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Determinantal formulae and loop equations

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Abstract

We prove that the correlations functions, generated by the determinantal process of the Christoffel-Darboux kernel of an arbitrary order 2 ODE, do satisfy loop equations.

1 Introduction

It is well known that matrix models satisfy both loop equations [12], and determinantal formulae [19, 13, 9, 20, 17]. However, both notions (loop equations and determinantal formulae) exist beyond matrix models. In this paper we show that the correlators obtained from determinants of the Christoffel-Darboux kernel of an arbitrary differential system of order 2, do satisfy loop equations.

For matrix models, the "fermionic" correlators are expectation values of ratios of characteristic polynomials:

$$\mathcal{K}_n(x_1,\dots,x_n;y_1,\dots,y_n) = \left\langle \frac{\prod_{i=1}^n \det(x_i - M)}{\prod_{i=1}^n \det(y_i - M)} \right\rangle \tag{1-1}$$

where $< . >$ is the expectation value with some matrix measure $dM\, \mathrm{e}^{-\operatorname{Tr} V(M)}$. It was found [16, 6, 3, 4] that there exists some kernels $\mathcal{K}_{i,j}$, such that these correlators satisfy Giambelli-type determinantal formulae:

$$\mathcal{K}_n(x_1,\dots,x_n;y_1,\dots,y_n) = \frac{\prod_{i,j}(x_i - y_j)}{\prod_{i>j}(x_i - x_j)\ \prod_{i>j}(y_i - y_j)}\, \det(\mathcal{K}_{i,j}(x_i;y_j)) \tag{1-2}$$

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On the other hand the "bosonic" correlators are expectation values of product of traces:

$$\mathcal{W}_n(x_1,\dots,x_n) = \left\langle \prod_{i=1}^n \mathrm{Tr}\, \frac{1}{x_i - M} \right\rangle \tag{1-3}$$

and in fact we prefer to consider their cumulants (also called connected parts):

$$W_n(x_1,\dots,x_n) = \left\langle \prod_{i=1}^n \mathrm{Tr}\, \frac{1}{x_i - M} \right\rangle_c \tag{1-4}$$

It is clear that traces can be obtained as some limits of determinants:

$$\lim_{y\to x} \frac{\partial}{\partial x} \frac{\det(x-M)}{\det(y-M)} = \mathrm{Tr}\, \frac{1}{x-M} \tag{1-5}$$

and one finds the determinantal formulae for the $\mathcal{W}_n$'s:

$$\mathcal{W}_n(x_1,\dots,x_n) = \text{"} \det(K(x_i,x_j)) \text{"} \tag{1-6}$$

where the bracketed determinant "det" introduced by [6], consists in renormalizing the 1 and 2 point function (the meaning of this bracket notation is explained below in def. 2.4), and where K is a kernel related to the kernels $\mathcal{K}_{i,j}$ in eq.(1-2), see def. 2.4 below.

In other words, knowing K, i.e. knowing fermionic correlators, we can find the W_n's, i.e. bosonic ones.

Conversely, one may reconstruct K from the W_n's, under the condition that the W_n's obey some loop equations. The formula which holds for matrix models, is an exponential formula:

$$K(x,y) = \exp\left(\sum_n \frac{1}{n!} \int_y^x \dots \int_y^x W_n \right) \tag{1-7}$$

This formula is nothing but a rewriting of the Heine formula [21] for matrix models, or also it can be viewed as the Sato formula of integrable systems [5, 18], it simply follows from

$$\frac{\det(x-M)}{\det(y-M)} = \mathrm{e}^{\mathrm{Tr}\,\left(\ln(x-M) - \ln(y-M)\right)} = \mathrm{e}^{\int_y^x \mathrm{Tr}\, \frac{dx'}{x'-M}} \tag{1-8}$$

Here, we are going to prove that those formulae continue to hold even without an underlying matrix model.

2 From determinants towards loop equations

In this section, we start from a 2×2 differential system $\Psi' = \mathcal{D}.\Psi$, we define its Christoffel-Darboux kernel K [19], and we define the correlation functions W_n from a determinantal formula. We prove that they satisfy loop equations.

All this section could hopefully be generalized to higher order $m\times m$ differential systems with $m>2$, but for simplicity, we consider only the 2×2 case here.

Again, we emphasize that all the present section is independent from any underlying random matrix problem.

2.1 Differential system of order 2

Consider a differential system of order 2:

$$\frac{d}{dx}\Psi(x) = \mathcal{D}(x)\,\Psi(x) \qquad , \qquad \Psi(x) = \begin{pmatrix} \psi(x) & \phi(x) \\ \tilde\psi(x) & \tilde\phi(x) \end{pmatrix} \qquad , \qquad \det\Psi = 1 \tag{2-1}$$

where $\mathcal{D}(x)$ is a traceless matrix, whose coefficients a,b,c,d are rational functions of x:

$$\mathcal{D}(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \qquad , \qquad \mathrm{Tr}\,\mathcal{D} = a+d = 0 \tag{2-2}$$

Notice that we have:

$$\mathcal{D} = \Psi'\,\Psi^{-1} = \begin{pmatrix} \psi'\tilde\phi - \phi'\tilde\psi & \phi'\psi - \psi'\phi \\ \tilde\psi'\tilde\phi - \tilde\phi'\tilde\psi & \tilde\phi'\psi - \tilde\psi'\phi \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2-3}$$

Notice that since $\mathrm{Tr}\,\mathcal{D}=0$, we have:

$$\mathcal{D}^t\,A = -A\,\mathcal{D} \qquad , \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2-4}$$

We define the spectral curve as the characteristic polynomial of $\mathcal{D}$:

Definition 2.1 *The spectral curve is:*

$$\hat{\mathcal{E}}(x,y) = \det(y-\mathcal{D}(x)) = y^2 - \frac{1}{2}\,\mathrm{tr}\,\mathcal{D}^2(x) = y^2 + a(x)d(x) - b(x)c(x) \tag{2-5}$$

The equation $\hat{\mathcal{E}}(x,y)=0$ is an algebraic equation, and it is hyperelliptical since it is of degree 2 in y.

Then we define the Christoffel-Darboux kernel associated to Ψ:

Definition 2.2 *We define the Christoffel-Darboux kernel $K(x_1,x_2)$ associated to Ψ, by:*

$$K(x_1,x_2) = \frac{\psi(x_1)\tilde\phi(x_2) - \tilde\psi(x_1)\phi(x_2)}{x_1-x_2} = \frac{1}{x_1-x_2}\,\vec\psi(x_1)^t\,A\,\vec\phi(x_2) \tag{2-6}$$

where:

$$\vec{\psi}(x_1) = \begin{pmatrix} \psi(x_1) \\ \tilde{\psi}(x_1) \end{pmatrix} \quad , \quad \vec{\phi}(x_2) = \begin{pmatrix} \phi(x_2) \\ \tilde{\phi}(x_2) \end{pmatrix} \quad , \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2-7}$$

A is called the Christoffel-Darboux matrix [8].

Then we define the correlation functions:

Definition 2.3 *We define the* **connected** *correlation functions by:*

$$\begin{aligned} W_1(x) &= \lim_{x'\to x} \left(K(x,x') - \frac{1}{x-x'} \right) = \psi'(x)\tilde{\phi}(x) - \tilde{\psi}'(x)\phi(x) \\ &= \vec{\psi}(x)^t \mathcal{D}(x)^t \, A \, \vec{\phi}(x) = -\vec{\psi}(x)^t \, A \, \mathcal{D}(x) \, \vec{\phi}(x) \end{aligned} \tag{2-8}$$

and for $n \geq 2$:

$$W_n(x_1,\dots,x_n) = -\frac{\delta_{n,2}}{(x_1-x_2)^2} - (-1)^n \sum_{\sigma=\text{cyles}} \prod_{i=1}^n K(x_i, x_{\sigma(i)}) \tag{2-9}$$

Definition 2.4 *We also define the* **non-connected** *correlation functions by the determinantal formulae:*

$$\mathcal{W}_n(x_1,\dots,x_n) = \text{"}\det\text{"} \left(K(x_i,x_j)\right) \tag{2-10}$$

where the quotation mark in " det" *means that, when we expand the determinant as a sum of permutations, each time we have a $K(x_i, x_{\sigma(i)})$ with $\sigma(i) = i$ we have to replace it by $W_1(x_i)$, and each time we have the product $K(x_i,x_j)K(x_j,x_i)$, we have to replace it by $K(x_i,x_j)K(x_j,x_i) + \frac{1}{(x_i-x_j)^2}$. This definition is inspired from [6].*

The connected correlation functions are the **cumulants** *of the non-connected ones.*

The purpose of these definitions, is that, for a matrix model $\int dM\, \mathrm{e}^{-\operatorname{Tr} V(M)}$, it coincides with correlation functions of eigenvalues. In that case, $\psi(x)\,\mathrm{e}^{\frac{1}{2}V(x)} = p_n(x)$ and $\tilde{\psi}(x)\,\mathrm{e}^{\frac{1}{2}V(x)} = p_{n-1}(x)$ are consecutive orthonormal polynomials of degrees n and $n-1$, orthonormal with respect to the measure $\mathrm{e}^{-V(x)}dx$. And $\phi\,\mathrm{e}^{-\frac{1}{2}V(x)} = \hat{p}_n$ and $\tilde{\phi}\,\mathrm{e}^{-\frac{1}{2}V(x)} = \hat{p}_{n-1}$ are their Hilbert transforms (see [6]):

$$\hat{p}_n(x) = \int \frac{dx'}{x-x'}\, p_n(x')\, \mathrm{e}^{-V(x')} \tag{2-11}$$

Thus in the case of matrix models we have:

$$K(x_1,x_2) = \mathrm{e}^{-\frac{1}{2}(V(x_1)-V(x_2))}\, \frac{p_n(x_1)\hat{p}_{n-1}(x_2) - p_{n-1}(x_1)\hat{p}_n(x_2)}{x_1-x_2} \tag{2-12}$$

2.1.1 Examples

For instance we have:

• The 1-point function is:

$$W_1(x) = \psi'(x)\tilde{\phi}(x) - \tilde{\psi}'(x)\phi(x) \tag{2-13}$$

• The 2-point function is:

$$\begin{aligned} & W_2(x_1,x_2) \\ = \quad & -K(x_1,x_2)K(x_2,x_1) - \frac{1}{(x_1-x_2)^2} \\ = \quad & -\frac{1}{(x_1-x_2)^2}\,\Big(\psi(x_1)\tilde{\psi}(x_2) - \tilde{\psi}(x_1)\psi(x_2)\Big)\,\Big(\phi(x_1)\tilde{\phi}(x_2) - \tilde{\phi}(x_1)\phi(x_2)\Big) \end{aligned}$$

(2 – 14)

One can check (see section 2.4 below) that:

$$\mathcal{W}_2(x,x) = W_2(x,x) + W_1(x)^2 = b(x)c(x) - a(x)d(x) = -\det \mathcal{D}(x) \tag{2-15}$$

which is a rational function of x.

• The 3-point function W_3 is:

$$W_3(x_1,x_2,x_3) = K(x_1,x_2)K(x_2,x_3)K(x_3,x_1) + K(x_1,x_3)K(x_3,x_2)K(x_2,x_1) \tag{2-16}$$

it is the cumulant of $\mathcal{W}_3$:

$$\begin{aligned} \mathcal{W}_3(x_1,x_2,x_3) \quad = \quad & \text{"}\det\text{"}\begin{pmatrix} K(x_1,x_1) & K(x_1,x_2) & K(x_1,x_3) \\ K(x_2,x_1) & K(x_2,x_2) & K(x_2,x_3) \\ K(x_3,x_1) & K(x_3,x_2) & K(x_3,x_3) \end{pmatrix} \\ = \quad & \det\begin{pmatrix} W_1(x_1) & K(x_1,x_2) & K(x_1,x_3) \\ K(x_2,x_1) & W_1(x_2) & K(x_2,x_3) \\ K(x_3,x_1) & K(x_3,x_2) & W_1(x_3) \end{pmatrix} \\ & -\frac{W_1(x_1)}{(x_2-x_3)^2} - \frac{W_1(x_2)}{(x_1-x_3)^2} - \frac{W_1(x_3)}{(x_1-x_2)^2} \end{aligned}$$

(2 – 17)

and so on...

