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TOPICAL REVIEW

Topological recursion in enumerative geometry and random matrices

Bertrand Eynard[1,2] **and Nicolas Orantin**[3]

[1] Institut de Physique Théorique, CEA, IPhT, F-91191 Gif-sur-Yvette, France
[2] CNRS, URA 2306, F-91191 Gif-sur-Yvette, France
[3] Theory Department, CERN, CH-1211 Geneva 23, Switzerland

E-mail: bertrand.eynard@cea.fr and nicolas.orantin@cern.ch

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Abstract

We review the method of symplectic invariants recently introduced to solve matrix models' loop equations in the so-called topological expansion, and further extended beyond the context of matrix models. For any given spectral curve, one defines a sequence of differential forms and a sequence of complex numbers F_g called symplectic invariants. We recall the definition of F_g's and we explain their main properties, in particular symplectic invariance, integrability, modularity, as well as their limits and their deformations. Then, we give several examples of applications, in particular matrix models, enumeration of discrete surfaces (maps), algebraic geometry and topological strings, and non-intersecting Brownian motions.

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Contents

1. Introduction

The theory of random matrices, since its introduction by Wigner [140], has been extremely prolific, thanks to its relation to numerous fields of physics and mathematics and the different approaches used to study it.

One of these different aspects is the possibility of enumerating maps, i.e. surfaces composed of polygons. Following the idea of 't Hooft [133], the partition function of a matrix model can have a $\frac{1}{N^2}$ expansion as the size of the considered matrix N goes to infinity and the terms of this expansion can be interpreted as the generating functions of maps with fixed

genus. Recently, it was understood how to solve, order by order in this so-called topological expansion, the loop equations (i.e. Schwinger–Dyson equations or ward identities) for matrix integrals [58]. This means that one is able to get a closed formula for all the terms in the extension in $\frac{1}{N^2}$ of the partition function and of all the correlation functions of the model.

The solution brought an unexpectedly rich structure [70], which not only solved the 1-matrix model but also solved multi-matrix models as well as their so-called double scaling limits [37, 52, 81] corresponding to the limit of maps composed of infinitely many polygons of infinitesimal size.

In all these cases, it was found that the topological expansion of the matrix integral could be computed by a common recursive formula starting from the data of the equilibrium density of eigenvalues of the random matrix under consideration. This equilibrium density $y(x)$ is called the spectral curve (in most matrix models $y(x)$ is an algebraic function, i.e. it satisfies an algebraic equation $\mathcal{E}(x, y(x)) = 0$).

This led to the definition of an underlying geometric structure which can be defined beyond the context of matrix models and which relies only on the intrinsic algebro-geometric properties of a plane curve $y(x)$ (which we call the spectral curve).

Later, it was understood that this structure also appears in problems of enumerative geometry, not known to be related to matrix models.

In other words, for any regular spectral curve $\mathcal{E} = \{y(x)\}$ (precise definition of spectral curves and examples are given in section 2.1), we can define a sequence of numbers $F_g(\mathcal{E})$, $g = 0, 1, 2, \ldots, \infty$. These numbers $F_g(\mathcal{E})$ (first introduced in [70]) are called the *symplectic invariants* of the spectral curve $\mathcal{E}$. This name comes from the fact that two spectral curves $\mathcal{E}$ and $\tilde{\mathcal{E}}$, which are related to one another by a symplectic transformation (i.e. an analytical map $\mathbb{C}^2 \to \mathbb{C}^2$, $(x, y) \mapsto (\tilde{x}, \tilde{y})$ which sends one spectral curve to the other and which conserves the wedge product $\mathrm{d}x \wedge \mathrm{d}y \to \mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y}$ on the spectral curve; see definition 2.4 for more precise definitions), have the same F_g's: $F_g(\mathcal{E}) = F_g(\tilde{\mathcal{E}})$. F_g is called the symplectic invariant of degree $2 - 2g$, because under a rescaling $y \to \lambda y$, F_g scales as $F_g \to \lambda^{2-2g} F_g$ (except F_1 which is logarithmic in λ); see section 4.1.

Moreover, for a spectral curve $\mathcal{E} = \{y(x)\}$, we define not only its symplectic invariants $F_g(\mathcal{E})$'s, but also a doubly infinite sequence of symmetric meromorphic n-forms $\omega_n^{(g)}(x_1, \ldots, x_n)[\mathcal{E}]$, $n \in \mathbb{N}$, $g \in \mathbb{N}$, such that $F_g = \omega_0^{(g)}$. For $n \geqslant 1$, these n-forms are not symplectically invariant, but they have many nice properties. They allow one to compute the derivatives of F_g's with respect to any parameter on which $\mathcal{E}$ could depend and they encode the deformation structure of F_g's; see section 4 for their properties.

As we shall see in this review, F_g's and $\omega_n^{(g)}$'s are interesting geometric objects, not only for their applications to various problems of enumerative geometry (each problem corresponding to a given spectral curve $\mathcal{E}$), but also on their own. Indeed they have remarkable properties for arbitrary spectral curves, even for spectral curves not known to correspond to any enumerative geometry problem. In particular, they are related to the Kodaira–Spencer field theory [50], to the WDVV special geometry [51, 141], topological strings [28] and Dijkgraaf–Vafa conjecture [49]. They are expected to be partition functions [28] of B model topological string theories on particular Calabi–Yau backgrounds, and through mirror symmetry, F_g's are thus expected to be the generating functions of Gromov–Witten invariants of genus g for some toric geometries [28].

As another example of interesting properties, F_g's have nice modular behavior and, for instance, they provide a solution to holomorphic anomaly equations [20, 73].

They also contain an integrable structure, related to the multi-component Kadamtsev–Petviashvili (KP) hierarchy [1], e.g. they satisfy determinantal formulae and Hirota equations [87].

Another nice property is that they can be computed by a simple diagrammatic method, which makes them really easy to use. For instance, the holomorphic anomaly equations can be proved only by drawing diagrams [2, 73].

Regarding the applications, we will consider the following examples:

(i) Enumeration of discrete surfaces, possibly carrying colors on their faces (Ising model), as well as the asymptotics of large discrete surfaces.
(ii) For the curve $y = \frac{1}{2\pi}\sin(2\pi\sqrt{x})$, F_g's compute the Weyl–Petersson volumes, i.e. enumerate Riemann surfaces with respect to a specific measure on their moduli space.
(iii) We will also consider the Kontsevich spectral curve, related to the Kontsevich integral, for which F_g's are generating functions for intersection numbers of Chern classes of cotangent bundles at marked points and $W_n^{(g)}$'s are generating functions of Mumford κ classes (in some sense by 'forgetting' some marked points).
(iv) For the curve $y = \mathrm{Argcosh}(x)$, and deformations of that curve, F_g's are generating functions for counting partitions with the Plancherel measure, related to the computation of Hurwitz numbers [30].
(v) q-deformed versions of Plancherel measure sums over partitions can also be computed with symplectic invariants of some appropriate spectral curve, which, not so surprisingly, is the (singular locus of the) mirror of a toric Calabi–Yau manifold. This is consistent with the conjecture [28] that F_g's are related to Gromov–Witten invariants. Indeed, Gromov–Witten invariants of toric Calabi–Yau threefolds can be computed, using the topological vertex, as sums over partitions, typically q-deformed Plancherel sums, for the simplest examples of toric Calabi–Yau threefolds.

The review is organized in three parts. Sections 2 and 3 introduce the definitions and the objects used in the general setup of the symplectic invariants respectively. Section 4 summarizes the main properties exhibited by the objects studied in the present review. Sections 5–11 present several applications of these objects to different fields of mathematics and physics.

2. Symplectic invariants of spectral curves

The main characters of the theory reviewed in this article, referred to as symplectic invariants, were originally introduced in [70], as a common framework for the solution of loop equations of several matrix models: 1-matrix, 2-matrix, matrix with external fields, as well as their double scaling limits. The further study of this common solution to the computation of the free energies of different matrix models showed that these objects have many nice properties by themselves, in particular symplectic invariance, and that they appear in other problems of enumerative geometry, not necessarily related to random matrices.

Here we only briefly summarize the construction of [70], without proofs, and we refer the reader to the original article for more details. This section is intended to bring all the tools necessary to the computation of the symplectic invariants associated with a given spectral curve.

2.1. Spectral curves

In this review, we define a spectral curve as follows[4].

[4] This definition is not exactly the one usually encountered in integrable systems [16]; in fact, it turns out that the plane curve we consider here is the 'classical limit' of the full spectral curve. We call it the spectral curve by abuse of language and because it has become customary to do so.

Definition 2.1. *A spectral curve $\mathcal{E} = (\mathcal{L}, x, y)$ is the data of a compact Riemann surface $\mathcal{L}$ and two analytical functions x and y on some open domain of $\mathcal{L}$.*

In some sense, we consider y as a (multivalued) function of x, given in a parametric form $(x(z), y(z))$, where the space of the parameter z is a Riemann surface $\mathcal{L}$.

Definition 2.2. *If $\mathcal{L}$ is a compact Riemann surface of genus $\bar{g}$, and x and y are meromorphic functions on $\mathcal{L}$, we say that the spectral curve is algebraic. If in addition, $\mathcal{L}$ is the Riemann sphere ($\mathcal{L} = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, i.e. of genus $\bar{g} = 0$), we say that the spectral curve is rational.*

Indeed, for an algebraic spectral curve, it is always possible to find a polynomial relationship between x and y:

$$\mathrm{Pol}(x, y) = 0.$$

For a rational spectral curve, the polynomial equation $\mathrm{Pol}(x, y) = 0$ can be parametrized with two rational functions $x(z)$ and $y(z)$ of a complex variable z.

Definition 2.3. *A spectral curve $(\mathcal{L}, x, y)$ is called regular if*

- *the differential form $\mathrm{d}x$ has a finite number of zeros $\mathrm{d}x(a_i) = 0$ and all zeroes of $\mathrm{d}x$ are simple zeros;*
- *the differential $\mathrm{d}y$ does not vanish at the zeros of $\mathrm{d}x$, i.e. $\mathrm{d}y(a_i) \neq 0$.*

This means that near $x(a_i)$, y behaves locally like a square-root $y(z) \sim y(a_i) + C\sqrt{x(z) - x(a_i)}$ or, in other words, the curve $y(x)$ has a vertical tangent at a_i.

From now on, we assume that we consider only regular spectral curves. We shall see below that symplectic invariants F_g's are defined only for regular spectral curves, and they diverge when the spectral curve becomes singular. Examples of singular spectral curves are considered in section 4.8; they play a central role in the double scaling limit in section 8.

Definition 2.4. *We say that two spectral curves $\mathcal{E} = (\mathcal{L}, x, y)$ and $\tilde{\mathcal{E}} = (\tilde{\mathcal{L}}, \tilde{x}, \tilde{y})$ are symplectically equivalent if there exists a conformal mapping $\phi : \mathcal{L} \to \tilde{\mathcal{L}}$ (bijective from the definition domain of x, y to the definition domain of $\tilde{x}$, $\tilde{y}$) and a bijective analytical map $f : \mathbb{C}^2 \to \mathbb{C}^2$, such that*

$$\forall z, \quad (\tilde{x}(\phi(z)), \tilde{y}(\phi(z))) = f(x(z), y(z)),$$

and such that f conserves the exterior product $\mathrm{d}x \wedge \mathrm{d}y$ in $\mathbb{C} \times \mathbb{C}$:

$$\mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y}.$$

The group of symplectomorphisms is generated by (choose $\tilde{\mathcal{L}} = \mathcal{L}$ and $\phi = \mathrm{Id}$):

- $\tilde{x} = x$, $\tilde{y} = y + R(x)$, $R(x) =$ *rational function of x.*
- $\tilde{x} = \frac{ax+b}{cx+d}$, $\tilde{y} = \frac{(cx+d)^2}{ad-bc}y$.
- $\tilde{x} = f(x)$, $\tilde{y} = \frac{1}{f'(x)}y$, *where f is analytical and injective in the image of x.*
- $\tilde{x} = y$, $\tilde{y} = -x$.

All these transformations conserve the symplectic form on $\mathcal{L}$:

$$\mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y} = \mathrm{d}x \wedge \mathrm{d}y,$$

whence the name.

The main property of F_g's we are going to define is that they are symplectic invariants: two curves which are symplectically equivalent have the same F_g's.

2.1.1. Examples of spectral curves. Interesting examples of spectral curves may come from several areas of physics or mathematics, and are related to some problems of enumerative geometry. We will study in detail some examples in sections 5 to 11. Here, in order to illustrate our notion of spectral curve, we give some examples of spectral curves of interest extracted from these applications.

For the readers familiar with matrix models, the spectral curve under consideration here can be thought of as the 'equilibrium density of eigenvalues of the random matrix'. It is not to be confused with the large N density of eigenvalues although, for many simple cases, the two may coincide[5]. In the most simple matrix models, the spectral curve is algebraic. For formal random matrix models, designed as combinatorics generating functions for counting discrete surfaces, the spectral curve is shown to be rational (see section 7).

In the context of string theory, the spectral curve is often given by a transcendental equation of the form $H(\mathrm{e}^x, \mathrm{e}^y) = 0$, where H is a polynomial. It is not an algebraic spectral curve, but is closely related to an algebraic curve. In this case, $\mathrm{d}x$ and $\mathrm{d}y$ are Abelian meromorphic differentials on the compact Riemann surface $\mathcal{L}$ corresponding to H (see section 11).

The origin of all the examples below is described in more detail in sections 5–11.

- The following curve is a rational spectral curve:
$$\mathcal{L} = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}, \qquad x(z) = z^2 - 2, \qquad y(z) = z^3 - 3z.$$
It satisfies the algebraic equation $y^2 - 2 = x^3 - 3x$. It is a hyperelliptical curve of genus $\bar{g} = 0$. This spectral curve is related to the so-called pure gravity Liouville field theory. It is often called the 'pure gravity' spectral curve or also the $(3, 2)$ spectral curve, because pure gravity is the $(3, 2)$ minimal conformal field theory; it has central charge $c = 0$ [46, 48]. See sections 4.8 and 8.
- The curve $y = \sqrt{x}$ is also a rational spectral curve, it satisfies $y^2 = x$ and can be parametrized by
$$\mathcal{L} = \mathbb{P}^1, \qquad x(z) = z^2, \qquad y(z) = z. \tag{2.1}$$
This spectral curve arises in the study of the extreme eigenvalue statistics of a random matrix, i.e. in the study of the Tracy–Widom law and of the Airy kernel [134]. It is often called the 'Airy' spectral curve. It is also called the $(1, 2)$ spectral curve in order to match the classification of minimal conformal field theories [46, 48]. The minimal model $(1, 2)$ has central charge $c = -2$. See section 8.
- The following spectral curve is also a rational spectral curve:
$$\mathcal{L} = \mathbb{P}^1, \qquad x(z) = \gamma\left(z + \frac{1}{z}\right), \qquad y(z) = -\frac{t}{\gamma z} + \frac{t_4 \gamma^3}{z^3},$$
where $\gamma^2 = \frac{1-\sqrt{1-12tt_4}}{6t_4}$. This spectral curve arises in the enumeration of quadrangulated surfaces, i.e. in the formal quartic matrix model. See section 7.4.
- The following spectral curve
$$\mathcal{L} = \mathbb{P}^1, \qquad x(z) = z^2, \qquad y(z) = \frac{1}{2\pi}\sin(2\pi z)$$

[5] The two notions coincide for example for matrix integrals with a polynomial potential. They do not coincide for example when the potential has an explicit dependence on N.

is related to the computation of Weyl–Petersson volumes. Note that it is not algebraic, but it can be parametrized by a complex variable, i.e. by a $\bar{g} = 0$ Riemann surface. See section 10.2. It is not known to be related to any matrix model.

- The following rational spectral curve

$$\mathcal{L} = \mathbb{P}^1, \qquad x(z) = z^2, \qquad y(z) = z^3 - 3tz$$

is singular at $t = 0$. Indeed at $t = 0$, the differential $\mathrm{d}y = 3(z^2 - t)\,\mathrm{d}z$ vanishes at $z = 0$ which is the zero of $\mathrm{d}x$. We will see below that F_g's diverge for singular curves, and thus the function $F_g(t)$ has a singularity at $t = 0$. By considering the change of parameter $z \to \sqrt{t}z$, the spectral curve can be rewritten as $x(z) = tz^2$, $y(z) = t^{3/2}(z^3 - 3z)$, and from the symplectic invariance $F_g(\{x, y\}) = F_g(\{x/t, ty\})$ and from the homogeneity $F_g(\{x, \lambda y\}) = \lambda^{2-2g} F_g(\{x, y\})$, we see that

$$F_g(t) = t^{5(1-g)} F_g(1),$$

which indeed diverges at $t = 0$. See section 4.8.

- The following spectral curve depends on two parameters $p \in \mathbb{Z}$ and $z_0 \in \mathbb{C}^*$:

$$\mathcal{L} = \mathbb{P}^1, \qquad \begin{cases} x(z) = \dfrac{\left(1 - \frac{z}{z_0}\right)\left(1 - \frac{1}{zz_0}\right)}{\left(1 + \frac{1}{z_0}\right)^2} \\ y(z) = \dfrac{1}{x(z)}\left(-\ln z + \dfrac{p}{2}\ln\left(\dfrac{1 - z/z_0}{1 - 1/zz_0}\right)\right). \end{cases}$$

It appears in the enumeration of q-deformed Plancherel sums of partitions, i.e. in the computation of the Gromov–Witten invariants of the Hirzebruch toric Calabi–Yau manifold $X_p = O(-p) \oplus O(p-2) \to \mathbb{P}^1$. See section 9.3.

This spectral curve is symplectically equivalent to (compute $\mathrm{d}x \wedge \mathrm{d}y$ in both cases)

$$\mathcal{L} = \mathbb{P}^1, \qquad \begin{cases} x(z) = \ln\left(\left(1 - \dfrac{z}{z_0}\right)\left(1 - \dfrac{1}{zz_0}\right)\right) \\ y(z) = \ln\left(\dfrac{1}{z}\left(\dfrac{1 - z/z_0}{1 - 1/zz_0}\right)^{\frac{p}{2}}\right). \end{cases}$$

This last spectral curve is such that e^x and e^y are rational functions of z, and thus by eliminating z, there exists a polynomial $H(\mathrm{e}^x, \mathrm{e}^y)$ such that

$$H(\mathrm{e}^x, \mathrm{e}^y) = 0.$$

This equation is precisely the singular locus of the mirror manifold of X_p. The full mirror manifold (not only its singular locus) is the three-dimensional submanifold of $\mathbb{C}^4$ locally given by $\{(x, y, \omega_+, \omega_-) \in \mathbb{C}^4 / H(\mathrm{e}^x, \mathrm{e}^y) = \omega_+\omega_-\}$. See section 11 or [28].

- The following spectral curve is of genus $\bar{g} = 1$; it is algebraic but not rational:

$$\mathcal{L} = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \qquad x(z) = \wp(z, \tau), \qquad y(z) = \wp'(z, \tau),$$

where $\wp$ is the Weierstrass function and $\mathcal{L}$ is the torus of modulus τ. It is algebraic because the Weierstrass function obeys the differential equation:

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3.$$

This spectral curve appears for instance in the Seiberg–Witten theory, related to a solution to the $\mathcal{N} = 2$ supersymmetric Yang–Mills theory [130].

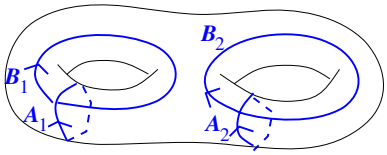
2.2. *Geometry of the spectral curve*

2.2.1. Genus and cycles. The only compact Riemann surface of genus $\bar{g} = 0$ is the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. It is simply connected.

A compact Riemann surface $\mathcal{L}$ of genus $\bar{g} \geqslant 1$ can be equipped with a symplectic basis (not unique) of $2\bar{g}$ non-contractible cycles such that

$$\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}, \qquad \mathcal{A}_i \cap \mathcal{A}_j = 0, \qquad \mathcal{B}_i \cap \mathcal{B}_j = 0.$$

They are such that $\mathcal{L} \setminus (\cup_i \mathcal{A}_i \cup_i \mathcal{B}_i)$ is a simply connected domain of $\mathcal{L}$, which we shall call the fundamental domain.



