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A matrix model for plane partitions

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Abstract. We construct a matrix model equivalent (exactly, not asymptotically) to the random plane partition model, with almost arbitrary boundary conditions. Equivalently, it is also a random matrix model for a TASEP-like process with arbitrary boundary conditions. Using the known solution of matrix models, this method allows us to find the large size asymptotic expansion of plane partitions, to *all* orders. It also allows us to describe several universal regimes. On the algebraic geometry point of view, this gives the Gromov–Witten invariants of $\mathbb{C}^3$ with branes, i.e. the topological vertex, in terms of the symplectic invariants of the mirror's spectral curve.

Keywords: rigorous results in statistical mechanics, matrix models, topology and combinatorics, free boundary problems (theory)

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1. Introduction

The statistical physics problem of counting plane partition configurations of some domain, as well as its various equivalent formulations, has become a very active and fascinating area of mathematical physics in the past few years, culminating with Okounkov's renowned work. Beyond a beautiful combinatorics problem, it has also many indirect applications, like a tiling problem similar to a discrete version of TASEP, i.e. the simplest model of out-of-equilibrium statistical physics, and algebraic geometry, as it plays a key role in the computation of Gromov–Witten invariants of some toric Calabi–Yau 3-folds, through the topological vertex method [3].

The works of Okounkov *et al* [48] have brought immense progress in the understanding of large size asymptotics behaviors of plane partitions. It was observed that, in many universal regimes, the statistical properties of large plane partitions is very similar to that of matrix models, and many works have taken advantage of that similarity.

Here, in this paper, we show that there is not only a 'similarity' between plane partition countings and matrix models, in fact, we show that plane partitions *is* a matrix model, even for finite size. As a consequence, we may use all the machinery developed for solving matrix models, and we are able to compute all order corrections to the large size asymptotics.

Our matrix model is a multi-matrix model, with non-polynomial potentials. It may look very complicated at first sight, and its spectral curve may look rather complicated too. However, the solution of matrix models is expressed in terms of symplectic invariants, and up to a symplectic transformation (which does not change the symplectic invariants), our complicated matrix model's spectral curve is equivalent to the Harnack curve of Kenyon–Okounkov–Sheffield [48].

Moreover, our formulation allows us to use the full toolbox of matrix model technology. For instance, the method of orthogonal polynomials gives determinantal formulae for correlation functions, the integrable structure, the Riemann–Hilbert problem and much more. Also the loop equation method allows us to compute the large size expansion order by order [55, 54].

1.1. Main results of this paper

Our main result is the theorem 1.1:

Theorem 1.1. *The tiling model-plane partition-tiling generating function can be written as a matrix integral. (A more precise result is written in theorem 1.1.) This identification is exact; it is not asymptotic.*

An immediate consequence is obviously.

Corollary. *All asymptotic limits of tiling model-plane partition-tilings are random matrix limit laws.*

(But it remains to classify all possible random matrix limit laws.)

Also, since our matrix model is a chain of matrices, classical results of matrix models apply:

Corollary. *The matrix model is integrable, the generating function is a Tau function and, for instance, correlation functions are given by determinantal formulae of Janossi density type [31].*

We would like also to emphasize that our 'tiling ↔ matrix model' identification works for very general cases, with almost any possible boundary conditions. We can also give weights to points of the domain to be tiled, some points can be forbidden (defects), or obliged, or just have an arbitrary weight.

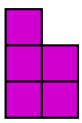
The second part of our paper, starting with section 4, consists in 'solving' the matrix model. We do it explicitly only for some not too complicated boundary conditions. We recover the Harnack curves of Kenyon–Okounkov–Sheffield.

We discuss many examples in sections 7 and after, and in particular we apply our method to the enumeration of TSSCPP's in section 8.

2. Some statistical physics models

2.1. Plane partitions

Consider three integers N_λ, N_μ, N_ν, and three partitions λ, μ, ν, for example:

$\lambda =$, $\mu =$, $\nu =$  . (2.1)

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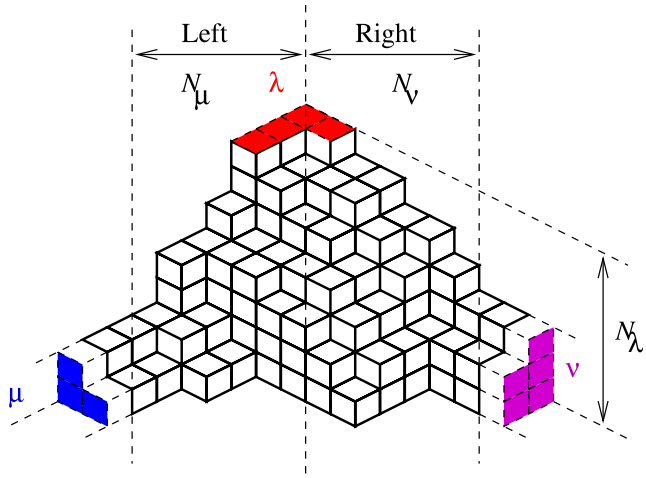
A plane partition π with boundaries λ, μ, ν, and of size N_λ, N_μ, N_ν, is a piling of cubic boxes in the corner of a room, with boundary conditions given by λ, μ, ν, for example:



N_λ is the height of the plane partition (height of the cubic boxes piling) and N_μ (resp. N_ν) is the extension towards left (resp. right), so that beyond N_μ (resp. N_ν) the section is frozen to μ (resp. ν).

The partition function we would like to compute is

$$Z_{N_\lambda,N_\mu,N_\nu}(q;\lambda,\mu,\nu) = \sum_{\pi,\partial\pi=(\lambda,\mu,\nu)} q^{|\pi|} \tag{2.2}$$

where $|\pi|$ is the number of boxes, i.e. the volume, which is called the 'weight' of π.

This partition function is the so-called 'topological vertex' in topological string theories [3], it is the building block to compute Gromov–Witten invariants of all toric Calabi–Yau 3-folds [7, 14, 3, 53, 57, 58, 62, 63, 68, 56].

From the combinatorics point of view, it is the generating function for counting plane partitions with given boundaries and weighted by their volume. From the statistical physics point of view, it can be viewed as a model for a growing three-dimensional crystal in the corner of a room. All those topics have remained important research areas in physics and mathematics, and it would be difficult to summarize all that has been done. Let us mention that Kasteleyn [44, 45] found an explicit expression for the partition function of a domino tiling, which can be rephrased as a plane partition, and since then the subject has been studied a lot; see, for example, [16, 47, 23].

2.1.1. Remark: semi-standard tableaux. If we slice our plane partition π at all integer times (time is the horizontal coordinate) $t = -N_\mu, \ldots, N_\nu$, at each time the slice is a two-dimensional partition $\lambda(t)$.

It is then clear that partitions $\lambda(t)$ are growing from $t = -N_\mu$ to 0, and then decreasing from $t = 0$ to N_ν:

$$\forall t < 0, \qquad \lambda(t) \prec \lambda(t+1), \qquad \forall t > 0, \qquad \lambda(t) \prec \lambda(t-1) \tag{2.3}$$

where $\prec$ is the partial ordering of partitions ($\lambda \prec \mu$ means that λ can be obtained from μ by removing boxes). Since we have $\mu \prec \lambda(-N_\mu) \prec \cdots \lambda(t-1) \prec \lambda(t) \prec \cdots \prec \lambda(1) \prec \lambda(0)$, we may draw all $\lambda(t)$ with $t \leq 0$ inside the Ferrer diagram of $\lambda(0)$, and we write in each

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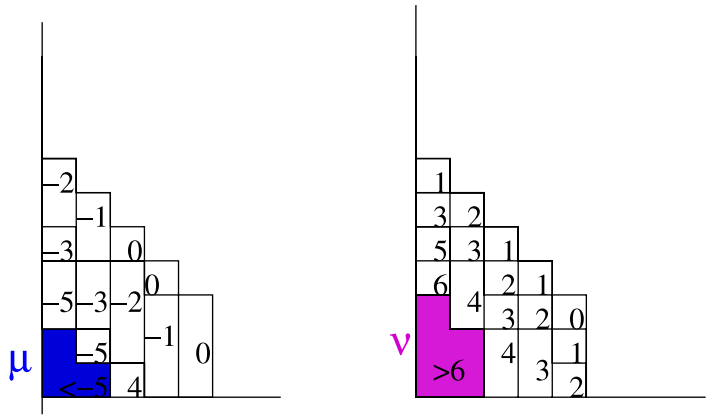

Figure 1. A plane partition is equivalent to the data of two semi-standard Young tableaux of the same shape $\lambda(0)$. Each of these two tableaux, is the superposition of growing (or decreasing) partitions $\lambda(t)$.

box the time t at which the box appears for the first time. We do the same for $t \geq 0$, and we have two semi-standard tableaux with the same shape $\lambda(0)$. A semi-standard tableau is a Ferrer diagram with integer entries decreasing along columns and strictly decreasing along rows. See figure 1.

When $\lambda = \mu = \nu = \emptyset$ and $N_\lambda = N_\mu = N_\nu = \infty$, the statistics of the partition $\lambda(0)$ is the sum over all pairs of semi-standard tableaux of shape $\lambda(0)$, i.e. it is the Plancherel measure [65, 66, 70]:

$$\mathcal{P}(\lambda) = \left(\frac{\dim(\lambda)}{|\lambda|!}\right)^2 = \frac{1}{(|\lambda|!)^2}(\#(\text{ semi-standard tableaux of shape}\lambda))^2. \tag{2.4}$$

We shall study this limit in section 7.4.

2.2. Jumping non-intersecting particles

TASEP means 'totally asymmetric exclusion process' [52]: it is the simplest model of statistical physics out of equilibrium, it has focused considerable amounts of work [18, 19, 64, 35, 36, 38] and it is still intensively studied. It is a model of self-avoiding particles which can either stay in their place or jump one step forward, provided that the next space is unoccupied. In the dynamics we shall be considering here, time is discrete, and at each unit of time several particles can jump.

It is well known that plane partitions can be rephrased in terms of a self-avoiding jumping particle model which is a kind of discrete TASEP [65, 66, 70, 69, 43, 6, 5, 2]: let us re-explain it here.

This rephrasing of plane partitions in terms of jumping particles is the standard 'diagonal slicing' used by many authors. We shall use a coordinate system, where the 'time' t goes from left to right, integer times corresponding to the positions of vertical edges of the plane partition, the position of particles h is measured from top to bottom and $h_i(t)$ is the position of the center of the ith vertical edge (the first one $i = 1$ is the uppermost one and the last one $i = N_\lambda$ is the bottom one). Times are integers $t \in \mathbb{Z}$,

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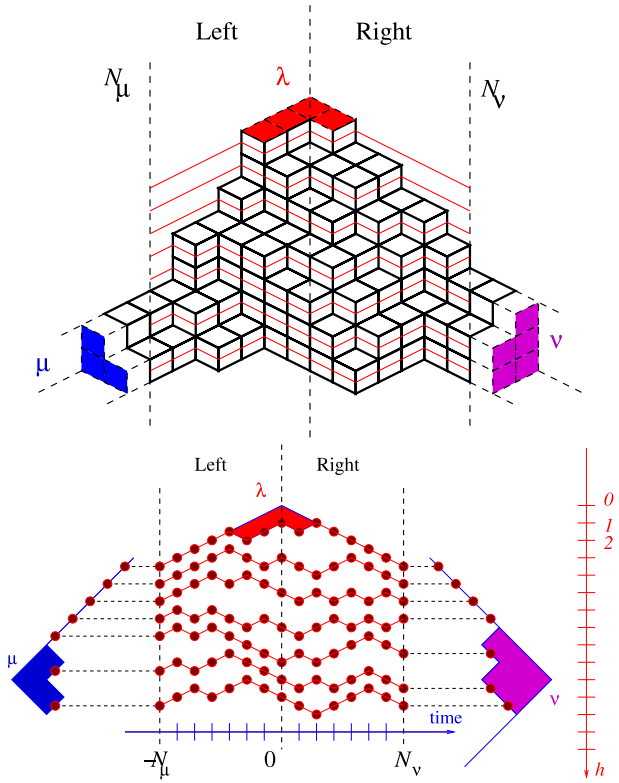

Figure 2. The level lines go only through upright or downright tiles. They form N_λ non-intersecting lines whose slopes are piecewise $\pm\frac{1}{2}$.

$-N_\mu \le t \le N_\nu$ and positions h are such that $h+|t|/2 \in \mathbb{N}$. In other words, our coordinate is on the 'rhombus lattice' (see figure 4).

Let us draw in red the N_λ non-intersecting lines going through tiles ⧄ and ⧅:

The tiles ⧄ correspond to upright lines with slope $+1/2$ and the tiles ⧅ correspond to downright lines with slope $-1/2$. The intersection of those red lines with integer time lines $t = -N_\mu, \ldots, N_\nu$ are interpreted as positions of some particles $h_i(t)$ ($h = 0$ is at the top). See figure 2.

Therefore, a plane partition π can also be described as the data of $1 + N_\mu + N_\nu$ sets of N_λ variables:

$$h_i(t), \qquad t \in \{-N_\mu, \ldots, N_\nu\}, \quad i = 1, \ldots, N_\lambda \tag{2.5}$$

such that, for every integer $t \in \{-N_\mu, \ldots, N_\nu\}$, the $h_i(t) - \frac{1}{2}|t|$ are non-negative ordered integers:

$$h_i(t) - \tfrac{1}{2}|t| \in \mathbb{N}, \qquad h_1(t) > h_2(t) > h_3(t) > \cdots > h_{N_\lambda}(t) \ge \tfrac{1}{2}|t| + r_t(\lambda) \tag{2.6}$$

where $r_t(\lambda)$ is the profile function of the partition λ.

At time $t = -N_\mu$ and time $t = N_\nu$ we have

$$h_i(-N_\mu) = \mu_i - i + N_\lambda + \tfrac{1}{2}N_\mu, \qquad h_i(N_\nu) = \nu_i - i + N_\lambda + \tfrac{1}{2}N_\nu \tag{2.7}$$

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and at each time t we have a partition $\lambda(t) = (\lambda_1(t) \geq \lambda_2(t) \geq \cdots \geq \lambda_{N_\lambda}(t))$:

$$\lambda_i(t) = h_i(t) + i - N_\lambda - \tfrac{1}{2}|t| \tag{2.8}$$

and we thus have $\lambda(-N_\mu) = \mu$ and $\lambda(+N_\nu) = \nu$.

Moreover we have

$$\forall t, \qquad h_i(t) - h_i(t+1) = \pm 1/2. \tag{2.9}$$

This is what we call here a discrete TASEP process:

- there are N particles at positions $h_i(t)$, and at each unit of time they jump by $\pm\frac{1}{2}$, and they can never occupy the same position. (In the usual formulation of TASEP, particles jump by 0 or 1, and here we have tilted the picture so that they jump by $\pm 1/2$, which is clearly the same thing up to $h_i(t) \to h_i(t) + t/2$.)

Notice that the total number of boxes in the partition is

$$|\pi| = \sum_{i,t} h_i(t) - \frac{N_\lambda}{2}(N_\lambda + N_\mu + N_\nu). \tag{2.10}$$

2.2.1. Height function and density. There is a relationship between the height function of the pile of cubes and the density of h_is. Define the density at time t as the Dirac-comb distribution:

$$\rho(h,t) = \frac{1}{N_\lambda} \sum_{i=1}^{N_\lambda} \delta(h - h_i(t)). \tag{2.11}$$

The profile of the partition at time t is recovered from the integral of ρ as follows. Define the integrated density:

$$I(h,t) = N_\lambda + N_\lambda \int_h^{(1/2)|t|} \rho(h',t)\,\mathrm{d}h', \tag{2.12}$$

where $I(h,t)$ computes the index $i = I(h,t)$ such that $h(x,t) = h_i$. Then, define the function

$$\lambda(h,t) = h - \tfrac{1}{2}|t| + I(h,t) - N_\lambda. \tag{2.13}$$

The plot of $\lambda(h,t)$ against $I(h,t)$ is the shape of the partition $\lambda(t)$ at time t. See figure 3.

The surface of the pile of cubes in $\mathbb{R}^3$ is recovered from plotting the partition $\lambda(t)$ at all times, i.e. it is given by

$$\begin{aligned} x_1 &= \lambda + \frac{1}{2}(|t| - t) = I - N_\lambda + h + \frac{t}{2} \\ x_2 &= \lambda + \frac{1}{2}(|t| + t) = I - N_\lambda + h - \frac{t}{2} \qquad x_3 = I. \end{aligned} \tag{2.14}$$

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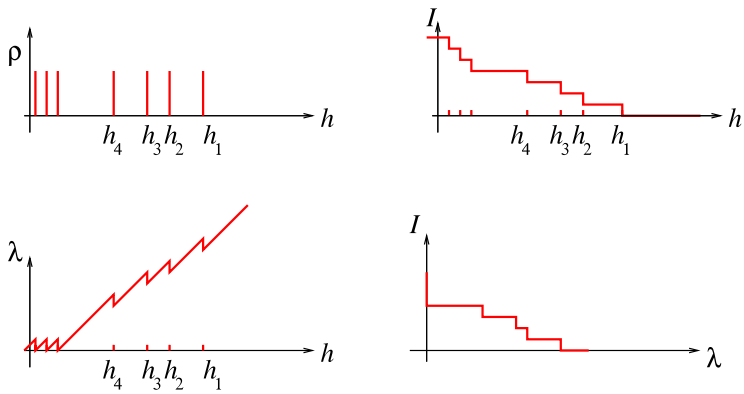

Figure 3. The density $\rho(h,t) = (1/N_\lambda)\sum_{i=1}^{N_\lambda} \delta(h - h_i(t))$ encodes the shape of the partition $\lambda(t)$.

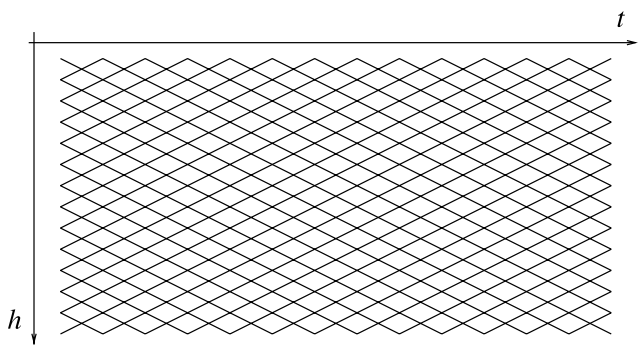

Figure 4. The rhombus lattice is a tiling of the plane, with lozenges whose centers are at the positions (h,t) such that $h - (t/2) \in \mathbb{Z}$ and $t \in \mathbb{Z}$. Notice that we orient h from top to bottom.

2.3. Lozenge tilings

Another representation of plane partitions and the self-avoiding particle model is with lozenge tilings of the rhombus lattice.

The rhombus lattice is a tiling of the plane, with lozenges ◇, whose centers are at the positions (h,t) such that $h - (t/2) \in \mathbb{Z}$ and $t \in \mathbb{Z}$, see figure 4.

A plane partition or a self-avoiding particle model configuration can be represented as $N = N_\lambda$ oriented self-avoiding walks in some domain $\mathcal{D}$ of the rhombus tiling of the plane. See figure 5. At a given time, each walker moves to the right $t \to t+1$, and either up $h \to h - \frac{1}{2}$ or down $h \to h + \frac{1}{2}$.

The centers of occupied lozenges are at positions $(h_i(t), t)$, $i = 1, \ldots, N$.

The domain $\mathcal{D}$ is the domain which contains all possible paths. $\mathcal{D}$ is a subdomain of the hexagon defined by the following six inequalities:

$$\begin{aligned} &t_{\min} \le t \le t_{\max} \qquad h_N(t_{\min}) - \frac{t - t_{\min}}{2} \le h \le h_1(t_{\min}) + \frac{t - t_{\min}}{2} \\ &h_N(t_{\max}) - \frac{t_{\max} - t}{2} \le h \le h_1(t_{\max}) + \frac{t_{\max} - t}{2}. \end{aligned} \tag{2.15}$$

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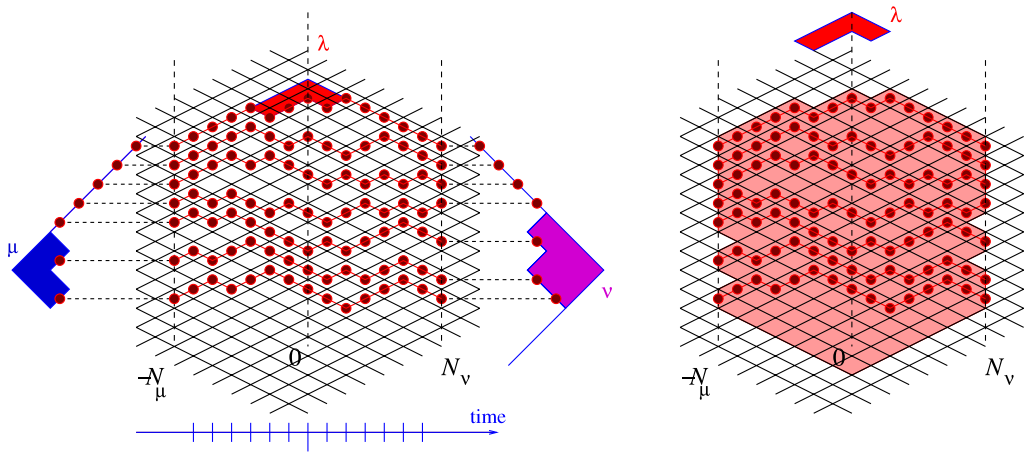

Figure 5. A plane partition configuration can be represented as N self-avoiding walks in some domain of the rhombus tiling of the plane. The white lozenges in the right figure are forbidden; they are related to the three boundary partitions λ, μ, ν.

Indeed, since at each time, and particularly at $t = t_{\min}$ and $t_{\max}$, we have $h_N \leq h_i \leq h_1$, no path starting at some $h_i(t_{\min})$ at $t = t_{\min}$ and ending at some $h_j(t_{\max})$ at $t = t_{\max}$ can go out of this hexagon.

Moreover, the domain $\mathcal{D}$ can be chosen such that some positions are forbidden. In fact we allow $\mathcal{D}$ to be any arbitrary subset of the maximal hexagon.

For example, for plane partitions, the positions corresponding to the boundaries λ, μ, ν are forbidden, i.e. $\mathcal{D}$ has holes corresponding to the three partitions λ, μ, ν at the boundaries.

- At $t = t_{\min} = -N_\mu$, we remove from $\mathcal{D}$ all lozenges which are not of the form $h_i(\mu)$ for some i.
- At $t = t_{\max} = N_\nu$, we remove from $\mathcal{D}$ all lozenges which are not of the form $h_i(\nu)$ for some i.
- At every time t we impose $h \geq |t|/2 + r_t(\lambda)$.

See figure 5.

2.3.1. Defects. It is interesting to generalize our model to include walks on more general domains, not only limited by three boundary partitions. In particular, we may allow defects and holes at almost any place.

Consider a connected compact domain $\mathcal{D}$ in the rhombus lattice, but not necessarily simply connected.

Definition. *The maximal domain $Max(\mathcal{D})$ of $\mathcal{D}$ is the intersection of $t_{\min} \leq t \leq t_{\max}$ with $h_N(t_{\max}) + (t - t_{\max})/2 \leq h \leq h_1(t_{\max}) - (t - t_{\max})/2$ (see the dashed region in figures 6 and 7).*

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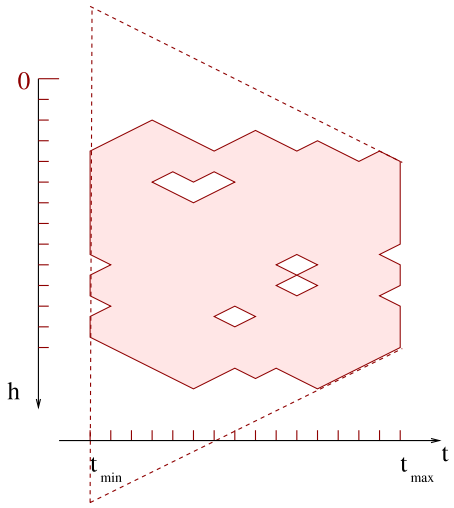

Figure 6. The maximal domain of $\mathcal{D}$ is the region comprised in the dashed line.

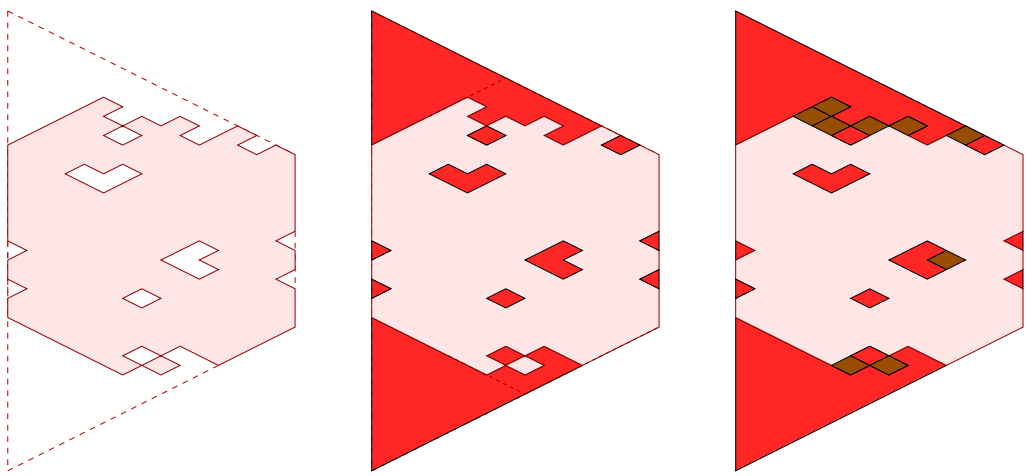

Figure 7. Consider a given domain $\mathcal{D}$. Its maximal domain $\mathrm{Max}(\mathcal{D})$ is represented by a dashed line in the left figure (it is obtained by lines of slopes $\pm\frac{1}{2}$ starting from the extremities of the domain at $t_{\max}$). In the middle figure we have represented the defect of $\mathcal{D}$, which is $\bar{\mathcal{D}} = \mathrm{Max}(\mathcal{D}) \setminus \mathcal{D}$ (the red region). In the right figure we have represented the shadow of $\bar{\mathcal{D}}$, which is the domain $\hat{\bar{\mathcal{D}}} \subset \mathrm{Max}(\mathcal{D})$ which is inaccessible to particles moving in $\mathcal{D}$, with slopes $\pm\frac{1}{2}$ (the brown + red region); it contains $\bar{\mathcal{D}}$. This means that particles moving in $\mathcal{D}$ can never enter the brown tiles.

The defect of $\mathcal{D}$ is $\bar{\mathcal{D}} = Max(\mathcal{D}) \setminus \mathcal{D}$. The shadow of $\bar{\mathcal{D}}$ is the domain $\hat{\bar{\mathcal{D}}} \subset Max(\mathcal{D})$, which is inaccessible to particles moving in $\mathcal{D}$, with slopes $\pm\frac{1}{2}$. We have $\bar{\mathcal{D}} \subset \hat{\bar{\mathcal{D}}}$.