2.2 Correlators in terms of a rank 1 matrix $M(x)$

Using the fact that K is a product of the form:

$$K(x_1,x_2) = \frac{1}{x_1-x_2}\,\vec{\psi}(x_1)^t\,A\,\vec{\phi}(x_2) \tag{2-18}$$

we define the following rank 1 matrix:

$$M(x) = \vec{\phi}(x)\,\vec{\psi}(x)^t\,A\; = \begin{pmatrix} -\phi(x)\tilde{\psi}(x) & \phi(x)\psi(x) \\ -\tilde{\phi}(x)\tilde{\psi}(x) & \tilde{\phi}(x)\psi(x) \end{pmatrix} \tag{2-19}$$

$M(x)$ is a rank 1 matrix, and it is a projector on state $\vec{\phi}(x)$:

$$M(x)^2 = M(x) \qquad , \qquad M(x)\vec{\phi}(x) = \vec{\phi}(x) \qquad , \qquad M(x)\vec{\psi}(x) = 0 \tag{2-20}$$

Its eigenvalues are 1 and 0, and in particular:

$$\mathrm{Tr}\, M(x) = 1 \tag{2-21}$$

We have the duality formula:

$$A\,M(x)^t\,A = M(x) - 1 \tag{2-22}$$

Notice, that thanks to property eq.(2-4), $M(x)$ satisfies the Lax equation:

$$\partial_x\,M(x) = [\mathcal{D}(x), M(x)] \tag{2-23}$$

A mere rewriting of the definition 2.3 of correlation functions gives:

Theorem 2.1

$$W_1(x) = -\,\mathrm{Tr}\,\mathcal{D}(x)M(x) \tag{2-24}$$

$$W_2(x_1,x_2) = -\frac{1-\,\mathrm{Tr}\,M(x_1)M(x_2)}{(x_1-x_2)^2} = -\,\frac{\mathrm{Tr}\,(M(x_1)-M(x_2))^2}{2\,(x_1-x_2)^2} \tag{2-25}$$

and if $n>2$:

$$\begin{aligned} & W_n(x_1,\dots,x_n) \\ = \quad & (-1)^{n+1}\,\mathrm{Tr}\,\sum_{\sigma=\text{cyclic}}\prod_i \frac{M(x_{\sigma(i)})}{x_{\sigma(i)}-x_{\sigma(i+1)}} \\ = \quad & \frac{(-1)^{n+1}}{n}\sum_{\sigma\in S_n}\frac{\mathrm{Tr}\,\left(M(x_{\sigma(1)})\,M(x_{\sigma(2)})\,\dots\,M(x_{\sigma(n)})\right)}{(x_{\sigma(1)}-x_{\sigma(2)})\,(x_{\sigma(2)}-x_{\sigma(3)})\,\dots\,(x_{\sigma(n)}-x_{\sigma(1)})} \end{aligned}$$
$$(2-26)$$

For Example:

$$W_3(x_1,x_2,x_3) = \frac{\mathrm{Tr}\,\Big(M(x_1)M(x_2)M(x_3)-M(x_1)M(x_3)M(x_2)\Big)}{(x_1-x_2)(x_2-x_3)(x_3-x_1)} \tag{2-27}$$

2.3 Loop operator

Definition 2.5 *The Loop operator is a derivation δ_x such that:*

$$\delta_{x_2}\psi(x_1) = -K(x_1,x_2)\psi(x_2) \qquad , \qquad \delta_{x_2}\tilde\psi(x_1) = -K(x_1,x_2)\tilde\psi(x_2) \tag{2-28}$$

$$\delta_{x_2}\phi(x_1) = -K(x_2,x_1)\phi(x_2) \qquad , \qquad \delta_{x_2}\tilde\phi(x_1) = -K(x_2,x_1)\tilde\phi(x_2) \tag{2-29}$$

and $\delta_x(uv) = v\,\delta_x u + u\,\delta_x v$, and $\frac{d}{dx_1}\delta_{x_2} = \delta_{x_2}\frac{d}{dx_1}$.

In the matrix models litterature, it coincides with the "loop insertion operator" [12, 1, 2], often denoted $\delta_x = \frac{\partial}{\partial V(x)}$.

2.3.1 Some properties of loop operators

The definitions of Def.2.5 can be rewritten as:

Theorem 2.2

$$\delta_y\vec\psi(x) = \frac{(1-M(y))}{y-x}\,\vec\psi(x) \qquad , \qquad \delta_y\vec\psi(x)^t\,A = \vec\psi(x)^t\,A\,\frac{M(y)}{y-x} \tag{2-30}$$

and:

$$\delta_y\vec\phi(x) = \frac{M(y)\,\vec\phi(x)}{x-y} \qquad , \qquad \delta_y\vec\phi(x)^t A = \vec\phi(x)^t A\,\frac{1-M(y)}{x-y} \tag{2-31}$$

From which it is easy to derive:

Theorem 2.3 *We have:*

$$\delta_y\,M(x) = \frac{1}{(x-y)}\,[M(y),M(x)] = \delta_x\,M(y) \tag{2-32}$$

And:

Theorem 2.4 *We have:*

$$\delta_{x_3}K(x_1,x_2) = -K(x_1,x_3)\,K(x_3,x_2) \tag{2-33}$$

From this last theorem, one easily finds:

Theorem 2.5 *We have:*

$$\delta_{x_{n+1}}W_n(x_1,\dots,x_n) = W_{n+1}(x_1,\dots,x_n,x_{n+1}) + \frac{\delta_{n,1}}{(x_1-x_2)^2} \tag{2-34}$$

Then, we have the following property, which is very useful:

Theorem 2.6 *We have:*

$$\delta_y \mathcal{D}(x) = \frac{1}{(x-y)^2} \left(\mathrm{Id} - M(x) - M(y)\right) + \frac{1}{x-y} \left[M(y), \mathcal{D}(x)\right] \qquad (2\text{-}35)$$

proof:

This theorem is quite technical and the proof is given in appendix A. The proof consists in applying δ_y to:

$$\vec{\phi}'(x) = \mathcal{D}(x)\, \vec{\phi}(x) \qquad (2\text{-}36)$$

□

2.4 Loop equations

In this section, we prove that the correlation functions W_n satisfy a set of "loop equations".

In that purpose we need to introduce some definitions.

Definition 2.6 *Definition of the rational functions P_n:*

$$P_1(x) = \det \mathcal{D}(x) = -\frac{1}{2}\, \mathrm{Tr}\, \mathcal{D}^2(x) \qquad (2\text{-}37)$$

$$P_2(x; x_2) = \mathrm{Tr}\, \frac{\mathcal{D}(x) - \mathcal{D}(x_2) - (x-x_2)\mathcal{D}'(x_2)}{(x-x_2)^2}\, M(x_2) \qquad (2\text{-}38)$$

and if $n \geq 2$:

$$Q_{n+1}(x; x_1, \ldots, x_n) = \sum_\sigma \frac{\mathrm{Tr}\, \mathcal{D}(x)\, M(x_{\sigma(1)}) M(x_{\sigma(2)}) \ldots M(x_{\sigma(n)})}{(x - x_{\sigma(1)})(x_{\sigma(1)} - x_{\sigma(2)}) \ldots (x_{\sigma(n-1)} - x_{\sigma(n)})(x_{\sigma(n)} - x)} \qquad (2\text{-}39)$$

$$\begin{aligned} & P_{n+1}(x; x_1, \ldots, x_n) \\ = \; & (-1)^n \Big[Q_{n+1}(x; x_1, \ldots, x_n) - \sum_{j=1}^n \frac{1}{x - x_j} \mathop{\mathrm{Res}}_{x' \to x_j} Q_{n+1}(x'; x_1, \ldots, x_n) \Big] \end{aligned} \qquad (2\text{-}40)$$

Theorem 2.7 *$P_{n+1}(x; x_1, \ldots, x_n)$ is a rational fraction of the variable x, whose only poles are at the poles of $a(x), b(x)$ or $c(x)$, and if $n \geq 1$, those poles are of degree at most* $\max(\deg a, \deg b, \deg c) - 2$.

proof:

It suffices to prove that P_{n+1} has no pole when $x = x_i$. Q_{n+1} has simple poles at those points, and P_{n+1} is constructed precisely by canceling the residues. □

Theorem 2.8 *For $n > 2$*

$$\begin{aligned} -\delta_{x_n} Q_n(x; x_1, \dots, x_{n-1}) &= (-1)^n\, P_{n+1}(x; x_1, \dots, x_{n-1}, x_n) \\ &\quad -(-1)^n \frac{\partial}{\partial x_n} \frac{W_n(x, x_1, \dots, x_{n-1}) + W_n(x_n, x_1, \dots, x_{n-1})}{x - x_n} \\ &\quad + R_n(x; x_1, \dots, x_{n-1}, x_n) \end{aligned} \tag{2-41}$$

where $R_n(x; x_1, \dots, x_{n-1}, x_n)$ is a rational fraction of x with only simple poles at $x = x_j, j = 1, \dots, n-1$.

It follows that for all $n \geq 1$:

$$\begin{aligned} & \delta_{x_n} P_n(x; x_1, \dots, x_{n-1}) \\ = \; & P_{n+1}(x; x_1, \dots, x_{n-1}, x_n) - \frac{\partial}{\partial x_n} \frac{W_n(x, x_1, \dots, x_{n-1}) + W_n(x_n, x_1, \dots, x_{n-1})}{x - x_n} \\ & -\delta_{n,2} \frac{\partial}{\partial x_n} \frac{1}{(x_1 - x_n)^2\, (x - x_n)} \end{aligned} \tag{2-42}$$

This theorem is quite technical and is proved in appendix B.

Theorem 2.9 *For all $n \geq 0$, the correlation functions W_k satisfy the loop equation:*

$$\begin{aligned} -P_{n+1}(x; L) &= W_{n+2}(x, x, L) + \sum_{J \subset L} W_{1+|J|}(x, J) W_{1+n-|J|}(x, L/J) \\ &\quad + \sum_{j=1}^{n} \frac{d}{dx_j} \frac{W_n(x, L/\{x_j\}) - W_n(L)}{x - x_j} \end{aligned} \tag{2-43}$$

where $L = \{x_1, \dots, x_n\}$.

proof:

This theorem is rather technical, and the proof is given in appendix C.

The proof consists in proving it for $n = 0$ and then apply recursively the loop operator.

□

We call equation eq.(2-43) the "loop equation", because it is the same as the loop equation of the 1-matrix model, it is the same which was used to compute recursively the topological expansion of matrix integrals in [14]. This implies that, if our system depends on some "large parameter" N in such a way that we have a topological expansion (see [12]), then all the terms in the expansion are those computed in [14], i.e. they are the correlation functions of the symplectic invariants of [15].

2.5 Topological expansion

Hypothesis 1: *Topological expansion.*

Let us assume that our differential system $\mathcal{D}_N(x)$ depends on some large parameter N, in such a way that the correlation functions have a so-called **topological expansion***:*

$$W_n(x_1,\dots,x_n) = \sum_g N^{2-2g-n} W_n^{(g)}(x_1,\dots,x_n) \tag{2-44}$$

where all $W_n^{(g)}$'s are algebraic functions.

We emphasize that not all differential systems have that property, and we are making a strong assumption here. However, this assumption holds for many systems which have practical applications in enumerative geometry and integrable systems for instance [10, 12].