The choice of cycles and of a fundamental domain is rather arbitrary and is not unique. Many of the quantities we are going to consider depend on this choice.

The quantities which do not depend on this choice are said to be modular invariant.

On a compact Riemann surface of genus $\bar{g} \geqslant 1$, there exist holomorphic differential forms (analytical everywhere on $\mathcal{L}$, in particular with no pole). These holomorphic forms clearly form a vector space (linear combinations are also holomorphic) over $\mathbb{C}$, and this vector space has dimension $\bar{g}$.

When we have a choice of cycles $\mathcal{A}_i, \mathcal{B}_j$, it is possible to choose a basis (which is unique), which we call $\mathrm{d}u_1, \ldots, \mathrm{d}u_{\bar{g}}$, and normalized such that

$$\oint_{\mathcal{A}_i} \mathrm{d}u_j = \delta_{i,j}.$$

Once we have defined these holomorphic forms $\mathrm{d}u_j$'s, we can compute their Riemann matrix of periods:

$$\tau_{i,j} = \oint_{\mathcal{B}_i} \mathrm{d}u_j.$$

This matrix $\tau_{i,j}$ is symmetric and its imaginary part is positive definite:

$$\tau_{i,j} = \tau_{j,i}, \qquad \mathrm{Im}\,\tau > 0.$$

2.2.2. Abel map. Consider an arbitrary origin o in the fundamental domain and fixed throughout this review.

For any point z in the fundamental domain, the vector $(u_1(z), \dots, u_{\bar{g}}(z))$

$$u_i(z) = \int_o^z \mathrm{d}u_i,$$

where the integration path is in the fundamental domain, is called the Abel map of z. It is a vector in $\mathbb{C}^{\bar{g}}$. It depends on the choice of o by an additive constant, and it depends on the choice of the fundamental domain, by a vector in the lattice $\mathbb{Z}^{\bar{g}} + \tau\mathbb{Z}^{\bar{g}}$. The quotient $\mathbb{C}^{\bar{g}}/(\mathbb{Z}^{\bar{g}} + \tau\mathbb{Z}^{\bar{g}})$ is called the Jacobian.

The Abel map sends points of $\mathcal{L}$ to points in the Jacobian.

2.2.3. Bergman kernel. Given a choice of cycles, we define the Bergman kernel $B(z_1, z_2)$ as the unique bilinear differential having one double pole at $z_1 = z_2$ (it is called 'second kind') and no other pole, and such that, in any local parameter z:

$$B(z_1, z_2) \underset{z_1 \to z_2}{\sim} \frac{\mathrm{d}z_1\, \mathrm{d}z_2}{(z_1 - z_2)^2} + \text{reg}, \qquad \forall\, i = 1, \dots, \bar{g}, \quad \oint_{\mathcal{A}_i} B(z_1, z_2) = 0.$$

One should keep in mind that the Bergman kernel depends only on $\mathcal{L}$ and not on the functions x and y.

The Bergman kernel can be seen as the derivative of the Green function, i.e. the solution of the heat kernel equation on $\mathcal{L}$.

The Bergman kernel is clearly unique because the difference of two Bergman kernels would have no pole and vanishing $\mathcal{A}$-cycle integrals; therefore, it would vanish.

It is also interesting to note that $B(z_1, z_2) = B(z_2, z_1)$ is symmetric.

Example.

- If $\mathcal{L} = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} =$ the Riemann sphere, the Bergman kernel is a rational expression:

  $$B(z_1, z_2) = \frac{\mathrm{d}z_1\, \mathrm{d}z_2}{(z_1 - z_2)^2}.$$

  Most of the applications between sections 5 and 11 are on $\mathcal{L} = \mathbb{P}^1$ and use this rational Bergman kernel.
- If $\mathcal{L} = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) =$ torus of modulus τ, the Bergman kernel is

  $$B(z_1, z_2) = \left(\wp(z_1 - z_2, \tau) + \frac{\pi}{\operatorname{Im}\tau}\right) \mathrm{d}z_1\, \mathrm{d}z_2, \tag{2.2}$$

  where $\wp$ is the Weierstrass elliptical function.
- If $\mathcal{L}$ is a compact Riemann surface of genus $\bar{g} \geqslant 1$, of the Riemann matrix of periods $\tau_{i,j}$, the Bergman kernel is

  $$B(z_1, z_2) = d_{z_1} d_{z_2} \ln(\theta(u(z_1) - u(z_2) - c, \tau)),$$

  where $u(z)$ is the Abel map, c is an odd characteristic and θ is the Riemann θ function of genus $\bar{g}$ (cf [76, 77] for θ functions).

2.2.4. Generalized Bergman kernel. Given an arbitrary symmetric matrix κ of size $\bar{g} \times \bar{g}$, we consider a 'deformed' Bergman kernel:

$$B_\kappa(z_1, z_2) = B(z_1, z_2) + 2\mathrm{i}\pi \sum_{i,j=1}^{\bar{g}} \kappa_{i,j}\, \mathrm{d}u_i(z_1)\, \mathrm{d}u_j(z_2).$$

If $\kappa = 0$, we recover the usual Bergman kernel $B_0 = B$.

The reason for introducing this κ is that a change of basis of cycles and fundamental domain can be rewritten as a change of κ.

Indeed, perform a $Sp_{2\bar{g}}(\mathbb{Z})$ change of symplectic basis of cycles ($C, D, \tilde{C}, \tilde{D}$ have coefficients in $\mathbb{Z}$ and $CD^t = DC^t, \tilde{C}\tilde{D}^t = \tilde{D}\tilde{C}^t, C\tilde{D}^t - D\tilde{C}^t = 1$):

$$\mathcal{A}_i = \sum_j C_{i,j}\mathcal{A}'_j + \sum_j D_{i,j}\mathcal{B}'_j, \qquad \mathcal{B}_i = \sum_j \tilde{C}_{i,j}\mathcal{A}'_j + \sum_j \tilde{D}_{i,j}\mathcal{B}'_j.$$

The Riemann matrix of periods τ' in the new basis $\mathcal{A}', \mathcal{B}'$ is related to the old one by the modular transformation:

$$\tau' = (\tilde{D} - \tau D)^{-1}(\tau C - \tilde{C})$$

and the Bergman kernel[6] changes as

$$B_0 \to B'_0 = B_0 + 2\mathrm{i}\pi \sum_{i,j=1}^{\bar{g}} \kappa_{i,j}\, \mathrm{d}u_i(z_1)\, \mathrm{d}u_j(z_2), \qquad \kappa = (\tilde{D}D^{-1} - \tau)^{-1}.$$

In other words, the change of cycles can be reabsorbed into a change of κ.

More generally, the kernel B_κ in a basis $\mathcal{A}, \mathcal{B}$ is equal to $B'_{\kappa'}$ in the basis $\mathcal{A}', \mathcal{B}'$, where

$$\kappa' = (\tilde{D}^t - D^t\tau)\kappa(\tilde{D} - \tau D) - (\tilde{D}^t - D^t\tau)D.$$

From now on, we will always consider B_κ, and we will write B instead of B_κ, i.e. we will omit the κ subscript, unless there is an ambiguity. However, for most of the practical applications, one often chooses $\kappa = 0$.

2.2.5. Schiffer kernel. In particular, if we choose κ to be

$$\kappa = (\overline{\tau} - \tau)^{-1} = \frac{\mathrm{i}}{2}(\mathrm{Im}\,\tau)^{-1},$$

we see that in the new basis $\mathcal{A}', \mathcal{B}'$, the matrix κ becomes

$$\kappa' = \frac{\mathrm{i}}{2}(\mathrm{Im}\,\tau')^{-1};$$

it takes the same form as in the initial basis. The Bergman kernel B_κ with this special value of κ is called the Schiffer kernel [19]; it is modular invariant: it does not depend on a choice of cycles. However, the price to pay to have modular invariance is to have a non-analytical dependence in τ and thus a non-analytical dependence in the spectral curve. This incompatibility between analyticity and modular invariance is the origin of the so-called holomorphic anomaly equation [2, 20]; see section 4.4.2.

Example. If $\mathcal{L} = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) =$ torus of modulus τ, the Schiffer kernel is

$$B(z_1, z_2) = \wp(z_1 - z_2, \tau)\, \mathrm{d}z_1\, \mathrm{d}z_2,$$

where $\wp$ is the Weierstrass elliptical function. Compare with equation (2.2).

2.2.6. Branch points. Branch points are the points with a vertical tangent; they are the *zeros of* $\mathrm{d}x$. Let us write them $a_i, i = 1, \ldots, \#\,\mathrm{bp}$:

$$\forall i, \quad \mathrm{d}x(a_i) = 0.$$

Since we consider a regular spectral curve, all branch points are simple zeroes of $\mathrm{d}x$, the curve $y(x)$ behaves locally like a square root $y(z) \sim y(a_i) + C_i\sqrt{x(z) - x(a_i)}$, near a branch point a_i, and thus, for any z close to a_i, there is exactly one point $\bar{z} \neq z$ in the vicinity of a_i such that

$$x(\bar{z}) = x(z).$$

[6] Remember that, by definition, it is normalized in such a way that it has vanishing $\mathcal{A}$-cycle integrals.

$\bar{z}$ is called the *conjugated point* of z. It is defined locally near each branch point a_i, and it is not necessarily defined globally.

Example.

- *Enumeration of maps ≈1-matrix model in the 1-cut case*.
  In this case we have $\mathcal{L}$ = Riemann sphere and (see section 7.3.2)

  $$x(z) = \alpha + \gamma(z + 1/z), \qquad \mathrm{d}x(z) = x'(z)\,\mathrm{d}z = \gamma(1 - z^{-2})\,\mathrm{d}z.$$

  The zeros of $\mathrm{d}x(z)$ are $z = \pm 1$, and we clearly have $\bar{z} = 1/z$:

  $$a_1 = 1, \qquad a_2 = -1, \qquad \bar{z} = 1/z.$$

  In this case, $\bar{z}$ is defined globally.
- *Pure gravity* (3, 2).
  In this case, we have $\mathcal{L}$ = Riemann sphere and

  $$x(z) = z^2 - 2, \qquad \mathrm{d}x(z) = 2z\,\mathrm{d}z.$$

  The only zero of $\mathrm{d}x(z)$ is $z = 0$, and we have $\bar{z} = -z$:

  $$a = 0, \qquad \bar{z} = -z.$$

  In this case, $\bar{z}$ is defined globally.
- *Ising model* (4, 3).
  In this case, we have $\mathcal{L}$ = Riemann sphere and

  $$x(z) = z^3 - 3z, \qquad \mathrm{d}x(z) = 3(z^2 - 1)\,\mathrm{d}z.$$

  The zeros of $\mathrm{d}x(z)$ are $a_i = \pm 1$, and near $a_i = \pm 1$ we have

  $$a_i = \pm 1, \qquad \bar{z} = -\tfrac{1}{2}(z - a_i\sqrt{12 - 3z^2}).$$

  In this case $\bar{z}$ is not defined globally, and it depends on a_i.

2.2.7. Recursion kernel. For any $z_0 \in \mathcal{L}$, and any z close to a branch point, we define the recursion kernel:

$$K(z_0, z) = \frac{-1}{2}\frac{\int_{z'=\bar{z}}^{z} B(z_0, z')}{(y(z) - y(\bar{z}))\,\mathrm{d}x(z)} \tag{2.3}$$

where the integral is taken in a small domain in the vicinity of the concerned branch point.

$K(z_0, z)$ is a meromorphic 1-form in variable z_0, is defined globally for all $z_0 \in \mathcal{L}$ and has simple poles at $z_0 = z$ and $z_0 = \bar{z}$.

In contrast, in variable z, the kernel $K(z_0, z)$ is defined only locally near branch points $z \sim a_i$, and it is the inverse of a differential. As we shall see below, $K(z_0, z)$ will always be used only in the vicinity of branch points, and it will always be multiplied by a quadratic differential in z, so that the product will be of a differential form.

Let us note that $K(z_0, z) = K(z_0, \bar{z})$ and that $K(z_0, z)$ has a simple pole when z approaches the branch point. Using De L'Hopital's rule, the leading behavior near the branch point is

$$K(z_0, z) \underset{z \to a_i}{\sim} -\frac{B(z, z_0)}{2\mathrm{d}y(z)\,\mathrm{d}x(z)} + \text{regular}.$$

2.3. Correlation functions

We start by defining a sequence of meromorphic n-forms $\omega_n^{(g)}$ with $n = 1, 2, \ldots$ and $g = 0, 1, 2, \ldots$, called correlators or correlation functions, by the following recursion.

Definition 2.5. *Given a spectral curve $\mathcal{E} = (\mathcal{L}, x, y)$, and a matrix κ (see section 2.2.4), we define recursively the following meromorphic forms:*

$$\omega_1^{(0)}(z) = -y(z)\,\mathrm{d}x(z) \qquad \omega_2^{(0)}(z_1, z_2) = B(z_1, z_2)$$

and if $2g - 2 + n \geqslant 0$ and J is a collective notation for n variables $J = \{z_1, \ldots, z_n\}$:

$$\omega_{n+1}^{(g)}(z_0, J) = \sum_i \operatorname*{Res}_{z \to a_i} K(z_0, z) \left[\omega_{n+2}^{(g-1)}(z, \bar{z}, J) + \sum_{h=0}^{g} {\sum_{I \subset J}}' \omega_{1+|I|}^{(h)}(z, I)\omega_{1+n-|I|}^{(g-h)}(\bar{z}, J \backslash I) \right] \tag{2.4}$$

where $\sum'$ on the right-hand side means that we exclude the terms with $(h, I) = (0, \emptyset)$, and (g, J).

This definition is indeed a recursive one, because all the terms on the right-hand side have a strictly smaller $2g - 2 + n$ than the left-hand side.

The functions $\omega_n^{(g)}$ with $2 - 2g - n < 0$ are called stable; the others are unstable (the only unstable ones are thus $\omega_1^{(0)}$ and $\omega_2^{(0)}$).

$\omega_n^{(g)}(z_1, \ldots, z_n)$ is a meromorphic 1-form on $\mathcal{L}$ in each variable z_i. It can be proved by recursion that it is in fact a symmetric form. Moreover, if $2 - 2g - n < 0$, its only poles are at branch points $z_i \to a_j$ and have no residues:

$$\operatorname*{Res}_{z_1 \to a_i} \omega_n^{(g)}(z_1, z_2, \ldots, z_n) = 0. \tag{2.5}$$

These properties can be proved by recursion, and we refer the reader to [70].

2.4. Free energies

The previous definition defines $\omega_n^{(g)}$ only for $n \geqslant 1$. Now, we define $F_g = \omega_0^{(g)}$, called *Free energy* or *symplectic invariant of degree* $2 - 2g$, by the following:

Definition 2.6. *Symplectic invariants.*

We define for $g \geqslant 2$

$$F_g(\mathcal{E}, \kappa) = \omega_0^{(g)} = \frac{1}{2 - 2g} \sum_i \operatorname*{Res}_{z \to a_i} \Phi(z)\omega_1^{(g)}(z) \tag{2.6}$$

where Φ is any function defined locally near branch points, such that $\mathrm{d}\Phi = y\,\mathrm{d}x$ (Φ is defined up to an additive constant, but thanks to equation (2.5), F_g does not depend on the choice of this constant).

The unstable cases $g = 0$ and $g = 1$ are special and one needs other definitions adapted from the matrix model case [23, 61].

Definition 2.7. *For $g = 1$, we define*

$$F_1 = \frac{1}{24} \ln \left(\tau_{\mathrm{B}}(\{x(a_i)\}) \prod_i y'(a_i) \right),$$

where we define

$$y'(a_i) = \lim_{z \to a_i} \frac{y(z) - y(a_i)}{\sqrt{x(z) - x(a_i)}}$$

and τ_{B} *is the Bergman* τ *function of Kokotov–Korotkin [105]. If* $x(z)$ *is a meromorphic function on* $\mathcal{L}$, τ_{B} *depends only on the values of* x *at its branch points, i.e.* $X_i = x(a_i)$. *It is defined by*

$$\frac{\partial \ln \tau_{\mathrm{B}}(\{X_i\})}{\partial X_i} = \operatorname*{Res}_{z \to a_i} \frac{B(z, \bar{z})}{\mathrm{d}x(z)}.$$

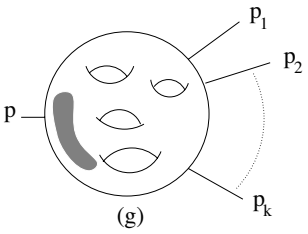
Note that B here stands for B_κ with arbitrary κ.

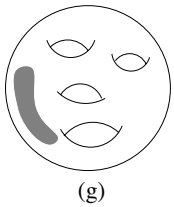
The definition of F_0 is more involved, and we refer the reader to [70]. A convenient way to define F_0 is through its third derivatives, using theorem 4.3. In fact, all F_g's with $g \geqslant 1$ are obtained in terms of local behavior of x and y around branch points, but F_0 depends on the behavior of x and y in the whole spectral curve, and not only in the vicinity of branch points. In the context of topological strings, F_0 is called the prepotential.

3. Diagrammatic representation

The recursive definitions of $\omega_k^{(g)}$ and $F^{(g)}$ can be represented graphically.

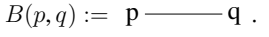
We represent the k-form $\omega_k^{(g)}(p_1, \ldots, p_k)$ as a 'blob-like surface' with g holes and k legs (or punctures) labeled with variables $p_1, \ldots, p_k$, and $F^{(g)} = \omega_0^{(g)}$ with 0 legs and g holes.

$\omega_{k+1}^{(g)}(p, p_1, \ldots, p_k) :=$ 

, $F^{(g)} :=$ 

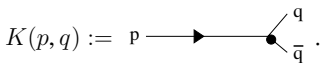
.

We represent the Bergman kernel $B(p, q)$ (which is also $\omega_2^{(0)}$, i.e. a blob with two legs and no hole) as a straight non-oriented line between p and q:

$B(p, q) :=$ 

.

We represent $K(p, q)$ as a straight arrowed line with the arrow from p toward q and with a trivalent vertex whose left leg is q and right leg is $\overline{q}$:

$K(p, q) :=$ 

.

Graphs.

Definition 3.1. *For any* $k \geqslant 0$ *and* $g \geqslant 0$ *such that* $k + 2g \geqslant 3$, *we define the following.*

Let $\mathcal{G}_{k+1}^{(g)}(p, p_1, \ldots, p_k)$ *be the set of connected trivalent graphs defined as follows.*

(1) There are $2g + k - 1$ *trivalent vertices called vertices.*
(2) There is one 1-valent vertex labeled by p, called the root.
(3) There are k 1-valent vertices labeled with $p_1, \ldots, p_k$ *called the leaves.*

(4) There are $3g + 2k - 1$ *edges.*

(5) Edges can be arrowed or non-arrowed. There are $k + g$ *non-arrowed edges and* $2g + k - 1$ *arrowed edges.*

(6) The edge starting at p has an arrow leaving from root p.

(7) The k edges ending at the leaves $p_1, \ldots, p_k$ *are non-arrowed.*

(8) The arrowed edges form a 'spanning[7] *planar*[8] *binary skeleton*[9] *tree' with root p. The arrows are oriented from root toward leaves. In particular, this induces a partial ordering of all vertices.*

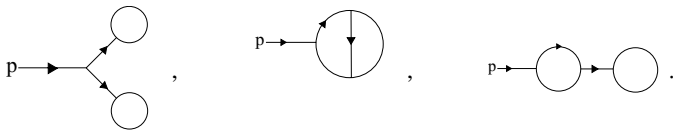
(9) There are k non-arrowed edges going from a vertex to a leaf and g non-arrowed edges joining two inner vertices. Two inner vertices can be connected by a non-arrowed edge only if one is the parent of the other following the arrows along the tree.

(10) If an arrowed edge and a non-arrowed inner edge come out of a vertex, then the arrowed edge is the left child. This rule only applies when the non-arrowed edge links this vertex to one of its descendants (not one of its parents).

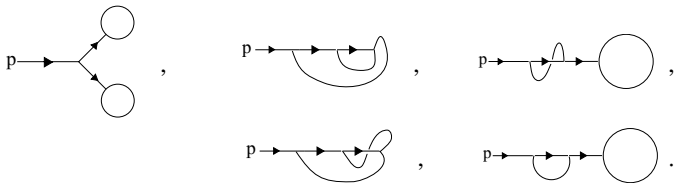
Example of $\mathcal{G}_1^{(2)}(p)$. As an example, let us build step by step all the graphs of $\mathcal{G}_1^{(2)}(p)$, i.e. $g = 2$ and $k = 0$.

