mathrm{Tr}\,\left[\hat{V}(\hat{M}) + (\hat{M}+\frac{c}{2})\sum_i A_i^2\right]} \tag{1-11}$$

where $\hat{M}$ is a "formal hermitian matrix" of size $N \times N$, $d\hat{M}$ and dA_i are the product of Lebesgue measures on the coefficients of the matrices, and where the potential $\hat{V}(\hat{M})$ is:

$$\begin{aligned} \hat{V}(\hat{M}) &= \frac{\hat{M}^2}{2} - \frac{\hat{t}_3\,\hat{M}^3}{3} - \frac{\hat{t}_4\,\hat{M}^4}{4} \cdots - \hat{t}_{d_{\max}} \frac{\hat{M}^{d_{\max}}}{d_{\max}} \\ &\stackrel{\text{def}}{=} \frac{\hat{M}^2}{2} - \hat{V}_{\geq 3}(\hat{M}) \end{aligned} \tag{1-12}$$

The notation $\int_{\text{formal}}$ actually means :

$$\begin{aligned} Z &= \frac{1}{\mathcal{Z}_0} \sum_{j=0}^{\infty} \frac{N^j t^{-j}}{j!} \int_{\mathbf{H}_N^{\mathfrak{n}+1}} d\hat{M}\; dA_1 \cdots dA_{\mathfrak{n}} \\ & \qquad e^{-\frac{N}{t}\,\mathrm{Tr}\,\left[\frac{\hat{M}^2}{2} + \frac{c}{2}\sum_{i=1}^{\mathfrak{n}} A_i^2\right]} \left(\mathrm{Tr}\,\left[-\hat{V}_{\geq 3}(\hat{\mathrm{M}}) - \sum_{i=1}^{\mathfrak{n}} \hat{M} A_i^2\right]\right)^j \end{aligned} \tag{1-13}$$

$$\begin{aligned} \text{where} \quad Z_0 &= \int_{\mathbf{H}_N^{\mathfrak{n}+1}} d\hat{M}\; dA_1 \cdots dA_{\mathfrak{n}}\; e^{-\frac{N}{2t}\,\mathrm{Tr}\,\left[\hat{M}^2 + c\sum_{i=1}^{\mathfrak{n}} A_i^2\right]} \\ &= 2^{\frac{(\mathfrak{n}+1)N}{2}}\, c^{-\frac{\mathfrak{n}N^2}{2}} \left(\frac{\pi t}{N}\right)^{\frac{(\mathfrak{n}+1)N^2}{2}} \end{aligned} \tag{1-14}$$

I.e to each power of t, we have a finite sum of polynomial moments of gaussian integrals over hermitian matrices of size $N \times N$. This model is defined for any $\mathfrak{n} \in \mathbf{R}$.

Z, $\ln Z$ and the various moments are well defined formal power series in t, and it was shown by I. Kostov [19] that they coincide with the $\mathcal{O}(\mathfrak{n})$ model generating functions. It is merely an application of Wick's theorem following the technique first introduced in [5]. With this notation we have:

$$W_k = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g-k} W_k^{(g)} = \left\langle \prod_{i=1}^{k} \mathrm{Tr}\, \frac{1}{x_i - \frac{c}{2} - \hat{M}} \right\rangle_{C,\text{formal}} \tag{1-15}$$

where the expectation value is taken with respect to the formal measure of Eqn. 1-11, where the subscript C means the cumulant, and:

$$\left(\frac{1}{x_i - \frac{c}{2} - \hat{M}}\right)_{\text{formal}} \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} \frac{1}{(x_i - \frac{c}{2})^{j+1}} \hat{M}^j \tag{1-16}$$

W_k and $W_k^{(g)}$ are called correlation functions in the language of statistical physics.

1.4 A quick look at the link with conformal field theories

▷ Polygonal large maps constructed by matrix models provide a discretization of Riemann surfaces. In the continuum limit, physically speaking, it is thought to define a theory of 2D quantum gravity: observables should be weighted sums over all possible surfaces endowed with a metric. The self avoiding paths present in the $\mathcal{O}(\mathfrak{n})$ model are in some sense matter fields added on the surface. The usual approach of 2D quantum gravity is Liouville field theory, which is a statistical model on the moduli space of Riemann surfaces endowed with a metric. Liouville theory can also be coupled to matter fields. Both approaches are conjecture to coincide and to enjoy conformal invariance.

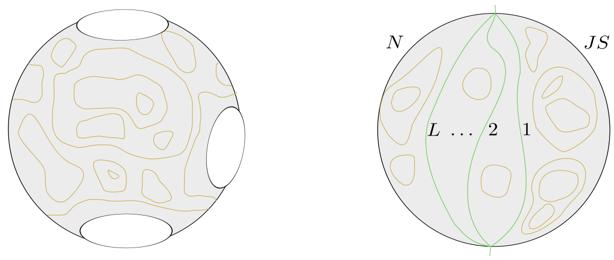
▷ "Double scaling limits" of matrix models are conjectured[4] to be conformal field theories (CFT) coupled to gravity [9]. This means in practice that the scaling exponents are given by Kac's table, that the double scaling limit (the definition is recalled in Section 6.3.6) W_k^* of the correlation functions satisfy PDE's on the spectral curve. These equations come from a representation of a Virasoro algebra with central charge c. In this correspondence, the double scaling limit of the $\mathcal{O}(\mathfrak{n})$ matrix model should be a conformal theory with central charge:

$$c = 1 - 6\left(\sqrt{\mathfrak{g}} - \frac{1}{\sqrt{\mathfrak{g}}}\right)^2 \tag{1-17}$$

where $\mathfrak{g}$ is such that $\mathfrak{n} = -2\cos(\pi\mathfrak{g})$. Various $\mathfrak{g}$ corresponding to the same $\mathfrak{n}$ define the various *phases* of the $\mathcal{O}(\mathfrak{n})$ model. Many studies on the $\mathcal{O}(\mathfrak{n})$ model with boundary operators [4, 27–29] support this proposal. See also the review [8] and the article [30].

▷ We bound ourselves to underline that, if these assumptions were correct, the critical exponents of the $\mathcal{O}(\mathfrak{n})$ model would be known for every genus from the KPZ relation [6, 7, 22]. KPZ is derived from Liouville conformal field theory and expresses how the critical exponents change when a CFT is coupled to gravity. Independently of CFT, rigorous results for the $\mathcal{O}(\mathfrak{n})$ model were derived by sheer analysis of the loop equations, and they agree with the CFT predictions. In the literature, the exponents and the scaling form of the following *genus* 0 functions of Fig. 2 were known.
In this article, we obtain (Section 6) the exponents and the scaling forms in all topologies of functions without loop insertion on the boundaries (Fig. 3).

[4] To our knowledge, this conjecture is currently proved only for the 1-matrix model, near an edge point where the equilibrium density of eigenvalues y(x) behaves as $x^{p/2}$ [3]



- $k \leq 3$ boundaries
without loop insertions
- loops have free color between 1 and n

- 1 boundary
- two times L consecutive insertions
of loops of free color between 1 and k
- Jacobsen-Saleur condition on the right side:
loops touching the boundary have free color between 1 and k
- Neumann condition on the left side:
loops touching the boundary have free color between 1 and n
- closed loops have free color between 1 and n

Figure 2:

- k boundaries without loop insertions
- genus g
- loops have free color between 1 and n

Figure 3:

1.5 Shift of the matrix model

It is more convenient to study the matrix model to perform the shift $M = \hat{M} + \frac{c}{2}$, and to rewrite:

$$\mathrm{Z} = \int_{\text{formal}} \mathrm{d}M\ \mathrm{d}A_1 \cdots \mathrm{d}A_{\mathfrak{n}}\ \mathrm{e}^{-\frac{N}{t}\,\mathrm{Tr}\left[\mathrm{V}(M)+\sum_{i=1}^{\mathfrak{n}} MA_i^2\right]} \tag{1-18}$$

where

$$\begin{aligned} \mathrm{V}(M) &= \hat{\mathrm{V}}(\hat{M}) \\ &= t_0 + \sum_{j=1}^{d_{\max}} \frac{t_j M^j}{j} \end{aligned} \tag{1-19}$$

With the shifted variable:

$$\mathrm{W}_k = \left\langle \prod_{i=1}^{k} \mathrm{Tr}\, \frac{1}{\mathrm{x}_i - M} \right\rangle_C \tag{1-20}$$

where the expectation value is taken with respect to the measure of Eqn. 1-18. If one wants to recover combinatorial quantities, one has to expand the correlation functions first as formal series in t, then as Laurent series in $\hat{\mathrm{x}}_i = \mathrm{x}_i - \frac{c}{2}$. In other words:

$$\mathrm{W}_k = \left\langle \prod_{i=1}^{k} \mathrm{Tr}\, \frac{1}{\hat{\mathrm{x}}_i - \hat{M}} \right\rangle_C \tag{1-21}$$

is defined as a formal series in t of Laurent series in $\hat{\mathrm{x}}$, whereas Eqn. 1-20 is already a resummation of the Laurent series.

The choice of the generating function of Eqn. 1-6 was done in order to coincide with the correlation function $\mathrm{W}_k^{(g)}$ of the matrix model in the form of Eqn. 1-18. The minus sign for c is common in statistical physics: we prefer to define matrix integrals which are convergent when c is positive, but for combinatorics, one needs formal matrix integrals, with $(-c)$ as formal parameter.

1.6 Loop insertion operator

Notice that, one may mark a face of size j, by taking a derivative with respect to t_j:

$$\mathrm{W}_{k+1}^{(g)}(\mathrm{x}_1, \dots, \mathrm{x}_k, \mathrm{x}_{k+1}) = \sum_{j \geq 0} \frac{1}{(\mathrm{x}_{k+1} - \frac{c}{2})^{j+1}} \frac{\partial \mathrm{W}_k^{(g)}(\mathrm{x}_1, \dots, \mathrm{x}_k)}{\partial \hat{t}_j} \tag{1-22}$$

Notice also that we have defined $\hat{t}_j$ only for $l \geq 3$ (this is the condition for Prop. 1.1 to hold), but it is easy to see that one may define $\hat{t}_2$, $\hat{t}_1$ and $\hat{t}_0$, and take the derivatives $\partial/\partial\hat{t}_0$ at $\hat{t}_0 = 0$, $\partial/\partial\hat{t}_1$ at $\hat{t}_1 = 0$, and $\partial/\partial\hat{t}_2$ at $\hat{t}_2 = -1$.

We define the "boundary insertion operator" as the formal series:

$$\frac{\partial}{\partial \hat{\mathrm{V}}(\hat{\mathrm{x}})} \stackrel{\mathrm{def}}{=} \sum_{j \geq 0} \frac{j}{\hat{\mathrm{x}}^{j+1}} \frac{\partial}{\partial \hat{t}_j} \tag{1-23}$$

In the new variable, $\mathrm{x} = \hat{\mathrm{x}} + \frac{c}{2}$, which is a formal series of the former one, we can do the resummation:

$$\begin{aligned} \frac{\partial}{\partial \hat{\mathrm{V}}(\hat{\mathrm{x}})} &= \sum_{j \geq 0} \frac{j}{\mathrm{x}^{j+1}} \frac{\partial}{\partial t_j} \\ &\stackrel{\mathrm{def}}{=} \frac{\partial}{\partial \mathrm{V}(\mathrm{x})} \end{aligned} \tag{1-24}$$

Then, we have:

$$\mathrm{W}_{k+1}^{(g)}(\mathrm{x}_1, \dots, \mathrm{x}_k, \mathrm{x}_{k+1}) = \frac{\partial}{\partial \mathrm{V}(\mathrm{x}_{k+1})} \mathrm{W}_k^{(g)}(\mathrm{x}_1, \dots, \mathrm{x}_k) \tag{1-25}$$

Again, this equality is to be understood order by order in powers of t, and to each power t^v, the sum over j is actually finite, since:

$$j \leq k + 1 + 2(2g - 2 + v) \qquad \text{from Eqn. 1-5.}$$

1.7 Loop equations

The loop equations provide relationships between the generating functions $\mathrm{W}_k^{(g)}$'s. They can be derived either by integration by parts in the matrix integral, or by direct combinatorial manipulations. For completeness we recall the two possibilities.

1.7.1 Derivation from the matrix integral

The derivation of loop equations from the matrix model is much faster than the bijective proof, but the reader not familiar with matrix integrals may prefer the bijective combinatorial proof in the next section.

The loop equations proceed from the invariance of an integral under an infinitesimal change of variable. This property is true for both formal integrals and convergent integrals, so we do not have to bother with variable $\hat{\mathrm{x}}$, and we can work directly with variable x. One has to compute the jacobian of the change of variable, which is typically a product of two terms, and the variation of the exponential term. The loop equation merely states that the jacobian cancels the variation of the exponential. The general method for computing jacobians of infinitesimal changes of matrix variables, is called "split and merge", and is exposed in many articles (in particular, [12]).

We indicate some changes of variables in the partition function 1-13, and we write directly the corresponding loop equations.

Let us define the following auxiliary functions (we write $I = \{\mathrm{x}_2, \ldots, \mathrm{x}_k\}$):

$$\mathrm{G}_k = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g-k} \mathrm{G}_k^{(g)} = \left\langle \mathrm{Tr}\, \frac{1}{\mathrm{x} - M} A_1^2 \prod_{\mathrm{x}_i \in I} \mathrm{Tr}\, \frac{1}{\mathrm{x}_i - M} \right\rangle_C \tag{1-26}$$

$$\tilde{\mathrm{G}}_k = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g-k} \tilde{\mathrm{G}}_k^{(g)} = \left\langle \mathrm{Tr}\, \frac{1}{\mathrm{x} - M} A_1 \frac{1}{\mathrm{x}' - M} A_1 \prod_{\mathrm{x}_i \in I} \mathrm{Tr}\, \frac{1}{\mathrm{x}_i - M} \right\rangle_C$$

$$\mathrm{P}_k = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g-k} \mathrm{P}_k^{(g)} = \left\langle \mathrm{Tr}\, \frac{\mathrm{V}'(\mathrm{x}) - \mathrm{V}'(M)}{\mathrm{x} - M} \prod_{\mathrm{x}_i \in I} \mathrm{Tr}\, \frac{1}{\mathrm{x}_i - M} \right\rangle_C$$

$\mathrm{P}_k^{(g)}$ is by construction the polynomial part of $\mathrm{V}'(\mathrm{x})\, \mathrm{W}_k^{(g)}(\mathrm{x}, \mathrm{x}_2, \ldots, \mathrm{x}_k)$ at large x:

$$\mathrm{P}_k^{(g)}(\mathrm{x}, \mathrm{x}_2, \ldots, \mathrm{x}_k) = \Big(\mathrm{V}'(\mathrm{x})\, \mathrm{W}_k^{(g)}(\mathrm{x}, \mathrm{x}_2, \ldots, \mathrm{x}_k)\Big)_+ \tag{1-27}$$

The invariance of the integral under the change of variable $M \to M + \epsilon \frac{1}{x-M}$, gives to the first order in ϵ:

$$\mathrm{W}_1(\mathrm{x})^2 + \mathrm{W}_2(\mathrm{x}, \mathrm{x}) = \frac{N}{t} \Big[\mathrm{V}'(\mathrm{x})\mathrm{W}_1(\mathrm{x}) - \mathrm{P}_1(\mathrm{x}) - \mathfrak{n}\, \mathrm{G}_1(\mathrm{x})\Big] \tag{1-28}$$

The invariance of the integral under the change of variable $A_1 \to A_1 + \epsilon \frac{1}{\mathrm{x}-M} A_1 \frac{1}{\mathrm{x}'-M}$, gives to the first order in ϵ:

$$\mathrm{W}_1(\mathrm{x})\mathrm{W}_1(\mathrm{x}') + \mathrm{W}_2(\mathrm{x},\mathrm{x}') = \frac{N}{t}\left[(\mathrm{x}+\mathrm{x}')\tilde{\mathrm{G}}_1(\mathrm{x},\mathrm{x}') - \mathrm{G}_1(\mathrm{x}) - \mathrm{G}_1(\mathrm{x}')\right] \tag{1-29}$$

Then, we combine those two equations by specializing to $\mathrm{x} = -\mathrm{x}'$:

$$\begin{aligned} & \mathrm{W}_2(\mathrm{x},\mathrm{x}) + \mathfrak{n}\mathrm{W}_2(\mathrm{x},-\mathrm{x}) + \mathrm{W}_2(-\mathrm{x},-\mathrm{x}) + \mathrm{W}_1(\mathrm{x})^2 + \mathfrak{n}\mathrm{W}_1(\mathrm{x},-\mathrm{x}) + \mathrm{W}_1(\mathrm{x},\mathrm{x}) \\ = & \frac{N}{t}\left[\mathrm{V}'(x)\mathrm{W}_1(\mathrm{x}) + \mathrm{V}'(-\mathrm{x})\mathrm{W}_1(\mathrm{x}) - \mathrm{P}_1(\mathrm{x}) - \mathrm{P}_1(-\mathrm{x})\right] \end{aligned} \tag{1-30}$$

Let us collect the powers of N. The highest one is N^2 and we obtain:

Theorem 1.1 ***Master loop equation.***

$$\begin{aligned} & \mathrm{W}_1^{(0)}(\mathrm{x})^2 + \mathfrak{n}\mathrm{W}_1^{(0)}(\mathrm{x})\mathrm{W}_1^{(0)}(-\mathrm{x}) + \mathrm{W}_1^{(0)}(-\mathrm{x})^2 \\ = & \mathrm{V}'(\mathrm{x})\mathrm{W}_1^{(0)}(\mathrm{x}) + \mathrm{V}'(-\mathrm{x})\mathrm{W}_1^{(0)}(\mathrm{x}) - \mathrm{P}_1^{(0)}(\mathrm{x}) - \mathrm{P}_1^{(0)}(-\mathrm{x}) \end{aligned} \tag{1-31}$$

We shall write the relations coming from N^{2-2g} term with $g > 0$ just below. Before, we notice that the relations 1-28 and 1-29 are true for any potential for which the quantities involved make sense. In particular, (t_0, t_1, t_2) is not restricted to $(0, 0, -1)$. Accordingly, one yields loop equations for any W_k by successive application of $\frac{\partial}{\partial \mathrm{V}}$. When we do this and collect the powers of N, we obtain the general loop equations:

Theorem 1.2 ***Loop equations.*** *We write $I = \{x_2, \dots, x_k\}$. For $k > 0$ and $g \geq 0$ (with the convention $\mathrm{W}_k^{(-1)} = 0$ for any k):*

$$\begin{aligned} & \mathrm{W}_{k+1}^{(g-1)}(\mathrm{x},\mathrm{x},I) + \mathfrak{n}\mathrm{W}_{k+1}^{(g-1)}(\mathrm{x},-\mathrm{x},I) + \mathrm{W}_{k+1}^{(g-1)}(-\mathrm{x},-\mathrm{x},I) \\ & + \sum_{\substack{J\subseteq I\ 0\leq h\leq g \\ (J,h)\neq(\emptyset,0),(I,g)}} \Big(\mathrm{W}_{|J|+1}^{(h)}(\mathrm{x},J)\mathrm{W}_{k-|J|}^{(g-h)}(\mathrm{x},I\setminus J) + \mathfrak{n}\mathrm{W}_{|J|+1}^{(h)}(\mathrm{x},J)\mathrm{W}_{k-|J|}^{(g-h)}(-\mathrm{x},I\setminus J) \\ & \qquad + \mathrm{W}_{|J|+1}^{(h)}(-\mathrm{x},J)\mathrm{W}_{k-|J|}^{(g-h)}(-\mathrm{x},I\setminus J)\Big) \\ & + \sum_{\mathrm{x}_i\in I}^{k} \frac{\partial}{\partial \mathrm{x}_i}\left(\frac{\mathrm{W}_k^{(g)}(\mathrm{x},I\setminus\{\mathrm{x}_i\}) - \mathrm{W}_k^{(g)}(I)}{\mathrm{x}-\mathrm{x}_i} - \frac{\mathrm{W}_k^{(g)}(-\mathrm{x},I\setminus\{\mathrm{x}_i\}) - \mathrm{W}_k^{(g)}(I)}{\mathrm{x}+\mathrm{x}_i}\right) \\ = & \mathrm{V}'(\mathrm{x})\mathrm{W}_k^{(g)}(\mathrm{x},I) + \mathrm{V}'(-\mathrm{x})\mathrm{W}_k^{(g)}(-\mathrm{x},I) - \mathrm{P}_k^{(g)}(\mathrm{x},I) - \mathrm{P}_k^{(g)}(-\mathrm{x},I) \end{aligned} \tag{1-32}$$

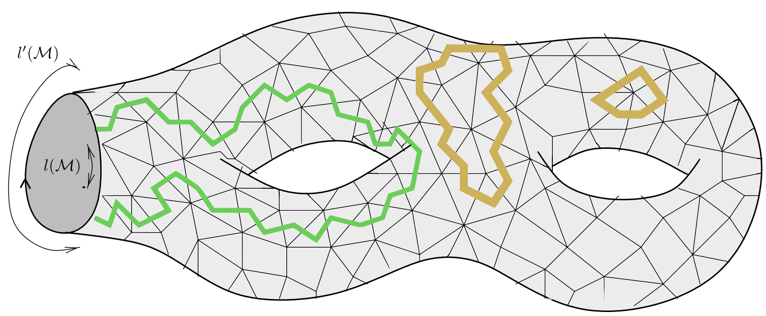
Eqn. 1-31 is just a specialization of Eqn. 1-32 for $k = 1$ and $g = 0$. Yet, as we will explain, it plays a special role and is often called the *master loop equation.* Let us notice that, if we set $\mathfrak{n} = 0$, we recover the loop equations of usual maps [12, 31, 32].

1.7.2 Derivation à la Tutte

Now, we give a bijective combinatorial proof of Theorem 1.2. Let us consider the auxiliary generating functions:

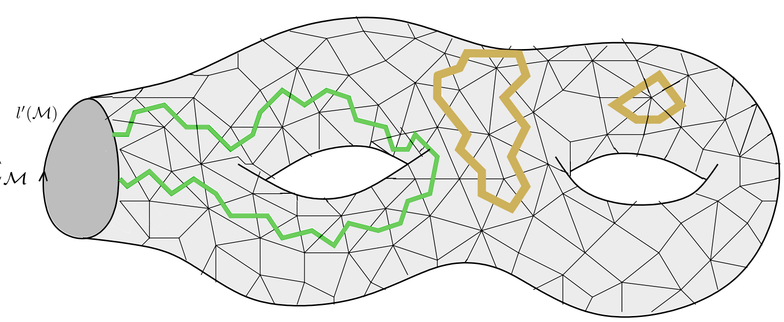
$$\tilde{\mathrm{G}}_k^{(g)} = \sum_{v=1}^{\infty} t^v \sum_{\mathcal{M}} \frac{(-c)^{\ell(\mathcal{M})}\mathfrak{n}^{\#\mathrm{loops}(\mathcal{M})}}{\#\mathrm{Aut}(\mathcal{M})} \frac{\hat{t}_3^{n_3(\mathcal{M})}\hat{t}_4^{n_4(\mathcal{M})}\cdots\hat{t}_{d_{\max}}^{n_{d_{\max}}(\mathcal{M})}}{(\mathrm{x}-\frac{c}{2})^{l(\mathcal{M})+1}(\mathrm{x}'-\frac{c}{2})^{l'(\mathcal{M})+1}\cdots(\mathrm{x}_k-\frac{c}{2})^{l_k(\mathcal{M})+1}} \tag{1-33}$$

where the sum ranges over the set $\mathbb{M}_{k-1|2}^{(g)}(v)$ of maps which carry a path of weight 1 (and not $\mathfrak{n}$) starting on the first boundary (of length $l(\mathcal{M})+l'(\mathcal{M})+2$), and ending on the same boundary, such that the distance between the starting point and ending point is arbitrary (and called $l(\mathcal{M})$). Pictorially:

$\tilde{\mathrm{G}}_k^{(g)} = \sum_{\mathcal{M}}$ 

$$\mathrm{G}_k^{(g)} = \sum_{v=0}^{\infty} t_v \sum_{\mathcal{M}} \frac{(-c)^{\ell(\mathcal{M})}\mathfrak{n}^{\#\mathrm{loops}(\mathcal{M})}}{\#\mathrm{Aut}(\mathcal{M})} \frac{\hat{t}_3^{n_3(\mathcal{M})}\hat{t}_4^{n_4(\mathcal{M})}\cdots\hat{t}_{d_{\max}}^{n_{d_{\max}}(\mathcal{M})}}{(\mathrm{x}'-\frac{c}{2})^{l'(\mathcal{M})+1}\cdots(\mathrm{x}_k-\frac{c}{2})^{l_k(\mathcal{M})+1}} \tag{1-34}$$

where the sum ranges over the subset $\widetilde{\mathbb{M}}_{k-1|2}^{(g)}(v) \subseteq \mathbb{M}_{k-1|2}^{(g)}(v)$ of maps with a path of weight 1 (and not $\mathfrak{n}$) starting on the first boundary (of length $l(\mathcal{M})+2$), and ending on an adjacent edge of the same boundary. Pictorially:

$\mathrm{G}_k^{(g)} = \sum_{\mathcal{M}}$ 

We see that $\mathrm{G}_k^{(g)}$ is a specialization of $\tilde{\mathrm{G}}_k^{(g)}$ when the distance between the two ends of the path is 1:

$$\tilde{\mathrm{G}}_k^{(g)}(\mathrm{x}') = \lim_{\mathrm{x}\to\infty} \mathrm{x}\,\tilde{\mathrm{G}}_k^{(g)}(\mathrm{x},\mathrm{x}') \qquad (1\text{-}35)$$

The combinatorial derivation proceeds like Tutte's equations [31], [32], i.e by recursion on the number of edges. We give two examples in Figs. 4-5.

$\triangleright$ Consider $\mathrm{W}_1^{(0)}$, which counts rooted planar maps: there is a marked polygon attached on the border of each drawing. If we remove the marked edge in a configuration counted by $\mathrm{W}_1^{(0)}$, we count $\left[\left(\mathrm{x}-\frac{c}{2}\right)\mathrm{W}_1^{(0)}(\mathrm{x})-t\right]$ (taking care of the term we added in the definition of $\mathrm{W}_1^{(0)}$). Three cases occur for the removal: the face on the other side of the marked edge could be an unmarked polygon, it could carry a piece of path, or it could be the marked face itself. Thus:

$$\left(\mathrm{x}-\frac{c}{2}\right)\mathrm{W}_1^{(0)}(\mathrm{x})-t=\left[\hat{\mathrm{V}}_{\geq 3}\left(\mathrm{x}-\frac{c}{2}\right)\mathrm{W}_1^{(0)}(\mathrm{x})\right]_-+\mathfrak{n}\mathrm{G}_1^{(0)}(\mathrm{x})+\mathrm{W}_1^{(0)}(\mathrm{x})^2 \qquad (1\text{-}36)$$

where $(\cdots)_-$ means the negative part of the Laurent expansion in $\mathrm{x}-\frac{c}{2}$. We see that the whole $\hat{\mathrm{V}}'\left(\mathrm{x}-\frac{c}{2}\right)=\mathrm{V}'(\mathrm{x})$ is restored in this equation: the quadratic term do not play a special role and the loop equation has a nice form because we added a term for $v=1$ to $\mathrm{W}_1^{(0)}$.

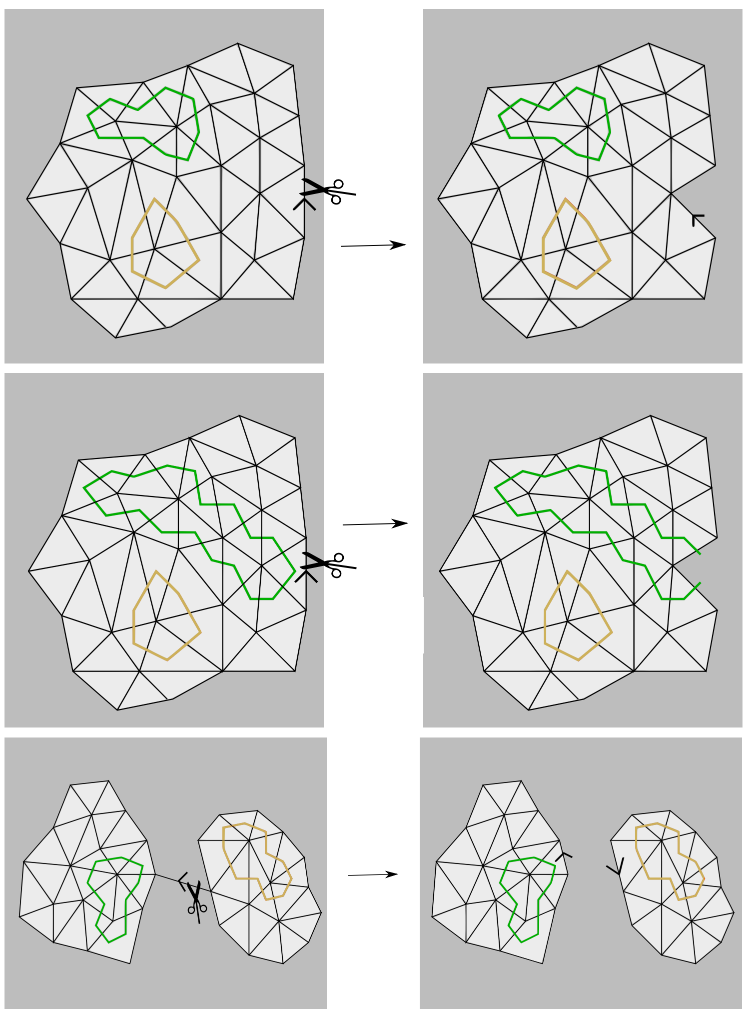


Figure 4: Illustration of Eqn. 1-36

The polynomial (in the variable $\hat{x}$ or x):

$$\mathrm{P}_1^{(0)}(\mathrm{x}) = \left\langle \frac{\hat{\mathrm{V}}'(\hat{\mathrm{x}}) - \hat{\mathrm{V}}'(\hat{M})}{\hat{\mathrm{x}} - \hat{M}} \right\rangle^{(0)} = \left\langle \frac{\mathrm{V}'(\mathrm{x}) - \mathrm{V}'(M)}{\mathrm{x} - M} \right\rangle^{(0)}$$

is precisely the nonnegative part of the Laurent expansion of $\mathrm{V}'(\mathrm{x})\mathrm{W}_1^{(0)}(\mathrm{x})$. Hence Eqn. 1-28:

$$\mathrm{W}_1^{(0)}(\mathrm{x})^2 = \mathrm{V}'(\mathrm{x})\mathrm{W}_1^{(0)}(\mathrm{x}) - \mathrm{P}_1^{(0)}(\mathrm{x}) - \mathfrak{n}\mathrm{G}_1^{(0)}(\mathrm{x})$$

▷ As a second example, consider a configuration counted in $\tilde{\mathrm{G}}_1^{(0)}(\mathrm{x},\mathrm{x}')$. If we remove the triangle of the border where the path starts, we are counting $\left[(\mathrm{x}+\mathrm{x}'-c)\,\tilde{\mathrm{G}}_1^{(0)}(\mathrm{x},\mathrm{x}') - \mathrm{G}_1^{(0)}(\mathrm{x}) - \mathrm{G}_1^{(0)}(\mathrm{x}')\right]$. Indeed, either l or l' is shortened by one, but we have to take care of the degenerate cases $\mathrm{G}_1^{(0)}(\mathrm{x})$ and $\mathrm{G}_1^{(0)}(\mathrm{x}')$ where l' (resp. l) shrinks to 0. One find two possibilities for this removal: either the path was of length 0 or not. Thus:

$$\left[(\mathrm{x}+\mathrm{x}'-c)\tilde{\mathrm{G}}_1^{(0)}(\mathrm{x},\mathrm{x}') - \mathrm{G}_1^{(0)}(\mathrm{x}) - \mathrm{G}_1^{(0)}(\mathrm{x}')\right] = \mathrm{W}_1^{(0)}(\mathrm{x})\mathrm{W}_1^{(0)}(\mathrm{x}') - c\tilde{\mathrm{G}}_1^{(0)}(\mathrm{x},\mathrm{x}') \tag{1-37}$$

Hence Eqn. 1-29.

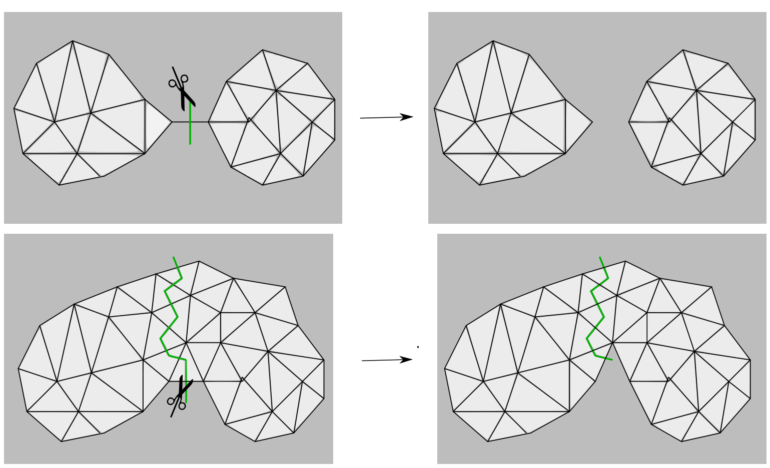


Figure 5: Illustration of Eqn. 1-37

One can derive the other loop equations of Theorem 1.2 for every $\mathrm{W}_k^{(g)}$ in a similar way.

1.8 Analyticity properties

To each order t^v, the sum over $\mathbb{M}_k^{(g)}(v)$ in $\mathrm{W}_k^{(g)}$ is a finite sum, and in particular, the coefficient of t^v in $\mathrm{W}_k^{(g)}$ is a rational fraction of each x_i, with poles only at $\mathrm{x}_i = \frac{c}{2}$, of maximal degree $4g-4+k+2v$. In Appendix C, we prove that this implies the following analytical properties for the $\mathrm{W}_k^{(g)}$'s:

Lemma 1.1 ***1-cut lemma*** *There exists $\rho_0 > 0$ (depending only on the degree d_{max} of the polynomial $\hat{\mathrm{V}}$, $\hat{t}_3, \dots, \hat{t}_{d_{\max}}, \mathfrak{n}$ and c, and a priori on k, g) such that, for all $r > 0$:*

$I = (\mathrm{x}_1, \dots, \mathrm{x}_k) \in \left(\mathbf{C} \setminus \mathcal{D}(\frac{c}{2}\,; r)\right)^k \Rightarrow$ the formal series in t, $\mathrm{W}_k^{(g)}(I)$ has a radius of convergence $\geq \frac{r}{\rho_0}$.