Those notions are illustrated in figure 7.

Definition. *The minimal defect $\bar{\mathcal{D}}_0$ of $\mathcal{D}$ is the smallest subdomain of $\bar{\mathcal{D}}$, such that*

$$\hat{\bar{\mathcal{D}}}_0 = \hat{\bar{\mathcal{D}}} \tag{2.16}$$

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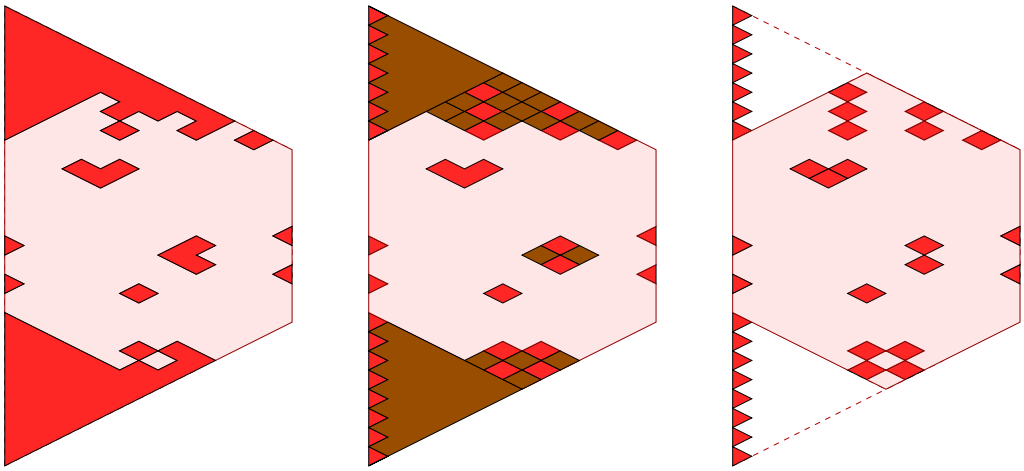

Figure 8. The left figure represents the domain $\mathcal{D}$ and its defect $\bar{\mathcal{D}}$ in red. The right figure represents (in red) the minimal defect $\bar{\mathcal{D}}_0$ of $\mathcal{D}$. The middle figure shows that both $\bar{\mathcal{D}}$ and $\bar{\mathcal{D}}_0$ have the same shadow (in brown + red).

2.3.2. Domain at time t. Let us call $\mathcal{D}(t)$ the slice of the domain $\mathcal{D}$ at time t, i.e. the set of allowed positions of particles at time t:

$$\mathcal{D}(t) = \left\{ h | h - \frac{t}{2} \in \mathbb{Z}, (h,t) \in \mathcal{D} \right\}. \tag{2.17}$$

Let us call $\bar{\mathcal{D}}(t)$ the slice of $\bar{\mathcal{D}}$ at time t, i.e. the position of holes:

$$\bar{\mathcal{D}}(t) = \left\{ h | h - \frac{t}{2} \in \mathbb{Z}, (h,t) \in \bar{\mathcal{D}} \right\}. \tag{2.18}$$

Let us assume that the number of particles N is such that

$$\forall t = t_{\min}, \ldots, t_{\max}, \qquad \#\mathcal{D}(t) \geq N, \tag{2.19}$$

in other words, such that it is indeed possible for N avoiding particles to move in $\mathcal{D}$.

Remark 2.1. Most often we will choose $N = \#\mathcal{D}(t_{\max})$, i.e. the position of all particles at time $t_{\max}$ are fixed. Very often we will also choose the domain such that $N = \#\mathcal{D}(t_{\max}) = \#\mathcal{D}(t_{\min})$, i.e. the position of all particles are also fixed at time $t_{\min}$.

But we emphasize that the method we present here works also without those assumptions.

2.3.3. Filling fractions. In general, the domain at time t, is a union of intervals:

$$\mathcal{D}(t) = \bigcup_{i=1}^{m_t} [a_i(t), b_i(t)]. \tag{2.20}$$

If the intervals are disconnected, no particle can jump from one interval to another, and thus the number of particles moving in an interval is constant in time, until intervals join. We may thus fix the number of particles in each interval. Call it $n_i(t)$ = filling fraction of the interval $[a_i, b_i]$ at time t. We have

$$\sum_{i=1}^{m_t} n_i(t) = N. \tag{2.21}$$

In general, if the domain has k holes, there are k independent filling fractions.

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2.4. Partition function

The partition function which we wish to compute is

$$Z_N(\mathcal{D};q,\alpha,\beta) = \sum_{h_1(t)>\cdots>h_N(t),h_i(t)\in\mathcal{D}(t)} q^{\sum_{i,t} h_i(t)} \prod_{t'=t_{\min}+1/2}^{t_{\max}-1/2} \alpha(t')^{\#(\nearrow)_{(t')}} \beta(t')^{\#(\searrow)_{(t')}} \tag{2.22}$$

where at each time t we have $h_1(t) > h_2(t) > \cdots > h_N(t)$, and where $h_i(t_{\min})$ and $h_i(t_{\max})$ have fixed values given by two partitions μ and ν (encoded by the boundaries of $\mathcal{D}$):

$$h_i(t_{\min}) = h_i(\mu), \qquad h_i(t_{\max}) = h_i(\nu). \tag{2.23}$$

We count the configuration with a weight $q^{|\pi|} \propto q^{\sum_{t,i} h_i(t)}$, and with weights $\alpha(t+\frac{1}{2})$ per number of upward jumps $\nearrow$ between times t and $t+1$, and with weights $\beta(t+\frac{1}{2})$ per number of downward jumps $\searrow$ between times t and $t+1$.

It can be rewritten:

$$\begin{aligned} Z_N(\mathcal{D};q,\alpha,\beta) = & \sum_{h_1(t)>\cdots>h_N(t),\, h_i(t)\in\mathcal{D}(t)} \prod_{i=1}^{N} \prod_{t=t_{\min}}^{t_{\max}} q^{h_i(t)} \\ & \times \prod_{t'=t_{\min}+1/2}^{t_{\max}-1/2} \prod_{i=1}^{N} [\alpha(t')\delta(h_i(t'+\tfrac{1}{2}) - h_i(t'-\tfrac{1}{2}) - \tfrac{1}{2}) \\ & + \beta(t')\delta(h_i(t'+\tfrac{1}{2}) - h_i(t'-\tfrac{1}{2}) + \tfrac{1}{2})] \end{aligned} \tag{2.24}$$

where here δ is Kroenecker's δ function.

2.5. Applications

- In topological string theory, Gromov–Witten invariants of toric Calabi–Yau 3-folds are computed with the topological vertex, which is the following sum of plane partitions [3, 59]:

$$Z = \sum_{\pi,\partial\pi=(\lambda,\mu,\nu)} q^{|\pi|}. \tag{2.25}$$

- When $\lambda = \mu = \nu = \emptyset$, $N_\lambda = N_\mu = N_\nu = \infty$ and $\alpha = \beta = 1$, this sum is known: it is the MacMahon formula:

$$Z = \sum_{\pi} q^{|\pi|} = \prod_{k=1}^{\infty} (1-q^k)^{-k} = 1 + q + 3q^2 + 6q^3 + 13q^4 + \cdots. \tag{2.26}$$

- The Razumov–Stroganov conjecture [4, 22] has put forward a problem of combinatorics of totally symmetric self-complementary plane partitions (TSSCPP), and the claim is about the relationship with the combinatorics of alternating sign matrices (ASM). There is a six-vertex matrix model formulation for ASM [51, 73], and it would be interesting to also have a matrix model for TSSCPP: this is what we address in section 8 below.

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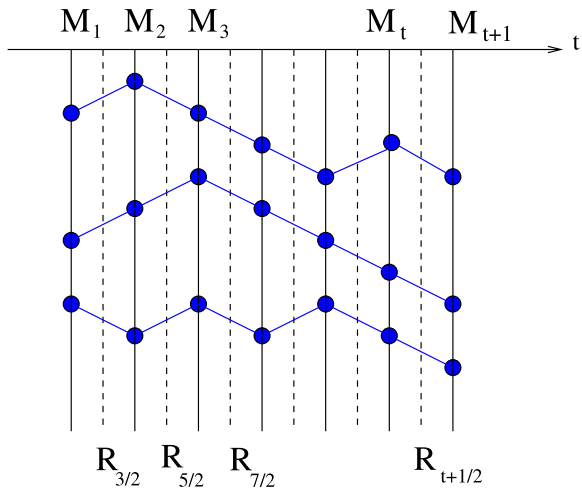

Figure 9. We introduce a chain of matrices. The eigenvalues of the random matrices M_t with t integer are the random $h_i(t)$. The $R_{t'}$ with t' half-integers are Lagrange multipliers which enforce the relations $h_i(t+1) - h_i(t) = \pm\frac{1}{2}$.

3. Matrix model

We are going to represent our self-avoiding particle process partition function equation (2.22) as a multi-matrix integral.

Let us sketch the idea of section 3.1: we shall introduce $t_{\max} - t_{\min} + 1$ normal matrices M_t of size $N \times N$ for all integer times t between $t_{\min}$ and $t_{\max}$, whose eigenvalues are $h_i(t)$. Moreover, we shall Fourier transform the δ functions which enforce $h_i(t+1) = h_i(t) \pm \frac{1}{2}$:

$$\begin{aligned}
&\left[\alpha(t+\tfrac{1}{2})\delta(h_i(t+1) - h_i(t) - \tfrac{1}{2}) + \beta(t+\tfrac{1}{2})\delta(h_i(t+1) - h_i(t) + \tfrac{1}{2})\right] \\
&\qquad = \int_{-\infty}^{+\infty} \mathrm{d}r_i \left[\alpha\left(t+\tfrac{1}{2}\right) \mathrm{e}^{2\mathrm{i}\pi r_i(h_i(t+1)-h_i(t)-1/2)} + \beta(t+\tfrac{1}{2})\mathrm{e}^{2\mathrm{i}\pi r_i(h_i(t+1)-h_i(t)+1/2)}\right] \\
&\qquad = \int_{-\infty}^{+\infty} \mathrm{d}r_i\, \mathrm{e}^{2\mathrm{i}\pi\, r_i(h_i(t+1)-h_i(t))} \left[\alpha\left(t+\tfrac{1}{2}\right) \mathrm{e}^{-\mathrm{i}\pi\, r_i} + \beta\left(t+\tfrac{1}{2}\right) \mathrm{e}^{\mathrm{i}\pi\, r_i}\right] \qquad (3.1)
\end{aligned}$$

i.e. we shall introduce some Lagrange multipliers $r_i(t+\frac{1}{2})$ at all half-integer times, and we will introduce an $N \times N$ Hermitian matrix $R_{t+(1/2)}$ whose eigenvalues are the Lagrange multipliers $r_i(t+\frac{1}{2})$, $i = 1, \ldots, N$, which implement the δ functions for the jumps between time t and $t+1$. See figure 9.

More generally, if we wanted to allow jumps of several steps $h_i(t+1) = h_i(t) + s, s \in \{s_1, \ldots, s_k\}$, we would take the Fourier transform of $\sum_j \alpha_j \mathrm{e}^{2\mathrm{i}\pi s_j r_i}$.

The non-intersecting condition for paths can be realized as a determinant, like the Gessel–Viennot formula [37], which allows us to rewrite

$$\prod_i \mathrm{e}^{2\mathrm{i}\pi r_i h_i} \to \det(\mathrm{e}^{2\mathrm{i}\pi r_i h_j}) \qquad (3.2)$$

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and we recognize that this expression is the Itzykson–Zuber–Harish–Chandra formula [39, 41]:

$$\det(\mathrm{e}^{2\mathrm{i}\pi r_i h_j}) = \Delta(R)\Delta(h)\int_{U(N)} \mathrm{d}U\, \mathrm{e}^{2\mathrm{i}\pi \mathrm{tr} RUhU^\dagger}. \tag{3.3}$$

This is the key to obtain a matrix integral; it introduces angular degrees of freedom in addition to eigenvalues:

$$M_t = Uh(t)U^\dagger, \qquad h = \mathrm{diag}(h_1(t), h_2(t), \ldots, h_N(t)). \tag{3.4}$$

We shall introduce some potential V_t for the matrices M_t to ensure that their eigenvalues are in $\mathcal{D}(t)$, and $V_{t_{\min}}$ and $V_{t_{\max}}$ enforce the initial values at time $t = t_{\min}$ or $t_{\max}$.

We shall find that the sum over $h_i(t)$s can be rewritten as a 'chain of matrices' matrix model.

So, let us describe the model now.

3.1. The multi-matrix model

Let us consider the following multi-matrix integral:

$$\mathcal{Z} = \int_{(H_N)^{t_{\max}-t_{\min}}} \prod_{t=t_{\min}}^{t_{\max}-1} \mathrm{d}M_t \int_{(\mathrm{i}H_N)^{t_{\max}-t_{\min}}} \prod_{t'=t_{\min}+1/2}^{t'=t_{\max}-1/2} \mathrm{d}R_{t'} \prod_{t=t_{\min}}^{t_{\max}} \mathrm{e}^{-\mathrm{Tr}V_t(M_t)} q^{\mathrm{Tr}M_t}$$
$$\times \prod_{t'=t_{\min}+1/2}^{t_{\max}-1/2} \mathrm{e}^{-\mathrm{Tr}U_{t'}(R_{t'})} \prod_{t'=t_{\min}+1/2}^{t_{\max}-1/2} \mathrm{e}^{\mathrm{Tr}R_{t'}(M_{t'+1/2}-M_{t'-1/2})}. \tag{3.5}$$

Each integral over M_t with $t = t_{\min}, \ldots, t_{\max}-1$ is over the set H_N of Hermitian matrices of size N, and each integral over $R_{t'}$ with $t' = t_{\min} + \frac{1}{2}, \ldots, t_{\max} - \frac{1}{2}$ is over the set $\mathrm{i}H_N$ of anti-Hermitian matrices of size N. Also there is no integration over $M_{t_{\max}}$, which is a fixed external field, which we choose equal to

$$M_{t_{\max}} = \mathrm{diag}(h_1(t_{\max}), \ldots, h_N(t_{\max})). \tag{3.6}$$

The potentials V_t or $U_{t'}$ are defined as follows:

- For t' half-integer:

$$\mathrm{e}^{-U_{t'}(r)} = \alpha(t')\mathrm{e}^{-(r/2)} + \beta(t')\mathrm{e}^{r/2}. \tag{3.7}$$

  This potential for $R_{t'}$ is the Fourier transform of the jumps, see equation (3.1): we have just rescaled r by $2\mathrm{i}\pi$.

- For t integer, $t_{\min} < t < t_{\max}$, we choose the potential $V_t(x)$ such that

$$\forall x: \qquad x - \frac{t}{2} \in \mathbb{Z} \cap \left[h_N(t_{\max}) + \frac{t_{\max}}{2}, h_1(t_{\max}) - \frac{t_{\max}}{2}\right],$$
$$\mathrm{e}^{-V_t(x)} = \begin{cases} 1 & \text{if } x \in \mathcal{D}(t) \setminus \hat{\bar{\mathcal{D}}} \\ 0 & \text{if } x \in \bar{\mathcal{D}}_0(t) \\ \text{arbitrary} & \text{otherwise} \end{cases} \tag{3.8}$$

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e^{-V_t} can be more or less interpreted as the characteristic function of the domain $\mathcal{D}(t)$ or, more precisely, the complementary domain of the defect $\bar{\mathcal{D}}(t)$.
However, this definition does not define a unique potential V_t, and many potentials V_t may have the property equation (3.8). In particular, we see that the value of $V_t(x)$ in the shadow of $\mathcal{D}$, more precisely in $\hat{\mathcal{D}} \setminus \bar{\mathcal{D}}_0$, is undetermined. We show below some rather canonical examples of V_t satisfying those constraints.
We will see below that the value of the partition function $\mathcal{Z}$ does not depend on the choice of V_t.

- For $t = t_{\min}$, we choose the potential $V_{t_{\min}}(x)$ such that

$$\forall x: \qquad x - \frac{t_{\min}}{2} \in \mathbb{Z} \cap \left[h_N(t_{\max}) + \frac{t_{\max}}{2}, h_1(t_{\max}) - \frac{t_{\max}}{2} \right],$$
$$\mathrm{e}^{-V_{t_{\min}}(x)} \begin{cases} \neq 0 & \text{if } x = h_i(\mu) \\ = 0 & \text{if } x \neq h_i(\mu) \\ \text{arbitrary} & \text{otherwise.} \end{cases} \tag{3.9}$$

In other words, we do not even require that $\mathrm{e}^{-V_{t_{\min}}} = 1$ in the domain; we only require that it is $\neq 0$.

Examples of V_t:

Since our domain is compact, we have to find a potential V_t with prescribed values at a finite number of points, and a possibility is to choose e^{-V_t} to be the Lagrange interpolating polynomial going through the prescribed values.

For example, if our domain at time t is contained in the interval

$$\mathcal{D}(t) \subset [k_{t-}, k_{t+}], \tag{3.10}$$

we may choose the potential V_t as the Lagrange interpolating polynomial:

$$\mathrm{e}^{-V_t(x)} = 1 - \sum_{i \in \bar{\mathcal{D}}(t)} \frac{P_{k_{t-},k_{t+}}(x)}{(x-i) P'_{k_{t-},k_{t+}}(i)}, \qquad P_{k_-,k_+}(h) = \prod_{j=k_-}^{k_+} (h-j). \tag{3.11}$$

Another possibility is to take the limit $k_+ \to \infty$:

$$\mathrm{e}^{-V_t(x)} = 1 - \sum_{i \in \bar{\mathcal{D}}(t)} \frac{(-1)^i}{i!(x-i)\Gamma(-x)} \tag{3.12}$$

or we may also choose

$$\mathrm{e}^{-V_t(x)} = \frac{\mathrm{e}^{\mathrm{i}\pi x} \sin \pi x}{\pi} \sum_{i \in \mathcal{D}(t)} \frac{1}{x-i}. \tag{3.13}$$

Depending on the type of applications we are interested in, it is sometimes more convenient to work with a potential of type equation (3.11) or a potential of type equation (3.12), or also their q deformations, or sometimes other potentials having the property equation (3.8).

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In general, we see that $-V_t(x)$ must have logarithmic singularities on $\bar{\mathcal{D}}_0(t)$, and thus we may write

$$V_t'(x) = -\sum_{i\in\bar{\mathcal{D}}_0(t)} \frac{1}{x-i} + f'(x) + g'\left(x - \frac{t}{2}\right) \tag{3.14}$$

where $g(x)$ can be any arbitrary entire function which vanishes on $x \in \mathbb{Z}$, and where $f(x)$ is an analytical function such that if $i \in \mathcal{D}(t)$ we have $V_t(x) = 0$. Since fixing the values of f and g does not fix the values of f' and g' at those points, we see that what characterizes V_t' is that it has simple poles with residue -1 in $\bar{\mathcal{D}}_0(t)$, plus an almost arbitrary analytical function.

3.1.1. Diagonalization. Let us diagonalize all matrices in equation (3.5). One needs to know that any normal matrix (and in particular Hermitian matrices) can be diagonalized by a unitary transformation:

$$M_t = \mathcal{U}_t X_t \mathcal{U}_t^\dagger, \qquad \mathcal{U}_t \in U(N), \qquad X_t = \mathrm{diag}(X_1(t), \ldots, X_N(t)) \tag{3.15}$$

and the matrix measures are

$$\mathrm{d}M_t = \frac{1}{N!}\,\mathrm{d}\mathcal{U}_t\,\mathrm{d}X_t \Delta(X_t)^2 \tag{3.16}$$

where $\mathrm{d}\mathcal{U}_t$ is the Haar measure on $U(N)$ and $\Delta(X_t)$ is the Vandermonde determinant:

$$\Delta(X_t) = \prod_{i>j}(X_i(t) - X_j(t)). \tag{3.17}$$

Similarly for the matrices $R_{t'}$

$$R_{t'} = \tilde{\mathcal{U}}_{t'} Y_{t'} \tilde{\mathcal{U}}_{t'}^\dagger, \qquad \tilde{\mathcal{U}}_{t'} \in U(N), \qquad Y_{t'} = \mathrm{diag}(Y_1(t'), \ldots, Y_N(t')). \tag{3.18}$$

Therefore we may rewrite equation (3.5) as

$$\begin{aligned}(N!)^{2(t_{\max}-t_{\min})+1}\mathcal{Z} = {} & \int_{\mathbb{R}} \prod_{t=t_{\min}}^{t_{\max}-1} \mathrm{d}X_t \int_{i\mathbb{R}} \prod_{t'=t_{\min}+1/2}^{t'=t_{\max}-1/2} \mathrm{d}Y_{t'}\, \mathrm{e}^{-\mathrm{Tr}V_{t_{\max}}(X_{t_{\max}})} q^{\mathrm{Tr}X_{t_{\max}}} \\ & \times \prod_{t=t_{\min}}^{t_{\max}-1} \mathrm{e}^{-\mathrm{Tr}V_t(X_t)} q^{\mathrm{Tr}X_t} \Delta(X_t)^2 \prod_{t'=t_{\min}+1/2}^{t_{\max}-1/2} \mathrm{e}^{-\mathrm{Tr}U_{t'}(Y_{t'})} \Delta(Y_{t'})^2 \\ & \times \prod_{t'=t_{\min}+1/2}^{t_{\max}-1/2} I(X_{t'+1/2}, Y_{t'}) I(-X_{t'-1/2}, Y_{t'})\end{aligned} \tag{3.19}$$

where $I(X, Y)$ is the Itzykson–Zuber integral:

$$I(X, Y) = \int_{U(N)} \mathrm{d}\mathcal{U}\, \mathrm{e}^{\mathrm{Tr}X\mathcal{U}Y\mathcal{U}^\dagger}. \tag{3.20}$$

It is well known that [39, 41]

$$I(X, Y) = \frac{\det\left(\mathrm{e}^{X_i Y_j}\right)}{\Delta(X)\Delta(Y)} \tag{3.21}$$

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and therefore [54] we get

$$(N!)^{2(t_{\max}-t_{\min})+1}\mathcal{Z} = \frac{1}{\Delta(X_{t_{\max}})}\int_{\mathbb{R}}\prod_{t=t_{\min}}^{t_{\max}-1}\mathrm{d}X_t\int_{i\mathbb{R}}\prod_{t'=t_{\min}+1/2}^{t'=t_{\max}-1/2}\mathrm{d}Y_{t'}\Delta(X_{t_{\min}})$$
$$\times\prod_{t=t_{\min}}^{t_{\max}}\mathrm{e}^{-\mathrm{Tr}V_t(X_t)}q^{\mathrm{Tr}X_t}\prod_{t'=t_{\min}+1/2}^{t_{\max}-1/2}\mathrm{e}^{-\mathrm{Tr}U_{t'}(Y_{t'})}$$
$$\times\prod_{t'=t_{\min}+1/2}^{t_{\max}-1/2}\det\left(\mathrm{e}^{X_i(t'+1/2)Y_j(t')}\right)\det\left(\mathrm{e}^{-X_i(t'-1/2)Y_j(t')}\right). \tag{3.22}$$

3.2. Relation between the matrix model and the self-avoiding particle model

3.2.1. Integrals over Lagrange multipliers. Then let us perform the integral over $Y_j(t')$. We have

$$\int\mathrm{d}Y_{t'}\det\left(\mathrm{e}^{X_i(t'+1/2)Y_j(t')}\right)\det\left(\mathrm{e}^{-X_i(t'-1/2)Y_j(t')}\right)\mathrm{e}^{-\mathrm{Tr}U_{t'}(Y(t'))}$$
$$= \sum_{\sigma_{t'},\tilde{\sigma}_{t'}}(-1)^{\sigma_{t'}\tilde{\sigma}_{t'}}\prod_i\int\mathrm{d}Y_i(t')\,\mathrm{e}^{Y_i(t')(X_{\sigma_{t'}(i)}(t'+1/2)-X_{\tilde{\sigma}_{t'}(i)}(t'-1/2))}\mathrm{e}^{-U_{t'}(Y_i(t'))}$$
$$= \sum_{\sigma_{t'},\tilde{\sigma}_{t'}}(-1)^{\sigma_{t'}\tilde{\sigma}_{t'}}\prod_i[\alpha(t')\delta(X_{\sigma_{t'}(i)}(t'+\tfrac{1}{2})-X_{\tilde{\sigma}_{t'}(i)}(t'-\tfrac{1}{2})-\tfrac{1}{2})$$
$$+\beta(t')\delta(X_{\sigma_{t'}(i)}(t'+\tfrac{1}{2})-X_{\tilde{\sigma}_{t'}(i)}(t'-\tfrac{1}{2})+\tfrac{1}{2})]$$
$$= N!\det[\alpha(t')\delta(X_i(t'+\tfrac{1}{2})-X_j(t'-\tfrac{1}{2})-\tfrac{1}{2})$$
$$+\beta(t')\delta(X_i(t'+\tfrac{1}{2})-X_j(t'-\tfrac{1}{2})+\tfrac{1}{2})]. \tag{3.23}$$

This term implies that there must exist some permutation such that

$$X_i(t'+\tfrac{1}{2}) = X_{\sigma_{t'}(i)}(t'-\tfrac{1}{2})\pm\tfrac{1}{2} \tag{3.24}$$

with respective probabilities $\beta(t'),\alpha(t')$. And, in particular, since this is true for $t=t_{\max}$, we have $\forall t$

$$X_i(t)-\frac{t}{2}\in\mathbb{Z}\cap\left[h_N(t_{\max})-\frac{t_{\max}-t}{2},h_1(t_{\max})+\frac{t_{\max}-t}{2}\right]. \tag{3.25}$$

In particular that implies that $\mathrm{e}^{-V_t(X_i(t))}=1$ if $X_i(t)\in\mathcal{D}(t)$ and $\mathrm{e}^{-V_t(X_i(t))}=0$ if $X_i(t)\in\bar{\mathcal{D}}_0(t)$. In particular, the matrix integral is indeed independent of the choice of V_t, provided that it satisfies equation (3.8).

Then, since the quantities we are summing are symmetric, we can assume, up to a multiplication by $N!$, that all $X_i(t)$ are ordered:

$$X_1(t)>X_2(t)>\cdots>X_N(t) \tag{3.26}$$

and this yields a factor $(N!)^{t_{\max}-t_{\min}}$, which we shall discard because we consider the partition function up to global trivial constants.