We first draw all planar binary skeleton trees with one root p and $2g + k - 1 = 3$ arrowed edges:

Then, we draw $g + k = 2$ non-arrowed edges in all possible ways such that every vertex is trivalent, also satisfying rule (9) of definition 3.1. There is only one possibility for the first tree and two for the second one:



It just remains to specify the left and right children for each vertex. The only possibilities in accordance with rule (10) of definition 3.1 are[10]



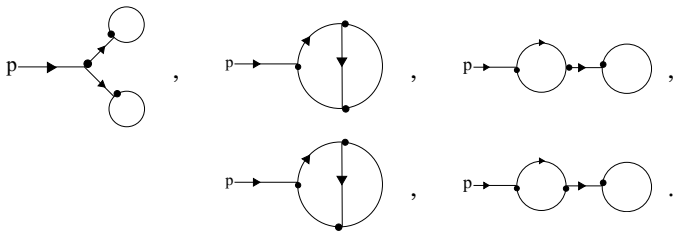
[7] It goes through all vertices.

[8] A planar tree means that the left child and right child are not equivalent. The right child is marked by a black disk on the outgoing edge.

[9] A binary skeleton tree is a binary tree from which we have removed the leaves, i.e. a tree with vertices of valence 1, 2 or 3.

[10] Note that the graphs are not necessarily planar.

In order to simplify the drawing, we can draw a black dot to specify the right child. This way, one gets only planar graphs:



Note that without prescriptions (9) and (10), one would get 13 different graphs whereas we only have 5.

Weight of a graph. Consider a graph $G \in \mathcal{G}_{k+1}^{(g)}(p, p_1, \ldots, p_k)$. Then, with each vertex $i = 1, \ldots, 2g + k - 1$ of G, we associate a label $q_i \in \mathcal{L}$ and associate q_i with the beginning of the left child edge and $\overline{q}_i$ with the right child edge. Thus, each edge (arrowed or not) links two labels which are points on the spectral curve $\mathcal{L}$.

- With an arrowed edge going from q' toward q, we associate a factor $K(q', q)$.
- With a non-arrowed edge going between q' and q, we associate a factor $B(q', q)$.
- Following the arrows backward (i.e. from leaves to root), for each vertex q, we take the sum over all branch points a_i of residues at $q \to a_i$.

After taking all the residues, we get the weight of the graph

$$w(G),$$

which is a multilinear form in $p, p_1, \ldots, p_k$.

Similarly, we define weights of linear combinations of graphs by

$$w(\alpha G_1 + \beta G_2) = \alpha w(G_1) + \beta w(G_2)$$

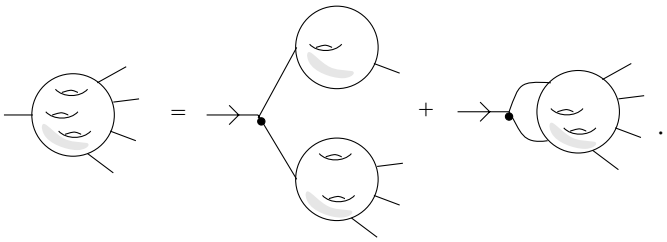
and for a disconnected graph, i.e. a product of two graphs,

$$w(G_1 G_2) = w(G_1) w(G_2).$$

Theorem 3.1. *We have*

$$\omega_{k+1}^{(g)}(p, p_1, \ldots, p_k) = \sum_{G \in \mathcal{G}_{k+1}^{(g)}(p, p_1, \ldots, p_k)} w(G) = w\left(\sum_{G \in \mathcal{G}_{k+1}^{(g)}(p, p_1, \ldots, p_k)} G \right).$$

Proof. This is precisely what the recursion equations (2.4) of definition 2.5 do. Indeed, one can represent them diagrammatically by



□

Such graphical notations are very convenient, and are a good support for intuition and even help in proving some relationships. It was immediately noted after [58] that these diagrams look very much like Feynman graphs, and there was a hope that they could be the Feynman's graphs for the Kodaira–Spencer quantum field theory. Some attempts were made to relate them to a quantum field theory, in particular by Flume–Grossehelweg–Klitz [75]. But they are not Feynman graphs, because Feynman graphs cannot have non-local restrictions like the fact that non-oriented lines can join only a vertex and one of its descendents.

These graphs are merely a notation for the recursive definition (2.4).

Lemma 3.1. *Symmetry factor.*

The weight of two graphs differing by the exchange of the right and left children of a vertex is the same. Indeed, the distinction between the right and left children is just a way of encoding symmetry factors.

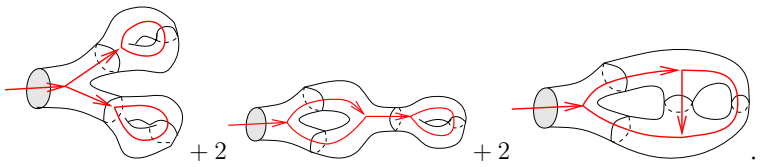
Proof. This property follows directly from the fact that $K(z_0, z) = K(z_0, \bar{z})$. □

3.1. Teichmuller pant gluings

Every Riemann surface of genus g with k boundaries can be decomposed into $2g - 2 + k$ pants whose boundaries are $3g - 3 + k$ closed geodesics (in the Poincaré metric with constant negative curvature) [88]. The number of ways (in the combinatorial sense) of gluing $2g-2+k$ pants by their boundaries is clearly the same as the number of diagrams of $\mathcal{G}_k^{(g)}$, and each diagram corresponds to one pants decomposition.

Indeed, consider the root boundary labeled by p and attach a pair of pants to this boundary. Draw an arrowed propagator from the boundary to the first pants. Then, choose one of the other boundaries of the pair of pants (there are thus two choices, left or right); it must be glued to another pair of pants (possibly not distinct from the first one). If this pair of pants was never visited, draw an arrowed propagator, and if it was already visited, draw a non-arrowed propagator. In the end, you get a diagram of $\mathcal{G}_k^{(g)}$. This procedure is bijective (up to symmetry factors), and with a diagram of $\mathcal{G}_k^{(g)}$, one may associate a gluing of pants.

Example with $k = 1$ and $g = 2$:

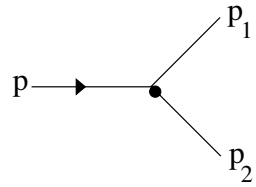
$\omega_1^{(2)} =$  $+ 2$ $+ 2$.

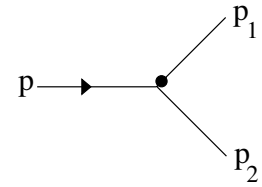
3.2. Example

Let us present some examples of correlation functions and free energy for low orders.

3.2.1. Three-point function.

$$\omega_3^{(0)}(p,p_1,p_2) = \quad + \quad$$



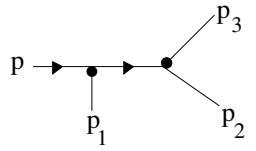


$$\begin{aligned}
&= \operatorname*{Res}_{q\to\mathbf{a}} K(p,q)\ [B(q,p_1)B(\overline{q},p_2)+B(\overline{q},p_1)B(q,p_2)]\\
&= -2\operatorname*{Res}_{q\to\mathbf{a}} K(p,q)\ [B(q,p_1)B(q,p_2)]\\
&= \operatorname*{Res}_{q\to\mathbf{a}} \frac{B(q,p)\,B(q,p_1)\,B(q,p_2)}{dx(q)\,dy(q)}\\
&= \sum_i \frac{B(a_i,p)\,B(a_i,p_1)\,B(a_i,p_2)}{2\,dy(a_i)\,dz_i(a_i)^2}
\end{aligned}$$

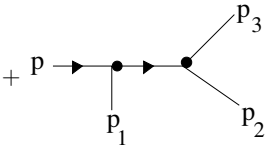
where $z_i(z)=\sqrt{x(z)-z(a_i)}$ is a local coordinate near a_i.

3.2.2. Four-point function.

$$\omega_4^{(0)}(p,p_1,p_2,p_3) = \quad + \quad 5 \text{ permutations of (1,2,3)}$$



$$+$$

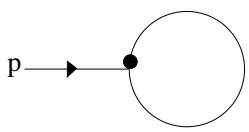


$$+ \quad 5 \text{ permutations of (1,2,3)}$$

$$\begin{aligned}
= \operatorname*{Res}_{q\to\mathbf{a}}\operatorname*{Res}_{r\to\mathbf{a}} & K(p,q)\,K(q,r)\ [B(\overline{q},p_1)B(r,p_2)B(\overline{r},p_3)\\
&+B(\overline{q},p_1)B(\overline{r},p_2)B(r,p_3)+B(\overline{q},p_2)B(r,p_1)B(\overline{r},p_3)\\
&+B(\overline{q},p_2)B(\overline{r},p_1)B(r,p_3)+B(\overline{q},p_3)B(r,p_2)B(\overline{r},p_1)\\
&+B(\overline{q},p_3)B(\overline{r},p_2)B(r,p_1)]\\
+\operatorname*{Res}_{q\to\mathbf{a}}\operatorname*{Res}_{r\to\mathbf{a}} & K(p,q)\,K(\overline{q},r)\ [B(q,p_1)B(r,p_2)B(\overline{r},p_3)\\
&+B(q,p_1)B(\overline{r},p_2)B(r,p_3)+B(q,p_2)B(r,p_1)B(\overline{r},p_3)\\
&+B(q,p_2)B(\overline{r},p_1)B(r,p_3)+B(q,p_3)B(r,p_2)B(\overline{r},p_1)\\
&+B(q,p_3)B(\overline{r},p_2)B(r,p_1)]\,.
\end{aligned}$$

3.2.3. One-point function to order 1.

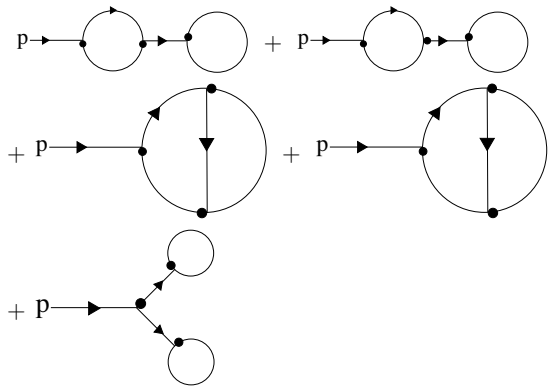
$$\omega_1^{(1)}(p) =$$



$$= \operatorname*{Res}_{q\to\mathbf{a}} K(p,q)\,B(q,\overline{q}).$$

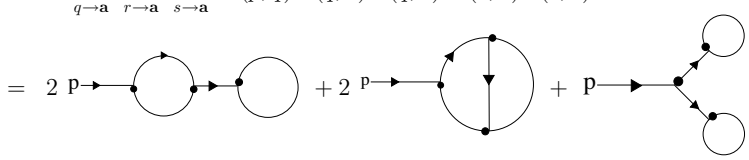
3.2.4. One-point function to order 2.

$$\omega_1^{(2)}(p) =$$



$$\begin{aligned}
&= \operatorname*{Res}_{q\to\mathbf{a}}\operatorname*{Res}_{r\to\mathbf{a}}\operatorname*{Res}_{s\to\mathbf{a}} K(p,q)K(q,r)K(r,s)\ B(\overline{q},\overline{r})B(s,\overline{s})\\
&\quad + \operatorname*{Res}_{q\to\mathbf{a}}\operatorname*{Res}_{r\to\mathbf{a}}\operatorname*{Res}_{s\to\mathbf{a}} K(p,q)K(q,r)K(\overline{r},s)\ B(\overline{q},r)B(s,\overline{s})\\
&\quad + \operatorname*{Res}_{q\to\mathbf{a}}\operatorname*{Res}_{r\to\mathbf{a}}\operatorname*{Res}_{s\to\mathbf{a}} K(p,q)K(q,r)K(r,s)\ B(\overline{q},\overline{s})B(s,\overline{r})\\
&\quad + \operatorname*{Res}_{q\to\mathbf{a}}\operatorname*{Res}_{r\to\mathbf{a}}\operatorname*{Res}_{s\to\mathbf{a}} K(p,q)K(q,r)K(r,s)\ B(\overline{q},s)B(\overline{s},\overline{r})\\
&\quad + \operatorname*{Res}_{q\to\mathbf{a}}\operatorname*{Res}_{r\to\mathbf{a}}\operatorname*{Res}_{s\to\mathbf{a}} K(p,q)K(\overline{q},r)K(q,s)\ B(\overline{r},r)B(\overline{s},s)
\end{aligned}$$

$$=$$



where the last expression is obtained using lemma 3.1.

3.2.5. Free energy F_2. The second free energy reads as

$$\begin{aligned}
-2F_2 = {} & 2\operatorname*{Res}_{p\to\mathbf{a}}\operatorname*{Res}_{q\to\mathbf{a}}\operatorname*{Res}_{r\to\mathbf{a}}\operatorname*{Res}_{s\to\mathbf{a}} \Phi(p)K(p,q)K(q,r)K(r,s)B(\overline{q},\overline{r})B(s,\overline{s})\\
& + 2\operatorname*{Res}_{p\to\mathbf{a}}\operatorname*{Res}_{q\to\mathbf{a}}\operatorname*{Res}_{r\to\mathbf{a}}\operatorname*{Res}_{s\to\mathbf{a}} \Phi(p)K(p,q)K(q,r)K(r,s)B(\overline{q},\overline{s})B(s,\overline{r})\\
& + \operatorname*{Res}_{p\to\mathbf{a}}\operatorname*{Res}_{q\to\mathbf{a}}\operatorname*{Res}_{r\to\mathbf{a}}\operatorname*{Res}_{s\to\mathbf{a}} \Phi(p)K(p,q)K(\overline{q},r)K(q,s)B(\overline{r},r)B(\overline{s},s).
\end{aligned}$$

4. Main properties

So, for every regular spectral curve $\mathcal{E} = (\mathcal{L}, x, y)$ (and matrix κ if $\mathcal{L}$ has genus $\bar{g} > 0$), we have defined some meromorphic n-forms $\omega_n^{(g)}$ and some complex numbers $F_g = \omega_0^{(g)}$. They have some remarkable properties as follows (see [70]).

- $\omega_n^{(g)}$ is symmetric in its n variables (this is proved by recursion).
- If $2g - 2 + n > 0$, then $\omega_n^{(g)}(z_1, \ldots, z_n)$ is a meromorphic form (in z_1 for instance) with poles only at the branch points, of degree at most $6g - 6 + 2n + 2$, and with vanishing residue.
- If two spectral curves $\mathcal{E} = (\mathcal{L}, x, y)$ and $\tilde{\mathcal{E}} = (\widetilde{\mathcal{L}}, \tilde{x}, \tilde{y})$ are symplectically equivalent (see definition 2.4), they have the same F_g's for $g > 1$ (although they do not have the same $\omega_n^{(g)}$'s in general)

$$\mathrm{d}x \wedge \mathrm{d}y = \pm \mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y} \quad \to \quad F_g(\mathcal{E}, \kappa) = F_g(\tilde{\mathcal{E}}, \kappa).$$

- If $\mathcal{L}$ is of genus $\bar{g} = 0$, then $\tau = \exp\left(\sum_{g=0}^{\infty} N^{2-2g} F_g\right)$ is a formal τ function; it obeys Hirota's equation. This theorem can be extended to $\bar{g} > 0$, with additional θ functions; see section 4.6.
- Dilaton equation, for $2g - 2 + n > 0$:

  $$\sum_i \operatorname*{Res}_{z_{n+1} \to a_i} \Phi(z_{n+1}) \omega_{n+1}^{(g)}(z_1, \ldots, z_n, z_{n+1}) = (2 - 2g - n)\omega_n^{(g)}(z_1, \ldots, z_n).$$

  This equation just reflects the homogeneity property, i.e. under a rescaling $y \to \lambda y$, we have $\omega_n^{(g)} \to \lambda^{2-2g-n}\omega_n^{(g)}$.
- The derivatives of $\omega_n^{(g)}$ with respect to many parameters on which the spectral curve may depend are computed in section 4.3.
- $\omega_n^{(g)}$'s have many other properties; for instance, their modular behavior satisfies the holomorphic anomaly equations.

Let us study these properties in deeper details.

4.1. Homogeneity

If one changes the function $y(z) \to \lambda y(z)$, i.e. just a rescaling of the curve, then it is clear from equation (2.3) that the kernel K is changed to K/λ and nothing else is changed. Thus, $\omega_n^{(g)}$ changes as

$$\omega_n^{(g)} \to \lambda^{2-2g-n}\omega_n^{(g)},$$

and in particular

$$F_g(\mathcal{L}, x, \lambda y) = \lambda^{2-2g} F_g(\mathcal{L}, x, y). \tag{4.1}$$

This implies that F_g is a homogeneous function of the spectral curve, of degree $2 - 2g$.

In particular, if one chooses $\lambda = -1$, one gets (for $g \geqslant 2$)

$$F_g(\mathcal{L}, x, -y, \kappa) = F_g(\mathcal{L}, x, y, \kappa). \tag{4.2}$$

4.2. Symplectic invariance

It is clear from the definitions that F_g and $\omega_n^{(g)}$ depend on the spectral curve only through the kernels B and $K(z_0, z) = -\frac{1}{2}\frac{\int_{z'=\bar{z}}^{z} B(z_0, z')}{(y(z) - y(\bar{z}))\,\mathrm{d}x(z)}$ and the number and position of branch points respectively.

The Bergman kernel B depends only on the underlying complex structure of the Riemann surface $\mathcal{L}$; thus it remains unchanged if we change the functions x and y as long as we do not change $\mathcal{L}$.

The kernel K depends on the functions x and y only through the combination $(y(z) - y(\bar{z}))\,\mathrm{d}x(z)$. Therefore, $K(z_0, z)$ remains unchanged if we do not change this combination.

In particular, the kernel K and therefore F_g and $\omega_n^{(g)}$ remain unchanged, if we change

- $y \to y + R(x)$, where $R(x)$ is some rational function of x;
- $y \to \lambda y$, $x \to x/\lambda$, where $\lambda \in \mathbb{C}^*$;
- $x \to \frac{ax+b}{cx+d}$, $y \to \frac{(cx+d)^2}{ad-bc} y$.

These transformations form a subgroup of the symplectomorphisms. Indeed, in all these cases, the symplectic form $\mathrm{d}x \wedge \mathrm{d}y$ is unchanged.

In order to have invariance under the full group of symplectomorphisms, we need to prove the invariance under the $\frac{\pi}{2}$ rotation in the x, y plane, i.e. $x \to y$, $y \to -x$, which

also conserves $\mathrm{d}x \wedge \mathrm{d}y$. Using homogeneity equation (4.2), we see that this is equivalent to considering the invariance under $x \to y$, $y \to x$.

This transformation does not conserve K, does not conserve the number of branch points and does not conserve $\omega_n^{(g)}$'s with $n \geqslant 1$. However, it was proved in [71] that it does conserve F_g's. The proof of [71] is very technical. It is inspired from the loop equations for the 2-matrix model. It amounts to defining some mixed $n+m$-forms $\omega_{n,m}^{(g)}$, where x and y play similar roles, and for which $\omega_{n,0}^{(g)}$ coincides with $\omega_n^{(g)}$ for the spectral curve $(\mathcal{L}, x, y)$ and $\omega_{0,m}^{(g)}$ coincides with $\omega_m^{(g)}$ for the spectral curve $(\mathcal{L}, y, x)$. In particular, $F_g = \omega_{0,0}^{(g)}$ is both $F_g = \omega_0^{(g)}$ for the spectral curve $(\mathcal{L}, x, y)$ and $F_g = \omega_0^{(g)}$ for the spectral curve $(\mathcal{L}, y, x)$. The proof of [71] relies on the fact that $\omega_{n+1,m}^{(g)} + \omega_{n,m+1}^{(g)}$ is an exact form.