The existence of such a topological expansion implies that the rational fraction $\frac{1}{N^2}\det(\mathcal{D}(x)) = -\frac{1}{N^2}\left(W_1^2(x) + W_2(x,x)\right)$ has a large N limit:

$$\lim_{N\to\infty} \frac{1}{N^2} \det(\mathcal{D}(x)) = -\mathcal{E}_\infty(x) \tag{2-45}$$

and we write:

$$Y(x) = -W_1^{(0)}(x) = -\sqrt{\mathcal{E}_\infty(x)} \tag{2-46}$$

$\mathcal{E}_\infty(x)$ is a rational function of x, and $Y(x)$ has branchcut singularities at all odd zeroes of $\mathcal{E}_\infty(x)$. It is regular at even zeroes, it just vanishes there. It can be seen easily that the fact that the W_n's satisfy loop equations, implies by recursion, that each $W_n^{(g)}$ is an algebraic function, with possible branchcut singularities at the odd zeroes, and poles at the even zeroes.

Let us then make a stronger additional hypothesis to the topological expansion property:

Hypothesis 2: *No pole at even zeroes.*

We assume that the $W_n^{(g)}$ have no poles at the even zeroes of $\mathcal{E}_\infty(x)$.

Again, this assumption does not hold for all systems, however, it holds for many of the most interesting systems concerning enumerative geometry and integrable systems [10, 12].

We can add a 3rd hypothesis. Our algebraic curve $Y^2 = \mathcal{E}_\infty(x)$ is an hyperelliptical curve, whose genus is related to the number of odd zeroes. If the genus is > 0, the Riemann surface $Y(x)$ is not simply connected, it has $2\times$ genus non-contractible cycles, and we may impose some further hypothesis on half those cycles, call them $\mathcal{A}_i$, $i = 1,\dots,$ genus:

Hypothesis 3: *Fixed filling fractions*

We assume that the $W_n^{(g)}$ with $n+g>1$ satisfy:

$$\oint_{\mathcal{A}_i} W_n^{(g)} = 0. \tag{2-47}$$

Notice that this hypothesis is automatically fulfilled if the genus is zero, i.e. if $Y(x)$ has only 1-cut.

If we make those three hypothesis, we can solve recursively the loop equations, and the loop equations have then a unique solution. That unique solution was first found in [14], and formalized in [15]. We get:

Corollary 2.1 *Relation to symplectic invariants*

If our differential system $\Psi(x) = \Psi(x,N)$ depends on some "large parameter" N, such that the correlation functions $W_n(x_1,\ldots,x_n)$ have a topological expansion of the form $W_n(x_1,\ldots,x_n) = \sum_g N^{2-2g-n} W_n^{(g)}(x_1,\ldots,x_n)$, and if the coefficients $W_n^{(g)}$'s are meromorphic functions on the limit spectral curve $Y(x)$ with singularities only at branchpoints, and fixed filling fractions (or 1-cut), then the coefficients $W_n^{(g)}(x_1,\ldots,x_n)$ in that expansion are obtained from the symplectic invariants defined in [15] for the spectral curve $y = Y(x)$.

proof:

The article [14], precisely consisted in finding the unique solution of loop equations having the topological expansion property $W_n = \sum_g N^{2-2g-n} W_n^{(g)}$, together with the condition that the $W_n^{(g)}$'s have singularities only at branchpoints and fixed filling fractions. In brief, it consisted in computing the only polynomials $P_n^{(g)}$'s compatible with the hypothesis, using Lagrange interpolation formula, and rewriting it in terms of contour integrals and residues on the spectral curve. And it proved [14, 11], that the unique solution of loop equations having the topological expansion property, together with this given analytical structure, is given by the "symplectic invariant correlators" of [15] for the spectral curve $y = Y(x)$.

We shall not enter the details here. A definition of symplectic invariants correlators is given in def 4.1 in section 4.3 below. The example of the Airy system, with spectral curve $Y(x) = \sqrt{x}$ is treated in further details below. □

3 Airy kernel

Let us now apply all the results of the preceding section to the example of the Airy system, which plays a very important role in the universal law of extreme values statistics [22], so, let us study it in details.

The Airy function $Ai(x)$ satisfies the Airy equation:

$$Ai''(x) = x\, Ai(x). \tag{3-1}$$

It can be rewritten as a differential system of order 2 (we introduce the other independent solution $Bi(x)$ called "Bairy"-function):

$$\frac{d}{dx}\Psi(x) = \mathcal{D}(x)\,\Psi(x) \quad , \quad \Psi(x) = \begin{pmatrix} Ai(x) & Bi(x) \\ Ai'(x) & Bi'(x) \end{pmatrix} \quad , \quad \det\Psi = 1 \tag{3-2}$$

The differential system $\mathcal{D}(x)$ is:

$$\mathcal{D}(x) = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \quad , \quad \mathrm{Tr}\,\mathcal{D} = 0 \tag{3-3}$$

The spectral curve is:

$$\hat{\mathcal{E}}(x,y) = \det(y - \mathcal{D}(x)) = y^2 - x \tag{3-4}$$

i.e.

$$y = \sqrt{x} \tag{3-5}$$

This spectral curve has only one branchpoint, located at $x = 0$, and there is only one cut $[0, -\infty)$.

Notice that x is not a good variable on the spectral curve. Instead, $z = \sqrt{x}$ is a good uniformizing variable, we write:

$$x(z) = z^2 \quad , \quad y(z) = z. \tag{3-6}$$

The corresponding Christoffel-Darboux kernel is the Airy kernel:

$$K_{\mathrm{Airy}}(x_1, x_2) = \frac{Ai(x_1)Bi'(x_2) - Ai'(x_1)Bi(x_2)}{x_1 - x_2} \tag{3-7}$$

This kernel plays an important role in the Tracy-Widom law, and in extreme values statistics [22].

The correlation functions are given by def.2.3:

$$W_1(x) = Ai'(x)Bi'(x) - x\, Ai(x)Bi(x) \tag{3-8}$$

and for $n \geq 2$:

$$\mathcal{W}_n(x_1, \ldots, x_n) = \text{"}\det\text{"}\,(K_{\mathrm{Airy}}(x_i, x_j)) \tag{3-9}$$

i.e.

$$W_n(x_1, \ldots, x_n) = -\frac{\delta_{n,2}}{(x_1 - x_2)^2} - (-1)^n \sum_{\sigma=\mathrm{cyles}} \prod_{i=1}^{n} K_{\mathrm{Airy}}(x_{\sigma(i)}, x_{\sigma(i+1)}) \tag{3-10}$$

3.1 Topological expansion

The Airy and Bairy functions have some well known BKW large x expansion (see []), in some sectors[3], of the form:

$$\begin{aligned} Ai(x) &\sim \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{\sqrt{2}\,x^{\frac{1}{4}}}\,(1+\sum_k c_k x^{-3k/2}) \\ &\sim \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{\sqrt{2}\,x^{\frac{1}{4}}}\,(1+\sum_{k\geq 1}\frac{(-1)^k\,(6k)!}{2^{6k}\,3^{2k}\,(2k)!\,(3k)!}\,x^{-3k/2}) \\ &\sim \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{\sqrt{2}\,x^{\frac{1}{4}}}\,(1-\frac{5}{3\ 2^4\ x^{3/2}}+\ldots) \end{aligned} \tag{3-11}$$

and similarly Bi is obtained by changing the sign of the square-root:

$$Bi(x)\sim \frac{e^{+\frac{2}{3}x^{\frac{3}{2}}}}{\sqrt{2}\,x^{\frac{1}{4}}}\,(1+\sum_k(-1)^k\,c_k x^{-3k/2})\sim \frac{e^{+\frac{2}{3}x^{\frac{3}{2}}}}{\sqrt{2}\,x^{\frac{1}{4}}}\,(1+\frac{5}{3\ 2^4\ x^{3/2}}+\ldots) \tag{3-12}$$

In order to expand the kernel K, it is convenient to introduce a scaling variable N, and rescale $x\to N^{\frac{2}{3}}x$. The kernel thus has a $1/N$ expansion of the form:

$$K(N^{\frac{2}{3}}x_1,N^{\frac{2}{3}}x_2)=N^{\frac{-2}{3}}\ e^{-\frac{2N}{3}(x_1^{\frac{3}{2}}-x_2^{\frac{3}{2}})}\ \sum_{g=0}^{\infty}N^{-g}\ K^{(g)}(x_1,x_2) \tag{3-13}$$

where

$$K^{(0)}(x_1,x_2)=\frac{1}{2\ (x_1x_2)^{\frac{1}{4}}}\ \frac{1}{\sqrt{x_1}-\sqrt{x_2}} \tag{3-14}$$

and every $K^{(g)}(x_1,x_2)$ for $g>0$ is an odd homogeneous rational fraction of $z_1=\sqrt{x_1}, z_2=\sqrt{x_2}$, with poles only at $z_1=0$ or $z_2=0$.

This implies that all correlation functions have a $1/N$ expansion (in fact $1/N^2$ expansion by parity):

$$W_n(N^{\frac{2}{3}}\,x_1,\ldots,N^{\frac{2}{3}}\,x_n)=N^{-\frac{2n}{3}}\ \sum_{g=0}^{\infty}N^{2-2g-n}\,W_n^{(g)}(x_1,\ldots,x_n) \tag{3-15}$$

where $W_n^{(g)}(x_1,\ldots,x_n)$ is a homogeneous rational fraction of the variables $z_i=\sqrt{x_i}$ of degree $6-6g-5n$, and it is easy to see that the only poles can be at $z_i=0$ (except $W_1^{(0)}$ and $W_2^{(0)}$).

[3]Due to the Stokes'phenomenon, the asymptotics may change from one sector to another, near the essential singularity at ∞, see []. The asymptotics we give here, are in the sector $|\mathrm{Arg}(x)|<\frac{2\pi}{3}$

For example the 1-point function has the expansion (see [7]):

$$W_1(z^2) = -z + \sum_{g=1}^{\infty} \frac{(6g-3)!!}{3^g\, 2^{5g}\, g!}\; z^{1-6g} \tag{3-16}$$

For example the 2-point function starts as:

$$W_2(z_1^2, z_2^2) = \frac{1}{4 z_1 z_2}\; \frac{1}{(z_1+z_2)^2} + \dots \tag{3-17}$$

and thus:

$$W_2^{(0)}(z_1^2, z_2^2) = \frac{1}{4 z_1 z_2}\; \frac{1}{(z_1+z_2)^2} \tag{3-18}$$

which can be rewritten:

$$W_2^{(0)}(x_1,x_2)dx_1 dx_2 = \frac{dx_1 dx_2}{4\sqrt{x_1 x_2}}\; \frac{1}{(\sqrt{x_1}+\sqrt{x_2})^2} = \frac{dz_1 dz_2}{(z_1-z_2)^2} - \frac{dx_1 dx_2}{(x_1-x_2)^2} \tag{3-19}$$

where $x(z) = z^2$, and thus $dx(z) = 2z dz$. In other words, $W_2^{(0)}(x_1,x_2)dx_1dx_2 + \frac{dx_1dx_2}{(x_1-x_2)^2} = B(z_1,z_2)$ is the Bergman kernel on the spectral curve $y=\sqrt{x}$.

The fact that the function $W_2^{(0)}$ is closely related to the Bergman kernel of its spectral curve is not surprising and fits with the general theory (see [15]).