Accordingly, if we hold $k-1$ variables $J = (\mathrm{x}_2, \dots, \mathrm{x}_k)$ fixed, at values different from $\frac{c}{2}$: for $r > 0$ small enough, for all $t \in \mathcal{D}\left(0\,; \frac{r}{\rho_0}\right)$, $\mathrm{W}_k^{(g)}(\mathrm{x}, J)$ is holomorphic for $\mathrm{x} \in \mathbf{C} \setminus \mathcal{D}(\frac{c}{2}\,; r)$, $\mathcal{C}^\infty$ in t, and $\mathcal{D}(\frac{c}{2}\,; r) \cap \overline{\mathcal{D}}(-\frac{c}{2}\,; r) = \emptyset$.

More precisely, there exists $t_0 > 0$ (depending only on $d_{\max}, \hat{t}_3, \dots, \hat{t}_{d_{\max}}, \mathfrak{n}$ and c) and two formal series $a(t)$, $b(t)$ in $\sqrt{t}$, whose radius of convergence in $\sqrt{t}$ is greater than $\sqrt{t_0}$ (non zero), such that (we write $\gamma(t)$ the segment $[a(t), b(t)]$):

- ▷ *$a(t) = \frac{c}{2} - c_1\sqrt{t} + O(\sqrt{|t|})$ and $b(t) = \frac{c}{2} + c_1\sqrt{t} + O(\sqrt{|t|})$ for some $c_1 \in \mathbf{C}$.*
- ▷ *$\mathrm{W}_1^{(0)}(\mathrm{x})$ is absolutely convergent for $t \in \mathcal{D}(0\,; t_0)$, holomorphic on $\mathbf{C} \setminus \gamma(t)$, and has a discontinuity on $\gamma(t)$. On a neighborhood of $\gamma(t)$, it takes the form $\mathrm{h}(\mathrm{x}) \times \sqrt{(\mathrm{x}-a)(\mathrm{x}-b)}$, where h is meromorphic in a neighborhood of $\gamma(t)$, has no pole except maybe in $a(t)$ and $b(t)$, and has no zeroes on $\gamma(t)$. We call this behavior a square root discontinuity.*
- ▷ *For all the other k, g's, $\mathrm{W}_k^{(g)}(\mathrm{x}_1, \dots, \mathrm{x}_k)$ has a square root discontinuity in each variable on $\gamma(t)$, and is holomorphic on $\mathbf{C} \setminus \gamma(t)$.*

proof:

This Lemma is quite technical, and we give its derivation in Appendix C. As a brief sketch of the proof, let us say that, by a very crude bound on $\#\mathbb{M}_1^{(0)}(v)$, we first prove that $\mathrm{W}_1^{(0)}$ is convergent in the domain $\mathbf{C} \setminus \mathcal{D}\left(\frac{c}{2}\,; \frac{|c|}{2}\right)$ when $|t| < t^*$ for some t^*, and thus $\mathrm{W}_1^{(0)}(\mathrm{x})$ can have singularities only in a small disc centered in c, of radius smaller than $\frac{|c|}{2}$. This implies that, $\mathrm{W}_1^{(0)}(-\mathrm{x})$ is analytical for x in this very disc. Then, from the loop equation 1-31, we see that $\mathrm{W}_1^{(0)}$ can only have (and must have) square-root discontinuity in the disc, at points $\mathrm{x} = a(t)$ and $\mathrm{x} = b(t)$. Eventually, the series $a(t)$ and $b(t)$ are determined by Eqn. 1-31. □

This property is called the 1-cut assumption in physics (although it is not an assumption here), and is closely related to Brown's lemma in combinatorics. It plays a very important role in the solution of the loop equations.

1.9 Remark: convergent matrix integrals

The 1-cut assumption holds for formal matrix integrals, i.e generating functions of the $O(\mathfrak{n})$ model configurations.

However, one could be interested in studying the matrix integral of Eqn. 1-18, not as a formal matrix integral, but as a genuine convergent integral. In this case, a 1-cut lemma can hold or not, depending on the choice of $\mathrm{V}(M)$, and in fact on the choice of the integration domain for the eigenvalues of M. In some sense, it holds if the integration domain is a "steepest descent" integration path for the potential V, but those considerations are beyond the scope of our article. When this is the case, for $|t|$ small enough according to the bound of the 1-cut lemma, $\left(\frac{t^2}{N^2}\right)^{2g-2+k} \mathrm{W}_k^{(g)}(t)$ is also the g-th order asymptotic of $\mathrm{W}_k(t|N)$ when $N \to \infty$ [].

In this article, we shall consider only the situation (realized in combinatorics) where the 1-cut property holds, and we leave the "multi-cut case" to a further work.

1.10 Main formulas for the 1-cut case

We sum up here the description of the correlation function in all topologies. The results for $\mathrm{W}_1^{(0)}$ [18] and $\mathrm{W}_2^{(0)}$ [15] were already known. We present a proof for them in the framework of our article in Sections 3.1 and 3.5: it allows us to introduce the basic notions with the help of which we find the general $\mathrm{W}_k^{(g)}$'s.

Theorem 1.3

$$\mathrm{W}_1^{(0)}(\mathrm{x}_0) = \frac{1}{2i\pi} \oint_{[a,b]} \mathrm{dx} \frac{\mathrm{x}\mathrm{V}'(\mathrm{x})}{x_0^2 - x^2} \left(\frac{\mathrm{f}_\mu(\mathrm{x}_0)}{\mathrm{R}_\mu(\mathrm{x}_0^2)} \mathrm{f}_\mu(\mathrm{x}) + \frac{\widehat{\mathrm{f}}_\mu(\mathrm{x}_0)}{\widehat{\mathrm{R}}_\mu(\mathrm{x}_0^2)} \widehat{\mathrm{f}}_\mu(\mathrm{x}) \right)$$

f_μ and $\hat{\mathrm{f}}_\mu$ are functions depending only on the endpoints a, b. R_μ and $\widehat{\mathrm{R}}_\mu$ are even rational functions constructed with them. a and b themselves are solutions in power series of $\sqrt{t}$ (for small t) of:

$$\begin{aligned} \operatorname*{Res}_{\mathrm{x}\to\infty} \left(\mathrm{dx}\,\mathrm{x}\mathrm{V}'(\mathrm{x})\mathrm{f}_\mu(\mathrm{x})\right) &= (-2+\mathfrak{n})t \\ \operatorname*{Res}_{\mathrm{x}\to\infty} \left(\mathrm{dx}\,\mathrm{x}\mathrm{V}'(\mathrm{x})\widehat{\mathrm{f}}_\mu(\mathrm{x})\right) &= (2+\mathfrak{n}) \operatorname*{Res}_{\mathrm{x}\to\infty} \left(\mathrm{dx}\,\mathrm{x}\mathrm{W}_1^{(0)}(\mathrm{x})\right) \end{aligned}$$

These conditions can also be recast as integrals over a closed path around $[a, b]$.

Theorem 1.4 *With the same special functions, we have:*

$$\mathrm{W}_2^{(0)}(\mathrm{x}_0,\mathrm{x}) = \frac{1}{4-\mathfrak{n}^2}\left(-\frac{2}{(\mathrm{x}_0-\mathrm{x})^2}+\frac{\mathfrak{n}}{(\mathrm{x}_0+\mathrm{x})^2}\right)+\frac{\mathrm{d}}{\mathrm{dx}}\left(\mathrm{H}(\mathrm{x}_0,\mathrm{x})\right)$$

where

$$\begin{aligned}\mathrm{H}(\mathrm{x}_0,\mathrm{x}) &= \frac{1}{2-\mathfrak{n}}\frac{\mathrm{x}\sigma(\mathrm{x})^2}{(\mathrm{x}_0^2-\mathrm{x}^2)(\mathrm{x}^2-e_\mu^2)}\mathrm{f}_\mu(\mathrm{x}_0)\mathrm{f}_\mu(\mathrm{x})+\frac{1}{2-\mathfrak{n}}\mathrm{f}_\mu(\mathrm{x}_0)\\ &+ \frac{ie_\mu\sigma(e_\mu)}{\sqrt{4-\mathfrak{n}^2}}\frac{\mathrm{x}}{(\mathrm{x}_0^2-e_\mu^2)(\mathrm{x}^2-e_\mu^2)}\mathrm{f}_\mu(\mathrm{x}_0)\widehat{\mathrm{f}}_\mu(\mathrm{x})\\ &+ \frac{1}{2+\mathfrak{n}}\frac{\mathrm{x}_0^2\mathrm{x}}{(\mathrm{x}_0^2-\mathrm{x}^2)(\mathrm{x}_0^2-e_\mu^2)}\widehat{\mathrm{f}}_\mu(\mathrm{x}_0)\widehat{\mathrm{f}}_\mu(\mathrm{x})\end{aligned}$$

We checked that the resulting expression for $\mathrm{W}_2^{(0)}(\mathrm{x}_0,\mathrm{x})$ *is symmetric in* x_0 *and* x.

The new result is that the other $\mathrm{W}_k^{(g)}$'s for $2g-2+k>0$ can be computed recursively by a residue formula:

Theorem 1.5

$$W_k^{(g)}(u_0,I) =$$
$$\operatorname*{Res}_{u\to \mathrm{u}(a),\mathrm{u}(b)} K(u_0,u)\left[\text{"}W_{k+1}^{(g-1)}(u,\overline{u},I)\text{"}+\sum_{J,h}{}' \overline{W}_{|J|+1}^{(h)}(u,J)\overline{W}_{k-|J|}^{(g-h)}(\overline{u},I\setminus J)\right]s(u)s(\overline{u})$$

- ▷ $x(u)=a\,\mathrm{dn}_{\sqrt{1-a^2/b^2}}\left(\frac{2u}{K'}\right)$ *is a change of variable, detailed and motivated in Sections 2.1-2.3. Locally at* $a_i\in\{a,b\}$, $u\propto\sqrt{\mathrm{x}-a_i}$.
- ▷ $\overline{u}$ *is defined locally :* $\overline{u}=2\tau-u$ *around* $u=\tau$, $\overline{u}=2\tau-u=1$ *around* $u=\tau+\frac{1}{2}$.
- ▷ $W_k^{(g)}(u_1,\ldots,u_k)=\mathrm{W}_k^{(g)}(x(u_1),\ldots,x(u_k))$ *are the correlation functions read in this new variable.*
- ▷ $\overline{W}_k^{(g)}(\mathrm{x}_1,\mathrm{x}_2,\ldots)=$

$$\frac{\delta_{k,1}\delta_{g,0}}{4-\mathfrak{n}^2}\left(2\mathrm{V}'(\mathrm{x}_1)-\mathfrak{n}\mathrm{V}'(-\mathrm{x}_1)\right)+\frac{\delta_{k,2}\delta_{g,0}}{4-\mathfrak{n}^2}\left(-\frac{2}{(\mathrm{x}_1-\mathrm{x}_2)^2}+\frac{\mathfrak{n}}{(\mathrm{x}_1+\mathrm{x}_2)^2}\right)$$
$$+\mathrm{W}_k^{(g)}(\mathrm{x}_1,\mathrm{x}_2,\ldots)$$

- ▷ $y(u)=W_1^{(0)}(u)-W_1^{(0)}(\overline{u})$. *When* $x(u)\in[a,b]$, *it coincides with* $(-2i\pi t)\times$ *(large* N *limit of the density of eigenvalues of* M*), see Appendix B.*
- ▷ $s(u)=\frac{\mathrm{d}x}{\mathrm{d}u}$.

▷ *The recursion kernel is:*

$$\mathcal{K}(u_0,u) = -\frac{1}{2}\frac{\int_{\overline{u}}^{u} \mathrm{d}u' s(u')\overline{W}_2^{(0)}(u_0,u')}{s(u)y(u)}$$

▷ $\sum'_{(J,h)}$ *is a sum over* $J \subseteq I, 0 \le h \le g$*, excluding* $(J,h) = (\emptyset,0)$ *and* (I,g)*.*

The method to compute the F_g's is explained in Sections 5.6-5.8.

2 The linear equation

We write $\mathfrak{n} = -2\cos(\pi\mu)$ and we assume $\mu \in\ ]0,1[\setminus\{0\}$. $\mu \in\ ]0,1[$ is in bijection with $\mathfrak{n} \in]-2,2[$. We discuss very briefly the cases $\mathfrak{n} = \pm 2$ in Section 6.4.

2.1 Saddle point equation

Due to the analytical structure of $\mathrm{W}_1^{(0)}$ (square root discontinuity with end points $a(t)$,$b(t)$), we can transform the non-linear loop equation Eqn. 1-31 into a linear one. The latter was originally called "saddle point equation" because it coincides with the saddle point approximation for the density of eigenvalues in the random matrix model approach (see for example, [18]).

Proposition 2.1 *"Saddle point equation"*

$$\boxed{\forall \mathrm{x} \in [a(t),b(t)] \qquad \mathrm{W}_1^{(0)}(\mathrm{x}+i\epsilon) + \mathfrak{n}\mathrm{W}_1^{(0)}(-\mathrm{x}) + \mathrm{W}_1^{(0)}(\mathrm{x}-i\epsilon) \underset{\epsilon\to 0}{=} \mathrm{V}'(\mathrm{x})} \tag{2-1}$$

proof:

We have the master loop equation 1-31:

$$\begin{aligned} & \mathrm{W}_1^{(0)}(\mathrm{x})^2 + \mathfrak{n}\mathrm{W}_1^{(0)}(\mathrm{x})\mathrm{W}_1^{(0)}(-\mathrm{x}) + \mathrm{W}_1^{(0)}(-\mathrm{x})^2 \\ = \ & \mathrm{V}'(\mathrm{x})\mathrm{W}_1^{(0)}(\mathrm{x}) + \mathrm{V}'(-\mathrm{x})\mathrm{W}_1^{(0)}(-\mathrm{x}) - \mathrm{P}_1^{(0)}(\mathrm{x}) - \mathrm{P}_1^{(0)}(-\mathrm{x}) \end{aligned}$$

where $\mathrm{P}_1^{(0)}(\mathrm{x})$ is a polynomial in x of degree $(\deg \mathrm{V} - 2)$. Because of the 1-cut lemma:

$$\forall \mathrm{x} \in [a(t),b(t)] \qquad \mathrm{W}_1^{(0)}(-\mathrm{x}+i\epsilon) - \mathrm{W}_1^{(0)}(-\mathrm{x}-i\epsilon) \underset{\epsilon\to 0}{=} 0$$

Besides, let us define:

$$\begin{aligned} & 2\mathrm{P}_e(\mathrm{x}) \\ = \ & \mathrm{P}_1^{(0)}(\mathrm{x}) + \mathrm{P}_1^{(0)}(-\mathrm{x}) \\ = \ & \mathrm{W}_1^{(0)}(\mathrm{x})^2 + \mathfrak{n}\mathrm{W}_1^{(0)}(\mathrm{x})\mathrm{W}_1^{(0)}(-\mathrm{x}) + \mathrm{W}_1^{(0)}(-\mathrm{x})^2 - \mathrm{V}'(\mathrm{x})\mathrm{W}_1^{(0)}(\mathrm{x}) - \mathrm{V}'(-\mathrm{x})\mathrm{W}_1^{(0)}(-\mathrm{x}) \end{aligned}$$

$\mathrm{P}_e(\mathrm{x})$ is a polynomial, so:

$$\forall \mathrm{x} \in [a(t), b(t)] \qquad \mathrm{P}_e(\mathrm{x}+i\epsilon) - \mathrm{P}_e(\mathrm{x}-i\epsilon) \underset{\epsilon\to 0}{=} 0$$

This equation factorizes into:

$$\left(\mathrm{W}_1^{(0)}(\mathrm{x}+i\epsilon) - \mathrm{W}_1^{(0)}(\mathrm{x}-i\epsilon)\right) \left[\mathrm{W}_1^{(0)}(\mathrm{x}+i\epsilon) + \mathfrak{n}\mathrm{W}_1^{(0)}(-\mathrm{x}) + \mathrm{W}_1^{(0)}(\mathrm{x}-i\epsilon) - \mathrm{V}'(\mathrm{x})\right] \underset{\epsilon\to 0}{=} 0$$

Since on $\mathrm{x} \in [a(t), b(t)]$ we have $\mathrm{W}_1^{(0)}(\mathrm{x}+i\epsilon) \neq \mathrm{W}_1^{(0)}(\mathrm{x}-i\epsilon)$ in the limit $\epsilon \to 0$, we find Eqn. 2-1. $\square$

Eqn. 2-1 is linear, provided that $a(t)$ and $b(t)$ are known. The non-linearity is hidden in the determination of $a(t)$ and $b(t)$ as we shall see. Given a segment $[a, b]$ of the positive real line, we shall study the general 1-cut solutions of the homogeneous linear equation:

$$\forall \mathrm{x} \in [a, b] \qquad \mathrm{W}(\mathrm{x}+i\epsilon) + \mathfrak{n}\mathrm{W}(-\mathrm{x}) + \mathrm{W}(\mathrm{x}-i\epsilon) \underset{\epsilon\to 0}{=} 0 \qquad (2\text{-}2)$$

The extension to an arbitrary path $[a, b]$ (maybe not connected) in the complex plane presents no difficulty, but is not needed for combinatorics.

2.2 Algebraic geometry construction

We look for a solution of Eqn. 2-1, with only one cut $[a, b]$, and in particular which is analytical on $[-b, -a]$. Since our equation involves both $\mathrm{x} \in [a, b]$ and $-\mathrm{x} \in [-b, -a]$, it is convenient to find a conformal mapping between the complex plane with two cuts $[a, b] \cup [-b, -a]$ and the hyperelliptical curve $\mathcal{S} : \sigma^2 = (\mathrm{x}^2 - a^2)(\mathrm{x}^2 - b^2)$.

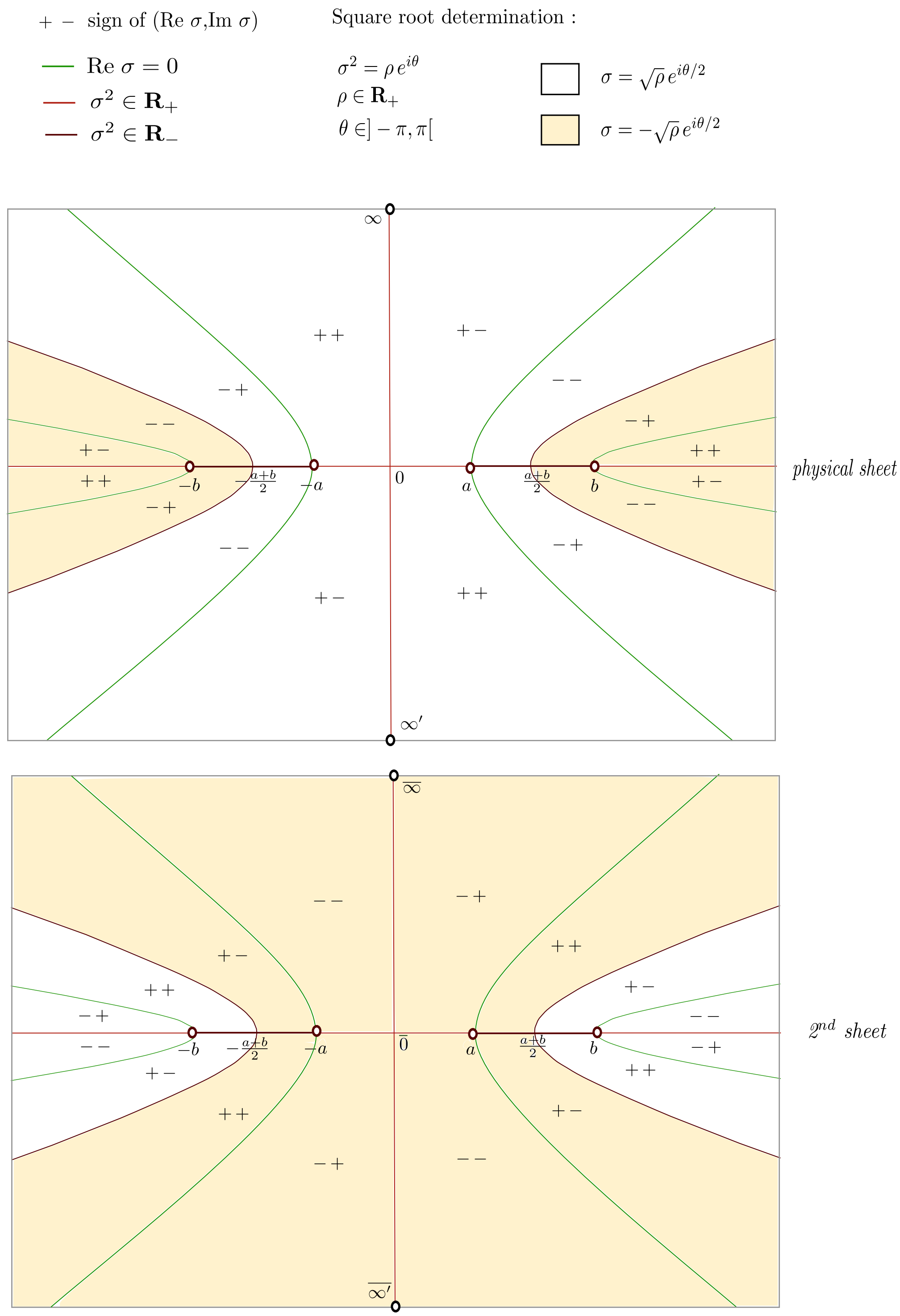


Figure 6: $\mathcal{S} : \sigma^2 = (\mathrm{x}^2 - a^2)(\mathrm{x}^2 - b^2)$ has two sheets, corresponding to opposite square roots, and we call one of them the *physical sheet*. The determination (white or yellow) of the square root changes when one crosses the lines where $\sigma^2 \in \mathbf{R}_-$, so that σ is an analytical function of x on each sheet, outside $[-b, -a] \cup [a, b]$. We have indicated the corresponding signs for $\mathrm{Re}(\sigma)$ and $\mathrm{Im}(\sigma)$ on the two sheets.

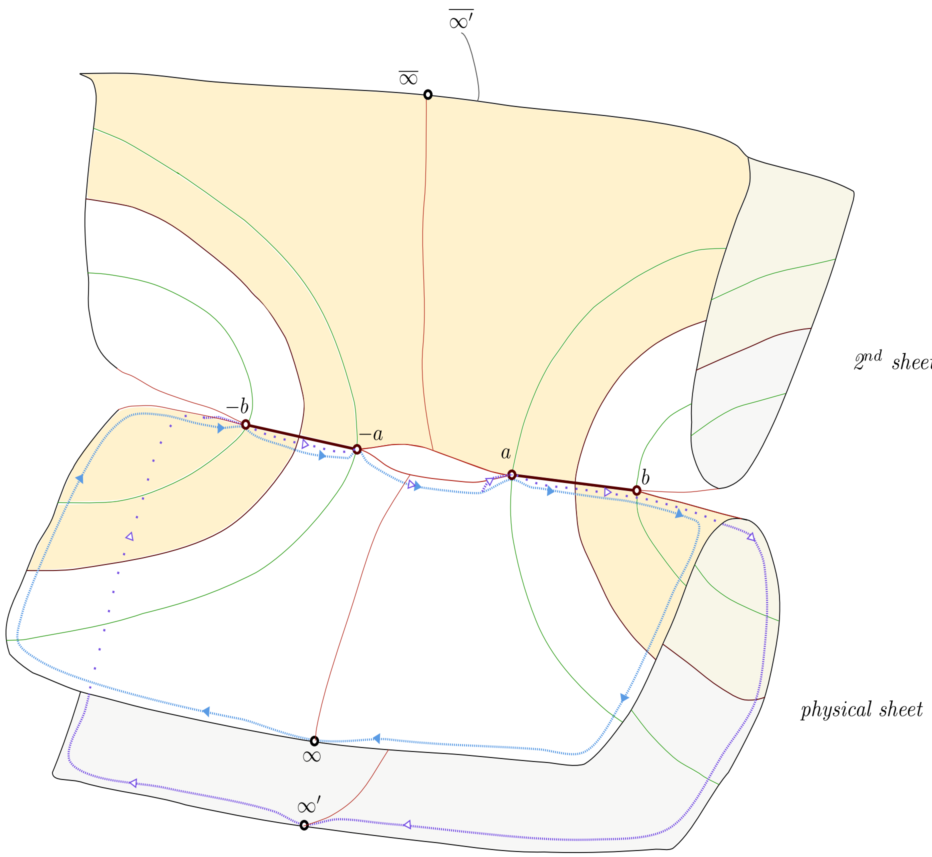


Figure 7:

$\mathcal{S}$ is topologically a torus. So, the space of holomorphic differential form is of complex dimension 1 and is generated by $\mathrm{d}x/\sigma$. To parametrize the surface, we define

$$\mathrm{u}(\mathrm{x}) = \frac{ib}{2K'} \int_{-a}^{\mathrm{x}} \frac{\mathrm{dx}'}{\sigma(\mathrm{x}')}$$

which is path dependant. We fix $K' \in \mathbf{R}_+^*$ such that $\mathrm{u}(-b) = \frac{1}{2}$ along the blue path. Then, $\mathrm{u}(a) = \tau$, shown in the next paragraph to be half of the modulus of the torus, lies in $i\mathbf{R}_+$ for $a, b \in \mathbf{R}_+$, $a < b$. We have drawn the paths followed on the first sheet only: they follow the real line, the blue one on the side $\mathrm{Im}\,\mathrm{x} > 0$, the purple one on the side $\mathrm{Im}\,\mathrm{x} < 0$. Because the square root on the second sheet is opposite to its determination on the physical sheet, the analogue integration paths on the second sheet lead to opposite u for the same value of x.

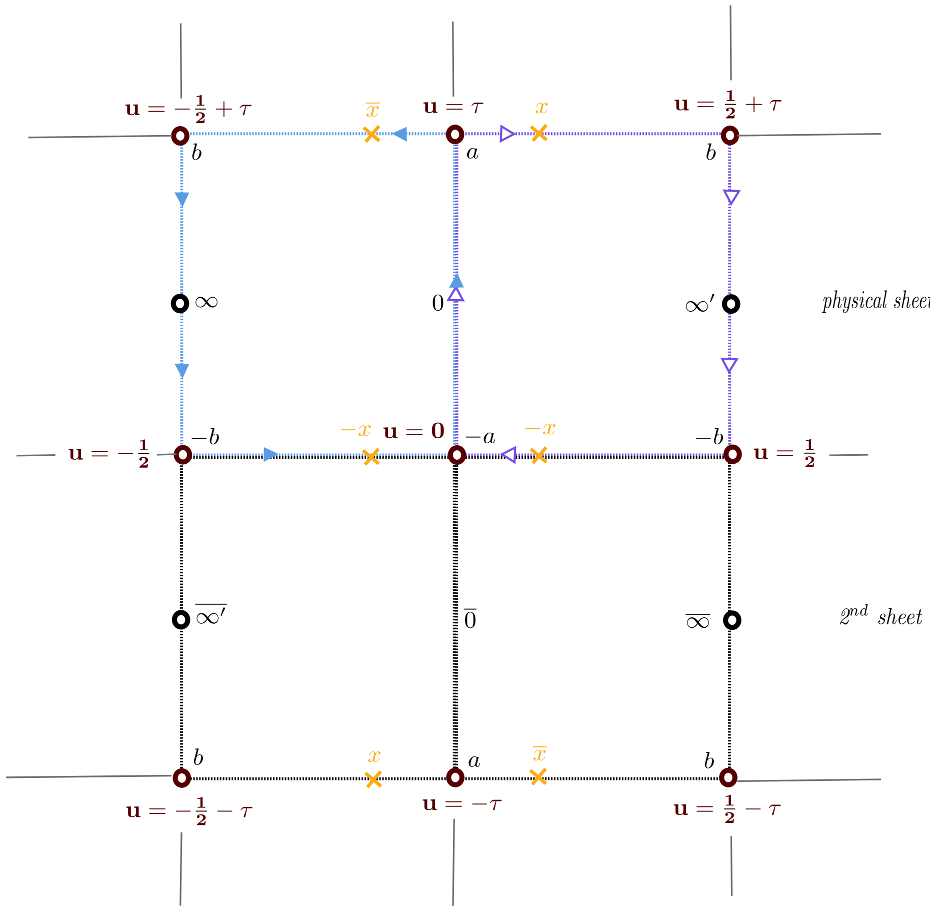


Figure 8:

We present the u-plane at the end of the construction. Circling only once along the paths in the two sheets gives the rectangular region $[-\frac{1}{2}, \frac{1}{2}] \times [-\tau, \tau]$. The path corresponding to $u \in [-\frac{1}{2}, \frac{1}{2}]$ or $u \in [-\tau, \tau]$ are non contractible loops in $\mathcal{S}$. Following these paths adds 1 (or 2τ) to u and leads to the same starting point. Then, it is Abel's theorem that u induces an isomorphism between $\mathcal{S}$ and the Riemann surface $\mathbf{C}/(\mathbf{Z} \oplus 2\tau\mathbf{Z})$. Moreover, we have marked the points corresponding to $\mathrm{x} + i\epsilon$, $\mathrm{x} - i\epsilon$ ($\epsilon > 0$), and $-\mathrm{x}$ for some $\mathrm{x} \in [a, b]$. We see that, in the variable u, they are merely τ-translations of each other:

generic point	**images in the torus**
x	$u,\ -u$
$-\mathrm{x}$	$\tau - u,\ -(\tau - u)$
$\overline{\mathrm{x}}$	$2\tau - u, u - 2\tau$

In a nutshell, our parametrization in terms of elliptic functions [20] reads:

$$\mathrm{x} = a\,\mathrm{sn}_k(\varphi), \qquad \varphi = \int_0^{\mathrm{x}/a} \frac{\mathrm{dx}}{\sqrt{(1-\mathrm{x}'^2)(1-k^2\mathrm{x}'^2)}}, \qquad u = \frac{i\varphi}{2K'} + \frac{\tau}{2}$$

where sn_k is the odd solution of $y' = \sqrt{(1-y^2)(1-k^2y^2)}$, and:

$$\begin{aligned}
\mathrm{cn}_k &= \sqrt{1-\mathrm{sn}_k^2} \\
\mathrm{dn}_k &= \sqrt{1-k^2\mathrm{sn}_k^2} \\
k &= a/b \\
K = K(k) &= \int_0^1 \frac{\mathrm{dx}'}{\sqrt{(1-\mathrm{x}'^2)(1-k^2\mathrm{x}'^2)}} \qquad \text{is the complete elliptic integral} \\
K' = K(\sqrt{1-k^2}) &= \int_0^\infty \frac{\mathrm{dx}'}{\sqrt{(1+\mathrm{x}'^2)(1+k^2\mathrm{x}'^2)}} \\
\tau &= \frac{iK}{K'} \qquad \text{is the modulus}
\end{aligned} \tag{2-3}$$

With help of the properties [20] under translation and rotation to imaginary argument of the elliptic functions sn, cn, dn, we obtain:

$$\mathrm{x} = a\,\mathrm{dn}_{\sqrt{1-a^2/b^2}}\left(\frac{2u}{K'}\right) \tag{2-4}$$

We mention it to be explicit, though it is not at all necessary to know this formula it in what follows.

2.3 Change of variable

This construction provides a nice change of variable to solve Eqn. 2-2. Let us note the inverse function of $\mathrm{x} \mapsto \mathrm{u}(x)$:

$$\begin{aligned} x \;:\; [-\tfrac{1}{2},\tfrac{1}{2}]\times[0,\tau] &\to \mathbf{C}, \text{physical sheet} \\ u &\mapsto x(u) \end{aligned}$$

Then, any function:

$$\begin{aligned} \mathrm{W} \;:\; \mathbf{C} &\to \mathbf{C} \\ \mathrm{x} &\mapsto \mathrm{W}(\mathrm{x}) \end{aligned}$$

which is at least analytic in the x-plane with two cuts $[-b,-a]$ and $[a,b]$, defines without ambiguity an analytic function:

$$\begin{aligned} W \;:\; [-\tfrac{1}{2},\tfrac{1}{2}]\times[0,\tau] &\to \mathbf{C} \\ u &\mapsto \mathrm{W}(x(u)) \end{aligned}$$

in the x plane	in the u plane
no cut on $[\infty, -b] \cup [b, \infty]$	1-translation invariant
no cut on $[-b, -a]$	even
(no cut on $[-b, -a]$) $\times\ \sigma(x)$	odd
$R_e(x) + \sigma(x) R_o(x)$ R_e even rational function of x R_o odd rational function of x	1- and τ-translation invariant

Figure 9: Dictionary between invariances in variable u and analyticity properties in variable x.

In some cases, W may be extended to the whole u-plane because values on the boundary of the initial region match. Eventually, properties of W(x) are translated into properties of $W(u)$ and vice versa.

Now, we apply this to any 1-cut solution W of Eqn. 2-2. We end up with a meromorphic function W defined on the whole u-plane, which is 1-translation invariant, u-even, and satisfies:

$$\forall u \in \tau + \left[-\frac{1}{2}, \frac{1}{2}\right] \qquad W(2\tau - u) + \mathfrak{n} W(\tau - u) + W(u) = 0 \tag{2-5}$$

Since W is analytic, it must be true on the whole u-plane. Using the parity property, and defining the operators of 1-translation $\mathbf{T}_1$, and of τ-translation $\mathbf{T}$, we rewrite Eqn. 2-5 as:

$$\forall u \in \mathbf{C} \qquad \begin{cases} (\mathbf{T}^2 + \mathfrak{n}\mathbf{T} + \mathrm{id})(W)(u) = 0 \\ (\mathbf{T}_1 - \mathrm{id})(W)(u) = 0 \end{cases} \tag{2-6}$$

2.4 Special 1-cut solutions

$\mathbf{T}$ is a linear operator on the space of meromorphic functions in the u-plane, over the field $\mathbf{k}$ of 1 and τ-translation invariant functions (the general form in the x variable of these biperiodic functions is described Fig. 9). With the notation $\mathfrak{n} = -2\cos(\pi\mu)$, the space of solutions of Eqn. 2-6 is the intersection of $\mathrm{Ker}\,(\mathbf{T}_1 - \mathrm{id})$ and:

$$\mathrm{Ker}\,\left(\mathbf{T}^2 + \mathfrak{n}\mathbf{T} + \mathrm{id}\right) = \mathrm{Ker}\,\left(\mathbf{T} - e^{i\pi\mu}\mathrm{id}\right) \oplus \mathrm{Ker}\,\left(\mathbf{T} - e^{-i\pi\mu}\mathrm{id}\right)$$

Thus, it has dimension two. Let us pick up in $\mathrm{Ker}(\mathbf{T}_1 - \mathrm{id}) \cap \mathrm{Ker}(\mathbf{T} - e^{i\pi\mu})$ a special function D_μ (i.e satisfying $D_\mu(u+1) = D_\mu(u)$ and $D_\mu(u+\tau) = e^{-i\pi\mu} D_\mu(u)$), having the asymptotic properties:

- ▷ $D_\mu(\mathrm{u}(\mathrm{x})) \sim 1/\mathrm{x}$ when $\mathrm{x} \to \infty$ in the physical sheet.
- ▷ $D_\mu(\mathrm{u}(\mathrm{x})) \propto 1/\sigma(\mathrm{x})$ when $\mathrm{x} \to a, b, -a, -b$.