That leads to

Theorem 4.1. *Symplectic invariance*

For $g > 1$, F_g's are invariant under the group of symplectomorphisms generated by

- $y \to y + R(x)$, *where $R(x)$ is some rational function of x;*
- $y \to \lambda y$, $x \to x/\lambda$, *where $\lambda \in \mathbb{C}^*$;*
- $x \to \frac{ax+b}{cx+d}$, $y \to \frac{(cx+d)^2}{ad-bc} y$;
- $x \to y$, $y \to -x$;

In addition, F_g's are also invariant under

- $x \to x$, $y \to -y$.

This theorem is a powerful tool which allows one to compare the F_g's of models which look *a priori* very different. We will see examples of applications in section 10.1.

$\omega_n^{(g)}$'s with $n \geqslant 1$ are not conserved under symplectic transformation; instead, they get shifted by exact forms.

4.3. Derivatives

In this section, we study how F_g's and $\omega_n^{(g)}$'s change under a change of spectral curve and, in particular, under infinitesimal holomorphic changes.

Consider an infinitesimal change $y \to y + \epsilon\delta y$ at fixed x, or in fact it is more appropriate to consider the variation of the differential form $y\,\mathrm{d}x$:

$$y\,\mathrm{d}x \to y\,\mathrm{d}x + \epsilon\delta(y\,\mathrm{d}x) + O(\epsilon^2) = y\,\mathrm{d}x + \epsilon\,\mathrm{d}\Omega + O(\epsilon^2),$$

where $\mathrm{d}\Omega$ is a meromorphic differential form[11] on an open subset of $\mathcal{L}$. If instead of working at fixed x, we prefer to work with some local parameter z, we write

$$\delta y(z)\,\mathrm{d}x(z) - \delta x(z)\,\mathrm{d}y(z) = \mathrm{d}\Omega(z).$$

This shows that the set of holomorphic deformations of the spectral curve is equipped with a Poisson structure, but we shall not study it in detail in this review; see [53, 54].

Classification of possible 1-forms $\mathrm{d}\Omega$. The deformation $\mathrm{d}\Omega$ is a 1-form. Here we shall consider only meromorphic deformations, and meromorphic 1-forms are classified as first kind (no pole), second kind (multiple poles, without residues) and third kind (only simple poles).

[11] Note that $y\,\mathrm{d}x$ does not need to be a meromorphic form itself to be able to consider such deformations.

- First kind deformations are holomorphic forms on $\mathcal{L}$, i.e. they are linear combinations of $\mathrm{d}u_i$'s (see section 2.2.1):
$$\mathrm{d}u_i(z) = \frac{1}{2\mathrm{i}\pi}\oint_{\overline{\mathcal{B}}_i} B(z,z'),$$
where $\overline{\mathcal{B}}_i = \mathcal{B}_i - \sum_j \tau_{i,j}\mathcal{A}_j$.
- Second kind deformations have double or multiple poles. They can be taken as linear combinations of Bergman kernels or of their derivatives. Choose a point $p \in \mathcal{L}$. If x is regular at p, choose the local parameter $\xi(z) = x(z) - x(p)$, and if x has a pole of degree d at p, choose $\xi(z) = x(z)^{-1/d}$, and define
$$B_k(z;\, p) = \operatorname*{Res}_{z'\to p} B(z,z')\xi(z')^{-k}. \tag{4.3}$$
All second kind differentials are linear combinations of such $B_k(z;\, p)$.
- Third kind differentials have only simple poles, and since the sum of residues must vanish, they must have at least two simple poles. Choose two points p_1 and p_2 in the fundamental domain, and define
$$\mathrm{d}S_{p_1,p_2}(z) = \int_{p_2}^{p_1} B(z,z'). \tag{4.4}$$
All third kind differentials are linear combinations of such $\mathrm{d}S$.

Theorem 4.2. *A general meromorphic differential form* $\mathrm{d}\Omega$ *with poles* p_k*'s can be written as*
$$\mathrm{d}\Omega(z) = 2\mathrm{i}\pi\sum_{i=1}^{\bar{g}}\delta\epsilon_i\,\mathrm{d}u_i(z) + \sum_k \delta t_k\,\mathrm{d}S_{p_k,o}(z) + \sum_k\sum_j \delta t_{k,j}B_j(z;\, p_k). \tag{4.5}$$

Proof. Indeed, let p_k be the poles of $\mathrm{d}\Omega$ and write the negative part of the Laurent series of $\mathrm{d}\Omega$ near its poles as
$$\mathrm{d}\Omega(z) \underset{z\to p_k}{\sim} \delta t_k\frac{\mathrm{d}\xi(z)}{\xi(z)} - \sum_{j\geqslant 1} j\delta t_{k,j}\frac{\mathrm{d}\xi(z)}{\xi(z)^{j+1}}.$$
We see that
$$\mathrm{d}\Omega(z) - \sum_k \delta t_k\,\mathrm{d}S_{p_k,o}(z) - \sum_k\sum_j \delta t_{k,j}B_j(z;\, p_k)$$
is a 1-form which has no poles; thus it is a holomorphic form, and therefore it must be a linear combination of $\mathrm{d}u_i$'s. □

If $\kappa = 0$, i.e. if B is the Bergman kernel, it is normalized on the $\mathcal{A}$-cycles, and we have $\oint_{\mathcal{A}} \mathrm{d}S = 0$, $\oint_{\mathcal{A}} B_\kappa = 0$. Thus, $\delta\epsilon_i$ are easily computed as $\delta\epsilon_i = \frac{1}{2\mathrm{i}\pi}\oint_{\mathcal{A}_i}\mathrm{d}\Omega$. However, if $\kappa \neq 0$, this is no longer true; we have $\delta\epsilon = \frac{1+\kappa\tau}{2\mathrm{i}\pi}\oint_{\mathcal{A}}\mathrm{d}\Omega - \frac{\kappa}{2\mathrm{i}\pi}\oint_{\mathcal{B}}\mathrm{d}\Omega$, and the variations $\delta t_{k,j}$ or δt_k get mixed with $\delta\epsilon_i$ through the variations of τ. The good way to undo this mixing is by defining a covariant variation.

Definition 4.1. *Covariant variation:*
$$D_{\mathrm{d}\Omega} \overset{\mathrm{def}}{=} \delta_{\mathrm{d}\Omega} + \mathrm{tr}\left(\kappa\delta_{\mathrm{d}\Omega}\tau\kappa\frac{\partial}{\partial\kappa}\right),$$
where $\delta_{\mathrm{d}\Omega}\tau$ *is the variation of the Riemann matrix of periods* τ *under the deformation* $y\,\mathrm{d}x \to y\,\mathrm{d}x + \epsilon\,\mathrm{d}\Omega$.

Derivatives with respect to κ are studied in detail in section 4.4.1.

The important point is that $\mathrm{d}\Omega$ can always be written as

$$\mathrm{d}\Omega(z) = \int_{\partial\Omega} B(z, z')\Lambda(z'),$$

where $\partial\Omega$ is some continuous path (a chain or a cycle, which is related to the Poincaré dual of $\mathrm{d}\Omega$) on $\mathcal{L}$ and $\Lambda(z')$ is an analytical function defined locally in a vicinity of $\partial\Omega$.

The theorem is then

Theorem 4.3. *Variation of the spectral curve.*

Under an infinitesimal deformation $\delta y\,\mathrm{d}x - \delta x\,\mathrm{d}y = \mathrm{d}\Omega(z) = \int_{\partial\Omega} B(z, z')\Lambda(z')$*, and at fixed* $x(z_i)$*'s,* $\omega_n^{(g)}$*'s change by*

$$D_{\mathrm{d}\Omega}\omega_n^{(g)}(z_1, \ldots, z_n) = \int_{\partial\Omega} \omega_{n+1}^{(g)}(z_1, \ldots, z_n, z')\Lambda(z').$$

For example, in particular with $n = 0$, we have

$$D_{\mathrm{d}\Omega}F_g = \int_{\partial\Omega} \omega_1^{(g)}(z')\Lambda(z').$$

4.3.1. The loop operator. This theorem can also be restated in terms of the 'loop operator', which corresponds to the second kind variation $\mathrm{d}\Omega(z) = B(z, z')$. We define

Definition 4.2. *The loop operator is*

$$D_{z'} \stackrel{\mathrm{def}}{=} D_{B(z,z')}.$$

It satisfies

$$D_{z'}\omega_n^{(g)}(z_1, \ldots, z_n) = \omega_{n+1}^{(g)}(z_1, \ldots, z_n, z').$$

The loop operator is a derivation, i.e. $D_{z'}(uv) = uD_{z'}v + vD_{z'}u$*, and it is such that* $D_{z_1}D_{z_2} = D_{z_2}D_{z_1}$ *and* $D_{z_1}\frac{\partial}{\partial z_2} = \frac{\partial}{\partial z_2}D_{z_1}$.

In random matrix theory, the loop operator [6] is most often written as a functional derivative with respect to the potential $V(x)$ (see section 5.2.6):

$$\frac{\partial}{\partial V(x(z'))} \stackrel{\mathrm{def}}{=} D_{z'}.$$

4.3.2. Inverse of the loop operator. The loop operator allows one to find $\omega_{n+1}^{(g)}$ in terms of $\omega_n^{(g)}$, i.e. increases n by 1. The inverse operator, which decreases n by 1, can also be written explicitly.

Theorem 4.4. *Let* Φ *be a primitive of* $y\,\mathrm{d}x$*, i.e. a function defined on the fundamental domain such that* $\mathrm{d}\Phi = y\,\mathrm{d}x$*, then we have, if* $2 - 2g - n < 0$*, the dilaton equation (referring to topological gravity's terminology)*

$$(2 - 2g - n)\omega_n^{(g)}(z_1, \ldots, z_n) = \sum_i \operatorname*{Res}_{z \to a_i} \omega_{n+1}^{(g)}(z_1, \ldots, z_n, z)\Phi(z).$$

(Φ is defined only up to an additive constant, but because of equation (2.5), the constant does not contribute to the residue.)

This theorem is easily proved by recursion on $2g + n - 2$. It is at the origin of definition 2.6, for $n = 0$.

4.4. Modular properties

In section 2.2.3, we have introduced a deformation of the Bergman kernel with a symmetric matrix κ of size $\bar{g} \times \bar{g}$. The reason to introduce this deformation was that it encodes the modular dependence of the Bergman kernel, i.e. how the Bergman kernel changes under a change of choice of cycles $\mathcal{A}, \mathcal{B}$. Thus, studying the modular dependence of F_g's and $\omega_n^{(g)}$'s amounts to studying their dependence on κ.

Also, in section 4.3, we have seen that the covariant derivative involves the computation of derivatives with respect to κ.

4.4.1. Dependence on κ. Since the kernels B and K depend linearly on κ, all the stable $\omega_n^{(g)}$'s and F_g's are polynomials in κ, of degree $3g-3+n$.

Note that $\partial B(z_1, z_2)/\partial\kappa_{i,j} = 2i\pi\, \mathrm{d}u_i(z_1)\, \mathrm{d}u_j(z_2)$ is factorized, i.e. a function of z_1 times a function of z_2. This simple observation, together with

$$\mathrm{d}u_i(z) = \frac{1}{2i\pi} \oint_{\overline{\mathcal{B}}_i} B(z, z'), \qquad \overline{\mathcal{B}}_i = \mathcal{B}_i - \sum_j \tau_{i,j} \mathcal{A}_j,$$

leads, by an easy recursion, to the following theorem.

Theorem 4.5. *For* $2-2g-n \leqslant 0$,

$$2i\pi\, \partial\omega_n^{(g)}(z_1, \ldots, z_n)/\partial\kappa_{i,j} = \frac{1}{2} \oint_{\overline{\mathcal{B}}_i} \mathrm{d}z' \oint_{\overline{\mathcal{B}}_j} \mathrm{d}z \left[\omega_{n+2}^{(g-1)}(z_1, \ldots, z_n, z, z') + \sum_{h=0}^{g} {\sum_{I \subset J}}' \omega_{1+|I|}^{(h)}(z, I)\omega_{1+n-|I|}^{(g-h)}(z', J/I) \right], \tag{4.6}$$

where $J = \{z_1, \ldots, z_n\}$ *and* $\sum_h {\sum_I}'$ *means as usual that we exclude* $(h, I) = (0, \emptyset), (g, J)$.

This theorem can be applied recursively to compute higher derivatives and eventually recover a polynomial of κ by its Taylor expansion at $\kappa = 0$, of the form

$$F_g(\kappa) = \sum_{k=0}^{3g-3} \frac{1}{k!} (\kappa)^k \left. \frac{\partial^k F_g}{\partial \kappa^k} \right|_{\kappa=0}. \tag{4.7}$$

According to theorem 4.3 of section 4.3, the $\overline{\mathcal{B}}$-cycle integrals, computed at $\kappa = 0$, are the derivatives with respect to coordinates ϵ_i of equation (4.5):

$$\kappa = 0 \quad \leftrightarrow \quad \frac{\partial\omega_n^{(g)}(z_1, \ldots, z_n)}{\partial\epsilon_i} = \oint_{\overline{\mathcal{B}}_i} \omega_{n+1}^{(g)}(z_1, \ldots, z_n, z),$$

and therefore we have

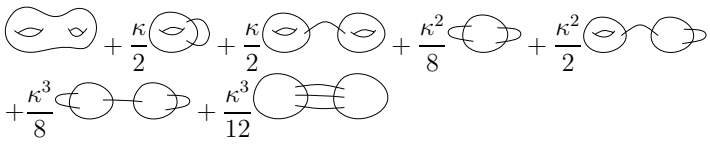
$$\partial\omega_n^{(g)}(z_1, \ldots, z_n)/\partial\kappa_{i,j}|_{\kappa=0} = \frac{1}{2} \frac{\partial}{\partial\epsilon_i} \frac{\partial}{\partial\epsilon_j} \omega_n^{(g-1)}(z_1, \ldots, z_n) + \frac{1}{2} \sum_{h=0}^{g} {\sum_{I \subset J}}' \frac{\partial}{\partial\epsilon_i} \omega_{1+|I|}^{(h)}(I) \frac{\partial}{\partial\epsilon_j} \omega_{n-|I|}^{(g-h)}(J/I).$$

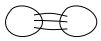
We can thus trade the κ dependence into derivatives with respect to the coordinates ϵ_i. For instance, we have at $g = 2$ and with $\bar{g} = 1$,

$$F_2(\kappa) = \overline{F_2} + \frac{\kappa}{2}(\partial^2\overline{F}_1 + \partial\overline{F}_1\partial\overline{F}_1) + \frac{\kappa^2}{8}(\partial^4\overline{F}_0 + 4\partial^3\overline{F}_0\partial\overline{F}_1) + \frac{\kappa^3}{48}(10\partial^3\overline{F}_0\partial^3\overline{F}_0),$$

where $\overline{F}_g = F_g(\kappa = 0)$ and $\partial = \partial/\partial\epsilon$.

This result is best interpreted graphically. For example, with $g = 2$, we have

$F_2(\kappa) \;=\;$  $+\frac{\kappa^3}{8}$ $+\frac{\kappa^3}{12}$

where each line with endpoints (i, j) is a factor $\kappa_{i,j}$ and each connected piece of Riemann surface of genus h, with k punctures $i_1, \ldots, i_k$, is $\frac{\partial^k \overline{F}_h}{\partial\epsilon_{i_1}\ldots\partial\epsilon_{i_k}}$. Each graph is a possible 'stable' degeneracy of a genus g Riemann surface (imagine each link contracted to a point); stability means that each connected component of genus h with k marked points must have $2 - 2h - k < 0$. The prefactor is $1/\#\mathrm{Aut}$, i.e. the inverse of the number of automorphisms of the graph; for instance, in the last graph  we have a $\mathbb{Z}_2$ symmetry by exchanging the two spheres and a σ_3 symmetry from permuting the three endpoints of the edges, i.e. $12 = \#(\mathbb{Z}_2 \times \sigma_3)$ automorphisms.

More generally, by a careful analysis of the combinatorics of ∂_{ϵ_i}'s, one can see (this was done in [73] and coincides with the diagrammatics of [2]) that the Taylor expansion equation (4.7) reconstructs the expansion of a formal Gaussian integral (i.e. order by order in powers of N):

$$\mathrm{e}^{\sum_g N^{2-2g} F_g(\epsilon,\kappa)} = \int \mathrm{d}\eta_1 \ldots \int \partial\eta_{\bar{g}}\, \mathrm{e}^{\sum_g N^{2-2g}\overline{F}_g(\eta)}\, \mathrm{e}^{-N^2\sum_i(\eta_i-\epsilon_i)\partial_{\epsilon_i}\overline{F}_g}$$
$$\times\, \mathrm{e}^{-\frac{N^2}{2}\sum_{i,j}(\eta_i-\epsilon_i)(\eta_j-\epsilon_j)\partial_{\epsilon_i}\partial_{\epsilon_j}\overline{F}_g}\, \mathrm{e}^{-\frac{N^2}{4i\pi}\sum_{i,j}(\eta_i-\epsilon_i)(\eta_j-\epsilon_j)(\kappa^{-1})_{i,j}}$$

and the graphical representation above is just Wick's expansion of the Gaussian integral.

This diagrammatic expansion of modular transformations was first derived in [2] in the context of topological strings.

4.4.2. Holomorphic anomaly. In particular, if we choose κ to be $\kappa = (\overline{\tau} - \tau)^{-1}$ we have seen in section 2.2.5 that the Bergman kernel becomes the Schiffer kernel and is modular invariant, which means that it is independent of the choice of cycles $\mathcal{A}, \mathcal{B}$. Since the only modular dependence of F_g's and $\omega_n^{(g)}$'s is in the Bergman kernel, we have

Theorem 4.6. *If κ is $\kappa = (\overline{\tau} - \tau)^{-1}$, then F_g and $\omega_n^{(g)}$'s are modular invariant.*

The price to pay to have modular invariant F_g's is that they are no longer analytical functions of τ, i.e. analytical functions of the spectral curve, and in particular they are no longer analytical functions of ϵ_i's.[12] However, since the only non-analytical dependence is a polynomial in κ, since κ^{-1} is linear in $\overline{\tau}$ which is the only non-analytical term and since we have relationships between derivatives with respect to κ and derivatives with respect to ϵ, by a simple chain rule, we obtain the following theorem [73].

Theorem 4.7. *If $\kappa = (\overline{\tau}-\tau)^{-1}$, F_g's are modular invariant and $\omega_n^{(g)}$'s satisfy the 'holomorphic anomaly equation'*

$$\frac{\partial\omega_n^{(g)}(J)}{\partial\overline{\epsilon}_i} = \frac{-1}{(2i\pi)^3}\kappa\frac{\partial^3\overline{F}_0}{\partial\overline{\epsilon}^3}\kappa\frac{1}{2}\left[\frac{\partial^2\omega_n^{(g-1)}(J)}{\partial\epsilon^2} + \frac{\partial\tau}{\partial\epsilon}\kappa\frac{\partial\omega_n^{(g-1)}(J)}{\partial\epsilon}\right.$$
$$\left. + \sum_{h=0}^{g}\sum_{I\subset J}\frac{\partial\omega_{|I|}^{(h)}(I)}{\partial\epsilon}\frac{\partial\omega_{n-|I|}^{(g-h)}(J\backslash I)}{\partial\epsilon}\right].$$

[12] This issue was already studied in the context of topological string theories in [2].

In particular, for $n = 0$,

$$\frac{\partial F_g}{\partial \overline{\epsilon}_i} = \frac{-1}{(2i\pi)^3}\kappa \frac{\partial^3 \overline{F}_0}{\partial \overline{\epsilon}^3}\kappa \frac{1}{2}\left[\frac{\partial^2 F_{g-1}}{\partial \epsilon^2} + \frac{\partial \tau}{\partial \epsilon}\kappa \frac{\partial F_{g-1}}{\partial \epsilon} + \sum_{h=1}^{g-1} \frac{\partial F_h}{\partial \epsilon}\frac{\partial F_{g-h}}{\partial \epsilon}\right].$$

This equation was first found by Bershadsky, Cecotti, Ooguri and Vafa (which we refer to as BCOV [20]) in the context of topological string theory. Here we see that the symplectic invariants F_g always satisfy this equation, and it is a strong hint that the symplectic invariants F_g should coincide with the string theory amplitudes, i.e. the Gromov–Witten invariants. This was formulated as a conjecture by Bouchard–Klemm–Mariño–Pasquetti in [28]; see section 11. Unfortunately, the holomorphic anomaly equations do not have a unique solution, and although this conjecture is almost surely correct, no general proof exists at the present time, apart from a very limited number of cases[13].

