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A MATRIX MODEL FOR SIMPLE HURWITZ NUMBERS, AND TOPOLOGICAL RECURSION

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Abstract. We introduce a new matrix model representation for the generating function of simple Hurwitz numbers. We calculate the spectral curve of the model and the associated symplectic invariants developed in [4]. As an application, we prove the conjecture proposed by Bouchard and Mariño [2], relating Hurwitz numbers to the spectral invariants of the Lambert curve $e^x = ye^{-y}$.

1. Summary

In [2], Bouchard and Mariño propose a new conjectural recursion formula to compute simple Hurwitz numbers, i.e. the weighted count of coverings of $\mathbb{C}\mathrm{P}^1$ with specified branching data. Their recursion is based on a new conjectured formalism for the type B topological string on mirrors of toric Calabi–Yau threefolds, called "remodeling the B-model", or "BKMP conjecture" [1]. The Bouchard-Mariño conjecture for Hurwitz numbers appears as a consequence of this general BKMP conjecture applied to the infinite framing limit of the open string theory of $\mathbb{C}^3$. In this limit, the amplitudes are known to give simple Hurwitz numbers.

They propose that the generating function for Hurwitz numbers can be recovered from the symplectic invariants (also called topological recursion) developed in [4], applied to the so called "Lambert curve" $y = L(e^x)$ defined by:

$$e^x = ye^{-y}.$$

In this paper, we make the link between Hurwitz numbers and the Lambert curve explicit. We introduce a new matrix model formula for the generating function of simple Hurwitz numbers

$$Z \propto \int_{\mathcal{H}_N(\mathcal{C})} \mathrm{d}M \exp\left(-\frac{1}{g_s}\,\mathrm{Tr}(V(M) - M\mathbf{R})\right), \tag{1}$$

where $V(x)$ is the potential

$$V(x) = -\frac{x^2}{2} + g_s(N - \frac{1}{2})x + x\ln(g_s/t) + i\pi x - g_s \ln\big(\Gamma(-x/g_s)\big).$$

The parameters g_s and the matrix $\mathbf{R}$ involved in the definition of Z are such that the weight of a covering of Euler characteristic χ is proportional to g_s^χ, and has a polynomial dependance in $v_i = \exp R_i$ which encodes the ramification data above branch points.

A method to compute topological expansion of matrix integrals with an external field was introduced in [4]. It consists in finding the spectral curve $\mathcal{S}$ (roughly speaking the equilibrium density of eigenvalues of the matrix, more precisely the planar part of the expectation value of the resolvent), then computing recursively a sequence of algebraic k-forms $\mathcal{W}_k^{(g)}(\mathcal{S})$, and some related algebraic quantities called symplectic invariants $\mathcal{F}_g(\mathcal{S}) = \mathcal{W}_0^{(g)}(\mathcal{S})$. Then, one of the main results of [4] is that

$$\ln Z = \sum_{g=0}^{\infty} g_s^{2g-2}\,\mathcal{F}_g(\mathcal{S}).$$

In our case, this implies that the generating function for simple Hurwitz numbers of genus g is precisely $\mathcal{F}_g(\mathcal{S})$, where $\mathcal{S}$ is the spectral curve of our matrix model.

It is rather easy to find the spectral curve of the matrix model Eqn 1. The result, after suitable symplectic transformations, reads

$$\widetilde{\mathcal{S}}(\mathbf{p}, g_s; t) = \begin{cases} x(z) = -z + \ln(z/t) + c_0 + \frac{c_1}{z} - \sum_{n=1}^{\infty} \frac{B_{2n}\,g_s^{2n}}{2n}\,f_{2n}(z) \\ y(z) = z + g_s \sum_{i=1}^{N} \frac{1}{(z-z_i)y_i} + \frac{1}{z_i y_i} \end{cases}, \tag{2}$$

where z_i, y_i, c_0 and c_1 are determined by consistency relations, and z_i and y_i are $O(1)$ when $g_s \to 0$, and c_0 and c_1 are $O(g_s)$. In particular, when we set the coupling constant $g_s = 0$, we recover the Lambert curve $y = L(te^x)$.

In [2], Bouchard and Mariño define another set of generating functions, denoted $H^{(g)}(x_1, \ldots, x_k)$, encoding genus g simple Hurwitz numbers, and which are derivatives of $\ln Z$, evaluated at $g_s = 0$. The statement of their conjecture is:

$$\frac{\mathcal{W}_k^{(g)}(z_1, \ldots, z_k)}{\mathrm{d}x(z_1)\cdots \mathrm{d}x(z_k)} = H^{(g)}(x(z_1), \ldots, x(z_k)),$$

where $\mathcal{W}_k^{(g)}$ are the k-forms of [4] computed for the Lambert curve.

We prove their conjecture by using the general properties of the invariants of [4], in particular the fact that derivatives of the $\mathcal{F}_g$'s with respect to almost any parameter, can be expressed in terms of the $\mathcal{W}_k^{(g)}$'s. Then it suffices to set $g_s = 0$, and this gives the Bouchard-Mariño conjecture.

Organization of the paper. In Section 2, we recall the definitions and derive a matrix model formula for the generating function of simple Hurwitz numbers. We recall the construction of the symplectic invariants and topological recursion of [4] in Section 3. In Section 4, we derive the spectral curve of our matrix model (the proof is presented in Appendix 7) and prove the Bouchard-Mariño conjecture following a method very close to [6]. In section 5, we briefly study the link with the Kontsevich integral. Section 6 addresses generalizations of our method and open questions.

2. Construction of the matrix model

Let $\mathrm{Cov}_n^*(C_1, \ldots, C_k)$ denote the weighted number of n-fold coverings (possibly disconnected) of $\mathbb{CP}^1$ ramified over k fixed points of $\mathbb{CP}^1$ with monodromies in the conjugacy classes $C_1, \ldots, C_k$. The weight is one over the order of the automorphism group of the covering. Similarly, let $\mathrm{Cov}_n(C_1, \ldots, C_k)$ denote the weighted number of n-fold **connected** coverings.

By a result of Burnside (see, eg [16]), we have

$$\mathrm{Cov}_n^*(C_1, \ldots, C_k) = \sum_{|\lambda|=n} \left(\frac{\dim \lambda}{n!}\right)^2 \prod_{i=1}^{k} f_\lambda(C_i), \tag{3}$$

where the sum ranges over all partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ of $|\lambda| = \sum \lambda_i = n$ boxes, $\dim \lambda$ is the dimension of the irreducible representation indexed by λ (with corresponding character χ_λ), and

$$f_\lambda(C_i) = \frac{|C_i|}{\dim \lambda} \chi_\lambda(C_i).$$

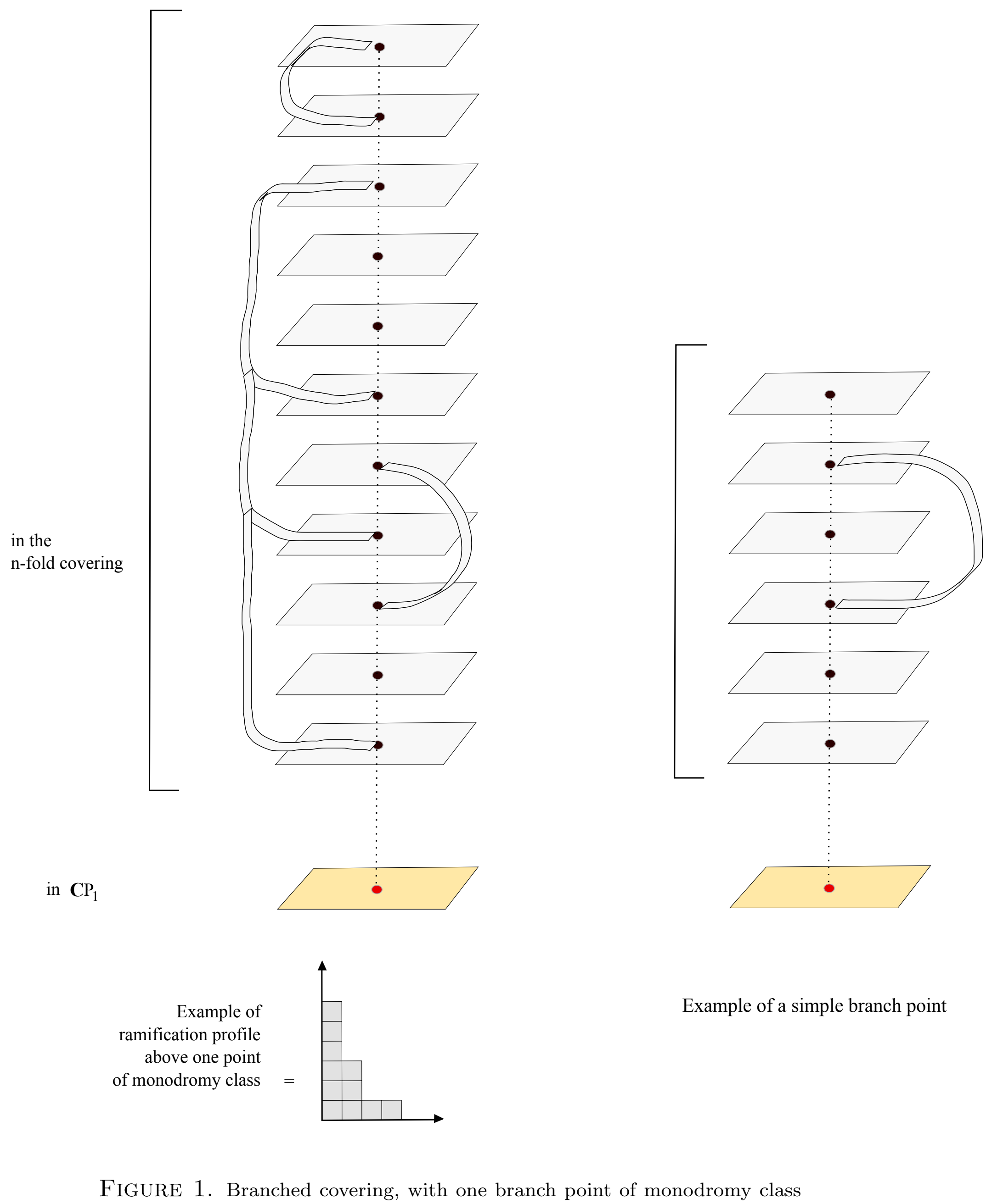

FIGURE 1. Branched covering, with one branch point of monodromy class μ, and simple branch points (for which the monodromy is a transposition).