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In other words, the result of the integral is a sum over $X_i(t) = h_i(t)$, where $h_i(t) - t/2$ are ordered integers:

$$\mathcal{Z} = \frac{1}{\Delta(h(t_{\max}))} \sum_{h_i(t), i=1,\dots,N, t=t_{\min}+1,\dots,t_{\max}-1} \Delta(h_i(t_{\min})) \prod_{i,t} \mathrm{e}^{-V_t(h_i(t))} q^{h_i(t)} \times \prod_{i,t'} [\alpha(t')\delta_{h_i(t'+1/2)-h_i(t'-1/2)-1/2} + \beta(t')\delta_{h_i(t'+1/2)-h_i(t'-1/2)+1/2}] \tag{3.27}$$

where the sum over $h_i(t)$, is such that

$$h_i(t) - \frac{t}{2} \in \mathbb{Z}, \qquad \text{and} \qquad h_1(t) > h_2(t) > \cdots > h_N(t) \tag{3.28}$$

and notice that at $t = t_{\max}$, the $h_i(t_{\max})$ are fixed.

Then, notice that $\mathrm{e}^{-V_t(h_i(t))} = 1$ if $h_i(t) \in \mathcal{D}(t)$ and $\mathrm{e}^{-V_t(h_i(t))} = 0$ if $h_i(t) \in \bar{\mathcal{D}}_0(t)$. In other words, the sum is only over $h_i(t)$s such that

$$h_i(t) - \frac{t}{2} \notin \bar{\mathcal{D}}_0(t). \tag{3.29}$$

In other words, we have a self-avoiding particle process on the rhombus lattice, which avoids a prescribed domain $\bar{\mathcal{D}}$, i.e. we recover our self-avoiding particle partition function.

Theorem 3.1. *The self-avoiding particle model partition function $Z_N(\mathcal{D}, q, \alpha, \beta)$ in a domain $\mathcal{D}$, is proportional to the matrix integral $\mathcal{Z}$:*

$$\mathcal{Z} = C_{N, t_{\max}-t_{\min}} \frac{\Delta(h_\mu)}{\Delta(h_\nu)} \mathrm{e}^{-\sum_i V_{t_{\min}}(h_i(\mu))} Z_N(\mathcal{D}, q, \alpha, \beta) \tag{3.30}$$

where the constant $C_{N,T}$ depends only on N and T, and nothing else. It contains the normalization factors, such as the volumes of unitary groups and the powers of $N!$.

This theorem implies immediately, as a tautology, that whatever limit we consider (for instance, large size limit, $q \to 1$ limit, bulk regime, behavior near edges, etc), the asymptotic statistical properties *are always matrix model limit laws*!

This explains why one finds sine-kernel laws in the bulk, Tracy–Widom laws near some boundaries, Pearcey laws and many more.

3.3. Determinantal formulae

Theorem 3.1 allows us to apply to the self-avoiding particle model and plane partitions all the technology developed for matrix models, in particular the methods of orthogonal polynomials [54, 55].

Consider a 'chain of matrices' integral:

$$\mathcal{Z} = \int_{\prod H_N(\mathcal{C}_i)} \prod_{i=1}^{p} \mathrm{d}M_i \, \mathrm{e}^{-Q\mathrm{Tr}[\sum_{i=1}^{p} V_i(M_i) + \sum_i c_i M_i M_{i+1}]} \tag{3.31}$$

where V_i are some potentials, where $H_N(\mathcal{C}_i)$ is the set of $N \times N$ normal matrices having their eigenvalues on contour $\mathcal{C}_i$ and where M_{p+1} is not integrated upon.

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The Eynard–Mehta theorem [31] shows that correlation functions of densities of eigenvalues $\rho_i(x) = \mathrm{Tr}\delta(x - M_i)$ are determinants. Namely, there exist some kernels $H_{i,j}(x, x')$ such that

$$\langle \rho_{k_1}(x_1) \cdots \rho_{k_n}(x_n) \rangle = \det \left(H_{k_i,k_j}(x_i, x_j) \right), \tag{3.32}$$

where the kernels $H_{i,j}$ are Christoffel–Darboux kernels for some families of biorthogonal polynomials.

From the point of view of self-avoiding particle model and plane partitions, this should allow us to recover many determinantal formulae in the TASEP literature, see, for instance, [50, 9, 60].

Another consequence of the 'orthogonal polynomial method' [55] is that matrix integrals of type equation (3.31) are Tau functions for the integrable Toda hierarchy [1]. This should allow us to recover many differential equations in the TASEP literature.

4. Matrix model's topological expansion

The good thing about theorem 3.1 is that the general expansion of matrix integrals of the chain of matrices type is known to all orders.

4.1. Generalities about the expansion of matrix integrals

See appendix C for a more detailed description.

Consider a 'chain of matrices' integral:

$$\mathcal{Z} = \int_{\prod H_N(\mathcal{C}_i)} \prod_{i=1}^{p} \mathrm{d}M_i \, \mathrm{e}^{-Q\mathrm{Tr}[\sum_{i=1}^{p} V_i(M_i) + \sum_i c_i M_i M_{i+1}]} \tag{4.1}$$

where V_i are some potentials, where $H_N(\mathcal{C}_i)$ is the set of $N \times N$ normal matrices having their eigenvalues on contour $\mathcal{C}_i$ and where M_{p+1} is not integrated upon.

In some good cases (depending on the choice of potentials V_i and paths $\mathcal{C}_i$), such an integral has a large Q expansion of the form:

$$\ln \mathcal{Z} \sim \sum_{g=0}^{\infty} Q^{2-2g} \mathcal{F}_g. \tag{4.2}$$

Such an expansion does not always exist. It exists only if the paths $\mathcal{C}_i$, which support the eigenvalues, are 'steepest descent paths' for the potentials V_i (see, e.g., [28, 25]). Finding the steepest descent paths associated with given potentials is an extremely difficult problem.

Fortunately, many applications of random matrices regard combinatorics, i.e. they are formal series in some formal parameter, and very often the corresponding so-called 'formal matrix integrals' do have a large Q expansion almost by definition[1], and equation (4.2) holds order by order in the formal parameter (a formal parameter which is not necessarily Q). Here, we are considering applications to statistical physics, our partition functions

[1] For formal matrix integrals, the integration paths $\mathcal{C}_i$ for eigenvalues are most often not known explicitly. They can be determined so that a large Q power series expansion does exist.

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are formal series and we shall assume that such an expansion exists (order by order in a suitable formal parameter).

The problem is then to compute the coefficients $\mathcal{F}_g$.

The answer was found in [33] by using loop equations (i.e. Schwinger–Dyson equations in the context of matrix models) and which just correspond to integration by parts.

The solution proceeds in two steps (which we explain below):

(1) Compute the 'spectral curve' $\mathcal{S}$ of the matrix model. The spectral curve $\mathcal{S} = (x, y)$ is a pair of two analytical functions $x(z), y(z)$ of a variable z living on a Riemann surface. The spectral curve is obtained from the 'classical limit' of the integrable system whose tau function is the matrix integral. Roughly speaking, if we eliminate z, the function $y(x)$ is more or less the equilibrium density of eigenvalues of the first matrix of the chain. We explain in appendix C how to find the spectral curve of a general chain of matrices. We emphasize that associating a spectral curve $\mathcal{S} = (x, y)$ to a given matrix model is something already done in the matrix model literature.

(2) Then compute the symplectic invariants $F_g(\mathcal{S})$ of that spectral curve (symplectic invariants of an arbitrary spectral curve $\mathcal{S}$ were first introduced in [27]: they are rather easy to compute and we recall their definition in appendix B) and the main result of [33] is that $\mathcal{F}_g = F_g(\mathcal{S})$, i.e.

$$\ln \mathcal{Z} = \sum_{g=0}^{\infty} Q^{2-2g} F_g(\mathcal{S}). \tag{4.3}$$

4.2. Spectral curve of the self-avoiding particle matrix model

We recall in appendix C the main results of [33], i.e. how to compute the spectral curve of an arbitrary chain of matrices. Here, in this section, we merely apply the general recipe of [33] (see appendix C) to our matrix integral equation (3.5), and we give a 'ready to use recipe'.

Recipe for finding the spectral curve of the matrix integral equation (3.5):

- Find $2(t_{\max} - t_{\min} + 1)$ analytical functions of a variable z (z belongs to a Riemann surface $\mathcal{L}$). There is one such analytical function for each matrix of the chain, plus one additional function at the end of the chain. Let us call them

$$\hat{X}(z,t), \qquad t = t_{\min}, \dots, t_{\max}, \qquad \hat{Y}(z,t'), \qquad t' = t_{\min} - \tfrac{1}{2}, \dots, t_{\max} - \tfrac{1}{2}. \tag{4.4}$$

Those functions are completely determined by the following constraints:

(1) Those functions must obey the following system of equations $\forall z$:

$$\begin{aligned} &\hat{X}(z, t' + \tfrac{1}{2}) - \hat{X}(z, t' - \tfrac{1}{2}) = U'_{t'}(\hat{Y}(z,t')), \qquad \forall t' = t_{\min} + \tfrac{1}{2}, \dots, t_{\max} - \tfrac{1}{2} \\ &\hat{Y}(z, t + \tfrac{1}{2}) - \hat{Y}(z, t - \tfrac{1}{2}) = \ln q - V'_t(\hat{X}(z,t)), \qquad \forall t = t_{\min}, \dots, t_{\max} - 1. \end{aligned} \tag{4.5}$$

(2) There exists a point in $\mathcal{L}$, which we call $\infty \in \mathcal{L}$, such that $\hat{X}(z, t_{\min})$ has a simple pole at $z \to \infty$, and we have

$$\hat{Y}\left(z, t_{\min} - \frac{1}{2}\right) \sim \frac{N}{\hat{X}(z, t_{\min})}. \tag{4.6}$$

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(3) There exist points $\zeta_i \in \mathcal{L}$, such that $\hat{X}(\zeta_i, t_{\max})$ is an eigenvalue of $M_{t_{\max}}$:

$$\hat{X}(\zeta_i, t_{\max}) = h_i(\nu), \tag{4.7}$$

and $\hat{Y}(z, t_{\max} - \frac{1}{2})$ has simple poles at the points ζ_i and behaves like

$$\hat{Y}\left(z, t_{\max} - \frac{1}{2}\right) \sim \frac{1}{\hat{X}(z, t_{\max}) - h_i(\nu)}. \tag{4.8}$$

(4) Define for $t = t_{\min}, \ldots, t_{\max}$:

$$W(x,t) = \operatorname*{Res}_{z\to\infty} \frac{\hat{Y}(z, t-1/2)\,\mathrm{d}\hat{X}(z,t)}{x - \hat{X}(z,t)}, \tag{4.9}$$

and call it 'the resolvent' of the matrix M_t. We require that $\forall t$, $W(x,t)$ is analytical in a vicinity of $x \to \infty$ and behaves like

$$W(x,t) \underset{x\to\infty}{\sim} \frac{N}{x}, \tag{4.10}$$

and $W(x,t)$ can be analytically continued to

$$\mathbb{C} \setminus \mathcal{D}(t) \tag{4.11}$$

where we recall that $\mathcal{D}(t) \subset \mathbb{R}$ is a compact region of $\mathbb{R}$.

(5) Typically, $W(x,t)$ may have branchcuts or isolated singularities, like poles or log singularities. The set of points at which $W(x,t)$ is not analytical is called the support:

$$\mathrm{supp}(t) = \{x, W(x,t) \text{ not analytical}\}. \tag{4.12}$$

The interior of $\mathrm{supp}(t)$ is called 'the liquid region' (it contains the cuts, it excludes the isolated singularities). We have $\mathrm{supp}(t) \subset \mathcal{D}(t)$ and

$$\overset{\circ}{\mathrm{supp}}\,(t) \subset \mathcal{D}(t). \tag{4.13}$$

If $\mathcal{D}(t) = \cup_i [a_i(t), b_i(t)]$, we require that $\forall i$: $\overset{\circ}{\mathrm{supp}}\,(t) \cap [a_i(t), b_i(t)]$ is connected.

(6) If the domain $\mathcal{D}(t)$ at time t is a disconnected union $\mathcal{D}(t) = \cup_{i=1}^{m_t} [a_i(t), b_i(t)]$, we require that $\forall i = 1, \ldots, m_t$:

$$\frac{1}{2\mathrm{i}\pi} \oint_{[a_i(t), b_i(t)]} W(x,t)\,\mathrm{d}x = n_i(t), \tag{4.14}$$

where the integration contour surrounds the interval $[a_i(t), b_i(t)]$ in the $\hat{X}(z,t)$ plane in the clockwise direction.

Finding functions satisfying all those requirements for a general domain, with general weights $\alpha(t'), \beta(t')$ is a difficult problem. But for not too complicated domains and weights, some simplifications may occur, and we will see many examples of explicit solutions below.

From now on, let us assume that we have found the functions $\hat{X}$ and $\hat{Y}$ satisfying all the requirements. Once we have found a solution to this problem, i.e. found the functions $\hat{X}(z,t)$ and $\hat{Y}(z,t')$, we define the spectral curves:

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Definition 4.1. *The spectral curve at time t is the pair of functions:*

$$\mathcal{S}_t = (\hat{X}(\cdot, t), \hat{Y}(\cdot, t - \tfrac{1}{2})). \tag{4.15}$$

Remark 4.1. Because of equation (4.5), the following 2-forms in $T^*\mathbb{C} \wedge T^*\mathbb{C}$, restricted to the spectral curve, are equal:

$$\begin{aligned} \mathrm{d}\hat{X}(z,t) \wedge \mathrm{d}\hat{Y}(z, t - \tfrac{1}{2}) &= \mathrm{d}\hat{X}(z,t) \wedge \mathrm{d}\hat{Y}(z, t + \tfrac{1}{2}) \\ &= \mathrm{d}\hat{X}(z,t+1) \wedge \mathrm{d}\hat{Y}(z, t + \tfrac{1}{2}) \end{aligned} \tag{4.16}$$

and therefore the spectral curves $\mathcal{S}_t$ and $\mathcal{S}_{t+1}$ are symplectically equivalent:

$$\mathcal{S}_t \equiv \mathcal{S}_{t+1}. \tag{4.17}$$

4.3. Symplectic invariants and topological expansion

The symplectic invariants $F_g(\mathcal{S})$ were introduced in [27]. To any spectral curve $\mathcal{S}$, one can associate, by simple algebraic computations, an infinite sequence of complex numbers $F_g(\mathcal{S})$, $g = 0, 1, 2, 3, \ldots$. We recall their definition in appendix B.

One of their main properties is that, if two spectral curves $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are symplectically equivalent, then we have $F_g(\mathcal{S}) = F_g(\tilde{\mathcal{S}})$.

In our case, because of equation (4.17), we have

$$F_g(\mathcal{S}_t) = F_g(\mathcal{S}_{t+1}), \tag{4.18}$$

and therefore $F_g(\mathcal{S}_t)$ is independent of t and the $F_g(\mathcal{S}_t)$s are conserved quantities.

It was proved in [33], for any chain of matrices, and here we apply it to our case, that:

Theorem 4.1.

$$\ln \mathcal{Z} = \sum_{g=0}^{\infty} F_g(\mathcal{S}_t) \tag{4.19}$$

where the right-hand side is independent of t.

This theorem holds order by order in some appropriate formal large parameter expansion. We will see examples below, where the formal parameter can be the size of the system, or $\ln q$, or $\alpha, \ldots$.

4.3.1. Arctic circle. For most interesting applications, there is some 'large parameter Q' in our problem (typically the size of the domain $\mathcal{D}$, or $Q = 1/\ln q$, or sometimes other parameters), such that the spectral curve scales like Q, typically:

$$\mathcal{S} = Q\mathcal{S}_\infty + o(Q) \tag{4.20}$$

(where we write $\lambda\mathcal{S} = (x, \lambda y)$ for a spectral curve $\mathcal{S} = (x, y)$).

The large Q spectral curve $\mathcal{S}_\infty$ was already computed in many works and in particular by Kenyon–Okounkov–Sheffield [48] who found the limit shape of the liquid region.

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From the homogeneity property of F_gs (see [27] and appendix B) we have $F_g(\mathcal{S}) = Q^{2-2g}F_g(Q^{-1}\mathcal{S})$, i.e. we find a large Q expansion:

$$\ln \mathcal{Z} = \sum_{g=0}^{\infty} Q^{2-2g} F_g(Q^{-1}\mathcal{S}). \tag{4.21}$$

Such an expansion is not very useful if $F_g(Q^{-1}\mathcal{S})$ depends on Q.

In fact, in many examples related to TASEP and plane partitions, we find that the spectral curve $Q^{-1}\mathcal{S}$ depends on Q, but up to a symplectic transformation we have miraculously (this happens, for instance, in the matrix model considered in [26])

$$Q^{-1}\mathcal{S} \equiv \mathcal{S}_\infty \qquad \text{modulo symplectomorphisms} \tag{4.22}$$

which implies

$$F_g(Q^{-1}\mathcal{S}) = F_g(\mathcal{S}_\infty), \tag{4.23}$$

and, in that case, $F_g(Q^{-1}\mathcal{S}) = F_g(\mathcal{S}_\infty)$ is independent of Q.

This miracle is deeply related to the structure of the self-avoiding particle model partition function, and with the so-called 'arctic circle phenomenon', i.e. the fact that the system freezes beyond a certain size [42]. This can also be related to the fact that we have some arbitrariness in choosing the potentials V_t and we could choose some potentials V_t which depend on Q in an appropriate way such that the spectral curve $Q^{-1}\mathcal{S}$ would not depend on Q. Although it is doable in theory, finding the corresponding V_ts seems horrendous. A rather explicit example of this phenomenon was discussed in [26].

Only in the case where we have this 'arctic circle phenomenon' do we have

$$\ln \mathcal{Z} = \sum_{g=0}^{\infty} Q^{2-2g} F_g(\mathcal{S}_\infty), \tag{4.24}$$

where the spectral curve $\mathcal{S}_\infty$ is the curve derived from the Harnack curve of Kenyon–Okounkov–Sheffield [48].

In that case, we can compute the large Q expansion of our self-avoiding particle model, to all orders in Q, not only the large Q leading order limit.

4.4. Reduced matrix model

For a fully general domain $\mathcal{D}$, there are as many equations of equation (4.5) to solve as the size of the domain, but, when the domain has very few defects, many of those equations simplify considerably, and we may consider a reduced problem.

Notice that every time $\bar{\mathcal{D}}(t) = \emptyset$, we may choose $V_t(x) = 0$ for all x, and thus $V_t' = 0$. Therefore, it is possible to simplify dramatically the equations determining the spectral curve.

Notice also that there can be many times t at which $\bar{\mathcal{D}}(t) \neq \emptyset$, but since particles can only follow lines with slopes $\pm\frac{1}{2}$, many places (the shadow of $\mathcal{D}$) are never visited. In other words, we may replace the defects $\bar{\mathcal{D}}$ by the minimal defect $\bar{\mathcal{D}}_0$ of $\mathcal{D}$, and since in general $\bar{\mathcal{D}}_0$ is a very small subset of $\bar{\mathcal{D}}$, this allows us to simplify the problem.

Only the minimal defect $\bar{\mathcal{D}}_0(t)$ needs to be specified.

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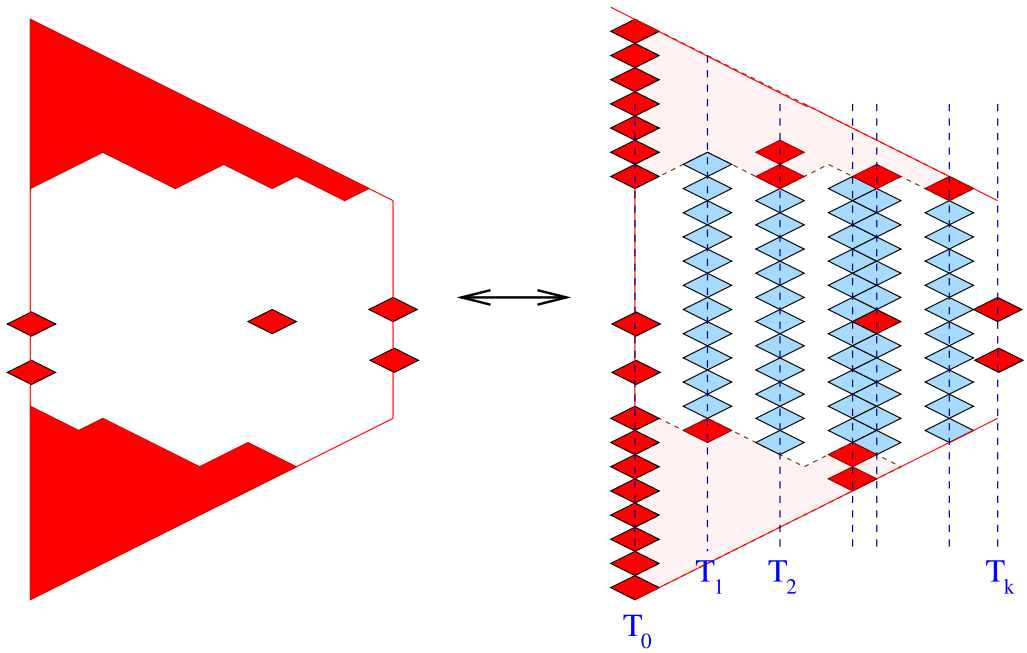

Figure 10. The left figure represents the domain $\mathcal{D}$ and its defect $\bar{\mathcal{D}}$ in red. The right figure represents the minimal defect $\bar{\mathcal{D}}_0$ in red. If there are $k+1$ times $T_0, \dots, T_k$ containing red lozenges, then the reduced matrix model is a $2k+1$ matrix model. The potentials of the reduced model must be such that $\mathrm{e}^{-V_{T_i}} = 0$ at the center of the red lozenges and $\mathrm{e}^{-V_{T_i}} = 1$ at the blue ones. The values of $\mathrm{e}^{-V_{T_i}}$ elsewhere are arbitrary.

The ordered times at which $\bar{\mathcal{D}}_0(t) \neq \emptyset$ are called

$$\{T_0, T_1, \dots, T_k\} \subset \{t_{\min}, \dots, t_{\max}\}, \tag{4.25}$$

where among those T_js we include the extremities $t_{\min}$ and $t_{\max}$:

$$T_0 = t_{\min}, \qquad T_k = t_{\max}. \tag{4.26}$$

The spectral curve equations (4.5) can be rewritten:

$$\begin{aligned} \hat{X}(z, T_{i+1}) - \hat{X}(z, T_i) &= \sum_{t'=T_i+1/2}^{T_{i+1}-1/2} U'_{t'}(\hat{Y}(z, T_{i+1} - \tfrac{1}{2}) + (t' - T_{i+1} + \tfrac{1}{2}) \ln q) \\ \hat{Y}(z, T_{i+1} - \tfrac{1}{2}) - \hat{Y}(z, T_i - \tfrac{1}{2}) &= (T_{i+1} - T_i) \ln q - V'_{T_i}(\hat{X}(z, T_i)). \end{aligned} \tag{4.27}$$

We thus define

$$\hat{X}_i(z) = \hat{X}(z, T_i), \qquad \hat{Y}_i(z) = \hat{Y}(z, T_i - \tfrac{1}{2}) \tag{4.28}$$

and the potentials

$$\tilde{U}'_i(y) = \sum_{t'=T_{i-1}+1/2}^{T_i-1/2} U'_{t'}(y + (t' - T_i + \tfrac{1}{2}) \ln q) \tag{4.29}$$

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i.e.

$$\mathrm{e}^{-\tilde{U}_i(y)} = \prod_{t'=T_{i-1}+1/2}^{T_i-1/2} (\alpha(t')q^{-(1/2)(t'-T_i+1/2)}\mathrm{e}^{-(y/2)} + \beta(t')q^{(1/2)(t'-T_i+1/2)}\mathrm{e}^{y/2}). \tag{4.30}$$

The loop equations (4.27) are equivalent to

$$\begin{aligned} \hat{X}_{i+1}(z) - \hat{X}_i(z) &= \tilde{U}'_{i+1}(\hat{Y}_{i+1}(z)), \qquad \forall i = 0, \ldots, k-1 \\ \hat{Y}_{i+1}(z) - \hat{Y}_i(z) &= (T_{i+1} - T_i)\ln q - V'_{T_i}(\hat{X}_i(z)), \qquad \forall i = 0, \ldots, k-1. \end{aligned} \tag{4.31}$$

One recognizes that these equations are exactly the equations of the spectral curve of another matrix model, which is a chain of matrices, but with only $2k+1$ matrices instead of $2(t_{\max} - t_{\min}) - 1$, and we get the following result:

Theorem 4.2. *The matrix model partition function can be rewritten:*

$$\begin{aligned} \mathcal{Z} = \int_{H_N(\mathcal{C})} \mathrm{d}M_0 \int_{(H_N)^k} \prod_{i=1}^{k-1} \mathrm{d}M_i \int_{(iH_N)^k} \prod_{i=1}^{k} \mathrm{d}R_i \\ \times \prod_{i=0}^{k-1} \mathrm{e}^{-\mathrm{Tr}V_{T_i}(M_i)} q^{(T_{i+1}-T_i)\mathrm{Tr}M_i} \prod_{i=1}^{k} \mathrm{e}^{-\mathrm{Tr}\tilde{U}_i(R_i)} \prod_{i=1}^{k} \mathrm{e}^{\mathrm{Tr}R_i(M_i - M_{i-1})} \end{aligned} \tag{4.32}$$

i.e. a chain of $2k+1$ matrices, instead of a chain of $2(t_{\max} - t_{\min}) - 1$ as we had before.

Notice that, knowing $\hat{X}_i$, we can reconstruct $\hat{X}(z,t)$ for all times $t \in [T_i, T_{i+1}]$:

$$\hat{X}(z,t) = \hat{X}_i(z) + \sum_{t'=T_i+1/2}^{t-1/2} U'_{t'}(\hat{Y}(z, T_{i+1} - \tfrac{1}{2}) + (t' - T_{i+1} + \tfrac{1}{2})\ln q), \tag{4.33}$$

and if $t' \in [T_i + \frac{1}{2}, T_{i+1} - \frac{1}{2}]$:

$$\hat{Y}(z,t') = \hat{Y}_{i+1}(z) + (T_{i+1} - \tfrac{1}{2} - t')\ln q. \tag{4.34}$$

5. Liquid region

Our spectral curve is thus given by a collection of functions $\hat{X}(z,t)$ and $\hat{Y}(z, t+\frac{1}{2})$, defined for all integer times t. We assume that the potentials V_t are non-zero only for times $t \in \{T_1, T_2, \ldots, T_k\}$, and the domain at time T_i is the union of m_i intervals:

$$\mathcal{D}(T_i) = \bigcup_{j=1}^{m_i} [a_{i,j}, b_{i,j}], \tag{5.1}$$

with filling fractions $n_{i,j}$, i.e. we require that there are $n_{i,j}$ particles in $[a_{i,j}, b_{i,j}]$.

This implies that the potentials $V(x, T_i)$ are of the form

$$V'(x, T_i) = f'(x, T_i) + \sum_{j=1}^{m_i} \psi(x - a_{i,j}) - \psi(x - b_{i,j}), \tag{5.2}$$

where $f'(x, T_i)$ is some analytical function and $\psi = \Gamma'/\Gamma$ is the digamma function, see appendix A, equation (A.7).