Let us briefly sketch the idea of BCOV. String theory partition functions represent 'path integrals' over the set of all Riemann surfaces with a conformal invariant weight. In other words, they are integrals over moduli spaces of Riemann surfaces of given topology, and topological strings are integrals with a topological weight. They compute intersection numbers of bundles over moduli spaces (see [111, 138] for an introduction to topological strings).

Moduli spaces can be compactified by adding their 'boundaries', which correspond to degenerate Riemann surfaces (for instance, when a non-contractible cycle gets pinched or when marked points come together). The integrals have thus boundary terms, which can be represented by δ functions, and δ functions are not holomorphic. In other words, string theory partition functions contain non-holomorphic terms which count degenerate Riemann surfaces.

On the other hand, if one decides to integrate only on non-degenerate surfaces, one gets holomorphic partition functions, but not modular invariant, because the boundaries of the moduli spaces are associated with a choice of pinched cycles. Modular invariant means independent of a choice of cycles.

To summarize, the holomorphic partition function is obtained after a choice of boundaries, i.e. a choice of a symplectic basis of non-contractible cycles $\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}$, and cannot be modular invariant. The modular invariance is restored by adding the boundaries, but this breaks holomorphicity.

There is thus a relationship between holomorphicity and modular invariance.

4.5. Background independence and nonperturbative modular invariance

We have seen in the previous section that F_g's are not modular invariant unless we choose $\kappa = (\overline{\tau} - \tau)^{-1}$, i.e. modular invariance can be restored by breaking holomorphicity.

In fact, there is another way of restoring modular invariance, without breaking holomorphicity. It exploits the fact that the modular transformations of F_g's, i.e. equation (4.6), are very similar to the modular transformation of θ functions. It was shown in [74] that certain combinations of θ functions and F_g's are modular and reconstruct a nonperturbative, modular partition function, which is also a τ function (see section 4.6) and has a background independence property.

Consider a characteristic (μ, ν) and a spectral curve $\mathcal{E} = (\mathcal{L}, x, y)$; choose $\kappa = 0$. Following [65, 74], we introduce a *nonperturbative partition function*, as a formal function of

[13] In [28], this conjecture is proposed as a new definition of the type IIB topological string theory, called 'remodeling of the B model'. The interested reader may find all the details of this conjecture as well as numerous checks in [28].

a *formal parameter N*, defined by

$$
\begin{aligned}
Z_{\mathcal{E}}(\mu,\nu;\epsilon) &= \mathrm{e}^{\sum_{g\geqslant 0} N^{2-2g} F_g(\epsilon)} \sum_{k} \sum_{l_i>0} \sum_{h_i>1-\frac{l_i}{2}} \frac{N^{\sum_i (2-2h_i-l_i)}}{k! l_1! \dots l_k!} F_{h_1}^{(l_1)} \dots F_{h_k}^{(l_k)} \Theta_{\mu,\nu}^{(\sum_i l_i)}(N F_0', \tau) \\
&= \mathrm{e}^{N^2 F_0 + F_1} \Theta_{\mu,\nu} \left\{ 1 + \sum_{j=1}^{\infty} N^{-j} Z_j(\mu,\nu;\epsilon) \right\} \\
&= \mathrm{e}^{N^2 F_0}\, \mathrm{e}^{F_1}\, \mathrm{e}^{(N^{-2}F_2 + N^{-4}F_3 + \cdots)} \left\{ \Theta_{\mu,\nu} + \frac{1}{N}\left(\Theta'_{\mu,\nu} F_1' + \frac{1}{6}\Theta'''_{\mu,\nu} F_0''' \right) + \frac{1}{N^2}\left(\frac{1}{2}\Theta''_{\mu,\nu} F_1'' \right.\right. \\
&\quad \left.\left. + \frac{1}{2}\Theta''_{\mu,\nu} F_1'^2 + \frac{1}{24}\Theta^{(4)}_{\mu,\nu} F_0'''' + \frac{1}{6}\Theta^{(4)}_{\mu,\nu} F_0''' F_1' + \frac{1}{72}\Theta^{(6)}_{\nu,\mu} F_0'''^2 \right) + \cdots \right\},
\end{aligned}
\tag{4.8}
$$

where Z_j is the sum of all terms contributing to order N^{-j}. In this partition function, F_g's are the symplectic invariants of the spectral curve $\mathcal{E}$; their derivatives are with respect to the background filling fraction ϵ and computed through theorem 4.3 at

$$
\epsilon_i = \frac{1}{2\mathrm{i}\pi} \oint_{\mathcal{A}_i} y\, \mathrm{d}x. \tag{4.9}
$$

Note that $F_g(\mathcal{E})$'s and their derivatives depend on the choice of a symplectic basis of $2\bar{g}$ one-cycles $\mathcal{A}_i, \mathcal{B}_j$ on $\mathcal{L}$. Finally, the θ function $\Theta_{\mu,\nu}$ of characteristics (μ,ν) is defined by

$$
\Theta_{\mu,\nu}(u,\tau) = \sum_{n\in\mathbb{Z}^{\bar{g}}} \mathrm{e}^{(n+\mu-N\epsilon)u}\, \mathrm{e}^{\mathrm{i}\pi(n+\mu-N\epsilon)\tau(n+\mu-N\epsilon)}\, \mathrm{e}^{2\mathrm{i}\pi n\nu} \tag{4.10}
$$

and is evaluated at

$$
u = N F_0', \qquad F_0' = \oint_{\mathcal{B}} y(x)\, \mathrm{d}x, \qquad \tau = \frac{1}{2\mathrm{i}\pi} F_0''. \tag{4.11}
$$

In (4.8), the derivatives of the Θ function (4.10) are with respect to u. The derivatives of Θ and the derivatives of F_g are written with tensorial notations. For instance, $\frac{1}{6}\Theta^{(4)}_{\mu,\nu} F_0''' F_1'$ actually means

$$
\frac{1}{6}\Theta^{(4)}_{\mu,\nu} F_0''' F_1' \equiv \frac{1}{2!3!1!} \sum_{i_1,i_2,i_3,i_4} \frac{\partial^4 \Theta_{\mu,\nu}}{\partial u_{i_1}\partial u_{i_2}\partial u_{i_3}\partial u_{i_4}} \frac{\partial^3 F_0}{\partial \epsilon_{i_1}\partial \epsilon_{i_2}\partial \epsilon_{i_3}} \frac{\partial F_1}{\partial \epsilon_{i_4}}
$$

and the symmetry factor (here, $\frac{1}{6} = \frac{2}{2!3!1!}$) is the number of relabelings of the indices, giving the same pairings and divided by the order of the group of relabelings, i.e. $k! l_1! \dots l_k!$, as usual in Feynmann graphs.

The Θ function above is closely related to the standard θ function, which is defined by

$$
\vartheta\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right](\xi|\tau) = \sum_{\mathbf{n}\in\mathbb{Z}^{\bar{g}}} \exp[\mathrm{i}\pi(n+\mu)\tau(n+\mu) + 2\mathrm{i}\pi(n+\mu)(\xi+\nu)].
$$

It it easy to see that these two functions are related as follows:

$$
\Theta_{\mu,\nu}(u,\tau) = \exp\left[-N^2\left(\epsilon F_0' + \tfrac{1}{2}\epsilon^2 F_0''\right)\right] \vartheta\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right](\xi|\tau), \tag{4.12}
$$

where

$$
\xi = \frac{N}{2\mathrm{i}\pi} \oint_{\mathcal{B}-\tau\mathcal{A}} y(x)\, \mathrm{d}x = N\left(\frac{F_0'}{2\mathrm{i}\pi} - \tau\epsilon \right). \tag{4.13}
$$

4.5.1. Modularity.

Theorem 4.8. *All the terms Z_j in equation (4.8) are modular, i.e. they transform as the characteristics (μ, ν). More precisely, if we make a modular change of cycles $\mathcal{A}, \mathcal{B} \to \tilde{\mathcal{A}}, \tilde{\mathcal{B}}$, we have*

$$e^{N^2\tilde{F}_0+\tilde{F}_1}\widetilde{\Theta}_{\tilde{\mu},\tilde{\nu}}(\tilde{u},\tilde{\tau}) = \zeta\begin{bmatrix}\mu\\ \nu\end{bmatrix} e^{N^2F_0+F_1}\Theta_{\mu,\nu}(u,\tau),$$

and for all $j \geqslant 1$

$$\tilde{Z}_j(\tilde{\mu},\tilde{\nu}) = Z_j(\mu,\nu).$$

This theorem was proved in [74], mostly using the diagrammatic representation of section 3 and the diagrammatic representation of [2].

For example, the following quantities are modular:

$$Z_1 = \frac{\Theta'_{\mu,\nu}}{\Theta_{\mu,\nu}}F'_1 + \frac{1}{6}\frac{\Theta'''_{\mu,\nu}}{\Theta_{\mu,\nu}}F'''_0,$$

$$Z_2 = F_2 + \frac{1}{2}\frac{\Theta''_{\mu,\nu}}{\Theta_{\mu,\nu}}F''_1 + \frac{1}{2}\frac{\Theta''_{\mu,\nu}}{\Theta_{\mu,\nu}}F'^2_1 + \frac{1}{24}\frac{\Theta^{(4)}_{\mu,\nu}}{\Theta_{\mu,\nu}}F''''_0 + \frac{1}{6}\frac{\Theta^{(4)}_{\mu,\nu}}{\Theta_{\mu,\nu}}F'''_0F'_1 + \frac{1}{72}\frac{\Theta^{(6)}_{\mu,\nu}}{\Theta_{\mu,\nu}}(F'''_0)^2.$$

4.5.2. Background independence.

Theorem 4.9. *The partition function (4.8) is independent of the background filling fraction ϵ, i.e. for any two filling fractions ϵ_1 and ϵ_2,*

$$Z_{\mathcal{E}}(\mu,\nu,\epsilon_1) = Z_{\mathcal{E}}(\mu,\nu,\epsilon_2).$$

This theorem is derived in [65, 74] directly from definition (4.8). It has important consequences which we shall not study here [74].

4.6. Integrability

Out of F_g's, one can define a 'formal τ function'. In this section, let us assume that $\mathcal{L} = \mathbb{P}^1$, i.e. it has genus $\bar{g} = 0$. The higher genus case is discussed in section 4.6.6.

Definition 4.3. *The formal τ function is defined as a formal series in a variable N:*

$$\ln \tau_N = \sum_{g=0}^{\infty} N^{2-2g}F_g.$$

Now, we shall explain why it makes sense to call it a τ function. τ functions are usually defined in the context of integrable systems, and they have several more or less equivalent definitions; see [16].

One possible definition of τ functions relies on Hirota equations [16, 87] and another one relies on free fermion representations, i.e. determinantal formulae [16, 84, 85, 90, 104].

4.6.1. Determinantal formulae. In the following of this section, most of the functions have an obvious formal N dependence. For the sake of brevity, we do not write it explicitly as long as it is not needed.

Out of $\omega_n^{(g)}$'s, it is convenient to define the formal series:

$$\omega_n(z_1,\dots,z_n) = -\frac{\delta_{n,2}\,\mathrm{d}x(z_1)\,\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2} + \sum_g N^{2-2g-n}\omega_n^{(g)}(z_1,\dots,z_n)$$

and also the 'non-connected' correlators:

$$\overline{\omega}_n(J) = \sum_{k=1}^{n}\sum_{I_1\cup\cdots\cup I_k=J}\prod_{i=1}^{k}\omega_{|I_i|}(I_i).$$

For example,

$$\begin{aligned}
\overline{\omega}_2(z_1,z_2) &= \omega_2(z_1,z_2)+\omega_1(z_1)\omega_1(z_2),\\
\overline{\omega}_3(z_1,z_2,z_3) &= \omega_3(z_1,z_2,z_3)+\omega_1(z_1)\omega_2(z_2,z_3)+\omega_1(z_2)\omega_2(z_1,z_3)\\
&\quad+\omega_1(z_3)\omega_2(z_1,z_2)+\omega_1(z_1)\omega_1(z_2)\omega_1(z_3).
\end{aligned}$$

In other words, ω_n are the cumulants of $\overline{\omega}_n$'s.

The following proposition is proved in some cases (hyperelliptical spectral curves [18]) and in all matrix models; however, it is *expected* to hold for any spectral curve.

Proposition 4.1. *There exists a (formal) kernel $H(z_1,z_2)$, such that*

$$\omega_1(z) = \lim_{z'\to z}\left(H(z,z') - \frac{\sqrt{\mathrm{d}z\,\mathrm{d}z'}}{z-z'}\right),$$

$$\omega_2(z_1,z_2) = -H(z_1,z_2)H(z_2,z_1) - \frac{\mathrm{d}x(z_1)\,\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}$$

and if $n\geqslant 3$,

$$\overline{\omega}_n(z_1,\dots,z_n) = \text{'det'}_{i,j=1,\dots,n}(H(z_i,z_j)),$$

where the quotation mark 'det' *means the following: write the determinant as a sum over permutations of products of H's:* $\det(H(z_i,z_j)) = \sum_\sigma(-1)^\sigma\prod_i H(z_i,z_{\sigma(i)})$. *Then, every time a permutation has a fixed point $\sigma(i)=i$ we must replace $H(z_i,z_i)$ by $\omega_1(z_i)$, and every time a permutation has a length 2 cycle $\sigma(i)=j,\sigma(j)=i$ we must replace the factor $H(z_i,z_j)H(z_j,z_i)$ by $-\omega_2(z_i,z_j)$.*

This is equivalent to saying that for $n\geqslant 3$, the cumulants are given by

$$\omega_n(z_1,\dots,z_n) = \sum_{\text{cyclic }\sigma}(-1)^\sigma\prod_{i=1}^{n}H(z_i,z_{\sigma(i)}).$$

Example.

$$\omega_3(z_1,z_2,z_3) = H(z_1,z_2)H(z_2,z_3)H(z_3,z_1)+H(z_1,z_3)H(z_3,z_2)H(z_2,z_1).$$

The determinantal formulae for correlation functions were first found by Dyson [56] and Mehta [117] in the context of random matrix theory, and have led to a huge number of applications.

Moreover, the kernel H can be written rather explicitly. In all matrix cases, the kernel H for the above determinantal formulae coincides with the kernel $\hat{H}$ which we define below.

Definition 4.4. *We define the formal kernel $\hat{H}$ as a formal spinor in z_1 and z_2, given by an 'exponential formula'*

$$\hat{H}(z_1, z_2) = \frac{\sqrt{\mathrm{d}x(z_1)\,\mathrm{d}x(z_2)}}{x(z_1) - x(z_2)}\, \mathrm{e}^{\sum_{n=1}^{\infty} \frac{1}{n!} \int_{z_2}^{z_1} \cdots \int_{z_2}^{z_1} \omega_n}. \tag{4.14}$$

This exponential formula for the kernel is to be understood order by order in powers of N, namely

$$\hat{H}(z_1, z_2) = \frac{\mathrm{e}^{-N \int_{z_2}^{z_1} y\,\mathrm{d}x}}{E(z_1, z_2)} \left[1 + N^{-1} \int_{z_2}^{z_1} \omega_1^{(1)} + \frac{N^{-1}}{6} \int_{z_2}^{z_1} \int_{z_2}^{z_1} \int_{z_2}^{z_1} \omega_3^{(0)} + O(N^{-2}) \right],$$

where $E(z_1, z_2)$ is the prime form:

$$E(z_1, z_2) = \frac{z_1 - z_2}{\sqrt{\mathrm{d}z_1\,\mathrm{d}z_2}}.$$

For example, one of the terms contributing to $\hat{H}$ to order N^{-1} is

$$\int_{z_2}^{z_1} \int_{z_2}^{z_1} \int_{z_2}^{z_1} \omega_3^{(0)} = \sum_i \frac{(z_1 - z_2)^3}{y'(a_i) x''(a_i) (a_i - z_1)^3 (a_i - z_2)^3}.$$

In all matrix model cases, the kernel H can be written as a Sato formula and coincides with $\hat{H}$, but this is not proved in general.

4.6.2. Baker–Akhiezer functions. Let $\alpha_1, \ldots, \alpha_m$ be the poles of the function $x(z)$, of respective degrees $d_1, \ldots, d_m$. Since x is a meromorphic form of degree $d = \sum_i d_i$, there are d sheets, i.e. d points on $\mathcal{L}$, $z^1(x), \ldots, z^d(x)$, such that $x(z^k) = x$. The following matrix,

$$\mathcal{H}(x_1, x_2) = (\hat{H}(z^j(x_1), z^i(x_2)))_{i,j=1,\ldots,d}$$

is a square matrix of size $d \times d$.

The Ψ function of the Lax system [16] is obtained by choosing $x_2 = \infty$, i.e. $z^i(x_2) = \alpha_i$. Since some poles α_i are multiple poles, in order to get an invertible matrix, we take linear combinations of rows, and we define

$$i = 1, \ldots, m, \quad j = 1, \ldots, d_i \qquad \psi_{i,j}(z) = \lim_{z_2 \to \alpha_i} \left[\left(\frac{\mathrm{d}}{\mathrm{d}\xi_i(z_2)} \right)^{j-1} \frac{\mathrm{e}^{-N \int_o^{z_2} y\,\mathrm{d}x}\, \hat{H}(z, z_2)}{\sqrt{\mathrm{d}\xi_i(z_2)}} \right], \tag{4.15}$$

where o is an arbitrary base point and $\xi_i(z_2) = x(z_2)^{-1/d_i}$ is the local parameter in the vicinity of α_i.

These functions are the Baker–Akhiezer (BA) functions.

We also have d couples $I = (i, j)$ with $i = 1, \ldots, m$, $j = 1, \ldots, d_i$, and thus the following matrix is a square matrix:

$$\Psi(x) = (\psi_I(z^k(x)))_{I=1,\ldots,d,k=1,\ldots,d}.$$

It is the Ψ function of the corresponding Lax system [16].

4.6.3. Examples. For example, if we consider the Airy curve $y = \sqrt{x}$, i.e. $\mathcal{E} = (\mathbb{P}^1, z^2, z)$, we find that $\hat{H}$ is the Airy kernel:

$$\hat{H}(z_1, z_2) = \frac{Ai(z_1^2) Bi'(z_2^2) - Ai'(z_1^2) Bi(z_2^2)}{z_1^2 - z_2^2} \sqrt{z_1 \mathrm{d}z_1 z_2\,\mathrm{d}z_2}.$$

The corresponding Baker–Akhiezer function is the Airy function and the correlators ω_n are the correlators given by the determinantal Airy process.

4.6.4. Sato formula. Theorem 4.3 implies that under an infinitesimal change of the spectral curve of the third kind (4.4): $\delta y\,\mathrm{d}x = t\,\mathrm{d}S_{z_1,z_2}$, we have

$$\frac{\partial \omega_n^{(g)}(z'_1,\dots,z'_n)}{\partial t} = \int_{z_2}^{z_1} \omega_{n+1}^{(g)}(z'_1,\dots,z'_n,z),$$

and thus,

$$\frac{\partial^n F_g}{\partial t^n} = \int_{z_2}^{z_1}\dots\int_{z_2}^{z_1} \omega_n^{(g)}(z'_1,\dots,z'_n).$$

The exponential formula of proposition 4.4 is nothing but the Taylor expansion of $F_g(t)$ computed at $t = N^{-1}$ in terms of derivatives taken at $t = 0$, i.e. $F_g(N^{-1}) = \sum_k \frac{N^{-k}}{k!}\partial_t^k F_g(0)$. In other words,

Theorem 4.10.

$$\hat{H}(z_1,z_2) = \frac{\sqrt{\mathrm{d}x(z_1)\,\mathrm{d}x(z_2)}}{x(z_1)-x(z_2)}\frac{\tau_N\left(\mathcal{L},x,y+\frac{1}{N}\frac{\mathrm{d}S_{z_1,z_2}}{\mathrm{d}x}\right)}{\tau_N(\mathcal{L},x,y)}.$$

This theorem can be interpreted as Sato's formula [129] for integrable systems. In the context of random matrix theory, it can be interpreted as Heine's formula [132].