2.1. **Simple Hurwitz numbers.** We are interested in counting n-fold coverings of genus g with 1 branch point of arbitrary profile μ, and only transpositions above other points (called simple branch points), c.f. Figure 1. We denote $C_{(2)}$ the conjugacy class of a transposition. For b simple branch points and one branch point of profile μ, the Euler characteristic of the $n-$fold covering reads from the Riemann-Hurwitz formula:

$$\chi = |\mu| + \ell(\mu) - b.$$

For connected coverings, we have $\chi = 2 - 2g$, where g is the genus, and we also define the simple Hurwitz numbers:

$$H_{g,\mu} = \mathrm{Cov}_n(C_\mu, \overbrace{C_{(2)}, \ldots, C_{(2)}}^{b}),$$

where $b = 2g - 2 + |\mu| + \ell(\mu)$.

2.2. **Generating function for simple Hurwitz numbers.** With the notations $p_\mu = \prod_i p_{\mu_i}$, and $\mathbf{p} = (p_1, p_2, \ldots)$, we shall study the generating function

$$Z\,(\mathbf{p}, g_s; t) = \sum_{n=0}^{\infty} t^n \sum_{|\mu|=n} \sum_{b=0}^{\infty} \frac{g_s^{b-|\mu|-\ell(\mu)}}{b!}\; p_\mu \,\mathrm{Cov}^*_n(C_\mu, \overbrace{C_{(2)}, \ldots, C_{(2)}}^{b}), \tag{4}$$

where $\mathrm{Cov}^*_n(C_\mu, C_{(2)}, \ldots, C_{(2)})$ is the number of (not necessarily connected) branched coverings of Euler characteristic $\chi = |\mu| + \ell(\mu) - b$.

In the language of string theory, $-g_s$ is the string coupling constant[1]. Let us emphasize that $Z(\mathbf{p}, g_s; t)$ is defined as a formal power series in t and g_s, i.e. it is merely a notation to collect all the coefficients. Each coefficient (for b and n fixed) is a finite sum, which is a polynomial function of $p_1, \ldots, p_n$. Notice also that the parameter t is redundant because we can change $p_j \to \rho^j p_j$ and $t \to t/\rho$ without changing the sum, i.e.

$$Z(\{p_1, p_2, p_3, \ldots, \}, g_s; t) = Z(\{\rho p_1, \rho^2 p_2, \rho^3 p_3, \ldots, \}, g_s; t/\rho).$$

In [2], t is chosen as $t = 1$, but we find more convenient to keep $t \neq 1$ for the moment, in order to have only two formal parameters g_s and t, instead of an infinite number of them g_s and $p_1, p_2, \ldots$, which would be the case if t were set to 1. The generating function of **connected** coverings is $F = \ln Z$ (in the sense of formal power series of t and g_s):

$$F(\mathbf{p}, g_s; t) = \ln Z = \sum_{b,n} \frac{t^n}{b!} \sum_{|\mu|=n} g_s^{2g-2}\; p_\mu H_{g,\mu},$$

where $b = 2g - 2 + |\mu| + \ell(\mu)$.

Therefore, we have a so-called *topological expansion* (equality of formal series):

$$F(\mathbf{p}, g_s; t) = \sum_{g=0}^{\infty} g_s^{2g-2}\; F_g(\mathbf{p}; t),$$

where F_g counts the number of connected coverings of genus g:

$$F_g(\mathbf{p}; t) = \sum_n t^n \sum_{|\mu|=n} \frac{p_\mu}{(2g - 2 + n + \ell(\mu))!}\, H_{g,\mu}.$$

[1] In the physics literature, it is customary to choose $-g_s$ instead of g_s as formal parameter.

Our goal in this article is to provide a recursive algorithm to compute the F_g's, and more precisely, prove that the F_g's are the symplectic invariants introduced in [4] for a spectral curve $\widetilde{\mathcal{S}}(\mathbf{p}, g_s; t)$ which we shall describe in Section 4. As a consequence, we shall prove the conjecture of Bouchard and Mariño [2].

2.3. **Partitions.** To make our notations clear, we recall some representations of partitions or Young tableaux. The set of all partitions λ of length $\leq N$ is in bijection with other interesting sets of objects:

- The decreasing finite series of N integers : $\lambda_1 \geq \ldots \geq \lambda_{\ell(\lambda)} \geq \lambda_{\ell(\lambda)+1} = \ldots = \lambda_N \geq 0$. The length $\ell(\lambda)$ of the partition is the number of non-vanishing λ_i's. $|\lambda| = \sum_{i=1}^{\ell(\lambda)} \lambda_i = n$ is the number of boxes of the partition.

- The strictly decreasing finite series of positive integers. They mark the positions (up to a translation) on the horizontal axis of the increasing jumps when the Young tableau (λ_i, i) is tilted anticlockwise by $\frac{\pi}{4}$ (c.f. Figure 2). The correspondence is given by

$$h_i = \lambda_i - i + N \qquad (i \in \{1, \ldots, N\}).$$

We have $h_1 > h_2 > \cdots > h_N \geq 0$.

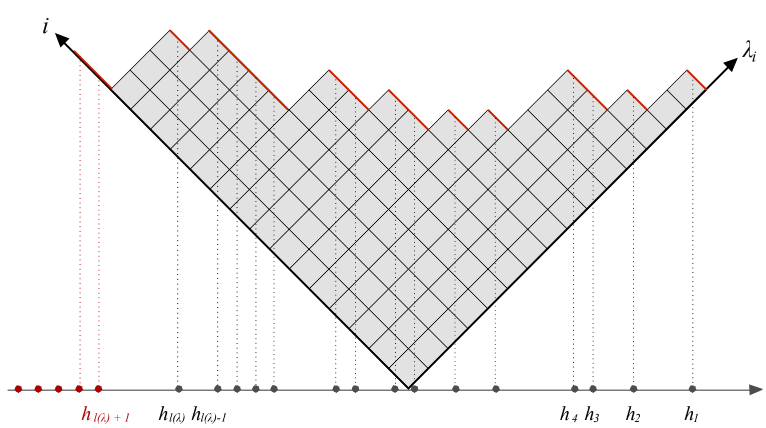

FIGURE 2. Rotate the partition by $\pi/4$. The h_i's mark the positions (up to a translation) on the horizontal axis of the increasing jumps in the Young tableau.

- The conjugacy classes of $\mathfrak{S}_n$. The class C_λ associated to λ is the one with $m_r = |\{i > 0 \quad \lambda_i = r\}|$ cycles of length r, and its cardinal is

$$|C_\lambda| = \frac{|\lambda|!}{\prod_{r\geq 1} m_r! \cdot r^{m_r}}.$$

- The equivalence classes of irreducible representations of $\mathfrak{S}_{|\lambda|}$.
- The equivalence classes of irreducible representations of $\mathrm{GL}_{\ell(\lambda)}(\mathbb{C})$, or of $\mathrm{U}(\ell(\lambda))$.

2.4. **Schur polynomials.** Recall the definition of Schur polynomials s_λ [13]: they coincide with the characters of the representation of $\mathrm{U}(k)$ indexed by λ such that $k = \ell(\lambda)$. As a matter of fact, if $\mathbf{v} = (v_1, \ldots, v_k)$ is an k-tuple of complex variables:

$$s_\lambda(\mathbf{v}) = \frac{\det\left(v_i^{\lambda_j - j + N}\right)}{\Delta(\mathbf{v})}, \tag{5}$$

where $\Delta(\mathbf{v}) = \prod_{1 \le j < i \le N}(v_i - v_j)$ is the Vandermonde determinant. This formula can be extended to a definition of s_λ with $N \ge k$ variables, $\mathbf{v} = (v_1, \ldots, v_N)$, provided that we take $\lambda_j = 0$ whenever $j > k$.

The Frobenius formula gives the expansion of a Schur polynomial in terms of the power-sum functions $\widetilde{p}_m = \sum_{i=1}^N v_i^m$,

$$s_\lambda(\mathbf{v}) = \frac{1}{n!} \sum_{|\mu| = n} |C_\mu|\, \chi_\lambda(C_\mu)\, \widetilde{p}_\mu, \tag{6}$$

which stresses the link between $\mathrm{U}(k)$ characters and $\mathfrak{S}_n$ characters. It is still valid with $N \ge k$ variables instead of $\ell(\lambda) = k$ variables.

From Eqn 5, one can obtain the classical result for the dimension of the representation indexed by λ:

$$s_\lambda(1, \ldots, 1) = \dim \lambda = \frac{\Delta(\mathbf{h})}{\prod_{i=1}^N h_i!},$$

while the Frobenius formula leads to the expression

$$f_\lambda(C_2) = \frac{1}{2}\sum_i h_i^2 - (N - \frac{1}{2})\sum_i h_i + \frac{N}{3}(N^2 - \frac{3}{2}N + 2).$$

2.5. Z **as a sum on partitions.** After Eqn 6, if we consider the variables p_m's to be power-sum functions of some N-upple parameter $\mathbf{v}$, we have

$$\left\{ \begin{array}{rcl} Z(\mathbf{p}, g_s; t) & = & \sum_{b,n} \frac{t^n\, g_s^{b-n}}{b!} \sum_{\ell(\lambda) \le N, |\lambda| = n} \frac{\dim \lambda}{n!}\ s_\lambda(\mathbf{v}) \left(f_\lambda(C_{(2)})\right)^b \\ p_m & = & g_s \sum_{i=1}^N v_i^m. \end{array} \right. \tag{7}$$

Hence,

$$Z(\mathbf{p}, g_s; t) = \sum_{\ell(\lambda) \le N} (t/g_s)^{|\lambda|}\ \frac{\dim \lambda}{|\lambda|!}\ s_\lambda(\mathbf{v})\ e^{g_s f_\lambda(C_{(2)})}.$$

Alternatively, given that s_λ is homogeneous of degree $|\lambda|$, we have

$$Z(\mathbf{p}, g_s; t) = \sum_{\ell(\lambda) \le N} \frac{\dim \lambda}{|\lambda|!}\ s_\lambda(t\mathbf{v}/g_s)\ e^{g_s f_\lambda(C_{(2)})}.$$

Again, we emphasize that the above expression for Z ought to be considered as a formal power series in t and g_s. For a given b and n, the coefficient of $t^n g_s^\chi$ is a polynomial in the p_m's, which involves only $p_1, \ldots, p_n$. Therefore, it is always possible to find N (independent of b) and $v_1, \ldots, v_N$ such that $\forall i \in \{1, \ldots, N\}$ we have $p_m = g_s \sum v_i^m$. The values of the p_m's for $m > n$ do not matter.