This determines D_μ uniquely, and we can actually construct it in terms of theta functions of modulus τ (see Appendix D for a reminder on theta functions):

$$D_\mu(u) = \frac{x(u)}{\sigma(x(u))} \frac{\vartheta_1\left(u - \frac{\tau}{2} + \frac{\mu}{2}\middle|\tau\right)}{\vartheta_1\left(u - \frac{\tau}{2}\middle|\tau\right)} \frac{\vartheta_1\left(-\frac{1}{2}\middle|\tau\right)}{\vartheta_1\left(-\frac{1}{2} + \frac{\mu}{2}\middle|\tau\right)} \tag{2-7}$$

By definition, it has two simple poles (mod $\mathbf{Z} \oplus \tau\mathbf{Z}$) at 0 and $\frac{1}{2}$, and a simple zero at $\mathrm{u}(\infty) = -\frac{1}{2} + \frac{\tau}{2}$. It also admits a second simple zero at $\mathrm{u}(e_\mu) = \frac{\tau-\mu}{2}$. This special function is our elementary brick to define a suitable basis of 1-cut solutions of Eqn. 2-6. Let us notice before that $u \mapsto D_\mu(-u)$ generates $\mathrm{Ker}(\mathbf{T}_1 - \mathrm{id}) \cap \mathrm{Ker}(\mathbf{T} - e^{-i\pi\mu}\mathrm{id})$.

To describe the space of 1-cut solutions of Eqn. 2-6, we rather work on the subfield of u-even functions of $\mathbf{k}$, which consists (read in the x variable) of rational functions of x^2. This space has again dimension two. We have found the following basis convenient for our purposes:

Theorem 2.1 ***Choice of basis***. *There exists a unique couple of meromorphic functions* $(f_\mu, \widehat{f}_\mu)$*, 1-cut solutions of Eqn. 2-6, such that:*

- ▷ $f_\mu(\mathrm{u}(\mathrm{x}))$ *is holomorphic for* $\mathrm{x} \in \mathbf{C} \setminus [a,b]$ *in the physical sheet (it has only one cut).*
- ▷ $f_\mu(\mathrm{u}(\mathrm{x})) \propto 1/\sigma(\mathrm{x})$ *when* $u \to \mathrm{u}(a) = \tau$*, or* $\mathrm{u}(b) = \frac{1}{2} + \tau$*.*
- ▷ $f_\mu(\mathrm{u}(\mathrm{x})) \sim 1/\mathrm{x}$ *when* $\mathrm{x} \to \infty$ *in the physical sheet.*
- ▷ $\widehat{f}_\mu(\mathrm{u}(\mathrm{x}))$ *is holomorphic for* $\mathrm{x} \in \mathbf{C} \setminus ([a,b] \cup \{0\})$ *in the physical sheet (it has only one cut).*
- ▷ $\widehat{f}_\mu(u)$ *is finite when* $u \to \mathrm{u}(a)$ *or* $\mathrm{u}(b)$*.*
- ▷ $\widehat{f}_\mu(\mathrm{u}(\mathrm{x})) \propto 1/\mathrm{x}$ *when* $\mathrm{x} \to 0$ *(in the two sheets).*
- ▷ $\widehat{f}_\mu(\mathrm{u}(\mathrm{x})) \sim 1$ *when* $\mathrm{x} \to \infty$ *in the physical sheet.*

The properties of these functions are listed in Appendix F. They are obtained as:

$$\begin{aligned} f_\mu(u) &= \frac{D_\mu(u)+D_\mu(-u)}{1-e^{i\pi\mu}} \\ \widehat{f}_\mu(u) &= \frac{\sigma(x(u))}{x(u)}\,\frac{D_\mu(u)-D_\mu(-u)}{1+e^{i\pi\mu}} \end{aligned} \tag{2-8}$$

We will use the notation $\mathrm{f}_\mu(\mathrm{x}) = f_\mu(\mathrm{u}(\mathrm{x}))$ and $\widehat{\mathrm{f}}_\mu(\mathrm{x}) = \widehat{f}_\mu(\mathrm{u}(\mathrm{x}))$ for x in the physical sheet. Then:

$$\begin{aligned} \mathrm{f}_\mu(\mathrm{x}) &= \frac{\mathrm{D}_\mu(\mathrm{x})+e^{i\pi\mu}\mathrm{D}_\mu(-\mathrm{x})}{1-e^{i\pi\mu}} \\ \widehat{\mathrm{f}}_\mu(\mathrm{x}) &= \frac{1}{\mathfrak{n}-2}\frac{\sigma(\mathrm{x})}{\mathrm{x}}\left(\mathfrak{n}\mathrm{f}_\mu(\mathrm{x})+2\mathrm{f}_\mu(-\mathrm{x})\right) \end{aligned} \tag{2-9}$$

2.5 General 1-cut solutions

We sum up the description we have obtained up to this point.

Theorem 2.2 *The 1-cut solutions of Eqn. 2-2 are of the form*

$$\mathrm{W}(\mathrm{x}) = \mathrm{A}(\mathrm{x}^2)\mathrm{f}_\mu(\mathrm{x}) + \mathrm{B}(\mathrm{x}^2)\widehat{\mathrm{f}}_\mu(\mathrm{x}) \tag{2-10}$$

where $\mathrm{A}(\mathrm{x}^2)$ *and* $\mathrm{B}(\mathrm{x}^2)$ *are rational functions of* x^2.

There exists a bilinear form $\perp$ (on the field $\mathbf{k}$) for which $\mathrm{f}_\mu\perp\widehat{\mathrm{f}}_\mu = 0$. It appears naturally in the loop equations 1-31 and 1-32, and has motivated the definition of $\widehat{\mathrm{f}}_\mu$ by Eqn. 2-8. It is defined, either in the x or the u variable, by:

$$\begin{aligned} (\mathrm{g}\perp\mathrm{h})(\mathrm{x}) &= \mathrm{g}(\mathrm{x})\mathrm{h}(\mathrm{x})+\mathrm{g}(-\mathrm{x})\mathrm{h}(-\mathrm{x})+\frac{\mathfrak{n}}{2}\left(\mathrm{g}(\mathrm{x})\mathrm{h}(-x)+\mathrm{g}(-\mathrm{x})\mathrm{h}(\mathrm{x})\right) \qquad (2\text{-}11) \\ (g\perp h)(u) &= g(u)h(u)+g(u-\tau)h(u-\tau)+\frac{\mathfrak{n}}{2}\left(g(u)h(u-\tau)+g(u-\tau)h(u)\right) \end{aligned}$$

Let us define $\mathrm{R}_\mu(\mathrm{x}^2) = (\mathrm{f}_\mu\perp\mathrm{f}_\mu)(\mathrm{x})$ and $\widehat{\mathrm{R}}_\mu(\mathrm{x}^2) = (\widehat{\mathrm{f}}_\mu\perp\widehat{\mathrm{f}}_\mu)(\mathrm{x})$, which are rational functions of x^2. We find their expressions in Appendix F, basically by studying their analytical properties:

$$\mathrm{R}_\mu(\mathrm{x}^2) = (2-\mathfrak{n})\frac{\mathrm{x}^2-e_\mu^2}{(\mathrm{x}^2-a^2)(\mathrm{x}^2-b^2)} \tag{2-12}$$

$$\widehat{\mathrm{R}}_\mu(\mathrm{x}^2) = (2+\mathfrak{n})\frac{\mathrm{x}^2-e_\mu^2}{\mathrm{x}^2} \tag{2-13}$$

Then, $\frac{\mathrm{f}_\mu}{\mathrm{R}_\mu}\perp\cdot$ and $\frac{\widehat{\mathrm{f}}_\mu}{\widehat{\mathrm{R}}_\mu}\perp\cdot$ project W on the basis:

$$\mathrm{A}(\mathrm{x}^2) = \frac{(\mathrm{W}\perp\mathrm{f}_\mu)(\mathrm{x})}{\mathrm{R}_\mu(\mathrm{x}^2)} \qquad \mathrm{B}(\mathrm{x}^2) = \frac{(\mathrm{W}\perp\widehat{\mathrm{f}}_\mu)(\mathrm{x})}{\widehat{\mathrm{R}}_\mu(\mathrm{x}^2)} \tag{2-14}$$

As rational functions of x^2, A and B can be determined from the required analytic properties (behavior at poles and zeroes) of the solution of Eqn. 2-2 we are looking for. We shall explain now how to find by this method $W_1^{(0)}$ and $W_2^{(0)}$. This was essentially done in [11, 15, 16] with fewer algebraic geometry language. However, the link with the topological recursion for all correlation functions that we describe now is expressed within the framework of algebraic geometry.

3 Genus 0, one and two points function

3.1 $W_1^{(0)}$ (Proof of Thm. 1.3)

$W_1^{(0)}(x)$ is a 1-cut solution of the complete linear equation with RHS $V'(x)$. Moreover:

- ▷ $W_1^{(0)}$ has no pole in the physical sheet (i.e in $\mathbf{C} \setminus [a, b]$).
- ▷ $W_1^{(0)}(x) \sim t/x$ when $x \to \infty$ in the physical sheet.
- ▷ $W_1^{(0)}(x)$ is finite when $x = a_i$ (a or b), and

$$W_1^{(0)}(x) - W_1^{(0)}(a_i) \propto \sqrt{(x - a_i)} \qquad \text{when } x \to a_i.$$

Thus, it takes the form:

$$W_1^{(0)}(x) \;=\; W^{(s)}(x) + A(x^2)\frac{f_\mu(x)}{R_\mu(x^2)} + B(x^2)\frac{\widehat{f}_\mu(x)}{\widehat{R}_\mu(x^2)} \tag{3-1}$$

$$\text{where} \quad W^{(s)}(x) \;=\; \frac{2V'(x) - \mathfrak{n}V'(-x)}{4 - \mathfrak{n}^2} \tag{3-2}$$

is a particular solution of the complete linear equation. One can prove that the properties of $W_1^{(0)}$ imply that the even rational functions $A(x^2) = (W_1^{(0)} - W_s)\perp f_\mu$ and $B(x^2) = (W_1^{(0)} - W_s)\perp \widehat{f}_\mu$ have no poles except at ∞, i.e are polynomials of x^2. Therefore:

$$\begin{aligned}
A(x^2) &= \frac{(2-\mathfrak{n})t}{x^2} - \frac{1}{2}\left(V'(x)f_\mu(x) + V'(-x)f_\mu(-x)\right) + o\left(\frac{1}{x^2}\right) && \text{when } x \to \infty \\
\text{but} &= \left((W_1^{(0)} - W^{(s)})\perp f_\mu(x)\right)_+ = -\frac{1}{2}\left(V'(x)f_\mu(x) + V'(-x)f_\mu(-x)\right)_+ && \text{exact} \\
B(x^2) &= -\frac{1}{2}\left(V'(x)\widehat{f}_\mu(x) + V'(-x)\widehat{f}_\mu(-x)\right) + O\left(\frac{1}{x^2}\right) && \text{when } x \to \infty \\
\text{but} &= \left[\left((W_1^{(0)} - W^{(s)})\perp \widehat{f}_\mu\right)(x)\right]_+ = -\frac{1}{2}\left(V'(x)\widehat{f}_\mu(x) + V'(-x)\widehat{f}_\mu(-x)\right)_+ && \text{exact}
\end{aligned}$$

and the degree in x^2 are:

$$\deg(A) = \left\lfloor \frac{d_{\max}}{2} \right\rfloor - 1, \qquad \deg(B) = \left\lfloor \frac{d_{\max} - 1}{2} \right\rfloor \tag{3-3}$$

We see that $W_1^{(0)}$ is entirely determined by the properties listed above, and that two constraints come from the necessary absence of a $\frac{1}{x^2}$ term in the expansion of A and B near infinity. They yield two equations for a and b:

$$\mathop{\mathrm{Res}}_{x\to\infty} dx\, (xV'(x)f_\mu(x)) = (-2+\mathfrak{n})t \tag{3-4}$$

$$\mathop{\mathrm{Res}}_{x\to\infty} dx \left(xV'(x)\widehat{f}_\mu(x)\right) = (2+\mathfrak{n}) \mathop{\mathrm{Res}}_{x\to\infty} dx \left(xW_1^{(0)}(x)\right) \tag{3-5}$$

Generically, they admit a unique solution $\{a(t), b(t)\}$ which is a power series in $\sqrt{t}$ for small t (this behavior is required for combinatorics, after the 1-cut lemma).

3.2 Integral representation for $W_1^{(0)}$

The polynomial parts can be rewritten with residue formulae:

$$\begin{aligned} A(x_0^2) &= -\frac{1}{2}\left(V'(x_0)f_\mu(x_0) + V'(-x_0)f_\mu(-x_0)\right)_+ \\ &= \mathop{\mathrm{Res}}_{x\to x_0} \frac{dx}{x-x_0} A(x^2) \\ &= -\mathop{\mathrm{Res}}_{x\to\infty} \frac{dx}{x-x_0} A(x^2) \\ &= \frac{1}{2}\mathop{\mathrm{Res}}_{x\to\infty} \frac{dx}{x-x_0}\left(V'(x)f_\mu(x) + V'(-x)f_\mu(-x)\right) \\ &= \mathop{\mathrm{Res}}_{x\to\infty} dx \frac{x}{x^2-x_0^2} V'(x)f_\mu(x) \\ &= -\frac{1}{2}\left(V'(x_0)f_\mu(x_0) + V'(-x_0)f_\mu(-x_0)\right) - \frac{1}{2i\pi}\oint_{[a,b]} dx \frac{xV'(x)}{x^2-x_0^2} f_\mu(x) \end{aligned} \tag{3-6}$$

and similarly:

$$\begin{aligned} B(x_0^2) &= -\frac{1}{2}\left(V'(x_0)\widehat{f}_\mu(x_0) + V'(-x_0)\widehat{f}_\mu(-x_0)\right)_+ \\ &= -\frac{1}{2}\left(V'(x_0)\widehat{f}_\mu(x_0) + V'(-x_0)\widehat{f}_\mu(-x_0)\right) - \frac{1}{2i\pi}\oint_{[a,b]} dx \frac{xV'(x)}{x^2-x_0^2} \widehat{f}_\mu(x) \end{aligned} \tag{3-7}$$

One may check easily that:

$$\begin{aligned} &\frac{1}{2}(V'(x_0)f_\mu(x_0) + V'(-x_0)f_\mu(-x_0))\frac{f_\mu(x_0)}{R_\mu(x_0)} \\ &+\frac{1}{2}(V'(x_0)\widehat{f}_\mu(x_0) + V'(-x_0)\widehat{f}_\mu(-x_0))\frac{\widehat{f}_\mu(x_0)}{\widehat{R}_\mu(x_0)} = \frac{2V'(x_0) - \mathfrak{n}V'(-x_0)}{4-\mathfrak{n}^2} = W^{(s)}(x_0) \end{aligned}$$

So, we have an expression for $W_1^{(0)}$ as an integral over a path enclosing the cut $[a, b]$:

Theorem 3.1

$$\boxed{\mathrm{W}_1^{(0)}(\mathrm{x}_0) = \frac{1}{2i\pi}\oint_{[a,b]} \mathrm{dx}\,\frac{\mathrm{x}\mathrm{V}'(\mathrm{x})}{\mathrm{x}_0^2 - \mathrm{x}^2}\left(\frac{\mathrm{f}_\mu(\mathrm{x}_0)}{\mathrm{R}_\mu(\mathrm{x}_0)}\mathrm{f}_\mu(\mathrm{x}) + \frac{\widehat{\mathrm{f}}_\mu(\mathrm{x}_0)}{\widehat{R}_\mu(\mathrm{x}_0)}\widehat{\mathrm{f}}_\mu(\mathrm{x})\right)} \tag{3-8}$$

3.3 Consistency relation for a and b

There are two requirements which are not automatic with Eqn. 3-8: $\mathrm{W}_1^{(0)}(\mathrm{x}_0) \sim t/\mathrm{x}_0$ when $\mathrm{x}_0 \to \infty$, and $\mathrm{W}_1^{(0)}(\mathrm{x}_0)$ should have no pole at $\mathrm{x}_0 = \pm e_\mu$. Therefore, we have the constraints:

$$-\frac{1}{2-\mathfrak{n}}\oint_{[a,b]} \mathrm{dx}\,\left(\mathrm{x}\mathrm{V}'(\mathrm{x})\mathrm{f}_\mu(\mathrm{x})\right) = t \tag{3-9}$$

$$\oint_{[a,b]} \mathrm{dx}\,\frac{\mathrm{x}\mathrm{V}'(\mathrm{x})}{\mathrm{x}^2 - e_\mu^2}\left[\sigma(e_\mu)\mathrm{f}_\mu(\mathrm{x}) + \frac{1+e^{i\pi\mu}}{1-e^{i\pi\mu}}e_\mu\widehat{\mathrm{f}}_\mu(\mathrm{x})\right] = 0 \tag{3-10}$$

We obtain only two independent equations, for the absence of pole at e_μ is equivalent to the absence of pole at $-e_\mu$ by use of Eqn. 6-10. If we move the contour away from the cut, we can rewrite them as residues at $\mathrm{x} = \infty$. The first one:

$$\operatorname*{Res}_{\mathrm{x}\to\infty} \mathrm{dx}\,\mathrm{x}\mathrm{V}'(\mathrm{x})\mathrm{f}_\mu(\mathrm{x}) = (-2+\mathfrak{n})t \tag{3-11}$$

is the condition already met in Eqn. 3-4. The second condition can also be expressed as a residue at infinity, and one can check that it is equivalent to Eqn. 3-5.

3.4 The spectral curve: y(x)

The object which plays the role of a spectral curve is the discontinuity of the resolvent $\mathrm{W}_1^{(0)}(\mathrm{x})$. We have:

$$\mathrm{y}(\mathrm{x}) \stackrel{\mathrm{def}}{=} \lim_{\epsilon\to 0}\left(\mathrm{W}_1^{(0)}(\mathrm{x}+i\epsilon) - \mathrm{W}_1^{(0)}(x-i\epsilon)\right) = 2\mathrm{W}_1^{(0)}(\mathrm{x}) + \mathfrak{n}\mathrm{W}_1^{(0)}(-\mathrm{x}) - \mathrm{V}'(\mathrm{x}) \tag{3-12}$$

A short computation gives:

$$\mathrm{y}(\mathrm{x}) = \mathrm{A}(\mathrm{x}^2)\mathrm{y}_\mu(\mathrm{x}) + \mathrm{B}(\mathrm{x}^2)\hat{\mathrm{y}}_\mu(\mathrm{x}) \tag{3-13}$$

where the two basic blocks are given by:

$$\mathrm{y}(\mathrm{x}) = \frac{\mathrm{x}\sigma(\mathrm{x})}{\mathrm{x}^2 - e_\mu^2}\hat{\mathrm{f}}_\mu(-\mathrm{x}), \qquad \hat{\mathrm{y}}(\mathrm{x}) = -\frac{\mathrm{x}\sigma(\mathrm{x})}{\mathrm{x}^2 - e_\mu^2}\mathrm{f}_\mu(-\mathrm{x}) \tag{3-14}$$

3.5 $\mathrm{W}_2^{(0)}$ and its primitive (Proof of Theorem 1.4)

3.5.1 Setting for $\mathrm{W}_2^{(0)}$

$\mathrm{W}_2^{(0)}(\mathrm{x}_0, \mathrm{x})$ is obtained by application of the loop insertion operator $\partial/\partial\mathrm{V}(\mathrm{x})$ to $\mathrm{W}_1^{(0)}(\mathrm{x}_0)$. Accordingly, it is a 1-cut solution (in both x_0 and x) of the complete linear equation with RHS $-1/(\mathrm{x}_0 - \mathrm{x})^2$. Now, let us sum up its properties and explain their origin:

- ▷ $\mathrm{W}_2^{(0)}(\mathrm{x}_0, \mathrm{x})$ is symmetric in x_0 and x.
- ▷ $\mathrm{W}_2^{(0)}(\mathrm{x}_0, \mathrm{x})$ has no pole when x_0 and x are in the physical sheet.
- ▷ $\mathrm{W}_2^{(0)}(\mathrm{x}_0, \mathrm{x}) \propto 1/\mathrm{x}_0^2$ when $\mathrm{x}_0 \to \infty$ in the physical sheet (x fixed).
- ▷ $\mathrm{W}_2^{(0)}(\mathrm{x}_0, \mathrm{x}) \propto 1/\sqrt{(\mathrm{x}_0 - a)(\mathrm{x}_0 - b)}$ when $\mathrm{x}_0 \to a, b$ (x fixed).

The first one is the invariance of the generating function of cylinder shaped maps carrying loops by exchange of the boundaries. The second property is the result of the 1-cut lemma, and the third one arises from the definition of $\mathrm{W}_2^{(0)}$ as a connected correlation function. The last property is a consequence of the application of $\partial/\partial\mathrm{V}(\mathrm{x})$ to $\mathrm{W}_1^{(0)}(\mathrm{x}_0)$, given that a and b are V dependant.

Let us decompose $\mathrm{W}_2^{(0)} = \mathrm{W}_2^{(s)} + \overline{\mathrm{W}}_2^{(0)}$, where:

$$\mathrm{W}_2^{(s)}(\mathrm{x}_0, \mathrm{x}) = \frac{1}{4 - \mathfrak{n}^2}\left(\frac{-2}{(\mathrm{x}_0 - \mathrm{x})^2} + \frac{\mathfrak{n}}{(\mathrm{x}_0 + \mathrm{x})^2}\right) \tag{3-15}$$

is a particular solution of the complete linear equation, and $\overline{\mathrm{W}}_2^{(0)}$ is the solution of the homogeneous linear equation.

3.5.2 Setting for its primitive

Because $\overline{\mathrm{W}}_2^{(0)}$ has double poles and no residues at $\mathrm{x}_0 = \mathrm{x}$, it is easier to find the function:

$$\mathrm{H}(\mathrm{x}_0, \mathrm{x}) = \int_\infty^{\mathrm{x}} \mathrm{dx}'\, \overline{\mathrm{W}}_2^{(0)}(\mathrm{x}_0, \mathrm{x}') \tag{3-16}$$

$\mathrm{H}(\mathrm{x}_0, \mathrm{x})$ is again a 1-cut solution (in x_0) of the homogeneous linear equation, whose properties inherited from $\mathrm{W}_2^{(0)}$ are:

- ▷ $\mathrm{H}(\mathrm{x}_0, \mathrm{x}) = \frac{2}{4-\mathfrak{n}^2}\frac{1}{\mathrm{x}_0 - \mathrm{x}} + O(1)$ when $\mathrm{x}_0 \to \mathrm{x}$.
- ▷ $\mathrm{H}(\mathrm{x}_0, \mathrm{x}) = \frac{\mathfrak{n}}{4-\mathfrak{n}^2}\frac{1}{\mathrm{x}_0 + \mathrm{x}} + O(1)$ when $\mathrm{x}_0 \to -\mathrm{x}$.
- ▷ $\mathrm{H}(\mathrm{x}_0, \mathrm{x}) \in O(1/\mathrm{x}_0)$ when $\mathrm{x}_0 \to \infty$ in the physical sheet.

▷ $\mathrm{H}(\mathrm{x}_0,\mathrm{x}) \propto 1/\sqrt{(\mathrm{x}_0-a)(\mathrm{x}_0-b)}$ when $\mathrm{x}_0 \to a, b$.

▷ $\mathrm{H}(\mathrm{x}_0,\mathrm{x})$ is holomorphic in x_0 elsewhere in the physical sheet.

▷ $\mathrm{H}(\mathrm{x}_0,\mathrm{x}) \in O(1/\mathrm{x})$ when $\mathrm{x} \to \infty$ in the physical sheet.

▷ $\frac{\partial}{\partial x}\left(\mathrm{H}(\mathrm{x}_0,\mathrm{x})\right)$ is symmetric in x_0 and x.

Firstly, we can write:

$$\mathrm{H}(\mathrm{x}_0,\mathrm{x}) = \mathrm{A}(\mathrm{x}_0^2,\mathrm{x})\frac{\mathrm{f}_\mu(\mathrm{x}_0)}{\mathrm{R}_\mu(\mathrm{x}_0^2)} + \mathrm{B}(\mathrm{x}_0^2,\mathrm{x})\frac{\widehat{\mathrm{f}}_\mu(\mathrm{x}_0)}{\widehat{\mathrm{R}}_\mu(\mathrm{x}_0^2)} \tag{3-17}$$

One can prove from the analytical structure of H that the rational functions of x_0^2, $\mathrm{A}(\mathrm{x}_0^2,\mathrm{x}) = \left(\mathrm{H}(\cdot,\mathrm{x})\perp\mathrm{f}_\mu\right)(\mathrm{x}_0)$ and $\mathrm{B}(\mathrm{x}_0^2,\mathrm{x}) = \left(\mathrm{H}(\cdot,\mathrm{x})\perp\widehat{\mathrm{f}}_\mu\right)(\mathrm{x}_0)$ must take the form:

$$\begin{aligned}
\mathrm{A}(\mathrm{x}_0^2,\mathrm{x}) = \left(\mathrm{H}(\cdot,\mathrm{x})\perp\mathrm{f}_\mu\right)(\mathrm{x}_0) &= \frac{\text{Even polynomial of degree 4 in } \mathrm{x}_0}{(\mathrm{x}_0^2-\mathrm{x}^2)(\mathrm{x}_0^2-a^2)(\mathrm{x}_0^2-b^2)} \\
\mathrm{B}(\mathrm{x}_0^2,\mathrm{x}) = \left(\mathrm{H}(\cdot,\mathrm{x})\perp\widehat{\mathrm{f}}_\mu\right)(\mathrm{x}_0) &= \frac{\mathrm{cte}(\mathrm{x})\,\mathrm{x}_0^2}{\mathrm{x}_0^2-\mathrm{x}^2}
\end{aligned}$$

Subsequently, we can write:

$$\begin{aligned}
\mathrm{H}(\mathrm{x}_0,\mathrm{x}) &= \frac{1}{\mathrm{x}_0^2-\mathrm{x}^2}\frac{1}{\mathrm{x}_0^2-e_\mu^2}\Bigg[\Bigg(g_1(\mathrm{x})(\mathrm{x}_0^2-\mathrm{x}^2)(\mathrm{x}_0^2-e_\mu^2) + g_2(\mathrm{x})\frac{\mathrm{x}_0^2-\mathrm{x}^2}{e_\mu^2-\mathrm{x}^2} \\
&\qquad + g_3(\mathrm{x})\frac{\mathrm{x}_0^2-e_\mu^2}{\mathrm{x}^2-e_\mu^2}\Bigg)\mathrm{f}_\mu(\mathrm{x}_0) + g_4(\mathrm{x})\mathrm{x}_0^2\widehat{\mathrm{f}}_\mu(\mathrm{x}_0)\Bigg]
\end{aligned}$$

The ascribed behavior at $\mathrm{x}_0 = \mathrm{x}$, $\mathrm{x}_0 = -\mathrm{x}$ (simple poles with known residue), $\mathrm{x}_0 = e_\mu$ and $\mathrm{x}_0 = -e_\mu$ (no poles), yields three equations determining g_2, g_3 and g_4:

$$\left\{\begin{aligned}
g_3(\mathrm{x})\,\mathrm{f}_\mu(\mathrm{x}) + g_4(\mathrm{x})\,\mathrm{x}^2\widehat{\mathrm{f}}_\mu(\mathrm{x}) &= \tfrac{2}{4-\mathfrak{n}^2}\,2\mathrm{x}(\mathrm{x}^2-e_\mu^2) \\
g_3(\mathrm{x})\,\mathrm{f}_\mu(-\mathrm{x}) + g_4(\mathrm{x})\,\mathrm{x}^2\widehat{\mathrm{f}}_\mu(-\mathrm{x}) &= \tfrac{-\mathfrak{n}}{4-\mathfrak{n}^2}\,2\mathrm{x}(\mathrm{x}^2-e_\mu^2) \\
g_2(\mathrm{x})\,\mathrm{f}_\mu(e_\mu) + g_4(\mathrm{x})\,e_\mu^2\widehat{\mathrm{f}}_\mu(e_\mu) &= 0
\end{aligned}\right. \tag{3-18}$$

Indeed, because of Eqn. 6-10 of Appendix F, the two equations expressing the absence of pole at e_μ and $-e_\mu$ are equivalent. Always with help of Eqn. 6-10, the solution of the system reads:

$$\left\{\begin{aligned}
g_2(\mathrm{x}) &= -i\tfrac{e_\mu\sigma(e_\mu)}{\sqrt{4-\mathfrak{n}^2}}\mathrm{x}\,\widehat{\mathrm{f}}_\mu(\mathrm{x}) \\
g_3(\mathrm{x}) &= \tfrac{1}{2-\mathfrak{n}}\mathrm{x}\sigma^2(\mathrm{x})\,\mathrm{f}_\mu(\mathrm{x}) \\
g_4(\mathrm{x}) &= \tfrac{1}{2+\mathfrak{n}}\mathrm{x}\,\widehat{\mathrm{f}}_\mu(\mathrm{x})
\end{aligned}\right. \tag{3-19}$$

Eventually, it is required that $\mathrm{H}(\mathrm{x}_0,\mathrm{x}) \in O(1/\mathrm{x})$ when $\mathrm{x} \to \infty$ in the physical sheet. Among the four terms in H, the leading contribution at $\mathrm{x} \to \infty$ is of order 1 and comes from g_1 and g_3. We can cancel it by choosing:

$$g_1(\mathrm{x}) = \frac{1}{2-\mathfrak{n}} + O\left(\frac{1}{\mathrm{x}}\right) \tag{3-20}$$

To sum up, we have proved:

Theorem 3.2

$$\mathrm{W}_2^{(0)}(\mathrm{x}_0,\mathrm{x}) = \frac{1}{4-\mathfrak{n}^2}\left(-\frac{2}{(\mathrm{x}_0-\mathrm{x})^2}+\frac{\mathfrak{n}}{(\mathrm{x}_0+\mathrm{x})^2}\right)+\frac{\partial}{\partial \mathrm{x}}\left(\mathrm{H}(\mathrm{x}_0,\mathrm{x})\right) \tag{3-21}$$

where

$$\begin{aligned}\mathrm{H}(\mathrm{x}_0,\mathrm{x}) &= \frac{1}{2-\mathfrak{n}}\frac{\mathrm{x}\sigma(\mathrm{x})^2}{(\mathrm{x}_0^2-\mathrm{x}^2)(\mathrm{x}^2-e_\mu^2)}\mathrm{f}_\mu(\mathrm{x}_0)\mathrm{f}_\mu(\mathrm{x})+\frac{1}{2-\mathfrak{n}}\mathrm{f}_\mu(\mathrm{x}_0)+O\left(\frac{1}{\mathrm{x}}\right)\mathrm{f}_\mu(\mathrm{x}_0)\\ &+ \frac{ie_\mu\sigma(e_\mu)}{\sqrt{4-\mathfrak{n}^2}}\frac{\mathrm{x}}{(\mathrm{x}_0^2-e_\mu^2)(\mathrm{x}^2-e_\mu^2)}\mathrm{f}_\mu(\mathrm{x}_0)\widehat{\mathrm{f}}_\mu(\mathrm{x})\\ &+ \frac{1}{2+\mathfrak{n}}\frac{\mathrm{x}_0^2\mathrm{x}}{(\mathrm{x}_0^2-\mathrm{x}^2)(\mathrm{x}_0^2-e_\mu^2)}\widehat{\mathrm{f}}_\mu(\mathrm{x}_0)\widehat{\mathrm{f}}_\mu(\mathrm{x})\end{aligned} \tag{3-22}$$

3.5.3 Solution for H

We derive $\mathrm{H}(\mathrm{x}_0,\mathrm{x})$ to obtain an expression of $\mathrm{W}_2^{(0)}$ in terms of f_μ and $\widehat{\mathrm{f}}_\mu$:

$$\begin{aligned}\mathrm{W}_2^{(0)}(\mathrm{x}_0,\mathrm{x}) &= \frac{1}{4-\mathfrak{n}^2}\left(\frac{-2}{(\mathrm{x}_0-\mathrm{x})^2}+\frac{\mathfrak{n}}{(\mathrm{x}_0+\mathrm{x})^2}\right)+O\left(\frac{1}{\mathrm{x}^2}\right)\\ &+ \left(\frac{e_\mu^2\sigma(e_\mu)^2}{(\mathrm{x}_0^2-e_\mu^2)(\mathrm{x}^2-e_\mu^2)^2}+\frac{2\sigma(\mathrm{x})^2\mathrm{x}^2}{(\mathrm{x}_0^2-\mathrm{x}^2)(\mathrm{x}^2-e_\mu^2)}+\right.\\ &\left.\frac{2\mathrm{x}^6-(a^2+b^2+3e_\mu^2)\mathrm{x}^4+2(a^2+b^2)e_\mu^2\mathrm{x}^2-a^2b^2e_\mu^2}{(\mathrm{x}_0^2-\mathrm{x}^2)(\mathrm{x}^2-e_\mu^2)^2}\right)\frac{1}{2-\mathfrak{n}}\mathrm{f}_\mu(\mathrm{x}_0)\mathrm{f}_\mu(\mathrm{x})\\ &+ \left(\frac{\alpha_1\mathrm{x}_0^2(\mathrm{x}^2-\widehat{e}_\mu^2)}{(\mathrm{x}_0^2-\mathrm{x}^2)(\mathrm{x}_0^2-e_\mu^2)(\mathrm{x}^2-e_\mu^2)}\right)\frac{1}{\sqrt{4-\mathfrak{n}^2}}\widehat{\mathrm{f}}_\mu(\mathrm{x}_0)\mathrm{f}_\mu(\mathrm{x})+(\mathrm{x}_0\leftrightarrow\mathrm{x})\\ &+ \left(\frac{\mathrm{x}_0^2+\mathrm{x}^2-2e_\mu^2}{(\mathrm{x}_0^2-\mathrm{x}^2)(\mathrm{x}_0^2-e_\mu^2)(\mathrm{x}^2-e_\mu^2)}\right)\frac{1}{2+\mathfrak{n}}\widehat{\mathrm{f}}_\mu(\mathrm{x}_0)\widehat{\mathrm{f}}_\mu(\mathrm{x})\end{aligned} \tag{3-23}$$

where α_1 and $\widehat{e}_\mu$ are constants introduced in Appendix F, and the $O(1/\mathrm{x}^2)$ is independent of x_0. It is not easy to see on the second and third line, but nevertheless true, that this expression is symmetric in x_0 and x. Accordingly, $g_1(\mathrm{x})$ must be the constant $1/(2-\mathfrak{n})$ and the $O(1/\mathrm{x}^2)$ must vanish. Collecting the pieces, we have constructed the unique function H which has all the required properties. Hence Theorem 1.4.