4.6.5. Hirota formula. Note that $\hat{H}(z_1,z_2)$ has a simple pole at $z_1 = z_2$ and behaves like

$$\hat{H}(z_1,z_2) \sim \frac{\sqrt{\mathrm{d}z_1\,\mathrm{d}z_2}}{z_1-z_2}$$

near $z_1 = z_2$, and this holds for any ($\bar{g} = 0$) spectral curve $\mathcal{E} = (\mathcal{L},x,y)$. In particular, we have, for any two such spectral curves $\mathcal{E}$ and $\tilde{\mathcal{E}}$,

Theorem 4.11.

$$\operatorname*{Res}_{z\to z_2} \hat{H}(z_1,z;\mathcal{E})\hat{H}(z,z_2;\tilde{\mathcal{E}}) = \hat{H}(z_1,z_2;\mathcal{E}).$$

If we consider that $\hat{H}$ is given by the Sato formula of theorem 4.10, this theorem is precisely the Hirota equation for the τ function τ_N [1, 16].

This theorem justifies that we can call τ_N a τ function. By expanding locally $\mathrm{d}S_{z_1,z_2}$ in the vicinity of poles of x, we can see that it is the multi-component KP τ function. There is one set of components for each pole α_i of x. In the case where $\mathcal{E}$ is an hyperelliptical curve, of type $y^2 = \mathrm{Pol}(x)$, the function x has two poles, which are symmetric with one another, and everything can be written in terms of the expansion near only one pole. In that case, τ_N reduces to the Kortweg–de Vries (KdV) τ function [16].

4.6.6. Higher genus spectral curves. So far, in this section, we considered genus zero spectral curves, i.e. $z \in \mathbb{C}$, and $x(z)$ and $y(z)$ are analytical functions of a complex variable.

For $\bar{g} = 0$ spectral curves, the integrability relied on the Sato formula, which gives the kernel $\hat{H}$ as the τ function of a shifted spectral curve, i.e. the exponential formula (4.14).

For higher genus $\bar{g} \geqslant 1$, the problem is that the exponential formula (4.14) does not define a well-defined spinor on $\mathcal{L}$. Indeed, $\mathcal{L}$ is not simply connected, and the Abelian integrals $\int_{z_2}^{z_1}\dots\int_{z_2}^{z_1}\omega_n$ are multivalued functions of z_1 and z_2 because there is not a unique integration path between z_1 and z_2. The exponential formula has to be modified. It was proposed to modify it with some θ functions (see section 4.5).

Definition 4.5. *Given a characteristic* (μ, ν)*, the '*τ *function' is defined by the nonperturbative partition function of section 4.5:*

$$\tau_N(\mu, \nu, \mathcal{E}) = Z_{\mathcal{E}}(\mu, \nu).$$

Then define the spinor kernel $\hat{H}_{(\mu,\nu)}$ through the Sato formula.

Definition 4.6.

$$\hat{H}_{(\mu,\nu)}(z_1, z_2) = \frac{\sqrt{\mathrm{d}x(z_1)\,\mathrm{d}x(z_2)}}{x(z_1) - x(z_2)} \frac{\tau_N\left(\mu, \nu, \mathcal{L}, x, y + \frac{1}{N}\frac{\mathrm{d}S_{z_1,z_2}}{\mathrm{d}x}\right)}{\tau_N(\mu, \nu, \mathcal{L}, x, y)}. \tag{4.16}$$

With this definition, $\hat{H}_{(\mu,\nu)}$ is closely related to the Szegö kernel [132].

Theorem 4.12. $\hat{H}_{(\mu,\nu)}(z_1, z_2)$ *is well defined for* $z_1, z_2 \in \mathcal{L}$.

Proof. Integrals of $\omega_n^{(g)}$'s are in principle defined only on the universal covering of $\mathcal{L}$, and one needs to check that after going around an $\mathcal{A}$-cycle or $\mathcal{B}$-cycle, $\hat{H}_{(\mu,\nu)}(z_1, z_2)$ takes the same value.

Note that, if z_1 goes around an $\mathcal{A}$-cycle, then $\mathrm{d}S_{z_1,z_2}$ is unchanged, and if z_1 goes around the cycle $\mathcal{B}_i$, then $\mathrm{d}S_{z_1,z_2}$ is shifted by a holomorphic differential:

$$\mathrm{d}S_{z_1+\mathcal{B}_i,z_2} = \mathrm{d}S_{z_1,z_2} + 2\mathrm{i}\pi\,\mathrm{d}u_i.$$

However, it was proved in [65] that the τ function above is background independent, which exactly means that, for any λ,

$$\tau_N(\mu, \nu, \mathcal{L}, x, y + \lambda\,\mathrm{d}u_i/\mathrm{d}x) = \tau_N(\mu, \nu, \mathcal{L}, x, y),$$

and therefore we see that $\hat{H}_{(\mu,\nu)}(z_1, z_2)$ is unchanged if z_1 goes around a $\mathcal{B}$-cycle. □

Then, we see that [74]

Theorem 4.13. $\hat{H}_{(\mu,\nu)}(z_1, z_2)$ *obeys the Hirota equation:*

$$\operatorname*{Res}_{z\to z_2} \hat{H}_{(\mu,\nu)}(z_1, z; \mathcal{E})\hat{H}_{(\mu,\nu)}(z, z_2; \tilde{\mathcal{E}}) = \hat{H}_{(\mu,\nu)}(z_1, z_2; \mathcal{E}).$$

If we consider that $\hat{H}_{(\mu,\nu)}$ is given by the Sato formula of theorem (4.16), this theorem is precisely the Hirota equation for the τ function τ_N.

4.7. Virasoro constraints

It has been understood for a long time that the random matrix integrals are fundamentally linked to Virasoro and $\mathcal{W}$-algebras through differential equations on their moduli called Virasoro or $\mathcal{W}$-constraints (see for example [114, 119] and references therein). The definition of the symplectic invariants and of the correlation functions themselves were inspired by these constraints since they mimic the solution of the loop equations of random matrix models, the latter being considered as equivalent to the Virasoro constraints.

In a series of papers [8–10], Alexandrov, Mironov and Morozov go even further and propose to generalize the notion of random matrix integrals by defining a general string partition function interpolating between different matrix models. This partition function is characterized as a 'D-module' solution of some Virasoro constraints.

It is natural to see the symplectic invariants and the τ function built from them as a good candidate for this string partition function. It is thus interesting to clarify the arising of Virasoro constraints in the theory of symplectic invariants by looking at the variations of the latter with respect to the moduli of the spectral curve. All the proofs of this section can be found in [44].

4.7.1. Virasoro at the branch points. One can slightly rewrite the recursive relations defining the correlation function (2.4) by moving all the terms to the same side of the equation. One gets

$$0 = \sum_i \operatorname*{Res}_{z\to a_i} K(z_0, z)\Bigg[\omega_{n+2}^{(g-1)}(z, \bar{z}, J) + \sum_{h=0}^{g}\sum_{I\subset J}\omega_{1+|I|}^{(h)}(z, I)\omega_{1+n-|I|}^{(g-h)}(\bar{z}, J\backslash I) + y\,\mathrm{d}x(z)\omega_{n+1}^{(g)}(\bar{z}, J) + y\,\mathrm{d}x(\bar{z})\omega_{n+1}^{(g)}(z, J)\Bigg].$$

By summing over the genus g and interpreting the correlation function as the result of the loop insertion operator on the symplectic invariants, this equation can be rephrased as a Virasoro constraint.

Theorem 4.14. *For any point z on the spectral curve, the partition function is a zero mode of the global Virasoro operator $\widehat{\mathcal{V}}(z)$:*

$$\widehat{\mathcal{V}}(z)\tau_N = 0$$

with

$$\widehat{\mathcal{V}}(z) = \sum_i \oint_{a_i} K(z, z') : \mathcal{J}(z')\mathcal{J}\left(\overline{z'}\right) :, \tag{4.17}$$

and the global current is defined by

$$\mathcal{J}(z) = Ny\,\mathrm{d}x(z) + \frac{1}{N}D_z$$

for any point z of the spectral curve where D_z is the loop operator.

This means that the recursive definition of the correlation functions is nothing but a Virasoro constraint on the τ function defined *globally* on the spectral curve.

Let us now approach a particular branch point a_i and blow up the spectral curve around this point (see section 4.8). A rational parametrization of the blown-up curve can read as

$$\begin{cases}\tilde{x}(z) = z^2\\ \tilde{y}(z) = \sum_{k=0}^{\infty} T_k^{(i)} z^k,\end{cases}$$

where $T_k^{(i)}$'s are the coefficients of the Taylor expansion of $y\,\mathrm{d}x$ around the branch points a_i:

$$y\,\mathrm{d}x(z) = \sum_{k=0}^{\infty} T_k^{(i)}\hat{\xi}_i^{k+1}(z)\,\mathrm{d}\hat{\xi}_i(z)$$

with the local coordinates

$$\hat{\xi}_i(z) = \sqrt{x(z) - x(a_i)}.$$

From section 4.8, one knows that the projection of the correlation functions in the local patch around a_i built from the local parameter $\hat{\xi}_i$ is given by the correlation functions of the blown-up curve, i.e. the correlation functions of the Kontsevich integral with times $T_k^{(i)}$, $k = 0, \ldots, \infty$. Since the recursive definition of these correlation functions is equivalent to the Kontsevich Virasoro constraints[14], this means that the global Virasoro operator projects to the continuous Virasoro operator in this local patch of coordinates around a_i.

[14] See [9, 38] for detail on these continuous Virasoro constraints.

Theorem 4.15. *For any branch point a_i, the partition function is a zero mode of a set local Kontsevich Virasoro operator $\widehat{\mathcal{V}}_i(\hat{\xi}_i(z))$ for any point z in a neighborhood of a_i:*

$$\forall i, \quad \widehat{\mathcal{V}}_i(\hat{\xi}_i(z))\tau_N = 0, \tag{4.18}$$

where $\hat{\xi}_i(z) = \sqrt{x(z) - x(a_i)}$ and the Virasoro operator annihilates the Kontsevich τ function:

$$\widehat{\mathcal{V}}_i(\hat{\xi}) Z_K\left(T_k^{(i)}\right) = 0,$$

where

$$Z_K\left(T_k^{(i)}\right) := \int_{\text{formal}} \mathrm{e}^{-N\,\mathrm{Tr}(\frac{M^3}{3} - \Lambda^2 M)}$$

with Kontsevich times

$$T_k^{(i)} := \frac{1}{N}\,\mathrm{Tr}\,\Lambda^{-k}.$$

Indeed, as is exhibited in section 10.1, the corresponding spectral curve has only one branch point and all the moduli of the integral are summed up in the Taylor expansion of the differential form $x\,\mathrm{d}y$ at this branch point.

4.7.2. Loop equations and Virasoro at the poles. In the preceding section, we have translated the recursive definition of the correlation functions into a set of Virasoro operators related to the moduli of the spectral curve at the branch points. One can proceed in a similar way for the moduli at the poles of the 1-form $y\,\mathrm{d}x$ by building a set of equations solved by the correlation functions called 'loop equations' since they mimic the loop equations of random matrix theory. Nevertheless, in order to obtain directly a set of Virasoro constraints, one has to restrict to the spectral curve which is hyperelliptical in this section[15].

Theorem 4.16. *The correlation functions ω_n are solutions of the loop equations:*

$$\sum_{l=0}^{k} \omega_{l+1}(z, \mathbf{z_L})\omega_{k-l+1}(z, \mathbf{z_{K\setminus L}}) + \frac{1}{N^2}\omega_{k+2}(z, z, \mathbf{z_K}) = P_{1,k}(z, \mathbf{z_K})\,\mathrm{d}x(z)^2,$$

where the function

$$P_{1,k}(z, \mathbf{z_K}) := \sum_i \oint_{\alpha_i} \frac{\sum_{l=0}^{k}\omega_{l+1}(z', \mathbf{z_L})\omega_{k-l+1}(z', \mathbf{z_{K\setminus L}}) + \frac{1}{N^2}\omega_{k+2}(z', z', \mathbf{z_K})}{(z_i(z) - z_i(z'))\,\mathrm{d}x(z')}$$

is a function of z with poles only at the poles of $y\,\mathrm{d}x$.

In the matrix model case, these loop equations are often referred to as Virasoro constraints. They indeed encode a set of Virasoro constraints built from the poles of $y\,\mathrm{d}x$. Let us make this assertion clear in the general framework of the symplectic invariants.

Theorem 4.17. *For any point $z \in \mathcal{L}$, the τ function satisfies*

$$\mathcal{V}(z)\tau_N = 0,$$

where one defines the global Virasoro operator:

$$\mathcal{V}(z) = \frac{1}{N^2} : \mathcal{J}^2(z) : + \sum_i \oint_{\alpha_i} \frac{: \mathcal{J}^2(z') :}{(\xi_i(z') - \xi_i(z))\,\mathrm{d}x(z')}, \tag{4.19}$$

[15] If the spectral curve is an arbitrary algebraic curve, one should get $\mathcal{W}$-algebra constraints, but this issue is still under investigation.

and $\xi_i(z) = \frac{1}{\xi(z)}$ *is a local parameter in the neighborhood of* α_i *(see equation (4.3) for the definition of* $\xi(z)$*).*

The τ function can thus be seen as the zero mode of another Virasoro operator globally defined on the spectral curve. This new operator, equivalent to the loop equations, can be easily projected to a set of local Virasoro operators in the vicinity of the poles of $y\,\mathrm{d}x$ instead of the branch points for the first one. In order to follow this procedure, one has to restrict to $y\,\mathrm{d}x$ which are holomorphic forms with poles α_i such that $y\,\mathrm{d}x(z) \sim_{z\to a_i} \sum_k k t_{i,k} \xi_i^k(z)\,\mathrm{d}\xi_i(z)$.

Theorem 4.18. *For any point* z *in the neighborhood of a pole* α_i *of* $y\,\mathrm{d}x$

$$\mathcal{V}_-^{(i)}(z)\tau_N = 0,$$

where the local Virasoro operator is defined as the loop operator:

$$\mathcal{V}_-^{(i)}(z) := \oint_{\alpha_i} \frac{\mathrm{d}\xi_i(z')}{\xi_i(z') - \xi_i(z)} : J^{(i)}(z')^2 :$$

with the local current

$$J^{(i)}(z) := \sum_{k\geqslant 0} \left[k t_{i,k} \xi_i(z)^{k-1}\,\mathrm{d}\xi_i(z) + \frac{\mathrm{d}\xi_i(z)}{\xi_i(z)^{k+1}} \frac{\partial}{\partial t_{i,k}} \right].$$

Note that these local Virasoro operators are indeed Laurent series in $\xi_i(z)$ with only negative powers whose coefficients are differential operators satisfying Virasoro commutation relations:

$$\mathcal{V}_-^{(i)}(z) = \sum_{k>0} \mathcal{V}_k^{(i)} \xi_i(z)^{-k} (\mathrm{d}\xi_i(z))^2$$

and

$$\left[\mathcal{V}_j^{(i)}, \mathcal{V}_l^{(k)}\right] = (j-l)\mathcal{V}_{j+l}\delta_{i,k}.$$

These local operators around the poles have also a natural solution: the 1-Hermitian matrix integral

$$Z_{\mathrm{1MM}} := \int_{\text{formal}} \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(M)}\,\mathrm{d}M$$

with a polynomial potential

$$V(x) := \frac{x^2}{2} - \sum_{k=3}^{d} t_k x^k$$

whose coefficients t_k are identified with the moduli at the poles $t_{i,k}$ (see section 5.2 for more details).

4.7.3. Givental decomposition formulae. Let us suppose in this short section that the spectral curve has genus 0, i.e. $\mathcal{L}$ = Riemann sphere. In this case, the only moduli of the curve are

- either the position of the poles and moduli $t_{i,k}$ at these poles,
- either the position of the branch point and the moduli $T_k^{(i)}$.

Let us first focus on the branch points of the spectral curve. The dependence of τ_N on the moduli at the branch points is constrained by the local Virasoro equations equation (4.18). Thus, this τ function can be decomposed as a product of the zero modes of the different local operators at the branch points, i.e. a product of Kontsevich integrals, up to a conjugation operator mixing the moduli at the different branch points.

Theorem 4.19. *τ_N can be decomposed into a product of Kontsevich integrals associated with the branch points a_i:*

$$\tau_N(\mathbf{T^{(1)}}, \mathbf{T^{(2)}}, \ldots) = \mathrm{e}^{\widehat{\mathcal{U}}} \prod_i \mathcal{Z}_K(\mathbf{T^{(i)}}),$$

where the symbol $\mathbf{T^{(i)}}$ stands for the infinite family $\{T_k^{(i)}\}_{k=0}^{\infty}$, with the intertwining operator $\widehat{\mathcal{U}}$ defined by

$$\widehat{\mathcal{U}} := \sum_{i,j} \oint_{a_j} \oint_{a_i} \widehat{A}^{(i,j)}(z, z')\widehat{\Omega}_j(z')\widehat{\Omega}_i(z),$$

where

$$\widehat{A}^{(i,j)}(z, z') := B(z, z') - \frac{\mathrm{d}\hat{\xi}_i(z)\,\mathrm{d}\hat{\xi}_j(z')}{(\hat{\xi}_i(z) - \hat{\xi}_j(z'))^2}$$

and

$$\widehat{\Omega}_i(z) := N \sum_k T_k^{(i)} \hat{\xi}_i^k(z)\,\mathrm{d}\hat{\xi}_i(z) - \frac{1}{N} \frac{\mathrm{d}\hat{\xi}_i(z)}{k\hat{\xi}_i^k(z)} \frac{\partial}{\partial T_k^{(i)}}. \tag{4.20}$$

One can proceed exactly in the same way by looking at the moduli at the poles: this time the decomposition is expressed as a product of 1-Hermitian matrix integrals.

Theorem 4.20. *τ_N can be decomposed into a product of 1-Hermitian matrix integrals associated with the poles α_i of the meromorphic form $y\,\mathrm{d}x$*

$$\tau_N(\mathbf{t_1}, \mathbf{t_2}, \ldots) = \mathrm{e}^{\mathcal{U}} \prod_i \mathcal{Z}_{1\mathrm{MM}}(\mathbf{t_i}),$$

where $\mathbf{t_i}$ stands for the infinite set $\{t_{i,k}\}_{k=0}^{\infty}$, with the intertwining operator $\mathcal{U}$ defined by

$$\mathcal{U} := \sum_{i,j} \oint_{\alpha_j} \oint_{\alpha_i} A^{(i,j)}(z, z')\Omega_j(z')\Omega_i(z),$$

where

$$A^{(i,j)}(z, z') = B(z, z') - \frac{\mathrm{d}\xi_i(z)\,\mathrm{d}\xi_j(z')}{(\xi_i(z) - \xi_j(z'))^2}$$

and

$$\Omega_i(z) := N \sum_k t_{i,k} \xi_i^k(z)\,\mathrm{d}\xi_i(z) - \frac{1}{N} \frac{\mathrm{d}\xi_i(z)}{k\xi_i^k(z)} \frac{\partial}{\partial t_{i,k}}. \tag{4.21}$$

Note that, in both cases, these decomposition formulae consist in writing a KP τ function as a product of KdV τ functions. Indeed, these formulae were already derived by Givental in the study of KP τ functions [79, 80].

4.7.4. Vertex operator and integrability. In this paragraph, we do not consider the Baker–Akhiezer functions as defined in section 4.6. In contrast, we define them as the images of the partition function under the action of some global operator on the spectral curve.

Let us define the equivalent of the Baker–Akhiezer (BA) functions:

Definition 4.7. *One defines the x-type global BA and dual BA functions as*

$$\Psi(z) := \exp(\Omega_x(z))\mathcal{Z} \qquad \textit{and} \qquad \Psi^*(z) := \exp(-\Omega_x(z))\,\mathcal{Z},$$

where

$$\Omega_x(q) := \int^q \mathcal{J}_x = N\int^q y\,\mathrm{d}x(z) - \frac{1}{N}\int^q D_z.$$

One also defines the x-type local BA functions as

$$\Psi_{i,0}(z) := \exp(\Omega_i(z))\,\mathcal{Z} \qquad \textit{and} \qquad \Psi^*_{i,0}(z) := \exp(-\Omega_i(z))\,\mathcal{Z},$$

where the operator Ω_i was defined in equation (4.21).