2.6. **Matrix integral representation of Z.** To write Z as a matrix integral we express s_λ in terms of the Itzykson-Zuber integral [12]:

$$\begin{aligned} \mathrm{I}(X,Y) &= \int_{\mathrm{U}(N)} \mathrm{d}U\, e^{\mathrm{Tr}(XUYU^\dagger)} \\ &= \frac{\det(e^{x_i y_j})}{\Delta(X)\Delta(Y)}. \end{aligned}$$

Here dU is the Haar measure on $U(N)$, normalized such that the second line holds without any constant prefactor.

Therefore, if we let $\mathbf{R} = \mathrm{diag}(\ln v_1, \dots, \ln v_N)$ and $\mathbf{h}_\lambda = \mathrm{diag}(h_1, \dots, h_N)$, we have from Eqn 5 :

$$s_\lambda(\mathbf{v}) = \Delta(\mathbf{h}_\lambda) \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} I(\mathbf{h}_\lambda, \mathbf{R}).$$

Then, the partition function looks like

$$\begin{aligned} Z(\mathbf{p}, g_s; t) &= \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \sum_\lambda I(\mathbf{h}_\lambda, \mathbf{R})\, \frac{\left(\Delta(\mathbf{h}_\lambda)\right)^2}{\prod_{i=1}^N h_i!} \prod_{i=1}^N e^{g_s A_2(h_i)}\, (g_s/t)^{-A_1(h_i)} \\ &= \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \sum_{h_1 > \cdots > h_N \geq 0} I(\mathbf{h}, \mathbf{R}) \left(\Delta(\mathbf{h})\right)^2 \prod_{i=1}^N \frac{e^{g_s A_2(h_i)} (g_s/t)^{-A_1(h_i)}}{\Gamma(h_i+1)} \\ &= \frac{1}{N!} \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \sum_{h_1, \dots, h_N \geq 0} I(\mathbf{h}, \mathbf{R}) \left(\Delta(\mathbf{h})\right)^2 \prod_{i=1}^N \frac{e^{g_s A_2(h_i)} (g_s/t)^{-A_1(h_i)}}{\Gamma(h_i+1)}, \end{aligned}$$

where $|\lambda| = \sum_i A_1(h_i)$ and $f_\lambda(C_{(2)}) = \sum_i A_2(h_i)$. The factorization of the weight with respect to the h_i's and the presence of a squared Vandermonde is the key for the representation of Z as a hermitian matrix model.

As it was done in [6], we represent the N sums as integrals over a contour, namely the contour $\mathcal{C}_0$ enclosing the non-negative integers, as pictured in Figure 3. We make use of a function which has simple poles with residue 1 at all integers:

$$f(\xi) = \frac{\pi e^{-i\pi\xi}}{\sin(\pi\xi)} = -\Gamma(\xi+1)\Gamma(-\xi) e^{-i\pi\xi}.$$

Thus, we have:

$$\begin{aligned} Z(\mathbf{p}, g_s; t) &= \frac{1}{N!} \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \oint_{\mathcal{C}_0^N} dh_1 \cdots dh_N \left(\Delta(\mathbf{h})\right)^2 I(\mathbf{h}, \mathbf{R}) \\ &\qquad \prod_{i=1}^N \frac{f(h_i) e^{g_s A_2(h_i)}\, (g_s/t)^{-A_1(h_i)}}{\Gamma(h_i+1)}. \end{aligned}$$

Actually, the ratio $f(\xi)/\Gamma(\xi+1)$ has only poles at non- negative integers, it has no pole at negative integers. So, we can replace C_0 by the contour $\mathcal{C}$ from Figure 3 which encloses all integers, and which we choose invariant under translation on the real axis. Therefore, we arrive at

$$\begin{aligned} Z(\mathbf{p}, g_s; t) &= \frac{1}{N!} \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \oint_{\mathcal{C}^N} dh_1 \cdots dh_N \left(\Delta(\mathbf{h})\right)^2 I(\mathbf{h}, \mathbf{R}) \\ &\qquad \prod_{i=1}^N \frac{f(h_i) e^{g_s A_2(h_i)}\, (g_s/t)^{-A_1(h_i)}}{\Gamma(h_i+1)}. \end{aligned}$$

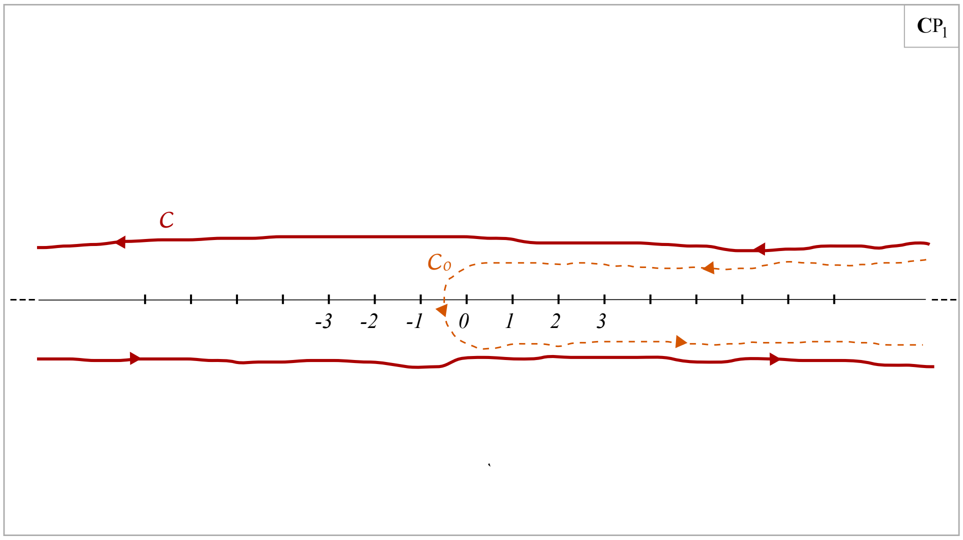

FIGURE 3. Contours enclosing the non-negative integers and the entire real line.

The set $\mathcal{H}_N(\mathcal{C})$ of normal matrices with eigenvalues on $\mathcal{C}$ is the set of matrices M which can be diagonalized by conjugation with a unitary matrix, and whose eigenvalues belong to $\mathcal{C}$:

$$M = U^\dagger\, X\, U,\ UU^\dagger = \mathrm{Id},\ X = \mathrm{diag}(x_1, \ldots, x_N),\ x_i \in \mathcal{C}.$$

$\mathcal{H}_N(\mathcal{C})$ is endowed with the measure

$$\mathrm{d}M = \Delta(X)^2\ \mathrm{d}X\, \mathrm{d}U,$$

where $\mathrm{d}U$ is the (up to normalization) Haar measure on $U(N)$ and $\mathrm{d}X$ is the product of Lebesgue curvilinear measures along $\mathcal{C}$.

From the above discussion, we can express our generating function as the matrix integral

$$Z(\mathbf{p}, g_s; t) = \frac{1}{N!}\,\frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \int_{\mathcal{H}_N(\mathcal{C})} \mathrm{d}M e^{-\operatorname{Tr}\hat{V}(M) + \operatorname{Tr}(M\mathbf{R})},$$

where

$$\hat{V}(\xi) = -g_s A_2(\xi) + \ln(g_s/t)\, A_1(\xi) + i\pi\xi - \ln\big(\Gamma(-\xi)\big).$$

Since we are interested in the expansion as a power series in g_s, we prefer to rescale $\xi = x/g_s$, i.e. $M \to M/g_s$, and rewrite

$$\boxed{Z(\mathbf{p}, g_s; t) = \frac{g_s^{-N^2}}{N!}\,\frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \int_{\mathcal{H}_N(\mathcal{C})} \mathrm{d}M e^{-\frac{1}{g_s}\operatorname{Tr}\left(V(M) - M\mathbf{R}\right)}}, \tag{8}$$

where now the potential reads

$$\begin{aligned} V(x) &= -g_s^2 A_2(x/g_s) + g_s \ln(g_s/t) A_1(x/g_s) + i\pi x - g_s \ln\big(\Gamma(-x/g_s)\big) \\ &= -\frac{x^2}{2} + g_s(N - \frac{1}{2})x + (\ln(g_s/t) + i\pi)x - g_s \ln\big(\Gamma(-x/g_s)\big) + C_t, \end{aligned}$$

with

$$C_t = -\frac{1}{3}g_s^2(N^2 - \frac{3}{2}N + 2) + \frac{1}{2}g_s(N-1)\ln(g_s/t)$$

To write its derivative, let $\psi = \Gamma'/\Gamma$, then

$$V'(x) = -x + g_s(N - \frac{1}{2}) + \ln(g_s/t) + i\pi + \psi(-x/g_s).$$

We have the well known formula that:

$$\psi(\xi) = \gamma + \frac{1}{\xi} + \sum_{k=1}^{\infty}\left(\frac{1}{\xi+k} - \frac{1}{k}\right)$$

where γ is the Euler-Mascheroni constant. In other words, $\psi(\xi)$ has simple poles at all negative integers, and an essential singularity (log singularity) at $\xi = \infty$. This gives:

$$V'(x) = -x + g_s(N - \frac{1}{2}) + \ln(g_s/t) + i\pi + \gamma - \frac{1}{x} - g_s\sum_{k=1}^{\infty}\left(\frac{1}{x-kg_s} + \frac{1}{kg_s}\right). \tag{9}$$

$V'(x)$ has simple poles at $x = X_j = jg_s$ for all $j \in \mathbb{N}$, and an essential singularity at $x = \infty$.