3.5.4 Explicit expressions in variables x

It was convenient to find H and $\mathrm{W}_2^{(0)}$ by decomposing them on the 1-cut basis $(\mathrm{f}_\mu,\widehat{\mathrm{f}}_\mu)$. Nevertheless, we can write now their decomposition on any basis, not necessarily made of 1-cut functions. Their expressions are more readable in the basis $(\mathrm{D}_\mu(\mathrm{x}),\mathrm{D}_\mu(-\mathrm{x}))$. We only give the result, which matches with the expression found in [11], and which makes the symmetry between x_0 and x explicit. The computations, starting from Theorem 3.2, are lengthy but without difficulties with the relations given in Appendix F.

Theorem 3.3 *We introduce the function I, which is related to the derivative of* D_μ *(see Eqn. 6-7, Appendix F):*

$$\mathrm{I}(\mathrm{x}) = \frac{\mathrm{x}\sigma(\mathrm{x}) + e_\mu\sigma(e_\mu)}{\mathrm{x}^2 - e_\mu^2} \tag{3-24}$$

Then:

$$\begin{aligned} \mathrm{H}(\mathrm{x}_0, \mathrm{x}) &= \frac{1}{4 - \mathfrak{n}^2}\left(e^{i\pi\mu}\mathrm{H}_+(\mathrm{x}_0, \mathrm{x}) + \mathrm{H}_+(-\mathrm{x}_0, \mathrm{x}) - \mathrm{H}_+(\mathrm{x}_0, \mathrm{x}) - e^{-i\pi\mu}\mathrm{H}_+(-\mathrm{x}_0, -\mathrm{x})\right) \\ \mathrm{H}_+(\mathrm{x}_0, \mathrm{x}) &= \mathrm{D}_\mu(\mathrm{x}_0)\,\sigma(\mathrm{x})\mathrm{D}_\mu(\mathrm{x})\,\frac{\mathrm{I}(\mathrm{x}) - \mathrm{I}(\mathrm{x}_0)}{\mathrm{x}^2 - \mathrm{x}_0^2} \end{aligned} \tag{3-25}$$

and:

$$\begin{aligned} \mathrm{W}_2^{(0)}(\mathrm{x}_0, \mathrm{x}) &= \frac{1}{4 - \mathfrak{n}^2}\Big(e^{i\pi\mu}\mathrm{W}_{2|+}^{(0)}(\mathrm{x}_0, \mathrm{x}) + \mathrm{W}_{2|+}^{(0)}(-\mathrm{x}_0, \mathrm{x}) \\ &\qquad + \mathrm{W}_{2|+}^{(0)}(\mathrm{x}_0, -\mathrm{x}) + e^{-i\pi\mu}\mathrm{W}_{2|+}^{(0)}(-\mathrm{x}_0, -\mathrm{x})\Big) \\ \mathrm{W}_{2|+}^{(0)}(\mathrm{x}_0, \mathrm{x}) &= \mathrm{D}_\mu(\mathrm{x}_0)\mathrm{D}_\mu(\mathrm{x})\left[1 - \left(\alpha_1 + \frac{\mathrm{x}\sigma(x) - \mathrm{x}_0\sigma(\mathrm{x}_0)}{\mathrm{x}^2 - \mathrm{x}_0^2}\right)\frac{\mathrm{I}(\mathrm{x}) - \mathrm{I}(\mathrm{x}_0)}{\mathrm{x}^2 - \mathrm{x}_0^2}\right] - \frac{1}{(\mathrm{x} + \mathrm{x}_0)^2} \end{aligned} \tag{3-26}$$

Eventually, we can compute from Eqn. 3-27 other quantities of importance:

Theorem 3.4 *At coinciding points,* $\mathrm{W}_2^{(0)}$ *is given by:*

$$\begin{aligned} \mathrm{W}_2^{(0)}(\mathrm{x}, \mathrm{x}) &= \frac{1}{4 - \mathfrak{n}^2}\left(\frac{\mathfrak{n}}{4\mathrm{x}^2} - \mathfrak{n}(\partial_2\mathrm{H})(\mathrm{x}, \mathrm{x}) + 2\mathrm{W}_{2|+}^{(0)}(\mathrm{x}, -\mathrm{x})\right) \\ \mathrm{W}_{2|+}^{(0)}(\mathrm{x}, -\mathrm{x}) &= \frac{1}{2}\frac{e_\mu^2 - \alpha_1^2}{\sigma^2(\mathrm{x})} + \frac{\mathrm{x}^2}{4}\frac{(b^2 - a^2)^2}{\sigma^4(\mathrm{x})} \end{aligned} \tag{3-27}$$

Let $\{a, b\} = \{a_i, a_j\}$*. Around the* a_i*'s,* $\mathrm{W}_2^{(0)}$ *has the following behavior:*

$$\lim_{\mathrm{x}_0 \to a_i}\lim_{\mathrm{x} \to a_j}\sigma(\mathrm{x}_0)\mathrm{W}_2^{(0)}(\mathrm{x}_0, \mathrm{x})\sigma(\mathrm{x}) = -\left(\sigma(e_\mu) + \alpha_1 e_\mu\right) \tag{3-28}$$

$$\lim_{\mathrm{x}_0 \to a_i}\lim_{\mathrm{x} \to a_i}\sigma(\mathrm{x}_0)\mathrm{W}_2^{(0)}(\mathrm{x}_0, \mathrm{x})\sigma(\mathrm{x}) = -\left(a_i^2 - e_\mu^2 + \alpha_1\frac{e_\mu\sigma(e_\mu)}{a_i^2 - e_\mu^2}\right) \tag{3-29}$$

3.6 Infinite series representation for $\mathrm{d}x(u_0)\mathrm{d}x(u)\overline{W}_2^{(0)}(u_0, u)$

We present here an alternative construction.

3.6.1 Setting

Let us define $s(u) = \frac{\mathrm{d}x}{\mathrm{d}u} = \frac{ib}{2K'}\sigma(x(u))$ (its properties are summarized in Appendix G). In this paragraph, we write $\mathbf{T_u}$ and $\mathbf{T_{u_0}}$ the operators of translation by τ, defined on a function $(u, u_0) \mapsto W(u, u_0)$ by:

$$(\mathbf{T_{u_0}}W)(u_0, u) = W(u_0 - \tau, u), \qquad (\mathbf{T_u}W)(u_0, u) = W(u_0, u - \tau)$$

We shall consider the function $\mathcal{B}$ such that:

$$\begin{aligned}\mathcal{B}(u_0,u) &\stackrel{\text{def}}{=} s(u_0)s(u)\left[\frac{\mathbf{T_{u_0}}-e^{-i\pi\mu}\text{id}}{e^{i\pi\mu}-e^{-i\pi\mu}}\,\frac{\mathbf{T_u}-e^{-i\pi\mu}\text{id}}{e^{i\pi\mu}-e^{-i\pi\mu}}\right]\left(\overline{W}_2^{(0)}\right)(u_0,u)\\ &= s(u_0)s(u)\overline{W}_{2|+}^{(0)}(u_0,u) \end{aligned} \tag{3-30}$$

$s(u_0)s(u)\overline{W}_2^{(0)}(u_0,u)$ can be decomposed in terms of $\mathcal{B}$ with help of Eqn. 3-26. Since s takes a minus sign under τ translation, we have:

$$\begin{aligned}s(u_0)s(u)\overline{W}_2^{(0)}(u_0,u) &= \frac{1}{4-\mathfrak{n}^2}\big(e^{i\pi\mu}\mathcal{B}(u_0,u)-\mathcal{B}(\tau-u_0,u)\\ &\quad -\mathcal{B}(u_0,\tau-u)+e^{-i\pi\mu}\mathcal{B}(\tau-u,\tau-u_0)\big)\end{aligned} \tag{3-31}$$

The function $\mathcal{B}$ has the following properties:

- ▷ $\mathcal{B}$ is a meromorphic function of $u_0, u \in \mathbf{C}$.
- ▷ $\mathcal{B}$ has for only pole $u = (\tau - u_0) \bmod \mathbf{Z} \oplus \tau\mathbf{Z}$, and:

$$\mathcal{B}(u) = \frac{1}{(u+u_0-\tau)^2} + O(1) \qquad \text{when } u \to \tau - u_0.$$

- ▷ $\mathcal{B}(u_0,u) = \mathcal{B}(u,u_0)$.
- ▷ $\mathcal{B}(u_0,u) = \mathcal{B}(u_0+1,u)$ and $\mathcal{B}(u_0+\tau,u) = e^{i\pi(1-\mu)}\mathcal{B}(u_0,u)$.

3.6.2 Construction of a solution

We can construct such a function explicitly by a deformation of the Weierstraß function $\wp$ (see Appendix D.0.5 for its definition). We assume $e^{i\pi\mu} \neq -1$ (i.e $\mathfrak{n} \neq 2$), and define a function $\wp_\mu$:

$$\wp_\mu(w) = \sum_{m\in\mathbf{Z}} e^{i\pi(\mu-1)m}\,\frac{\pi^2}{\sin^2\pi(w+m\tau)} \tag{3-32}$$

This series converges, and $\wp_\mu$ is 1-translation invariant and take a phase $e^{i\pi(1-\mu)}$ when $w \mapsto w+\tau$. It coincides with $\wp + c_0$ when $\mu = 1 \bmod 2\mathbf{Z}$ (i.e. $\mathfrak{n} = 2$), where c_0 is a constant depending on τ. It has for only pole $w = 0 \bmod (\mathbf{Z} \oplus \tau\mathbf{Z})$, which is a double pole without residues. As for D_μ, one can represent $\wp_\mu$ as a quotient of theta functions, and we find that it has two distinct zeroes :

$$w_1 \qquad \text{and} \qquad w_2 = -w_1 + \frac{1-\mu}{2}. \tag{3-33}$$

$\wp_\mu$ enjoys properties generalizing those of $\wp$. We present them in Appendix E. For example, $\wp_1 = \wp + c_0$ satisfies first and second order ODE's:

$$\begin{aligned}(\wp_1')^2 &= 4\wp_1^3 - 12c_0\wp_1^2 + (12c_0^2 - g_2)\wp_1 - (g_3 + 4c_0^3) \\ \wp_1'' &= 6\wp_1^2 - 12c_0\wp_1 + 6c_0^2 - \frac{g_2}{2}\end{aligned}$$

for some constants g_2 and g_3 depending on τ. For $\wp_\mu$, they become:

$$\begin{aligned}\wp_1'\wp_\mu &= 4\wp_1^2\wp_\mu - 4(2c_0 + c_0^{(\mu)})\wp_1\wp_\mu + \left[4c_0^2 + c_0c_0^{(\mu)} + (c_0^{(\mu)})^2 - g_2^{(\mu)}\right]\wp_\mu + \lambda q_\mu \\ \wp_\mu'' &= 6\wp_1\wp_\mu - 6(c_0 + c_0^{(\mu)})\wp_\mu + \lambda' q_\mu\end{aligned}$$

where:

$$q_\mu(w) = \left[\wp_1(w)\wp_\mu'(w) - \wp_1'(w)\wp_\mu(w) + (c_0^{(\mu)} - c_0)\wp_\mu'(w) - 3c_1^{(\mu)}\wp_\mu(w)\right]$$

and λ, λ', $c_0^{(\mu)}$ and $g_2^{(\mu)}$ are constants depending on μ and τ, such that:

$$\begin{aligned}&\lim_{\mu\to 1}\lambda = \infty, \qquad \lim_{\mu\to 1}\lambda' = \infty, \qquad \lim_{\mu\to 1}\frac{\lambda}{\lambda'} = \frac{g_3 + 4c_0^3}{\frac{g_2}{2} - 6c_0^2} \\ &\lim_{\mu\to 1} c_0^{(\mu)} = c_0, \qquad \lim_{\mu\to 1} g_2^{(\mu)} = g_2\end{aligned}$$

3.6.3 Solution

Now, $\mathcal{B}(u_0, u) - \wp_\mu(u + u_0 - \tau)$ is holomorphic, 1-translation invariant and takes a phase $e^{i\pi(1-\mu)}$ when one of the arguments is shifted by τ. Since $\mu \in \mathbf{R}$, this difference is a entire bounded function, so is constant, and this constant must vanish for $e^{i\pi\mu} \neq -1$. Therefore, $\mathcal{B}(u_0, u) = \wp_\mu(u + u_0 - \tau)$. Now, we shall collect the four terms in Eqn. 3-31.

Theorem 3.5

$$s(u_0)s(u)\overline{W}_2^{(0)}(u_0, u) = \sum_{m\in\mathbf{Z}} \tilde{r}_m \frac{\pi^2}{\sin^2\pi(u + u_0 + m\tau)} + r_m \frac{\pi^2}{\sin^2\pi(u - u_0 + m\tau)} \tag{3-34}$$

where:

$$r_m = \frac{2\cos\pi(\mu - 1)m}{4 - \mathfrak{n}^2} \qquad \textit{and} \qquad \tilde{r}_m = -r_{2+m}$$

As a function of $\mathfrak{n}$, $r_m = r_{-m} = \frac{2}{4-\mathfrak{n}^2}T_m(-\mathfrak{n}/2)$ *where the* T_m*'s are the Chebyshev polynomials of the first kind* ($m \in \mathbf{N}$). *The first values are:*

$$r_0 = \frac{2}{4 - \mathfrak{n}^2}, \quad r_1 = \frac{\mathfrak{n}}{4 - \mathfrak{n}^2}, \quad r_2 = \frac{\mathfrak{n}^2 - 2}{4 - \mathfrak{n}^2}, \quad r_3 = \frac{\mathfrak{n}^3 - 3\mathfrak{n}}{4 - \mathfrak{n}^2}, \ \ldots$$

□

It is usual in matrix models to find that the differential form $\mathrm{dx_0 dx}\overline{\mathrm{W}}_2^{(0)}(x_0, x)$ is a universal object. Physically, it characterizes the correlation of density of eigenvalues. The universality is striking : it only depends on τ and on the parameter $\mu \in \mathbf{R}/2\mathbf{Z}$. If we regard them as independent parameters, formula 3-34 is the Fourier series in μ of $\overline{W}_2^{(0)}$. Besides, it only depends on $\mathrm{u(x)} - \mathrm{u(x_0)}$ and $\mathrm{u(x)} - \mathrm{u(-x_0)}$. This fact has a simple interpretation. Consider two charges/eigenvalues located at $\mathrm{x_0}$ and x. In the $\mathcal{O}(\mathfrak{n})$ model, $\mathrm{x_0}$ also feels the mirror charge located at $-\mathrm{x}$. We found that $\mathrm{dxdx_0}\overline{W}_2^{(0)}(x_0, x)$ only depends on the 'differences' between the position of the interacting sources, which is a very natural conclusion. 'Position' and 'difference' have a natural meaning on the spectral curve, which is equipped with the addition law $(u_0, u) \mapsto u_0 + u$.

3.7 Double integration of $\mathrm{dx_0 dx}\overline{W}_2^{(0)}(\mathrm{x_0}, \mathrm{x})$

3.7.1 Generalized theta function

There exists a function Θ_μ, such that $\wp_\mu(w) = -\partial_w^2 \ln \Theta_\mu(w)$. We can define it (by a choice of integration constants) as:

$$\begin{aligned} \Theta_\mu(w) &\stackrel{\mathrm{def}}{=} \frac{1}{\pi} \exp\left[\int_0^w \mathrm{d}w_1 \int_0^{w_1} \mathrm{d}w_2 \left(\frac{1}{w_2^2} - \wp_\mu(w_2)\right)\right] \\ &= \frac{\sin \pi w}{\pi w} \prod_{m=1}^{\infty} \left(\frac{\sin \pi(m\tau + w) \sin(m\tau - w)}{\sin^2 \pi m \tau}\right)^{\exp(i\pi(\mu-1)m)} \end{aligned} \quad \text{(3-35)}$$

This identity generalizes $\wp(w) = -\partial^2 \ln \vartheta_1(w) + \mathrm{cte}$ in our problem, and we see it as the generalization of the Jacobi triple product identity for the ϑ_1 function.

3.7.2 Double primitive of $\overline{\omega}_2^{(0)}$

Eventually, one can find a function $\Psi(u_0, u)$, such that:

$$\begin{aligned} \mathrm{d}x(u_0) H(u_0, u) &= -\left(\mathrm{d}_{u_0} \ln \Psi\right)(u_0, u) \\ \overline{W}_2^{(0)}(u, u_0) &= -\left(\mathrm{d}_u \mathrm{d}_{u_0} \ln \Psi\right)(u_0, u) \end{aligned} \quad \text{(3-36)}$$

in the form of a infinite product:

$$\begin{aligned} \Psi(u_0, u) &\stackrel{\mathrm{def}}{=} \exp\left[-\int^{u_0} \mathrm{d}u_0' s(u_0') \int_{\mathrm{u}(\infty)}^{u} \mathrm{d}u' s(u)\, \overline{W}_2^{(0)}(u_0', u')\right] \\ &= \mathrm{cte} \prod_{m \in \mathbf{Z}} \left(\frac{\sin \pi(u_0 - \overline{\mathrm{u}(\infty)} + m\tau)\, \sin \pi(u_0 - \mathrm{u}(\infty) + m\tau)}{\sin \pi(u_0 - \overline{u} + m\tau)\, \sin \pi(u_0 - u + m\tau)}\right)^{r_m} \end{aligned} \quad \text{(3-37)}$$

Here, we define[5] $\overline{u} = 2\tau - u \bmod \mathbf{Z}$, which will also play a role in the following section.

4 Other correlation functions at all genera

4.1 Preliminaries

4.1.1 Outline

We will see that the loop equations 1-32 imply that $\mathrm{W}_k^{(g)}$ for $2g-2+k>0$ satisfies the homogeneous linear equation 2-2. Two methods can be used from this point to prove that they satisfy the topological recursion. The first one consists in identifying the even rational functions of x, $\mathrm{A}_k^{(g)}(\mathrm{x}^2, J)$ and $\mathrm{B}_k^{(g)}(\mathrm{x}^2, J)$ in the decomposition:

$$\mathrm{W}_k^{(g)}(\mathrm{x}, I) = \mathrm{A}_k^{(g)}(\mathrm{x}^2, I)\mathrm{f}_\mu(\mathrm{x}) + \mathrm{B}_k^{(g)}(\mathrm{x}^2, I)\widehat{\mathrm{f}}_\mu(\mathrm{x}) \tag{4-1}$$

as we did for $\mathrm{W}_1^{(0)}$ and $\mathrm{W}_2^{(0)}$ in Section 3. We chose a second one, that consists in finding a Cauchy-type residue formula for 1-cut solutions of Eqn. 2-2, only in terms of residues at branch points. Also, we construct the Cauchy kernel $\mathrm{G}(\mathrm{x}_0, \mathrm{x})$ accurately replacing $1/(\mathrm{x}_0 - \mathrm{x})$ for our spectral curve.

4.1.2 Technical lemma

We first prove two properties of the $\perp$-product, defined in Eqn. 2-12 and appearing throughout this article.

Lemma 4.1

- ▷ *If f and g are solutions of the homogeneous linear equation 2-2, then $(f \perp g)(u)$ is* 1- *and* 2τ*-translation invariant.*
- ▷ *Besides, if f and g have only one cut, then* $\mathrm{f}\perp\mathrm{g}$ *is an even rational function of* x.
- ▷ *If f is a 1-cut solution of 2-2, g has one cut, and $f \perp g$ has no cut on $[a, b]$, then g is a 1-cut solution of 2-2 as well.*

proof:

[5] NB: in Figs. 6-8, we defined a point $\overline{\infty}$, and we had $\mathrm{u}(\overline{\infty}) = \frac{-1-\tau}{2}$. But this notation is a priori different from that one:

$$\overline{\mathrm{u}(\infty)} = \frac{1+3\tau}{2} = \mathrm{u}(\overline{\infty}) + 2\tau \bmod \mathbf{Z}.$$

For the first point, we rather work with the u variable than with x, and use the dictionary of Section 9: recall for example that "having one cut" translates into "being u-even". If f and g are solution of the loop equation:

$$\begin{aligned}\mathbf{T}(f\perp g) &= \mathbf{T}f\cdot\mathbf{T}g+\mathbf{T}^2f\cdot\mathbf{T}^2g+\frac{\mathfrak{n}}{2}\left(\mathbf{T^2}f\cdot\mathbf{T}g+\mathbf{T}f\cdot\mathbf{T}^2g\right)\\ &= (f\perp g)\end{aligned}$$

If on the top of that f and g have only one cut, so does $f\perp g$ for it is obviously invariant under $u\mapsto\tau-u$. Hence the second point. Now, assume f is a 1-cut solution of Eqn. 2-2 and f$\perp$g has no cut on $[a,b]$ (for example, f$\perp$g is an even rational function of x). Then, for $\mathrm{x}\in[a,b]$:

$$\begin{aligned}0 &=_{\epsilon\to 0} (\mathrm{f}\perp\mathrm{g})(\mathrm{x}+i\epsilon)-(\mathrm{f}\perp\mathrm{g})(\mathrm{x}-i\epsilon)\\ &=_{\epsilon\to 0} \left(f(\mathrm{x}+i\epsilon)+\frac{\mathfrak{n}}{2}\mathrm{f}(-\mathrm{x})\right)\mathrm{g}(\mathrm{x}+i\epsilon)-\left(\mathrm{f}(\mathrm{x}-i\epsilon)+\frac{\mathfrak{n}}{2}\mathrm{f}(-\mathrm{x})\right)\mathrm{g}(\mathrm{x}-i\epsilon)\\ &\quad+\frac{\mathfrak{n}}{2}\left(\mathrm{f}(\mathrm{x}+i\epsilon)-\mathrm{f}(\mathrm{x}-i\epsilon)\right)\mathrm{g}(-\mathrm{x})\\ &=_{\epsilon\to 0} \left(\mathrm{f}(\mathrm{x}+i\epsilon)-\mathrm{f}(\mathrm{x}-i\epsilon)\right)\left[\mathrm{g}(\mathrm{x}+i\epsilon)+\mathrm{g}(\mathrm{x}-i\epsilon)+\frac{\mathfrak{n}}{2}\mathrm{g}(-\mathrm{x})\right]\end{aligned}$$

Since f has a discontinuity on $]a,b[$, g must satisfy the homogeneous linear equation. □

4.1.3 A Cauchy formula

Let us introduce now the Cauchy kernel:

$$\begin{aligned}\mathrm{G}(\mathrm{x}_0,\mathrm{x}) &= \int_{\infty}^{\mathrm{x}}\mathrm{dx}'\left(2\overline{\mathrm{W}}_2^{(0)}(\mathrm{x}_0,\mathrm{x}')+\mathfrak{n}\overline{\mathrm{W}}_2^{(0)}(\mathrm{x}_0,-\mathrm{x}')\right)\\ G(\mathrm{x}_0,u) &= \mathrm{G}(\mathrm{x}_0,x(u))\end{aligned} \tag{4-2}$$

and recall the definition of the auxiliary kernel:

$$\begin{aligned}\mathrm{H}(\mathrm{x}_0,\mathrm{x}) &= \int_{\infty}^{\mathrm{x}}\mathrm{dx}'\overline{\mathrm{W}}_2^{(0)}(\mathrm{x}_0,\mathrm{x}')\\ H(\mathrm{x}_0,u) &= \mathrm{H}(\mathrm{x}_0,x(u))\end{aligned} \tag{4-3}$$

One can show that G has the following properties:

- ▷ $G(\mathrm{x}_0,u)$ is a meromorphic function of $u\in\mathbf{C}$, with $u=\mathrm{u}(\mathrm{x})\bmod(\mathbf{Z}\oplus2\tau\mathbf{Z})$ as only pole.
- ▷ $\mathrm{G}(\mathrm{x}_0,\mathrm{x})=\frac{1}{\mathrm{x}_0-\mathrm{x}}+O(1)$ when $\mathrm{x}_0\to\mathrm{x}$.
- ▷ $G(\mathrm{x}_0,-u)=-G(\mathrm{x}_0,u)+\mathrm{cte}(x_0)$.

Theorem 4.1 ***Cauchy formula****. Let* W *(in the* x*-plane) or* W *(in the* u*-plane) be a 1-cut solution of the homogeneous linear equation. If* W *is holomorphic on* $\mathbf{C}\setminus[a,b]$*, and has no residue at* $\mathrm{x}=\infty$*, we have:*

$$\mathrm{W}(\mathrm{x}_0)=\frac{1}{2}\mathop{\mathrm{Res}}_{u\to\tau,\tau+\frac{1}{2}} \mathrm{d}u s(u)\, G(\mathrm{x}_0,u)W(u) \tag{4-4}$$

where $s(u)=\frac{\mathrm{d}x}{\mathrm{d}u}$*. One can add to* G *a constant* $\gamma_i(\mathrm{x}_0)$*, depending on the branch point* $\mathrm{u}(a_i)$*,* $a_i\in\{a,b\}$*, without changing the formula.*

proof:

We write the usual Cauchy formula and recast it as residues on the branch points a and b only. Let $u_0=\mathrm{u}(\mathrm{x}_0)$:

$$\begin{aligned}
\mathrm{W}(\mathrm{x}_0) &= -\mathop{\mathrm{Res}}_{\mathrm{x}\to\mathrm{x}_0} \mathrm{dx}\,\mathrm{G}(\mathrm{x}_0,\mathrm{x})\mathrm{W}(\mathrm{x}) \\
&= -\mathop{\mathrm{Res}}_{\mathrm{x}\to\mathrm{x}_0} \mathrm{dx}\left(\mathrm{G}(\mathrm{x}_0,\mathrm{x})\mathrm{W}(\mathrm{x}) - \underbrace{\mathrm{G}(\mathrm{x}_0,-\mathrm{x})\mathrm{W}(-\mathrm{x})}_{\text{no pole at } \mathrm{x}_0}\right) \\
&= -\frac{1}{2}\mathop{\mathrm{Res}}_{\mathrm{x}\to\mathrm{x}_0,-\mathrm{x}_0} \mathrm{dx}\left(\mathrm{G}(\mathrm{x}_0,\mathrm{x})\mathrm{W}(\mathrm{x}) - \mathrm{G}(\mathrm{x}_0,-\mathrm{x})\mathrm{W}(-\mathrm{x})\right) \qquad \text{by parity}
\end{aligned}$$

We now switch to u variable (the properties of $s(u)$ are listed in Appendix G):

$$\begin{aligned}
\mathrm{W}(\mathrm{x}_0) &= -\frac{1}{4}\mathop{\mathrm{Res}}_{\substack{u\to u_0,-u_0\\ \tau-u_0,-(\tau-u_0)}} \mathrm{d}u s(u)\left(G(\mathrm{x}_0,u)W(u)-G(\mathrm{x}_0,u-\tau)W(u-\tau)\right) \\
&= -\frac{1}{4}\mathop{\mathrm{Res}}_{\substack{u\to u_0,-u_0\\ \tau-u_0,-(\tau-u_0)}} \mathrm{d}u\left(s(u)G(\mathrm{x}_0,u)W(u)+s(u-\tau)G(\mathrm{x}_0,u-\tau)W(u-\tau)\right) \\
&= -\frac{1}{4}\mathop{\mathrm{Res}}_{\substack{u\to u_0,-u_0\\ \tau-u_0,-(\tau-u_0)}} \mathrm{d}u\left(sH(\mathrm{x}_0,\cdot)\perp W\right)(u)
\end{aligned}$$

where H is defined in Eqn. 3-16. One knows that $s(u)H(\mathrm{x}_0,u)$ (up to a constant in u, irrelevant in the residue) satisfies the linear equation in u. So, Lemma 4.1 tells us that $(sH(\mathrm{x}_0,\cdot)\perp W)(u)$ is 1- and 2τ-translation invariant, i.e is elliptic. Since the sum of residues of an elliptic function vanishes:

$$\mathrm{W}(\mathrm{x}_0)=\frac{1}{4}\mathop{\mathrm{Res}}_{\substack{u\to 0,\frac{1}{2}\\ \tau,\tau+\frac{1}{2}}} \mathrm{d}u\left(sH(\mathrm{x}_0,\cdot)\perp W\right)(u)$$

Eventually, the assumption of u-parity/1-cut property for W allows us to rewrite:

$$\mathrm{W}(\mathrm{x}_0)=\frac{1}{2}\mathop{\mathrm{Res}}_{u\to\tau,\tau+\frac{1}{2}} \mathrm{d}u s(u)\, G(\mathrm{x}_0,u)W(u)$$

□

4.2 Application to the $\mathrm{W}_k^{(g)}$'s

4.2.1 Recursive residue formula for $W_k^{(g)}$, $2g-2+k>0$

In the matrix model formulation, $W_k^{(g)}(\mathrm{x}_1,\dots,\mathrm{x}_k)$ represents correlation of densities of eigenvalues, hence have to be integrated on some interval of $[a,b]$ to give physical results. So, the natural objects are rather the differential forms:

$$\mathrm{d}x_1\cdots\mathrm{d}x_k\,\mathrm{W}_k^{(g)}(\mathrm{x}_1,\dots,\mathrm{x}_k) \tag{4-5}$$

These forms can be pushed backwards under $u\mapsto x(u)$ to define differential forms $\omega_k^{(g)}$ on the u-domain $\left]-\frac{1}{2},\frac{1}{2}\right[\times]-\tau,\tau[$.

$$\begin{aligned}\omega_k^{(g)}(u_1,\dots,u_k) &= \mathrm{d}u_1 s(u_1)\cdots\mathrm{d}u_k s(u_k)\,W(u_1,\dots,u_k)\\ &= \mathrm{d}x(u_1)\cdots\mathrm{d}x(u_k)\,\mathrm{W}(x(u_1),\dots,x(u_k))\end{aligned} \tag{4-6}$$

and we recall that $s(u)=\frac{\mathrm{d}x}{\mathrm{d}u}$.

We are in position to prove our main result, which is more conveniently written with these differential forms $\omega_k^{(g)}$ than with the functions $\mathrm{W}_k^{(g)}$ or $W_k^{(g)}$.

Theorem 4.2 *For $2g-2+k>0$, $\mathrm{W}_k^{(g)}$ is a 1-cut solution of the homogeneous linear equation in each variable:*

$$\mathrm{W}_k^{(g)}(\mathrm{x}+i\epsilon,I)+\mathfrak{n}\mathrm{W}_k^{(g)}(-\mathrm{x},I)+\mathrm{W}_k^{(g)}(\mathrm{x}-i\epsilon,I)\underset{\epsilon\to 0}{=}0 \tag{4-7}$$

Thus, $\omega_k^{(g)}$ can be extended on the whole u-plane. It is a meromorphic form in each variable, with poles only at points where x is equal to a or b, and of order $2g-2+k$. We have the recursive formula:

$$\boxed{\omega_k(u_0,I)=\mathop{\mathrm{Res}}_{u\to\tau,\tau+\frac{1}{2}}\mathrm{d}u\,\mathcal{K}(u_0,u)\left[\text{"}\omega_{k+1}^{(g-1)}(u,\overline{u},I)\text{"}+\sum_{(J,h)}{}'\,\overline{\omega}_{|J|+1}^{(h)}(u,J)\,\overline{\omega}_{k-|J|}^{(g-h)}(\overline{u},I\setminus J)\right]} \tag{4-8}$$

- ▷ *$\sum'_{(J,h)}$ is a sum over $J\subseteq I, 0\leq h\leq g$, excluding $(J,h)=(\emptyset,0)$ and (I,g).*
- ▷ *$\overline{u}\neq u$ is defined locally at τ and $\tau+\frac{1}{2}$ by $x(u)=x(\overline{u})$:*

$$\begin{aligned}\overline{u}&=2\tau-u &&\textit{around}\quad \tau\\ \overline{u}&=2\tau+1-u &&\textit{around}\quad \tau+\frac{1}{2}\end{aligned} \tag{4-9}$$

▷ *The recursion kernel is:*

$$\mathcal{K}(u_0, u) = -\frac{1}{2} \frac{\int_{\overline{u}}^{u} \mathrm{d}x(u')\, \overline{\omega}_2^{(0)}(u_0, u')}{(y\mathrm{d}x)(u)} \tag{4-10}$$

It is a differential form in u_0, and the inverse of a differential form in u.

▷ *"$\omega_{k'}^{(g')}(u, \overline{u}, I)$" differs from $\omega_k'^{(g')}$ only for $(k', g') = (2, 0)$, and:*

$$\text{"}\omega_2^{(0)}(u, \overline{u})\text{"} = -\Big(\omega_2^{(0)}(u, u) + \mathfrak{n}\omega_2^{(0)}(u, \tau - u) + \omega_2^{(0)}(\tau - u, \tau - u)\Big) \tag{4-11}$$

▷ *When $(k, g) \neq (1, 1)$, the same formula is also true if one replaces $\omega_{k+1}^{(g-1)}(u, \overline{u}, I)$ by $\omega_{k+1}^{(g-1)}(u, u, I)$. However, one cannot in general replace $\overline{u}$ by u in the terms of $\sum_{J,h}'$.*

We have found that the stable correlation functions, i.e the stable generating functions for genus g maps, carrying closed loops, with k boundaries, fit in the same recursion structure as the stable[6] maps without loops (which is the case $\mathfrak{n} = 0$). Such a structure is called "topological recursion" [17]. As usual in this formalism, the computation of stable correlation functions is more uniform than the computation of the unstable ones, $\omega_1^{(0)}$ and $\omega_2^{(0)}$. This provides also a recursive algorithm to compute the $\mathrm{W}_k^{(g)}$'s: we need a stack of residues of length $2g - 2 + k$ in order to reach $\mathrm{W}_k^{(g)}$.