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We shall now describe the spectral curve in a more physical way, made of two regions: a liquid region and a frozen region. In the frozen region, the resolvents $W(x,t)$ have poles, i.e. the densities $\rho(x,t)$ are Dirac combs, which according to section 2.2.1 correspond to fixed ('frozen') eigenvalues for the matrix M_t. The liquid region is the region where the resolvents have other types of singularities, namely algebraic cuts. The limit between the liquid and solid region is often called the 'arctic circle' [42, 48]. All this shall be made more precise in the upcoming paragraphs.

5.1. Interpolation to real times

It is, in fact, better to enlarge these definitions to all times (not necessarily integers or half-integers) by interpolation:

$$\forall t' \in]T_{i-1}, T_i], \qquad \hat{Y}(z,t') = \hat{Y}(z, T_i - \tfrac{1}{2}) + (T_i - \tfrac{1}{2} - t') \ln q \tag{5.3}$$

and we write

$$y(z,t) = \mathrm{e}^{\hat{Y}(z,t)} \tag{5.4}$$

i.e.

$$\forall t' \in]T_{i-1}, T_i], \qquad y(z,t') = y(z, T_i - \tfrac{1}{2}) q^{T_i - 1/2 - t'} = y_i(z) q^{T_i - 1/2 - t'}. \tag{5.5}$$

The function $y(z,t')$ is discontinuous at time T_i:

$$\frac{y_i(z)}{q^{T_i - T_{i-1}} y_{i-1}(z)} = \mathrm{e}^{-V'(\hat{X}(z,T_i),T_i)}. \tag{5.6}$$

Similarly we define for $t \in]T_{i-1}, T_i]$

$$\hat{X}(z,t) = \hat{X}(z,T_i) - \frac{1}{2} \sum_{t'=t+1/2}^{T_i - 1/2} \frac{q^{t'} \alpha(t')/\beta(t') - y_i(z) q^{T_i - 1/2}}{q^{t'} \alpha(t')/\beta(t') + y_i(z) q^{T_i - 1/2}}. \tag{5.7}$$

If α/β is constant over some interval, then we can find explicitly the interpolating function:

$$\hat{X}(z,t) = \hat{X}(z,T_i) + \frac{t - T_i}{2} + \psi_q\left(-\frac{\alpha}{\beta}\frac{1}{y_i(z)} q^{t+1-T_i}\right) - \psi_q\left(-\frac{\alpha}{\beta}\frac{q}{y_i(z)}\right) \tag{5.8}$$

where $\psi_q(x)$ is the 'quantum digamma' function, see appendix A, equation (A.12).

If $q = 1$, we have

$$\hat{X}(z,t) = \hat{X}(z,T_i) + \frac{t - T_i}{2} \frac{(\alpha/\beta) - y_i(z)}{(\alpha/\beta) + y_i(z)}. \tag{5.9}$$

Notice that this expression depends linearly on t.

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5.2. Densities of particles

Consider the density of particles at time t (in section 2.2.1, we have seen that these densities are related to the profile of the plane partitions):

$$\bar{\rho}(x,t) = \sum_{i=1}^{N} \langle \delta(x - h_i(t)) \rangle = \langle \mathrm{Tr}\delta(x - M_t) \rangle \,. \tag{5.10}$$

We also consider their Stieltjes transforms, i.e. the resolvents

$$\bar{W}(x,t) = \sum_{i=1}^{N} \left\langle \frac{1}{x - h_i(t)} \right\rangle = \left\langle \mathrm{Tr}\frac{1}{x - M_t} \right\rangle = \int_{\mathbb{R}} \frac{\bar{\rho}(x',t)\,\mathrm{d}x'}{x - x'}. \tag{5.11}$$

Let us assume that our model has a formal large Q expansion in some parameter Q, it follows from the general solution of the chain of matrices [33], that one can write

$$\bar{W}(x,t) = \sum_{g=0}^{\infty} Q^{1-2g}\, W^{(g)}(x,t), \qquad \bar{\rho}(x,t) = \sum_{g=0}^{\infty} Q^{1-2g}\, \rho^{(g)}(x,t) \tag{5.12}$$

where the first term $W^{(0)}(x,t) = W(x,t)$ is given by equation (4.9):

$$W^{(0)}(x,t) = W(x,t) = \mathrm{Res}_{z\to\infty} \frac{\hat{Y}(z,t-1/2)\,\mathrm{d}\hat{X}(z,t)}{x - \hat{X}(z,t)}, \tag{5.13}$$

and the density is the discontinuity of $W(x,t)$ across $\mathrm{supp}(t)$:

$$\frac{W(x - \mathrm{i}0, t) - W(x + \mathrm{i}0, t)}{2\mathrm{i}\pi} = \rho(x,t). \tag{5.14}$$

All the higher corrections $W^{(g)}(x,t)$ for $g \geq 1$ were also computed in [33], and they are the correlators $W_1^{(g)}$ of the symplectic invariants (see appendix B or [27]) of the spectral curve $\mathcal{S}_t$. In other words, if we know the spectral curve $\mathcal{S}_t$ at time t, there is a simple recursive algebraic algorithm to compute all the corrections, of any correlation function, to all orders in Q. We shall not detail this here.

Remark 5.1. Let us mention that, if the arctic circle property does not hold (this may be the case for complicated domains), then the spectral curve $\mathcal{S}_t$ may depend on Q, and thus all the $W^{(g)}$ depend on Q. In other words, equation (5.12) is true as an equality of a formal series, but it is not really a large Q expansion. In particular, $W^{(0)}$ is not necessarily the large Q limit. When the arctic circle property holds, then the $W^{(g)}$ are independent of Q, and we really have a large Q expansion. We will see examples where this holds below in section 7.

Let us now study the leading term.

$W(x,t)$ can have various sorts of singularities: it can have isolated singularities, like poles, or log singularities, and it can have branchcuts. In terms of the density $\rho(x,t)$, the poles of $W(x,t)$ correspond to δ-Dirac distributions and cuts correspond to smooth positive densities. Log singularities of $W(x,t)$ correspond to discontinuous jumps of the density.

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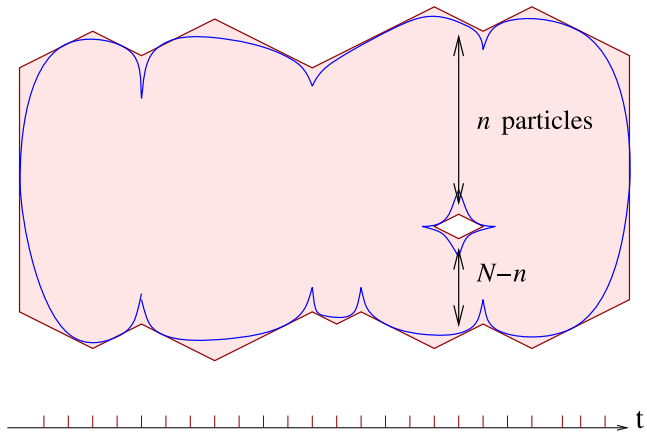

Figure 11. Given a domain $\mathcal{D}$, we compute its spectral curve $\mathcal{S}_t$ for all times t. From the spectral curves we compute the densities $\rho(x,t)$, whose supports are contained in $\mathcal{D}(t)$. The interior of the support is called the 'liquid region'. The liquid region is delimited by the envelope, which is a curve $X_c(t)$ contained in $\mathcal{D}$. The points on the envelope are branchpoints of the functions $\hat{X}(z,t)$, i.e. zeroes of $\mathrm{d}\hat{X}(z,t)$. We shall see below that the envelope has some special properties of tangency to the domain $\mathcal{D}$.

Notice that equation (4.10) implies that $\rho(x,t)$ is normalized, with total weight N:

$$\int_{\mathbb{R}} \rho(x,t)\,\mathrm{d}x = N, \tag{5.15}$$

but, if we exclude the isolated singularities, i.e. if we integrate only in the liquid region (the interior of the support), we have

$$\int_{\overset{\circ}{\mathrm{supp}}(t)} \rho(x,t)\,\mathrm{d}x \leq N. \tag{5.16}$$

Notice also that, if $\mathcal{D}(t)$ is disconnected, i.e. a union of intervals $\mathcal{D}(t) = \cup_{i=1}^{m_t}[a_i(t), b_i(t)]$, we have required that the filling fraction is fixed:

$$\int_{a_i(t)}^{b_i(t)} \rho(x,t)\,\mathrm{d}x = n_i(t). \tag{5.17}$$

5.3. The envelope of the liquid region

The liquid region is the interior of the support of the density, i.e. it excludes all the isolated singularities of $W(x,t)$ and it contains only the cuts.

The complement of the liquid region in $\mathcal{D}$ is called the 'solid' region, it contains the isolated singularities of $W(x,t)$ and possibly their accumulation points.

The boundary of the liquid region is called the envelope of the liquid region. It is given by the branchpoints.

The branchpoints $z_c(t)$ at time t, are zeroes of the differential form $\mathrm{d}\hat{X}$, i.e. they are the points where the density has a vertical tangent:

$$\mathrm{d}\hat{X}(z_c(t),t) = 0 \tag{5.18}$$

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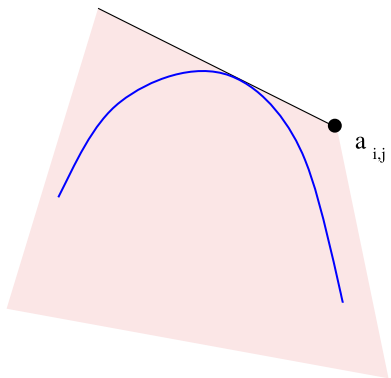

Figure 12. A point of the envelope at which the tangent has a slope $\pm 1/2$ is necessarily on the boundary of the shadow of $\mathcal{D}$. This means that there is a point $a_{i,j}$ of the defect $\bar{\mathcal{D}}$ on the tangent.

where d is the differential with respect to z. Not all branchpoints are boundaries of the liquid region. Let us consider only those which are on the envelope. The boundary of the liquid region is thus at $X_c(t)$ such that

$$X_c(t) = \hat{X}(z_c(t), t). \tag{5.19}$$

By definition, the envelope $X_c(t)$ is a curve contained in $\mathcal{D}$. Let us study some of its properties.

5.3.1. Tangency

Theorem 5.1. *If a point of the envelope has a tangent of slope $\pm 1/2$, then this point must be on the boundary of the shadow of $\mathcal{D}$.*

Proof. Let $\bar{t} \in]T_{i-1}, T_i]$ be such that the point $(\bar{t}, X_c(\bar{t}))$ is a point of the envelope at which the tangent has slope $+1/2$ (resp. $-1/2$). Let $\alpha = z_c(\bar{t})$, i.e. we have $X_c(\bar{t}) = \hat{X}(\alpha, \bar{t})$.

Since on the envelope we have $\mathrm{d}\hat{X} = 0$, we see that at all times t we have

$$\frac{\mathrm{d}X_c(t)}{\mathrm{d}t} = \left.\frac{\partial \hat{X}(z,t)}{\partial t}\right|_{z=z_c(t)} \tag{5.20}$$

and therefore we must have

$$\frac{\partial \hat{X}(\alpha,t)}{\partial t} = \frac{1}{2} \qquad \left(\text{resp.} -\frac{1}{2}\right), \tag{5.21}$$

and, from equation (5.7), this implies that

$$y_i(\alpha) = 0 \qquad (\text{resp. } y_i(\alpha) = \infty). \tag{5.22}$$

Then, if $y_i(\alpha) = 0$ (resp. ∞), we see from equation (5.7) that $\forall t \in]T_{i-1}, T_i]$:

$$\hat{X}(\alpha,t) = X_i(\alpha) + \frac{t - T_i}{2} \qquad \left(\text{resp. } X_i(\alpha) - \frac{t - T_i}{2}\right) \tag{5.23}$$

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and in particular at $t = \bar{t}$, we have

$$X_c(t) = X_i(\alpha) + \frac{\bar{t} - T_i}{2} \qquad \left(\text{resp. } X_i(\alpha) - \frac{\bar{t} - T_i}{2}\right), \tag{5.24}$$

which means that the tangent to the envelope at point $\bar{t}$ passes through the point $a = \hat{X}(\alpha, T_i)$ at time T_i:

$$a = \hat{X}(\alpha, T_i). \tag{5.25}$$

Then, notice that $y_i(\alpha) = \mathrm{e}^{\hat{Y}(\alpha, T_i - 1/2)}$ can be equal to 0 or ∞ only if $\hat{Y}(z, T_i - \frac{1}{2})$ has a singularity at $z = \alpha$. Because of equation (4.31), we see that singularities of $\hat{Y}(z, T_i - \frac{1}{2})$ can occur only if $V'(a, T_i)$ has a singularity.

Remember that $V'(x, t)$ has no singularity on $\mathcal{D}(t)$, it has singularities only at the defects $\bar{\mathcal{D}}(t)$, and thus this means that the point (a, T_i) belongs to the defect $\bar{\mathcal{D}}$. But we know that the point $(X_c(\bar{t}), t)$ belongs to $\mathcal{D}$, and the slope between the points (a, T_i) and $(X_c(\bar{t}), t)$ is equal to $\pm\frac{1}{2}$, i.e. the point $(X_c(\bar{t}), t)$ is in the shadow of the defect (a, T_i). □

5.3.2. Local convexity. In the classical case, where $q^t\alpha(t)/\beta(t)$ is constant, we have a convexity property:

Theorem 5.2. *If $q^t\alpha/\beta$ is constant in time, the liquid region is locally convex.*

Proof. The density at time t must have a real support, starting at a branchpoint, i.e. at $X_c(t)$. Consider that, up to a reparameterization of z, and up to trivial translations in t and $\hat{X}$, we can write locally, near the branchpoint point $X_c(0)$:

$$\hat{X}(z, t) = X_c + X_t t + \tfrac{1}{2} X_{z,z} z^2 + X_{z,t} z t + \cdots \tag{5.26}$$

and up to a rescaling of z we assume $X_{z,z} = 1$. Notice from equation (5.9) that the assumption $q^t\alpha/\beta$ constant guarantees that $\hat{X}(z, t)$ is linear in t; there is no t^2 term.

The branchpoint at time t is at $z_c(t)$ given by $\mathrm{d}\hat{X}(z, t)/\mathrm{d}z = 0$, i.e.

$$z_c(t) = -X_{z,t} t + \cdots \tag{5.27}$$

i.e. the envelope $X_c(t) = \hat{X}(z_c(t), t)$ is, up to order 2:

$$X_c(t) = X_c + X_t t - \frac{X_{z,t}^2}{2} t^2 + \mathrm{O}(t^2). \tag{5.28}$$

Our assumption on the reality of the supports implies that $X_c(t) \in \mathbb{R}$ for all t, and therefore X_t and $X_{z,t}^2$ must be real quantities.

The point $\bar{z}$ such that $X(z, t) = X(\bar{z}, t)$ is

$$\bar{z} = 2z_c(t) - z + \cdots. \tag{5.29}$$

The function $y(z)$ is then given by solving equation (5.7):

$$\begin{aligned} y(z, t') &= q^{t' - T_i + 1/2} \frac{\alpha(t')}{\beta(t')} \frac{1 - 2(X_t + X_{z,t} z + \cdots)}{1 + 2(X_t + X_{z,t} z + \cdots)} \\ &= q^{t' - T_i + 1/2} \frac{\alpha(t')}{\beta(t')} \frac{1 - 2X_t}{1 + 2X_t} \left(1 - \frac{4X_{z,t}\, z}{1 - 4X_t^2} + \cdots\right) \end{aligned} \tag{5.30}$$

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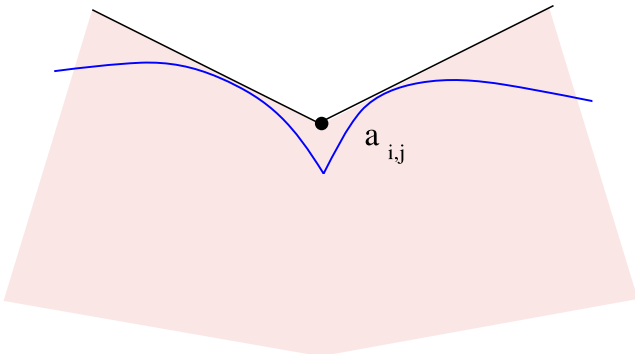

Figure 13. The envelope has a cusp at $t = T_i$.

and therefore

$$\frac{y(z,t')}{y(\bar{z},t')} \sim 1 - \frac{8X_{z,t}}{1-4X_t^2}(z - z_c(t)) + \cdots \tag{5.31}$$

and the density $\rho(x,t) = (1/2\mathrm{i}\pi)\ln(y(\bar{z})/y(z))$, with $x = \hat{X}(z,t)$ is, up to order 2:

$$\rho(x,t) = \frac{4\,X_{z,t}}{\mathrm{i}\pi(1-4X_t^2)}(z - z_c(t)) + \cdots = \frac{4\sqrt{2}}{\pi(1-4X_t^2)}\sqrt{X_{z,t}^2\,(X_c(t) - x)} + \cdots. \tag{5.32}$$

This implies that $X_{z,t}^2\,(X_c(t) - x) \geq 0$ in the liquid region. If the liquid region is above $X_c(t)$, we must have $X_{z,t}^2 < 0$, and if the liquid region is below $X_c(t)$, we must have $X_{z,t}^2 > 0$. In all cases, the liquid region is locally convex. □

Remark 5.2. If we are not in the classical case, i.e. if $q^t\alpha/\beta$ is not constant, then $\hat{X}(z,t)$ is not linear in t, there may be a second derivative $X_{t,t} \neq 0$, and the envelope is locally

$$X_c(t) = X_c + X_t t + \frac{X_{t,t} - X_{z,t}^2}{2}t^2 + \mathrm{O}(t^2) \tag{5.33}$$

and, although $X_{z,t}^2$ has a constant sign, $X_{t,t} - X_{z,t}^2$ can have any sign and the envelope is not necessarily convex.

5.3.3. Cusps. Notice that $\hat{Y}(z,t)$ is discontinuous at $t = T_i$, i.e. there is a left value and a right value. This implies that $\mathrm{d}X_c/\mathrm{d}t$ has a right tangent and a left tangent, and thus generically, the envelope has cusps at $t = T_i$.

5.3.4. Genus and holes. Condition 6 of section 4.2, i.e. equation (4.14), tells us that the spectral curve must have at least as many nontrivial cycles as the number of holes in our domain, i.e. the genus of our spectral curve is at least the number of holes, and generically it coincides with the number of holes.

In particular, if we consider a simply connected domain $\mathcal{D}$ with no holes, we can expect to have genus 0, which means that the functions $\hat{X}(z,t)$ and $\hat{Y}(z,t)$ are analytical functions of a complex variable z. The Riemann surface in which z lives can be chosen as a domain of the complex plane.

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6. Asymptotic regimes

Of course, our model depends on so many parameters (shape of the domain, coefficients $\alpha(t'), \beta(t')$, potentials V_t, etc) that it is almost impossible to classify all possible asymptotic regimes. However, a few asymptotic regimes are more relevant for applications in statistical physics or algebraic geometry.

We classify them into two kinds:

- Macroscopic asymptotic regimes, which describe the behavior of the partition function, i.e. the statistics of our self-avoiding particle model at the scale of the size of the domain. Typically, we shall consider asymptotic behaviors for large sizes or for small $\ln q$.
- Microscopic asymptotic regimes, which describe the statistics of our self-avoiding particle model particles in a very small region of the domain, typically the behavior in the bulk, or near the edges, especially near special points, like near the envelope, or near cusps of the envelope.

Let us comment on a few of them and let us emphasize that our method, works for all possible asymptotic regimes, and it gives not only the leading order asymptotics, it also gives the full asymptotic expansion to all orders.

6.1. Classical case $q = 1$ and large size asymptotics

Classical means that we choose $q = 1$. Here, we shall assume that the weights $\alpha(t') = \beta(t') = 1$ are constant, although the general case is doable by the same methods.

Consider a domain $\mathcal{D}$ whose size is $T = t_{\max} - t_{\min}$. Assume that there are k defects and k does not depend on T:

$$k = \mathrm{O}(1). \tag{6.1}$$

The defects are at times $T_i = T\tau_i$, where τ_i are independent of T, and the defect at time T_i is the union of m_i intervals:

$$\bar{\mathcal{D}}(T_i) = \cup_{j=1,\dots,m_i}[Ta_{i,j}, Tb_{i,j}] \tag{6.2}$$

where again $a_{i,j}$ and $b_{i,j}$ do not depend on T.

Assume also that the number of particles N scales like T:

$$N = \mathcal{N}T, \tag{6.3}$$

and the number of particles $n_{i,j}$ in each interval $[a_{i,j}, b_{i,j}]$ scales like T:

$$n_{i,j} = \mathcal{N}_{i,j}T, \tag{6.4}$$

and of course $\forall i$

$$\sum_j \mathcal{N}_{i,j} = \mathcal{N}. \tag{6.5}$$

See figure 14.

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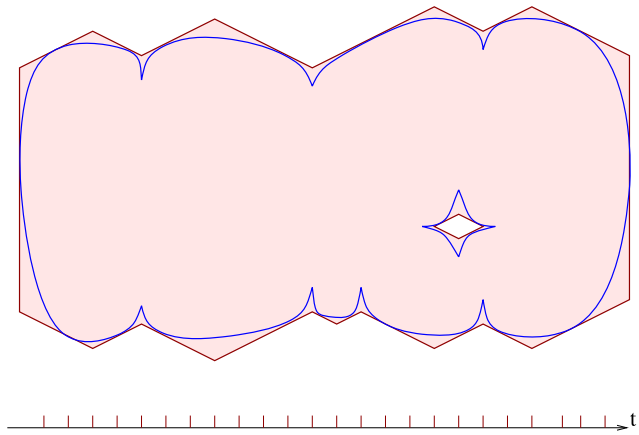

Figure 14. A polygonal domain with k defect at times $T_i = T\tau_i$. The envelope of the spectral curve is the algebraic curve of smallest degree, tangent to all boundaries.

6.1.1. The spectral curve. The spectral curve equations (4.31) are

$$\hat{X}_i - \hat{X}_{i-1} = -\frac{T}{2}(\tau_i - \tau_{i-1}) \tanh \frac{\hat{Y}_i}{2} \qquad \hat{Y}_{i+1} - \hat{Y}_i = -V_i'(\hat{X}_i) \tag{6.6}$$

where we choose the potentials V_i according to equation (3.14):

$$V_i'(X) = f'(X) + \sum_{j=1}^{m_i} \psi(X - Ta_{i,j}) - \psi(X - Tb_{i,j}) \tag{6.7}$$

where f' has no singularity and $\psi = \Gamma'/\Gamma$ is the digamma function (see appendix A).

Moreover, the last matrix of the chain $M_k = M(t_{\max}) = \mathrm{diag}(\lambda_j)$ is not integrated upon. Its spectrum is $\mathcal{D}(t_{\max})$, i.e. it is delimited by the intervals $[Ta_{k,j}, Tb_{k,j}]$. Then, equation (4.8) is

$$\hat{Y}_k \underset{X_k \to \text{e.v. of } M_k}{\sim} \sum_{j=1}^{m_k} \sum_{i=Ta_{k,j}}^{Tb_{k,j}} \frac{1}{X_k - i} = \sum_{j=1}^{m_k} \psi(X_k - Ta_{k,j}) - \psi(X_k - Tb_{k,j}). \tag{6.8}$$

6.1.2. Rescaling. Since our model is defined as a formal power series, and we are looking for the large T expansion, we should try to find the spectral curve in the large T expansion. To that purpose we rescale the variables:

$$\hat{X}_i = Tx_i, \qquad \hat{Y}_i = \ln y_i, \tag{6.9}$$

and we use Stirling's asymptotic formula for the Γ and ψ functions, see appendix A:

$$\psi(x) \sim \ln x - \frac{1}{2x} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nx^{2n}}, \tag{6.10}$$

where B_n are the Bernoulli numbers.

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That gives

$$\frac{y_{i+1}}{y_i} = \prod_{j=1}^{m_i} \frac{x_i - b_{i,j}}{x_i - a_{i,j}} (1 + \text{regular} + \mathrm{O}(1/T)) \tag{6.11}$$

where the $\mathrm{O}(1/T)$ term is, to each power of T^{-1}, a rational function of x_i with poles at $a_{i,j}$ or $b_{i,j}$, plus possibly an arbitrary analytical function with no singularity. We also have

$$y_k \sim \prod_{j=1}^{m_k} \frac{x_k - a_{k,j}}{x_k - b_{k,j}} (1 + \text{regular} + \mathrm{O}(1/T)) \tag{6.12}$$

where the $\mathrm{O}(1/T)$ term is, to each power of T^{-1}, a rational function of x_k with poles at $a_{k,j}$ or $b_{k,j}$, plus possibly an arbitrary analytical function with no singularity.

We also have from equation (6.6)

$$x_i - x_{i-1} = \frac{\tau_i - \tau_{i-1}}{2} \frac{1 - y_i}{1 + y_i} \tag{6.13}$$

i.e.

$$y_i = \frac{\tau_i - \tau_{i-1} - 2(x_i - x_{i-1})}{\tau_i - \tau_{i-1} + 2(x_i - x_{i-1})}. \tag{6.14}$$

6.1.3. Singularities. We see from equations (6.11) and (6.12) that, if $x_i = a_{i,j}$, the ratio y_i/y_{i+1} has a zero and thus either y_i has a zero or y_{i+1} has a pole, or both.

In fact, since y_i and y_{i+1} are multivalued functions of x_i, there might be several points on the spectral curve such that $x_i = a_{i,j}$. In that case, it may happen that $y_i = 0$ corresponds to $x_i = a_{i,j}$ in one sheet and $y_{i+1} = \infty$ corresponds to $x_i = a_{i,j}$ in another sheet. Or, if both y_i has a zero and y_{i+1} has a pole at the same point, this means that $x_i - a_{i,j}$ has a double zero, i.e. $\mathrm{d}x_i$ has a zero, and this means that we are at a branchpoint, i.e. the point where two sheets meet.

Therefore, we shall assume that generically both possibilities occur and we conclude that:

- y_i has a zero when $x_i = a_{i,j}$ or $x_{i-1} = b_{i-1,j}$, and has a pole when $x_i = b_{i,j}$ or $x_{i-1} = a_{i-1,j}$.
  Since the x_is and y_is have only meromorphic singularities, it is natural to look for an algebraic solution, i.e. such that x_i and y_i are meromorphic functions on an algebraic curve $\mathcal{L}$. Indeed, our freedom to choose the potential V_t can be used to eliminate all the singularities except those described above.
  The problem then consists in finding algebraic functions x_i, y_i such that:
- $x_i - x_{i-1} = ((\tau_i - \tau_{i-1})/2)((1 - y_i)/(1 + y_i))$,
- y_i has a simple zero when $x_i = a_{i,j}$ or $x_{i-1} = b_{i-1,j}$, and has a simple pole when $x_i = b_{i,j}$ or $x_{i-1} = a_{i-1,j}$,
- the density measures $1/(2\mathrm{i}\pi) \ln (y_i(\bar{z})/y_i(z))\, \mathrm{d}x_i(z)$ (where $x_i(z) = x_i(\bar{z})$ are the two points on each side of the support) must have their supports in $\mathcal{D}$ (in particular the supports are real). Those reality conditions for an algebraic curve are closely related to the conditions which define the Harnack curve in Kenyon–Okounkov–Sheffield [48].