Using Stirling's formula for the large ξ asymptotic expansion of ψ

$$\psi(\xi) \underset{\xi\to\infty}{=} \ln\xi - \frac{1}{2\xi} - \sum_{l\geq 1}\frac{B_{2l}}{2l}\frac{1}{\xi^{2l}},$$

we have order by order in the small g_s expansion

$$\boxed{V'(x) = -x + \ln(x/t) + g_s(N - \frac{1}{2}) + \frac{g_s}{2x} - \sum_{l=1}^{\infty}\frac{B_{2l}}{2l}\frac{g_s^{2l}}{x^{2l}},} \tag{10}$$

where the B_l's are the Bernoulli number. We recall their generating function

$$\frac{\xi}{e^{\xi}-1} = 1 + \sum_{l=1}^{\infty}\frac{B_l}{l!}\xi^l.$$

The first few are $B_2 = 1/6,\ B_4 = -1/30, \dots$ while $B_{2l+1} = 0$ for $l \geq 1$.

3. Generalities on matrix models

Matrix integrals of the form

$$Z = \int_{\mathcal{H}_N(\mathcal{C})} \mathrm{d}M e^{-\frac{1}{g_s}\operatorname{Tr}\left[V(M) - M\mathbf{R}\right]}$$

are called "1-matrix model in an external field". They can be computed for any potential V, any contour $\mathcal{C}$ and any external matrix $\mathbf{R}$. Here, our task is even simpler because we regard this integral as a formal integral, i.e. a formal power series in powers of t and g_s, and therefore all our computations are to be performed order by order in powers of t and g_s.

In our case V depends on g_s, but for the moment, let us assume that V is an arbitrary potential, and in particular we assume that there is no relationship between the coefficients of V and g_s.

Typically, in our case (see Eqn 9), we choose V of the form:

$$V'(x) = -x + C + \sum_{j=1}^{n} \frac{u_j}{x - x_j}, \tag{11}$$

where for the moment we assume that there is no relationship between the coefficients of u_j, x_j of V and g_s.

3.1. **Topological expansion.** For some choices of V, $\mathbf{R}$ and $\mathcal{C}$, it may or may not happen that the convergent integral Z has a power series expansion in g_s

$$\ln Z = \sum_{g=0}^{\infty} g_s^{2g-2} F_g.$$

This happens only if $\mathcal{C}$ is a "steepest descent path" for the potential V and $\mathbf{R}$. In general, it is rather difficult to compute the steepest descent paths of a given arbitrary potential.

Fortunately, when Z is defined as a formal integral we don't need to find the steepest descent paths, and very often, formal series do have a topological expansion almost by definition, order by order in the formal parameters, which is the case here as we argued in Paragraph 2.2. In other words,

$$\ln Z(\mathbf{p}, g_s; t) = \sum_{g=0}^{\infty} g_s^{2g-2} F_g(\mathbf{p}; t)$$

holds order by order in powers of g_s and t.

3.2. **Loop equations and spectral curve.** We introduce the resolvent W_1 and auxiliary quantities P_1 and $P_{i,j}$:

$$\begin{aligned} W_1(x) &= \left\langle \mathrm{Tr}\, \frac{1}{x - M} \right\rangle = \sum_{g=0}^{\infty} g_s^{2g-1}\, W_1^{(g)}(x) \\ P_1(x,y) &= \left\langle \mathrm{Tr}\, \frac{V'(x) - V'(M)}{x - M}\, \frac{1}{y - \mathbf{R}} \right\rangle = \sum_{g=0}^{\infty} g_s^{2g-1}\, P_1^{(g)}(x,y), \\ P_{i,j} &= \left\langle ((x_j - M)^{-1})_{i,i} \right\rangle = \sum_{g=0}^{\infty} g_s^{2g-1}\, P_{i,j}^{(g)}, \end{aligned}$$

which we assume to have topological expansions in powers of g_s^{2g-1}. In our case, their precise definition, order by order in t as a power series of g_s, is given in Appendix 7.

Loop equations, also called Schwinger-Dyson equations, is a general technique, which merely reflects the fact that an integral is invariant by change of variable. It is a standard matrix model exercise, (see [4] for the 1-matrix model in an external fiels), to prove that the invariance of Z under the infinitesimal change of variable

$$M \mapsto M + \frac{\epsilon}{x - M} \frac{1}{y - \mathbf{R}} + O(\epsilon^2)$$

implies that the following *loop equation* is satisfied:

$$\left\{ \begin{array}{l} W_1^{(0)}(x) = P_1^{(0)}(x, Y(x)) \\ Y(x) = V'(x) - W_1^{(0)}(x). \end{array} \right. \tag{12}$$

Notice that $P_1^{(0)}(x,y)$ is a rational function of y, of degree N. If V' were a rational function of x (finite n in Eqn 11), then $P_1^{(0)}(x,y)$ would be a rational function of x of degree n:

$$P_1^{(0)}(x,y) = -g_s \operatorname{Tr} \frac{1}{y-\mathbf{R}} + \sum_{j=1}^{n} \sum_{i=1}^{N} \frac{u_j}{x-x_j} \frac{1}{y-R_i} \, P_{i,j}^{(0)}.$$

Thus the loop equation would be an algebraic equation, i.e. $Y(x)$ would be an algebraic function of x. Determining the rational function $P_1^{(0)}(x,y)$ (i.e. determining all the coefficients $P_{i,j}$) is possible but very tedious, and in fact, it is better to characterize an algebraic function $Y(x)$ by its singularities and its periods. In general, one would find that $W_1^{(0)}(x) = V'(x) - Y(x)$ has no singularity at the singularities of V', and the inverse function $x(Y)$ has poles of residue g_s at the eigenvalues of $\mathbf{R}$. The genus of the algebraic function $Y(x)$ and the periods $\oint Y dx$ are related to the integration path $\mathcal{C}$.

In general, the relationship between the periods and $\mathcal{C}$ is quite complicated, but, for many applications to combinatorics, we are considering only formal matrix integrals, i.e formal perturbation with parameter t of a gaussian integral. In that case, the spectral curve $Y(x)$ is always a genus 0 curve (for the Hurwitz matrix integral, we prove it in Appendix 7). This means that there exists a parametrization of $Y(x)$ with a complex variable z:

$$Y(x) \leftrightarrow \left\{ \begin{array}{l} x = x(z) \\ Y = y(z), \end{array} \right.$$

where x and y are two analytical functions of z. The functions $x(z)$ and $y(z)$ are monovalued functions of z, but $Y(x) = y(z)$ is multivalued, because there might exist several z such that $x(z) = x$. One of the determinations of $Y(x)$ is called the physical sheet.

The functions x and y are fully determined by their singularities. More precisely:

- $W_1^{(0)}(x) = V'(x) - Y(x)$ is analytical in the physical sheet; it can have no singularity except at branchpoints.
- From the definition of $W_1^{(0)}$, we have that at $x = \infty$ in the physical sheet
$$W_1^{(0)}(x) \sim \frac{N g_s}{x(z)} + o(\frac{1}{x(z)}).$$
- As a consequence of Eqn 12, $x(z)$ must have simple poles when $z \to z_i$ such that $y(z_i) = R_i$, i.e.
$$x(z) \sim \frac{g_s}{(z-z_i)\; y'(z_i)}.$$

In our case, for a potential of type 11, this implies:

$$\left\{ \begin{array}{l} x(z) = z + C - g_s \sum_{i=1}^{N} \frac{1}{(z-z_i)\, y'(z_i)} \\ y(z) = -z + \sum_{j=1}^{n} \frac{u_j}{(z-\hat{z}_j)\, x'(\hat{z}_j)} \end{array} \right. \tag{13}$$

where $x(\hat{z}_j) = x_j$, $y(z_i) = R_i$.

The above characterization of the spectral curve $Y(x)$ is valid even if the potential V' is not rational, for instance if $n \to \infty$. From now on, the function $Y(x)$, or more precisely the pair of analytical functions

$$\mathcal{S} = z \mapsto (x(z), y(z))$$

is called the "spectral curve" of our matrix model.

3.3. **Topological recursion.** We recall in this section the construction of the topological recursion from [4]. For our purposes, we only deal with spectral curves of genus 0. Hence, we adapt the definitions of [4] to this case.

A *spectral curve* $\mathcal{S}$ is a pair (x, y) of analytic functions on $\mathbb{C}\mathrm{P}_1$. Let a_i be the zeroes of $\mathrm{d}x$, and assume they are simple. Then, locally at a_i, $y \propto \sqrt{(x - x(a_i))}$. Let us denote $\overline{z} \neq z$, the unique point corresponding to the other branch of the squareroot, such that $x(z) = x(\overline{z})$. $\overline{z}$ is defined locally near the a_i's.

A tower of k-forms $\mathcal{W}_k^{(g)}(z_1, \ldots, z_k)$ is constructed as follows.

- $\mathcal{W}_1^{(0)} = -y\mathrm{d}x$.
- $\mathcal{W}_2^{(0)}$ is defined as the Bergman[2] kernel:
$$\mathcal{W}_2^{(0)}(z_1, z_2) = B(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}.$$
- We define a recursion kernel
$$K(z', z) \frac{-\frac{1}{2}\int_{\overline{z}}^{z} B(z', \cdot)}{(y(z) - y(\overline{z}))\mathrm{d}x(z)}.$$
- For $k + 2g - 2 > 0$, we define recursively the k-forms $\mathcal{W}_k^{(g)}(z_1, \ldots, z_k)$ by:
$$\begin{aligned} \mathcal{W}_k^{(g)}(z_1, \underbrace{z_2, \ldots, z_k}_{K}) \quad &= \quad \sum_i \operatorname*{Res}_{z \to a_i} K(z_1, z) \Big[\mathcal{W}_{k+1}^{(g-1)}(z, \overline{z}, K) \\ &\quad + \sum_{J \subseteq K,\ 0 \leq h \leq g}{}^{'} \mathcal{W}_{|J|+1}^{(h)}(z, J) \mathcal{W}_{k-|J|}^{(g-h)}(\overline{z}, K \setminus J) \Big], \end{aligned} \tag{14}$$
where $\sum'$ ranges over $(J, h) \neq (\emptyset, 0), (I, g)$.
- $\mathcal{W}_0^{(g)} = \mathcal{F}_g$ are defined for $g \geq 2$ by
$$\mathcal{F}_g = \frac{1}{2 - 2g} \sum_i \operatorname*{Res}_{z \to a_i} \left(\mathcal{W}_1^{(g)}(z) \Phi(z) \right),$$
where Φ is a primitive of $y\mathrm{d}x$ locally at the a_i's, i.e. $d\Phi = ydx$.
- The definition of $\mathcal{F}_1$ and $\mathcal{F}_0$ is more involved, and we refer the reader to [4].