3.2 Loop equations and symplectic invariants

We have seen from theorem 2.9, that the W_n's satisfy the loop equations eq.(2-43) (where it can be seen from the degrees, that $\hat{P}_n(x;x_1,\dots,x_n) = x\delta_{n,0}$):

$$\begin{aligned} W_{n+2}(x,x,L) &+ \sum_{J\subset L} W_{1+|J|}(x,J) W_{1+n-|J|}(x,L/J) \\ &+ \sum_j \frac{d}{dx_j} \frac{W_n(x,L/\{j\}) - W_n(L)}{x-x_j} = x\,\delta_{n,0} \end{aligned} \tag{3-20}$$

Moreover the W_n's have a topological expansion:

$$W_n(N^{\frac{2}{3}}\, x_1, \dots, N^{\frac{2}{3}}\, x_n) = N^{-\frac{2n}{3}} \sum_{g=0}^{\infty} N^{2-2g-n}\, W_n^{(g)}(x_1,\dots,x_n) \tag{3-21}$$

where $W_n^{(g)}(x_1,\dots,x_n)$ is a rational fraction of the $z_i = \sqrt{x_i}$, with poles only at $z_i = 0$.

In other words, the Airy system satisfies the 3 asumptions of section 2.5, and thus corollary 2.1 applies:

Theorem 3.1 $W_n^{(g)}(x_1,\dots,x_n)$ *are the symplectic invariant correlators of [15] for the spectral curve* $y^2 = x$.

This result was stated in [15] without a proof, and thus we just provide here the missing proof in [15] In fact in [15] it was stated in the opposite way, i.e. it was claimed that the symplectic invariant correlators for the spectral curve $y^2 = x$, were the Airy correlation functions.

A summarized definition of symplectic invariants and their correlators is recalled in section 4.3 below. Let us just give here the examples of some of the first $W_n^{(g)}$'s, which are written explicitely in section 10.5. of [15] for the Airy curve:

$$W_1^{(1)}(x) = \frac{1}{(2\sqrt{x})^5} \quad , \quad W_1^{(2)}(x) = \frac{2\ \ 9!!}{2!\,3^2\,(2\sqrt{x})^{11}} \quad , \quad W_1^{(3)}(x) = \frac{2^2\ \ 15!!}{3!\,3^3\,(2\sqrt{x})^{17}} \tag{3-22}$$

$$W_3^{(0)}(x_1, x_2, x_3) = \frac{1}{2\ (4x_1x_2x_3)^{3/2}} \tag{3-23}$$

$$W_4^{(0)}(x_1, x_2, x_3, x_4) = \frac{3}{2^6\ (x_1x_2x_3x_4)^{3/2}}\ \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right) \tag{3-24}$$

and it was claimed in [15] that:

$$W_n^{(0)}(x_1, \dots, x_n) = \frac{(n-3)!}{2^{2n-2}}\ \frac{1}{\prod_{j=1}^n x_j^{3/2}}\ \sum_{|\lambda|=n-3} M_\lambda(1/x_i) \prod_j \frac{(2\lambda_j + 1)!!}{\lambda_j!} \tag{3-25}$$

where $M_\lambda(z_1, \dots, z_n) = \mathrm{Sym}(z_1^{\lambda_1} \dots z_n^{\lambda_n})$ is the elementary symmetric monomial function indexed by the partition λ. This claim was later proved by Michel Bergère [7].

And so on...

3.3 Exponential formulae

Let us define the symmetric primitives $\Phi_n(x_1, \dots, x_n)$ such that $\partial_{x_1} \dots \partial_{x_n} \Phi_n = W_n$, more precisely:

$$\Phi_1(x) = -\frac{2}{3} x^{\frac{3}{2}} + \int_\infty^x (W_1(x') + \sqrt{x'})\, dx' \tag{3-26}$$

$$\begin{aligned} \Phi_2(x_1, x_2) \quad = \quad & -\ln\left(\sqrt{x_1} + \sqrt{x_2}\right) \\ & + \int_\infty^{x_1} dx_1' \int_\infty^{x_2} dx_2'\ \ (W_2(x_1', x_2') - \frac{1}{4\sqrt{x_1'}\sqrt{x_2'}\,(\sqrt{x_1'} + \sqrt{x_2'})^2}) \end{aligned}$$

$$(3-27)$$

and if $n > 2$:

$$\Phi_n(x_1, \dots, x_n) = \int_\infty^{x_1} dx_1' \dots \int_\infty^{x_n} dx_n'\ \ W_n(x_1', \dots, x_n') \tag{3-28}$$

If we integrate eq.(3-20) with respect to $x_1, \dots, x_n$, and write collectively $L = \{x_1, \dots, x_n\}$, we find:

$$\partial_1 \partial_2 \Phi_{n+2}(x, x, L) + \sum_{J \subset L} \partial_1 \Phi_{1+|J|}(x, J) \partial_1 \Phi_{1+n-|J|}(x, L/J)$$

$$+\sum_j \frac{\partial_1 \Phi_n(x, L/\{j\}) - \partial_1 \Phi_n(x_j, L/\{j\})}{x - x_j} = x\,\delta_{n,0} \tag{3-29}$$

where ∂_i means derivative with respect to the i^{th} variable.

3.3.1 Exponential formula for the Airy function

Then we define:

$$f(x) = \sum_l \frac{1}{l!} \partial_1 \Phi_{l+1}(x, x, \ldots, x) \tag{3-30}$$

where ∂_1 means that we take the derivative with respect to the first variable only, and then set all variables $x_i = x$.

Let us compute:

$$\frac{d}{dx} f(x) + f(x)^2 \tag{3-31}$$

We have:

$$\begin{aligned}
& \frac{d}{dx} f(x) + f(x)^2 \\
= & \sum_l \frac{1}{l!} \partial_1\partial_2 \Phi_{l+2}(x, x, \ldots, x) + \sum_l \frac{1}{l!} \partial_1^2 \Phi_{l+1}(x, x, \ldots, x) \\
& + \sum_{l_1} \sum_{l_2} \frac{1}{l_1!} \frac{1}{l_2!} \partial_1 \Phi_{l_1+1}(x, x, \ldots, x) \partial_1 \Phi_{l_2+1}(x, x, \ldots, x) \\
= & \sum_l \frac{1}{l!} \partial_1\partial_2 \Phi_{l+2}(x, x, \ldots, x) + \sum_l \frac{1}{l!} \partial_1^2 \Phi_{l+1}(x, x, \ldots, x) \\
& + \sum_l \frac{1}{l!} \sum_{J \subset L} \partial_1 \Phi_{1+|J|}(x, J) \partial_1 \Phi_{1+l-|J|}(x, L/J) \\
= & \; x
\end{aligned} \tag{3-32}$$

where we used eq.(3-29) with $L = \{x, \ldots, x\}$.

Therefore $f(x)$ satisfies the Ricatti equation $f' + f^2 = x$, which implies that:

$$f(x) = \frac{Ai'(x)}{Ai(x)} \tag{3-33}$$

and this proves that:

Theorem 3.2 *The Airy function satisfies the exponential formula*

$$\boxed{Ai(x) = \mathrm{e}^{\sum_l \frac{1}{l!} \Phi_l(x, \ldots, x)} = \mathrm{e}^{\sum_l \frac{1}{l!} \int^x \ldots \int^x W_l}} \tag{3-34}$$

This equality makes sense order by order in the large x expansion, because to any given power in x, the sum is finite.

Similarly, we find:

$$g(x) = \sum_l \frac{(-1)^l}{l!} \partial_1 \Phi_{l+1}(x, x, \ldots, x) = \frac{Bi'(x)}{Bi(x)} \tag{3-35}$$

i.e.

$$Bi(x) = \mathrm{e}^{\sum_l \frac{(-1)^l}{l!} \Phi_l(x,\ldots,x)} = \mathrm{e}^{\sum_l \frac{(-1)^l}{l!} \int^x \ldots \int^x W_l} \tag{3-36}$$

3.3.2 Exponential formula for the Airy kernel

Then we define:

$$U(x, y) = \sum_{l=1}^{\infty} \frac{1}{l!} \int_y^x \ldots \int_y^x W_l \tag{3-37}$$

In other words we have:

$$U(x, y) = \sum_{n+m>0} \frac{(-1)^m}{n!\, m!}\ \Phi_{n+m}(\overbrace{x, \ldots, x}^{n}, \overbrace{y, \ldots, y}^{m}) \tag{3-38}$$

and write:

$$U(x, y) = \ln Ai(x) + \ln Bi(y) + \ln\left(g(y) - f(x)\right) + \ln h(x, y) \tag{3-39}$$

We are now going to determine the function $h(x, y)$.

The loop equation eq.(3-29) with $L = (\overbrace{x, \ldots, x}^{n}, \overbrace{y, \ldots, y}^{m})$ implies:

$$U_{xx} + U_x^2 - \frac{U_x + U_y}{x - y} = x \tag{3-40}$$

and

$$U_{yy} + U_y^2 + \frac{U_x + U_y}{x - y} = y \tag{3-41}$$

That implies for $h(x, y)$:

$$h_{xx} + 2h_x(f + \frac{f'}{f - g}) = \frac{h_x + h_y}{x - y} \tag{3-42}$$

$$h_{yy} + 2h_y(g - \frac{g'}{f - g}) = -\frac{h_x + h_y}{x - y} \tag{3-43}$$

We shall prove that $h(x, y) = 1$. From its very definition, we see that the function $U(x, y)$ has a power series expansion at large y, and thus we have:

$$h(x, y) = 1 + \sum_{k=1}^{\infty} \frac{h_k(x)}{y^{k/2}} \tag{3-44}$$

where the leading coefficient $h_0 = 1$ is easily obtained from the leading order behaviors of Ai and Bi.

Now, assume that there exists $k>0$ such that $h_k \neq 0$, and let $k = \min\{\tilde{k} \,/\, h_{\tilde{k}} \neq 0\}$, this implies that

$$h_{yy} + 2h_y(g - \frac{g'}{f-g}) + \frac{h_x + h_y}{x-y} \sim -\frac{k\,h_k(x)}{y^{\frac{k+1}{2}}}(1 + O(y^{-1/2})) \tag{3-45}$$

and thus equation eq.(3-43) implies that $h_k = 0$ which is a contradiction. Therefore, $\forall k \geq 1$, $h_k = 0$, and therefore $h = 1$, and we recognize the Airy kernel:

$$K(x,y) = \frac{\mathrm{e}^{U(x,y)}}{x-y} \tag{3-46}$$

and thus:

Theorem 3.3 *The Airy kernel satisfies the exponential formula*

$$\boxed{K(x,y) = \frac{1}{x-y}\,\mathrm{e}^{\sum_{l=1}^{\infty}\frac{1}{l!}\int_y^x \cdots \int_y^x W_l}} \tag{3-47}$$

4 Exponential formula, conjecture

In this section, we conjecture and argue that the exponential formula:

$$K(x,y) \stackrel{?}{=} \frac{1}{x-y}\,\mathrm{e}^{\sum_{l=1}^{\infty}\frac{1}{l!}\int_y^x \cdots \int_y^x W_l} \tag{4-1}$$

should hold not only for the Airy system, but also for a much larger class of differential systems, namely those which have a topological expansion satisfying the 3 hypothesis of section 2.5, and which have a **1-cut** spectral curve.

First, in order for this exponential formula to make sense, we need to know what the infinite sum in the exponential means, i.e. we need a large parameter to make a power series expansion.

This is the case when we have a topological expansion property, i.e. when W_n has a large N expansion of the type:

$$W_n(x_1,\dots,x_n) = \sum_{g=0}^{\infty} N^{2-2g-n} W_n^{(g)}(x_1,\dots,x_n) \tag{4-2}$$

then, the exponential formula means:

$$\tilde{K}(x,y) = \mathrm{e}^{N\int_y^x W_1^{(0)}(x')dx'}\;\frac{\frac{1}{2}\mathrm{e}^{\int_y^x\int_y^x W_2^{(0)}(x_1',x_2')dx_1'dx_2'}}{x-y}\;\exp\Big($$

$$\sum_{g=0}^{\infty} \sum_{l\geq 1+2\delta_{g,0}} \frac{N^{2-2g-l}}{l!} \int_y^x \dots \int_y^x W_l^{(g)} \Big)$$

$$= \quad \mathrm{e}^{N\int_y^x W_1^{(0)}(x')dx'} \,\frac{\frac{1}{2}\mathrm{e}^{\int_y^x\int_y^x W_2^{(0)}(x_1',x_2')dx_1'dx_2'}}{x-y} \Big[1+$$

$$+\frac{1}{N}\Big(\int_y^x W_1^{(1)} + \frac{1}{6}\int_y^x\int_y^x\int_y^x W_3^{(0)}\Big) + O(1/N^2)\Big] \qquad (4-3)$$

where the last exponential contains only negative powers of N and can be expanded at large N.