We finally define the corresponding y-type BA functions:

$$\widetilde{\Psi}(z) := \exp(\Omega_y(z))\mathcal{Z} \qquad \textit{and} \qquad \widetilde{\Psi}^*(z) := \exp(-\Omega_y(z))\mathcal{Z},$$

where

$$\Omega_y(q) := \int^q \mathcal{J}_y = N\int^q x\,\mathrm{d}y(z) - \frac{1}{N}\int^q D_z.$$

These functions correspond to deformations of the spectral curve and thus coincide with the Baker–Akhiezer functions of equation (4.15).

Lemma 4.1. *The BA functions can be written in terms of the partition function as*

$$\Psi(z) := \exp(\Omega_x(z))\mathcal{Z}(\mathbf{t_i}) = \frac{\mathcal{Z}\left(\mathbf{t_i} + \left[z_i^{-1}\right]\right)}{\mathcal{Z}(\mathbf{t_i})},$$

$$\Psi_{i,0}(z) := \exp(\Omega_i(z))\mathcal{Z}(t_i) = \frac{\mathcal{Z}\left(t_i + \left[z_i^{-1}\right]\right)}{\mathcal{Z}(t_i)},$$

$$\widetilde{\Psi}(z) := \exp(\Omega_y(z))\mathcal{Z}(\tilde{\mathbf{t}}_\mathbf{i}) = \frac{\mathcal{Z}\left(\tilde{\mathbf{t}}_\mathbf{i} + \left[z_i^{-1}\right]\right)}{\mathcal{Z}(\tilde{\mathbf{t}}_\mathbf{i})}$$

and

$$\widetilde{\Psi}_{i,0}(z) := \exp(\Omega_{y,i}(z))\mathcal{Z}(\tilde{t}_i) = \frac{\mathcal{Z}\left(\tilde{t}_i + \left[z_i^{-1}\right]\right)}{\mathcal{Z}(\tilde{t}_i)},$$

where $\tilde{t}_i$ are the coefficients of the Taylor expansion of $x\,\mathrm{d}y(z)$ as $z \to \alpha_i$ and $[z^{-1}]$ is the usual Hirota symbol [1].

On the other hand, thanks to the pole structure of the BA functions, one gets

Theorem 4.21. *The Baker–Akhiezer functions satisfy the bilinear Hirota equation*

$$\sum_i \oint_{\alpha_i} \Psi(p|\mathbf{t})\,\mathrm{e}^{-N\int^p y\,\mathrm{d}x + N\int^p y'\mathrm{d}x'}\Psi^*(p|\mathbf{t}') = \sum_i \oint_{\alpha_i} \widetilde{\Psi}(p|\mathbf{t})\,\mathrm{e}^{-N\int^p x\,\mathrm{d}y + N\int^p x'\mathrm{d}y'}\widetilde{\Psi}^*(p|\mathbf{t}'),$$

where x' and y' are functions on $\mathcal{L}$ satisfying another algebraic equation

$$\mathcal{E}'(x'(z), y'(z)) = 0$$

compared to

$$\mathcal{E}(x(z), y(z)) = 0.$$

Proof. The proof relies on the simple observation that

$$\Omega_x - N\int^z y\,\mathrm{d}x = \Omega_y - N\int^z x\,\mathrm{d}y = D_z.$$

□

Corollary 4.1. *If x has q poles and y has p poles, the partition function $\mathcal{Z}(\mathbf{t})$ is a τ function of the multi-component $p+q$ KP hierarchy since it satisfies the Hirota equations [1]:*

$$\sum_i \oint_{\alpha_i} \frac{\mathcal{Z}\left(t_i + \left[z_i^{-1}\right]\right)}{\mathcal{Z}(t_i)} \frac{\mathcal{Z}\left(t_i' - \left[z_i^{-1}\right]\right)}{\mathcal{Z}(t_i')} \mathrm{e}^{N\sum_k (t_{i,k}-t'_{i,k})z_i^{k-1}(p)}$$
$$= \oint_{\alpha_i} \frac{\mathcal{Z}\left(\tilde{t}_i + \left[\tilde{z}_i^{-1}\right]\right)}{\mathcal{Z}(\tilde{t}_i)} \frac{\mathcal{Z}\left(\tilde{t}_i' - \left[\tilde{z}_i^{-1}\right]\right)}{\mathcal{Z}(\tilde{t}_i')} \mathrm{e}^{N\sum_k (\tilde{t}_{i,k}-\tilde{t}'_{i,k})\tilde{z}_i^{k-1}(p)}.$$

4.8. Singular limits

F_g's and $\omega_n^{(g)}$'s can be computed for any regular spectral curve, i.e. as long as the branch points are simple. When the spectral curve is singular, F_g's are not defined.

Nevertheless, consider a one-parameter family of spectral curves $\mathcal{E}(t)$, such that $\mathcal{E}(t_c)$ is singular; we prove below that $F_g(t)$ diverges as $t \to t_c$ in the following form:

$$F_g(t) \sim (t - t_c)^{(2-2g)\mu} \tilde{F}_g.$$

The goal of this section is to prove this divergent behavior and compute the exponent μ and the prefactor $\tilde{F}_g$. These asymptotics are very important in many applications in mathematics and physics, for instance Witten's conjecture relates the asymptotics of large discrete surfaces to integrals over moduli spaces of continuous Riemann surfaces. Asymptotic formulae also play a key role in the universal limits of random matrix eigenvalue statistics or in the study of universality in the statistics of non-intersecting Brownian motions (see section 6).

In the context of matrix models quantum gravity, the prefactor $\tilde{F}_g$ is called the double scaling limit of F_g and the exponent $2-2\mu$ is called

$$2 - 2\mu = \gamma_{\text{string}} = \text{string susceptibility exponent.}$$

It is such that F_0'' formally diverges with the exponent $-\gamma_{\text{string}}$ [37, 81]:

$$\frac{\mathrm{d}^2 F_0}{\mathrm{d}t^2} \sim (t - t_c)^{-\gamma_{\text{string}}}.$$

4.8.1. Blow-up of a spectral curve. Consider a one-parameter family of spectral curves $\mathcal{E}(t) = (\mathcal{L}(t), x(z,t), y(z,t))$, such that $\mathcal{E}(t)$ is regular in an interval $]t_c, t_0]$. For the moment, we do not assume that $\mathcal{E}(t_c)$ is singular, i.e. it may be either regular or singular. In a small vicinity of t_c, we can, to leading orders, parametrize $\mathcal{E}(t)$ in terms of $\mathcal{L} = \mathcal{L}(t_c)$.

Moreover, let a be a branch point and let us study the correlators $\omega_n^{(g)}(z_1, \ldots, z_n)$ in the vicinity of a. We choose a rescaled local coordinate ζ in the vicinity of a. Let us write

$$z = a + (t - t_c)^{\nu}\zeta + o((t - t_c)^{\nu}).$$

We want to compute the asymptotic behavior of $\omega_n^{(g)}(a + (t-t_c)^{\nu}\zeta_1, \ldots, a + (t-t_c)^{\nu}\zeta_n; t)$ in the limit $t \to t_c$.

First, let us study the behavior of x and y, by Taylor expansion. Let q be the first non-trivial power in the Taylor expansion of x, i.e.

$$x(a + (t - t_c)^{\nu}\zeta; t) = x(a; t_c) + (t - t_c)^{q\nu}\tilde{x}(\zeta) + o((t - t_c)^{q\nu})$$

and similarly, there is an exponent p such that

$$y(a + (t - t_c)^{\nu}\zeta; t) = y(a; t_c) + (t - t_c)^{p\nu}\tilde{y}(\zeta) + o((t - t_c)^{p\nu}).$$

This means that at $t = t_c$, the curve $\mathcal{E}(t_c)$ behaves like $y \sim y(a) + (x - x(a))^{p/q}$. It is regular if $\frac{p}{q} = \frac{1}{2}$ and singular otherwise.

The rescaled curve $\tilde{\mathcal{E}} = (\tilde{\mathcal{L}}, \tilde{x}, \tilde{y})$ is called the blow-up of the spectral curve near the branch point a, in the limit $t \to t_c$.

The choice of the exponent ν must be such that $\tilde{\mathcal{E}}$ is a regular spectral curve. We cannot give a general formula for ν, since it depends on the explicit choice of a one-parameter family of spectral curves $\mathcal{E}(t)$ and how it is parametrized. Also, here we consider only algebraic singularities of type $y \sim x^{p/q}$, but the method could certainly be extended to other types of singularities.

Examples.

- The following spectral curve arises in the enumeration of quadrangulated surfaces (see section 7.4):

$$\mathcal{L} = \mathbb{P}^1, \qquad x(z) = \gamma\left(z + \frac{1}{z}\right), \qquad y(z) = \frac{-1}{\gamma z} + \frac{t_4\gamma^3}{z^3},$$

where

$$\gamma^2 = \frac{1 - \sqrt{1 - 12t_4}}{6t_4}.$$

The branch points $x'(a) = 0$ are $a = \pm 1$. Consider the branch point $a = 1$ and introduce an auxiliary scaling variable t:

$$z = 1 + t\zeta,$$

such that x and y are independent of t, and let us study the vicinity of $t \to t_c = 0$.

We wish to study the behavior of $\omega_n^{(g)}(1 + t\zeta_1, \ldots, 1 + t\zeta_n)$ in the vicinity of $t \to 0$, i.e. the behavior of $\omega_n^{(g)}$ in the vicinity of the branch point $a = 1$.

In the limit $t \to 0$, we Taylor expand x and y:

$$x(z) = 2\gamma + \gamma t^2\zeta^2 + O(t^3),$$
$$y(z) = \frac{-1}{\gamma} + t_4\gamma^3 + t\zeta\left(\frac{1}{\gamma} - 3t_4\gamma^3\right) + O(t^2).$$

Note that we have $q = 2$ and $p = 1$, which means that our curve is not singular at $t = 0$ (which was expected since it is actually independent of t).

The blow-up is

$$\tilde{x}(\zeta) = \gamma\zeta^2, \qquad \tilde{y}(\zeta) = \left(\frac{1}{\gamma} - 3t_4\gamma^3\right)\zeta.$$

Up to a rescaling, it is the Airy spectral curve (see section 8 and example (2.1)).

- In the previous example, at $t_4 = \frac{1}{12}$, we have $(1 - 3t_4\gamma^4) = 0$, and thus one needs to go further in the Taylor expansion. Let us now choose $t = 1 - 12t_4$, $t_c = 0$ and $a = 1$. We have in the limit $t \to t_c$

$$x\left(1 + \frac{1}{\sqrt{2}}t^{\frac{1}{4}}\zeta\right) = 2\sqrt{2} + \frac{\sqrt{t}}{\sqrt{2}}(\zeta^2 - 2) + o(\sqrt{t}),$$
$$y\left(1 + \frac{1}{\sqrt{2}}t^{\frac{1}{4}}\zeta\right) = -\frac{\sqrt{2}}{3} + \frac{t^{\frac{1}{2}}}{2\sqrt{2}}(\zeta^2 - 2) - \frac{t^{\frac{3}{4}}}{12}(7\zeta^3 - 12\zeta) + o\left(t^{\frac{3}{4}}\right),$$

and, in fact, what we really need is the asymptotic behavior of $y(z) - y(\bar{z})$, i.e.

$$y\left(1 + \frac{1}{\sqrt{2}}(t_c - t)^{\frac{1}{4}}\zeta\right) - y\left(\frac{1}{(1 + \frac{1}{\sqrt{2}}(t_c - t)^{\frac{1}{4}}\zeta)}\right) = -\frac{2t^{\frac{3}{4}}}{3}(\zeta^3 - 3\zeta) + o\left(t^{\frac{3}{4}}\right).$$

Thus, the blow-up of the spectral curve in this limit is

$$\tilde{x}(\zeta) = \tfrac{1}{\sqrt{2}}(\zeta^2 - 2), \qquad \tilde{y}(\zeta) = -\tfrac{1}{3}(\zeta^3 - 3\zeta).$$

Note that it is proportional to the pure gravity spectral curve $(p, q) = (3, 2)$ [46]; see section 8 and the first example in section 2.1.1. It is the spectral curve which arises every time we have a $y \sim x^{3/2}$ cusp singularity.

4.8.2. Asymptotics. In order to study the asymptotics of $\omega_n^{(g)}$'s, we need to study the asymptotics of the kernels B and K, in the limit $t \to t_c$.

We have

$B(z_0, z) \sim$

	z near a	z far from a
z_0 near a	$\tilde{B}(\zeta_0, \zeta)$	$O((t - t_c)^{\nu})$
z_0 far from a	$O((t - t_c)^{\nu})$	$O(1)$

$\times (1 + O((t - t_c)^{\nu}))$,

where $\tilde{B}(\zeta_0, \zeta)$ is the Bergman kernel of the blown-up spectral curve $(\tilde{\mathcal{L}}, \tilde{x}, \tilde{y})$:

$$\tilde{B}(\zeta_0, \zeta) = \frac{d\zeta_0 d\zeta}{(\zeta - \zeta_0)^2}.$$

Similarly, the kernel K behaves like

$K(z_0, z) \sim$

	z near a	z far from a
z_0 near a	$(t - t_c)^{-(p+q)\nu}\tilde{K}(\zeta_0, \zeta)$	$O(1)$
z_0 far from a	$O((t - t_c)^{-(p+q-1)\nu})$	$O(1)$

$\times (1 + O((t - t_c)^{\nu}))$,

where $\tilde{K}(\zeta_0, \zeta)$ is the recursion kernel of the blown-up spectral curve $(\tilde{\mathcal{L}}, \tilde{x}, \tilde{y})$:

$$\tilde{K}(\zeta_0, \zeta) = \frac{1}{2}\left(\frac{1}{\zeta_0 - \zeta} - \frac{1}{\zeta_0 + \zeta}\right)\frac{1}{2\tilde{y}(\zeta)\tilde{x}'(\zeta)}.$$

Therefore, we see that the leading contribution to $\omega_n^{(g)}(1 + \delta\zeta_0, \ldots, 1 + \delta\zeta_n)$ is given by the terms where all residues are taken near a and the leading contribution can be computed only in terms of $\tilde{B}$ and $\tilde{K}$. By an easy recursion, one gets

Theorem 4.22. *Singular limit of $\omega_n^{(g)}$ with $2 - 2g - n < 0$ and $n \geqslant 1$.*

If a is a branch point, the asymptotics of $\omega_n^{(g)}$ in the vicinity of $t \to t_c$ and $z_i \to a$ are given by

$$\omega_n^{(g)}(a + (t - t_c)^{\nu}\zeta_1, \ldots, a + (t - t_c)^{\nu}\zeta_n) \sim (t - t_c)^{(2-2g-n)(p+q)\nu}\tilde{\omega}_n^{(g)}(\zeta_1, \ldots, \zeta_n),$$

where $\tilde{\omega}_n^{(g)}$ are the correlators of the blown-up spectral curve $\tilde{\mathcal{E}} = (\tilde{\mathcal{L}}, \tilde{x}, \tilde{y})$.

In this theorem, the exponent ν is unspecified; it depends on our choice of a one-parameter family of curves, i.e. it depends on each example. We recall that it must be chosen such that the blown-up curve $\tilde{\mathcal{E}}$ is regular. We see examples in section 8.

This theorem implies in particular that all correlation functions in the vicinity of a regular branch point are, to leading order, the same as the Airy process correlation functions; we recover the universal Airy law near regular branch points. This is related to the universal Tracy–Widom law [134].

One may extend this theorem to $\omega_0^{(g)} = F_g$'s.

Theorem 4.23. *Singular limit of F_g.*

If at $t = t_c$, the branch point a becomes singular and no other branch point becomes singular, then the asymptotics of F_g, $g \geqslant 2$, in the vicinity of $t \to t_c$ are given by

$$F_g \sim (t - t_c)^{(2-2g)(p+q)\nu} \widetilde{F}_g + o((t - t_c)^{(2-2g)(p+q)\nu}),$$

where $\widetilde{F}_g$ are the symplectic invariants of the blown-up spectral curve $\tilde{\mathcal{E}} = (\tilde{\mathcal{L}}, \tilde{x}, \tilde{y})$.

In fact, this theorem also holds in the case where the branch point a is not singular, but it becomes useless. Indeed, if a is not singular, the blown-up spectral curve is the Airy curve, and $\tilde{F}_g$'s of the Airy spectral curve vanish for all $g \geqslant 1$, and therefore all that the theorem says in this case is that F_g does not diverge as $(t - t_c)^{(2-2g)(p+q)\nu}$. This is obvious since F_g is not divergent at all.

Also, the condition that a is the only singular branch point is not so necessary. In fact, if several branch points become singular simultaneously at t_c, with the same exponents ν and (p, q), then the asymptotics of F_g is the sum of contributions of most singular branch points. The most basic example is a symmetric spectral curve $y(x) = y(-x)$, for which branch points come in pairs. The leading order of F_g then gets a factor 2: $F_g \sim 2(t - t_c)^{(2-2g)(p+q)\nu} \widetilde{F}_g + o((t - t_c)^{(2-2g)(p+q)\nu})$.

5. Application to matrix models

The recursion relations defining the symplectic invariants and their correlation functions were originally found in the study of the 1-Hermitian random matrix model [40, 58], where they appeared as the solution to the so-called loop equations. It is thus interesting to recall to what extent the symplectic invariants give a solution to the computation of the free energies and correlation functions' topological expansions in different matrix models. It is also interesting to emphasize the special properties of the spectral curves obtained from matrix models to recall that they represent only a particular subcase in the whole framework for symplectic invariants.

5.1. Generalities about matrix models

Random matrices were introduced by Wigner in 1951 [140], in the study of energy levels of heavy nuclei. Wigner had observed that the distribution of heavy nuclei energy levels could be very well approximated by the statistical distribution of the eigenvalues of a large random matrix. Wigner was thus the first to realize that random matrix laws are universal laws, a kind of generalization of the Gauss law, but for the limit of a large number of interacting random variables (in contrast to the Gauss law or Poisson law which holds only for independent variables). The same idea was successfully used in the study of quantum chaos [26, 82, 108] and disordered mesoscopic systems [82], and random matrix laws were also found in the distribution of time between buses [14, 106] and in many other physical phenomenon [82, 108].

Many methods have been invented to compute matrix integrals and correlation functions of eigenvalues. In particular, Dyson and Mehta have initiated the study of random matrices through the theory of orthogonal polynomials [56, 116]. This method is an exact method, and it has brought an incredibly rich structure, related to the theory of integrable systems [113].

The goal was to study the limit of large random matrices, and the method of orthogonal polynomials was not so successful for this purpose. An important breakthrough came with

the so-called Riemann–Hilbert method, introduced by Its and Bleher [25] and reached full maturity with Deift and collaborators [45] (see also [24] for a review of the subject). So far it is the most used method to prove universality of random matrix laws. It allows one to prove the existence of a certain form for the large size asymptotic expansion for various correlation functions and expectation values, but it is not very efficient in computing the coefficients in the expansion. As we shall see below, once the existence of an asymptotic expansion is proved, the most efficient method so far to get the coefficients is the loop equation method.

In 1974, 'tHooft was studying QCD, whose fields are matrices, and he noted that the power of the size of the matrix is the Euler characteristics of the Feynman graphs [133]. This gave the idea to Brezin *et al* [36] in 1978 to use matrix models to count discrete surfaces of given topology. This was the start of a huge activity in physics, called 2D quantum gravity [46, 47, 78].

It is in this context that Migdal invented the loop equation method [118], which is in fact the Schwinger–Dyson equation applied to matrix integrals. It was called loop equations because it could be interpreted as splitting or gluing Feynman graphs along loops drawn on the graphs. This method was then used by many people and, under the assumption of a large N expansion, it was revealed to be extremely efficient. In particular, in 1995, Ambjorn *et al* invented a recursive method to compute all the expansion [12]. Their method was however limited only to the simplest matrix models and was not so easily generalized. For the 2-matrix model, the first subleading correction was found in 2002 [59] and the method to compute the full expansion is now the one presented in this review.

Let us emphasize that a matrix integral (or in fact any integral), defined by its Feynman graphs, is not really an integral over a group of matrices; it is only a 'formal' integral (see the definition in section 5.2 or see [64]). There was thus a discrepancy between results found from the orthogonal polynomial methods for convergent integrals and results found from the loop equation method for formal integrals [7, 31, 93]. This discrepancy was understood in [27], and it was realized that convergent integrals and formal integrals are different objects, but related by a very simple relationship (see section 5.6).

Let us also mention that many other methods have been used to study matrix integrals.