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6.1.4. The large size limit: Harnack curve. So, we have to find a spectral curve, and its envelope, satisfying many equations (which give the poles and zeroes), as well as many reality conditions. Since we have only meromorphic types of singularities (poles), it is natural to look for real algebraic spectral curves.

It is also natural to look for minimal degree algebraic curves, i.e. having just the number of poles and zeroes implied by our equations and no other poles. In fact, introducing other poles would most probably break the condition on the homology of steepest descent paths. Also we can again use our freedom to choose the potential V_t in order to reduce the degree, so that we eliminate all the singularities except those described above.

An envelope and spectral curve having all the required properties can be found from the work of Kenyon–Okounkov–Sheffield [48], and they proved that their envelope is indeed the limit shape of the liquid domain to leading order at large T.

Their spectral curve is a Harnack curve and the envelope $x_c(\tau)$ is an algebraic curve given by a polynomial equation:

$$P(x_c, \tau) = 0 \tag{6.15}$$

where P is a real polynomial, which has some very special properties.

In particular, it satisfies all the properties we want for our envelope and also, being a maximal Harnack curve means that the area enclosed by the envelope is maximal. Another property is that the genus of P is exactly the number of holes of the domain in the envelope.

6.1.5. Recipe for an algebraic spectral curve. Therefore, let us make the assumption that the spectral curve is the algebraic curve of smallest degree satisfying all the constraints. We shall discuss the consistency of that assumption in section 6.1.7 below.

The problem we have to solve in order to find the spectral curve is then:

- Find a Riemann surface whose genus is equal to the number of holes in $\mathcal{D}$.
- Find two meromorphic functions $u(z)$ and $v(z)$, with only simple poles, and let us denote the set of poles as $\infty_{i,j}$, $i = 0, \ldots, k$, $j = 1, \ldots, m_i$.
- We denote
$$x(z, \tau) = u(z) + \tau v(z), \tag{6.16}$$
and $\forall i$:
$$x_i(z) = u(z) + \tau_i v(z) \tag{6.17}$$
and we require that $\forall i, j$ (cusp condition):
$$\mathrm{Res}_{\infty_{i,j}} x_i(z) = 0. \tag{6.18}$$
- The meromorphic functions $u(z)$ and $v(z)$ must be such that $\forall i, j$:
$$\text{if } i < k \quad \exists z \quad \text{such that } x_i(z) = a_{i,j} \quad \text{and} \quad v(z) = -\tfrac{1}{2}, \tag{6.19}$$
$$\text{if } i > 0 \quad \exists z \quad \text{such that } x_i(z) = a_{i,j} \quad \text{and} \quad v(z) = +\tfrac{1}{2}, \tag{6.20}$$
$$\text{if } i < k \quad \exists z \quad \text{such that } x_i(z) = b_{i,j} \quad \text{and} \quad v(z) = +\tfrac{1}{2}, \tag{6.21}$$
$$\text{if } i > 0 \quad \exists z \quad \text{such that } x_i(z) = b_{i,j} \quad \text{and} \quad v(z) = -\tfrac{1}{2}, \tag{6.22}$$

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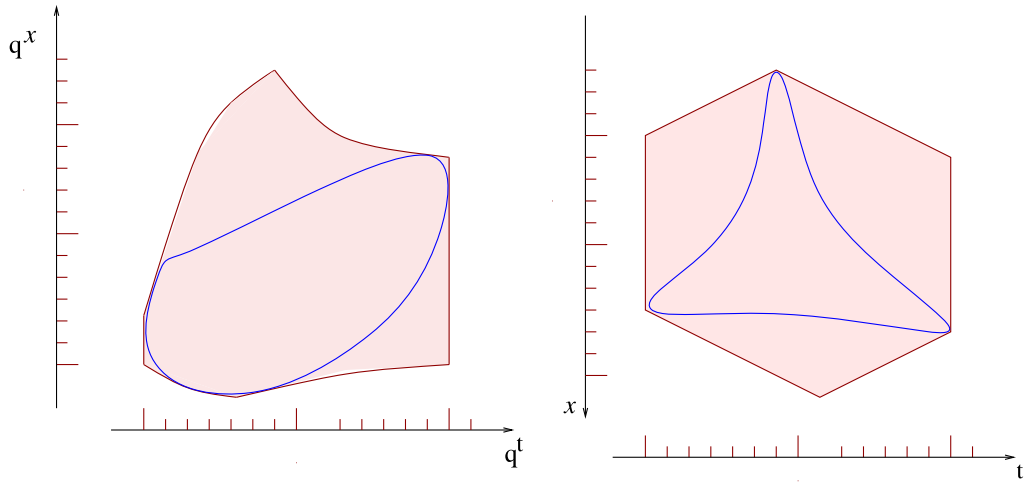

Figure 15. The envelope of the liquid region is the smallest degree algebraic plane curve, tangent to the image of the hexagon under $(x,t) \to (q^x, q^t)$.

- and we have the filling fractions:

$$\frac{1}{2\mathrm{i}\pi}\int_{[a_{i,j},b_{i,j}]} \frac{x_i(z)}{y(z)}\,\mathrm{d}y(z) = \mathcal{N}_{i,j} \tag{6.23}$$

where

$$y(z) = \frac{1-2v(z)}{1+2v(z)}, \tag{6.24}$$

and where we assume that the number of particles in $[a_{i,j}, b_{i,j}]$ is $n_{i,j} = T\mathcal{N}_{i,j}$.

Explicit examples for given domains (hexagon, cardioid) are treated in section 7.

6.1.6. Envelope of the liquid region. The envelope of the liquid region corresponding to the spectral curve above is obtained as follows:

the branchpoints $z_c(\tau)$ are solutions of $\mathrm{d}x(z_c,\tau) = 0 = \mathrm{d}u(z_c) + \tau \mathrm{d}v(z_c)$, and in fact, it is much easier to compute τ as a function of z_c, namely

$$\tau = -\frac{\mathrm{d}u(z_c)}{\mathrm{d}v(z_c)}. \tag{6.25}$$

Then, one computes $x_c(\tau) = x(z_c(\tau),\tau)$, and again it is easier to parameterize this equation by z_c, which gives

$$x_c(\tau) = \begin{cases} \tau = -\dfrac{\mathrm{d}u(z_c)}{\mathrm{d}v(z_c)} \\ x_c = u(z_c) - v(z_c)\dfrac{\mathrm{d}u(z_c)}{\mathrm{d}v(z_c)}. \end{cases} \tag{6.26}$$

We see that the envelope is an algebraic curve, and from section 5.3.1, we know that it is an algebraic curve tangent to the boundary of the shadow of $\mathcal{D}$, with cusps at $\tau = \tau_i$, and whose genus is the same as the number of holes of $\mathcal{D}$.

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Again, explicit examples of envelopes are given in section 7. For example, the envelope of a hexagonal domain is the ellipse tangent to all sides of the hexagon, see figure 16.

Recovering the spectral curve from the envelope:

Assume that we know the envelope $x_c(\tau_c)$, i.e. the algebraic curve tangent to the boundaries of $\mathcal{D}$. Let us explain how to recover the full spectral curve $x(z,\tau)$.

Given the equation of the envelope $x_c(\tau_c)$ (which is a multivalued algebraic function), one can choose locally $z=\tau_c$ as a local parameter. One finds that the spectral curve is (at least in the domain of τ, where τ_c can be chosen as a local parameter)

$$\begin{aligned} &x(z,\tau) = x_c(z) + (\tau - z)v(z), \\ &y(z) = \frac{1-2v(z)}{1+2v(z)}, \qquad \text{where } v(z) = \frac{\mathrm{d}x_c(z)}{\mathrm{d}z}. \end{aligned} \tag{6.27}$$

Therefore, knowing the envelope allows us to recover the full spectral curve, i.e. the functions $x(z,\tau)$ and $y(z)$.

6.1.7. Large size asymptotic expansion. Now, suppose that we have found those functions, and that we have indeed found the correct spectral curve. The spectral curve is then the pair of functions $\mathcal{S}_t = (\hat{X}(z,t), \hat{Y}(z,t))$, i.e., up to a symplectic transformation:

$$\mathcal{S} = (Tu(z), \ln(y(z))), \tag{6.28}$$

and we notice that $u(z)$ and $y(z)$ do not depend on T. Therefore we have $F_g(Tu, \ln y) = T^{2-2g}F_g(u, \ln y)$.

Thus, from theorem 4.1, we have the full large T asymptotic expansion:

Conjecture 6.1. $\ln \mathcal{Z}$ *has the large size* T *expansion:*

$$\ln \mathcal{Z} \sim \sum_{g=0}^{\infty} T^{2-2g} F_g(u, \ln y) \tag{6.29}$$

where the meromorphic functions $u(z)$ *and* $y(z)$ *are computed by solving the requirements of section 6.1.5 and* F_g *is the* gth *symplectic invariant defined in [27].*

Sketch of a possible proof:

A hint of that conjecture is that the leading large T densities are governed by $F_0(u, \ln y)$, and it can be seen that this coincides with the limit shape found by Kenyon–Okounkov–Sheffield [48].

In order to prove this conjecture, relying on the work of [33], we only have to prove that we have indeed found the correct spectral curve, and that our guess (that u and v are algebraic functions of smallest possible degree) is correct.

In principle, this means proving that the integration paths in our self-avoiding particle matrix model are indeed the steepest descent paths for our potentials, but this seems too difficult. For matrix models with polynomial potentials, this is usually proved by the Riemann–Hilbert method of Deift *et al* [17].

Another possible proof is to prove this order by order in some formal parameter, especially if the model tends to a Gaussian matrix integral in the small parameter limit.

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One suggestion is to notice that, in our matrix model, the potentials V_t may depend on T, and our spectral curve can be expected to depend on T, and somehow the Harnack curve gives only the leading term:

$$\mathcal{S}(T) = \mathcal{S}_\infty + \mathrm{O}(1/T) \tag{6.30}$$

where $\mathcal{S}_\infty$ is given by the Harnack curve of [48]. Indeed, we have not really taken into account the $\mathrm{O}(1/T)$ term in equations (6.11) and (6.12).

However, we have some freedom in the choice of V_t and we can choose any V_t provided that the spectral curve satisfies equations (6.11) and (6.12), and in particular we may choose the spectral curve $\mathcal{S}_\infty$, which does satisfy equations (6.11) and (6.12), with the $\mathrm{O}(1/T)$ term vanishing. In other words, we use the freedom of choosing the potentials V_t, and we define V_t from the spectral curve through equation (6.11), instead of the contrary.

Somehow, we go backwards. We first construct the spectral curve $\mathcal{S}_\infty$ and then we construct the potentials V_t which correspond to it.

This very special choice of V_t guarantees that $\mathcal{S}_\infty$ is the spectral curve of our model, and then it is independent of T. □

6.2. Quantum case

Now we consider $q \neq 1$. We also assume for simplicity that the weights $\alpha(t') = \beta(t') = 1$ are constant.

The potential $\tilde{U}_i(Y)$ appearing in equation (4.30) is

$$\mathrm{e}^{-\tilde{U}_i(Y)} = \mathrm{e}^{-(T_i - T_{i-1})(Y/2)} \frac{g(-\mathrm{e}^{-Y} q^{T_i - T_{i-1}})}{g(-\mathrm{e}^{-Y})} \tag{6.31}$$

where $g(y) = \prod_{j=1}^{\infty}(1 - q^j/y)$ is the q-product (it is a quantum deformation of the Γ-function, see appendix A). Thus

$$\tilde{U}_i'(Y) = \frac{T_i - T_{i-1}}{2} + \psi_q(-\mathrm{e}^{-Y}) - \psi_q(-\mathrm{e}^{-Y} q^{(T_i - T_{i-1})}) \tag{6.32}$$

where $\psi_q(x) = x g'(x)/g(x) = \sum_{j=1}^{\infty} q^j/(x - q^j)$.

Thus, equation (4.31) gives

$$\hat{X}_i - \hat{X}_{i-1} = \frac{T_i - T_{i-1}}{2} + \sum_{j=0}^{T_i - T_{i-1} - 1} \frac{1}{1 + q^j \mathrm{e}^{-\hat{Y}_i}} \tag{6.33}$$

and at intermediate times $T_{i-1} \leq t \leq T_i$:

$$\hat{X}(z,t) = \hat{X}_i(z) + \frac{t - T_i}{2} - \sum_{j=0}^{T_i - t - 1} \frac{1}{1 + q^j \mathrm{e}^{-\hat{Y}_i(z)}}. \tag{6.34}$$

We choose the same domain $\mathcal{D}$ as in the classical case, i.e. a domain which scales with a factor $T = t_{\max} - t_{\min}$.

Several possibilities may occur:

- The regime $T \ll (1/\ln q)$ is more or less the same as $q = 1$, which we have studied in section 6.1, i.e. there is a liquid region of typical size T.

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- In the regime $T \gg (1/|\ln q|)$, there is a liquid region of typical size $1/|\ln q|$, and most of the domain $\mathcal{D}$ is in a frozen phase.
- In the intermediate regime $q^T \sim \mathrm{O}(1)$, the liquid phase is of typical size $T \sim (1/\ln q)$. This is the most interesting regime.

6.3. Case $q^T \sim \mathrm{O}(1)$

We consider the regime where T is large and $\ln q$ is small, and $q^T \sim \mathrm{O}(1)$. We define

$$q^T = \mathfrak{q}. \tag{6.35}$$

Then we shall repeat most of the steps of the classical case $q = 1$. First we rescale:

$$x_i = q^{\hat{X}_i}, \qquad y_i = \mathrm{e}^{\hat{Y}_i}, \qquad \tau = t/T. \tag{6.36}$$

6.3.1. Equation of the spectral curve. Equations (4.31) are

$$\frac{y_{i+1}}{y_i} = \mathfrak{q}^{(\tau_{i+1}-\tau_i)} \prod_{j=1}^{m_i} \frac{x_i - \mathfrak{q}^{b_{i,j}}}{x_i - \mathfrak{q}^{a_{i,j}}} (1 + \text{regular} + \mathrm{O}(\ln q)) \tag{6.37}$$

$$y_k = \prod_{j=1}^{m_k} \frac{x_i - \mathfrak{q}^{a_{i,j}}}{x_i - \mathfrak{q}^{b_{i,j}}} (1 + \text{regular} + \mathrm{O}(\ln q)). \tag{6.38}$$

And to leading order at small $\ln q$ we have

$$\frac{x_{i-1}}{x_i} = \mathfrak{q}^{-(\tau_i - \tau_{i-1})/2} \frac{y_i + \mathfrak{q}^{\tau_i - \tau_{i-1}}}{y_i + 1} (1 + \mathrm{O}(\ln q)). \tag{6.39}$$

Notice that those equations imply only meromorphic singularities for y_i and x_i, and again it is natural to look for an algebraic spectral curve. Notice that if y_i has a zero (resp. a pole) at z, then we have $x_{i+1}(z) = \mathfrak{q}^{-(\tau_{i+1}-\tau_i)} x_i(z)$ (resp. $x_{i-1}(z) = \mathfrak{q}^{(\tau_{i-1}-\tau_i)} x_i(z)$).

6.3.2. Recipe for an algebraic spectral curve. Therefore, let us make the assumption that our functions $x_i(z)$ are algebraic of smallest possible degree satisfying all the constraints. The consistency of that assumption can be discussed like in section 6.1.7 for $q = 1$, and we discuss it again in section 6.3.4 below.

The problem we have to solve in order to find the spectral curve is then:

- Find a Riemann surface whose genus is equal to the number of holes in $\mathcal{D}$.
- Find two meromorphic functions $u(z)$ and $v(z)$, with only simple poles, and let us denote the set of poles as $\infty_{i,j}$, $i = 0, \ldots, k$, $j = 1, \ldots, m_i$.
- We denote

$$x(z, \tau) = \mathfrak{q}^{\tau/2} u(z) + \mathfrak{q}^{-\tau/2} v(z), \tag{6.40}$$

and $\forall i$

$$x_i(z) = x(z, \tau_i) = \mathfrak{q}^{\tau_i/2} u(z) + \mathfrak{q}^{-\tau_i/2} v(z). \tag{6.41}$$

and we require that $\forall i, j$ (cusp condition):

$$\mathrm{Res}_{\infty_{i,j}} x_i(z) = 0. \tag{6.42}$$

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- The meromorphic functions $u(z)$ and $v(z)$ must be such that $\forall i, j$ (tangency conditions):

$$\text{if } i < k \quad \exists z \quad \text{such that } x_i(z) = \mathfrak{q}^{a_{i,j}} \quad \text{and} \quad x_{i+1}(z) = \mathfrak{q}^{a_{i,j}-(\tau_{i+1}-\tau_i)/2}, \tag{6.43}$$

$$\text{if } i > 0 \quad \exists z \quad \text{such that } x_i(z) = \mathfrak{q}^{a_{i,j}} \quad \text{and} \quad x_{i-1}(z) = \mathfrak{q}^{a_{i,j}+(\tau_{i-1}-\tau_i)/2}, \tag{6.44}$$

$$\text{if } i < k \quad \exists z \quad \text{such that } x_i(z) = \mathfrak{q}^{b_{i,j}} \quad \text{and} \quad x_{i+1}(z) = \mathfrak{q}^{b_{i,j}+(\tau_{i+1}-\tau_i)/2}, \tag{6.45}$$

$$\text{if } i > 0 \quad \exists z \quad \text{such that } x_i(z) = \mathfrak{q}^{b_{i,j}} \quad \text{and} \quad x_{i-1}(z) = \mathfrak{q}^{b_{i,j}-(\tau_{i-1}-\tau_i)/2}, \tag{6.46}$$

- and we have the filling fractions:

$$\frac{1}{2\mathrm{i}\pi}\int_{[a_{i,j},b_{i,j}]} \hat{Y}_i(z)\,\mathrm{d}\hat{X}_i(z) = n_{i,j} \tag{6.47}$$

  where $x_i = q^{\hat{X}_i}$ and $\hat{Y}_i$ is obtained by solving equation (6.33).

  Explicit examples for given domains (hexagon) are treated in section 7.

6.3.3. Envelope. The envelope is given by the branchpoints $z_c(t)$ solutions of $\mathrm{d}x(z_c,\tau) = 0$, and again it is easier to find τ as a function of z_c than the contrary. We have

$$q^t = \mathfrak{q}^\tau = -\frac{\mathrm{d}v(z)}{\mathrm{d}u(z)}. \tag{6.48}$$

The envelope $X_c(t)$ is also better given in a parametric form with the parameter z_c as

$$q^t = -\frac{\mathrm{d}v(z_c)}{\mathrm{d}u(z_c)} \qquad q^{X_c} = q^{t/2}u(z_c) + q^{-t/2}v(z_c). \tag{6.49}$$

Since u and v are meromorphic, this implies that q^t and q^{X_c} are related by a polynomial equation:

$$P(q^t, q^{X_c}) = 0. \tag{6.50}$$

Again, we claim that this polynomial is the same as the Harnack curve of Kenyon–Okounkov–Sheffield [48].

In other words, the plane curve $x_c(q^t) = q^{X_c(t)}$, is an algebraic plane curve inscribed in the image of the domain $\mathcal{D}$ under the map $(x,t) \to (q^x, q^t)$.

Recovering the spectral curve from the envelope:

Assume that we know the envelope $X_c(t)$, or, writing $x_c = q^{X_c}$ and $z = q^{t/2}$, assume that we know the plane curve $x_c(z)$, i.e. an algebraic curve tangent to the boundaries of the image of $\mathcal{D}$ under $(x,t) \to (q^x, q^{t/2})$. Let us explain how to recover the full spectral curve $x(z,\tau)$.

Given the equation of the envelope $x_c(q^{t/2})$ (which is a multivalued algebraic function), one can choose locally $z = q^{t/2}$ as a local parameter.

One finds that the spectral curve is (at least in the domain of t where q^t can be chosen as a local parameter)

$$x(z,t) = \frac{q^{t/2}}{2}\left(\frac{x_c(z)}{z} + x_c'(z)\right) + \frac{q^{-t/2}}{2}\left(zx_c(z) - z^2x_c'(z)\right). \tag{6.51}$$

Therefore, knowing the envelope allows us to recover the full spectral curve, i.e. the functions $x(z,\tau) = q^{\hat{X}(z,t)}$ and also $\hat{Y}_i(z)$ from equation (6.33).

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6.3.4. Small $\ln q$ *asymptotic expansion.* Now, suppose that we have found those functions, and that we have indeed found the correct spectral curve. The spectral curve is then the pair of functions $\mathcal{S}_t = (\hat{X}(z,t), \hat{Y}(z,t))$, i.e. up to a symplectic transformation:

$$\mathcal{S} = \left(\frac{1}{\ln q} \ln\left(x(z,\tau)\right), \ln\left(y(z,\tau)\right) \right), \tag{6.52}$$

and we notice that $x(z,\tau)$ and $y(z,\tau)$ do not depend on q, they depend only on $\mathfrak{q}$. Therefore we have $F_g((1/\ln q)\ln x, \ln y) = (\ln q)^{2g-2} F_g(\ln x, \ln y)$.

Thus, from theorem 4.1, we have the full large T asymptotic expansion:

Conjecture 6.2. $\ln \mathcal{Z}$ *has the small* $\ln q$ *expansion:*

$$\ln \mathcal{Z} \sim \sum_{g=0}^{\infty} (\ln q)^{2g-2} F_g(\ln x, \ln y) \tag{6.53}$$

where the meromorphic functions $x(z)$ *and* $y(z)$ *are computed by solving the requirements of section 6.3.2 and* F_g *is the* gth *symplectic invariant defined in [27].*

A possible proof could follow the same ideas which we discussed in section 6.1.7 for $q = 1$.

6.4. Microscopic asymptotics

It is a truism to say that, since our model is a matrix model, it has all the local universal behaviors of matrix models.

6.4.1. Zoom near a point. Let us choose a point (x_0, t_0) anywhere in the plane (it can be inside or outside the domain, or on the border).

Let us rescale x and t with some scaling parameter s, with some exponents α and δ, i.e. we write

$$x = x_0 + s^\alpha \xi, \qquad t = t_0 + s^\delta \tau. \tag{6.54}$$

Also, on the spectral curve, we choose a point z_0 such that $\hat{X}(z_0, t_0) = x_0$, and we rescale it by choosing a rescaled local parameter ζ:

$$z = z_0 + s^\gamma \zeta. \tag{6.55}$$

We rewrite the asymptotics of the functions $\hat{X}(z,t)$ and $\hat{Y}(z,t)$ in terms of those rescaled variables:

$$\hat{X}(z,t) = x_0 + s^\alpha \hat{\xi}(\zeta,\tau) + o(s^\alpha), \qquad \hat{Y}(z,t) = y_0 + s^\beta \hat{\eta}(\zeta,\tau) + o(s^\beta). \tag{6.56}$$

If the exponents $\alpha, \beta, \gamma, \delta$ are well chosen, then the spectral curve $\mathcal{S}_{\text{zoom}} = (\xi(\zeta,\tau), \eta(\zeta,\tau))$ is a regular spectral curve (it has only branchpoints of square root type). We call this curve the *blow up* of the spectral curve $(\hat{X}, \hat{Y})$ near the point (x_0, t_0). In practice, finding the blow up is a rather trivial task, it merely consists in finding the first non-vanishing terms in the Taylor expansion near a point.

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From [27], we have that the correlation functions $W_n^{(g)}(x_1,\dots,x_n;t)$:

$$\sum_g W_n^{(g)}(x_1,\dots,x_n;t) = \left\langle \mathrm{tr}\frac{1}{x_1-M_t}\mathrm{tr}\frac{1}{x_2-M_t}\cdots\mathrm{tr}\frac{1}{x_n-M_t}\right\rangle_c \tag{6.57}$$

behave at small s like

$$W_n^{(g)}(x_1,\dots,x_n;t) \sim s^{(2-2g-n)(\alpha+\beta)-n\alpha}\omega_n^{(g)}(\xi_1,\dots,\xi_n;\tau)(1+o(s^0)), \tag{6.58}$$

where $\omega_n^{(g)}$ are the 'symplectic invariants' correlation functions of [27] associated with the spectral curve $\mathcal{S}_{\mathrm{zoom}}$.

In other words, since the correlation functions are symplectic invariant correlators of a spectral curve $\mathcal{S}$, their local behavior is, to leading order, given by the symplectic invariant correlators of the blown-up spectral curve. This theorem found in [27] is very easy to prove by recursion on n and g (see appendix B).

Therefore, it suffices to find the blown-up spectral curve to characterize the leading behavior of the correlation functions near a point.

6.4.2. Airy kernel near regular boundaries of the liquid region. Near a regular point of the envelope $X_c(t_0)$, we have $\mathrm{d}\hat{X}/\mathrm{d}z = 0$ at $t = t_0$ and thus we have a Taylor expansion in z of the form

$$\hat{X}(z,t) \sim X_0(t) + (t-t_0)X_1(t)z + X_2(t)z^2 + \cdots, \tag{6.59}$$

where each $X_i(t)$ has a regular Taylor expansion in $t-t_0$. Let us rescale:

$$z = s\zeta, \qquad t = t_0 + s\tau. \tag{6.60}$$

We have to the first few orders in s:

$$\hat{X}(z,t) = X_0(t_0) + s(\dot{X}_0\tau) + s^2(X_1\tau\zeta + X_2\zeta^2) + \mathrm{O}(s^3). \tag{6.61}$$

We also have

$$\dot{\hat{X}}(z,t) = \dot{X}_0 + s\,(X_1\zeta\tau) + \mathrm{O}(s^2), \tag{6.62}$$

and therefore

$$\hat{Y}(z,t) \sim Y_0 + sX_1\zeta + \mathrm{O}(s^2). \tag{6.63}$$

The blown-up curve is thus

$$\mathcal{S}_{\mathrm{Airy}} = \begin{cases} x(\zeta,\tau) = \zeta\tau + \zeta^2 \\ y(\zeta,\tau) = \zeta \end{cases} \tag{6.64}$$

which satisfies

$$(y+\tau/2)^2 = x + \tau^2/4. \tag{6.65}$$

The spectral curves such that $\mathrm{Pol}_q(y) = \mathrm{Pol}_p(x)$ where Pol_q is a polynomial of degree q and Pol_p is a polynomial of degree p, appear in the so-called (p,q)-minimal models, i.e. in the classification of conformal field theories [49, 20]. It has central charge $c = 1-6(p-q)^2/pq$. Here, this is the $(1,2)$ minimal model, with central charge $c=-2$, which is well known to be generated by the Airy differential system $Ai'' = xAi$, and the correlation functions are determinants of the Airy kernel. It is also well known to be related to the Tracy–Widom law of extreme eigenvalue statistics [71].