3.4. **Main properties of the $\mathcal{W}_k^{(g)}$.**

- Symmetry $\quad \forall k, g, \quad \mathcal{W}_k^{(g)}(z_1, \ldots, z_k)$ is symmetric in $z_1, \ldots, z_k$.
- Invariance $\quad \forall k, g, 2g + k - 2 > 0, \quad \mathcal{W}_k^{(g)}$ is unchanged if we add to y a rational function of x.
- Exchange invariance $\quad \forall g \geq 2 \quad \mathcal{F}_g$ is unchanged if we exchange x and y.

[2]After Stefan Bergman (1895-1977), mathematician of Polish origin.

- Deformation If we perform an infinitesimal deformation $(\delta x, \delta y)$ of the spectral curve, the $\mathcal{W}_k^{(g)}$'s change. Let us introduce

$$\Omega(z) = \delta x(z) \mathrm{d}y(z) - \delta y(z) \mathrm{d}x(z).$$

This form does not depend on the parametrization z, and can always be represented as $\int_{z' \in \gamma} B(z, z') \Lambda(z')$ for some path γ and some meromorphic function Λ defined in its neighborhood (this data is called the dual of Ω). Then

$$\delta \mathcal{W}_k^{(g)}(z_1, \dots, z_k) = \int_{z \in \gamma} \Lambda(z) \mathcal{W}_{k+1}^{(g)}(z, z_1, \dots, z_k). \tag{15}$$

- Limits $W_k^{(g)}(\mathcal{S})$ is compatible with limits of curves.
- Link to matrix models It was proved in [4] that, if our matrix model has a topological expansion property:

$$W_k(z_1, \dots, z_k) = \left\langle \prod_{i=1}^{k} \mathrm{Tr} \, \frac{1}{x(z_i) - M} \right\rangle_c = \sum_{g=0}^{\infty} g_s^{2g-2+k} W_k^{(g)}(z_I),$$

and

$$\ln Z = \sum_{g=0}^{\infty} g_s^{2g-2} \, F_g,$$

then, loop equations imply that:

$$W_k^{(g)}(z_1, \dots, z_k) dx(z_1) \dots dx(z_k) = \mathcal{W}_k^{(g)}(z_1, \dots, z_k)$$

where $\mathcal{W}_k^{(g)}$ are computed using the spectral curve $\mathcal{S} = (x(z), y(z))$ presented in Section 3.2 (except for $(k, g) = (1, 0), (2, 0)$, which receive simple additional contributions). Similarly, for $g \geq 2$:

$$F_g = \mathcal{F}_g.$$

(F_0 and F_1 also receive simple corrections).

Since the $\mathcal{F}_g(\mathcal{S})$'s are invariant under transformations of $\mathcal{S}$ which leave $|\mathrm{d}x \wedge \mathrm{d}y|$ unchanged, we call them *symplectic invariants* of $\mathcal{S}$.

In the following section we shall apply the general theory of [4] to our Hurwitz matrix integral Eqn 8.

4. Spectral curve of the Hurwitz matrix model

In our case, the spectral curve of our matrix model must be determined order by order in powers of g_s and t. Here, we only give the result. The proof is quite technical, and is deferred to Appendix 7. It relies on the fact that, after a suitable shift, to leading order in g_s and t, we have a Gaussian matrix integral. In particular, this implies that the spectral curve has genus 0, i.e. it can be parametrized with a uniformization variable $z \in \mathbb{C}$. By composition with some homographic map, we can choose a parametrization for which $x(z) \sim z$ when $z \to \infty$ and $x(0) = 0$.

Since our potential V is of the form Eqn. 11 with $n \to \infty$, our spectral curve is of the form Eqn. 13. Therefore we guess that the spectral curve must be of the form:

(16)
$$\mathcal{S}(\mathbf{p};t) : \quad \begin{cases} x(z) = z + g_s \sum_{i=1}^N \frac{1}{(z-z_i)y_i} + \frac{1}{z_i y_i} \\ y(z) = -z + \ln(z/t) + c_0 + \frac{c_1}{z} - \sum_{l=1}^\infty \frac{B_{2l}\, g_s^{2l}}{2l} \left(f_{2l}(z) - \frac{f_{2l,1}}{z}\right) \end{cases}$$

where

$$\begin{cases} y(z_i) = R_i = \ln v_i \\ y_i = y'(z_i) \end{cases}$$

and where

$$f_l(z) = \operatorname*{Res}_{z'\to 0} \frac{dz'}{z-z'}\, (x(z'))^{-l} = \sum_{j=1}^{l} f_{l,j}\, z^{-j}$$

is such that $x(z)^{-l} - f_l(z)$ has a finite limit when $z \to 0$. c_0 is chosen such that $V'(x(z)) - y(z) \sim O(1/z)$ at large z:

$$c_0 = (N - \frac{1}{2})\, g_s + g_s \sum_{i=1}^N \frac{1}{z_i\, y_i}.$$

The coefficient c_1 is

$$c_1 = \frac{g_s}{2} - \sum_{l=1}^\infty \frac{B_{2l}\, g_s^{2l}}{2l}\, f_{2l,1},$$

and one can check (this comes from the fact that the sum of all residues of $V'(x)$ must vanish) that it is such that $V'(x(z)) - y(z) \sim N g_s/z$ at large z i.e.:

$$c_1 = (N - \frac{1}{2})\, g_s - g_s \sum_{i=1}^N \frac{1 - z_i}{z_i\, y_i}.$$

Each term z_i, y_i, c_0, c_1, is to be viewed as a power series in g_s and in t. To the first few orders in g_s we have:

$$z_i = L(tv_i) + \frac{g_s}{1 - L(tv_i)} \left(\frac{1 + L(tv_i)}{2} + L(tv_i) \sum_j \frac{L(tv_j)}{1 - L(tv_j)} \right) + O(g_s^2)$$

$$c_0 = g_s \left(-\frac{1}{2} - \sum_{i=1}^N \frac{L(tv_i)}{1 - L(tv_i)} \right) + O(g_s^2)$$

$$c_1 = \frac{-g_s}{2} + O(g_s^2)$$

and notice that $L(tv_i)$ is a power series in t:

$$L(tv_i) = \sum_{m=1}^\infty \frac{m^{m-1}\, t^m\, v_i^m}{m!}.$$

The proof that $\mathcal{S}(\mathbf{p};t)$ is the correct spectral curve for our problem is given in Appendix 7. It is obtained by computing the spectral curve order by order in the small g_s and t expansion.

The computation of $\mathcal{W}_k^{(g)}$'s in the topological recursion formula Eqn 14 involve taking residues at all zeroes of $x'(z)$, i.e. involves symmetric rational functions of the z_i's, and one can see that the coefficients of $\mathcal{W}_k^{(g)}$ are, order by order in g_s and t, polynomials of $p_m = g_s \sum_i v_i^m$, as required for the computation of Hurwitz numbers indeed.

4.1. **Spectral curve at $g_s = 0$.** At $g_s = 0$, the spectral curve reduces to:

$$\mathcal{S}_0 : \quad \begin{cases} x(z) = z \\ y(z) = -z + \ln(z/t) \end{cases} \tag{17}$$

i.e. $x = L(te^y)$, where L is the Lambert function. Up to exchanging the roles of x and y, this is the Lambert curve $\mathcal{S}_{\text{Lambert}}$ appearing in Bouchard-Mariño [2].

4.2. **The symplectic invariants.** So far, from the general theory of matrix models and from general properties of the topological recursion, we have proved that the generating function of simple Hurwitz numbers of genus g is the symplectic invariant

$$F_g(\mathbf{p}; t) = \sum_n t^n \sum_{|\mu|=n} \frac{p_\mu}{(2g-2+n+\ell(\mu))!} H_{g,\mu} = \mathcal{F}_g(\mathcal{S}(\mathbf{p}; t)), \tag{18}$$

where $\mathcal{S}(\mathbf{p}; t)$ is the spectral curve of Eqn 16 (in fact we have proved it only for $g \geq 2$, and we consider that the cases $g = 0$ and $g = 1$ are easier).

The computation of symplectic invariants $\mathcal{F}_g$ involves computing residues at the zeroes of $x'(z)$, and there are $2N$ such zeroes, which makes the computation complicated.

One may use the invariance properties of $\mathcal{F}_g$, under the exchange of x and y. Indeed, the zeroes of $y'(z)$ are much simpler to compute, order by order in powers of g_s. At $g_s = 0$, $y'(z)$ has only one zero located at $z = 1$.

Therefore, we introduce a new spectral curve satisfying

$$F_g(\mathbf{p}; t) = \mathcal{F}_g(\widetilde{\mathcal{S}}(\mathbf{p}; t)),$$

defined by exchanging x and y:

$$\boxed{\widetilde{\mathcal{S}}(\mathbf{p}, g_s; t) : \begin{cases} x(z) = -z + \ln(z/t) + c_0 + \frac{c_1}{z} - \sum_{l=1}^{\infty} \frac{B_{2l}\, g_s^{2l}}{2l} f_{2l}(z) \\ y(z) = z + g_s \sum_{i=1}^{N} \frac{1}{(z-z_i)y_i} + \frac{1}{z_i y_i} \end{cases} ,}$$

where now $x(z_i) = \ln v_i$ and $y_i = x'(z_i)$.