4.2.2 Commentary

Let us comment this result before the proof. First, it provides an example of the topological recursion with a deformation of the notion of spectral curve.

$$\mathcal{L}_{\mathfrak{n}} = (\mathcal{T} = \mathbf{C}/(\mathbf{Z} \oplus \tau\mathbf{Z}), x, y)$$

$\mathfrak{n} = -2\cos(\pi\mu)$ is the deformation parameter. The set of branch points is $\{0, \frac{1}{2}\} \subseteq \mathcal{T}$. But $y(u)$, $\mathcal{K}(u_0, u)$, … are multivalued function on $\mathcal{T}$, basically constructed with the function D_μ which takes a phase $e^{i\pi\mu}$ under τ-translation. We shall give a univocal meaning to the notions of "function" or "differential form on the spectral curve" in Section 5.3, where we need it for computations. Though, we keep their polysemy in the next paragraph.

Associated to a spectral curve, there is a Bergman kernel $\mathrm{d}u_0\mathrm{d}u\, \mathcal{B}(u_0, u)$, which is a meromorphic form on the spectral curve with an only pole when $x(u) = x(u_0)$, which is double. It is defined up to a holomorphic form on the spectral curve, and it allows

[6]A Riemann surface with k marked points and genus g is called stable if it has only a finite number of automorphisms. This happens iff $2g - 2 + k > 0$. We carry this notion to discrete maps and correlation function.

to express $\omega_2^{(0)}$. For example, in a 1-matrix model with two cuts, the spectral curve is a torus with x, y two meromorphic functions on it. In this case, a Bergman kernel is the $\mathrm{d}u_0\mathrm{d}u\,\wp(u-u_0)$, and it is equal to $2\overline{\omega}_2^{(0)}(u_0,u)+h\,\mathrm{d}u_0\mathrm{d}u$ for some $h\in\mathbf{C}$. Here, for $\mathcal{L}_{\mathfrak{n}}$, a deformed Bergman kernel can be defined as $\mathcal{B}(u,u_0)=\wp_\mu(u-u_0)$ with the deformed Weierstraß function of Eqn. 3-32, and it enters $\omega_2^{(0)}$ through Eqn. 3-31.

Eventually, we point out that the introduction of $\overline{u}$ (especially in the bracket term of Eqn. 4-8) is necessary in order to have the canonical form for the topological recursion. This fact is encountered in the two-matrix model as well, and its origin is still mysterious. Here, the $^{-}$ defined at the neighborhood of $\mathrm{u}(a_i)\in\{\mathrm{u}(a),\mathrm{u}(b)\}$ maps u to its symmetric with respect to $\mathrm{u}(a_i)$, i.e $\overline{u}=2\tau-u$ locally at $\mathrm{u}(a)=\tau$, and $\overline{u}=2\tau+1-u$ locally at $\mathrm{u}(b)=\tau+\frac{1}{2}$.

4.2.3 Proof of Theorem 4.2

proof:

$\mathrm{W}_k^{(g)}$ satisfies the loop equation:

$$2\left(\mathrm{W}_k^{(g)}(\cdot,I)\perp\overline{\mathrm{W}}_1^{(0)}\right)(\mathrm{x})=-\mathrm{E}_k^{(g)}(\mathrm{x},I)-\mathrm{Q}_k^{(g)}(\mathrm{x},I)$$

where

$$\begin{aligned}\mathrm{E}_k^{(g)}(\mathrm{x},I) &= \mathrm{W}_{k+1}^{(g-1)}(\mathrm{x},\mathrm{x},I)+\mathfrak{n}\mathrm{W}_{k+1}^{(g-1)}(\mathrm{x},-\mathrm{x},I)+\mathrm{W}_{k+1}^{(g-1)}(-\mathrm{x},-\mathrm{x},I)\\ &+\sum_{\substack{J\subseteq I,0\leq h\leq g\\(J,h)\neq(\emptyset,0),(I,g)}}\left(\mathrm{W}_{|J|+1}^{(h)}(\cdot,J)\perp\mathrm{W}_{k-|J|}^{(g-h)}(\cdot,I\setminus J)\right)(\mathrm{x})\end{aligned}$$

and

$$\mathrm{Q}_k^{(g)}(\mathrm{x},I)=\mathrm{P}_k^{(g)}(\mathrm{x},I)+P_k^{(g)}(-\mathrm{x},I)+\sum_{\mathrm{x}_i\in I}\frac{\partial}{\partial\mathrm{x}_i}\left[\mathrm{W}_{k-1}^{(g)}(I)\left(\frac{1}{\mathrm{x}-\mathrm{x}_i}-\frac{1}{\mathrm{x}+\mathrm{x}_i}\right)\right]$$

It is obvious that:

- ▷ $\mathrm{Q}_k^{(g)}(\cdot,I)$ is finite when $\mathrm{x}=a$ or b.
- ▷ $\mathrm{Q}_k^{(g)}$ has no cut on $[a,b]$.

One can check easily as well (using symmetry in all variables of correlation functions) that the combination of $\mathrm{W}_{k+1}^{(g-1)}(\cdot,\cdot,I)$ entering in $\mathrm{E}_k^{(g)}(\mathrm{x}_0,I)$ has no cut on $[a,b]$ as well. Since $\mathrm{Q}_k^{(g)}$ has no cut on $[a,b]$ as well, an easy recursion on $2g-2+k$ and the second point in Lemma 4.1 show that $\mathrm{E}_k^{(g)}(\cdot,I)$ has no cut on $[a,b]$. Hence, $\left(\mathrm{W}_k^{(g)}(\cdot,I)\perp\overline{\mathrm{W}}_1^{(0)}\right)$ has no cut on $[a,b]$. Furthermore, according to the 1-cut lemma 1.1, $\mathrm{W}_k^{(g)}(\cdot,I)$ has only one cut. Now, the assumptions in the third point of Lemma 4.1 are satisfied, with

$f = W_k^{(g)}(\cdot, I)$ and $g = \overline{W}_1^{(0)}$. So, $W_k^{(g)}(\cdot, I)$ must be a 1-cut solution of the homogeneous linear equation.

Afterwards, we apply the Cauchy formula (Eqn. 4-4), with arbitrary constants $\gamma_i(x_0)$:

$$\begin{aligned}
& W_k^{(g)}(x_0, I) \\
= \; & \frac{1}{2} \operatorname*{Res}_{u\to\tau,\tau+\frac{1}{2}} du s(u) \left(G(x_0,u) + \gamma_i(x_0)\right) W_k^{(g)}(u, I) \\
= \; & \frac{1}{2} \operatorname*{Res}_{u\to\tau,\tau+\frac{1}{2}} du s(u) \frac{G(x_0,u) + \gamma_i(x_0)}{y(u)} \left(W_k^{(g)}(u,I)y(u) + \underbrace{W_k^{(g)}(u-\tau,I)y(u-\tau)}_{\text{no pole in } \tau,\tau+\frac{1}{2}} \right) \\
= \; & -\frac{1}{2} \operatorname*{Res}_{u\to\tau,\tau+\frac{1}{2}} du s(u) \frac{G(x_0,u) + \gamma_i(x_0)}{y(u)} \left(E_k^{(g)}(u,I) + \underbrace{Q_k^{(g)}(u,I)}_{\text{is finite in } \tau,\tau+\frac{1}{2}} \right) \\
= \; & \operatorname*{Res}_{u\to\tau,\tau+\frac{1}{2}} du \left[-\frac{1}{2} \frac{G(x_0,u) + \gamma_i(x_0)}{s(u)y(u)} \right] E_k^{(g)}(u,I) \left(s(u)\right)^2
\end{aligned}$$

Besides, we have:

$$\begin{aligned}
G(x_0,u) &= \int_{u(\infty)}^{u} du' s(u') \left(2\overline{W}_2^{(0)}(x_0,u') + \mathfrak{n}\overline{W}_2^{(0)}(x_0,\tau-u') \right) \\
&= \int_{u(\infty)}^{u} du' s(u') \left(\overline{W}_2^{(0)}(x_0,u') - \overline{W}_2^{(0)}(x_0,u'-2\tau) \right) \\
&= \left(\int_{u(\infty)}^{u} du' s(u') \overline{W}_2^{(0)}(x_0,u') \right) - \left(\int_{2\tau-u(\infty)}^{2\tau-u} du' s(u') \overline{W}_2^{(0)}(x_0,u') \right) \\
&= \left(\int_{2\tau-u}^{u} du' s(u') \overline{W}_2^{(0)}(x_0,u') \right) + \left(\int_{u(\infty)}^{2\tau-u(\infty)} du' s(u') \overline{W}_2^{(0)}(x_0,u') \right) \\
\text{or} \quad &= \left(\int_{2\tau+1-u}^{u} du' s(u') \overline{W}_2^{(0)}(x_0,u') \right) + \left(\int_{u(\infty)}^{2\tau-u(\infty)} du' s(u') \overline{W}_2^{(0)}(x_0,u') \right) \\
&\quad - \left(\int_{0}^{1} du' s(u') \overline{W}_2^{(0)}(x_0,2\tau-u') \right)
\end{aligned}$$

We used the 1-translation invariance for the last line. We see that, in a neighborhood of $u(a_i) = \tau$ (resp. $u(a_i) = \tau + \frac{1}{2}$):

$$G(x_0,u) = \int_{\overline{u}}^{u} du' s(u') \overline{W}_2^{(0)}(x_0,u') + \text{cte}_i(x_0)$$

and we can cancel this constant by the choice of $\gamma_i(x_0)$. Thus, the quantity in bracket (the recursion kernel) is indeed given by Eqn. 4-10 (written completely in variable u).

We assume now $(k,g)\neq(1,1)$. Some terms in $E_k^{(g)}$ have $u-\tau$ as arguments, hence are regular at $\mathrm{u}(a)=\tau$ and $\mathrm{u}(b)=\tau+\frac{1}{2}$ and do not contribute to the residue. We make use of the homogeneous linear equation to write them apart. Since $(k+1,g-1)\neq(2,0)$, $W_{k+1}^{(g-1)}(u,\overline{u},I)$ exists, and:

$$\begin{aligned} E_k^{(g)}(u,I) &= -W_{k+1}^{(g-1)}(u,\overline{u},I) - \sum_{\substack{J\subseteq I, 0\le h\le g \\ (J,h)\neq(\emptyset,0),(I,g)}} \overline{W}_{|J|+1}^{(h)}(u,J)\overline{W}_{k-|J|}^{(g-h)}(\overline{u},I\setminus J) \\ &+ \sum_{\substack{J\subseteq I, 0\le h\le g \\ (J,h)\neq(\emptyset,0),(I,g)}} \overline{W}_{|J|+1}^{(h)}(u-\tau,J)\overline{W}_{k-|J|}^{(g-h)}(u-\tau,I\setminus J) \\ &+ W_{k+1}^{(g-1)}(u-\tau,u-\tau,I) \end{aligned}$$

Accordingly, we end up with:

$$\begin{aligned} W_k^{(g)}(u_0,I) &= \operatorname*{Res}_{u\to\tau,\tau+\frac{1}{2}} \mathrm{d}u\,\mathcal{K}(u_0,u)\Bigg[W_{k+1}^{(g)}(u,\overline{u},I)+ \\ &\qquad \sum_{\substack{J\subseteq I, 0\le h\le g \\ (J,h)\neq(\emptyset,0),(I,g)}} \overline{W}_{|J|+1}^{(h)}(u,J)\overline{W}_{k-|J|}^{(g-h)}(\overline{u},I\setminus J)\Bigg] s(u)(-s(u)) \end{aligned}$$

and we know that $s(\overline{u})=-s(u)$. This can be rewritten with differential forms and we find Eqn. 4-8.

Eventually, we have to write separately the case $(k,g)=(1,1)$, because the polar structure of $W_2^{(0)}$ is different. In $E_k^{(g)}(u,I)$, there is only one term, $-$"$W_2^{(0)}(u,\overline{u})$", defined by

$$\text{"}W_2^{(0)}(u,\overline{u})\text{"} \stackrel{\text{def}}{=} -\left(W_2^{(0)}(u,u)+\mathfrak{n}W_2^{(0)}(u,\tau-u)+W_2^{(0)}(\tau-u,\tau-u)\right)$$

If we set:

$$\begin{aligned} \text{"}\omega_2^{(0)}(u,\overline{u})\text{"} &\stackrel{\text{def}}{=} -(\mathrm{d}x(u))^2\left(W_2^{(0)}(u,u)+\mathfrak{n}W_2^{(0)}(u,\tau-u)+W_2^{(0)}(\tau-u,\tau-u)\right) \\ &= -\left(\omega_2^{(0)}(u,u)+\mathfrak{n}\omega_2^{(0)}(u,\tau-u)+\omega_2^{(0)}(\tau-u,\tau-u)\right), \end{aligned}$$

then Eqn. 4-8 is still correct. □

4.3 Examples

Finding $\mathrm{W}_3^{(0)}$ and $\mathrm{W}_1^{(1)}$ involve only one residue computation. Let us call v_i the branch points $\mathrm{u}(a)=\tau$ and $\mathrm{u}(b)=\tau+\frac{1}{2}$. We shall use the following notation:

$$\begin{aligned} \overline{\omega}_2^{(0)}(u_0,v_i) &\equiv \left(\frac{\overline{\omega}_2^{(0)}(u_0,u)}{\mathrm{d}u}\right)\Big|_{u=v_i} \\ \text{and } (\partial_2^m\overline{\omega}_2^{(0)})(u_0,v_i) &\equiv \frac{\partial^m}{\partial u^m}\left(\frac{\overline{\omega}_2^{(0)}(u_0,u)}{\mathrm{d}u}\right)\Big|_{u=v_i} \qquad \text{for } m\ge 1 \end{aligned}$$

I.e., we identify differentials forms and families of functions indexed by an atlas of **C** and having the proper transformation under a change of local coordinate, and we read all differential forms in the local coordinate u.

4.3.1 Behavior of the recursion kernel near a branch point

As a preliminary to the computations, we give the Laurent expansion of $\mathcal{K}(u, u_0)$ when $\delta = u - v_i \to 0$. In the local coordinate u:

$$\begin{aligned} \mathcal{K}(u_0, v_i + \delta) &= -\frac{\overline{\omega}_2^{(0)}(u_0, v_i)}{y'(v_i)s'(v_i)}\frac{1}{\delta} + \\ & \left[\frac{1}{4}\frac{\overline{\omega}_2^{(0)}(u_0, v_i)}{y'(v_i)s'(v_i)}\left(\frac{y'''(v_i)}{y'(v_i)} + \frac{s'''(v_i)}{s'(v_i)}\right) - \frac{1}{6}\frac{\partial_2^2\overline{\omega}_2^{(0)}(u_0, v_i)}{y'(v_i)s'(v_i)}\right]\delta + o(\delta) \end{aligned} \tag{4-12}$$

We notice $\mathcal{K}(u_0, v_i + \delta)$ is odd in δ since $\mathcal{K}$ is changed in $-\mathcal{K}$ when $u \leftrightarrow \overline{u}$. All the same, only the odd order derivative (with respect to u) of s and y do not vanish a priori at $u = v_i$.

4.3.2 Computation of $\mathrm{W}_3^{(0)}$

$\mathrm{W}_3^{(0)}$ is the generating function of pairs of pants. We have:

Theorem 4.3

$$\boxed{\omega_3^{(0)}(u_0, u_1, u_2) = \sum_i \frac{-2}{y'(v_i)s'(v_i)}\,\overline{\omega}_2^{(0)}(u_0, v_i)\,\overline{\omega}_2^{(0)}(u_1, v_i)\,\overline{\omega}_2^{(0)}(u_2, v_i)} \tag{4-13}$$

This trilinear identity is usual in matrix models. This structure appears in various situations (finite group theory, fusion rules in conformal field theory, Frobenius manifolds ...), sometimes under the name "WDVV equation" (Witten-Dijkgraaf-Verlinde-Verlinde).

4.3.3 Computation of $\mathrm{W}_1^{(1)}$

$\mathrm{W}_1^{(1)}$ is the generating function for rooted toroidal maps. According to the topological recursion:

$$\omega_1^{(1)}(u_0) = \sum_i \operatorname*{Res}_{u \to v_i} \mathcal{K}(u, u_0)\,"\omega_2^{(0)}(u, \overline{u})" \tag{4-14}$$

Using the Laurent expansion of $\mathcal{K}(u_0,u)$ when $u\to v_i$ (Eqn. 4-12), we find:

$$\begin{aligned}
\omega_1^{(1)}(u_0) &= -\frac{\overline{\omega}_2^{(0)}(u_0,v_i)}{y'(v_i)s'(v_i)}\mathop{\mathrm{Res}}_{u\to v_i}\frac{\text{"}\omega_2^{(0)}(u,\overline{u})\text{"}}{u-v_i} \\
&+\frac{1}{4}\frac{\overline{\omega}_2^{(0)}(u_0,v_i)}{y'(v_i)s'(v_i)}\left(\frac{y'''(v_i)}{y'(v_i)}+\frac{s'''(v_i)}{s'(v_i)}\right)\mathop{\mathrm{Res}}_{u\to v_i}(u-v_i)\text{"}\omega_2^{(0)}(u,\overline{u})\text{"} \\
&-\frac{1}{6}\frac{\partial_2^2\overline{\omega}_2^{(0)}(u_0,v_i)}{y'(v_i)s'(v_i)}\mathop{\mathrm{Res}}_{u\to v_i}(u-v_i)\text{"}\omega_2^{(0)}(u,\overline{u})\text{"}
\end{aligned}$$

Besides, we can make use of the representation of $\overline{\omega}_2^{(0)}$ as an infinite series (Eqn. 3-34) and compute explicitly:

$$\begin{aligned}
\frac{\text{"}\omega_2^{(0)}(u,\overline{u})\text{"}}{(\mathrm{d}u)^2} &= -\frac{1}{(\mathrm{d}u)^2}\left(\omega_2^{(0)}(u,u)+\mathfrak{n}\omega_2^{(0)}(u,\tau-u)+\omega_2^{(0)}(\tau-u,\tau-u)\right) \\
&= \lim_{u\to u_0}\left(\frac{s(u)s(u_0)}{(x(u)-x(u_0))^2}-\sum_{m\in\mathbf{Z}}\frac{(2r_m-\mathfrak{n}r_{m+1})\pi^2}{\sin^2\pi(u-u_0+m\tau)}\right) \\
&= \frac{1}{6}(\mathrm{S}x)(u)-\frac{\pi^2}{3}-\sum_{m\in\mathbf{Z}\setminus\{0\}}\frac{q_m\,\pi^2}{\sin^2\pi m\tau}
\end{aligned}$$

where we have set:

$$\begin{aligned}
q_m &= 2r_m-\mathfrak{n}r_{m+1}=\frac{\sin\pi(\mu-1)(m+1)}{\sin\pi\mu}=-q_{-(m+2)} \\
q_m &= (-1)^{m+1}U_m\left(-\frac{\mathfrak{n}}{2}\right) \qquad \text{if } m\geq 0
\end{aligned} \tag{4-15}$$

Here come into play the Chebyshev polynomials of the second kind U_m ($m\in\mathbf{N}$).

The *schwartzian derivative* of x with respect to u also appears:

$$(\mathrm{S}x)(u) \stackrel{\text{def}}{=} \frac{x'''(u)}{x'(u)}-\frac{3}{2}\left(\frac{x''(u)}{x'(u)}\right)^2 \tag{4-16}$$

When $u\to v_i$, the first term is finite whereas the second one has a double pole:

$$(\mathrm{S}x)(u)=-\frac{3}{2}\frac{1}{(u-v_i)^2}+O((u-v_i)^2) \qquad \text{when } u\to v_i \tag{4-17}$$

Hence, $\mathrm{Res}_{\,u\to v_i}(u-v_i)\text{"}\omega_2^{(0)}(u,\overline{u})\text{"}=-\frac{1}{4}$.

Eventually, let us introduce the so-called *connective projection*:

$$\begin{aligned}
S_{B,\mathrm{x}}(v_i) &\stackrel{\text{def}}{=} -6\mathop{\mathrm{Res}}_{u\to v_i}\frac{\text{"}\omega_2^{(0)}(u,\overline{u})\text{"}}{\mathrm{d}x(u)} \\
&= \frac{1}{s'(v_i)}\left(2\pi^2+\sum_{m\in\mathbf{Z}\setminus\{0\}}\frac{6q_m\pi^2}{\sin^2\pi m\tau}\right)
\end{aligned} \tag{4-18}$$

We have in terms of $S_{B,\mathrm{x}}$:

$$\operatorname*{Res}_{u\to v_i} \frac{\text{"}\omega_2^{(0)}(u,\overline{u})\text{"}}{(u-v_i)} = -\frac{1}{6} S_{B,\mathrm{x}}(v_i) \tag{4-19}$$

When we gather the results, we obtain:

Theorem 4.4

$$\omega_1^{(1)}(u_0) = \sum_i \frac{1}{24}\frac{(\partial_2^2\overline{\omega}_2^{(0)})(u_0,v_i)}{y'(v_i)s'(v_i)} + \frac{\overline{\omega}_2^{(0)}(u_0,v_i)}{y'(v_i)}\left(\frac{S_{B,\mathrm{x}}(v_i)}{6} - \frac{1}{16}\frac{s'''(v_i)}{s'(v_i)^2} - \frac{1}{16}\frac{y'''(v_i)}{y'(v_i)s'(v_i)}\right)$$

This expression is, again, similar to the 1-matrix model. We have given the details of the computation to illustrate that the steps are the same as in the 1-matrix model, provided the definition of "$\omega_2^{(0)}(u,\overline{u})$" is adapted to the $\mathcal{O}(\mathfrak{n})$ model according to Eqn. 4-11.

4.3.4 Useful identities

If one is interested in more explicit expressions in terms of a and b, let us define η_i such that:

$$\mathrm{y}(\mathrm{x}) \sim 2\eta_i\sqrt{\mathrm{x}-a_i} \qquad \text{when } \mathrm{x}\to a_i \tag{4-20}$$

Let us denote the other branch point with an index j, and note $\tilde{K} = ib/2K'$, such that $s(\mathrm{u}(\mathrm{x})) = \tilde{K}\sigma(\mathrm{x})$. We have the following expressions:

Theorem 4.5

$$\eta_i = -\frac{2}{\frac{\partial a_i}{\partial t}}\sqrt{\frac{(2-\mathfrak{n})(a_i^2-e_\mu^2)}{2a_i(a_i^2-a_j^2)}} \tag{4-21}$$

$$y'(v_i) = \lim_{\mathrm{x}\to a_i}\left(s(\mathrm{u}(\mathrm{x}))\frac{\mathrm{dy}}{\mathrm{dx}}\right) = 2\eta_i\tilde{K}\sqrt{2a_i(a_i^2-a_j^2)} \tag{4-22}$$

$$s'(v_i) = \tilde{K}^2 a_i(a_i^2-a_j^2) \tag{4-23}$$

proof:

We will see later (Theorem 5.2) that $\partial_t \mathrm{W}_1^{(0)}(\mathrm{x}) = \mathrm{f}_\mu(\mathrm{x})$. By definition of η_i:

$$\frac{\partial \mathrm{W}_1^{(0)}}{\partial t} \sim -\frac{\eta_i\frac{\partial a_i}{\partial t}}{\sqrt{\mathrm{x}-a_i}} \qquad \text{when } \mathrm{x}\to a_i$$

This can be compared to the behavior of $\mathrm{f}_\mu(\mathrm{x})$ at a_i given by Eqn. 6-15, and yields Eqn. 4-21. The other expressions are independent of the first and straightforward. □

5 Properties of the $W_k^{(g)}$'s

In this section, we show that the properties found with the topological recursion in the 1-matrix model are completely extendable to the $\mathcal{O}(\mathfrak{n})$ model. The special geometry structure is present, the stable F_g's ($g \geq 2$) are computed by the same integration formula, and (almost) the same formula exists for F_0 and F_1.

Once the properties for unstable quantities are checked, many proofs are recursions identical to the case of the 1-matrix model. They only use the residue recursion formula Eqn. 4-8 and a few other simple properties which are satisfied here. We choose not to reproduce the derivation of these known results, for sake of brevity, and to stress what really needs to be proven in the $\mathcal{O}(\mathfrak{n})$ model. Also, we refer to [17] for the complete proofs.

We refer to Appendix G for a summary of the notations introduced so far.

5.1 Symmetry

Though it is not obvious, the residue formula Eqn. 4-8 yields symmetric $\omega_k^{(g)}$'s. This statement is based on the symmetry of $\omega_2^{(0)}$, and the loop equations Eqn. 1-32. The proof is similar to the case of the one hermitian matrix model.

5.2 Homogeneity

The partition function of the $\mathcal{O}(\mathfrak{n})$ model (Eqn. 1-13), $\mathrm{Z}(t, t_3, t_4, \ldots)$, is invariant under

$$(t, t_0, t_1, t_2, t_3, \ldots) \mapsto (\lambda t, \lambda t_0, \lambda t_1, \lambda t_2, \lambda t_3, \ldots) \qquad \lambda > 0$$

Therefore:

$$\forall k, g \in \mathbf{N}^2 \qquad \left(t\frac{\partial}{\partial t} + \sum_{j\geq 0} t_j \frac{\partial}{\partial t_j}\right) \mathrm{W}_k^{(g)} = (2-2g-k)\mathrm{W}_k^{(g)} \tag{5-1}$$

5.3 Special geometry

What we call special geometry is the data of a non degenerate pairing, which to a differential form $\widehat{\Omega}$ on the spectral curve, associates:

- $\triangleright$ a cycle $\Omega^* \subseteq \mathbf{C}$, i.e in the u-plane,
- $\triangleright$ a germ of holomorphic function Λ_Ω on Ω^*,

with the following property, for all $k, g \geq 0$:

$$\begin{aligned} \delta_\epsilon \overline{\omega}_k^{(g)}(I) &= \int_{\Omega^*} \mathrm{d}u \Lambda_\Omega(u) \overline{\omega}_{k+1}^{(g)}(u, I) \\ \text{when } y\mathrm{d}x &\to y\mathrm{d}x + \epsilon\widehat{\Omega} \qquad \epsilon \to 0 \end{aligned} \tag{5-2}$$

Though, for the $\mathcal{O}(\mathfrak{n})$ model, we understand the notion of "differential form on the spectral curve" in a slightly deformed way. The manifold for the spectral curve can be considered as $\mathcal{T}$ or $\mathbf{C}$. If $\widehat{\Omega}$ is a differential meromorphic form on $\mathbf{C}$, we define:

$$\Omega = \frac{1}{4-\mathfrak{n}^2}\left(2\widehat{\Omega}(u) + \mathfrak{n}\widehat{\Omega}(\tau - u)\right) \tag{5-3}$$

For us, $\widehat{\Omega}$ is a differential form on the spectral curve if $\Omega/\mathrm{d}u$ is a u-even solution of Eqn. 2-6 (that is, $\widehat{\Omega}$ has the same properties as $y\mathrm{d}x$). A consequence of Eqn. 5-2 is that the pairing is given by integration of $\overline{\omega}_2^{(0)}$ against Ω^*:

$$\Omega(u_0) = \int_{\Omega^*} \mathrm{d}u\, \Lambda_\Omega(u)\overline{\omega}_2^{(0)}(u_0, u) \tag{5-4}$$

$$\widehat{\Omega}(u_0) = \int_{\Omega^*} \mathrm{d}u\, \Lambda_\Omega(u)\left(2\overline{\omega}_2^{(0)}(u_0, u) - \mathfrak{n}\overline{\omega}_2^{(0)}(\tau - u_0, u)\right) \tag{5-5}$$

In the next paragraphs, we compute the variations of the $\omega_k^{(g)}$'s with respect to the t_j's (which is a simple task) and to t (which gives an interesting result). Roughly speaking, we hold x fixed, but the value u(x) changes since a and b are determined by the consistency relations and depend on t and on the t_j's. To avoid confusions, we note $\partial_t|_x$, $\partial_{t_j}|_x$ these variations. By definition:

$$\left(\frac{\partial}{\partial t_\bullet}\Big|_{\mathrm{x}} \omega_k^{(g)}\right)(u_1, \ldots, u_k) = \mathrm{d}x(u_1)\cdots\mathrm{d}x(u_k)\left(\frac{\partial}{\partial t_\bullet}\mathrm{W}\right)(x(u_1), \ldots, x(u_k)) \tag{5-6}$$

In $\mathrm{W}_1^{(0)}(\mathrm{x})$ had several cuts in the x plane, variations with respect to filling fractions of the cuts would also be interesting. but they do not exists in the 1-cut case. We prove below that these variations fit in the special geometry structure.

5.4 Variations of the t_j's

Theorem 5.1

$$\forall k, g \geq 0,\ \forall j \geq 3 \qquad \left(\frac{\partial}{\partial t_j}\Big|_{\mathrm{x}} \omega_k^{(g)}\right)(I) = \mathop{\mathrm{Res}}_{u \to \mathrm{u}(\infty)} \mathrm{d}u\, \frac{x(u)^j}{j}\, \omega_{k+1}^{(g)}(u, I) \tag{5-7}$$

The cycle associated to t_j is $(\Omega_j^*, \Lambda_j) = (\frac{1}{2i\pi}\mathcal{C}_\infty, \frac{x(u)^j}{j})$, where $\mathcal{C}_\infty$ is a contour surrounding $\mathrm{u}(\infty)$ *and no other special point.*

proof:

We start from the definition of the correlation function before the topological expansion:

$$\begin{aligned}\left(\frac{\partial}{\partial t_1}\mathrm{W}_k\right)(I) &= -\Big\langle \frac{N}{t}\frac{\mathrm{Tr}M^j}{j}\prod_{\mathrm{x}_i \in I}\mathrm{Tr}\frac{1}{\mathrm{x}_i - M}\Big\rangle_C \\ &= \mathop{\mathrm{Res}}_{\mathrm{x}\to\infty} \mathrm{dx}\,\Big\langle \frac{N}{t}\frac{\mathrm{x}^j}{j}\mathrm{Tr}\frac{1}{\mathrm{x}-M}\prod_{\mathrm{x}_i\in I}\mathrm{Tr}\frac{1}{\mathrm{x}_i - M}\Big\rangle_C\end{aligned}$$

After expansion in powers of $\frac{N}{t}$, and translation into the language of differential forms, we obtain Eqn. 5-7. $\square$

5.5 Variation of t

We begin with the obvious remark that $\partial_t|_{\mathrm{x}}\,\overline{\omega}_k^{(g)} = \partial_t|_{\mathrm{x}}\,\omega_k^{(g)}$. Let us compute first the variations of the stable correlation forms $\omega_1^{(0)}$ and $\omega_2^{(0)}$.

Theorem 5.2

$$\left(\frac{\partial}{\partial t}\Big|_{\mathrm{x}} \overline{\omega}_1^{(0)}\right)(u) = \int_{\mathrm{u}(\infty)}^{\mathrm{u}(\overline{\infty})} \overline{\omega}_2^{(0)}(\cdot, u) \tag{5-8}$$

We have defined the point $\overline{\infty}$ on $\mathcal{S}$ such that $u(\overline{\infty}) = \frac{-1-\tau}{2}$, see Figs. 7-8. The cycle associated to t is $(\Omega_t^*, \Lambda_t) = ([\frac{-1+\tau}{2}, \frac{-1-\tau}{2}], 1)$. There is a corresponding form on the spectral curve: $\Omega_t(u) = \mathrm{d}x(u)\, f_\mu(u)$.

Theorem 5.3

$$\begin{aligned}\left(\frac{\partial}{\partial t}\Big|_{\mathrm{x}} \overline{\omega}_2^{(0)}\right)(u_1,u_2) &= \int_{\Omega_t^*} \omega_3^{(0)}(\cdot, u_1, u_2) && \text{(5-9)}\\ &= \sum_i \operatorname*{Res}_{u\to v_i} \mathrm{d}u \left[\mathcal{K}(u_1,u)\overline{\omega}_2^{(0)}(u,u_2)\left(\frac{\partial}{\partial t}\Big|_{\mathrm{x}} \overline{\omega}_1^{(0)}\right)(u)\right] + (u_1 \leftrightarrow u_2) && \text{(5-10)}\end{aligned}$$

To prove the formula 5-2 for $2g-2+k>0$, we need first a lemma for the variation of the recursive kernel $\mathcal{K}$ (Eqn. 4-10).

Lemma 5.1

$$\left[\left(\frac{\partial}{\partial t}\Big|_{\mathrm{x}} + \frac{\partial \ln y}{\partial t}(u)\right)\mathcal{K}\right](u_0,u) = \sum_i \operatorname*{Res}_{u'\to v_i} \mathrm{d}u' \left[\mathcal{K}(u_0,u')\mathcal{K}(u',u)\left(\frac{\partial}{\partial t}\Big|_{\mathrm{x}} \overline{\omega}_1^{(0)}\right)(u')\right] \tag{5-11}$$

(5-12)

Then, the general result follows:

Theorem 5.4 *For all $k, g \neq (0,0)$:*

$$\left(\frac{\partial}{\partial t}\Big|_{\mathrm{x}} \overline{\omega_k^{(g)}}\right)(I) = \int_{\Omega_t^*} \overline{\omega}_{k+1}^{(g)}(\cdot, I) \tag{5-13}$$

This will be completed in Section 5.7 with expressions for the derivatives of F_0.

Proof of Theorem 5.2

We derive the properties of $\partial_t|_{\mathrm{x}}\,\omega_1^{(0)}$ from those of $\mathrm{W}_1^{(0)}$ listed in Section 3.1. They coincide with those of f_μ listed in Section 2.4, and both of these functions are 1-cut solution of the homogeneous linear equation. Hence $\left(\frac{\partial}{\partial t}\Big|_{\mathrm{x}}\mathrm{W}_1^{(0)}\right)(\mathrm{x}) = \mathrm{f}_\mu(\mathrm{x})$. Now, we would like to represent $\mathrm{f}_\mu(\mathrm{x}_0)\mathrm{d}\mathrm{x}_0$ as an integral of $\overline{\omega}_2^{(0)}$ over some path in the u-plane. Having a look at $H(x_0,x)$ in Theorem 3.3, it is possible to write:

$$f_\mu(u_0)\mathrm{d}x(u_0) = \mathrm{cte}\times\int_{\frac{-1+\tau}{2}}^{u^*}\overline{\omega}_2^{(0)}(\cdot,u_0) \tag{5-14}$$

if we take $u^*\neq \mathrm{u}(\infty)$ such that $f_\mu(u^*)=0$. An accurate value is $u^* = \mathrm{u}(\overline{\infty}) = \mathrm{u}(\infty)-\tau = \frac{-1-\tau}{2}$. Using the homogeneous linear equation, one computes:

$$f_\mu(u_0)\sim\frac{\mathfrak{n}-1}{x(u_0)}\qquad\text{when } u_0\to\mathrm{u}(\overline{\infty}) \tag{5-15}$$

and find that $f_\mu(u_0)\mathrm{d}x(u_0) = H(u_0,\overline{\infty})\mathrm{d}x(u_0)$.