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6.4.3. Pearcey kernel near cusps. We have a cusp each time a pole disappears, generically a simple pole. Thus we have a Laurent expansion in z starting at z^{-1}, and such that the residue of the pole vanishes at $t = t_0$, i.e.

$$\hat{X}(z,t) \sim \frac{(t-t_0)v(t)}{z} + x_0(t) + u(t)z + \cdots, \tag{6.66}$$

and all the coefficients $v(t), x_0(t), u(t), \ldots$ have a regular Taylor expansion near $t = t_0$. Let us rescale $z = s\zeta/u(t_0)$ and $t = t_0 + s^2\tau/u(t_0)v(t_0)$, to order s, and up to constant terms, we have

$$\hat{X}(z,t) = x_0 + s\left(\zeta + \frac{\tau}{\zeta}\right) + \mathrm{O}(s^2), \tag{6.67}$$

and generically, $\hat{Y}$ behaves like

$$\hat{Y}(z,t) = y_0 - \frac{z}{v_0} + \cdots. \tag{6.68}$$

The blown-up curve is thus

$$\mathcal{S}_{\text{Pearcey}} = \begin{cases} x(\zeta,\tau) = \zeta + \dfrac{\tau}{\zeta} \\ y(\zeta,\tau) = \zeta, \end{cases} \tag{6.69}$$

and the local behaviors of correlation functions are the correlation functions of that universal curve.

This is the spectral curve whose correlators are generated by the Pearcey kernel.

We thus recover the well-known Pearcey kernel behavior [72].

6.4.4. Critical points. If we consider a point on the boundary such that more derivatives vanish, typically we find that the spectral curve behaves locally like

$$y(x) \sim y(a) + (x - x(a))^{p/q} + \cdots. \tag{6.70}$$

It was shown in [27] that the correlation functions tend towards those of the (p,q) reduction of the KP hierarchy, i.e. the (p,q) minimal model of central charge $c = 1 - 6(p-q)^2/pq$.

The model $(3,2)$ of central charge $c = 0$ is called 'pure gravity', the model $(5,2)$ of central charge $c = -22/5$ is called Lee–Yang, the model $(4,3)$ of central charge $c = \frac{1}{2}$ is called the Ising model, etc. Their correlators are generated by determinantal formulae from a kernel involving the ψ-system associated with a nonlinear equation of Gelfand–Dikii type (Painlevé I is the Gelfand–Dikii equation for pure gravity $(3,2)$). Some details can be found in [20, 8].

6.4.5. Other local behaviors. Also, we expect that locally in the bulk of the liquid region the behavior is given by the sine-kernel [10, 6, 43], and near vertical boundaries of the liquid region we have $y(x) \sim y(a) + (x-a)^2$, i.e. we expect the $(2,1)$ model, described by the Hermit kernel of the 'birth of a cut' (see [13, 34]). Also we expect to find the 'Bead model' limit in the tentacles of the amoeba when $|T \ln q|$ is large, see [12].

However, these cases are such that the blown-up curve is not regular, and we cannot directly apply the method of [27].

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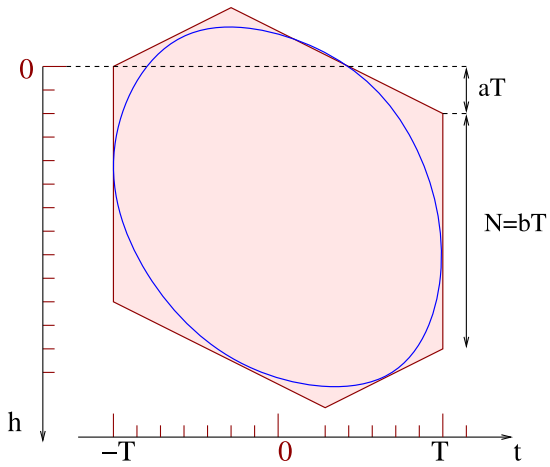

Figure 16. Domain for the hexagon.

6.4.6. Arbitrary local behaviors. Then, one could easily invent some domains for which a local blown-up curve could be any spectral curve specified in advance, and by choosing sufficiently complicated domains one can obtain any limit law.

The classifications of all possible laws is more or less the classification of singularities of spectral curves, and it is more or less the classification of spectral curves themselves.

7. Examples

In this section, we illustrate our method by applying it to several classical examples, which were already studied in the literature with other methods. Here, however, we have a method to obtain not only the large size limit, but also all corrections to all orders.

7.1. The hexagon

Our domain $\mathcal{D}$ is the hexagon of figure 16, with slopes $\pm\frac{1}{2}$.

We choose $t_{\max} = -t_{\min} = T$. We choose the weighs $\alpha(t) = \beta(t) = 1$ at all times. We write $N = bT$.

There is no defect, therefore we have $k = 1$. The reduced matrix integral is a 2-matrix model with an external field:

$$\mathcal{Z}_{\text{hexagon}} = \int_{H_N} \mathrm{d}M_0 \int_{iH_N} \mathrm{d}R_1\, \mathrm{e}^{-\mathrm{Tr}V_0(M_0)} \mathrm{e}^{-\mathrm{Tr}\tilde{U}_1(R_1)} \mathrm{e}^{\mathrm{Tr}R_1\,(M_1-M_0)} \tag{7.1}$$

where M_1 is the fixed following matrix:

$$M_1 = \mathrm{diag}(Ta+1, Ta+2, \ldots, Ta+N) = T\,\mathrm{diag}\left(a+\frac{1}{T}, a+\frac{2}{T}, \ldots, a+b\right). \tag{7.2}$$

For V_0, we can choose

$$\mathrm{e}^{-V_0(X)} = \frac{1}{\Gamma(X)\Gamma(N+1-X)}. \tag{7.3}$$

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7.1.1. The classical hexagon $q = 1$. We apply the recipe of section 6.1.5.

First, the domain has no hole and we look for an algebraic curve of genus 0. Therefore it can be parameterized by a uniformizing variable in the complex plane $z \in \mathbb{C}$. Let us look for two rational fractions $u(z)$ and $v(z)$, with two poles, and we write

$$x(z,\tau) = u(z) + \tau v(z). \tag{7.4}$$

Up to a reparameterization of z, we choose the poles to be at $z = 0$ and ∞. Moreover, we require that at $\tau = -1$ the pole at $z = 0$ disappears, and that at $\tau = +1$ the pole at $z = \infty$ disappears. This implies that u and v and x are of the form

$$x(z,\tau) = c + r\tau + \gamma\left((1-\tau)z + \frac{1+\tau}{z}\right), \tag{7.5}$$

$$u(z) = c + \gamma\left(z + \frac{1}{z}\right), \qquad v(z) = r + \gamma\left(-z + \frac{1}{z}\right). \tag{7.6}$$

We define

$$x_0(z) = x(z,-1) = c - r + 2\gamma z, \qquad x_1(z) = x(z,1) = c + r + 2\gamma/z. \tag{7.7}$$

Then we find the coefficients r, c, γ by solving the following system:

$$\begin{array}{llll} \exists z & \text{such that } x_0(z) = 0 & \text{and} & v(z) = -\frac{1}{2}, \\ \exists z & \text{such that } x_0(z) = b & \text{and} & v(z) = \frac{1}{2}, \\ \exists z & \text{such that } x_1(z) = a & \text{and} & v(z) = \frac{1}{2}, \\ \exists z & \text{such that } x_1(z) = a+b & \text{and} & v(z) = -\frac{1}{2}. \end{array} \tag{7.8}$$

We have three unknowns and four equations, but one can check that there are only three independent equations and the system has a solution.

We easily find

$$c = \frac{a+b}{2}, \qquad r = \frac{a(1+b)}{2}, \qquad 16\gamma^2 = (1-a)(1+a)b(b+2). \tag{7.9}$$

Envelope

The branchpoints $z_c(\tau)$ are found from $x'(z_c,\tau) = 0$, and we find that there are two branchpoints:

$$z_c(\tau) = \pm\sqrt{\frac{1+\tau}{1-\tau}}. \tag{7.10}$$

That gives

$$x_c(\tau) = c + r\tau \pm 2\gamma\sqrt{1-\tau^2}, \tag{7.11}$$

or explicitly

$$x_c(\tau) = \tfrac{1}{2}\left(a + b + \tau a(1+b) \pm \sqrt{(1-\tau^2)(1-a^2)b(b+2)}\right). \tag{7.12}$$

One can easily check that this is the equation of the ellipse tangent to all sides of the hexagon. See the ellipse in figure 16.

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7.1.2. The quantum hexagon $q \neq 1$. Now consider $q^T = \mathrm{O}(1)$ and write

$$\mathfrak{q} = q^T \tag{7.13}$$

and we shall define the $\mathfrak{q}$-numbers:

$$[\tau] = \frac{\mathfrak{q}^{-\tau/2} - \mathfrak{q}^{\tau/2}}{\mathfrak{q}^{-1/2} - \mathfrak{q}^{1/2}}. \tag{7.14}$$

Let us now apply the recipe of section 6.3.2.

We write

$$x(z,\tau) = \mathfrak{q}^{\tau/2} u(z) + \mathfrak{q}^{-\tau/2} v(z) \tag{7.15}$$

where u and v are rational fractions with two poles. Up to a reparameterization of z, we choose the poles to be at $z = 0$ and ∞. Moreover, we require that at $\tau = -1$ the pole at $z = 0$ disappears, and that at $\tau = +1$ the pole at $z = \infty$ disappears. This implies that u and v and x are of the form:

$$x(z,\tau) = \frac{[1-\tau]}{[2]}(c+z) + \frac{[1+\tau]}{[2]}\left(r + \frac{d}{z}\right). \tag{7.16}$$

We define

$$x_0(z) = x(z,-1) = c + z, \qquad x_1(z) = x(z,1) = r + \frac{d}{z}. \tag{7.17}$$

Then we find the coefficients r, c, d by solving the following system:

$$\begin{aligned}
&\exists z \quad \text{such that } x_0(z) = 1 \quad \text{and} \quad x_1(z) = \mathfrak{q}^{-1},\\
&\exists z \quad \text{such that } x_0(z) = \mathfrak{q}^{b} \quad \text{and} \quad x_1(z) = \mathfrak{q}^{b+1},\\
&\exists z \quad \text{such that } x_1(z) = \mathfrak{q}^{a} \quad \text{and} \quad x_0(z) = \mathfrak{q}^{a-1},\\
&\exists z \quad \text{such that } x_1(z) = \mathfrak{q}^{a+b} \quad \text{and} \quad x_0(z) = \mathfrak{q}^{a+b+1}.
\end{aligned} \tag{7.18}$$

We have three unknowns and four equations, but one can check that there are only three independent equations, and the system has a solution. We find

$$c = \frac{\mathfrak{q}^{-1} + \mathfrak{q}^{a+b} - \mathfrak{q}^{a} - \mathfrak{q}^{b+1}}{\mathfrak{q}^{-1} - \mathfrak{q}}, \qquad r = -\frac{1 + \mathfrak{q}^{a+b+1} - \mathfrak{q}^{a-1} - \mathfrak{q}^{b}}{\mathfrak{q}^{-1} - \mathfrak{q}}, \tag{7.19}$$

$$d = \frac{(1-\mathfrak{q}^{b})(\mathfrak{q}^{a} - \mathfrak{q})(\mathfrak{q}^{a} - \mathfrak{q}^{-1})(\mathfrak{q}^{b+1} - \mathfrak{q}^{-1})}{(\mathfrak{q}^{-1} - \mathfrak{q})^2}. \tag{7.20}$$

Envelope

The branchpoints are at $z_c(\tau) = \pm\sqrt{d\,[1+\tau]/[1-\tau]}$ and thus the envelope is

$$x_c(\tau) = \frac{[1-\tau]c + [1+\tau]r \pm 2\sqrt{d\,[1+\tau]\,[1-\tau]}}{[2]}. \tag{7.21}$$

See some examples plotted in figure 17.

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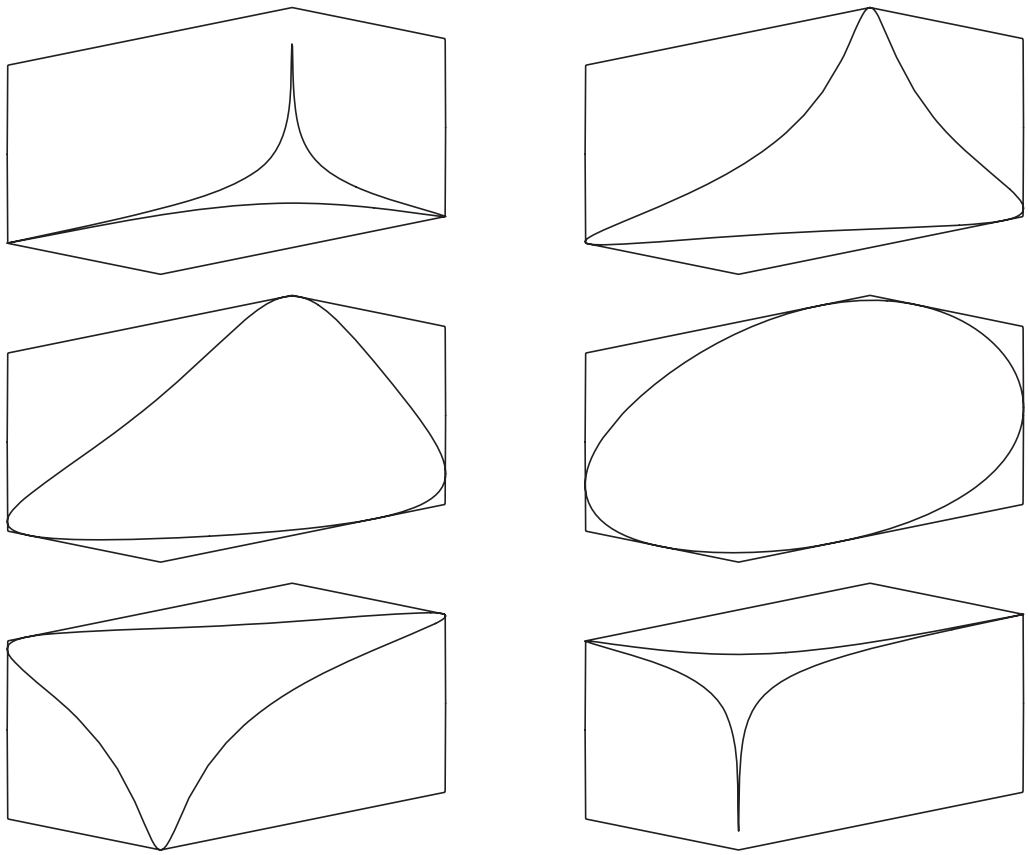

Figure 17. The envelope of the hexagonal domain. The plots of equation (7.21) are for the hexagon $a = 0.3$, $b = 2$, and for values of q respectively $q = 0.001, 0.1, 0.3, 0.8, 10, 1000$. Notice that the liquid region is convex only for q close to 1.

7.2. The cardioid

Consider the classical case $q = 1$. The domain is the one represented in figure 18. We assume $0 < a < 1$ and $b > 0$.

We find the spectral curve by applying the recipe of section 6.1.5.

Consider

$$x(z, \tau) = u(z) + \tau v(z), \tag{7.22}$$

where u and v are rational fractions with three poles.

Up to a reparameterization of z, we choose the poles to be at $z = -1, 0, 1$. Moreover, we require that at $\tau = -1$ the pole at $z = -1$ disappears, at $\tau = 0$ the pole at $z = 0$ disappears, and at $\tau = +1$ the pole at $z = +1$ disappears. Moreover, since the domain has a symmetry $\tau \to -\tau$, we choose u and v and x of the form

$$x(z, \tau) = a + b - \frac{1}{2} - 2w + \frac{\gamma(1-\tau)}{1-z} + \frac{s\tau}{z} + \frac{\gamma(1+\tau)}{1+z}, \tag{7.23}$$

$$u(z) = a + b - \frac{1}{2} - 2w + \frac{2\gamma}{1-z^2}, \qquad v(z) = \frac{2\gamma z}{z^2-1} + \frac{s}{z}. \tag{7.24}$$

The symmetry $\tau \to -\tau$ is such that $x(-z, -\tau) = x(z, \tau)$.

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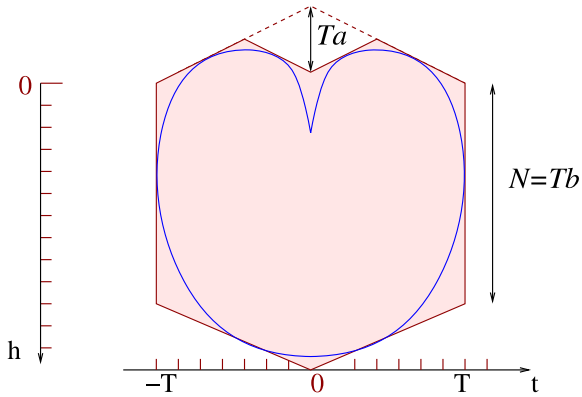

Figure 18. The domain of the cardioid. The envelope is a cardioid.

We write

$$x_0(z) = x(z,-1) = a + b - \frac{1}{2} - 2w + \frac{2\gamma}{1-z} - \frac{u}{z}, \tag{7.25}$$

$$x_1(z) = x(z,0) = a + b - \frac{1}{2} - 2w + \frac{2\gamma}{1-z^2}, \tag{7.26}$$

$$x_2(z) = x(z,1) = a + b - \frac{1}{2} - 2w + \frac{u}{z} + \frac{2\gamma}{1+z}. \tag{7.27}$$

The coefficients w, s, γ are determined from the following system (we have written only the independent equations):

$$\begin{array}{llll} \exists z & \text{such that } x_2(z) = 0 & \text{and} & v(z) = \frac{1}{2}, \\ \exists z & \text{such that } x_0(z) = b & \text{and} & v(z) = \frac{1}{2}, \\ \exists\, z & \text{such that } x_1(z) = a - \frac{1}{2} & \text{and} & v(z) = \frac{1}{2}. \end{array} \tag{7.28}$$

Those equations imply that

$$3w^2 - w(2a + 2b - 1) + b(a-1) = 0, \tag{7.29}$$

$$\gamma = 2w(b + 1 - w)(a - w) = 2(w+1)(b-w)(a-w-1), \tag{7.30}$$

$$2s = -(2w+1)(2b+1-2w)(2a-1-2w). \tag{7.31}$$

And we have

$$3w = a + b - \tfrac{1}{2} + 2\gamma + s. \tag{7.32}$$

Envelope

Let $s/2\gamma = e$.

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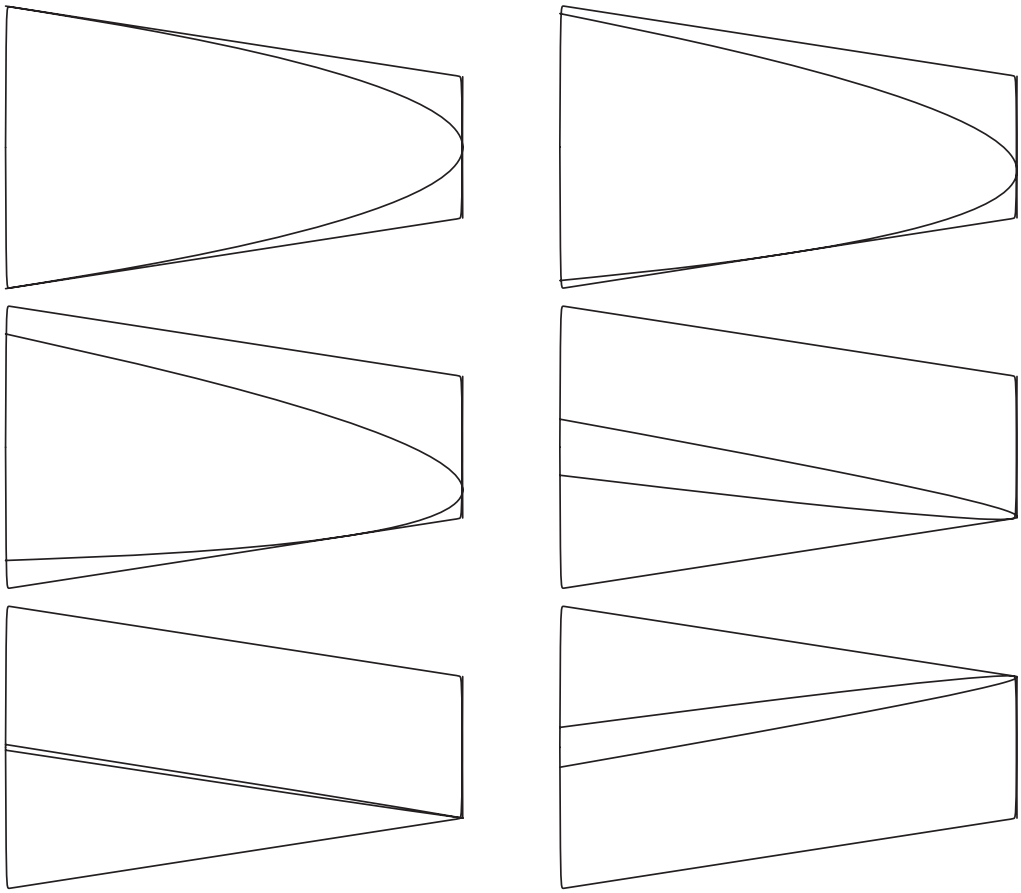

Figure 19. The envelope of the trapezoidal domain. The plots are for $\alpha = 1, 0.5, 0.25, 0.01, 0.0001, 200$.

Parametrically

$$\tau = \frac{2z^3}{z^2(1+z^2) + e(z^2-1)^2} \tag{7.33}$$

$$X_c = a + b - \frac{1}{2} - 2w + 2\gamma \frac{z^2 + e(1+z^2)}{z^2(z^2+1) + e(1-z^2)^2}. \tag{7.34}$$

One can check that it is the equation of the cardioid inscribed in the domain, see figure 18.

The same domain can also be considered in the quantum case, but we do not do it here.

7.3. The trapezoid

Take $N = T$, $\mathcal{D}$ is the domain $\mathcal{D}(t_{\min}) = [0, 2T]$ at $t_{\min} = 0$ and $\mathcal{D}(t_{\max}) = [T/2, 3T/2]$ at $t_{\max} = T$. We choose the weights $\beta = 1$ and $\alpha \neq 1$ with α constant in time. We study the classical case $q = 1$.

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Notice that $\#\mathcal{D}(t_{\min}) = 2N \neq N = \#\mathcal{D}(t_{\max})$. Therefore the initial matrix $M_{t_{\min}}$ is not fixed, it must have a potential $V_{t_{\min}}$ satisfying condition equation (3.8) instead of equation (3.9), but in fact it suffices to choose

$$V_{t_{\min}} = 0. \tag{7.35}$$

Let us look for an algebraic genus-zero spectral curve of minimal degree. First, notice that we have no cusp condition at $t_{\min}$. We thus need to have only two poles, let us say at $z = 0$ and ∞, and the pole at $z = 0$ disappears at $t = T$. Thus we write (with $\tau = t/T$)

$$x(z,\tau) = 1 + c + r\tau + (u\tau + v)z - \frac{1-\tau}{z}. \tag{7.36}$$

That means

$$x_0(z) = x(z,0) = 1 + c + vz - \frac{1}{z}, \qquad x_1(z) = x(z,1) = 1 + c + r + (u+v)z \tag{7.37}$$

and

$$y_0(z) = y_1(z) = \alpha\frac{(1/2) - (r + uz + (1/z))}{(1/2) + (r + uz + (1/z))}. \tag{7.38}$$

The coefficients c, r, u, v can be determined by requiring that:

- $\exists z$ such that $x_0(z) = 0$ and $x_1(z) = \frac{1}{2}$.
- $\exists z$ such that $x_0(z) = 2$ and $x_1(z) = \frac{3}{2}$.
- $\ln(y_0(z)) \sim 1/x_0(z)$ at $z \to \infty$.

The last condition implies $u = 0$ and $r = \frac{1}{2}((\alpha-1)/(\alpha+1))$. Then, the two other conditions imply $c = 0$ and $v = (r - \frac{1}{2})(r + \frac{1}{2})$.

Finally we find the following spectral curve:

$$x_0(z) = 1 - \frac{\alpha z}{(\alpha+1)^2} - \frac{1}{z}, \qquad x_1(z) = 1 + \frac{1}{2}\frac{\alpha-1}{\alpha+1} - \frac{\alpha z}{(\alpha+1)^2} \tag{7.39}$$

$$x(z,t) = 1 + \frac{t}{2}\frac{\alpha-1}{\alpha+1} - \frac{\alpha z}{(\alpha+1)^2} - \frac{1-t}{z} \tag{7.40}$$

$$y_1(z) = \frac{z - 1 - \alpha}{z + 1 + (1/\alpha)}. \tag{7.41}$$

The envelope of the liquid region is

$$x_c(t) = 1 + \frac{t}{2}\frac{\alpha-1}{\alpha+1} \pm \frac{2}{\alpha+1}\sqrt{(1-t)\alpha} \tag{7.42}$$

which is a parabola shifted by a straight line and tangent to at least two of the trapezoid boundaries.

Notice that, when $\alpha = 1$, particles have the same probability to go upward $h_i \to h_i + \frac{1}{2}$ or downward $h_i \to h_i - \frac{1}{2}$, and the spectral curve is symmetric with respect to $x \to 2 - x$. When α is very small, the probability to go downwards is very small, and therefore almost all the particles are in the solid region going upwards, and the liquid region becomes a narrow region around the line $x = 1 - t/2$. In contrast, when α is very large, the probability to go downwards is very large, and almost all the particles are in the solid region going downwards, and the liquid region becomes a narrow region around the line $x = 1 + t/2$. See figure 20.

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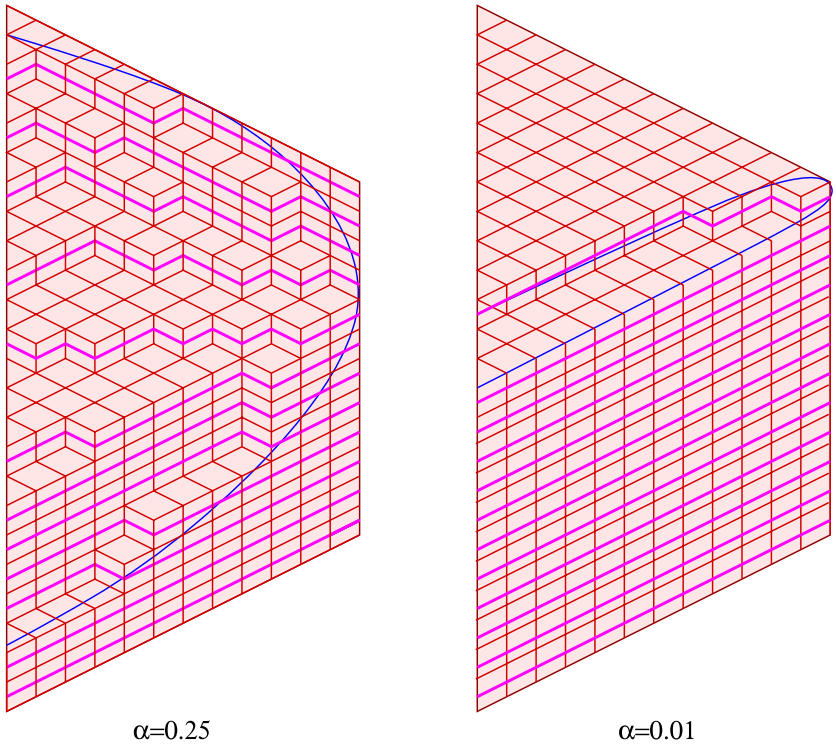

Figure 20. Typical self-avoiding particle model, or typical plane partitions in the trapezoid. The tiling outside the liquid region is regular; it is frozen. For small α, the probability to go downwards is very small, and therefore almost all the particles are in the solid region going upwards, and the liquid region becomes a narrow region around the line $x = 1 - t/2$.