4.3. **The correlation forms.** So far, we have introduced the F_g's as generating functions which encode the simple Hurwitz numbers $H_{g,\mu}$ by expansion on a proper basis of polynomials of the p_m's. Bouchard and Mariño [2] define another generating function, namely the function of k-variables:

$$H^{(g)}(x_1, \ldots, x_k) = \sum_{\ell(\mu)=k} t^{|\mu|} \frac{\prod_{i=1}^k \mu_i \cdot M_\mu(x_1, \ldots, x_k)}{(2g-2+|\mu|+k)!} H_{g,\mu},$$

where $M_\mu(\mathbf{x}) = \sum_{\sigma \in \mathfrak{S}_k} \prod_{i=1}^k x_{\sigma(i)}^{\mu_i}$ are the (un-normalized) symmetric monomials.

It is easy to relate both. If we recall the combinatorial definition Eqn 18

$$F_g(\mathbf{p}; t) = \sum_\mu t^{|\mu|} \frac{p_\mu}{(2g-2+n+\ell(\mu))!} H_{g,\mu},$$

we see that

$$H^{(g)}(v_1, \ldots, v_k) = \left(\frac{v_1 \ldots v_k}{g_s^k} \frac{\partial^k F_g}{\partial v_1 \ldots \partial v_k} \right)\Bigg|_{g_s=0} = \left(\frac{1}{g_s^k} \frac{\partial^k F_g}{\partial R_1 \ldots \partial R_k} \right)\Bigg|_{g_s=0} .$$

We recall that $R_i = \ln v_i$.

The deformation property Eqn 15 of symplectic invariants allows us to calculate their derivatives. When we perform an infinitesimal variation $v_i \to v_i + \delta v_i$, i.e. a variation $R_i \to R_i + \delta R_i$ on the spectral curve, we need to compute

$$\Omega_i(z) = \delta x(z) \mathrm{d}y(z) - \delta y(z) \mathrm{d}x(z).$$

First notice that the form $y\mathrm{d}x$ is a meromorphic form (its singularities are poles). It has simple poles of constant residues g_s near $z = z_j$, i.e. locally near z_j we have

$$y \sim \frac{g_s}{x - R_j},$$

which implies that, locally near z_j,

$$\Omega_i = -g_s \, \delta_{i,j} \delta R_j \, \frac{\mathrm{d}x}{(x - R_j)^2} + O(1)$$

Then, observe that the other poles of $y\mathrm{d}x$ are independent of R_i. For example, $(V'(x) - y)\mathrm{d}x$ has a simple pole at ∞, with residue $N g_s$ independent of R_i. Therefore, Ω_i has no residue, and thus no pole at ∞. Similarly there is no pole at $z = 0$. The final result is that Ω_i is a meromorphic form with poles only at z_i:

$$\frac{1}{g_s} \, \Omega_i(z) = -\frac{\delta R_i \, \mathrm{d}z}{(z - z_i)^2 \, y_i}.$$

$\Omega_i(z)$ can be written in term of the Bergman kernel $B(z, z') = \frac{\mathrm{d}z \, \mathrm{d}z'}{(z - z')^2}$ as

$$\Omega_i(z) = -g_s \, \delta R_i \, \operatorname*{Res}_{z' \to z_i} \, B(z, z') \, \frac{1}{(z' - z_i) \, y_i}.$$

Then, the general theorems about the $\mathcal{F}_g$'s tell us that

$$\begin{aligned} \delta \mathcal{F}_g &= g_s \delta R_i \operatorname*{Res}_{z' \to z_i} \mathcal{W}_1^{(g)}(z') \, \frac{1}{(z' - z_i) \, y_i} \\ &= g_s \delta R_i \, \frac{\mathcal{W}_1^{(g)}(z_i)}{\mathrm{d}x(z_i)}, \end{aligned}$$

and more generally

$$\delta \mathcal{W}_k^{(g)}(z_1, \ldots, z_k) = g_s \delta R_i \, \operatorname*{Res}_{z' \to z_i} \, \mathcal{W}_{k+1}^{(g)}(z_1, \ldots, z_k, z') \left[\frac{1}{(z' - z_i) y_i} \right].$$

The result is that

$$\frac{1}{g_s^k} \frac{\partial^k \mathcal{F}_g}{\partial v_1 \ldots \partial v_k} = \left(\frac{\mathcal{W}_k^{(g)}(z_1, \ldots, z_k)}{\mathrm{d}x(z_1) \cdots \mathrm{d}x(z_k)}. \right) \Bigg|_{R_1 = x(z_1), \ldots, R_k = x(z_k)}$$

As a final step, we take the limit $g_s \to 0$. The limit of the spectral curve $\widetilde{\mathcal{S}}(\mathbf{v}; t)$ is simply

$$\mathcal{S}_{\mathrm{Lambert}} \ : \ \left\{ \begin{array}{l} x(z) = -z + \ln(z/t) \\ y(z) = z, \end{array} \right.$$

i.e. it is the Lambert spectral curve: $y = L(te^x)$.

In other words, we have proved that the function $H^{(g)}(v_1,\dots,v_k)$ is the correlation form $\mathcal{W}_k^{(g)}$ of the Lambert spectral curve $\mathcal{S}_{\text{Lambert}}$:

$$\boxed{H^{(g)}(v_1,\dots,v_k) = \left[\frac{\mathcal{W}_k^{(g)}(z_1,\dots,z_k)}{\mathrm{d}x(z_1)\cdots\mathrm{d}x(z_k)}\right](\mathcal{S}_{\text{Lambert}}),}$$

where $x(z_i)=\ln v_i = R_i$, i.e. $z_i = L(tv_i)$.

This is precisely the Bouchard-Mariño conjecture.

5. Relationship with intersection numbers, Kontsevich integral and ELSV formula

Notice that the Lambert spectral curve

$$\mathcal{S}_{\text{Lambert}} \;:\; \begin{cases} x(z) = -z + \ln(z/t) \\ y(z) = z \end{cases}$$

has only one branchpoint (solution of $x'(z)=0$), given by $z=1$. This is the reason why the Bouchard-Mariño conjecture is so efficient to compute Hurwitz numbers.

Since the topological recursion for computing the $\mathcal{W}_k^{(g)}$'s and $\mathcal{F}_g$'s, involves only the computation of residues at the branch point, we may perform a Taylor expansion near $z=1$: Let us define:

$$y = 1+\zeta$$

and

$$\xi^2 = -2(x+1+\ln t)$$

We have, in the limit $\zeta\to 0$:

$$\frac{1}{2}\xi^2 = \frac{\zeta^2}{2} - \frac{\zeta^3}{3} + \frac{\zeta^4}{4} + \cdots = \sum_{m\geq 2}\frac{(-1)^m\,\zeta^m}{m}$$

and we invert that expansion $y = 1+\xi+\frac{\xi^2}{3}+\frac{\xi^3}{36}-\frac{\xi^4}{270}+\frac{\xi^5}{6\,.6!}+\cdots$, which we write:

$$y = 1-2\xi+\sum_{m\geq 1} t_{m+2}\xi^m$$

In other words, the $\mathcal{W}_k^{(g)}$'s and $\mathcal{F}_g$'s of the Lambert curve $\mathcal{S}_{\text{Lambert}}$, are the same as the $\mathcal{W}_k^{(g)}$'s and $\mathcal{F}_g$'s of the following spectral curve $\mathcal{S}_K$:

$$\mathcal{S}_K \;:\; \begin{cases} x(\xi) = -1-\ln t - \frac{1}{2}\,\xi^2 \\ y(\xi) = 1-2\xi+\sum_{m\geq 1} t_{m+2}\xi^m \end{cases}$$

This spectral curve is exactly the Kontsevich spectral curve for times t_m's (see [5]). Here the t_m's satisfy the following recursion $t_2=0$, $t_3=3$, $t_4=\frac{1}{3}$, and for $m\geq 4$:

$$t_{m+1} = \frac{t_m}{m} - \frac{1}{2}\sum_{l=2}^{m-2} t_{l+2}\,t_{m+2-l} \tag{19}$$

We form the following series:

$$f(z) = \sum_{m=1}^{\infty}\frac{(2m+1)!}{m!}\,\frac{t_{2m+3}}{2-t_3}\,z^m$$

and

$$g(z) = -\ln(1 - f(z)) = \sum_{m=1}^{\infty} \tilde{t}_m \, z^m$$

we find to the first orders:

$$g(z) = -\frac{z}{6} + \frac{z^3}{45} - \frac{8z^5}{315} + \frac{8z^7}{105} + \cdots$$

Then, it was found in [5] that:

$$\begin{aligned} \mathcal{W}_k^{(g)}(z_1, \ldots, z_k) \quad = \quad & \frac{1}{2^{3g-3+k}} \sum_{d_0+\cdots+d_k=3g-3+k} \sum_{j=1}^{d_0} \frac{1}{j!} \sum_{m_1+\cdots+m_j=d_0, m_i>0} \\ & \prod_{i=1}^{k} \frac{(2d_i+1)!}{d_i!} \frac{\mathrm{d}z_i}{z_i^{2d_i+2}} \prod_{i=1}^{j} \tilde{t}_{m_i} \left\langle \prod_{i=1}^{j} \kappa_{m_i} \prod_{i=1}^{k} \psi_i^{d_i} \right\rangle_{\overline{\mathcal{M}}_{g,k}} \end{aligned}$$

where $\overline{\mathcal{M}}_{g,k}$ is the stable compact moduli space of Riemann surfaces of genus g with k marked points, and κ_j is the j^{th} Mumford's tautological class, and $\psi_i = c_1(\mathcal{L}_i)$ is the first Chern class of the cotangent bundle at the i^{th} marked point.

In other words, just by looking at the Lambert spectral curve, we see that there is a relationship between the generating function for Hurwitz numbers of genus g with a monodromy of length k, and the generating function for intersection numbers of tautological classes on $\overline{\mathcal{M}}_{g,k}$.