Proof of Theorem 5.3-Lemma 5.1

We compute with help successively of Eqns. 5-1,5-7,4-13,5-8, and comparison with Eqn. 4-12:

$$\begin{aligned}
\left(t\frac{\partial}{\partial t}\Big|_{\mathrm{x}}\overline{\omega}_2^{(0)}\right)(u_0,u_1) &= -\sum_j\left(t_j\frac{\partial}{\partial t_j}\Big|_{\mathrm{x}}\overline{\omega}_2^{(0)}\right)(u_0,u_1)\\
&= -\operatorname*{Res}_{u\to\mathrm{u}(\infty)}\mathrm{V}(x(u))\omega_3^{(0)}(u,u_0,u_1)\\
&= 2\sum_i\frac{\overline{\omega}_2^{(0)}(v_i,u_0)\,\overline{\omega}_2^{(0)}(v_i,u_1)}{y'(v_i)s'(v_i)}\operatorname*{Res}_{u\to\mathrm{u}(\infty)}\Big(\mathrm{V}(x(u))\overline{\omega}_2^{(0)}(v_i,u)\Big)\\
&= 2\sum_i\frac{\overline{\omega}_2^{(0)}(v_i,u_0)\,\overline{\omega}_2^{(0)}(v_i,u_1)}{y'(v_i)s'(v_i)}\left[-\left(t\frac{\partial}{\partial t}\Big|_{\mathrm{x}}\overline{\omega}_1^{(0)}\right)(v_i)+\overline{\omega}_1^{(0)}(v_i)\right]\\
&= -2\sum_i\frac{\overline{\omega}_2^{(0)}(v_i,u_0)\,\overline{\omega}_2^{(0)}(v_i,u_1)}{y'(v_i)s'(v_i)}\left(t\int_{\frac{-1+\tau}{2}}^{\frac{-1-\tau}{2}}\overline{\omega}_2^{(0)}(v_i,u)\right)\\
&= t\int_{\frac{-1+\tau}{2}}^{\frac{-1-\tau}{2}}\omega_3^{(0)}(u,u_0,u_1)\\
\text{or}\quad &= 2\sum_i\operatorname*{Res}_{u\to v_i}\left[\mathcal{K}(u_0,u)\,\overline{\omega}_2^{(0)}(u,u_1)\,\left(t\frac{\partial}{\partial t}\Big|_{\mathrm{x}}\overline{\omega}_1^{(0)}\right)(u)\right]
\end{aligned}$$

The last line is an application of the lemma above. By symmetry of $(\partial_t|_{\mathrm{x}}\,\overline{\omega}_2^{(0)})(u_0,u_1)$, we can also distribute u_0 and u_1 in two ways, as in Eqn. 5-10.

Then, it is easy to prove Theorem 5.1 from the expression of $\partial_t|_{\mathrm{x}}\,\overline{\omega}_2^{(0)}$ in the last line.

Proof of Theorem 5.4

For $(k,g) \neq (0,0)$, the proof is similar to the 1-matrix model. We shall consider the case of $\partial_t \mathrm{F}_0$ later, in Section 5.7.

5.6 Integration formula

The inverse operation, consisting in recovering $\omega_k^{(g)}$ from $\omega_{k+1}^{(g)}$, can also be performed by a residue calculation at the branch points.

Theorem 5.5 *For all k, g such that $2g-2+k \geq 0$:*

$$\boxed{\sum_i \operatorname*{Res}_{u\to v_i} \phi(u)\, \omega_{k+1}^{(g)}(u, I) = (2-2g-k)\omega_k^{(g)}(I)} \tag{5-16}$$

where $\mathrm{d}\phi = y\mathrm{d}x$*. In particular, for* $g \geq 2$*:*

$$\mathrm{F}_g = \frac{1}{2-2g} \sum_i \operatorname*{Res}_{u\to v_i} \phi(u)\, \omega_1^{(g)}(u) \tag{5-17}$$

proof:

Similar to the 1-matrix model. □ Though, F_0 and F_1 cannot be found by this method.

5.7 Computation of F_0

5.7.1 Formal results

As a formal series, F_0 is the generating function of spherical maps. It is also the free energy of the model in the thermodynamic limit. In other words, it gives the leading order of $\ln \mathrm{Z}$ modulo the remark about convergent matrix integrals (Section 1.9). Heuristically, one can find a formula to compute it directly, which one may find convenient or not. It comes from the saddle point technique applied to the partition function Z of Eqn. 1-13 (see Appendix B for details).

Proposition 5.1 *Heuristics for* F_0

$$\begin{aligned} \mathrm{F}_0 &= \lim_{N\to\infty} \frac{t^2}{N^2} \ln \mathrm{Z}_N \\ &= \frac{1}{2}\int_a^b \mathrm{d}x_0 \int_a^b \mathrm{d}x \varrho(x)\varrho(x_0)\left(2\ln(x-x_0) - \mathfrak{n}\ln(x+x_0)\right) - \frac{1}{t}\int_a^b \mathrm{d}x\varrho(x)\mathrm{V}(x) \end{aligned} \tag{5-18}$$

where:

$$\varrho(\mathrm{x}) = -\frac{1}{2i\pi t}\mathrm{y}(\mathrm{x}) \tag{5-19}$$

is the density of eigenvalues (which has $[a,b]$ as support on the real line) in the thermodynamic limit.

In fact, we are going to prove:

Theorem 5.6

$$\frac{\partial^3 \mathrm{F}_0}{\partial t^3} = \sum_i \frac{-2\left(\lim_{u\to v_i} f_\mu(u)s(u)\right)^3}{y'(v_i)s'(v_i)} \tag{5-20}$$

$$\begin{aligned}\frac{\partial^2 \mathrm{F}_0}{\partial t^2} &= (2-\mathfrak{n})\ln x(u_0) + \mathrm{C}_2 + \\ &+ \int_{\mathrm{u}(\infty)}^{u_0}\left(f_\mu(u) - \frac{1}{x(u)}\right)\mathrm{d}x(u) + \int_{u_0}^{\mathrm{u}(\overline{\infty})}\left(f_\mu(u) + \frac{1-\mathfrak{n}}{x(u)}\right)\mathrm{d}x(u)\end{aligned} \tag{5-21}$$

$$\begin{aligned}\frac{\partial \mathrm{F}_0}{\partial t} &= (2-\mathfrak{n})t\ln x(u_0) - \mathrm{V}(x(u_0)) + \mathrm{C}_2 t \\ &+ \int_{\mathrm{u}(\infty)}^{u_0}\left(\omega_1^{(0)}(u) - \frac{t\,\mathrm{d}x(u)}{x(u)}\right) \\ &+ \int_{u_0}^{\mathrm{u}(\overline{\infty})}\left(\omega_1^{(0)}(u) - \mathrm{V}'(x(u))\mathrm{d}x(u) + \frac{(1-\mathfrak{n})t\,\mathrm{d}x(u)}{x(u)}\right)\end{aligned} \tag{5-22}$$

$$\begin{aligned}\mathrm{F}_0 &= \left(1-\frac{\mathfrak{n}}{2}\right)t^2\ln x(u_0) - \frac{t}{2}\mathrm{V}(x(u_0)) + \frac{1}{2}\mathrm{C}_2 t^2 + \frac{1}{2}\operatorname*{Res}_{\mathrm{x}\to\infty}\mathrm{dx}\,\mathrm{V}(\mathrm{x})\mathrm{W}_1^{(0)}(\mathrm{x}) \\ &+ \frac{t}{2}\int_{\mathrm{u}(\infty)}^{u_0}\left(\omega_1^{(0)}(u) - \frac{t\,\mathrm{d}x(u)}{x(u)}\right) \\ &+ \frac{t}{2}\int_{u_0}^{\mathrm{u}(\overline{\infty})}\left(\omega_1^{(0)}(u) - \mathrm{V}'(x(u))\mathrm{d}x(u) + \frac{(1-\mathfrak{n})t\,\mathrm{d}x(u)}{x(u)}\right)\end{aligned} \tag{5-23}$$

u_0 can be any point (in particular, one can choose a branch point for u_0), and these expressions do not depend on u_0. C_2 is a constant depending only on $\mathfrak{n}$.

Let us make a few comments on these results. They show that the relation of special geometry (Theorem 5.4) can be extended with some care to F_0, i.e $\partial_t|_{\mathrm{x}}$ on correlation forms can be replaced by $\int_{\Omega_t^*}\mathrm{d}x(u)\frac{\partial}{\partial\mathrm{V}(x(u))}$. This is directly true for $\partial_t^3\mathrm{F}_0$, and it is true for $\partial_t^2\mathrm{F}_0$ and $\partial_t\mathrm{F}_0$ modulo a regularization of the integral. The final formula comes from the homogeneity formula Eqn. 5-1, which expresses F_0 in terms of $t\partial_t\mathrm{F}_0$. It is rigorous and gives a precise sense to the heuristic formula. These formulas are very close to those for the 1-matrix model (i.e $\mathfrak{n} = 0$), and the principle of their proof is the same (the saddle point equation Eqn. 2-2 plays the key role).

The integration constant in $\partial_t \mathrm{F}_0$ is included in the choice of a primitive "V" of V$'$. Since we really defined the $\mathcal{O}(\mathfrak{n})$ matrix model with a potential V such that $\hat{\mathrm{V}}(0) = 0$, the formula is correct when we take the initial V as primitive. Eventually, we have normalized the generating function such that $\mathrm{Z} \equiv 1$ when $t \to 0$. This implies that $\mathrm{F}_0 \in O(t)$ when $t \to 0$, so there is no integration constant in the last step (in F_0).

5.7.2 Explicit results

As usual in matrix models and in the topological recursion, $\partial_t^3 \mathrm{F}_0$ and $\partial_t^2 \mathrm{F}_0$ are universal and depend only on a and b, whereas $\partial_t \mathrm{F}_0$ and F_0 really depend on the whole potential. We can compute explicitly $\partial_t^3 \mathrm{F}_0$ with help of the relations of Section 4.3.4. This is essentially a result of [11, 16].

Theorem 5.7 *The third derivative of* F_0 *can be expressed in terms of* $(\partial_t a, \partial_t b)$ *as:*

$$\boxed{\frac{\partial^3 \mathrm{F}_0}{\partial t^3} = \frac{(2-\mathfrak{n})}{a^2 - b^2}\left[\frac{a^2 - e_\mu^2}{a}\frac{\partial a}{\partial t} - \frac{b^2 - e_\mu^2}{b}\frac{\partial b}{\partial t}\right]} \tag{5-24}$$

It can be in principle integrated as:

$$\frac{\partial^2 \mathrm{F}_0}{\partial t^2} = \left(1 - \frac{\mathfrak{n}}{2}\right) \ln\left[a^2\, g\left(1 - \frac{a^2}{b^2}\right)\right] + \hat{\mathrm{C}}_2 \tag{5-25}$$

where $\hat{C}_2$ *is a constant independent of* t, *and* g *a function defined by:*

$$\frac{\mathrm{d}\ln g}{\mathrm{d}m} = \frac{1}{m(1-m)\,\mathrm{cd}^2_{\sqrt{m}}(\mu K(\sqrt{m}))} \tag{5-26}$$

and $\mathrm{cd}_k(w) = \frac{\mathrm{dn}_k(w)}{\mathrm{cn}_k(w)}$.

This expression is the best suited for numerics.

Besides, the infinite series representation we developed in Section 3.6 provides a new way to find $\partial_t^2 \mathrm{F}_0$. When one computes carefully the regularized integrals of Eqn. 5-21, one obtains:

Theorem 5.8

$$\frac{\partial^2 \mathrm{F}_0}{\partial t^2} = \ln\left[e^{\mathrm{C}_2 - \frac{i\pi\mathfrak{n}}{2}}\left(\frac{\pi b}{2K'}\right)^{2-\mathfrak{n}} \frac{\text{"}\Psi(\mathrm{u}(\infty), \mathrm{u}(\infty))\text{"}}{\text{"}\Psi(\mathrm{u}(\overline{\infty}), \mathrm{u}(\infty))\text{"}}\right] \tag{5-27}$$

$"\Psi(\cdot,\cdot)"$ *stands for the infinite product of Eqn. 3-37, where the vanishing factors like* $"\sin(0)"$ *are omitted. Namely:*

$$\boxed{\frac{\partial^2 \mathrm{F}_0}{\partial t^2} = \ln\left[e^{\mathrm{C}_2 - \frac{i\pi\mathfrak{n}}{2}}\left(\frac{\pi b}{2K'}\right)^{2-\mathfrak{n}} \text{"}\prod_{m\in\mathbf{Z}}\text{"}\left(\frac{\sin\pi m\tau\,\sin\pi(m-2)\tau}{\sin^2\pi m\tau}\right)^{r_m}\right]} \tag{5-28}$$

where $r_m = \frac{2\cos\pi(\mu-1)m}{4-\mathfrak{n}^2}$ *was introduced in Theorem 3.5. Since* $\partial_t^2\mathrm{F}_0$ *is real,* e^{C_2} *must cancel the* $\mathfrak{n}$ *dependant phase.*

A formula like 5-27, basically a ln of ratio of theta functions taken at $\mathrm{x} = \infty$ points, is not unknown in matrix models: it can be encountered for $\partial_t^2\mathrm{F}_0$ in the 1-matrix model with two cuts (when the spectral curve is a torus). The definition of the "theta function" is only deformed in our case (cf. Section 3.7).

5.7.3 Proof of Theorem 5.6

A formula like in Theorem 5.4 for $k = 0$ cannot be true because the integral of $\omega_1^{(0)}$ over Ω_t^* is divergent. The idea is to compute higher derivatives of F_0, and integrate them. We first notice that $\int_{\Omega_t^*}$ and $\partial_t|_{\mathrm{x}}$ commutes. Indeed, if $\widehat{\omega}$ is a differential form on the spectral curve, the integral of ω over Ω_t^* in the u-plane is also an integral over the path $\infty \longrightarrow \overline{\infty}$ in the double-sheeted x-plane.

- ▷ The simplest object to compute is $\partial_t^3\mathrm{F}_0$, for $\iiint_{(\Omega_t^*)^3}\omega_3^{(0)}$ is finite. One can prove, by derivating the homogeneity relation 5-1 and using the previous results about variations of correlation forms, that:

$$\frac{\partial^3\mathrm{F}_0}{\partial t^3} = \iiint_{(\Omega_t^*)^3}\omega_3^{(0)}$$

  We found the expression for $\omega_3^{(0)}$ in Eqn. 4-13. Hence:

$$\begin{aligned}\frac{\partial^3\mathrm{F}_0}{\partial t^3} &= \sum_i \frac{-2\left(\partial_t|_{\mathrm{x}}\omega_1^{(0)}\right)^3(v_i)}{y'(v_i)s'(v_i)} \\ &= \sum_i \frac{-2\left(\lim_{u\to v_i} f_\mu(u)s(u)\right)^3}{y'(v_i)s'(v_i)}\end{aligned}$$

- ▷ Next, we want to give a sense to:

$$\frac{\partial^2\mathrm{F}_0}{\partial t^2} = \text{"}\int_{\Omega_t^*}\partial_t|_{\mathrm{x}}\omega_1^{(0)}\text{"}$$

such that its derivative with respect to t at x is the correct expression:

$$\frac{\partial^3 \mathrm{F}_0}{\partial t^3} = \int_{\Omega_t^*} \partial_t^2|_{\mathrm{x}} \omega_1^{(0)}$$

This is possible if we add a (function of x independent of t)$\times$(dx) to $\partial_t|_{\mathrm{x}}\omega_1^{(0)}(u) = f_\mu(u)\mathrm{d}x(u)$, such that the integral is finite. We have to subtract the $\frac{1}{\mathrm{x}}$ terms of f_μ at both points $\mathrm{x} \to \infty$ and $\mathrm{x} \to \overline{\infty}$. To this purpose, we cut the integral in two with an arbitrary point u_0: from $\mathrm{u}(\infty)$ to u_0, and from u_0 to $\mathrm{u}(\overline{\infty})$. The well-defined and correct expression for $\partial_t^2 \mathrm{F}_0$ is:

$$\begin{aligned} \frac{\partial^2 \mathrm{F}_0}{\partial t^2} &= (2-\mathfrak{n}) \ln x(u_0) \\ &\quad + \int_{\mathrm{u}(\infty)}^{u_0} \left(f_\mu(u) - \frac{1}{x(u)} \right) \mathrm{d}x(u) + \int_{u_0}^{\mathrm{u}(\overline{\infty})} \left(f_\mu(u) + \frac{1-\mathfrak{n}}{x(u)} \right) \mathrm{d}x(u) \end{aligned}$$

up to an integration constant C_2.

- ▷ We would like to integrate this relation once more with respect to t, i.e. to give a sense in a similar way to:

$$\begin{aligned} \frac{\partial \mathrm{F}_0}{\partial t} &= (2-\mathfrak{n})t \ln x(u_0) + \mathrm{C}_2 t + \int_{\mathrm{u}(\infty)}^{u_0} \left(\omega_1^{(0)}(u) - \frac{t\,\mathrm{d}x(u)}{x(u)} \right) + \\ &\quad +\text{"} \int_{u_0}^{\mathrm{u}(\overline{\infty})} \left(\omega_1^{(0)}(u) + \frac{t(1-\mathfrak{n})\,\mathrm{d}x(u)}{x(u)} \right) \text{"} \end{aligned}$$

but the last integral is divergent. As a matter of fact, with help of Eqn. 2-1, we find that:

$$\mathrm{W}_1^{(0)}(\mathrm{x}) = \mathrm{V}'(\mathrm{x}) + \frac{t(\mathfrak{n}-1)}{\mathrm{x}} + o\left(\frac{1}{\mathrm{x}}\right) \qquad \text{when } \mathrm{x} \to \overline{\infty}$$

so the correct answer is:

$$\begin{aligned} \frac{\partial \mathrm{F}_0}{\partial t} &= (2-\mathfrak{n})t \ln x(u_0) + \mathrm{C}_2 t + \int_{\mathrm{u}(\infty)}^{u_0} \left(\omega_1^{(0)}(u) - \frac{t\,\mathrm{d}x(u)}{x(u)} \right) + \\ &\quad -\mathrm{V}(x(u_0)) + \int_{u_0}^{\mathrm{u}(\overline{\infty})} \left(\omega_1^{(0)}(u) - \mathrm{V}'(x(u))\mathrm{d}x(u) + \frac{t(1-\mathfrak{n})\,\mathrm{d}x(u)}{x(u)} \right) \end{aligned}$$

- ▷ Eventually, we reach F_0 thanks to the homogeneity relation:

$$\begin{aligned} \mathrm{F}_0 &= \frac{1}{2}\left(t\frac{\partial \mathrm{F}_0}{\partial t} + \sum_{j\geq 0} t_j \frac{\partial \mathrm{F}_0}{\partial t_j} \right) \\ &= \left(1 - \frac{\mathfrak{n}}{2}\right) t^2 \ln x(u_0) - \frac{t}{2}\mathrm{V}(x(u_0)) + \operatorname*{Res}_{\mathrm{x}\to\infty} \mathrm{dx}\, \mathrm{V}(\mathrm{x})\mathrm{W}_1^{(0)}(\mathrm{x}) \\ &\quad + \frac{t}{2} \int_{\mathrm{u}(\infty)}^{u_0} \left(\omega_1^{(0)}(u) - \frac{t\,\mathrm{d}x(u)}{x(u)} \right) \\ &\quad + \frac{t}{2} \int_{u_0}^{\mathrm{u}(\overline{\infty})} \left(\omega_1^{(0)}(u) - \mathrm{V}(x(u))\mathrm{d}x(u) + \frac{t(1-\mathfrak{n})\,\mathrm{d}x(u)}{x(u)} \right) \end{aligned}$$

▷ The homogeneity relation also implies that C_2, which is independent of t by definition, must be independent of V. So, it can only depend on $\mathfrak{n}$.

5.8 Computation of F_1

As a formal series, F_1 is the generating functions of toroidal maps. It is also the finite size correction to the free energy (modulo Section 1.9). It can be reached by Theorem 5.4, i.e. integration of $\omega_1^{(1)}$ (given by Theorem 4.4):

$$\begin{aligned}
\frac{\partial \mathrm{F}_1}{\partial t} &= \int_{\Omega_t^*} \omega_1^{(1)} \\
&= \sum_i \frac{1}{24} \frac{\int_{u_0\in\Omega_t^*} (\partial_2^2 \overline{\omega}_2^{(0)})(u_0, v_i)}{y'(v_i)s'(v_i)} + \\
&\quad + \frac{\int_{u_0\in\Omega_t^*} \overline{\omega}_2^{(0)}(u_0, v_i)}{y'(v_i)} \left(\frac{S_B(v_i)}{6} - \frac{1}{16}\frac{s'''(v_i)}{s'(v_i)^2} - \frac{1}{16}\frac{y'''(v_i)}{y'(v_i)s'(v_i)}\right) \\
&= \sum_i \frac{1}{24}\frac{(sf_\mu)''(v_i)}{y'(v_i)s'(v_i)} + \frac{(sf_\mu)(v_i)}{y'(v_i)} \left(\frac{S_B(v_i)}{6} - \frac{1}{16}\frac{s'''(v_i)}{s'(v_i)^2} - \frac{1}{16}\frac{y'''(v_i)}{y'(v_i)s'(v_i)}\right)
\end{aligned} \tag{5-29}$$

The integration of the right hand side has been studied in the literature [14], and relies on the computation of $\partial_t|_{\mathrm{x}}(y'(v_i))$ and the comparison with Eqn. 5-29. This equation is essentially the same as the one encountered in the one matrix model. So, we give directly the result:

Theorem 5.9

$$\boxed{\mathrm{F}_1 = \frac{1}{12}\ln[\tau_B((a_i)_i)] + \frac{1}{24}\ln\left(\prod_i y'(a_i)\right)} \tag{5-30}$$

τ_B is the Bergman tau function of our problem, which is a function of the position of the branch points in the x*-plane, $(a_i)_i$, and which is defined by:*

$$\forall i, \qquad \frac{\partial \ln \tau_B}{\partial a_i} = S_B(\mathrm{u}(a_i)) \tag{5-31}$$

The peculiar expression (Eqn. 4-13) of $\omega_3^{(0)}$ ensures that the form $\sum_i \mathrm{d}a_i\, S_B(\mathrm{u}(a_i))$ is closed.

6 Large maps

As an application of the topological recursion, we can derive the critical exponents for the $\mathcal{O}(\mathfrak{n})$ model without loop at boundaries, thus extending the proofs existing for planar maps (genus 0). In this section, we recall known facts about the critical points, outline the method to take the critical limit in the formalism we used, and give the main results.

We recall that:

$$\mathfrak{n} = -2\cos(\pi\mu), \qquad \text{with } \mu \in \,]0, 1[$$

6.1 Principles

6.1.1 Asymptotic of maps

Let $\hat{\mathrm{V}}$ be a fixed potential. The analysis of singularities (also called critical points) in t of $\mathrm{W}_k^{(g)}(t)$ allows to find the asymptotic of the (weighted) number of $\mathcal{O}(\mathfrak{n})$ maps with genus g and k boundaries. We illustrate this on $\mathrm{F}_0(t)$.

$$\mathrm{F}_0(t) = \sum_{v \geq 0} \mathrm{F}_{0|v}\, t^v$$

was defined as a generating function of spherical $\mathcal{O}(\mathfrak{n})$ maps with v vertices. It is a formal power series in t, and we know (see next paragraph) by a corollary of the 1-cut lemma that its radius of convergence $\rho_0(\hat{\mathrm{V}})$ is strictly positive. Hence, there exists $t^* \in \mathcal{C}(0\,;\,\rho_0(\hat{\mathrm{V}}))$ such that $\mathrm{F}_0(t)$ has a singularity when $t \rightarrow t^*$. We can decompose $\mathrm{F}_0(t) = f_r(t) + f_s(t)$ with f_r analytic in a neighborhood of t^* and f_s nonanalytic (maybe divergent) at $t = t^*$. Then, we have the well-known relation:

$$\begin{array}{llll} \text{if } \; f_s(t) & \sim A\left(1 - \frac{t}{t^*}\right)^{\alpha} & & \text{when } t \rightarrow t^* \\ \text{then } \; \mathrm{F}_{0|v} & \sim \frac{A}{\Gamma(-\alpha)} v^{-(\alpha+1)} (t^*)^{-v} & & \text{when } v \rightarrow \infty \end{array} \qquad (6\text{-}1)$$

6.1.2 Critical points of the $\mathcal{O}(\mathfrak{n})$ model

We have to find the singularities of the $\mathrm{W}_k^{(g)}$'s which are the closest to 0 in order to study observables on large maps. We claim that this singularity is common to k, g. Indeed, according to the 1-cut lemma, $\mathrm{W}_1^{(0)}(t)$ has a strictly positive radius $\rho_0(\hat{\mathrm{V}})$ and a singularity at some $t^* \in \mathcal{C}(0\,;\,\rho_0(\hat{\mathrm{V}}))$. Successive applications of the loop insertion operator $\frac{\partial}{\partial \hat{\mathrm{V}}(x)}$ preserve the radius and the singularity, and yield $\mathrm{W}_k^{(0)}$ $(k \geq 1)$. We have to argue that the residue formula do not change the radius either, but this granted, t^* is common to all stable $\mathrm{W}_k^{(g)}$'s. The unstable functions, F_0 and F_1 are obtained by an integration formula, which preserves the radius and the singularity at t^*.

In general, when one solves the saddle point equation, one finds $\mathrm{y}(\mathrm{x}) \sim 2\eta_i\sqrt{\mathrm{x} - a_i}$ for some $\eta_i \in \mathbf{C}$ when $x \rightarrow a_i$. This means that, in the local parameter on the spectral curve $u \propto \sqrt{\mathrm{x} - a_i}$, y has a simple zero at $u = \mathrm{u}(a_i)$. This property falls down for some exceptional potentials at some value $t = t^*$, and we rather have $\mathrm{y}(\mathrm{x}) \propto (\mathrm{x} - a_i)^{\lambda}$ for

$\lambda \geq \frac{1}{2}$. Then, the stable $W_k^{(g)}$'s expressed by the residue formula diverge when $t \to t^*$. The intuition can be supported by the Coulomb gas picture and one finds [11] different kinds of critical points t^*.

- ▷ Pure gravity. $a_i^2(t^*)$ is a zero of order p of $\mathrm{A}(x^2)$ and $\mathrm{B}(x^2)$ in Eqn. 3-1. Then, $\mathrm{y(x)} \propto (\mathrm{x}-a_i)^{p+\frac{1}{2}}$ when $\mathrm{x} \to a_i$. It happens when the liquid of eigenvalues (in the thermodynamic limit) crosses a potential barrier, i.e. at a point a_i where $\mathrm{y}(a_i) = 0$ and where the effective potential for an eigenvalues, $\mathrm{V_{eff}(x)} = \frac{2\mathrm{V'(x)} - \mathfrak{n}\mathrm{V'(-x)}}{4-\mathfrak{n}^2}$, safisfies $\mathrm{V}_{\mathrm{eff}}^{(l)}(a_i) = 0$ for $1 \leq l \leq p+1$. Then, the model has a well-defined limit for x close to a_i, which does not feel the presence of the interface at $\mathrm{x} = 0$. Back to combinatorics, such a singularity is associated with large maps without macroscopic loops. This limit exists already in the limit of the 1-matrix model and describes pure gravity. In the CFT classification, it is described by the $(p, 2)$ minimal model.

- ▷ $a(t^*) = 0$. The cuts $[-b, -a]$, $[a, b]$ merge. It happens when the eigenvalues can be as close as they want to their mirrors. Then, the method of Section 2 is not valid stricto sensu, because the torus is pinched at the point $x = 0$. The singularity lie in D_μ, and not in the polynomials A and B. These critical points are specific to the $\mathcal{O}(\mathfrak{n})$ model and associated to large maps where macroscopic loops are densely drawn.

- ▷ Combinations of the two situations. Merging cuts ($a = 0$) and tuning of V such that $\mathrm{A}(x^2)$ and $\mathrm{B}(x^2)$ have zeroes of order m. These multicritical points are also associated to large maps where loops are macroscopic. Although, they are not dense : we rather have cohabitation of regions dominated by gravity, and regions dominated by macroscopic loops.

We let aside pure gravity, and concentrate ourselves on the critical point $t \to t^*$ corresponding to $a \to 0$.

6.1.3 Taking the limit $a \to 0$

When $a \to 0$:

$$K \sim \frac{\pi}{2}, \qquad K' \simeq \ln\left(\frac{4b}{a}\right) \to \infty$$

In this limit $k \to 0$, the sn_k, cn_k and dn_k elliptic functions degenerate in the trigonometric functions sin, cos, and 1. We are interested in the limit where $a \to 0$ while $x^* = x/a$ remains finite. Then, φ defined by

$$x^* = \sin\varphi = \mathrm{ch}\,\chi \tag{6-2}$$

is a well-defined parametrization. We first have to determine the rescaled limit of the elementary function D_μ. It was expressed in Eqn. 2-7 in terms of ϑ_1 functions of modulus $\tau = iK/K'$. To take the limit $\tau \to 0$, it is easier to change $\vartheta_1(\cdot|\tau)$ into a $\vartheta_1(\cdot|\tau')$ with $\tau' = -1/\tau$ by a modular transformation. Then, in the sum:

$$\vartheta_1(w|\tau') = i \sum_{m\in\mathbf{Z}} (-1)^m e^{i\pi\left(m-\frac{1}{2}\right)^2\tau' + i\pi(2m-1)w},$$

only one or two terms (depending on the scaling form of w we are interested in) are dominant when $i\pi\tau' \to -\infty$. Performing this modular transformation, we obtain:

$$\mathrm{D}_\mu(u) = \frac{\mathrm{x}(u)}{\sigma(\mathrm{x}(u))} \frac{\vartheta_1\left(\frac{\varphi}{2K} - \frac{\mu\tau'}{2}\middle|\tau'\right)}{\vartheta_1\left(\frac{\varphi}{2K}\middle|\tau'\right)} \frac{\vartheta_1\left(\frac{\tau'}{2}\middle|\tau'\right)}{\vartheta_1\left((1-\mu)\frac{\tau'}{2}\middle|\tau'\right)} \exp^{i\pi\mu\left(-\frac{\varphi}{2K}+\frac{\tau'}{2}\right)} \tag{6-3}$$

where $u = \frac{i\varphi}{2K'} + \frac{\tau}{2}$. For x in the first quadrant of the physical sheet, $\sigma(\mathrm{x}) = a^2\cos\varphi$. Taking the limit, we have[7]

$$\mathrm{D}_\mu(\mathrm{x}) \sim \left(\frac{a}{4b}\right)^{-\mu} \frac{e^{-i\mu\varphi}}{2ib\,\cos\varphi} \tag{6-4}$$

The function for x arbitrary is given by analytic continuation of the expression above. One can also derive:

$$\alpha_1 \sim ib \tag{6-5}$$

Besides, the comparison between a direct calculation of $D^{(2)}_\mu(x) = \mathrm{D}_\mu(x)\mathrm{D}_\mu(-x)$ and the expression:

$$\mathrm{D}^{(2)}_\mu(x) = -\frac{x^2 - e_\mu^2}{(x^2-a^2)(x^2-b^2)} \qquad \text{from Appendix F}$$

tells us the behavior of e_μ when $a \to 0$:

$$e_\mu^2 \sim -\frac{1}{4}\, a^{2(1-\mu)}(4b)^{2\mu} \tag{6-6}$$

Furthermore, one can derive expressions of interest (valid for x_0, x in the first quadrant, or to be analytically continued):

▷ Basis of 1-cut solution of Eqn. 2-6.

$$\begin{aligned} \frac{\mathrm{f}_\mu}{\mathrm{R}_\mu} &\sim \ \mathrm{cte}\,\cos\varphi\cos\mu\left(\varphi - \frac{\pi}{2}\right) = -i\mathrm{cte}\;\mathrm{ch}\chi\,\mathrm{sh}\mu\chi \\ \frac{\widehat{\mathrm{f}}_\mu}{\widehat{\mathrm{R}}_\mu} &\sim \ \widehat{\mathrm{cte}}\,\sin\varphi\sin\mu\left(\varphi - \frac{\pi}{2}\right) = i\widehat{\mathrm{cte}}\;\mathrm{ch}\chi\,\mathrm{sh}\mu\chi \end{aligned} \tag{6-7}$$

[7] One could wonder what happen for another value of μ such that $\mathfrak{n} = -2\cos\pi\mu$ in this limit procedure. In fact, the limit of $\mathcal{D}_\mu$ depends only on the determination of μ between $]-1,1[$. Then, it turns out that choosing the negative determination and taking the limit leads to a regular free energy, i.e. one do not describe the critical point we look for.