7.4. The Plancherel law

Let us choose a partition μ and $N \geq n(\mu)$. We write

$$h_i(\mu) = \mu_i - i + N \tag{7.43}$$

and we have

$$h_1(\mu) > h_2(\mu) > \cdots > h_N(\mu) \geq 0. \tag{7.44}$$

Consider the domain $\mathcal{D}$, comprised between $t_{\min} = 0$, and $t_{\max} = T$, and such that at $t = T$ the particles are at $h_i(T) = T/2 + N - i$ (in some sense the boundary is the trivial partition shifted by $T/2$), and at $t = 0$, the particles are at $h_i(0) = h_i(\mu)$ (the boundary is the partition μ). See figure 21.

Let us call $\mathcal{P}_T(\mu)$ the plane partitions generating function in this domain:

$$\mathcal{P}_T(\mu) = Z_T(\mu) = \sum_{\pi, \partial\pi|_0 = \mu, \partial\pi|_T = \emptyset} q^{|\pi|}. \tag{7.45}$$

Let us define the 'Plancherel law' as the limit $T \to \infty$:

$$\text{Plancherel}(\mu) = \mathcal{P}(\mu) = \mathcal{P}_\infty(\mu). \tag{7.46}$$

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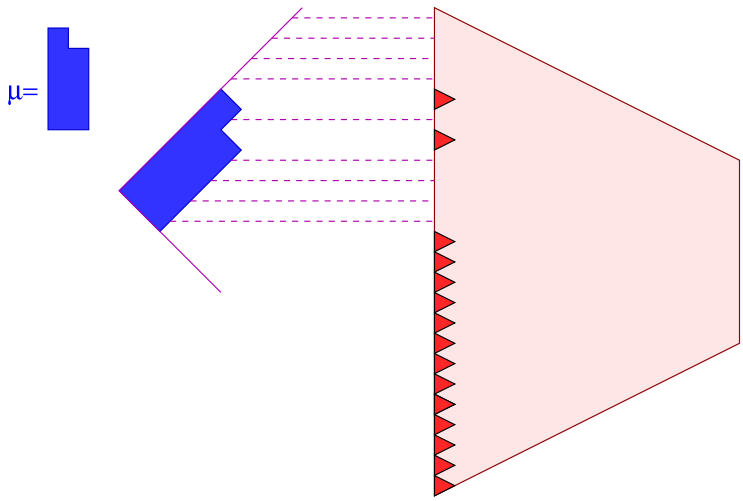

Figure 21. The domain for the Plancherel law. At time $t = t_{\max} = T$, we have $h_i(t_{\max}) = T/2 + N - i$ and at time $t = 0 = t_{\min}$ we have $h_i(t_{\min}) = h_i(\mu)$.

It has been well known from a really long time [69] that

$$\mathcal{P}(\mu) = \frac{\prod_{i>j}\left(q^{(h_i(\mu)-h_j(\mu))/2} - q^{(h_j(\mu)-h_i(\mu))/2}\right)}{\prod_{i=1}^{N}\prod_{j=1}^{h_i(\mu)}(q^{-(h_i(\mu))/2} - q^{(h_i(\mu))/2})} = \frac{\prod_{i>j}[h_i(\mu) - h_j(\mu)]}{\prod_i [h_i(\mu)]!}, \tag{7.47}$$

and if $q = 1$, that reduces to the classical Plancherel law:

$$\mathcal{P}(\mu) = \frac{\Delta(h_i(\mu))}{\prod_{i=1}^{N} h_i(\mu)!}. \tag{7.48}$$

As a check of our matrix model approach, let us recover this classical result from the matrix model.

7.4.1. The classical Plancherel law $q = 1$. As presented in section 3, the domain $\mathcal{D}$ is characterized by:

- the matrix M_T:

$$M_T = \frac{T}{2}\mathrm{Id} + \mathrm{diag}(0, 1, 2, \ldots, N-1), \tag{7.49}$$

- a potential at $t = 0$, which satisfies the conditions of equation (3.9). We choose

$$\mathrm{e}^{-V_0(x)} = \frac{\mathrm{e}^{\mathrm{i}\pi x}}{\Gamma(-x)} \prod_{i=1}^{N} \frac{1}{x - h_i(\mu)}. \tag{7.50}$$

  Notice that we have

$$\mathrm{e}^{-V_0(h_i(\mu))} = \frac{h_i(\mu)!}{\prod_{j\neq i}(h_i(\mu) - h_j(\mu))} \tag{7.51}$$

  and thus

$$\mathrm{e}^{-\sum_i V_0(h_i(\mu))} = (-1)^{N(N-1)/2} \frac{\prod_i h_i(\mu)!}{\Delta^2(h_i(\mu))} \tag{7.52}$$

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- and

$$\tilde{U}(y) = -T \ln\left(2\cosh\frac{y}{2}\right). \tag{7.53}$$

Theorem 3.1, or more precisely its reduced version theorem 4.2, gives the relationship between the self-avoiding particle model partition function $\mathcal{P}_T(\mu)$ and the matrix model:

$$\begin{aligned}\mathcal{Z} &= \int \mathrm{d}M_0\,\mathrm{d}R_1 \mathrm{e}^{-\mathrm{Tr}V_0(M_0)}\mathrm{e}^{-\mathrm{Tr}\tilde{U}(R)}\mathrm{e}^{\mathrm{Tr}R_1(M_T-M_0)} \\ &= C_{N,T,2}\frac{\Delta(h_i(\mu))}{\Delta(M_T)}\mathrm{e}^{-\sum_i V_0(h_i(\mu))}\mathcal{P}_T(\mu)\end{aligned} \tag{7.54}$$

i.e.

$$\mathcal{P}_T(\mu) = C\frac{\Delta(h_i(\mu))}{\prod_i h_i(\mu)!}\mathcal{Z}. \tag{7.55}$$

We can already guess that, in the large T limit, the role of the $h_i(\mu)$s in the spectral curve is going to be subleading, i.e. to large T leading order $\mathcal{Z}$ is going to be independent of $h_i(\mu)$.

In principle, our matrix model could be used to find the asymptotics of the Plancherel measure [61].

8. Obliged places and TSSCPPs

So far, we have considered a self-avoiding particle model with defects, i.e. forbidden places for the particles. Defects were introduced by choosing $\mathrm{e}^{-V_t}(x) = 0$ at the corresponding place (x,t).

One could also be interested in a constrained self-avoiding particle model, where we want to oblige some places to be visited at some given times. This cannot be achieved directly by tuning the potentials V_t, but this can be achieved as follows.

8.1. Obliged places

We choose a potential V_t such that $\mathrm{e}^{-V_t(x)} = 1-\eta$ at the place (x,t), i.e. we enforce a probability $1-\eta$ that the place (x,t) can be visited. In other words, the contribution of self-avoiding particle configurations such that one particle visits (x,t) will be proportional to $1-\eta$, and the contribution of self-avoiding particle configurations such that no particle visits (x,t) will be independent of η (notice that no more than one particle can visit (x,t)). Therefore the partition function $\mathcal{Z}$ is made of two terms:

$$\mathcal{Z} = \mathcal{Z}_0 + (1-\eta)\mathcal{Z}_1 \tag{8.1}$$

where $\mathcal{Z}_1$ is the partition function of self-avoiding particle configurations which visit (x,t). We have

$$\mathcal{Z}_1 = -\frac{\mathrm{d}}{\mathrm{d}\eta}\mathcal{Z}\Big|_{\eta=0} = \mathcal{Z}\left\langle\frac{\mathrm{d}}{\mathrm{d}\eta}\mathrm{Tr}V_t(M_t)\right\rangle_{\eta=0}. \tag{8.2}$$

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In other words, $\mathcal{Z}_1$ can be realized as some expectation value in our matrix integral. We write

$$A(x) = \frac{\mathrm{d}}{\mathrm{d}\eta}\mathrm{Tr}V_t(M_t)\Big|_{\eta=0}. \tag{8.3}$$

We have that

$$\frac{\mathcal{Z}_1}{\mathcal{Z}} = \langle \mathrm{Tr}A(M_t)\rangle. \tag{8.4}$$

More generally, if we want to have several obliged places (x_i, t_i), $i = 1, \ldots, s$, we introduce a function $A(x, t_i)$ for each of them:

$$\frac{\mathcal{Z}_{\{(x_i,t_i)\}}}{\mathcal{Z}} = \left\langle \prod_{i=1}^{s} \mathrm{Tr}A(M_{t_i}, t_i) \right\rangle. \tag{8.5}$$

Again, we have some arbitrariness in the choice of $A(x, t)$. The only requirements are:

- and for t integer, $t_{\min} < t < t_{\max}$, we choose $A(x, t)$ such that

$$\forall x \in \mathcal{D}(t)/\hat{\hat{\mathcal{D}}}, \qquad A(x,t) = \begin{cases} 1 & \text{if } \exists i \quad (x,t) = (x_i, t_i) \\ 0 & \text{otherwise, in } \mathcal{D}(t)/\hat{\hat{\mathcal{D}}}, \end{cases} \tag{8.6}$$

  and arbitrary values everywhere else.

  A possible choice could be

$$A(x,t) = \sum_i \delta_{t,t_i} \frac{\sin(\pi(x - x_i))}{\pi(x - x_i)}, \tag{8.7}$$

but many other choices could also be made.

8.2. TSSCPP

An application of what precedes is the partition function for counting TSSCPP (totally symmetric self-complementary plane partitions), see figure 22. Counting TSSCPP has become a famous combinatorics problem, due to its link with alternating sign matrices, Razumov–Stroganoff conjecture, Hecke algebras and qKZ relations [21, 4].

A TSSCPP configuration is completely determined by a partition of 1/12th of the hexagon.

In terms of a self-avoiding particle process, see figure 23, this can be viewed as N self-avoiding particles $h_i(t)$ jumping by $\pm\frac{1}{2}$, and such that particle i has to follow a straight line after time $t \geq N - i$:

$$h_i(t) = N - i + \frac{t}{2} \qquad \text{if } t \geq N - i. \tag{8.8}$$

In other words, we have a self-avoiding particle process with some obliged positions. Also, we do not fix the positions of the particles at time $t = 0$ (although we could easily do it in our matrix model).

Notice that if we fix $h_i(N-1) = N - i + (N-1)/2$ at time $t = N - 1$, and if we oblige $h_i(N-i) = \frac{3}{2}(N-i)$ only at time $t = N - i$, then the self-avoiding particle process necessarily evolves in a way such that $h_i(t) = N - i + t/2$ if $t \geq N - i$. In other words, it is sufficient to oblige only one position at each time: we oblige the position $(3t/2, t)$ at time t.

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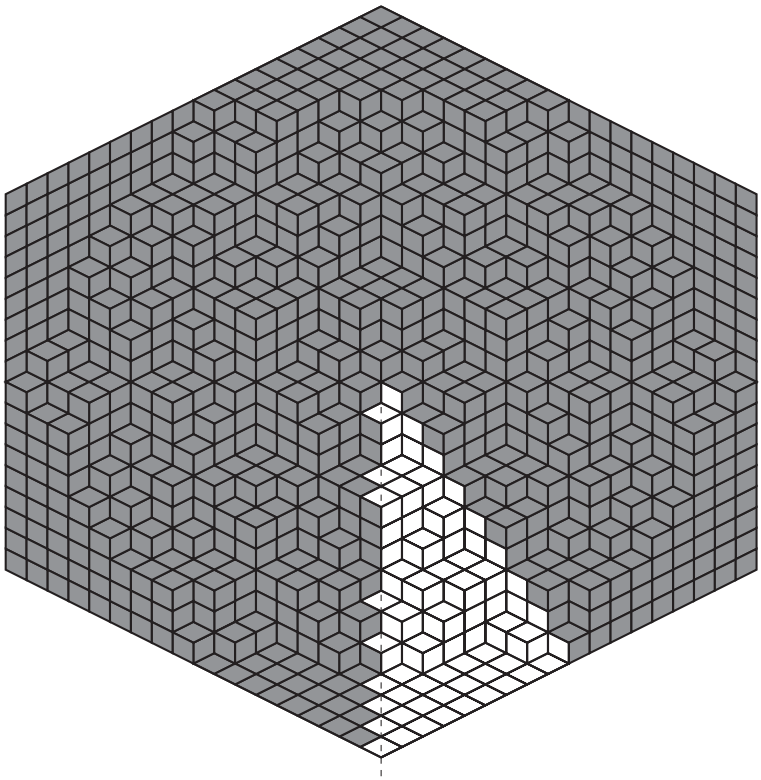

Figure 22. A TSSCPP is a plane partition with all symmetries of the hexagon + self complementarity (i.e. imagine that it is a pile of cubes within a big cube, then it must be equal to its complement). A TSSCPP configuration is completely determined by a partition of 1/12th of the hexagon (the white region).

8.2.1. The matrix model. We choose for $t = 0, \ldots, N-1$

$$A_t(x) = \frac{C_t}{x - 3t/2} \prod_{j=0}^{2N-2-t} \left(x - \frac{t}{2} - j \right) \tag{8.9}$$

where the constant $C_t = 1/t!(2N-2-2t)!$ is such that $A_t(3t/2) = 1$.

We may also choose

$$A_t(x) = \frac{C_t}{(x - 3t/2)\Gamma(x - t/2)} \tag{8.10}$$

where the constant $C_t = (-1)^t/t!$ is such that $A_t(3t/2) = 1$.

For our matrix model, we choose the potential $V_t = 0$ for all t (there is no defect).

The TSSCPP partition function is thus realized by the matrix model:

$$\mathcal{Z}_{\mathrm{TSSCPP}}(N; q) = \int \prod_{t=0}^{N-2} \mathrm{d}M_t \prod_{t'=1/2}^{N-3/2} \mathrm{d}R_{t'} \left(\prod_{t=0}^{N-1} \mathrm{Tr} A_t(M_t) \right)$$
$$\times \prod_{t=0}^{N-2} q^{\mathrm{Tr} M_t} \prod_{t'=1/2}^{N-3/2} \mathrm{e}^{-\mathrm{tr} U_{t'}(R_{t'})} \mathrm{e}^{\mathrm{tr} R_{t'}(M_{t'+1/2} - M_{t'-1/2})} \tag{8.11}$$

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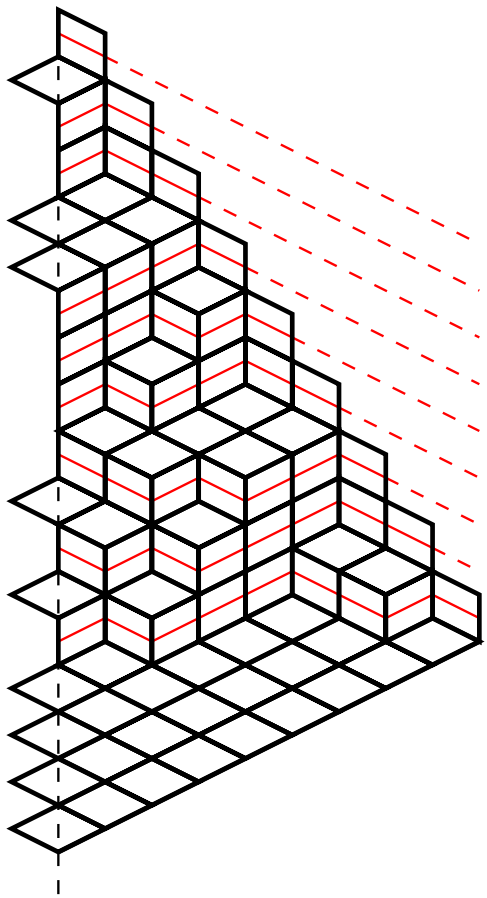

Figure 23. A plane partition of 1/12th of the hexagon can be realized as a self-avoiding particle process such that $h_i(t) = N - i + t/2$ if $t \geq N - i$.

where we integrate $M_0, \ldots, M_{N-2}$ and $R_{1/2}, \ldots, R_{N-3/2}$, and $M_{N-1} = \mathrm{diag}((N-1)/2, \ldots, ((N-1)/2) + N - 1)$ is not integrated upon. All matrices are of size $N \times N$.

8.2.2. Examples.

- $N = 2$:

$$\mathcal{Z}_{\mathrm{TSSCPP}}(2) = \int \mathrm{d}M_0\, \mathrm{d}R_{1/2} \mathrm{Tr} A_0(M_0) q^{\mathrm{Tr} M_0} \mathrm{e}^{-\mathrm{tr} U_{1/2}(R_{1/2})} \mathrm{e}^{\mathrm{tr} R_{1/2}(M_1 - M_0)} \tag{8.12}$$

$$M_1 = \mathrm{diag}(\tfrac{1}{2}, \tfrac{3}{2}), \qquad A_0(x) = \tfrac{1}{2}(x-1)(x-2). \tag{8.13}$$

- $N = 3$:

$$\mathcal{Z}_{\mathrm{TSSCPP}}(3) = \int \mathrm{d}M_0\, \mathrm{d}M_1\, \mathrm{d}R_{1/2} \mathrm{d}R_{3/2} \mathrm{Tr} A_0(M_0) \mathrm{Tr} A_1(M_1) q^{\mathrm{Tr} M_0} q^{\mathrm{Tr} M_1} \times \mathrm{e}^{-\mathrm{tr} U_{1/2}(R_{1/2})} \mathrm{e}^{\mathrm{tr} R_{1/2}(M_1 - M_0)}\, \mathrm{e}^{-\mathrm{tr} U_{3/2}(R_{3/2})} \mathrm{e}^{\mathrm{tr} R_{3/2}(M_2 - M_1)} \tag{8.14}$$

$$M_2 = \mathrm{diag}(1, 2, 3), \qquad A_0(x) = \tfrac{1}{24}(x-1)(x-2)(x-3)(x-4) \tag{8.15}$$

$$A_1(x) = \tfrac{1}{2}(x - \tfrac{1}{2})(x - \tfrac{5}{2})(x - \tfrac{7}{2}). \tag{8.16}$$

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8.2.3. The spectral curve. Since we have chosen all $V_t = 0$, we have the same spectral curve as for the trapezoid of section 7.3. Our partition function is computed as an expectation value:

$$\mathcal{Z}_{\mathrm{TSSCPP}}(N) = \sum_g (\ln q)^{2g-2+N} \mathrm{Res}_{\infty_1} \cdots \mathrm{Res}_{\infty_N} \times W_N^{(g)}(z_1, \ldots, z_N) A_1(\hat{X}_1(z_1)) \cdots A_N(\hat{X}_N(z_N)). \tag{8.17}$$

8.2.4. Fixed position at time $t = 0$. If we want to fix the positions of particles at time $t = 0$, it suffices to introduce a potential:

$$\mathrm{e}^{-V_0(x)} = \frac{\prod_{i=0}^{2N-2}(x-i)}{\prod_{i=1}^{N}(x-h_i(0))}. \tag{8.18}$$

It would be interesting to relate those partition functions with fixed initial conditions for the particles, and see if we have some qKZ equations, and if they can be related to alternating sign matrices partition functions.

It was already noted [73] that there is a matrix model formulation for the six-vertex model counting alternating sign matrices. It would be interesting to compare the spectral curve of the six-vertex matrix model and the spectral curve of the matrix model we introduced here.

9. Application to topological strings

Regarding the applications to algebraic geometry and topological strings, our method allows us to see that Gromov–Witten invariants of the toric Calabi–Yau 3-folds $\mathfrak{X} = \mathbb{C}^3$ with branes (called the topological vertex) are the symplectic invariants of their mirror's spectral curve $\mathcal{S}_{\tilde{\mathfrak{X}}}$. In other words, we confirm that the 'remodeling the B-model' proposal of BKMP [11] holds for those $\mathbb{C}^3$ toric CY 3-folds:

$$\text{BKMP's claim: } \mathrm{GW}_g(\mathfrak{X}) = F_g(\mathcal{S}_{\tilde{\mathfrak{X}}}). \tag{9.1}$$

This claim is in the spirit of the Dijkgraaf–Vafa conjecture, saying that B-model topological strings should be equivalent to some matrix model. Here, we have proved that the topological vertex itself can be written as a matrix model.

In order to prove BKMP for general toric CY 3-folds, It remains to find a 'matrix model way' to glue topological vertices together.

10. Conclusion

We have introduced a new formulation of a self-avoiding particle model in statistical physics, or lozenge tilings, dimer models or plane partitions, in terms of matrix models. Then we briefly discussed how to apply the rich technology of matrix models.

However, this is only a starting point. It is only a proposed framework and we expect that, exploiting the immense knowledge of matrix integrals developed since Wigner–Dyson–Mehta, we can find new consequences for the self-avoiding particle model.

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It should be possible to look at many extensions of this approach, for instance taking appropriate limits, it should be possible to apply this formalism to Dyson processes, or vicious walkers, and recover many results in the matrix model framework.

As an application to algebraic geometry, we have only found the 'topological vertex', i.e. the building block for computing Gromov–Witten invariants of all toric CY 3-folds, but unfortunately it is, at the present time, not known how to perform the gluing of vertices, in the language of symplectic invariants of [27]. This question needs to be addressed.

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Appendix A. Gamma-function and q-product

A.1. Gamma-function

The Gamma-function $\Gamma(x)$ is such that

$$\Gamma(x) = \int_0^\infty t^{x-1}\,\mathrm{d}t\,\mathrm{e}^{-t}. \tag{A.1}$$

If $x = n+1$ is a positive integer:

$$\Gamma(n+1) = n! = n(n-1)(n-2)\cdots \tag{A.2}$$

Γ has poles at all negative integers:

$$\Gamma(-n+\epsilon) \underset{\epsilon\to 0}{\sim} \frac{(-1)^n}{n!\epsilon}. \tag{A.3}$$

We have

$$\Gamma(x)\Gamma(-x) = \frac{-\pi}{x\sin(\pi x)}. \tag{A.4}$$

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Stirling formula:

$$\frac{\Gamma'(x)}{\Gamma(x)} = \psi(x) \sim \ln x - \frac{1}{2x} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nx^{2n}} \tag{A.5}$$

where B_n are the Bernoulli numbers:

$$B_1 = -\tfrac{1}{2}, \qquad B_2 = \tfrac{1}{6}, \qquad B_4 = -\tfrac{1}{30}, \ldots. \tag{A.6}$$

The digamma function $\psi(x)$ is such that

$$\sum_{i=1}^{n} \frac{1}{x-i} = \psi(x) - \psi(x-n). \tag{A.7}$$

A.2. q-product

The q-product is defined as an infinite product:

$$g(x) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{x} q^n\right) \stackrel{\text{def}}{=} \frac{1}{(1-1/x)\Gamma_q(1/x)}. \tag{A.8}$$

The function $\Gamma_q(x)$ shares many similarities with the function $\Gamma(x)$ by replacing integers with q-numbers. It is a quantum Γ-function. q-numbers are

$$[n] = \frac{q^{-n/2} - q^{n/2}}{q^{-1/2} - q^{1/2}}. \tag{A.9}$$

They tend to usual numbers $[n] \to n$ in the limit $q \to 1$.

If n is an integer we have

$$g(q^n) = 0,$$

$$g'(q^n) = g(1) q^{-(n(n+2)-1)/2} (1-q)^{n-1} \prod_{m=1}^{n-1} [m] = g(1) q^{-(n(n+2)-1)/2} [n-1]!. \tag{A.10}$$

Stirling formula at small $\ln q$:

$$\ln g(x) = \frac{1}{\ln q} \sum_{n=0}^{\infty} \frac{(-1)^n B_n}{n!} (\ln q)^n Li_{2-n}(1/x). \tag{A.11}$$

For the quantum digamma function ψ_q that gives:

$$\begin{aligned} \psi_q(x) := \frac{x g'(x)}{g(x)} &= \sum_n \frac{q^n}{x - q^n} \\ &\sim -\frac{1}{\ln q} \sum_{n=0}^{\infty} \frac{(-1)^n B_n}{n!} (\ln q)^n Li_{1-n}(1/x) \end{aligned} \tag{A.12}$$

where Li_n is the polylogarithm:

$$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n} \tag{A.13}$$

$$Li_1(x) = -\ln(1-x), \qquad Li_0(x) = \frac{x}{1-x}, \qquad Li_n'(x) = \frac{1}{x} Li_{n-1}(x). \tag{A.14}$$

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That is:

$$\psi_q(x) \sim \frac{1}{\ln q}\left[\ln\left(1-\frac{1}{x}\right) - \frac{\ln q}{2(x-1)} - \sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}(\ln q)^{2n} Li_{1-2n}(x)\right]. \tag{A.15}$$

Appendix B. Introduction to symplectic invariants

Here, we briefly recall the definition of symplectic invariants, but we refer the reader to [27, 28] for more details.

They were first introduced in [24, 15] and further formalized in [27] as a solution of the so-called 'topological expansion' of matrix integrals. But then, in [27], they were defined as algebraic quantities for spectral curves, independent of the existence of an underlying matrix model.

B.1. Definition of symplectic invariants

Consider a spectral curve $\mathcal{S} = (\mathcal{L}, x, y)$, where $\mathcal{L}$ is a compact Riemann surface, and x and y are two analytical functions, such that $\mathrm{d}x$ and $\mathrm{d}y$ are meromorphic forms on $\mathcal{L}$.

The branchpoints a_i are defined as the zeroes of $\mathrm{d}x$:

$$\mathrm{d}x(a_i) = 0. \tag{B.1}$$

We assume that the spectral curve is regular, i.e. we have a finite number of branchpoints, and they are simple branchpoints, which means that each a_i is a simple zero of $\mathrm{d}x$, and $\mathrm{d}y(a_i) \neq 0$. This is equivalent to saying that, near a_i, we have a square root behavior:

$$y(z) \sim y(a_i) + C\sqrt{x(z) - x(a_i)}. \tag{B.2}$$

Since we have a square root branchpoint near a_i, this means that in the vicinity of a_i, there exists $\bar{z} \neq z$ on the other branch, i.e. $x(\bar{z}) = x(z)$:

$$\text{near } a_i, \quad \exists! \bar{z} \neq z, \qquad x(\bar{z}) = x(z). \tag{B.3}$$

$\bar{z}$ is called the conjugate of z near a_i. It is defined only locally near branchpoints and it is not necessarily defined globally.