4.1 1-cut spectral curves

From now on, we make the 3 hypothesis of section 2.5, and we assume that we have a **1-cut** spectral curve.

4.1.1 Rational parametrization

Any algebraic curve of genus 0 can be parametrized with some rational functions of a complex variable, i,e. there exist two rational functions $x(z)$ and $y(z)$ such that $Y(x(z)) = y(z)$. Here, our equation $Y^2 = -\mathcal{E}_\infty(x)$ is of degree 2 in Y, and it implies that the function $x(z)$ is a rational function of degree 2 of z, it has either 2 simple poles, or a double pole.

In other words, the data of the function $Y(x)$, is equivalent to the data of two rational functions $x(z)$ and $y(z)$, where $x(z)$ is of degree 2:

$$\begin{cases} x(z) \\ y(z) \end{cases} \tag{4-4}$$

• If $x(z)$ has a double pole, we may always reparametrize z such that the double pole is at ∞, and the zeroe of x' is at zero, i.e. we may always choose:

$$x(z) = z^2 + c \tag{4-5}$$

This is the case for the Airy system.

• If $x(z)$ has a two simple poles, we may always reparametrize z such that the simple poles are at 0 and ∞, i.e. we may always choose:

$$x(z) = \gamma(z + \frac{1}{z}) + c \tag{4-6}$$

This is the case for matrix models differential systems.

4.1.2 Branchpoints

The branchpoints a_i are defined as the zeroes of $x'(z)$. There are either 1 or 2 branchpoints.

- If $x(z)$ has a double pole, $x(z) = z^2+c$, there is only one branchpoint at $z = a = 0$.
- If $x(z)$ has a two simple poles, $x(z) = \gamma(z + \frac{1}{z}) + c$, there are two branchpoints at $a_1 = +1$ and $a_2 = -1$.

4.1.3 Conjugated points

All hyperelliptical curves have an involution $Y(x) \to -Y(x)$, i.e. for each z, there exists a point $\bar{z}$, such that:

$$x(\bar{z}) = x(z) \qquad , \qquad y(\bar{z}) = -y(z) \tag{4-7}$$

- If $x(z)$ has a double pole, $x(z) = z^2 + c$, we have $\bar{z} = -z$.
- If $x(z)$ has a two simple poles, $x(z) = \gamma(z + \frac{1}{z}) + c$, we have $\bar{z} = 1/z$.

4.2 Bergman kernel

The notion of a Bergman kernel exists for all algebraic curves, but for curves of genus 0, it is particularly simple:

We define the Bergman kernel:

$$B(z_1, z_2) = \frac{1}{(z_1 - z_2)^2} \tag{4-8}$$

it is the only rational fraction having a double pole on the diagonal, and which is integrable (it has no residue, and it decreases as $1/z^2$ at ∞).

It can be proved [15], that the function $W_2^{(0)}$ satisfying our 3 hypothesis, is always:

$$W_2^{(0)}(x(z_1), x(z_2)) + \frac{1}{(x(z_1) - x(z_2))^2} = \frac{B(z_1, z_2)}{x'(z_1)x'(z_2)} \tag{4-9}$$

Let us compute:

$$\begin{aligned}
& \int_{z_2}^{z_1} \int_{z_2}^{z_1} W_2^{(0)}(x(z_1), x(z_2))\, dx(z_1)\, dx(z_2) \\
= & \int_{z_2}^{z_1} \int_{z_2}^{z_1} \frac{dz_1\, dz_2}{(z_1 - z_2)^2} - \frac{dx(z_1)\, dx(z_2)}{(x(z_1) - x(z_2))^2} \\
= & -2 \ln \frac{(z_1 - z_2)}{x(z_1) - x(z_2)} - \ln x'(z_1)x'(z_2)
\end{aligned} \tag{4-10}$$

and therefore:

$$\frac{1}{x(z_1) - x(z_2)}\, \mathrm{e}^{\frac{1}{2} \int_{z_2}^{z_1} \int_{z_2}^{z_1} W_2^{(0)}(x(z_1),x(z_2))\, dx(z_1)\, dx(z_2)} = \frac{1}{(z_1 - z_2)\, \sqrt{x'(z_1)x'(z_2)}} \tag{4-11}$$

4.3 Definitions of the symplectic invariants

Definition 4.1 *We define (see [15]) the symplectic invariants correlators:*

$$\omega_1^{(0)}(z) = -y(z) \tag{4-12}$$

$$\omega_2^{(0)}(z_1, z_2) = B(z_1, z_2) - \frac{x'(z_1)\, x'(z_2)}{(x(z_1) - x(z_2))^2} \tag{4-13}$$

$$\omega_{n+1}^{(g)}(z_0, J) = \sum_i \mathop{\rm Res}_{z \to a_i} dz\ \mathbb{K}(z_0, z) \Big[\omega_{n+2}^{(g-1)}(z, z, J) + \sum_{h=0}^{g} \sum_{I \subset J}{}' \omega_{1+|I|}^{(h)}(z, I)\omega_{1+j-|I|}(z, J/I)\Big] \tag{4-14}$$

where J is a collective notation for the n variables $J = \{z_1, \ldots, z_n\}$, where $\Sigma_{I\subset J}$ is the sum over subsets of J, and $\sum'$ means that we exclude $(h, I) = (0, \emptyset)$ and $(h, I) = (g, J)$, and where a_i are the branch points $x'(a_i) = 0$, and the recursion kernel $\mathbb{K}$ is:

$$\mathbb{K}(z_0, z) = \frac{1}{2y(z)\, x'(\bar{z})\ (z - z_0)} \tag{4-15}$$

We also define the "full" correlators, as formal series:

$$\omega_n(z_1, \ldots, z_n) = \sum_{g=0}^{\infty} N^{2-2g-n}\, \omega_n^{(g)}(z_1, \ldots, z_n) \tag{4-16}$$

Each correlator $\omega_n^{(g)}(z_1, \ldots, z_n)$ is a rational fucntion of the $z_i \in \mathcal{L}$, and they have poles only at the branchpoints a_i (except $\omega_1^{(0)}$ and $\omega_2^{(0)}$).

We define the scalar correlators $\hat{W}_n^{(g)}$ by dividing by the dx_i's:

Definition 4.2

$$\hat{W}_n^{(g)}(x(z_1), \ldots, x(z_n)) = \frac{\omega_n^{(g)}(z_1, \ldots, z_n)}{x'(z_1) \ldots x'(z_n)} \tag{4-17}$$

and their "full" resummed version as formal series:

$$\hat{W}_n(x_1, \ldots, x_n) = \sum_{g=0}^{\infty} N^{2-2g-n}\, \hat{W}_n^{(g)}(x_1, \ldots, x_n) \tag{4-18}$$

The corollary 2.1 means that:

Corollary 4.1

$$\hat{W}_n^{(g)}(x_1, \ldots, x_n) = W_n^{(g)}(x_1, \ldots, x_n) \tag{4-19}$$

4.4 Exponential formulae

Let us define the formal Baker-Akhiezer kernel:

Definition 4.3

$$\tilde{K}(z_1,z_2) = \frac{e^{-N\int_{z_2}^{z_1} y dx}}{(z_1-z_2)\sqrt{x'(z_1)\,x'(z_2)}}\,\exp\left[\sum_{g,l}{}' \frac{N^{2-2g-l}}{l!}\underbrace{\int_{z_2}^{z_1}\dots\int_{z_2}^{z_1}}_{l}\omega_l^{(g)}\right] \tag{4-20}$$

where $\sum'$ means that we exclude all terms such that $2-2g-l\geq 0$. This definition makes sense order by order in the $1/N$ expansion: $\ln\tilde{K}$ is a formal series in $1/N$, whose coefficients are rational functions of z_1 and z_2.

This kernel clearly has the property that:

$$\sum_g N^{1-2g}\, W_1^{(g)}(x(z)) = \lim_{z'\to z}\tilde{K}(z,z') - \frac{1}{x(z)-x(z')} \tag{4-21}$$

i.e. it satisfies the determinantal formula for W_1.

We conjecture that:

Conjecture 4.1

$$\tilde{K}(z_1,z_2) = K(x(z_1),x(z_2)) \tag{4-22}$$

This conjecture is true for the Airy spectral curve, as we have seen in section 3.3.

A way to prove that conjecture would be to prove that:

$$-\,\delta_{z_3}\,\tilde{K}(z_1,z_2) \stackrel{?}{=} \tilde{K}(z_1,z_3)\,\tilde{K}(z_3,z_2) \tag{4-23}$$

where Indeed, by recursively acting with δ_{z_j} on eq.(4-21), we would generate the determinantal formula for W_n.

Let us verify that this conjecture holds to the leading orders in $1/N$. To leading orders we have:

$$\begin{aligned}\tilde{K}(z_1,z_2) &= \frac{e^{-N\int_{z_2}^{z_1} y dx}}{(z_1-z_2)\sqrt{x'(z_1)\,x'(z_2)}}\,\Big[1 \\ &\quad +\frac{1}{N}\left(\int_{z_2}^{z_1}\omega_1^{(1)} + \frac{1}{6}\int_{z_2}^{z_1}\int_{z_2}^{z_1}\int_{z_2}^{z_1}\omega_3^{(0)}\right) + O(1/N^2)\Big]\end{aligned} \tag{4-24}$$

therefore (using $x'(z_{n+1})\,\delta_{z_{n+1}}\,\omega_n^{(g)}(z_1,\dots,z_n) = \frac{1}{N}\omega_{n+1}^{(g)}(z_1,\dots,z_n,z_{n+1})$), we have:

$$-\,x'(z_3)\,\delta_{z_3}\ln\tilde{K}(z_1,z_2)$$

$$
\begin{aligned}
&= -\int_{z_2}^{z_1} \omega_2^{(0)}(z', z_3) - \frac{1}{2N} \int_{z_2}^{z_1}\int_{z_2}^{z_1} \omega_3^{(0)}(z_1', z_2', z_3) + O(1/N^2) \\
&= -\int_{z_2}^{z_1} \frac{dz'}{(z'-z_3)^2} - \frac{1}{2N} \int_{z_2}^{z_1}\int_{z_2}^{z_1} \omega_3^{(0)}(z_1', z_2', z_3) + O(1/N^2) \\
&= \frac{(z_1-z_2)}{(z_1-z_3)(z_3-z_2)} - \frac{1}{2N} \int_{z_2}^{z_1}\int_{z_2}^{z_1} \omega_3^{(0)}(z_1', z_2', z_3) + O(1/N^2).
\end{aligned}
\tag{4-25}
$$

On the other hand we have:

$$
\begin{aligned}
\frac{\tilde{K}(z_1,z_3)\tilde{K}(z_3,z_2)}{\tilde{K}(z_1,z_2)} = \frac{(z_1-z_2)}{(z_1-z_3)(z_3-z_2)\,x'(z_3)} \Big[1+ \\
+\frac{1}{6N}\Big(\int_{z_3}^{z_1}\int_{z_3}^{z_1}\int_{z_3}^{z_1} \omega_3^{(0)} + \int_{z_2}^{z_3}\int_{z_2}^{z_3}\int_{z_2}^{z_3} \omega_3^{(0)} \\
- \int_{z_2}^{z_1}\int_{z_2}^{z_1}\int_{z_2}^{z_1} \omega_3^{(0)}\Big) + O(1/N^2)\Big].
\end{aligned}
\tag{4-26}
$$

The equality eq.(4-23) clearly holds to order N^0.