▷ Basic blocks of the spectral curve.

$$\begin{aligned} \mathrm{y}_\mu &\sim \text{ cte} \sin\varphi \cos\mu\left(\varphi+\frac{\pi}{2}\right) = \text{cte}' \, \mathrm{ch}\chi\, \mathrm{ch}\mu(\chi+i\pi) \\ \hat{\mathrm{y}}_\mu &\sim \hat{\text{cte}}' \cos\varphi \sin\mu\left(\varphi+\frac{\pi}{2}\right) = -i\hat{\text{cte}}' \, \mathrm{sh}\chi\, \mathrm{sh}\mu(\chi+i\pi) \end{aligned} \tag{6-8}$$

▷ Brick of the auxiliary Cauchy kernel.

$$H_+(\chi_0,\chi) \sim e^{-i\pi\mu}\frac{\mathrm{d}\chi_0}{\mathrm{dx}^*(\chi_0)}\, e^{\mu(\chi+\chi_0)}(-1-i\coth(\chi+\chi_0)) \tag{6-9}$$

▷ Bergman kernel.

$$\mathcal{B}(\chi_0,\chi) \sim \mathrm{d}\chi_0\mathrm{d}\chi e^{\mu(\chi+\chi_0)}\left[\mu+i\mu\coth(\chi+\chi_0)-\frac{1}{\mathrm{sh}^2(\chi+\chi_0)}\right] \tag{6-10}$$

Eventually, the primitive of the limit of $H_+(x_0,x)\mathrm{dx}_0$ is a related to a well-known special function, the Lerch $\Phi_1(z)$ function. It is defined (for $|z|<1$) as a power series around $z=0$ which is a deformation of the logarithm series with a parameter κ:

$$\Phi_1(z\,;\,\kappa) = \sum_{n=0}^{\infty}\frac{z^n}{n+\kappa}$$

and can be extended by analytic continuation. Then, we have:

$$\begin{aligned} \mathrm{dx}_0\, H_+(x_0,x) &\sim e^{-i\pi\mu}(\partial\Psi_+^*)(\chi+\chi_0) \qquad (6\text{-}11) \\ \mathcal{B}(u_0,u) &\sim -(\partial^2\Psi_+^*)(\chi+\chi_0) \\ \text{where} \qquad \Psi_+^*(w) &= e^{\mu w}\left[\frac{1\mp i}{\mu} - i\Psi_1\left(e^{-2w}\,;\,\mp\frac{\mu}{2}\right)\right] \quad \text{with } \pm = \mathrm{sign}(\mathrm{Re}\, w) \end{aligned}$$

We called this function Ψ_+^* since it is the elementary brick for the limit Ψ^* of the function introduced in Eqn. 3-37.

6.2 Spectral curve and topological recursion when $a \to 0$

6.2.1 Description of the critical point

	At the critical point : $a = 0$	
λ	$\mathrm{y}(\mathrm{x}) \sim C\,\mathrm{x}^{\lambda}$ when $\mathrm{x} \to 0$	

	Approach of the critical point : $a \to 0$	
α	$\mathcal{L}_{a,b}\ :\ \mathrm{y}(\mathrm{x}) \sim a^{\alpha}\mathrm{y}^*(\mathrm{x}/a)$	x/a finite
$\alpha(k,g)$	$\mathrm{W}_k^{(g)}[\mathcal{L}_{a,b}](\mathrm{x}_1,\dots,\mathrm{x}_k) \sim a^{\alpha(k,g)}\mathrm{W}_k^{(g)}[\mathcal{L}^*]\left(\frac{\mathrm{x}_1}{a},\dots,\frac{\mathrm{x}_k}{a}\right)$	$2g-2+k \geq 0$
β	$\left(1-\frac{t}{t^*}\right) \propto a^{2\beta}$	
γ_{str}	$\mathrm{U}(t) = \frac{\partial^2 \mathrm{F}_0}{\partial t^2} \propto \left(1-\frac{t}{t^*}\right)^{-\gamma_{str}}$	

We define the critical exponents of the $\mathcal{O}(\mathfrak{n})$ model without loop at boundaries. $U(t)$ is a fundamental quantity to many aspects. In the correspondence of the critical model with a CFT, the "string susceptibility" γ_{str} gives the necessary central charge:

$$\mathrm{c} = 1 - 6\frac{\gamma_{str}^2}{1-\gamma_{str}} \tag{6-12}$$

Furthermore, the conformal dimensions (from which the scaling exponents can be extracted) of a conformal field theory coupled to gravitation ($\widetilde{\Delta}$), or not (Δ) are related:

$$\widetilde{\Delta} = \frac{\Delta(\Delta - \gamma_{str})}{1-\gamma_{str}} \tag{6-13}$$

These are the KPZ equations : they relate exponents of a statistical model of a regular lattice to those on the random lattice.

6.2.2 Limit of the spectral curve

It is easy to see that the residue formula is compatible with limits of curves. So, we ask ourselves what is the spectral curve $\mathcal{L}_{a,b}$ in the limit $a \to 0$, and we call it $\mathcal{L}^*$. We find, by linear combination of Eqn. 6-7 and 6-7, that $w_0(\chi) = \mathrm{ch}(\mu+1)\chi$ and $w_{-1}(\chi) = \mathrm{ch}(\mu-1)\chi$ are solutions. By choosing appropriate polynomials $\mathrm{A}(\mathrm{ch}^2\chi)$

and $\mathrm{B}(\mathrm{ch}^2\chi)$ of maximal degree D, which are also linear combinations of $\mathrm{ch}(2m\chi)$ for $0 \leq l \leq D$ integer, we find recursively that the general solution for w is a linear combination of:

$$w_m(\chi) = \mathrm{ch}(\mu + 2m + 1)\chi, \qquad \text{for } -(D+1) \leq m \leq D \tag{6-14}$$

The spectral curve associated to a homogeneous part of the resolvent which would be w_m is:

$$y_m(\chi) = 2w_m(\chi) + \mathfrak{n} w_m(-(\chi + i\pi)) = -2\,\mathrm{sh}\left[(\mu + 2m + 1)\chi + 2i\pi\mu\right] \tag{6-15}$$

When we are careful about the maximal degree of the polynomials A and B (see Eqn. 3-3), we obtain:

Theorem 6.1 *When $a \to 0$ and $\mathrm{x}^* = \mathrm{x}/a$ is kept finite, the rescaled limit of the spectral curve is of the form:*

$$\boxed{\begin{cases} x^*(\chi) = \mathrm{ch}\chi \\ y^*(\chi) = \sum_{m=-(D+1)}^{D} c_m\ \mathrm{sh}\left((\mu + 2m + 1)\chi + 2i\pi\mu\right) \end{cases}} \tag{6-16}$$

If the potential V has maximal degree d_{max}, then $D = \lfloor (d_{max} - 1)/2 \rfloor$ and $c_m \in \mathbf{C}$. In case d_{max} is odd, one has in addition $c_D = c_{-(D+1)}$. The coefficients are subjected to the extra condition $y(\chi = 0) = 0$.

□

For example, in the fully packed case ($d_{\max} = 2$, $D = 0$), the limit spectral curve must be:

$$y^*_{\mathrm{FPL}}(\chi) = \mathrm{cte}\ \mathrm{ch}\chi\,\mathrm{sh}\mu\chi \tag{6-17}$$

Eventually, we gather in Fig. 10 the objects associated to this curve, namely the recursion kernel $\mathcal{K}^*$ (from Eqn. 6-9) and the Bergman kernel $\mathcal{B}^*$ (Eqn. 6-10).

	Double scaling limit
$\mathcal{L}^*$	$\mathcal{L}^* \ : \ \begin{cases} x^*(\chi) & = \ \mathrm{ch}\,\chi \\ y^*(\chi) & = \ \sum_{m=-(D+1)}^{D} c_m \,\mathrm{sh}\left((\mu+2m+1)\chi + 2i\pi\mu\right) \end{cases}$
H_+	$H_+^*(\chi_0,\chi) = e^{-i\pi\mu}\dfrac{\mathrm{d}\chi_0}{\mathrm{d}x^*(\chi_0)}\ e^{\mu(\chi+\chi_0)}\left(-1 - i\coth(\chi+\chi_0)\right)$
$\mathcal{B}^*$	$\mathcal{B}^*(\chi_0,\chi) = \mathrm{d}\chi_0\mathrm{d}\chi\, e^{\mu(\chi+\chi_0)}\left[\mu + i\mu\coth(\chi+\chi_0) - \dfrac{1}{\mathrm{sh}^2(\chi+\chi_0)}\right]$
$\mathcal{K}^*$	$\mathcal{K}^*(\chi_0,\chi) = \dfrac{1}{\sin\pi\mu}\dfrac{\mathrm{d}\chi_0}{[y^*\mathrm{d}x^*](\chi)}\left[\begin{array}{r} \mathrm{ch}\,\mu(\chi-i\pi)\,\mathrm{ch}\,\mu\chi_0 - \frac{i}{2}\frac{\mathrm{sh}\,\mu(\chi+\chi_0-i\pi)}{\mathrm{sh}\,(\chi+\chi_0)}\,\mathrm{ch}\,(\chi+\chi_0) \\ -\frac{i}{2}\frac{\mathrm{sh}\,\mu(\chi-\chi_0-i\pi)}{\mathrm{sh}\,(\chi-\chi_0)}\,\mathrm{ch}\,(\chi-\chi_0) \end{array}\right]$

Figure 10: Summary on the critical $\mathcal{O}(\mathfrak{n})$ model.

6.2.3 Limit of the $\mathrm{W}_k^{(g)}$'s

We see on Eqn. 6-10 that $\mathcal{B}$ has a well-defined limit, without scaling factor, when $a \to 0$, so:

$$\mathrm{dx}_0\mathrm{dx}\overline{\mathrm{W}}_2^{(0)}(x_0,x) \sim \mathrm{dx}_0^*\mathrm{dx}^*\overline{\mathrm{W}}_2^{(0)*}(x_0^*,x^*) \tag{6-18}$$

Subsequently, for the recursion kernel:

$$\mathrm{K}(\mathrm{x}_0,\mathrm{x}) \sim a^{-(\alpha+1)}\,\mathrm{K}^*(\mathrm{x}_0^*,\mathrm{x}^*) \tag{6-19}$$

For g,k such that $2g-2+k>0$, $\mathrm{W}_k^{(g)}$ is a string of $2g-2+k$ residues against a recursion kernel, of a product of $\overline{\mathrm{W}}_2^{(0)}(\cdot,\cdot)$ blocks. Hence:

$$\mathrm{dx}_1\cdots\mathrm{dx}_k\,\mathrm{W}_k^{(g)}(x_1,\ldots,x_k) \sim a^{(\alpha+1)(2-2g-k)}\,\mathrm{dx}_1^*\cdots\mathrm{dx}_k^*\,\mathrm{W}_k^{(g)*}(x_1^*,\ldots,x_k^*) \tag{6-20}$$

This yields:

$$\alpha(k,g) = (2-2g-k)(\alpha+1) - k \tag{6-21}$$

6.3 Results for $a \to 0$

6.3.1 Determination of λ

We have to solve the saddle point equation for $a=0$ and x finite. This limit can be found either by a direct guess of the parametrization and the general solution [11,18,19], or by computing with limits of theta functions D_μ and then the basis $(\mathrm{f}_\mu/\mathrm{R}_\mu, \widehat{\mathrm{f}}_\mu/\widehat{\mathrm{R}}_\mu)$, as we did when x was of order a.

	Table of exponents	Example : triangular lattice	
	Phase $(\varepsilon, m) \in \{\pm 1\} \times \mathbf{N}$	Dense phase $(\varepsilon, m) = (-1, 0)$	Dilute phase $(\varepsilon, m) = (1, 0)$
λ	$\varepsilon(1-\mu) + 2m + 1$	μ	$2 - \mu$
α	$\varepsilon(1-\mu) + 2m + 1$	μ	$2 - \mu$
$\alpha(k, g)$	$(2-2g)(\alpha+1) - k\alpha$		
β	$\frac{1-\varepsilon}{2}\mu + m + \frac{1+\varepsilon}{2}$	μ	1
γ_{str}	$\frac{\mu-1}{\frac{1-\varepsilon}{2}\mu+m+\frac{1+\varepsilon}{2}}$	$1 - \frac{1}{\mu}$	$-(1-\mu)$

Figure 11: Summary on the critical $\mathcal{O}(\mathfrak{n})$ model.

Theorem 6.2 *We introduce the parametrization* $\mathrm{x} = \frac{b}{\mathrm{ch}\,\zeta}$. *Then, the general spectral curve when* $a = 0$ *and* x *is kept finite, is of the form:*

$$\boxed{\mathrm{y}(\mathrm{x}) = \mathrm{A}\left(\frac{1}{\mathrm{ch}^2\zeta}\right) \mathrm{th}\zeta\, \mathrm{ch}\mu\,(\zeta + i\pi) + \mathrm{B}\left(\frac{1}{\mathrm{ch}^2\zeta}\right) \mathrm{sh}\,\mu(\zeta + i\pi)} \tag{6-22}$$

where A *and* B *are polynomials, subjected to the extra condition* $y(\zeta = 0) = 0$. *If the potential* V *has maximal degree* d_{max}, *we have:*

$$\deg(\mathrm{A}) \leq \left\lfloor \frac{d_{max}}{2} \right\rfloor - 1, \qquad \deg(\mathrm{B}) \leq \left\lfloor \frac{d_{max}-1}{2} \right\rfloor$$

□

When $\mathrm{x} \to 0$, $\mathrm{x} \sim \frac{b}{2} e^{-\zeta}$. Since $\mu \in \,]0, 1[$, the two terms (y_μ and $\hat{\mathrm{y}}_\mu$) admit $\mathrm{x}^{-\mu}$ as leading order, and:

$$\mathrm{x}^{\mu},\ \mathrm{x}^{-\mu+2},\ \ldots,\ \mathrm{x}^{(\mu-1)+2m+1},\ \mathrm{x}^{(1-\mu)+2m+1},\ \ldots$$

as subleading orders. If we demand[8] $\mathrm{y}(0) = 0$, A and B are such that this leading order disappears. Hence, the first possible term is x^{μ}, and other subleading order can

[8]It is true for all $a \neq 0$ that $\mathrm{y}_a(a) = 0$. Though, we did not find a convincing argument to rule out from the case $\mathrm{y}_{a=0}(0) = \infty$ for combinatorics. In this case, y could be divergent as $x^{-\mu}$ but integrable on $[0, b]$ since $\mu \in \,]0, 1[$.

be canceled by a special choice of A and B. In general, we have:

$$\lambda = \varepsilon(1-\mu) + 2m + 1, \qquad \varepsilon \in \{\pm 1\},\ m \in \mathbf{N} \tag{6-23}$$

The admissible values for ε and m depends on $d_{\max}$. They determine the "phases" of the $\mathcal{O}(\mathfrak{n})$ model. For example :

- ▷ On a triangular lattice ($d_{\max} = 3$), there exists only two phases, in agreement with [19]:

$$\begin{array}{lll} \text{Dense phase} & (\varepsilon, m) = (-1, 0) & \lambda = \mu \\ \text{Dilute phase} & (\varepsilon, m) = (1, 0) & \lambda = 2 - \mu \end{array}$$

- ▷ In the fully packed case ($d_{\max} = 2$), only the dense phase $(\varepsilon, m) = (-1, 0)$ is present, and the limit spectral curve is:

$$\mathrm{y}_{\mathrm{FPL}}(\mathrm{x}) = \mathrm{cte}\,\mathrm{th}\zeta\,\mathrm{ch}\mu(\zeta + i\pi) \tag{6-24}$$

- ▷ With the general potential of degree $d_{\max} = 2d' + 1$ ($d' > 1$), there are $2d'$ phases described by $(\varepsilon, m) \in \{\pm 1\} \times [1, d']$.
- ▷ With the general potential of degree $d_{\max} = 2d'$, $(\varepsilon, m) = (1, d')$ cannot be reached, so there are $2d' - 1$ phases.'

6.3.2 Determination of α

α is defined such that $\mathrm{y}_a(\mathrm{x}) \sim a^{\alpha}\mathrm{y}^*(\mathrm{x}/a)$ for finite x/a and $a \to 0$. On the other hand, $\mathrm{y}_{a=0}(\mathrm{x})$ behaves as x^{λ} when $\mathrm{x} \to 0$. The two solutions match if $\mathrm{y}^*(\mathrm{x}^*) \propto \mathrm{x}^{\alpha'}$ when $\mathrm{x}^* \to \infty$ with $\alpha = \alpha'$, and $\alpha' = \lambda$. Theorems 6.1 and 6.2 are compatible in the sense that they produce the same possible values for α' and λ.

6.3.3 Determination of β

We have proved that $\left(\partial_t \mathrm{W}_1^{(0)}\right)(x) = \mathrm{f}_\mu(x)$. Independently, we have in the limit $a \to 0$:

$$\begin{array}{rll} \partial_t(\mathrm{W}_1^{(0)})(\mathrm{x}) \propto & a^{\alpha - 2\beta} \times (\text{function of } \mathrm{x}/a) & \text{almost by definition} \\ \mathrm{f}_\mu(\mathrm{x}) \sim & a^{-\mu}\,\mathrm{f}_\mu^*(\mathrm{x}/a) & \text{from Eqn. 6-4} \end{array}$$

Hence:

$$2\beta = \alpha + \mu \tag{6-25}$$

6.3.4 Determination of γ_{str}

We take the limit $a \to 0$ in Eqn. 5-24 giving the third derivative of F_0 and keep the nonanalytic part at $t = t^*$:

$$\left(\frac{\partial^3 \mathrm{F}_0}{\partial t^3}\right)_{\text{singular}} \sim \text{cte}\, a^{2(1-\mu)-2\beta} \tag{6-26}$$

Hence:

$$\boxed{\gamma_{str} = \frac{\mu - 1}{\beta} = \frac{\mu - 1}{\frac{1+\varepsilon}{2}\mu + m}} \tag{6-27}$$

We notice that γ_{str} is always negative. Furthermore, the limit of Eqn. 5-24 yields a regular part for $\partial_t^3 \mathrm{F}_0$, i.e which is analytic at $t = t^*$. This regular part is dominant in $\partial_t^3 \mathrm{F}_0$ when $t \to t^*$. Also, the same phenomenon occurs for $U(t)$ and $\mathrm{F}_0(t)$.

6.3.5 Critical behavior of F_1

With expression Eqn. 5-30, it is easy to see that $\mathrm{F}_1 \propto \ln(a)$ when $a \to 0$.

6.3.6 Remark: double scaling limit

We have reviewed the fact that, for a given value of $\mathfrak{n}$, the model has many possible continuum limits, described by $(\varepsilon, m) \in \{\pm 1\} \times \mathbf{N}$. We refer to [11] or [8] for a discussion of the case "$\mu = p/q$ rational" in relation with the (p, q) minimal models of CFT.

For any of these limits, we have seen that the stable $\mathrm{W}_k^{(g)}(t)$ diverge (at least for $g \geq 2$) as $a^{\alpha(k,1)-(2g-2)(\alpha+1)}$ when $t \to t^*$. Take $k \neq 0$ to avoid unstable maps, and consider the formal power series $\mathrm{W}_k(t)$, depending on N:

$$\begin{aligned}
\mathrm{W}_k(t|N) &= \sum_{g \geq 0} \left(\frac{N}{t}\right)^{2-2g} \mathrm{W}_k^{(g)}(t) \\
&= a^{-\alpha(k,1)} \sum_{g \geq 0} \left(\frac{N a^{-(\alpha+1)}}{t}\right)^{2-2g} a^{\alpha(k,g)} \mathrm{W}_k^{(g)}(t) \qquad (6\text{-}28)
\end{aligned} \tag{6-29}$$

We can also define a function:

$$\mathrm{W}_k^*(N^*) = \sum_{g \neq 0} (N^*)^{2-2g} \mathrm{W}_k^{(g)*} \tag{6-30}$$

If we send $a \to 0$ and $N \to \infty$ while keeping $N a^{-(\alpha+1)} = N^*$ finite, then:

$$\mathrm{W}_k(t|N) \sim \mathrm{W}_k^*(N^*) \tag{6-31}$$

For this reason, W_k^*, and by extension $W_k^{(g)*}$, are called in matrix model context the "double scaling limit" of W_k, resp. $W_k^{(g)}$. Within the conjecture relating limits of matrix models to CFT (see Section 1.4), W_k^* should be solution of PDE's coming from the conformal field theory of central charge:

$$c = 1 - 6\left(\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}}\right) \tag{6-32}$$

Still, it demands a proof.

6.4 Special values of $\mathfrak{n}$

The $\mathcal{O}(0)$ model is the 1-matrix model with potential $x \mapsto V(x)$, and the $\mathcal{O}(-2)$ model is the 1-matrix model with potential $x' \mapsto V(\sqrt{x'})$. The $\mathcal{O}(2)$ model is a statistical model of 2D spins on a random lattice. On the regular lattice, the model admits critical points described by Kosterlitz and Thouless in the 70s [23]. It can be shown by techniques similar to those we presented, that $\gamma_{str} = 0$ in this case. The critical theory of $\mathcal{O}(2)$ model is found to agree with a KT-transition.

7 Conclusion

We have extended the topological recursion to a non compact spectral curve. The $W_k^{(g)}$'s of the $\mathcal{O}(\mathfrak{n})$ matrix model fit in this formalism. We have investigated here the 1-cut case, which is relevant for combinatorics. It provides an algorithm to compute number of maps with self avoiding loops of $\mathfrak{n}$ possible colors in any topology and with an arbitrary number of boundaries. In particular, the cylinder function $W_2^{(0)}$ is universal (as expected) and closely related to a deformation of the Weierstraß $\wp$ function. We also extended the procedure to compute F_0 and F_1. The generalization of the results to the multi-cut case seems straightforward: one just pick up residue to every endpoint of a cut. Though, before arriving to this point, one would have to find the spectral curve (i.e $W_1^{(0)}$) and $W_2^{(0)}$, i.e to solve Eqn. 2-2 on multiple cuts. This is the matter of another work.

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A Appendix:

B Coulomb gas interpretation of the matrix model

In the partition function of the matrix model 1-18, integrals over A matrices are gaussian and can be performed:

$$\begin{aligned} \mathrm{Z} &= \int_{\mathbf{H}_N^{\mathfrak{n}+1}} \mathrm{d}M \mathrm{d}A_1 \cdots \mathrm{d}A_{\mathfrak{n}}\, e^{-\frac{N}{t}\mathrm{Tr}\left[\mathrm{V}(M) + \sum_{i=1}^{\mathfrak{n}} M A_i^2\right]} \\ &= \int_{\mathbf{H}_N} \frac{\mathrm{d}M}{\det(\mathbf{L}_M + \mathbf{R}_M)^{\mathfrak{n}/2}} e^{-\frac{N}{t}\mathrm{Tr}\left[\mathrm{V}(M)\right]} \end{aligned} \tag{2-1}$$

where $\mathbf{L}_M$ (or $\mathbf{R}_M$) is the linear operator of left (or right) multiplication by M. Let us make a change of variables through the polar decomposition:

$$\begin{aligned} M &= U \mathrm{diag}(\mathrm{x}_1, \ldots, \mathrm{x}_N) U^{\dagger}, \qquad U \in \mathbf{U}(N), \qquad \mathrm{x}_i \in \mathbf{R} \\ \mathrm{d}M &= \left[\prod_{1 \le i < j \le N} (\mathrm{x}_i - \mathrm{x}_j)\right]^2 \mathrm{d}U \mathrm{d}\mathrm{x}_1 \cdots \mathrm{d}\mathrm{x}_N \end{aligned} \tag{2-2}$$

The integral over U can be performed and we are left with a statistical model for the eigenvalues:

$$\mathrm{Z} = \int_{\mathbf{R}^N} \mathrm{d}\mathrm{x}_1 \cdots \mathrm{d}\mathrm{x}_N \frac{\left[\prod_{1 \le i < j \le N} (\mathrm{x}_i - \mathrm{x}_j)\right]^2}{\left[\prod_{1 \le i, j \le N} (\mathrm{x}_i + \mathrm{x}_j)\right]^{\mathfrak{n}/2}} e^{-\frac{N}{t}\sum_{i=1}^{N} \mathrm{V}(\mathrm{x}_i)} \tag{2-3}$$

When $\mathfrak{n} = 0$, this is the partition function of a system of N unit charges (positions x_i), constrained to live on the real line, repelling each other by Coulomb interaction in two dimensions, trapped in the external potential V. When $\mathfrak{n} \neq 0$, a charge located at x feels the attraction of a fictive charge of strength $-\mathfrak{n}$ located at $-\mathrm{x}$. Physically, it occurs when the imaginary axis is a dielectric interface, separating $\mathrm{Re} > 0$ of relative dielectric constant ϵ_{r+} and $\mathrm{Re} < 0$ with ϵ_{r-}, such that $\epsilon_{r-}/\epsilon_{r+} = \tan\left(\frac{\pi\mu}{2}\right)$ (where $\mathfrak{n} = -2\cos(\pi\mu)$).

Heuristically, the eigenvalues density $\frac{1}{N}\left\langle \sum_{i=1}^{N} \delta(\mathrm{x} - \mathrm{x}_i)\right\rangle$ becomes a measure with continuous density ϱ in the thermodynamic limit. It is related to the discontinuity of the resolvent on the real line:

$$\begin{aligned} \mathrm{W}_1^{(0)}(\mathrm{x} + i\epsilon) - \mathrm{W}_1^{(0)}(\mathrm{x} - i\epsilon) &= \frac{t}{N}\left\langle \sum_{i=1}^{N} \frac{1}{\mathrm{x} - \mathrm{x}_i + i\epsilon} - \frac{1}{\mathrm{x} - \mathrm{x}_i - i\epsilon}\right\rangle \\ &= \frac{t}{N}\left\langle \sum_{i=1}^{N} \frac{-2i\epsilon}{(\mathrm{x} - \mathrm{x}_i)^2 + \epsilon^2}\right\rangle \\ \text{i.e} \quad \mathrm{y}(\mathrm{x}) &= -2i\pi t \rho(\mathrm{x}) + o(\epsilon) \qquad \text{when } \epsilon \to 0^+ \end{aligned} \tag{2-4}$$

Then, the free energy in the thermodynamic limit should be the energy associated to this density:

$$\begin{aligned}
\mathrm{F}_0 &= \frac{t^2}{N^2}\lim_{N\to\infty}\ln \mathrm{Z}_N \\
&= \frac{t^2}{2}\int_a^b \mathrm{dx}_0 \int_a^b \mathrm{dx}\, \varrho(\mathrm{x})\varrho(\mathrm{x}_0)\left(2\ln(\mathrm{x}-\mathrm{x}_0) - \mathfrak{n}\ln(\mathrm{x}+\mathrm{x}_0)\right) - t\int_a^b \mathrm{dx}\, \varrho(\mathrm{x})\mathrm{V}(\mathrm{x})
\end{aligned}$$

C Proof of the 1-cut lemma

Lemma C.1 *For all $\mathfrak{n}$, k, g, there exists $\rho > 0$ such that $\#\mathbb{M}_k^{(g)}(v) \in O(\rho^v)$ when $v \to \infty$.*

proof:

This is based on a rough upper bound on the cardinality of $\mathbb{M}_k^{(g)}$. If $\mathfrak{n} = 0$ and $t_j = 0$ for $j \geq 4$, $\mathbb{M}_k^{(g)}(v)$ are sets of usual triangulations, and it is known [31] that the lemma is true with $\overline{\rho} = \frac{1}{6\sqrt{3}}$. We assume $\mathfrak{n}$ an integer, for it is enough to prove the lemma for $\lceil \mathfrak{n} \rceil$. Now, we describe in Fig. C a map from $\mathbb{M}_k^{(g)}(v)$ to the set of triangulations with v vertices where every face carries a label among $\{\mathrm{B}, \mathrm{I}, \mathrm{M}, 1, \ldots, \mathfrak{n}\}$.

This association is injective:

- ▷ If a decorated triangulation comes from $\mathcal{M} \in \mathbb{M}_k^{(g)}(v)$, the unmarked edges in $\mathcal{M}$ are those which glue two "B"-triangles, while the marked edges glue a "B" and a "M" triangle. So, we have the skeleton of $\mathcal{M}$.
- ▷ The polygons of this skeleton admitting an inner triangle labeled $j \in \{1, \ldots, \mathfrak{n}\}$ instead of "I" are necessarily triangles. They must carry a loop, and the j-label indicates where it should be drawn.

By construction:

$$v(\mathcal{M}') \leq (d_{\max} + 1)v(\mathcal{M}), \qquad g(\mathcal{M}) = g(\mathcal{M}'), \qquad k(\mathcal{M}) = k(\mathcal{M}')$$

Hence:

$$\#\mathbb{M}_k^{(g)}(v) \in O\left(\overline{\rho}^{(\mathfrak{n}+3)(d_{\max}+1)v}\right)$$

□

We proceed in detail with the proof of the 1-cut lemma, which is now the same as for usual maps and the 1-matrix model. We recall that the notion of the color of a loop does not intervene in $\mathbb{M}_k^{(g)}$, but only in the weight of a map: the previous upper bound on $\#\mathbb{M}_k^{(g)}$ is valid uniformly for $\mathfrak{n}$ bounded. Moreover, we notice that the association of Fig. C preserves the number of automorphism. We conclude that the coefficient of t^v is a $O(\rho_0/r)$ where

$$\begin{aligned} \rho_0 &= \max(t_j^{\frac{2}{j-2}}, \mathfrak{n}c^2) \\ r &= \min_i \sqrt{\left|\mathrm{x}_i - \frac{c}{2}\right|} \end{aligned}$$

which proves the first part of the 1-cut lemma.

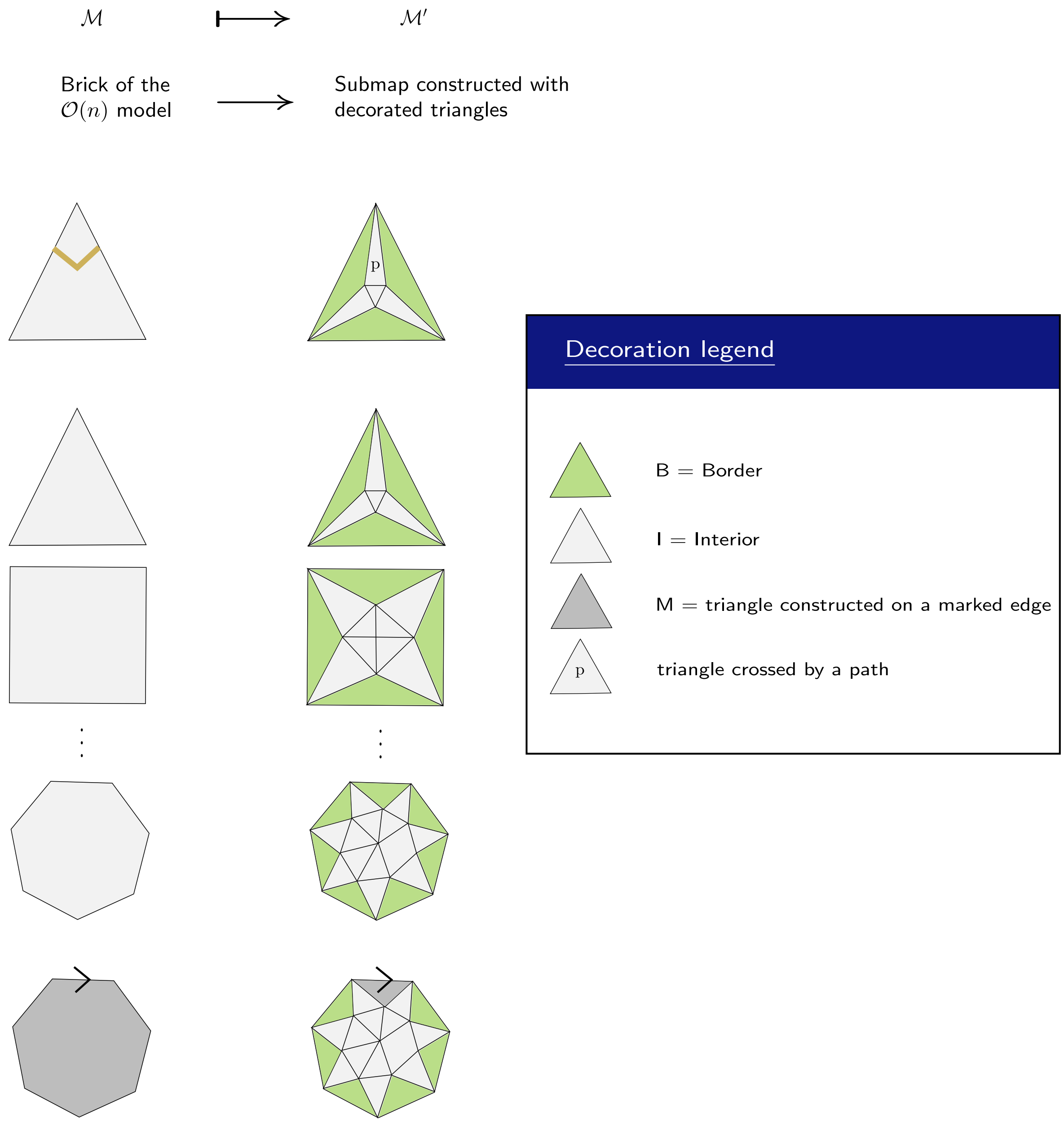

Figure 12: Every j-gon (with or without loops, marked or unmarked) is itself triangulated with $3j$ decorated triangles: construct a triangle on each edge, and a triangle at each corner. In the inside, there remains a j-gon which you cut in j triangular sectors from its center, that you label "I". The other labels are distributed as follows: for a triangle carrying a loop, the corner triangle where the loop is drawn is decorated by the color of the loop $j \in \{1, \dots, \mathfrak{n}\}$; for a marked face, the triangle constructed on the marked edge is decorated by "M" ; in any other case, the triangles constructed on the edges are decorated by "B", while the other are labelled "I".

Next, we solve the master loop equation in $\mathrm{W}_1^{(0)}(\mathrm{x})$ which is quadratic:

$$\begin{aligned}\mathrm{W}_1^{(0)}(\mathrm{x}) &= \frac{1}{2}\left(\mathrm{V}'(\mathrm{x}) - \mathfrak{n}\mathrm{W}_1^{(0)}(-\mathrm{x}) + \sqrt{\Delta(\mathrm{x})}\right) \\ \Delta(\mathrm{x}) &= \left(\mathrm{V}'(\mathrm{x}) - \mathfrak{n}\mathrm{W}_1^{(0)}(-\mathrm{x})\right)^2 \\ &\quad -4\left(\mathrm{P}_1^{(0)}(\mathrm{x}) + \mathrm{P}_1^{(0)}(-\mathrm{x}) - \mathrm{V}'(-\mathrm{x})\mathrm{W}_1^{(0)}(-\mathrm{x}) + \mathrm{W}_1^{(0)}(-\mathrm{x})^2\right)\end{aligned}$$

We have to describe the zeroes of Δ for t small enough. There exists r small enough such that the discs of centers $\pm\frac{c}{2}$ and radius r do not intersect and such that V' has no other zero than $\frac{c}{2}$ in $\mathcal{D}(\frac{c}{2}\,;\,r)$. Then, for $t \in \mathcal{D}(0\,;\,\frac{\rho_0}{r})$, we see that $\Delta(\mathrm{x})$ in holomorphic on $\mathcal{D}(\frac{c}{2}\,;\,r)$. For $t = 0$, $\Delta(\mathrm{x}) = (\mathrm{V}'(\mathrm{x}))^2$ is not the zero function and has a double zero in c. By continuity in t and analyticity for $\mathrm{x} \in \mathcal{D}(\frac{c}{2}\,;\,r)$, Δ has still two zeroes (with multiplicity), which limit is $\frac{c}{2}$ when $t \to 0$. Thus, there exists an holomorphic function h_t in $\mathcal{D}(\frac{c}{2}\,;\,r)$, without zeroes, and a continuous pair $\{a(t), b(t)\}$ such that $\Delta(\mathrm{x}) = \mathrm{h}_t^2(\mathrm{x})(\mathrm{x} - a(t))(\mathrm{x} - b(t))$.