On a Riemann surface $\mathcal{L}$, there exists a 'Bergman kernel', i.e. a symmetric meromorphic 2-form, having a double pole, with no residue, on the diagonal, i.e. which behaves like

$$B(z_1, z_2) \underset{z_1 \to z_2}{\sim} \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1 - z_2)^2} + \mathrm{O}(1) \tag{B.4}$$

where z is any local parameter.

The Bergman kernel is unique if we also fix its periods $\oint_{\mathcal{A}_i} B$, but this is not necessary for the definition of symplectic invariants. Let us assume that we have chosen one Bergman kernel $B(z_1, z_2)$.

Definition B.1. *We define the following n-forms:*

$$W_1^{(0)}(z) = -y(z)\,\mathrm{d}x(z) \tag{B.5}$$

$$W_2^{(0)}(z_1, z_2) = B(z_1, z_2) \tag{B.6}$$

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and by recursion for $2-2g-n<0$ *and* $n\geq 1$, *and if we denote collectively* $J=\{z_2,\dots,z_n\}$:

$$W_n^{(g)}(z_1,J)=\sum_i \mathrm{Res}_{z\to a_i}K(z_1,z)\Bigg[W_{n+1}^{(g-1)}(z,\bar z,J)+\sum_{h=0}^{g}{}'\sum_{I\subset J}W_{1+|I|}^{(h)}(z,I)W_{n-|I|}^{(g-h)}(\bar z,J\setminus I)\Bigg] \tag{B.7}$$

where

$$K(z_1,z)=\frac{\int_z^{\bar z}B(z_1,z')}{2(y(z)-y(\bar z))\,\mathrm{d}x(z)}. \tag{B.8}$$

For $n=0$, *and* $g\geq 2$, *we define*

$$F_g=W_0^{(g)}=\frac{1}{2-2g}\sum_i \mathrm{Res}_{z\to a_i}W_1^{(g)}(z)\Phi(z) \tag{B.9}$$

where Φ *is any function such that* $\mathrm{d}\Phi=y\,\mathrm{d}x$.

The $W_n^{(g)}$s defined in this way are always symmetric meromorphic n-forms, having poles only at branchpoints (except $W_1^{(0)}$ and $W_2^{(0)}$). Also F_g does not depend on the choice of integration constant for Φ.

There is also a definition for F_0 and F_1, but we refer the reader to [27].

B.2. Symplectic invariants of genus-zero spectral curves

An interesting example is when $\mathcal{L}$ is the complex plane $\mathbb{C}$. x and y are functions of a complex variable z. In that case we have

$$B(z_1,z_2)=\frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1-z_2)^2}. \tag{B.10}$$

For example we have

$$W_3^{(0)}(z_1,z_2,z_3)=\sum_i\frac{1}{x''(a_i)y'(a_i)}\frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3}{(z_1-a_i)^2(z_2-a_i)^2(z_3-a_i)^2}. \tag{B.11}$$

B.3. Examples of spectral curves and their symplectic invariants

Let us give a few examples of spectral curves:

- The spectral curve $\mathcal{S}_{\mathrm{WP}}$ defined by the functions $x(z)=z^2, y(z)=(1/2\pi)\sin(2\pi z)$, i.e. $y=(1/2\pi)\sin(2\pi\sqrt{x})$ appears in the Weil–Petersson volumes of moduli spaces, its F_gs are the Weil–Petersson volumes: $F_g(\mathcal{S}_{\mathrm{WP}})=\mathrm{Vol}(\mathcal{M}_g)$, see [29].
- The spectral curve $\mathcal{S}_{\mathrm{Airy}}$ defined by $y^2-x=0$, i.e. $y=\sqrt{x}$ is associated with the Airy kernel law, and to the universal behavior of extreme eigenvalue distribution, i.e. to the Tracy–Widom law [71]. The F_gs of the Airy curve are equal to zero: $F_g(\mathcal{S}_{\mathrm{Airy}})=0$.

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- The spectral curve $y = \sqrt{\mathrm{Pol}_{2m+1}(x)}$, where Pol_{2m+1} is a polynomial of degree $2m+1$, is associated with the $(2m+1, 2)$ minimal model in conformal field theory (with central charge $c = 1 - 3((2m-1)^2/(2m+1))$), i.e. a reduction of the KdV hierarchy. Its F_gs are related to the KdV Tau-function, and they can be computed by the asymptotic expansion of the solution of a Painlevé-type equation (in fact the $m+1$st Gelfand–Dikii equation $R_{m+1}(u(t)) = t$). The case $m = 0$, i.e. the $(1, 2)$ minimal model, is the Airy case. The case $m = 1$, i.e. the $(3, 2)$ minimal model, is called pure gravity, and the case $m = 2$, i.e. the $(5, 2)$ minimal model, is called Lee–Yang singularity.
- Spectral curves of type $\mathrm{Pol}(\mathrm{e}^x, \mathrm{e}^y) = 0$ appear in topological strings, where Pol is a polynomial. More precisely, consider a Toric Calabi–Yau 3-fold $\mathfrak{X}$. Through mirror symmetry [40], it has a mirror $\tilde{\mathfrak{X}}$, which is also a toric Calabi–Yau 3-fold, and which is given by a submanifold of $\mathbb{C}^4$ of equation $H(\mathrm{e}^x, \mathrm{e}^y) = uv$ where H is a polynomial. The spectral curve $\mathcal{S}_{\tilde{\mathfrak{X}}}$ is defined as the singular locus of $\tilde{\mathfrak{X}}$, which satisfies the equation $H(\mathrm{e}^x, \mathrm{e}^y) = 0$. The 'remodeling the B-model' conjecture of BKMP [11] is that the generating function of Gromov–Witten invariants $\mathrm{GW}_g(\mathfrak{X})$ of genus g of $\mathfrak{X}$ is equal to the symplectic invariant $F_g(\mathcal{S}_{\tilde{X}})$ of the spectral curve $\mathcal{S}_{\tilde{\mathfrak{X}}}$ of the mirror:

$$\mathrm{GW}_g(\mathfrak{X}) \stackrel{?}{=} F_g(\mathcal{S}_{\tilde{\mathfrak{X}}}). \tag{B.12}$$

  This conjecture was proved in a few cases, and in particular for the family of Hirzebruch manifolds $\mathfrak{X}_p = \mathrm{O}(-p) \oplus \mathrm{O}(p-2) \to \mathbb{P}^1$ (which include the connifold), see [26]. It was also partially proved for $SU(n)$ Seiberg–Witten theories [46], and for 2^* geometries [67].

B.4. Some properties of symplectic invariants

Let us now give a few properties of symplectic invariants. We refer the reader to [27, 28] for the general theory, and for proofs and details.

Properties:

- *Homogeneity.* If we rescale $y \to \lambda y$, which we denote $\lambda\mathcal{S} = (x, \lambda y)$, we have

$$F_g(\lambda\mathcal{S}) = \lambda^{2-2g} F_g(\mathcal{S}), \qquad W_n^{(g)}(\lambda\mathcal{S}) = \lambda^{2-2g-n} W_n^{(g)}(\mathcal{S}). \tag{B.13}$$

  In particular that implies $F_g(-\mathcal{S}) = F_g(\mathcal{S})$.
- *Symplectic invariance.* If two spectral curves $\mathcal{S} = \{y(x)\}$ and $\tilde{\mathcal{S}} = \{\tilde{y}(\tilde{x})\}$ are symplectically equivalent, i.e. if there is a map from $\mathbb{C} \times \mathbb{C}$ to $\mathbb{C} \times \mathbb{C}$ which sends one spectral curve to the other and conserves the symplectic form $\mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y}$, then

$$F_g(\mathcal{S}) = F_g(\tilde{\mathcal{S}}). \tag{B.14}$$

  For instance, we can change $y \to y + R(x)$, where $R(x)$ is a rational function, or $(y \to x, x \to -y)$, or $(x \to \ln x, y \to xy), \ldots$, without changing F_g.
  The $W_n^{(g)}$ with $n \geq 1$ are not symplectic invariants; they change under symplectic transformations, but they change in a covariant way. For example, $W_1^{(g)}(\mathcal{S}) - W_1^{(g)}(\tilde{\mathcal{S}})$ is an exact form.

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- *Variations.* Consider an infinitesimal deformation of the spectral curve $(x, y) \to (x + \epsilon\delta x, y + \epsilon\delta y)$, such that $\Omega = \delta y\,\mathrm{d}x - \delta x\,\mathrm{d}y$ is a meromorphic differential form on the spectral curve. Any meromorphic form Ω is dual to a cycle Ω^* on the curve. The duality pairing is realized through the Bergman kernel:

$$\Omega(z) = \oint_{\Omega^*} B(z, z'). \tag{B.15}$$

Then, the infinitesimal variation of F_g and $W_n^{(g)}$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} W_n^{(g)}(z_1, \ldots, z_n) = \oint_{\Omega^*} W_{n+1}^{(g)}(z_1, \ldots, z_n, z'). \tag{B.16}$$

The infinitesimal variation of F_g is the case $n = 0$: $\mathrm{d}F_g/\mathrm{d}\epsilon = \oint_{\Omega^*} W_1^{(g)}(z')$.

- *Limits.* Consider a one-parameter family of spectral curves $\mathcal{S}(t)$, such that, at $t = t_c$, the spectral curve $\mathcal{S}(t_c)$ becomes singular. Consider its blow up: $\mathcal{S}(t) \sim (t - t_c)^\nu\, \mathcal{S}_c + o((t - t_c)^\nu)$, where the exponent ν is chosen such that the curve $\mathcal{S}_c$ is regular. Then we have

$$F_g(\mathcal{S}(t)) \sim (t - t_c)^{(2-2g)\nu} F_g(\mathcal{S}_c) + o((t - t_c)^{(2-2g)\nu}) \tag{B.17}$$

and more generally

$$W_n^{(g)}(\mathcal{S}(t)) \sim (t - t_c)^{(2-2g-n)\nu} W_n^{(g)}(\mathcal{S}_c) + o((t - t_c)^{(2-2g-n)\nu}). \tag{B.18}$$

This theorem is very useful and gives the asymptotic behaviors. For instance, if we zoom near a regular branch point, the spectral curve always behaves like $y = \sqrt{x}$, i.e. $\mathcal{S}_c = \mathcal{S}_{\mathrm{Airy}}$, and we find the Airy law, and Tracy–Widom. Near an algebraic cusp singularity $y \sim x^{p/q}$, we find the spectral curve $\mathcal{S}_c = \mathcal{S}_{(p,q)}$ of the (p, q) minimal model of conformal field theories (of central charge $c = 1 - 6((p - q)^2/pq)$), which is a reduction of the KP hierarchy.

These are just a few of the properties satisfied by the F_gs. See [27] for more. For instance, there are also modular properties and holomorphic anomaly equations, Hirota equations, and diagrammatic representations.

Appendix C. Solution of matrix models

Let us present a brief review about matrix models.

C.1. Generalities on the solution of matrix models

Consider a matrix integral of type

$$\mathcal{Z} = \int_{\prod H_N(\mathcal{C}_i)} \prod_{i=1}^{p} \mathrm{d}M_i\, \mathrm{e}^{-Q\mathrm{Tr}[\sum_{i=1}^{p} V_i(M_i) + \sum_i c_i M_i M_{i+1}]} \tag{C.1}$$

where $H_N(\mathcal{C}_i)$ is the set of normal matrices having their eigenvalues on contour $\mathcal{C}_i$, and where M_{p+1} is not integrated upon.

If one wants to find a large Q expansion of $\ln \mathcal{Z}$, the answer is that one has first to compute the 'spectral curve' $\mathcal{S}$ (defined below). Then, in [33], it is proved that:

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Theorem C.1. *If $\mathcal{S}$ is the spectral curve associated with potentials V_i and paths $\mathcal{C}_i$, and if $\mathcal{Z}$ has a large Q expansion of the form*

$$\ln \mathcal{Z} = \sum_{g=0}^{\infty} Q^{2-2g} F_g(\mathcal{S}) \tag{C.2}$$

then the coefficient $F_g = F_g(\mathcal{S})$ is the symplectic invariant of degree $2-2g$ of the spectral curve $\mathcal{S}$. Symplectic invariants $F_g(\mathcal{S})$ of a spectral curve $\mathcal{S}$ were defined and introduced in [27].

We shall explain the meaning of this theorem below in further detail. We shall explain how to compute the spectral curve $\mathcal{S}$ of a matrix model, and how to compute its symplectic invariants F_g. We recall the definition of symplectic invariants in appendix B.

Let us just mention that finding the spectral curve is rather automatic. It is mostly an algebraic task, and for most examples the spectral curve is a rather simple object. Here, we have the spectral curves of Kenyon–Okounkov–Sheffield [48].

Then, computing the symplectic invariants of a spectral curve is rather easy. The definition of symplectic invariants [27] involves computing residues of rather standard functions (exponentials, logs, rational functions, etc), and can be completely automated, see appendix B. It is really an efficient method. For example, F_1 can often be computed by hand. Moreover, symplectic invariants satisfy many properties, which make them really convenient to use. For example, there are formulae for computing their derivatives with respect to any parameter and formulae for finding their limits near singularities.

C.2. Generalities about loop equations

Theorem C.1 for the chain of matrices was proved from loop equations [33].

Loop equations are obtained by integration by parts in the matrix integral, or equivalently by writing that an integral is invariant under change of variables. For example, assuming that we make an infinitesimal change of variable in equation (C.1):

$$M_3 \to M_3 + \epsilon M_2^5 M_3^3 + \mathrm{O}(\epsilon^2) \tag{C.3}$$

one finds the Jacobian to first order in ϵ:

$$\mathrm{d}M_3 \to \mathrm{d}M_3(1 + \epsilon(\mathrm{Tr}M_2^5\mathrm{Tr}M_3^2 + \mathrm{Tr}M_2^5 M_3\mathrm{Tr}M_3 + \mathrm{Tr}M_2^5 M_3^2\mathrm{Tr}\,\mathrm{Id}) + \mathrm{O}(\epsilon^2)). \tag{C.4}$$

Writing that the integral is invariant to first order in ϵ implies

$$\begin{aligned}\langle \mathrm{Tr}M_2^5\mathrm{Tr}M_3^2 + \mathrm{Tr}M_2^5 M_3\mathrm{Tr}M_3 + \mathrm{Tr}M_2^5 M_3^2\mathrm{Tr}\,\mathrm{Id}\rangle \\ = Q\langle \mathrm{Tr}V_3'(M_3)M_2^5 M_3^3 + c_2\mathrm{Tr}M_2^6 M_3^3 + c_3\mathrm{Tr}M_2^5 M_3^2 M_4\rangle.\end{aligned} \tag{C.5}$$

Similarly, by considering other appropriate changes of variables, one can write a family of relationships of the type equation (C.5) between expectation values. This was done systematically in [33].

Instead of considering expectation values of powers $\langle \mathrm{Tr}M^k\rangle$, it is more convenient to group them in a formal generating function $\bar{W}_1(x) = \langle \mathrm{Tr}(1/(x-M))\rangle = \sum_{k=0}^{\infty}(\langle \mathrm{Tr}M^k\rangle/x^{k+1})$. This leads us to introduce

$$\bar{W}_n(x_1,\ldots,x_n) = \left\langle \mathrm{Tr}\frac{1}{x_1-M_1}\mathrm{Tr}\frac{1}{x_2-M_1}\cdots\mathrm{Tr}\frac{1}{x_n-M_1}\right\rangle_c \tag{C.6}$$

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(where the subscript $\langle\cdot\rangle_c$ means the cumulant, or connected part). Since we made the hypothesis that there exists a large Q expansion we write

$$\bar{W}_n(x_1,\dots,x_n)=\sum_{g=0}^{\infty}Q^{2-2g-n}\bar{W}_n^{(g)}(x_1,\dots,x_n). \tag{C.7}$$

The problem consists in finding a family of relationships among those quantities, and which allow us to find the solution, i.e. compute all of them. This was done in [33, 32].

The result of [33] is that the $W_n^{(g)}(x_1,\dots,x_n)=\bar{W}_n^{(g)}(x_1,\dots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n$s are the n-forms of theorem C.1, i.e. the $W_n^{(g)}$s of [27], and where the spectral curve $\mathcal{S}=(x,y)$ is $y=-\bar{W}_1^{(0)}(x)$.

In some sense, $\bar{W}_1^{(0)}(x)$ can be viewed as the 'large Q' limit of the resolvent of the matrix M_1 (the first matrix in the chain equation (C.1)). It is often called the 'equilibrium distribution of eigenvalues of M_1', but one should take this description with some care. Indeed, $\bar{W}_1^{(0)}$ can be shown to be the 'weak' large Q limit of the resolvent, for some potentials (in particular in the potentials do not depend on Q), and some integration paths $\mathcal{C}_i$, but this is probably wrong in general. In fact, in our case, the potentials do depend on Q, and saying that the spectral curve is the 'large Q' limit of the resolvant is slightly wrong (and at least one needs to make precise the notion of large Q limit).

The only correct definition of the spectral curve, which corresponds to theorem C.1, is that $y=\bar{W}_1^{(0)}(x)$ is the first term in the formal large Q expansion, doing as if the potentials were independent of Q.

Working to order $g=0$ in the expansion equation (C.7) within the loop equations is formally equivalent to replacing expectation values of a product of traces, by a product of expectation values of each trace (indeed, the connected part comes with a factor Q^{2-2g-n}, whereas the non-connected term, i.e. the factorized term, comes with a factor Q^{2-2g+n}, i.e. all terms which are not disconnected do not contribute to the highest power of Q):

$$\langle\mathrm{Tr}A_1\mathrm{Tr}A_2\cdots\mathrm{Tr}A_k\rangle\longrightarrow\langle\mathrm{Tr}A_1\rangle\langle\mathrm{Tr}A_2\rangle\cdots\langle\mathrm{Tr}A_k\rangle. \tag{C.8}$$

This formal manipulation allows us to find an equation which determines the spectral curve $\bar{W}_1^{(0)}(x)$.

C.3. Spectral curve of the chain of matrices

Here, we shall just give a 'ready to use' recipe of how to find the spectral curve for a chain of matrices matrix model of type equation (C.1). This recipe is extracted from [33], and it is really technical to explain how to obtain it. However, it is easy to use.

C.4. General case

The spectral curve of a matrix integral of type equation (C.1):

$$\mathcal{Z}=\int_{\prod H(\mathcal{C}_i)}\prod_{i=1}^{p}\mathrm{d}M_i\,\mathrm{e}^{-Q\mathrm{Tr}[\sum_{i=1}^{p}V_i(M_i)+\sum_i c_iM_iM_{i+1}]} \tag{C.9}$$

is characterized by a set (see [32, 33]) of $p+2$ analytical functions of a variable z (z belongs to a Riemann surface $\mathcal{L}$). There is one such analytical function for each matrix

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M_i, $i = 1, \ldots, p+1$ of the chain, plus one additional function at the end of the chain. Let us call them

$$\hat{X}_i(z), \qquad i = 0, \ldots, p+1. \tag{C.10}$$

Those functions are completely determined by the following system of equations:

$$\forall i = 2, \ldots, p, \qquad c_{i-1}\hat{X}_{i-1}(z) + c_i\hat{X}_{i+1}(z) + V_i'(\hat{X}_i(z)) = 0 \tag{C.11}$$

and

$$\bar{W}_1^{(0)}(\hat{X}_1(z)) = \hat{X}_0(z) = V_1'(\hat{X}_1(z)) - c_1\hat{X}_2(z), \tag{C.12}$$

together with the conditions:

- $\hat{X}_p(z)$ has simple poles at the values of z such that $\hat{X}_{p+1}(z)$ is an eigenvalue of M_{p+1}. Let us call ζ_i, the value of z such that $\hat{X}_{p+1}(z) = \lambda_i$ the ith eigenvalue of M_{p+1}. Then we have in the vicinity of $z \to \zeta_i$:

$$\hat{X}_p(z) \sim \frac{1}{\hat{X}_{p+1}(z) - \lambda_i}. \tag{C.13}$$

- The function $\bar{W}_1^{(0)}(x)$ is analytical outside of the cuts $[a_i, b_i]$, the endpoints of the cuts being zeroes of $\mathrm{d}\hat{X}_1$. Near $\hat{X}_1 \to \infty$ (only in the physical sheet), we have

$$\hat{X}_0(z) \sim \frac{N/Q}{\hat{X}_1(z)} + \mathrm{O}(1/\hat{X}_1(z)^2). \tag{C.14}$$

- The genus of the Riemann surface $\mathcal{L}$, and the period integrals $\oint_{\mathcal{A}_i} \hat{X}_0\,\mathrm{d}\hat{X}_1$ on non-contractible cycles $\mathcal{A}_i$ of $\mathcal{L}$, are related to the choice of contours $\mathcal{C}_i$ in the matrix integral equation (C.9). The following quantity is called the 'filling fraction':

$$\epsilon_i = \frac{1}{2\mathrm{i}\pi}\oint_{\mathcal{A}_i} \hat{X}_0\,\mathrm{d}\hat{X}_1, \tag{C.15}$$

  and it is such that $Q\epsilon_i$ is the number of eigenvalues of M_1 contained in the domain surrounded by the projection in $\mathbb{C}$ of the cycle $\mathcal{A}_i$ by the application $\hat{X}_1$.

More generally, $(Q/2\mathrm{i}\pi)\oint_{\mathcal{A}_i} \hat{X}_{j-1}\,\mathrm{d}\hat{X}_j$ is the number of eigenvalues of M_j contained in the projection in $\mathbb{C}$ of the cycle $\mathcal{A}_i$ by the application $\hat{X}_j$.

Finding the spectral curve for arbitrary potentials V_i and arbitrary contours $\mathcal{C}_i$ can be extremely tedious. However, here we are interested in formal matrix integrals, which are defined as formal power series expansions in some formal parameter, and in particular, the spectral curve we are looking for must also be found as a formal power series. In particular, the genus is the genus of the spectral curve when we send the formal parameter to 0, and in general that simplifies the computation of the spectral curve a lot. Very often the genus is, in fact, zero.

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C.4.1. Topological expansion. When we have determined the spectral curve $\mathcal{S}$, i.e. the two functions $x(z) = \hat{X}_1(z)$ and $y(z) = \hat{X}_0(z)$:

$$\mathcal{S} = (x, y) = \mathcal{S}_{1,0} = (\hat{X}_1, \hat{X}_0), \qquad x(z) = \hat{X}_1(z), \qquad y(z) = \hat{X}_0(z), \tag{C.16}$$

we have

$$\ln \mathcal{Z} = \sum_{g=0}^{\infty} F_g(\mathcal{S}). \tag{C.17}$$

It is then useful to use the symplectic invariance of the F_gs.

Consider the following spectral curves:

$$\mathcal{S}_{i,j} = (\hat{X}_i, \hat{X}_j). \tag{C.18}$$

Due to equation (C.11), we have

$$c_i \, \mathrm{d}\hat{X}_{i+1} \wedge \mathrm{d}\hat{X}_i = -c_{i-1} \, \mathrm{d}\hat{X}_{i-1} \wedge \mathrm{d}\hat{X}_i = c_{i-1} \, \mathrm{d}\hat{X}_i \wedge \mathrm{d}\hat{X}_{i-1} \tag{C.19}$$

i.e. all the spectral curves $\mathcal{S}_{i,i+1}$ are symplectically equivalent:

$$c_i \mathcal{S}_{i,i+1} \equiv c_{i-1} \mathcal{S}_{i-1,i} \equiv -c_{i-1} \mathcal{S}_{i,i-1} \tag{C.20}$$

and therefore the theorem of symplectic invariance of the F_gs [27, 30] gives $\forall i$:

$$c_i^{2-2g} F_g(\mathcal{S}_{i,i+1}) = c_{i-1}^{2-2g} F_g(\mathcal{S}_{i-1,i}) = c_{i-1}^{2-2g} F_g(\mathcal{S}_{i,i-1}). \tag{C.21}$$

In other words, the F_gs can be computed with the spectral curve $\mathcal{S} = (x, y)$, where x and y are any two consecutive $\hat{X}_i, \hat{X}_{i+1}$. We do not need to choose the pair $\hat{X}_1, \hat{X}_0$.

C.4.2. Densities and correlation functions. For the general chain of matrices, the $W_n^{(g)}$s were defined as the Stieltjes transforms of the density correlation functions of the first matrix M_1 in the chain, i.e. the resolvent equation (C.6). The densities can be recovered by taking the discontinuities:

$$\rho(h) = \sum_g Q^{1-2g} \rho^{(g)}(h) = \left\langle \sum_i \delta(h - h_i) \right\rangle \tag{C.22}$$

where

$$\rho^{(g)}(h) = \frac{1}{2\mathrm{i}\pi} (\bar{W}_1^{(g)}(h - \mathrm{i}0) - \bar{W}_1^{(g)}(h + \mathrm{i}0)). \tag{C.23}$$

We recall that the spectral curve used to compute $W_n^{(g)}$ here is the spectral curve $\mathcal{S}_{1,0} = (\hat{X}_1, \hat{X}_0)$, and the $W_n^{(g)}$s are not invariant under symplectic transformations.

However, just by applying equation (B.16), one can see that, for any matrix M_j of the chain, one has that the density of eigenvalues of M_j:

$$\rho_j(h) = \sum_g Q^{1-2g} \rho_j^{(g)}(h) = \left\langle \sum_i \mathrm{Tr}\delta(h - M_j) \right\rangle \tag{C.24}$$

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is given by

$$\rho_j^{(g)}(h) = \frac{1}{2\mathrm{i}\pi\,\mathrm{d}\hat{X}_j(h)}(W_1^{(g)}(h-\mathrm{i}0) - W_1^{(g)}(h+\mathrm{i}0))_{\mathcal{S}_{j,j-1}}. \tag{C.25}$$

Therefore, the density of eigenvalues of each matrix of the chain can be computed to all orders in the large Q expansion, using the $W_n^{(g)}$s defined in [27].

In the general chain of matrices, the support of the densities must be related to the integration paths $\mathcal{C}_i$s: one defines the support of densities as the loci where densities are real. The support of densities, together with their analytical continuations, must be homologicaly equivalent to the integration paths $\mathcal{C}_i$s. Also, saying that the paths $\mathcal{C}_i$ are steepest descent paths is more or less equivalent to saying that the densities ρ_i must be real and positive on their supports, and i times the analytical continuation of the densities, along the analytical continuations of the supports, must be real and strictly positive (stability condition). This is a highly nontrivial condition in general.

Fortunately, for formal matrix integrals, defined as formal power series in a formal parameter, the small limit of the parameter often gives a very simple matrix model, with a very simple spectral curve, and it is often easy to check positivity to leading order, and then that the formal power series expansion does not destroy the supports and positivity.

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