To order $1/N$, the equality to prove is thus:

$$
\begin{aligned}
\frac{(z_1-z_2)}{6\,(z_1-z_3)(z_3-z_2)} \left(\int_{z_3}^{z_1}\int_{z_3}^{z_1}\int_{z_3}^{z_1} \omega_3^{(0)} + \int_{z_2}^{z_3}\int_{z_2}^{z_3}\int_{z_2}^{z_3} \omega_3^{(0)} - \int_{z_2}^{z_1}\int_{z_2}^{z_1}\int_{z_2}^{z_1} \omega_3^{(0)}\right) \\
\overset{?}{=} -\frac{1}{2} \int_{z_2}^{z_1}\int_{z_2}^{z_1} \omega_3^{(0)}(z_1', z_2', z_3)\,.
\end{aligned}
\tag{4-27}
$$

or in other words:

$$
\begin{aligned}
&\frac{(z_1-z_2)}{3\,(z_1-z_3)(z_3-z_2)} \left(\int_{z_3}^{z_1}\int_{z_3}^{z_1}\int_{z_3}^{z_1} \omega_3^{(0)} + \int_{z_2}^{z_3}\int_{z_2}^{z_3}\int_{z_2}^{z_3} \omega_3^{(0)} - \int_{z_2}^{z_1}\int_{z_2}^{z_1}\int_{z_2}^{z_1} \omega_3^{(0)}\right) \\
&+ \int_{z_2}^{z_1}\int_{z_2}^{z_1} \omega_3^{(0)}(z_1', z_2', z_3) \\
&\overset{?}{=} 0
\end{aligned}
\tag{4-28}
$$

For any rational spectral curve, $\omega_3^{(0)}$ is given by eq.(4-14):

$$
\begin{aligned}
&\omega_3^{(0)}(z_1,z_2,z_3) \\
&= 2\sum_i \mathop{\mathrm{Res}}_{z\to a_i} \mathbb{K}(z_1,z)\,\omega_2^{(0)}(z,z_2)\omega_2^{(0)}(z,z_3) \\
&= \sum_i \frac{1}{y'(a_i)\;x''(a_i)}\,\frac{1}{(a_i-z_1)^2(a_i-z_2)^2(a_i-z_3)^2}
\end{aligned}
\tag{4-29}
$$

It can also be rewritten:

$$
\omega_3^{(0)}(z_1,z_2,z_3) = \sum_i \mathop{\mathrm{Res}}_{z\to a_i} \frac{dz}{y'(z)\,x'(z)}\;\frac{1}{(z-z_1)^2\,(z-z_2)^2\,(z-z_3)^2}
\tag{4-30}
$$

Then, let us define:

$$S_{z_1,z_2}(z) = \int_{z_2}^{z_1} B(z,z')\,dz' = \frac{1}{z-z_1} - \frac{1}{z-z_2} = \frac{(z_1-z_2)}{(z-z_1)(z-z_2)} \tag{4-31}$$

We thus have:

$$\int_{z_2}^{z_1}\int_{z_2}^{z_1}\int_{z_2}^{z_1} \omega_3^{(0)}(z_1',z_2',z_3') = \sum_i \mathop{\mathrm{Res}}_{z\to a_i} \frac{(S_{z_1,z_2}(z))^3}{x'(z)\,y'(z)}\,dz \tag{4-32}$$

and

$$\int_{z_2}^{z_1}\int_{z_2}^{z_1} \omega_3^{(0)}(z_1',z_2',z_3) = \sum_i \mathop{\mathrm{Res}}_{z\to a_i} \frac{(S_{z_1,z_2}(z))^2\,B(z,z_3)}{x'(z)\,y'(z)}\,dz \tag{4-33}$$

The equality eq.(4-28) is thus:

$$\sum_i \mathop{\mathrm{Res}}_{z\to a_i} \frac{dz}{x'(z)\,y'(z)} \Big[\frac{(z_1-z_2)}{3\,(z_1-z_3)(z_3-z_2)} \Big((S_{z_1,z_3}(z))^3 + (S_{z_3,z_2}(z))^3 - (S_{z_1,z_2}(z))^3\Big) + (S_{z_1,z_2}(z))^2\,B(z,z_3)\Big] \stackrel{?}{=} 0 \tag{4-34}$$

The expression inside the bracket is:

$$\begin{aligned}
& \frac{(z_1-z_2)}{3\,(z_1-z_3)(z_3-z_2)} \Big((S_{z_1,z_3}(z))^3 + (S_{z_3,z_2}(z))^3 - (S_{z_1,z_2}(z))^3\Big) \\
& +(S_{z_1,z_2}(z))^2\,B(z,z_3) \\
= \; & \frac{(z_1-z_2)}{3\,(z_1-z_3)(z_3-z_2)} \Big((S_{z_1,z_3}(z))^3 + (S_{z_3,z_2}(z))^3 - (S_{z_1,z_3}(z) + S_{z_3,z_2}(z))^3\Big) \\
& +(S_{z_1,z_2}(z))^2\,B(z,z_3) \\
= \; & \frac{(z_1-z_2)}{(z_1-z_3)(z_3-z_2)}\, S_{z_1,z_3}(z)\, S_{z_3,z_2}(z)\, S_{z_2,z_1}(z) + (S_{z_1,z_2}(z))^2\,B(z,z_3) \\
= \; & S_{z_2,z_1}(z) \Big[\frac{(z_1-z_2)}{(z_1-z_3)(z_3-z_2)}\, S_{z_1,z_3}(z)\, S_{z_3,z_2}(z) + S_{z_2,z_1}(z)\,B(z,z_3)\Big] \\
= \; & S_{z_2,z_1}(z) \; \Big[\frac{(z_1-z_2)}{(z_1-z_3)(z_3-z_2)}\, \frac{(z_1-z_3)}{(z-z_1)(z-z_3)}\, \frac{(z_3-z_2)}{(z-z_3)(z-z_2)} \\
& + \frac{(z_2-z_1)}{(z-z_2)(z-z_1)}\, \frac{1}{(z-z_3)^2}\Big] \\
= \; & S_{z_2,z_1}(z)\, dz^2 \Big[\frac{(z_1-z_2)}{(z-z_1)(z-z_2)(z-z_3)^2} \\
& + \frac{(z_2-z_1)}{(z-z_2)(z-z_1)\,(z-z_3)^2}\Big] \\
= \; & 0
\end{aligned} \tag{4-35}$$

Therefore we have proved that the conjecture holds to order $1/N$.

However, this method is not the good method to prove it to all orders, and we leave it for a further work.

5 Conclusion

In this article, we have proved that for any 2×2 differential system, the determinantal correlation functions do obey loop equations.

This is particularly useful if in addition, our differential system has a topological expansion property, because in that case, the solution of loop equations is known.

We have thus completed some of the claims in [15], regarding integrability, and we have made the link between those two formulations of integrability.

Moreover, we have conjectured an exponential formula for the formal kernel defined from the correlation functions of a spectral curve. We have proved this formula for the Airy kernel, and we have proved that it holds to the first two orders in $1/N$. It remains to be proved in the general case.

Also, we have considered only 2×2 systems for simplicity, but it seems that our method could be extended to higher rank systems.

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Appendix

A Proof of theorem 2.6 $\delta_y\mathcal{D}(x)$

Theorem 2.6:

$$-\delta_y\mathcal{D}(x)=\frac{1}{(x-y)^2}(M(x)+M(y)-1)+\frac{1}{x-y}[\mathcal{D}(x),M(y)] \tag{A-1}$$

proof:

We start from

$$\vec{\phi}'(x)=\mathcal{D}(x)\,\vec{\phi}(x) \tag{A-2}$$

and we apply δ_y, i.e.

$$\delta_y\vec{\phi}'(x) \qquad = \qquad d_x\delta_y\vec{\phi}(x)$$

$$\begin{aligned} \delta_y(\mathcal{D}(x)\vec{\phi}(x)) &= d_x\delta_y\vec{\phi}(x) \\ (\delta_y\mathcal{D}(x))\,\vec{\phi}(x) + \mathcal{D}(x)\delta_y\vec{\phi}(x) &= d_x\delta_y\vec{\phi}(x) \end{aligned} \quad (A-3)$$

and therefore:

$$\begin{aligned} \delta_y\mathcal{D}(x)\,.\vec{\phi}(x) &= (d_x - \mathcal{D}(x))\,\delta_y\vec{\phi}(x) \\ &= (d_x - \mathcal{D}(x))\,\frac{M(y)\,\vec{\phi}(x)}{x-y} \\ &= \left(\frac{[M(y),\mathcal{D}(x)]}{x-y} - \frac{M(y)}{(x-y)^2}\right)\vec{\phi}(x) \end{aligned} \quad (A-4)$$

Which means that $\delta_y\mathcal{D}(x)$ is the matrix in the RHS modulo a matrix which projects $\vec{\phi}(x)$ to $\vec{0}$, and thus, there exists a function $f(x,y)$ such that:

$$\delta_y\mathcal{D}(x) = \frac{[M(y),\mathcal{D}(x)]}{x-y} - \frac{M(y)}{(x-y)^2} + f(x,y)(1-M(x)) \tag{A-5}$$

Since $\mathrm{Tr}\,\mathcal{D}(x) = 0$, by taking the trace we find $f(x,y) = 1/(x-y)^2$, and thus:

$$\delta_y\mathcal{D}(x) = \frac{[M(y),\mathcal{D}(x)]}{x-y} + \frac{1-M(x)-M(y)}{(x-y)^2} \tag{A-6}$$

□

B Proof of theorem 2.8 $\delta_{x_n}P_n(x;x_1,\dots,x_{n-1})$

Theorem 2.8:

For $n > 2$

$$\begin{aligned} & -\delta_{x_n}Q_n(x;x_1,\dots,x_{n-1}) \\ = \; & (-1)^n\,P_{n+1}(x;x_1,\dots,x_{n-1},x_n) \\ & -(-1)^n\frac{\partial}{\partial x_n}\,\frac{W_n(x,x_1,\dots,x_{n-1}) + W_n(x_n,x_1,\dots,x_{n-1})}{x-x_n} \\ & + R_n(x;x_1,\dots,x_{n-1},x_n) \end{aligned} \tag{B-1}$$

where $R_n(x;x_1,\dots,x_{n-1},x_n)$ is a rational fraction of x with only simple poles at $x = x_j, j = 1,\dots,n-1$.

It follows that for all n:

$$\begin{aligned} & \delta_{x_n}P_n(x;x_1,\dots,x_{n-1}) \\ = \; & P_{n+1}(x;x_1,\dots,x_{n-1},x_n) - \frac{\partial}{\partial x_n}\,\frac{W_n(x,x_1,\dots,x_{n-1}) + W_n(x_n,x_1,\dots,x_{n-1})}{x-x_n} \end{aligned}$$

$$-\delta_{n,2}\,\frac{\partial}{\partial x_n}\,\frac{1}{(x_1-x_n)^2\,(x-x_n)} \tag{B-2}$$

Proof:

Let us start with the case $n>2$, and then we consider the two special cases $n=1$ and $n=2$.