Computing $\Delta(a_i(t))$ for a zero $a_i(t)$ of Δ from Eqn. 3-1 allows us to find $a_i(t)$ as a formal series in $\sqrt{t}$, and yields two solutions $a(t)$ and $b(t)$, such that $a(t) + b(t) \in O(t)$. The property of having a finite radius of convergence in t for $\mathrm{W}_1^{(0)}$ implies a finite radius of convergence for a and b.

Eventually, the analytical properties of $\mathrm{W}_k^{(g)}$ are derived by a straightforward recursion using the loop equations and Lemma 4.1.

D Elliptical functions and biperiodicity

D.0.1 Basic facts

Let us say a few words about biperiodic functions, i.e. satisfying:

$$f(w) = f(w+1) = f(w+\tau) \tag{4-1}$$

where $(1, \tau)$ is $\mathbf{R}$-free. Such functions are also called "elliptical functions". They satisfy a few properties:

- ▷ If f has no pole, then f is constant.
- ▷ The number of zeroes of f in the rectangle $(0, 1, 1+\tau, \tau)$ equals the number of its poles.
- ▷ The sum of residues of f at its poles in the rectangle, vanishes. In particular, this shows that f cannot have only one simple pole, it must have at least two simple poles, or one multiple pole.

D.0.2 ϑ_1 function

Let $\tau \in \mathbf{C}$ such that $\mathrm{Im}\tau > 0$. The first Jacobi theta function is defined by:

$$\vartheta_1(w|\tau) = i \sum_{m\in\mathbf{Z}} (-1)^m e^{i\pi\left(m-\frac{1}{2}\right)^2\tau + i\pi(2m-1)w} \tag{4-2}$$

This series is absolutely convergent for all $w \in \mathbb{C}$, so $\vartheta_1(\cdot|\tau)$ is an entire function. It satisfies:

$$\vartheta_1(w|\tau) = -\vartheta_1(w+1|\tau), \qquad \vartheta_1(w+\tau|\tau) = \mathrm{e}^{-2i\pi\left(w+\frac{\tau}{2}\right)} \tag{4-3}$$

and:

$$\vartheta_1(w|\tau) = -\vartheta_1(-w|\tau) \tag{4-4}$$

It has a unique zero mod $(\mathbf{Z} \oplus \tau\mathbf{Z})$, located at 0. Besides:

$$(\vartheta_1)'(1/2\,|\tau\,) = 0 \tag{4-5}$$

D.0.3 Description of the meromorphic functions on the torus

An elliptical function f with poles $w_1, \dots, w_L$ and poles $w'_1, \dots, w'_{L'}$, can always be written[9] as:

$$f(w) = A \frac{\prod_{l'=1}^{L'} \vartheta_1(w - w_l|\tau)}{\prod_{l=1}^{L} \vartheta_1(w - w'_{l'}|\tau)} \tag{4-6}$$

for some constant A, and the poles and zeroes are such that:

$$\sum_{l'} w'_{l'} - \sum_{l} w_l = 0 \mod \mathbf{Z} \oplus \tau\mathbf{Z} \tag{4-7}$$

D.0.4 Inversion of modulus

There is a relation between $\vartheta_1(\cdot|\tau)$ and $\vartheta_1(\cdot|\tau')$, where $\tau' = -1/\tau$, which is an application of Poisson's summation formula:

$$\vartheta_1(w|\tau) = \frac{-1}{\sqrt{i\tau}} e^{\frac{-i\pi w^2}{\tau}} \vartheta_1\left(\frac{w}{\tau}\Big| - \frac{1}{\tau}\right) \tag{4-8}$$

[9] Indeed, the ratio of f and this expression would be an elliptical function with no pole, i.e a constant.

D.0.5 Weierstraß function

Derivatives of $\ln \vartheta_1$ are in relation with well-known special functions. Let us introduce the Weierstraß $\wp$ function:

$$\begin{aligned}
\wp(w|\tau) &= \frac{1}{w^2} + \sum_{(l,m)\in\mathbf{Z}^2\setminus\{(0,0)\}} \left(\frac{1}{(w+l+m\tau)^2} - \frac{1}{(l+m\tau)^2} \right) \\
&= \sum_{m\in\mathbf{Z}} \frac{\pi^2}{\sin^2 \pi(w+m\tau)} - \sum_{m\in\mathbf{Z}\setminus\{0\}} \frac{\pi^2}{\sin^2 \pi m\tau} - \frac{\pi^2}{3} \qquad (4\text{-}9)
\end{aligned}$$

This function is elliptic, and related to the second derivative of $\ln \vartheta_1$:

$$\wp(w|\tau) = -(\ln \vartheta_1)''(w|\tau) + \tilde{c}_0 \qquad (4\text{-}10)$$

for some constant $\tilde{c}_0$ depending on τ.

E Properties of the μ-deformed Weierstraß function

E.1 Definition and basic facts

We have defined:

$$\wp_\mu(w) = \sum_{m\in\mathbf{Z}} e^{-i\pi(1-\mu)m} \frac{\pi^2}{\sin^2 \pi(w+m\tau)} \qquad (5\text{-}1)$$

We shall derive the differential equations it satisfies for $\mu \neq 1 \mod 2\mathbf{Z}$. Let us recall two essential facts about functions which, like $\wp_\mu$, belong to $\mathrm{Ker}(\mathbf{T}_1 - \mathrm{id}) \cap \mathrm{Ker}(\mathbf{T} - e^{i\pi\nu}\mathrm{id})$.

(a) Any quotient of them is elliptic, and any elliptic function is a rational function of $\wp$ and $\wp'$, modulo $\wp'^2 = (4\wp^3 - g^{(2)}\wp - g^{(3)})$.

(b) Any such function has at least a pole and a zero (just represent it by the accurate ratio and products of theta functions).

(a) could be used directly to find $\wp'_\mu/\wp_\mu$ and $\wp''_\mu/\wp_\mu$ by studying their zeroes and poles. We found easier to take another route.

E.2 First order equation

For $\mu = 1$, $\wp_1 = \wp + c_0$ with c_0, a constant depending on τ, and it is well-known that :

$$\begin{aligned}
(\wp_1')^2 &= 4\wp_1^3 - 12c_0\wp_1^2 + (12c_0^2 - g_2)\wp_1 - (g_3 + 4c_0^3) \qquad (5\text{-}2) \\
\wp_1'' &= 6\wp_1^2 - 12c_0\wp_1 + 6c_0^2 - \frac{g_2}{2} \qquad (5\text{-}3)
\end{aligned}$$

where g_2, g_3 are constants depending on τ. In the limit $\mu \to 1$, we ought to recover these equations. Yet, the differential equation involving only $\wp_\mu$ that we look for must be linear. We guess that the correct generalization of $(\wp_1'(w))^2$ when $\mu \neq 0$ is $\wp_1'(w)\wp_\mu'(w)$, and we try to match its polar part when $w \to 0$ with a generalization of the RHS of Eqn. 5-2. $\wp_1$ is even and behaves as:

$$\wp_1(w) = \frac{1}{w^2} + c_0 + \frac{1}{2}c_2 w^2 + O(w^4) \tag{5-4}$$

where it turns out (from Eqn. 5-2) that $c_2 = g_2/10$. On the other hand, $\wp_\mu$ is not even. So, there exists constants $c_j^{(\mu)}$ $(0 \leq j \leq 3)$ such that:

$$\wp_\mu(w) = \frac{1}{w^2} + c_0 + c_1^{(\mu)} w + \frac{1}{2}c_2^{(\mu)} w^2 + \frac{1}{6}c_3^{(\mu)} w^3 + O(w^4) \tag{5-5}$$

We obtain:

$$\wp'(w)\wp_\mu'(w) = \frac{4}{w^6} - \frac{2c_1^{(\mu)}}{w^3} - \frac{2c_2^{(\mu)} + 2c_2}{w^2} + \frac{c_3^{(\mu)}}{w} + O(1) \tag{5-6}$$

The simplest candidate in $\mathrm{Ker}(\mathbf{T}_1 - \mathrm{id}) \cap \mathrm{Ker}(\mathbf{T} - e^{i\pi\nu}\mathrm{id})$ to match this polar part is a linear combination of

$$\wp_1^2\wp_\mu, \quad \wp_1\wp_\mu, \quad \wp_\mu, \quad \wp_1'\wp_\mu, \quad \wp_1\wp_\mu' \quad \text{and} \quad \wp_\mu'. \tag{5-7}$$

The polar part of these six functions are independent, so they are enough to fix the terms up to w^{-6}. As an application of (b), this linear combination must be equal to $\wp'\wp_\mu'$. We give directly the result:

$$\wp_1{}'\wp_\mu = 4\wp_1^2\wp_\mu - 4(2c_0 + c_0^{(\mu)})\wp_1\wp_\mu + \left[4c_0^2 + c_0c_0^{(\mu)} + (c_0^{(\mu)})^2 - g_2^{(\mu)}\right]\wp_\mu + \lambda q_\mu \tag{5-8}$$

where q_μ is the function defined by:

$$q_\mu(w) = \left[\wp_1(w)\wp_\mu'(w) - \wp_1'(w)\wp_\mu(w) + (c_0^{(\mu)} - c_0)\wp_\mu'(w) - 3c_1^{(\mu)}\wp_\mu(w)\right] \tag{5-9}$$

and:

$$g_2^{(\mu)} \stackrel{\text{def}}{=} 6c_2 + 4c_2^{(\mu)} \tag{5-10}$$

$$\lambda \stackrel{\text{def}}{=} \frac{-\frac{5}{6}c_3^{(\mu)} + (2c_0 + c_0^{(\mu)})c_1^{(\mu)}}{c_2^{(\mu)} - c_2} \tag{5-11}$$

E.3 Second order equation

Similarly, we can match the polar part when $w \to 0$ of $\wp_\mu''$ by a linear combination of

$$\wp_1\wp_\mu, \quad \wp_\mu, \quad \wp_1'\wp_\mu, \quad \wp_1\wp_\mu' \quad \text{and} \quad \wp_\mu'. \tag{5-12}$$

The result is:

$$\wp_\mu'' = 6\wp_1\wp_\mu - 6(c_0 + c_0^{(\mu)})\wp_\mu + \lambda' q_\mu(w) \tag{5-13}$$

where:

$$\lambda' = \frac{-3c_1^{(\mu)}}{c_2^{(\mu)} - c_2} \tag{5-14}$$

E.4 Consistency of the limit $\mu \to 1$

Let us consider the limit $\mu \to 1$ of these differential equations. $c_j^{(\mu)}$ (for $0 \leq j \leq 3$) is equal to c_j (which is zero when j is odd) in the limit $\mu \to 1$ since we can commute the limit and the residue in:

$$c_j^{(\mu)} = \operatorname*{Res}_{w\to 0} \frac{\mathrm{d}w}{w^{l+1}}\, \wp_\mu(w) \tag{5-15}$$

Comparing to Eqn. 5-8 to 5-2, and Eqn. 5-13 to 5-3, we find:

$$\begin{aligned} \forall w \in \mathbf{C} \qquad & \lim_{\mu\to 1} \lambda q_\mu(w) = -(4c_0^3 + g_3) \\ & \lim_{\mu\to 1} \lambda' q_\mu(w) = 6c_0^2 - \frac{g_2}{2} \end{aligned} \tag{5-16}$$

But one can compute independently:

$$\lambda' q_\mu(w) = -\frac{6c_1^{(\mu)}}{w} + \lambda' \left(\frac{5}{6} c_3^{(\mu)} - 2c_1^{(\mu)} c_0^{(\mu)} \right) + O(w) \tag{5-17}$$

Hence λ and λ' diverge when $\mu \to 1$ in a way such that:

$$\lim_{\mu\to 1} \frac{\lambda}{\lambda'} = \frac{g_3 + 4c_0^2}{\frac{g_2}{2} - 6c_0^3} \tag{5-18}$$

F Properties of the special solutions of Eqn. 2-6

▷ Theta expression for D_μ:

$$D_\mu(u) = \frac{x(u)}{\sigma(x(u))} \frac{\vartheta_1\left(u - \frac{\tau}{2} + \frac{\mu}{2}\middle|\tau\right)}{\vartheta_1\left(u - \frac{\tau}{2}\,|\tau\right)} \frac{\vartheta_1\left(-\frac{1}{2}\,|\tau\right)}{\vartheta_1\left(-\frac{1}{2} + \frac{\mu}{2}\middle|\tau\right)} \tag{6-1}$$

$u_e = \frac{\tau-\mu}{2}$ is the second zero (mod $\mathbf{Z} \oplus \tau\mathbf{Z}$) of D_μ. We call $e_\mu = x(u_e)$, the corresponding point in the physical x-sheet:

$$e_\mu = a\,\mathrm{sn}_k(i\mu K') \tag{6-2}$$

We use the notation $\mathrm{D}_\mu(\mathrm{x}) = D_\mu(\mathrm{u}(\mathrm{x}))$.

▷ Relation between D_μ and $D_{-\mu}$. $D^{(2)} : u \mapsto D_\mu(u)D_\mu(\tau - u)$ is an u-even and biperiodic function. Hence $\mathrm{D}_\mu^{(2)}(\mathrm{x}) = D_\mu^{(2)}(\mathrm{u}(\mathrm{x}))$ is a rational fraction of x^2. In this variable, it has simple poles at $\mathrm{x}^2 = a^2$ and $\mathrm{x}^2 = b^2$, simple zeroes at $\mathrm{x}^2 = \infty$, $\mathrm{x}^2 = e_\mu^2$, and is equivalent to $-1/\mathrm{x}^2$ when $\mathrm{x}^2 \to \infty$. Thus:

$$\mathrm{D}_\mu^{(2)}(\mathrm{x}) = -\frac{\mathrm{x}^2 - e_\mu^2}{(\mathrm{x}^2 - a^2)(\mathrm{x}^2 - b^2)} \tag{6-3}$$

▷ Asymptotic of $\mathrm{D}_\mu(\mathrm{x})$. We define the constants α_1 and α_2 by:

$$\mathrm{D}_\mu(\mathrm{x}) = \frac{1}{\mathrm{x}} + \frac{\alpha_1}{\mathrm{x}^2} + \frac{\alpha_2}{\mathrm{x}^3} + O\left(\frac{1}{\mathrm{x}^4}\right) \qquad \text{when x} \to \infty \text{ in the physical sheet} \tag{6-4}$$

Studying Eqn. 6-1 when $u \to \frac{-1+\tau}{2}$ yields the expression:

$$\alpha_1 = \frac{-ib}{2K'}(\ln \vartheta_1)' \left(\frac{1-\mu}{2}\Big|\tau\right). \tag{6-5}$$

Studying Eqn. 6-3 when $x \to \infty$ yields the expression of α_2:

$$\alpha_2 = a^2 + b^2 - e_\mu^2 - \alpha_1^2 \tag{6-6}$$

The terms up to α_2 are involved in the fully packed case.

▷ Derivative of D_μ. D_μ and $(\partial_u D_\mu)$ are both in $\mathrm{Ker}\,(\mathbf{T} - e^{i\pi\mu})$, so $\partial_u \ln D_\mu$ is 1- and τ- translation invariant. By studying its analytical properties, we obtain:

$$\frac{\mathrm{dD}_\mu}{\mathrm{dx}} = \mathrm{L}(\mathrm{x})\mathrm{D}_\mu(\mathrm{x}) \tag{6-7}$$

where:

$$\begin{aligned} \mathrm{L}(\mathrm{x}) &= \frac{-\alpha_1 + \mathrm{I}(\mathrm{x})}{\sigma(\mathrm{x})} - \frac{\mathrm{d}\ln\sigma}{\mathrm{dx}} \\ \text{and } \mathrm{I}(\mathrm{x}) &= \frac{\mathrm{x}\sigma(\mathrm{x}) + e_\mu \sigma(e_\mu)}{\mathrm{x}^2 - e_\mu^2} \end{aligned}$$

▷ Definition of f_μ and $\widehat{f}_\mu$.

$$\begin{aligned} f_\mu(u) &= \frac{D_\mu(u) + D_\mu(-u)}{1 - e^{i\pi\mu}} \\ \widehat{f}_\mu(u) &= \frac{\sigma(x(u))}{x(u)} \frac{D_\mu(u) - D_\mu(-u)}{1 + e^{i\pi\mu}} \end{aligned} \tag{6-8}$$

We recall that $u(-x) = \tau - u(x)$. So, in the x variable:

$$\begin{aligned}
\mathrm{f}_\mu(x) &= \frac{\mathrm{D}_\mu(\mathrm{x}) + e^{i\pi\mu}\mathrm{D}_\mu(-\mathrm{x})}{1 - e^{i\pi\mu}} \\
\widehat{\mathrm{f}}_\mu(\mathrm{x}) &= \frac{1}{\mathfrak{n} - 2}\sigma(\mathrm{x})\mathrm{x}\left(\mathfrak{n}\mathrm{f}_\mu(\mathrm{x}) + 2\mathrm{f}_\mu(-\mathrm{x})\right)
\end{aligned} \tag{6-9}$$

▷ Wronskien. The wronskien function is defined by $\Xi_\mu = (\mathbf{T} f_\mu)\, \widehat{f_\mu} - f_\mu\, (\mathbf{T}\widehat{f_\mu})$. After computation:

$$\begin{aligned}
\Xi_\mu(\mathrm{u}(\mathrm{x})) &= \mathrm{f}_\mu(\mathrm{x})\widehat{\mathrm{f}}_\mu(-\mathrm{x}) - \mathrm{f}_\mu(-\mathrm{x})\widehat{\mathrm{f}}_\mu(\mathrm{x}) \\
&= -\frac{2\sigma(\mathrm{x})}{\mathrm{x}}\mathrm{D}_\mu^{(2)}(\mathrm{x}) \\
&= \frac{2(\mathrm{x}^2 - e_\mu^2)}{\mathrm{x}\sigma(\mathrm{x})}
\end{aligned} \tag{6-10}$$

▷ $\perp$-Norms of f_μ and $\widehat{\mathrm{f}}_\mu$. From the definitions:

$$\begin{aligned}
\mathrm{R}_\mu(\mathrm{x}^2) = (\mathrm{f}_\mu \perp \mathrm{f}_\mu)(\mathrm{x}) &= -(2 - \mathfrak{n})\,\mathrm{D}_\mu^{(2)}(\mathrm{x}) \\
&= (2 - \mathfrak{n})\frac{\mathrm{x}^2 - e_\mu^2}{(\mathrm{x}^2 - a^2)(\mathrm{x}^2 - b^2)}
\end{aligned} \tag{6-11}$$

$$\begin{aligned}
\widehat{\mathrm{R}}_\mu(\mathrm{x}^2) = \left(\widehat{\mathrm{f}}_\mu \perp \widehat{\mathrm{f}}_\mu\right)(\mathrm{x}) &= -(2 + \mathfrak{n})\frac{\sigma(\mathrm{x})^2}{\mathrm{x}^2}\,\mathrm{D}_\mu^{(2)}(\mathrm{x}) \\
&= (2 + \mathfrak{n})\frac{\mathrm{x}^2 - e_\mu^2}{\mathrm{x}^2}
\end{aligned} \tag{6-12}$$

▷ Asymptotic of f_μ and $\widehat{\mathrm{f}}_\mu$.

$$\begin{aligned}
\mathrm{f}_\mu(\mathrm{x}) &= \frac{1}{\mathrm{x}} + \frac{1 + e^{i\pi\mu}}{1 - e^{i\pi\mu}}\frac{\alpha_1}{\mathrm{x}^2} + \frac{\alpha_2}{\mathrm{x}^3} + O\left(\frac{1}{\mathrm{x}^4}\right) \\
\widehat{\mathrm{f}}_\mu(\mathrm{x}) &= 1 + \frac{1 - e^{i\pi\mu}}{1 + e^{i\pi\mu}}\frac{\alpha_1}{\mathrm{x}} + \frac{\alpha_2}{\mathrm{x}^2} + O\left(\frac{1}{\mathrm{x}^3}\right)
\end{aligned} \tag{6-13}$$

when $\mathrm{x} \to \infty$ in the physical sheet.

▷ Values at e_μ. Since $\mathrm{D}_\mu(e_\mu) = 0$, $\mathrm{f}_\mu(e_\mu)$ and $\widehat{\mathrm{f}}_\mu(e_\mu)$ are both proportional to $\mathrm{D}_\mu(-e_\mu)$ (everything is read in the variable x). More precisely:

$$\frac{\mathrm{f}_\mu(e_\mu)}{\widehat{\mathrm{f}}_\mu(e_\mu)} = -\frac{1+e^{i\pi\mu}}{1-e^{i\pi\mu}}\frac{e_\mu}{\sigma(e_\mu)} \tag{6-14}$$

▷ Behavior at branch points. Thanks to Eqn. 6-11, we can find the behavior of f_μ near $a_i \in \{a, b\}$:

$$\lim_{\mathrm{x}\to a_i} \sigma(\mathrm{x})\mathrm{f}_\mu(\mathrm{x}) = \sqrt{(2-\mathfrak{n})(a_i^2 - e_\mu^2)} \tag{6-15}$$

▷ First order 2×2 differential system. The differential equation Eqn. 6-7 can be turned into a first order differential system defining $(\mathrm{f}_\mu, \widehat{\mathrm{f}}_\mu)$ intrinsically. The solution is unique if we require the asymptotic to be given by Eqn. 6-13 up to $O(\mathrm{x}^{-3})$ when $\mathrm{x}\to\infty$ in the physical sheet. The data of e_μ (Eqn. 6-2) and α_1 (Eqn. 6-5) ensures that this functions have the correct monodromy, i.e are one cut solutions of the saddle point equation 2-2.

$$\frac{\mathrm{d}}{\mathrm{dx}}\begin{pmatrix} \mathrm{f}_\mu(\mathrm{x}) \\ \frac{\mathrm{x}\widehat{\mathrm{f}}_\mu(\mathrm{x})}{\sigma(\mathrm{x})} \end{pmatrix} = \begin{pmatrix} \mathrm{L}_o(\mathrm{x}) & \frac{1+e^{i\pi\mu}}{1-e^{i\pi\mu}}\mathrm{L}_e(\mathrm{x}) \\ \frac{1-e^{i\pi\mu}}{1+e^{i\pi\mu}}\mathrm{L}_e(\mathrm{x}) & \mathrm{L}_o(\mathrm{x}) \end{pmatrix}\cdot\begin{pmatrix} \mathrm{f}_\mu(\mathrm{x}) \\ \frac{\mathrm{x}\widehat{\mathrm{f}}_\mu(\mathrm{x})}{\sigma(\mathrm{x})} \end{pmatrix} \tag{6-16}$$

where L_o and L_e and the odd and even part of L:

$$\mathrm{L}_o(\mathrm{x}) = \frac{1}{2}\frac{\mathrm{d}\ln \mathrm{R}_\mu}{\mathrm{dx}} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\frac{1}{2}\ln(\mathrm{x}^2 - e_\mu^2) - \ln\sigma(\mathrm{x})\right] \tag{6-17}$$

$$\mathrm{L}_e(\mathrm{x}) = \frac{1}{\sigma(\mathrm{x})}\left[-\alpha_1 + \frac{e_\mu\sigma(e_\mu)}{\mathrm{x}^2 - e_\mu^2}\right] = \frac{-\alpha_1}{\sigma(\mathrm{x})}\frac{\mathrm{x}^2 - \widehat{e}_\mu^2}{\mathrm{x}^2 - e_\mu^2} \tag{6-18}$$

We have called $\widehat{e}_\mu = \sqrt{e_\mu^2 + \frac{e_\mu\sigma(e_\mu)}{\alpha_1}}$, the zero of $\mathrm{L}_e(\mathrm{x})$.

G Summary of equations and notations

▷ Change of variable. Roman characters (like W) are used for functions defined in the x variable. Slanted characters (like W) are used for the same functions read in the u variable, defined on the whole u-plane by extension. In particular, $\mathrm{x}\mapsto\mathrm{u}(\mathrm{x})$ and $u\mapsto x(u)$ are reciprocal functions in some domains, and:

$$W(u) = \mathrm{W}(x(u))$$

when $x(u)$ is in the physical x-sheet.

▷ Differential form $\mathrm{d}x$.

$$\begin{array}{rcl} s & \stackrel{\text{def}}{=} & \frac{\mathrm{d}x}{\mathrm{d}u} = \frac{ib}{2K'}\sqrt{(x^2(u)-a^2)(x^2(u)-b^2)} \\ s(u) & \text{is} & 1\text{ - and } 2\tau\text{ -translation invariant} \\ s(-u) & = & -s(u) \\ s(u-\tau) & = & -s(u) \end{array}$$

▷ Correlation functions.

$$\begin{array}{rl} \forall \mathrm{x}\in[a,b] & \mathrm{W}_1^{(0)}(\mathrm{x}+i\epsilon)+\mathrm{W}_1^{(0)}(\mathrm{x}-i\epsilon)+\mathfrak{n}\mathrm{W}_1^{(0)}(-\mathrm{x}) \underset{\epsilon\to 0}{=} \mathrm{V}'(\mathrm{x}) \\ & \mathrm{W}_2^{(0)}(\mathrm{x}+i\epsilon)+\mathrm{W}_2^{(0)}(\mathrm{x}-i\epsilon)+\mathfrak{n}\mathrm{W}_2^{(0)}(-\mathrm{x}) \underset{\epsilon\to 0}{=} -\frac{1}{(\mathrm{x}-\mathrm{x}')^2} \\ \text{else} & \mathrm{W}_k^{(g)}(\mathrm{x}+i\epsilon,I)+\mathrm{W}_k^{(g)}(\mathrm{x}-i\epsilon,I)+\mathfrak{n}\mathrm{W}_k^{(g)}(-\mathrm{x},I) \underset{\epsilon\to 0}{=} 0 \end{array}$$

$\overline{\mathrm{W}}_k^{(g)}$ is the part of $\mathrm{W}_k^{(g)}$ satisfying the homogeneous linear equation. $W(I)$ or $\overline{W}(I)$ indicates the same functions, but depending on the new variable u_i instead of x_i for some i's.

▷ Correlation forms.

$$\omega_k^{(g)}(\mathrm{u}(\mathrm{x}_1),\ldots,\mathrm{u}(\mathrm{x}_k)) = \mathrm{d}x_1\cdots\mathrm{d}x_k\, W(\mathrm{u}(\mathrm{x}_1),\ldots,\mathrm{u}(\mathrm{x}_k))$$

for x_i 's in the physical sheet.

▷ Spectral curve.

$$\begin{array}{rcl} y(u) & \stackrel{\text{def}}{=} & 2\overline{W}_1^{(0)}(u)+\mathfrak{n}\overline{W}_1^{(0)}(u-\tau) \\ y(-u) & = & -y(u) \end{array}$$

▷ Auxiliary Cauchy kernel H.

$$\begin{array}{rcl} \mathrm{H}(\mathrm{x}_0,\mathrm{x}) & \stackrel{\text{def}}{=} & \int_\infty^{\mathrm{x}} \mathrm{d}\mathrm{x}'\overline{\mathrm{W}}_2^{(0)}(\mathrm{x}_0,\mathrm{x}') \\ H(\mathrm{x}_0,u) & = & \int_{\mathrm{u}(\infty)}^{u} \mathrm{d}u' s(u')\overline{\mathrm{W}}_2^{(0)}(\mathrm{x}_0,u) \end{array}$$

The following properties hold for all $u \in \mathbf{C}$, $\mathrm{x}_0 \in \mathbf{C}$:

$$\begin{aligned} H(\mathrm{x}_0, u-2\tau) - \mathfrak{n} H(\mathrm{x}_0, u-\tau) + H(x_0, u) &= \mathrm{cte}_1(\mathrm{x}_0) \\ H(\mathrm{x}_0, u) &= H(\mathrm{x}_0, -u) + \mathrm{cte}_2(\mathrm{x}_0) \end{aligned}$$

▷ Cauchy kernel G.

$$\begin{aligned} \mathrm{G}(\mathrm{x}_0, \mathrm{x}) &\stackrel{\text{def}}{=} \int_\infty^{\mathrm{x}} \mathrm{dx}' \left(2\overline{\mathrm{W}}_2^{(0)}(\mathrm{x}_0, \mathrm{x}') + \mathfrak{n}\overline{\mathrm{W}}_2^{(0)}(\mathrm{x}_0, -\mathrm{x}')\right) \\ G(\mathrm{x}_0, u) &= 2H(\mathrm{x}_0, u) - \mathfrak{n} H(\mathrm{x}_0, u-\tau) \\ G(\mathrm{x}_0, -u) &= -G(\mathrm{x}_0, u) + \mathrm{cte}_3(\mathrm{x}_0) \end{aligned}$$

References

[1] R.J. Baxter, *Exactly solved models in statistical mechanics*, Dover Publication (last edition 2007)

[2] C. Berge, *Théorie des graphes et ses applications*, Dunod, 1958. Translation re-published as : *The theory of graphs*, Dover Publication (2001)

[3] M. Bergère, B. Eynard, *Universal scaling limits of matrix models, and* (p,q)*-Liouville gravity*, `arXiv:math-ph/0909.0854` (2009)

[4] J.-E. Bourgine, K. Hosomichi, *Boundary operators in the* $\mathcal{O}(\mathfrak{n})$ *and RSOS matrix models*, JHEP, `arXiv:hep-th/08113252` (2009)

[5] E. Brézin, C. Itzykson, G. Parisi, J.-B. Zuber, *Planar diagrams*, Comm. in Math. Phys. **59**, 35 (1978)

[6] F. David, *Conformal field theories coupled to 2d gravity in the conformal gauge*, Mod. Phys. Lett. **A3**, 1651 (1988)

[7] J. Distler, H. Kawai, *Conformal field theory and 2d quantum gravity, or who's afraid of Joseph Liouville ?*, Nucl. Phys. **B321**, 509 (1989)

[8] P. Di Francesco, P. Ginsparg, J. Zinn-Justin, *2d gravity and random matrices*, Phys. Rept. **254**, 1-133, `arXiv:hep-th/9306153v2` (1994)

[9] P. Di Francesco, P. Mathieu, D. Sénéchal, *Conformal field theory*, Springer, (corrected edition, 1999). On the $\mathcal{O}(\mathfrak{n})$ model : see p229-231.

[10] B. Duplantier, I. Kostov, *Conformal spectra of polymers on a random surface*, Phys. Lett. Rev. **61**, 13 (1988)

[11] B. Eynard, *Gravité quantique bidimensionnelle et matrices aléatoires*, Thèse de doctorat (1995)

[12] B. Eynard, *Formal matrix integrals and combinatorics of maps*, `arXiv:math-ph/0611087` (1996)

[13] B. Eynard, *Topological expansion for the 1-hermitian matrix model correlation functions*, JHEP/024A/0904, `arXiv:hep-th/0407261` (2004)

[14] B. Eynard, A. Kokotov, D. Korotkin, *Genus one contribution to free energy in hermitian two-matrix model*, Nucl.Phys. **B694**, p443-472, `arXiv:hep-th/0403072` (2004)

[15] B. Eynard, C. Kristjansen, *Exact solution of the $\mathcal{O}(\mathfrak{n})$ model on a random lattice*, Nucl. Phys. **B455**, p577-618, `arXiv:hep-th/9506193` (1995)

[16] B. Eynard, C. Kristjansen, *More on the exact solution of the $\mathcal{O}(\mathfrak{n})$ model on a random lattice and an investigation of the case $|\mathfrak{n}| > 2$*, Nucl.Phys. **B466**, p463-487, `arXiv:hep-th/9512052` (1996)

[17] B. Eynard, N. Orantin, *Invariants of algebraic curves and topological expansion*, Communications in Number Theory and Physics, Vol 1, Number 2, p347-452, `arXiv:math-ph/0702045` (2007)

[18] B. Eynard, J. Zinn-Justin, *The $\mathcal{O}(\mathfrak{n})$ model on a random surface: critical points and large order behavior*, Nucl. Phys. **B386**, p558-591, `arXiv:hep-th/9204082` (1992)

[19] M. Gaudin, I. Kostov, *$\mathcal{O}(\mathfrak{n})$ on a fluctuating planar lattice. Some exact results*, Phys. Lett. **B220**, 200 (1989)

[20] Z.X. Wang, D.R. Guo, *Special functions*, ed. World Scientific (1989)

[21] V. Kazakov, I. Kostov, *Loop gas model for open strings*, Nucl. Phys. **B386**, 520 (1992)

[22] V. G. Knizhnik, A. M. Polyakov, A. B. Zamolodchikov, *Fractal structure of 2d quantum gravity*, Mod. Phys. Lett. **A3**, 819 (1988)

[23] J. M. Kosterlitz, *The critical properties of the two dimensional $x - y$ model*, J. Phys. **C7**, 1046 (1974)

[24] I. Kostov, M.L. Mehta, *Random surfaces of arbitrary genus: exact results for $D = 0$, and -2 dimensions*, Phys. Lett. **B189**, p118-124 (1987)

[25] I. Kostov, *$\mathcal{O}(\mathfrak{n})$ vector model on a planar random lattice: spectrum of anomalous dimensions*, Mod. Phys. Lett. **A4**, 217 (1989)

[26] I. Kostov, *Loop amplitude for nonrational string theories*, Phys. Lett. **B266**, 312 (1991)

[27] I. Kostov, M. Staudacher, *Multicritical phases of the $\mathcal{O}(\mathfrak{n})$ model on a random lattice*, Nucl. Phys. **B384**, p459-483, `arXiv:hep-th/9203030` (1992)

[28] I. Kostov, *Boundary loop models and 2D quantum gravity*, J. Stat. Mech. 07-08 : P08023, `arXiv:hep-th/0703221` (2007)

[29] J. L. Jacobsen, H. Saleur, *Conformal boundary loop models*, Nucl. Phys. **B788**, p137-166 (2008), `arXiv:math-ph/0611078` (2007)

[30] G.W. Moore, N. Seiberg, M. Staudacher, *From loops to states in 2d quantum gravity*, Nucl. Phys. **B362**, p665-709 (1991)

[31] W.T. Tutte, *A census of planar triangulations*, Can. J. Math. **14**, p21-38 (1962)

[32] W.T. Tutte, *A census of planar maps*, Can. J. Math. **15**, p249-271 (1963)