This type of relationship is completely natural and expected. The link coming from the ELSV formula [3], relating Hurwitz numbers to Hodge integrals:

$$H_{g,\mu} = \frac{(2g-2+\ell(\mu)+|\mu|)!}{|\operatorname{Aut}\mu|} \prod_{i=1}^{\ell(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,\ell(\mu)}} \frac{\Lambda_g^{\vee}(1)}{\prod_{i=1}^{\ell(\mu)}(1-\mu_i\psi_i)},$$

where $\Lambda_g^{\vee}(t) = \sum_{i=0}^{g} \lambda_i t^i$ is the total Chern class of the Hodge bundle $\mathbb{E}$ over $\overline{\mathcal{M}}_{g,n}$. In fact, the change of variables taking the Hurwitz generating function to the generating function of all Hodge integrals with a single λ factor is given by the Lambert curve itself [2, 9, 11]. Furthermore, Mumford's formula [7, 15]

$$\mathrm{ch}(\mathbb{E}) = g + \sum_{l=1}^{\infty} \frac{B_{2l}}{(2l)!} \left(\kappa_{2l-1} + \frac{1}{2} \iota_* \sum_{i=0}^{2l-2} (-1)^i \psi^i \bar{\psi}^{2l-2-i} \right)$$

allows one to express Hodge integrals in terms of ψ class intersections. The time parameters t_m appearing in Eqn 19 are closely related to these topics. Mironov and Morozov [14] have previously considered similar constructions.

An other natural question arising is the relationship between the Bouchard–Mariño conjecture and the cut–and–join equation [8]. Since they are both recursive algorithms for computing Hurwitz numbers, it seems likely that they should be related. In fact, they are, appropriately interpreted, completely equivalent. This topic, as well as a more detailed discussion of the deformation of the Kontsevich model to the Hurwitz model, are deferred to future papers.

6. Conclusion

With the integral representation of $\mathrm{U}(N)$ characters, it is possible to express in general the partition function Z of Hurwitz numbers as a matrix model. In the case of simple Hurwitz numbers, we obtain a 1-matrix model with external field, whose spectral curve is found by solving the master loop equation:

$$\widetilde{\mathcal{S}}(\mathbf{p};t):\left\{\begin{array}{l} x(z)=-z+\ln(z/t)+c_0+\frac{c_1}{z}-\sum_{n=1}^{\infty}\frac{B_{2n}\,g_s^{2n}}{2n}\,f_{2n}(z) \\ y(z)=z+g_s\sum_{i=1}^{N}\frac{1}{(z-z_i)y_i}+\frac{1}{z_iy_i}\end{array}\right. .$$

Our main results are:

$$\begin{aligned} F_g(\mathbf{p};t) &= \mathcal{F}_g(\widetilde{\mathcal{S}}(\mathbf{p};t)) \\ H^{(g)}(v_1,\ldots,v_k) &= \left[\frac{\mathcal{W}_k^{(g)}(z_1,\ldots,z_k)}{\mathrm{d}x(z_1)\cdots\mathrm{d}x(z_k)}\right](\widetilde{\mathcal{S}}_{\text{Lambert}}) \qquad \text{where}\left\{\begin{array}{l} x(z_i)=\ln v_i \\ y(z_i)=L(tv_i)\end{array}\right. \end{aligned}$$

It provides an algorithm, namely the topological recursion of matrix models, to compute the $H_{g,\mu}$ by the residue formula of Paragraph 3.3, with only one branchpoint involved. As a matter of fact, this recursion relation between simple Hurwitz numbers is understood to be equivalent to the Laplace transform of the cut-and-join equation with help of the ELSV formula. Besides, Z for simple Hurwitz numbers is the time evolution of the KP τ-function, a fact agreeing with its one-matrix model representation. We also see explicitly on the Lambert curve $\mathcal{S}_{\text{Lambert}}=\lim_{g_s\to 0}\widetilde{\mathcal{S}}(\mathbf{p};t)$ the relation between Z and the Kontsevich τ-function.

We hope that our matrix model-minded methods could help investigating double Hurwitz numbers (where Z is a Toda τ-function) and further.

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7. Appendix: proof of the spectral curve

The proof works order by order in g_s and t, and it relies on the fact that, to leading order, we have a Gaussian matrix integral.

7.1. Shift of the matrix model.

We start from:

$$Z(\mathbf{p},g_s;t)=\frac{g_s^{-N^2}}{N!}\frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})}\int_{\mathcal{H}_N(\mathcal{C})}dM\,e^{-\frac{1}{g_s}\operatorname{Tr}\left[V(M)-M\mathbf{R}\right]},$$

where the potential $V(x)$ is:

$$V(x)=-\frac{x^2}{2}+g_s(N-\frac{1}{2})x+(\ln(g_s/t)+i\pi)\,x-g_s\ln\big(\Gamma(-x/g_s)\big)+C_t,$$

and C_t does not depend on x. Order by order in g_s we have the Stirling expansion:

$$V'(x)=-x+\ln(x/t)+g_s(N-\frac{1}{2})+\frac{g_s}{2x}-\sum_{l=1}^{\infty}\frac{B_{2l}\,g_s^{2l}}{2l\,x^{2l}}.$$

We need to compute this matrix integral in the small g_s and t expansion (up to a constant factoring out of the integral).

First let us perform a shift
$$\widetilde{M} = M - \tilde{\mathbf{R}}$$
where $\tilde{\mathbf{R}} = \mathrm{diag}(\tilde{R}_1, \ldots, \tilde{R}_N)$ is such that:
$$V'(\tilde{R}_i) = R_i.$$
The equation $V'(\tilde{R}_i) = R_i$ has several solutions, we choose the one which is a power series in g_s and t:
$$\tilde{R}_i = L(tv_i) - g_s(\frac{1}{2} + \frac{NL(tv_i)}{1 - L(tv_i)}) + \cdots = \sum_{l \geq 0} g_s^l \tilde{R}_{i,l}(v_i).$$
and we choose the determination of the Lambert function such that $L(tv_i)$ has a small t expansion $L(tv_i) = tv_i + t^2 v_i^2 + \cdots = \sum_{m \geq 1} m^{m-1}(tv_i)^m/m!$.

Then we have:
$$Z(\mathbf{p}, g_s; t) = \frac{g_s^{-N^2}}{N!} \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} e^{-\frac{1}{g_s} \mathrm{Tr}\left[V(\tilde{\mathbf{R}}) - \tilde{\mathbf{R}}\mathbf{R}\right]} \int_{\mathcal{H}_N(\mathcal{C})} d\widetilde{M}\, e^{-\frac{1}{g_s} \tilde{\mathcal{V}}(\widetilde{M})},$$
where we decompose $\tilde{\mathcal{V}}(\widetilde{M})$
$$\tilde{\mathcal{V}}(\widetilde{M}) = \tilde{\mathcal{V}}_2(\widetilde{M}) + \tilde{\mathcal{V}}_{\geq 3}(\widetilde{M})$$
into a quadratic function of $\widetilde{M}$
$$\tilde{\mathcal{V}}_2(\widetilde{M}) = \frac{1}{2} \mathrm{Tr} \left(-\widetilde{M}^2 + g_s \sum_{j=0}^{\infty} \frac{1}{\tilde{\mathbf{R}} - jg_s} \widetilde{M} \frac{1}{\tilde{\mathbf{R}} - jg_s} \widetilde{M} \right)$$
and $\tilde{\mathcal{V}}_{\geq 3}$ contains all the higher degree terms (we will not need explicit expressions, however, the interested reader can derive them easily).

Notice that $\tilde{\mathcal{V}}_2(\widetilde{M})$ and $\tilde{\mathcal{V}}_{\geq 3}(\widetilde{M})$ have a small g_s expansion (for instance approximate the sum $\sum_j$ in $\tilde{\mathcal{V}}_2$ by a Riemann integral).

Order by order in g_s we have:
$$\begin{aligned} Z(\mathbf{p}, g_s; t) \quad = \quad & \frac{g_s^{-N^2}}{N!} \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} e^{-\frac{1}{g_s} \mathrm{Tr}\left[V(\tilde{\mathbf{R}}) - \tilde{\mathbf{R}}\mathbf{R}\right]} \\ & \int_{\mathcal{H}_N(\mathcal{C})} \mathrm{d}\widetilde{M}\, e^{-\frac{1}{g_s} \tilde{\mathcal{V}}_2(\widetilde{M})} \sum_{m=0}^{\infty} \frac{(-1)^m}{g_s^m\, m!} (\tilde{\mathcal{V}}_{\geq 3}(\widetilde{M}))^m, \end{aligned}$$
If we rescale $\widetilde{M} = \sqrt{g_s}\, A$, we have
$$\begin{aligned} Z(\mathbf{p}, g_s; t) \quad = \quad & \frac{1}{N!} \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} e^{-\frac{1}{g_s} \mathrm{Tr}\left[V(\tilde{\mathbf{R}}) - \tilde{\mathbf{R}}\mathbf{R}\right]} \\ & \int_{\mathcal{H}_N(\mathcal{C})} \mathrm{d}A\, e^{-\tilde{\mathcal{V}}_2(A)} \sum_{m=0}^{\infty} \frac{(-1)^m}{g_s^m\, m!} (\tilde{\mathcal{V}}_{\geq 3}(A/\sqrt{g_s}))^m. \end{aligned}$$
Since $\tilde{\mathcal{V}}_{\geq 3}(A/\sqrt{g_s}) = O(\sqrt{g_s})$, we see that order by order in powers of g_s, we may exchange the sum and integral. Therefore:
$$\begin{aligned} Z(\mathbf{p}, g_s; t) \quad = \quad & \frac{g_s^{-N^2}}{N!} \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} e^{-\frac{1}{g_s} \mathrm{Tr}\left[V(\tilde{\mathbf{R}}) - \tilde{\mathbf{R}}\mathbf{R}\right]} \\ & \sum_{m=0}^{\infty} \frac{(-1)^m}{g_s^m\, m!} \int_{\mathcal{H}_N(\mathcal{C})} \mathrm{d}\widetilde{M}\, e^{-\frac{1}{g_s} \tilde{\mathcal{V}}_2(\widetilde{M})}\, (\tilde{\mathcal{V}}_{\geq 3}(\widetilde{M}))^m, \end{aligned}$$