B.1 Case $n>2$

proof:

We write:

$$X_{i_1,i_2,\dots,i_k}=(x_{i_1}-x_{i_2})(x_{i_2}-x_{i_3})\dots(x_{i_{k-1}}-x_{i_k}) \tag{B-3}$$

Let us work modulo rational fractions of x having only simple poles at $x=x_1,\dots,x_{n-1}$. We have:

$$\begin{aligned}
& -\delta_{x_n}Q_n(x;x_1,\dots,x_{n-1}) \\
= \quad & \sum_{j=1}^{n-1}\sum_\sigma \frac{1}{(x_j-x_n)}\,\frac{\mathrm{Tr}\,\mathcal{D}(x)M(x_{\sigma(1)})\dots[M(x_{\sigma(j)}),M(x_n)]\dots M(x_{\sigma(n-1)})}{(x-x_{\sigma(1)})\,X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x)} \\
& +\sum_\sigma \frac{1}{(x-x_n)}\,\frac{\mathrm{Tr}\,[\mathcal{D}(x),M(x_n)]\,M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x-x_{\sigma(1)})\,X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x)} \\
& +\sum_\sigma \frac{1}{(x-x_n)^2}\,\frac{\mathrm{Tr}\,M(x)M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x-x_{\sigma(1)})\,X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x)} \\
& +\sum_\sigma \frac{1}{(x-x_n)^2}\,\frac{\mathrm{Tr}\,M(x_n)M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x-x_{\sigma(1)})\,X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x)} \\
& -\sum_\sigma \frac{1}{(x-x_n)^2}\,\frac{\mathrm{Tr}\,M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x-x_{\sigma(1)})\,X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x)}
\end{aligned}$$

$(B-4)$

The first 2 lines rearrange into:

$$\begin{aligned}
& \sum_{j=1}^{n-1}\sum_\sigma \frac{1}{(x_j-x_n)}\,\frac{\mathrm{Tr}\,\mathcal{D}(x)M(x_{\sigma(1)})\dots[M(x_{\sigma(j)}),M(x_n)]\dots M(x_{\sigma(n-1)})}{(x-x_{\sigma(1)})\,X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x)} \\
& +\sum_\sigma \frac{1}{(x-x_n)}\,\frac{\mathrm{Tr}\,[\mathcal{D}(x),M(x_n)]\,M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x-x_{\sigma(1)})\,X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x)} \\
= \quad & Q_{n+1}(x;x_1,\dots,x_{n-1},x_n)
\end{aligned} \tag{B-5}$$

In the 4th line, we write that:

$$\begin{aligned}
& \frac{1}{(x-x_n)^2}\,\frac{(x_n-x_{\sigma(1)})(x_{\sigma(n-1)}-x_n)}{(x-x_{\sigma(1)})(x_{\sigma(n-1)}-x)} \\
\equiv \quad & \frac{1}{(x-x_n)^2}-\frac{1}{(x-x_n)(x_n-x_{\sigma(1)})}-\frac{1}{(x-x_n)(x_n-x_{\sigma(n-1)})}
\end{aligned} \tag{B-6}$$

Therefore we have:

$$
\begin{aligned}
& -\delta_{x_n} Q_n(x;x_1,\dots,x_{n-1}) \\
\equiv\quad & Q_{n+1}(x;x_1,\dots,x_{n-1},x_n) \\
& -(-1)^n \frac{W_n(x,x_1,\dots,x_{n-1})+W_n(x_n,x_1,\dots,x_{n-1})}{(x-x_n)^2} \\
& -\sum_\sigma \frac{1}{x-x_n}\,\frac{\mathrm{Tr}\, M(x_n)M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x_n-x_{\sigma(1)})^2\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x_n)} \\
& +\sum_\sigma \frac{1}{x-x_n}\,\frac{\mathrm{Tr}\, M(x_n)M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x_n-x_{\sigma(1)})\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x_n)^2} \\
& -\sum_\sigma \frac{1}{(x-x_n)^2}\,\frac{\mathrm{Tr}\, M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x-x_{\sigma(1)})\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x)}
\end{aligned}
$$

$(B-7)$

and notice that:

$$
\begin{aligned}
& Q_{n+1}(x;x_1,\dots,x_{n-1},x_n) \\
\equiv\quad & (-1)^n\, P_{n+1}(x;x_1,\dots,x_{n-1},x_n) + \frac{1}{x-x_n}\mathop{\mathrm{Res}}_{x'\to x_n} Q_{n+1}(x;x_1,\dots,x_{n-1},x_n) \\
\equiv\quad & (-1)^n\, P_{n+1}(x;x_1,\dots,x_{n-1},x_n) \\
& +\frac{1}{x-x_n}\mathop{\mathrm{Res}}_{x'\to x_n}\sum_\sigma \frac{\mathrm{Tr}\,\mathcal{D}(x')M(x_n)\,M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x'-x_n)(x_n-x_{\sigma(1)})\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x')} \\
& +\frac{1}{x-x_n}\mathop{\mathrm{Res}}_{x'\to x_n}\sum_\sigma \frac{\mathrm{Tr}\, M(x_n)\mathcal{D}(x')\,M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x'-x_{\sigma(1)})\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x')(x_n-x')} \\
\equiv\quad & (-1)^n\, P_{n+1}(x;x_1,\dots,x_{n-1},x_n) \\
& +\sum_\sigma \frac{\mathrm{Tr}\,\mathcal{D}(x_n)M(x_n)\,M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x-x_n)(x_n-x_{\sigma(1)})\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x_n)} \\
& -\sum_\sigma \frac{\mathrm{Tr}\, M(x_n)\mathcal{D}(x_n)\,M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x-x_n)(x_n-x_{\sigma(1)})\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x_n)} \\
\equiv\quad & (-1)^n\, P_{n+1}(x;x_1,\dots,x_{n-1},x_n) \\
& +\sum_\sigma \frac{\mathrm{Tr}\,[\mathcal{D}(x_n),M(x_n)]\,M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x-x_n)(x_n-x_{\sigma(1)})\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x_n)} \\
\equiv\quad & (-1)^n\, P_{n+1}(x;x_1,\dots,x_{n-1},x_n) \\
& +\sum_\sigma \frac{\mathrm{Tr}\,\frac{\partial}{\partial x_n} M(x_n)\,M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x-x_n)(x_n-x_{\sigma(1)})\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x_n)} \\
\equiv\quad & (-1)^n\, P_{n+1}(x;x_1,\dots,x_{n-1},x_n) \\
& +\frac{1}{x-x_n}\frac{\partial}{\partial x_n}\sum_\sigma \frac{\mathrm{Tr}\, M(x_n)\,M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x_n-x_{\sigma(1)})\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x_n)} \\
& +\sum_\sigma \frac{\mathrm{Tr}\, M(x_n)\,M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x-x_n)(x_n-x_{\sigma(1)})^2\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x_n)} \\
& -\sum_\sigma \frac{\mathrm{Tr}\, M(x_n)\,M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x-x_n)(x_n-x_{\sigma(1)})\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x_n)^2}
\end{aligned}
$$

$$\begin{aligned}
\equiv\quad & (-1)^n\, P_{n+1}(x;x_1,\dots,x_{n-1},x_n) \\
& -\frac{(-1)^n}{x-x_n}\,\frac{\partial}{\partial x_n}\,W_n(x_1,\dots,x_n) \\
& +\sum_\sigma \frac{\mathrm{Tr}\, M(x_n)\, M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x-x_n)(x_n-x_{\sigma(1)})^2\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x_n)} \\
& -\sum_\sigma \frac{\mathrm{Tr}\, M(x_n)\, M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x-x_n)(x_n-x_{\sigma(1)})\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x_n)^2}
\end{aligned} \qquad (B-8)$$

And therefore:

$$\begin{aligned}
& -\delta_{x_n} Q_n(x;x_1,\dots,x_{n-1}) \\
\equiv\quad & (-1)^n\, P_{n+1}(x;x_1,\dots,x_{n-1},x_n) \\
& -(-1)^n \frac{\partial}{\partial x_n}\,\frac{W_n(x,x_1,\dots,x_{n-1})+W_n(x_n,x_1,\dots,x_{n-1})}{x-x_n} \\
& -\sum_\sigma \frac{1}{(x-x_n)^2}\,\frac{\mathrm{Tr}\, M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x-x_{\sigma(1)})\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x)}
\end{aligned} \qquad (B-9)$$

Consider the term in the last line:

$$\begin{aligned}
& -\sum_\sigma \frac{1}{(x-x_n)^2}\,\frac{\mathrm{Tr}\, M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x-x_{\sigma(1)})\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x)} \\
\equiv\quad & -\frac{\partial}{\partial x_n}\left(\frac{1}{x-x_n}\sum_\sigma \frac{\mathrm{Tr}\, M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x_n-x_{\sigma(1)})\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x_n)}\right)
\end{aligned} \qquad (B-10)$$

We have:

$$\begin{aligned}
& \sum_\sigma \frac{\mathrm{Tr}\, M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x_n-x_{\sigma(1)})\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x_n)} \\
=\quad & \sum_\sigma \frac{\mathrm{Tr}\, M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x_n-x_{\sigma(1)})\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x_{\sigma(1)})} \\
& -\sum_\sigma \frac{\mathrm{Tr}\, M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x_n-x_{\sigma(n-1)})\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x_{\sigma(1)})}
\end{aligned} \qquad (B-11)$$

and in the last line, we replace $\sigma=\tilde\sigma.S$ where S is the shift $i\to i+1$, thus:

$$\begin{aligned}
& \sum_\sigma \frac{\mathrm{Tr}\, M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x_n-x_{\sigma(1)})\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x_n)} \\
=\; & \sum_\sigma \frac{\mathrm{Tr}\, M(x_{\sigma(1)})\dots M(x_{\sigma(n-1)})}{(x_n-x_{\sigma(1)})\, X_{\sigma(1),\sigma(2),\dots,\sigma(n-1)}\,(x_{\sigma(n-1)}-x_{\sigma(1)})}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\tilde\sigma} \frac{\mathrm{Tr}\, M(x_{\tilde\sigma(1)}) \dots M(x_{\tilde\sigma(n-1)})}{(x_n - x_{\tilde\sigma(1)})\, X_{\tilde\sigma(1),\tilde\sigma(2),\dots,\tilde\sigma(n-1)}\, (x_{\tilde\sigma(n-1)} - x_{\tilde\sigma(1)})} \\
&= 0
\end{aligned} \tag{B-12}
$$

And Finally:

$$
\begin{aligned}
& -\delta_{x_n} Q_n(x; x_1, \dots, x_{n-1}) \\
\equiv \quad & (-1)^n\, P_{n+1}(x; x_1, \dots, x_{n-1}, x_n) \\
& -(-1)^n \frac{\partial}{\partial x_n}\, \frac{W_n(x, x_1, \dots, x_{n-1}) + W_n(x_n, x_1, \dots, x_{n-1})}{x - x_n}
\end{aligned}
$$

$(B-13)$

This implies:

$$
\begin{aligned}
& \delta_{x_n} P_n(x; x_1, \dots, x_{n-1}) \\
= \quad & P_{n+1}(x; x_1, \dots, x_{n-1}, x_n) \\
& - \frac{\partial}{\partial x_n}\, \frac{W_n(x, x_1, \dots, x_{n-1}) + W_n(x_n, x_1, \dots, x_{n-1})}{x - x_n}
\end{aligned}
$$

$(B-14)$

□

B.2 Case $n = 1$

$$
\delta_y\, P_1(x) = P_2(x; y) - \frac{\partial}{\partial y}\, \frac{W_1(x) + W_1(y)}{x - y} \tag{B-15}
$$

proof:

We start from

$$
P_1(x) = -\frac{1}{2}\, \mathrm{Tr}\, \mathcal{D}^2(x) \tag{B-16}
$$

$$
\begin{aligned}
& \delta_y\, P_1(x) \\
= \quad & -\mathrm{Tr}\, \mathcal{D}(x)\, \delta_y\, \mathcal{D}(x) \\
= \quad & \frac{1}{(x-y)^2}\, \mathrm{Tr}\, \mathcal{D}(x)\, (M(x) + M(y) - 1) \\
= \quad & \frac{1}{(x-y)^2}\, (\, \mathrm{Tr}\, \mathcal{D}(x) M(x) + \, \mathrm{Tr}\, \mathcal{D}(x)\, M(y)) \\
= \quad & \frac{1}{(x-y)^2}\, \mathrm{Tr}\, (\mathcal{D}(x) M(x) + \, \mathrm{Tr}\, \mathcal{D}(y) M(y) + (x-y)\, \mathrm{Tr}\, \mathcal{D}'(y) M(y)) \\
& + \frac{1}{(x-y)^2}\, \mathrm{Tr}\, (\mathcal{D}(x) - \mathcal{D}(y) - (x-y) \mathcal{D}'(y))\, M(y) \\
= \quad & \frac{1}{(x-y)^2}\, \mathrm{Tr}\, (\mathcal{D}(x) M(x) + \, \mathrm{Tr}\, \mathcal{D}(y) M(y) + (x-y)\, \mathrm{Tr}\, \mathcal{D}'(y) M(y) \\
& \quad + \mathcal{D}(y) M'(y)\,) \\
& + \frac{1}{(x-y)^2}\, \mathrm{Tr}\, (\mathcal{D}(x) - \mathcal{D}(y) - (x-y) \mathcal{D}'(y))\, M(y)
\end{aligned}
$$

$$
\begin{aligned}
= &\quad \frac{\partial}{\partial y}\, \frac{\mathrm{Tr}\,\mathcal{D}(x)M(x) + \mathrm{Tr}\,\mathcal{D}(y)M(y)}{x-y} \\
&\quad + \frac{1}{(x-y)^2}\, \mathrm{Tr}\,(\mathcal{D}(x) - \mathcal{D}(y) - (x-y)\mathcal{D}'(y))\, M(y) \\
= &\quad -\frac{\partial}{\partial y}\, \frac{W_1(x)+W_1(y)}{x-y} + P_2(x;y)
\end{aligned}
\qquad (B-17)
$$

□

B.3 Case $n = 2$

$$
\delta_y\, P_2(x;x_1) = P_3(x;x_1,y) - \frac{\partial}{\partial y}\, \frac{W_2(x,x_1) + W_2(y,x_1) + \frac{1}{(y-x_1)^2}}{x-y} \tag{B-18}
$$

proof:

We define:

$$
Q_2(x;x_1) = \frac{\mathrm{Tr}\,\mathcal{D}(x)\, M(x_1)}{(x-x_1)^2} \tag{B-19}
$$

We start from

$$
P_2(x;x_1) = Q_2(x;x_1) - \frac{1}{(x-x_1)^2} \mathop{\mathrm{Res}}_{x'\to x_1} (x'-x_1)\, Q_2(x';x_1) - \frac{1}{x-x_1} \mathop{\mathrm{Res}}_{x'\to x_1} Q_2(x';x_1) \tag{B-20}
$$

Let us compute $\delta_y Q_2(x;x_1)$, modulo rational fractions of x having only a simple or a double pole at $x = x_1$:

$$
\begin{aligned}
&\quad -\delta_y Q_2(x;x_1) \\
= &\quad \frac{1}{(x-x_1)^2\,(x-y)^2}\, \mathrm{Tr}\,(M(x)+M(y)-1)M(x_1) \\
&\quad + \frac{1}{(x-x_1)^2\,(x-y)}\, \mathrm{Tr}\,[\mathcal{D}(x), M(y)]\, M(x_1) \\
&\quad + \frac{1}{(x-x_1)^2\,(x_1-y)}\, \mathrm{Tr}\,\mathcal{D}(x)\, [M(x_1), M(y)] \\
= &\quad \frac{1}{(x-x_1)^2\,(x-y)^2}\, \mathrm{Tr}\, M(x)M(x_1) \\
&\quad + \frac{1}{(x-x_1)^2\,(x-y)^2}\, \mathrm{Tr}\, M(y)M(x_1) - \frac{1}{(x-x_1)^2\,(x-y)^2} \\
&\quad - \frac{1}{(x-x_1)\,(x_1-y)\,(y-x)}\, \mathrm{Tr}\,\mathcal{D}(x)\, [M(x_1), M(y)] \\
= &\quad \frac{W_2(x,x_1)}{(x-y)^2} + \frac{(y-x_1)^2 W_2(y,x_1)}{(x-x_1)^2\,(x-y)^2} + \frac{1}{(x-x_1)^2(x-y)^2} \\
&\quad - \frac{1}{(x-x_1)\,(x_1-y)\,(y-x)}\, \mathrm{Tr}\,\mathcal{D}(x)\, [M(x_1), M(y)] \\
= &\quad \frac{W_2(x,x_1) + W_2(y,x_1)}{(x-y)^2} + \frac{W_2(y,x_1)}{(x-x_1)^2} - \frac{2\, W_2(y,x_1)}{(x-x_1)\,(x-y)} \\
&\quad + \frac{1}{(x-x_1)^2(x-y)^2} + \frac{1}{(x-x_1)\,(x_1-y)\,(x-y)}\, \mathrm{Tr}\,\mathcal{D}(x)\, [M(x_1), M(y)]
\end{aligned}
$$

$$
\begin{aligned}
\equiv \quad & \frac{W_2(x,x_1)+W_2(y,x_1)}{(x-y)^2} - \frac{2\,W_2(y,x_1)}{(y-x_1)\,(x-y)} + \frac{\partial}{\partial y}\,\frac{1}{(x-y)\,(x_1-y)^2} \\
& -Q_3(x;x_1,y)
\end{aligned}
$$
$(B-21)$

Notice that:

$$
\begin{aligned}
& Q_3(x;x_1,y) \\
\equiv \quad & P_3(x;x_1,y) + \frac{1}{x-y}\,\mathop{\rm Res}_{x'\to y}\, Q_3(x';x_1,y) \\
\equiv \quad & P_3(x;x_1,y) \\
& +\frac{1}{x-y}\,\mathop{\rm Res}_{x'\to y}\,\frac{1}{(x'-x_1)\,(x_1-y)\,(y-x')}\,{\rm Tr}\,\mathcal{D}(x')\,[M(x_1),M(y)] \\
\equiv \quad & P_3(x;x_1,y) + \frac{1}{x-y}\,\frac{1}{(y-x_1)^2}\,{\rm Tr}\,\mathcal{D}(y)\,[M(x_1),M(y)] \\
\equiv \quad & P_3(x;x_1,y) - \frac{1}{x-y}\,\frac{1}{(y-x_1)^2}\,{\rm Tr}\,[\mathcal{D}(y),M(y)]\,M(x_1) \\
\equiv \quad & P_3(x;x_1,y) - \frac{1}{x-y}\,\frac{1}{(y-x_1)^2}\,\frac{\partial}{\partial y}\,{\rm Tr}\,M(y)\,M(x_1) \\
\equiv \quad & P_3(x;x_1,y) - \frac{1}{x-y}\,\frac{1}{(y-x_1)^2}\,\frac{\partial}{\partial y}\,\Big((y-x_1)^2 W_2(y,x_1)+1\Big) \\
\equiv \quad & P_3(x;x_1,y) - \frac{1}{x-y}\,\frac{\partial}{\partial y}\,W_2(y,x_1) - \frac{2}{(x-y)\,(y-x_1)}\,W_2(y,x_1)
\end{aligned}
$$
$(B-22)$

and therefore:

$$
-\,\delta_y Q_2(x;x_1) \equiv \frac{\partial}{\partial y}\,\frac{W_2(x,x_1)+W_2(y,x_1)+\frac{1}{(y-x_1)^2}}{x-y} - P_3(x;x_1,y) \qquad \text{(B-23)}
$$

which implies:

$$
\delta_y P_2(x;x_1) = P_3(x;x_1,y) - \frac{\partial}{\partial y}\,\frac{W_2(x,x_1)+W_2(y,x_1)+\frac{1}{(y-x_1)^2}}{x-y} \qquad \text{(B-24)}
$$

□

C Proof of theorem 2.9 loop equations

Theorem 2.9:

For all $n \geq 0$, the correlation functions W_n satisfy the loop equation:

$$
\begin{aligned}
0 \quad = \quad & W_{n+2}(x,x,L) + \sum_{J\subset L} W_{1+|J|}(x,J)W_{1+n-|J|}(x,L/J) \\
& +\sum_{j=1}^{n}\frac{d}{dx_j}\,\frac{W_n(x,L/\{x_j\})-W_n(L)}{x-x_j} + P_{n+1}(x;L) \qquad \text{(C-1)}
\end{aligned}
$$

where $L=\{x_1,\ldots,x_n\}$.

C.1 Case $n = 0$

proof:

we have:

$$\begin{aligned} W_1(x) &= -\operatorname{Tr} \mathcal{D}(x) M(x) \\ &= -\operatorname{Tr} \mathcal{D}(x) \vec{\phi}(x) \vec{\psi}(x)^t A \\ &= -\vec{\psi}(x)^t A \mathcal{D}(x) \vec{\phi}(x) \end{aligned} \quad (C-2)$$

and thus:

$$\begin{aligned} W_1(x)^2 &= \vec{\psi}(x)^t A \mathcal{D}(x) \vec{\phi}(x) \vec{\psi}(x)^t A \mathcal{D}(x) \vec{\phi}(x) \\ &= \operatorname{Tr} \mathcal{D}(x) \vec{\phi}(x) \vec{\psi}(x)^t A \mathcal{D}(x) \vec{\phi}(x) \vec{\psi}(x)^t A \\ &= \operatorname{Tr} \mathcal{D}(x) M(x) \mathcal{D}(x) M(x) \end{aligned} \quad (C-3)$$

Beside, from eq.(2-25), we have:

$$\begin{aligned} W_2(x,x) &= -\frac{1}{2} \operatorname{Tr} M'(x)^2 \\ &= -\frac{1}{2} \operatorname{Tr} [\mathcal{D}(x), M(x)]^2 \\ &= \operatorname{Tr} \mathcal{D}(x)^2 M(x)^2 - \operatorname{Tr} \mathcal{D}(x) M(x) \mathcal{D}(x) M(x) \end{aligned} \quad (C-4)$$

Therefore

$$W_2(x,x) + W_1(x)^2 = \operatorname{Tr} \mathcal{D}(x)^2 M(x)^2 \quad \text{(C-5)}$$

Moreover we have:

$$\begin{aligned} \operatorname{Tr} \mathcal{D}^2 M^2 &= \operatorname{Tr} \mathcal{D}^2 M \\ &= \operatorname{Tr} \mathcal{D}^2 \vec{\phi} \vec{\psi}^t A \\ &= \vec{\psi}^t A \mathcal{D}^2 \vec{\phi} \\ &= -\vec{\psi}'^t A \vec{\phi}' \\ &= \tilde{\psi}' \phi' - \psi' \tilde{\phi}' \\ &= (c\psi + d\tilde{\psi})(a\phi + b\tilde{\phi}) - (a\psi + b\tilde{\psi})(c\phi + d\tilde{\phi}) \\ &= (bc - ad)\,(\psi\tilde{\phi} - \tilde{\psi}\phi) \\ &= bc - ad \\ &= -\det \mathcal{D} = \frac{1}{2} \operatorname{Tr} \mathcal{D}^2 \end{aligned} \quad \text{(C-6)}$$

and finally:

$$W_2(x,x) + W_1(x)^2 = -\det \mathcal{D}(x) = \frac{1}{2} \operatorname{Tr} \mathcal{D}^2(x) = -P_1(x) \quad \text{(C-7)}$$

□

C.2 Case $n = 1$

We have:

$$W_2(x,x) + W_1(x)^2 = \frac{1}{2}\,\mathrm{Tr}\,\mathcal{D}(x)^2 \tag{C-8}$$

Apply δ_y to both sides, that gives:

$$W_3(x,x,y) + 2W_1(x)\Big(W_2(x,y) + \frac{1}{(x-y)^2}\Big) = -P_2(x;y) + \frac{\partial}{\partial y}\,\frac{W_1(x)+W_1(y)}{x-y} \tag{C-9}$$

which is equivalent to:

$$W_3(x,x,y) + 2W_1(x)W_2(x,y) = -P_2(x;y) - \frac{\partial}{\partial y}\,\frac{W_1(x)-W_1(y)}{x-y} \tag{C-10}$$

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