More generally, expectation values of polynomials $Q_p(\widetilde{M})$ are computed as formal power series, whose coefficients are polynomial moments of a gaussian integral:

$$< Q_p(\widetilde{M}) > \quad = \quad \frac{\sum_{m=0}^{\infty} \frac{(-1)^m}{g_s^m\, m!} \int_{\mathcal{H}_N(\mathcal{C})} \mathrm{d}\widetilde{M}\, e^{-\frac{1}{g_s}\tilde{\mathcal{V}}_2(\widetilde{M})}\ (\tilde{\mathcal{V}}_{\geq 3}(\widetilde{M}))^m\ Q_p(\widetilde{M})}{\sum_{m=0}^{\infty} \frac{(-1)^m}{g_s^m\, m!} \int_{\mathcal{H}_N(\mathcal{C})} \mathrm{d}\widetilde{M}\, e^{-\frac{1}{g_s}\tilde{\mathcal{V}}_2(\widetilde{M})}\ (\tilde{\mathcal{V}}_{\geq 3}(\widetilde{M}))^m}.$$

In this form, we can use Wick's theorem. It shows that, if $Q_p(\widetilde{M})$ is any homogeneous polynomial of total degree p in the entries of the matrix $\widetilde{M}$, the expectation value $\left\langle Q_p(\widetilde{M} - \widetilde{\mathbf{R}})\right\rangle$ is a power series in g_s and t. It is expressed as a finite sum of connected[3] fat-graphs. When we restrict the sum to the fatgraphs of genus g, we note it with a superscript $^{(g)}$. We claim that, for our matrix model, this expectation value is $O(g_s^{p/2})$, and $O(g_s^{2g-1})$. It can be seen as follows : fatgraphs contributing to the sum have one vertex of degree p, and v internal vertices of degree ≥ 3 coming from $(\tilde{\mathcal{V}}_{\geq 3})^v$. If we call f the number of faces, and e the number of edges in the fatgraph, its Euler characteristic is $2 - 2g = (v + 1) - e + f$. The power of g_s coming from the gaussian integral is $j \geq e - v$ (indeed, we have g_s^e coming from the gaussian integral with $e^{-\frac{1}{g_s}\tilde{\mathcal{V}}_2(\widetilde{M})}$, and g_s^{-v} accompanying $(\tilde{\mathcal{V}}_{\geq 3})^v$, and in addition $\tilde{\mathcal{V}}_2$ and $\tilde{\mathcal{V}}_{\geq 3}$ themselves have a g_s expansion). So, we have:

$$j \geq e - v = 2g - 1 + f \geq 2g - 1.$$

On the other hand, the number of half-edges is $2e = p + \sum_i i n_i$ where n_i is the number of internal vertices of degree $i \geq 2$, and we have $v = \sum_i n_i$. So:

$$j \geq e - v = \sum_i (\frac{i}{2} - 1)n_i + \frac{p}{2} \geq \frac{p}{2}.$$

In particular, let us show how to define the $W_1^{(g)}$'s, the topological expansion of the one-point correlation function. Consider :

$$T_p(x) = \left\langle \mathrm{Tr}\left[\left(\frac{1}{x - \widetilde{\mathbf{R}}}\,\widetilde{M}\right)^p \frac{1}{x - \widetilde{\mathbf{R}}}\right]\right\rangle.$$

It is a double power series, whose coefficients are rational functions of x, and are polynomial gaussian expectation values of $\widetilde{M}$. In its representation as a sum over fatgraphs, we collect those of genus g to define $T_p^{(g)}(x)$. Its coefficients are still rational functions of x. Since $T_p^{(g)}(x) = O(g_s^{p/2})$, we write

$$T_p^{(g)}(x) = \sum_{j \geq p/2} g_s^j\ T_{p,j}^{(g)}(x),$$

and thus we have, in the sense of formal power series of g_s:

$$\sum_{p=0}^{\infty} T_p^{(g)}(x) = \sum_{j \geq 2g-1} g_s^j\ \sum_{p=0}^{2j} T_{p,j}^{(g)}(x).$$

We are in position to define $W_1^{(g)}(x)$ as:

$$g_s^{2g-1}\, W_1^{(g)}(x) = \sum_{j=2g-1}^{\infty} g_s^j \sum_{p=0}^{2j} T_{p,j}^{(g)}(x).$$

[3]Connected because the normalization factor $\widetilde{Z}^{-1}$ is included in the expectation value.

$W_1^{(g)}(x)$ is thus a formal power series in powers of g_s and t, whose coefficients are rational functions of x. These definitions give a meaning to the equality between formal double power series:

$$W_1(x) = \sum_{g=0}^{\infty} g_s^{2g-1} W_1^{(g)}(x),$$

where $W_1(x)$ is the resolvent :

$$W_1(x) \overset{\text{formal}}{=} \left\langle \mathrm{Tr}\, \frac{1}{x-M} \right\rangle.$$

In a similar manner, one can define $g_s^{2g-2+k}\, W_k^{(g)}(x_1,\dots,x_k)$ as the formal double power series computing the sum over (c for connected) fatgraphs of genus g arising in the correlation function:

$$W_k(x_1,\dots,x_k) \overset{\text{formal}}{=} \left\langle \prod_{i=1}^{k} \mathrm{Tr}\left(\frac{1}{x_i - M}\right)\right\rangle_c .$$

By construction we have, in the sense of formal series:

$$W_k(x_1,\dots,x_k) = \sum_{g=0}^{\infty} g_s^{2g-2+k}\, W_k^{(g)}(x_1,\dots,x_k).$$

To sum things up, the correlation functions $W_k^{(g)}(x_1,\dots,x_k)$ can be defined as formal power series in g_s and t, such that the coefficients are rational functions of the x_i's. It is defined by collecting the fatgraphs of genus g in the Wick theorem's expansion of gaussian integrals.

The loop equations of gaussian matrix integrals are well known, and they imply that the $W_k^{(g)}$ satisfy the topological recursion of [4].

For $W_1^{(0)}$, or more precisely for $Y(x) = V'(x) - W_1^{(0)}(x)$, the loop equations read (see [4]):

$$\begin{aligned}
V'(x) - Y &= \left\langle \mathrm{Tr}\, \frac{V'(x)-V'(M)}{x-M}\, \frac{1}{Y-\mathbf{R}} \right\rangle^{(0)} \\
&= -g_s\, \mathrm{Tr}\, \frac{1}{Y-\mathbf{R}} + g_s \sum_{i=1}^{N}\sum_{j=0}^{\infty} \frac{1}{x-jg_s}\, \frac{1}{Y-R_i} \left\langle \left(\frac{1}{jg_s - M}\right)_{i,i} \right\rangle^{(0)} \\
&= -g_s\, \mathrm{Tr}\, \frac{1}{Y-\mathbf{R}} + g_s \sum_{i=1}^{N}\sum_{j=0}^{\infty}\sum_{p=0}^{\infty} \frac{1}{x-jg_s}\, \frac{1}{Y-R_i}\, T_{j,p;i}
\end{aligned}$$

(20)

$$T_{j,p;i} = \left\langle \left(\frac{1}{jg_s - \tilde{\mathbf{R}}} \left(\widetilde{M}\, \frac{1}{jg_s - \tilde{\mathbf{R}}}\right)^{p} \right)_{i,i} \right\rangle^{(0)}$$

where as usual $< . >^{(0)}$ means that we shift $M = \tilde{\mathbf{R}} + \widetilde{M}$, and keep only the genus zero fatgraphs in the gaussian expectation value. Notice that the sum over j is absolutely convergent. Moreover, to a given order in g_s, the sum over p is finite.

This equation is sufficient to determine $Y(x)$ order by order in g_s. To leading order we find $Y(x) = -x + \ln{(x/t)} + O(g_s)$. To subleading orders, we recursively have to determine a finite number of coefficients of the g_s expansion of $T_{j,p;i}$. Those coefficients are completely determined by the condition that, to each order in g_s, $W_1^{(0)}(x)$ is a rational function of x with poles only at $x = \tilde{R}_i$, and in particular it must have no pole at $x = jg_s$, or at $Y(x) = R_i$.

7.2. **Asymptotic expansion of the spectral curve.** To leading order in g_s, we have $V'(x) = -x + \ln(x/t)$, and:

$$Y(x) = -x + \ln(x/t) + O(g_s)$$

This leading order spectral curve has a rational uniformization:

$$\begin{cases} x(z) = z \\ y(z) = -z + \ln(z/t) \end{cases}$$

Since the higher order g_s corrections to $V'(x)$ are all rational, the corrections to $P_1^{(0)}(x, y)$ are also rational functions of x and y. This implies that all corrections to $Y(x)$ can be written with the uniformizing variable z, i.e. to all orders in g_s the spectral curve is of genus 0.

To all orders in g_s, the equation Eqn 20 is algebraic of degree $N + 1$ in the variable Y, this implies that the function $x(z)$ is rational of degree $N+1$. It is easy to see that it has N simple poles at z_i such that $y(z_i) = R_i$, and one simple pole at ∞. Up to a homographical change of variable z, we assume that $x(0) = 0$ and $x(z) \sim z$ at large z, i.e.:

$$x(z) = z + g_s \sum_{i=1}^{N} \frac{1}{(x - z_i)\, y_i} + \frac{1}{z_i y_i}$$

Moreover, one sees directly from Eqn 20, that the residue of $x dy$ is g_s, i.e. $y_i = y'(z_i)$. The function $y(z)$ starts to leading order in g_s as $y(z) = -z + \ln(z/t) + O(g_s)$, and all the higher g_s corrections are rational functions of z. Let $x = \infty$ and $X_j = jg_s$, $j \in \mathbb{N}$ be the singularities of $V'(x)$. For each X_j, let us choose $\hat{z}_j$ such that $x(\hat{z}_j) = X_j$ (and we must choose the value of $\hat{z}_j$ which has a small t power series expansion). Since $V'(x)$ has a simple pole of residue g_s at $x = X_j$, one sees from Eqn 20, that $y dx$ has a simple pole of residue g_s at $z = \hat{z}_j$. Another way of saying this is to write:

$$y(z) = \frac{1}{2i\pi} \oint_{\mathcal{C}_0} \frac{dz'}{z - z'}\, V'(x(z')),$$

where $\mathcal{C}_0$ is a contour surrounding ∞ and all the $\hat{z}_j$'s.

Order by order in g_s, using the Stirling expansion of $V'(x)$, this gives the spectral curve $\mathcal{S}(\mathbf{p}; t)$ of Eqn 16.

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