(i) The character expansion method, developed originally by Kazakov and collaborators [99], was then extensively studied by Orlov and Harnad [84, 85].
(ii) A geometric signification of F_g's was found by Wiegmann and Zabrodin in the study of the so-called normal matrix model [139].
(iii) A supersymmetric method was invented by Efetov [57] and has met many new developments with Zirnbauer [143].
(iv) Riemann–Hilbert methods [24, 25, 45].
(v) Large deviation methods [17].
(vi) Double scaling limits [37, 95].

5.2. *1-matrix model*

The formal 1-matrix integral is defined as a formal power series in a variable t.

Consider a polynomial $V(x)$ (called 'potential') of degree $d+1 > 2$ as well as its d stationary points ξ_i:

$$\forall i = 1, \dots, d, \qquad V'(\xi_i) = 0$$

and the non-quadratic part of its Taylor expansion around these points

$$\delta V_i(x) = V(x) - V(\xi_i) - \frac{V''(\xi_i)}{2}(x - \xi_i)^2.$$

Let us also consider a d-partition of N, i.e. a set of d integers n_i satisfying

$$\sum_{i=1}^{d} n_i = N.$$

Definition 5.1. *Formal 1-hermitian matrix integral.*

The formal 1-matrix integral is defined as a formal power series in a variable t:

$$\begin{aligned} Z_{1\mathrm{MM}} &= \mathrm{e}^{-\frac{N}{t}\sum_i n_i V(\xi_i)} \sum_{k=0}^{\infty} \frac{(-1)^k N^k}{t^k k!} \int_{H_{n_1}} \cdots \int_{H_{n_d}} \prod_{i=1}^{d} \mathrm{d}M_i \left(\sum_i \mathrm{Tr}\, \delta V_i(M_i) \right)^k \\ &\quad \times \mathrm{e}^{-\frac{N}{2t}\sum_i V''(x_i)\mathrm{Tr}\left(M_i - \xi_i \mathbf{1}_{n_i}\right)^2} \prod_{i>j} \det\left(M_i \otimes \mathbf{1}_{n_j} - \mathbf{1}_{n_i} \otimes M_j\right)^2 \\ &= \sum_{k=0}^{\infty} t^k A_k, \end{aligned} \tag{5.1}$$

where each A_k is a (well-defined) polynomial moment of a Gaussian integral.

It is denoted by

$$Z_{1\mathrm{MM}} = \int_{\mathrm{formal}} \mathrm{e}^{-\frac{N}{t}\mathrm{Tr}\, V(M)}\, \mathrm{d}M.$$

This last notation comes from the exchange of the Taylor expansion of $\mathrm{e}^{-\frac{N}{t}\sum_i \mathrm{Tr}\, \delta V_i(M_i)}$ and the integral. Once this commutation is performed, the integral obtained corresponds to the formal expansion of the integral $\int \mathrm{e}^{-\frac{N}{t}\mathrm{Tr}\, V(M)}\, \mathrm{d}M$ around a saddle point $\tilde{M}$ solution of $V'(\tilde{M}) = 0$, which we choose as

$$\tilde{M} = \mathrm{diag}\left(\overbrace{\xi_1, \ldots, \xi_1}^{n_1}, \ldots, \overbrace{\xi_i, \ldots, \xi_i}^{n_i}, \ldots, \overbrace{\xi_d, \ldots, \xi_d}^{n_d} \right).$$

The integers n_i thus correspond to choosing the number of eigenvalues of the saddle matrix located at a particular solution of the saddle-point equation $V'(\xi_i) = 0$.

However, in general, the Taylor expansion and the integral do not commute, and the formal matrix integral is different from the usual convergent matrix integral:

$$\int_{\mathrm{formal}} \mathrm{e}^{-\frac{N}{t}\mathrm{Tr}\, V(M)}\, \mathrm{d}M \neq \int \mathrm{e}^{-\frac{N}{t}\mathrm{Tr}\, V(M)}\, \mathrm{d}M.$$

In fact, typically, convergent matrix integrals are obtained for V bounded from below, whereas formal matrix integrals have combinatorical interpretations for V unbounded from below.

Remark 5.1. The definition of the matrix integral not only depends on the potential (i.e. the coefficients of this polynomial) but also depends on the filling fractions

$$\epsilon_i := \frac{t n_i}{N}, \qquad i = 1, \ldots, d, \qquad \sum_{i=1}^{d} \epsilon_i = t.$$

The formal logarithm of $Z_{1\mathrm{MM}}$ is also a formal power series in t of the form

$$\ln Z_{1\mathrm{MM}} = \sum_{k=0}^{\infty} t^k \widetilde{A}_k,$$

and it can be seen from the general properties of polynomial moments of Gaussian matrix integrals (observation first made by 't Hooft [133]) that each coefficient $N^{-2}\widetilde{A}_k$ is a *polynomial* in $1/N^2$:

$$\widetilde{A}_k = N^2 \sum_{g=0}^{g_{\max}(k)} \widetilde{A}_k^{(g)} N^{-2g}.$$

Therefore, we may collect together the coefficients of given powers of N and define a formal power series

$$F_g = t^{2-2g} \sum_{k=1}^{\infty} \tilde{A}_k^{(g)} t^k.$$

We have

Theorem 5.1.

$$\ln Z_{1\mathrm{MM}} = \sum_{g=0}^{\infty} (N/t)^{2-2g} F_g.$$

This theorem is an equality between formal power series of t. This means that the coefficients in the small t power series expansions of both sides are the same. For a given power of t, the sum on the right-hand side is in fact a finite sum.

5.2.1. Loop equations. We may also define the following formal correlation functions:

$$W_n(x_1, \ldots, x_n) = \left\langle \mathrm{Tr} \frac{1}{x_1 - M} \ldots \mathrm{Tr} \frac{1}{x_n - M} \right\rangle_c,$$

where the subscript c means the cumulant and the notation $\mathrm{Tr}\frac{1}{x-M}$ stands for the formal series:

$$\mathrm{Tr} \frac{1}{x - M} \equiv \sum_{i=1}^{d} \sum_{k=0}^{\infty} \mathrm{Tr} \frac{(M_i - \xi_i \mathbf{1}_{n_i})^k}{(x - \xi_i)^{k+1}} \tag{5.2}$$

to be inserted in the integrand of equation (5.1). Again, $W_n(x_1, \ldots, x_n)$ is defined as a formal power series in t, whose coefficients are polynomial moments of Gaussian integrals. Moreover, one may note that the coefficient of t^k is a polynomial in $1/N$ and is a rational fraction of $x_1, \ldots, x_n$ with poles at ξ_i's.

Remark 5.2. Each coefficient of $W_n(x_1, \ldots, x_n)$ is a rational fraction of x_j's with poles at ξ_i's, and from equation (5.2), one sees that simple poles can appear only when $k = 0$ in equation (5.2), i.e. terms independent of M_i. This implies that the cumulants $W_n(x_1, \ldots, x_n)$ can have no simple poles when $n > 1$, and for W_1, the only residue is

$$\operatorname*{Res}_{x \to \xi_i} W_1(x)\, \mathrm{d}x = n_i, \tag{5.3}$$

and when $n > 1$,

$$\operatorname*{Res}_{x_1 \to \xi_i} W_n(x_1, x_2, \ldots, x_n)\, \mathrm{d}x_1 = 0. \tag{5.4}$$

Again, these two equalities are equalities between the coefficients of formal series of t.

We may collect together coefficients with the same power of N. This allows one to write

$$W_n(x_1, \ldots, x_n) = \sum_{g \geqslant 0} \left(\frac{N}{t}\right)^{2-2g-n} W_n^{(g)}(x_1, \ldots, x_n), \tag{5.5}$$

where each $W_n^{(g)}$ is a formal power series in t, whose coefficients are rational fractions of $x_1, \ldots, x_n$ with poles at ξ_i's. Equation (5.5) is an equality of formal power series of t.

For further convenience, we also define in a similar manner:

$$P_n(x_1; x_2, \ldots, x_n) = \left\langle \mathrm{Tr}\frac{V'(x_1) - V'(M)}{x_1 - M}\mathrm{Tr}\frac{1}{x_2 - M} \ldots \mathrm{Tr}\frac{1}{x_n - M}\right\rangle_c,$$

which is a polynomial in the variable x_1. Again, we may collect together coefficients with the same power of N^{-1} and define $P_n^{(g)}$ such that

$$P_n(x_1; x_2, \ldots, x_n) = \sum_{g \geqslant 0} \left(\frac{N}{t}\right)^{2-2g-n} P_n^{(g)}(x_1; x_2, \ldots, x_n).$$

Then, the Schwinger–Dyson equations imply

Theorem 5.2. *We have the loop equations, $\forall n, g$:*

$$\begin{aligned} W_{n+2}^{(g-1)}(x, x, J) &+ \sum_{h=0}^{g} \sum_{I \subset J} W_{1+|I|}^{(h)}(x, I) W_{1+n-|I|}^{(g-h)}(x, J \backslash I) \\ &+ \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{W_n^{(g)}(x, J\backslash\{x_j\}) - W_n^{(g)}(J)}{x - x_j} \\ &= V'(x) W_{n+1}^{(g)}(x, J) - P_{n+1}^{(g)}(x; J), \end{aligned} \tag{5.6}$$

where $J = \{x_1, \ldots, x_n\}$.

Proof. This theorem can be proved by integrating by parts the Gaussian integrals for each power of t; see [58]. It is called the Schwinger–Dyson equations, loop equations or sometimes Ward identities (cf [44, 46, 97, 118]). □

Also note that this theorem corresponds to the global Virasoro constraints of theorem 4.16.

Loop equations were initially used to find the topological expansion of the 1-matrix model in the special case of 1-cut [12] and also to the first-order 2-cuts [5].

5.2.2. Spectral curve. For $n = 0$ and $g = 0$, the loop equation (5.6) reduces to an algebraic equation for $W_1^{(0)}(x)$, sometimes known as the master loop equation:

$$\left(W_1^{(0)}(x)\right)^2 = V'(x) W_1^{(0)}(x) - P_1^{(0)}(x), \tag{5.7}$$

where $P_1^{(0)}(x)$ is a polynomial of x of degree at most $d - 1$. The one-point function is thus given by

$$W_1^{(0)}(x) = \frac{V'(x)}{2} - \sqrt{\frac{V'(x)^2}{4} - P_1^{(0)}(x)}.$$

We define

$$y = \frac{V'(x)}{2} - W_1^{(0)}(x) = \sqrt{\frac{V'(x)^2}{4} - P_1^{(0)}(x)},$$

and the master loop equation implies that the function y is a solution of the algebraic equation (hyperelliptical) $H_{1\mathrm{MM}}(x, y) = 0$ where

$$H_{1\mathrm{MM}}(x, y) := y^2 - \left(\frac{V'(x)}{2}\right)^2 + P_1^{(0)}(x),$$

which is called the *spectral curve* associated with the 1-Hermitian matrix model.

From equation (5.3), we see that if one chooses the $\mathcal{A}_i$-cycle to be a circle, independent of t, around ξ_i, then we have (order by order in t)

$$\frac{1}{2i\pi}\oint_{\mathcal{A}_i} W_1^{(0)}(x)\,dx = \frac{tn_i}{N} = \epsilon_i, \qquad i = 1, \dots, d.$$

This last equation gives d constraints, i.e. the same number as the coefficients of the polynomial $P_1^{(0)}$ (which is of degree $d-1$); therefore, it determines $P_1^{(0)}(x)$. Just by looking at the first terms in the small t expansion, one has

$$P_1^{(0)}(x) = \sum_{i=1}^{d} \epsilon_i \frac{V'(x)}{x-\xi_i} + O(t^2). \tag{5.8}$$

The data n_i/N are thus equivalent to the data of $P_1^{(0)}$. Note that $\epsilon_i = tn_i/N$ is of order $O(t)$ in the small t expansion.

In case some ϵ_i's are vanishing, we define $\bar{g}+1 =$ the number of non-vanishing ϵ_i's and assume that $\epsilon_1, \dots, \epsilon_{\bar{g}+1}$ are non-vanishing and $\epsilon_{\bar{g}+2}, \dots, \epsilon_d$ are zero.

Order by order in t, we have

$$y = \prod_{i=\bar{g}+2}^{d} (x - \xi_i - A_i(t)) \sqrt{\prod_{i=1}^{\bar{g}+1} ((x-\xi_i-B_i(t))^2 - 4C_i(t))}, \tag{5.9}$$

where $A_i(t)$, $B_i(t)$, $C_i(t)$ are formal power series of t. To the first orders,

$$A_i = \frac{4}{V''(\xi_i)} \sum_{j=1}^{\bar{g}+1} \frac{\epsilon_j}{\xi_i - \xi_j} + O(t^2),$$

$$B_i = \frac{1}{2V''(\xi_i)} \sum_{j\neq i} \frac{\epsilon_j}{\xi_i - \xi_j} \quad - \frac{\epsilon_i}{4} \frac{V'''(\xi_i)}{V''^2(\xi_i)} + O(t^2),$$

$$C_i = \frac{\epsilon_i}{V''(\xi_i)} + O(t^2).$$

Let us study the specificities of the 1-matrix model spectral curve $(\mathcal{L}_{1\mathrm{MM}}, x, y)$.

Genus of $\mathcal{L}_{1\mathrm{MM}}$. The Riemann surface $\mathcal{L}_{1\mathrm{MM}}$ has genus $\bar{g}$ lower than $d-1$:

$$\bar{g} \leqslant d-1.$$

Sheeted structure. The polynomial $H_{1\mathrm{MM}}(x, y)$ has degree 2 in y. This means that the embedding of $\mathcal{L}_{1\mathrm{MM}}$ is composed of two copies of the Riemann sphere, called sheets, glued by $\bar{g}+1$ cuts so that the resulting Riemann surface $\mathcal{L}_{1\mathrm{MM}}$ has genus $\bar{g}$. Each copy of the Riemann sphere corresponds to one particular branch of the solutions of the equation $H_{1\mathrm{MM}}(x, y) = 0$. Since there are only two sheets in involution, this spectral curve is said to be hyperelliptic. It also means that the application $z \to \overline{z}$ is globally defined since it is the map which exchanges both sheets:

$$y(\overline{z}) = -y(z).$$

Pole structure. The function $x(z)$ on the Riemann surface $\mathcal{L}_{1\mathrm{MM}}$ has two simple poles (call them α_+ and α_-), one in each sheet. Near $\alpha_\pm$, $y(z)$ behaves like

$$y(z) \underset{z\to\alpha_\pm}{\sim} \pm\frac{1}{2}V'(x(z)) \mp \frac{t}{x(z)} + O(1/x(z)^2).$$

5.2.3. The two-point function. For $n = 1$ and $g = 0$, the loop equation (5.6) reads as

$$\frac{\partial}{\partial x_1}\frac{W_1^{(0)}(x) - W_1^{(0)}(x_1)}{x - x_1} = \big(V'(x) - 2W_1^{(0)}(x)\big)W_2^{(0)}(x, x_1) - P_2^{(0)}(x; x_1),$$

i.e.

$$\begin{aligned} W_2^{(0)}(x, x_1) &= \frac{\frac{\partial}{\partial x_1}\frac{W_1^{(0)}(x) - W_1^{(0)}(x_1)}{x - x_1} + P_2^{(0)}(x; x_1)}{2y(x)} \\ &= -\frac{1}{2(x - x_1)^2} + \frac{\frac{1}{2}\frac{\partial}{\partial x_1}\frac{V'(x) - V'(x_1) + 2y(x_1)}{x - x_1} + P_2^{(0)}(x; x_1)}{2y(x)}. \end{aligned} \tag{5.10}$$

This equation shows that $W_2^{(0)}(x, x_1)$ is a meromorphic function of z and z_1 on the spectral curve $\mathcal{L}_{1\mathrm{MM}}$. It is a multivalued function of x, x_1, but it is a monovalued function in the variables z, z_1. Therefore, let us write

$$W_2^{(0)}(x(z), x(z_1))\,\mathrm{d}x(z)\,\mathrm{d}x(z_1) = \bar{\omega}_2^{(0)}(z, z_1),$$

where $\bar{\omega}_2^{(0)}(z, z_1)$ is a meromorphic form of z and z_1. It is clear from the first line of equation (5.10) that $\bar{\omega}_2^{(0)}(z, z_1)$ has no pole at $z = z_1$, but has a pole at $z = \bar{z}_1$. Moreover, since $\mathrm{d}x(z)/y(z)$ has no pole at branch points, we see that $\bar{\omega}_2^{(0)}(z, z_1)$ has no pole when z approaches a branch point. By looking at the behavior at large x, we see, from $\deg P_2^{(0)} \leqslant d - 2$, that $\bar{\omega}_2^{(0)}(z, z_1)$ has no pole at the two infinities $\alpha_\pm$.

From equation (5.10), it may seem that $\bar{\omega}_2^{(0)}(z, z_1)$ could have simple poles at the zeros of $y(z)$, but the residues are computed by equation (5.4) and they vanish. Thus, we see that the only possible pole of $\bar{\omega}_2^{(0)}(z, z_1)$ can be at $z = \bar{z}_1$.

Then, note that the second line of equation (5.10) is the sum of a term which is even under $z \to \bar{z}$ and a term which is odd under $z \to \bar{z}$. Since the sum of these two terms must have no pole at $z = z_1$, we see that the pole at $z = \bar{z}_1$ must be twice the pole of the even part. Therefore, we find that $\bar{\omega}_2^{(0)}(z, z_1)$ has a double pole at $z = \bar{z}_1$, with no residue, and no other pole. Moreover, equation (5.4) implies that on every $\mathcal{A}$-cycle, we have

$$\oint_{z \in \mathcal{A}_i} \bar{\omega}_2^{(0)}(z, z_1) = 0.$$

The only meromorphic differential having all these properties is the Bergmann kernel:

$$\bar{\omega}_2^{(0)}(z, z_1) = -B(z, \bar{z}_1) = B(z, z_1) - \frac{\mathrm{d}x(z)\,\mathrm{d}x(z_1)}{(x(z) - x(z_1))^2}$$

(we choose $\kappa = 0$).

5.2.4. Higher correlators. Similar to what we just did with $W_2^{(0)}$, we are going to compute every $W_n^{(g)}$ and relate it to the symplectic invariants of the curve $y(x)$.

First, note that the loop equations (5.6) imply recursively that each $W_n^{(g)}$ is in fact a meromorphic function on the spectral curve, and thus we prefer to rewrite:

Definition 5.2.

$$\omega_n^{(g)}(z_1, \ldots, z_n) = W_n^{(g)}(x(z_1), \ldots, x(z_n))\,\mathrm{d}x(z_1) \ldots \mathrm{d}x(z_n) + \delta_{n,2}\delta_{g,0}\frac{\mathrm{d}x(z_1)\,\mathrm{d}x(z_2)}{(x(z_1) - x(z_2))^2}.$$

Indeed, correlation functions $W_n^{(g)}$ are multivalued functions of the complex variable x whereas $\omega_n^{(g)}$ are monovalued n-forms on the spectral curve $\mathcal{L}$. Somehow, these forms are built to choose one particular branch of a solution of the master loop equation (5.7).

Structure of the form $\omega_n^{(g)}$. One clearly sees from loop equation (5.6) that $\omega_{n+1}^{(g)}(z, z_1, \dots, z_n)$ can possibly have poles only at branch points, or at coinciding points $x(z) = x(z_j)$, i.e. at $z = z_j$ or $z = \bar{z}_j$, or also at the zeros of $y(z)$.

From the degrees $\deg P_n^{(g)} \leqslant d - 2$, one can see that there is no pole at the infinities $\alpha_\pm$. Also, there is manifestly no pole at $z = z_j$, and one may note that $\omega_{n+1}^{(g)}$ is an odd function when $z \to \bar{z}$; thus it can also have no pole when $z = \bar{z}_j$.

The zeros of $y(z)$ are either the branch points or double points, which are, order by order in t, of the form (see equation (5.9)

$$\xi_i + A_i(t) + O(t^2), \qquad A_i(t) = O(t).$$

Let us assume by recursion on $2g + n$ that each $\omega_n^{(g)}$ has no poles at these double points. This is true for $\omega_1^{(0)}$ and $\omega_2^{(0)}$. Assume that it is true for every $\omega_{n'}^{(g')}$ with $2g' + n' \leqslant 2g + n$; let us prove it for $2g + n + 1$. From the loop equation (5.6), one sees that $\omega_{n+1}^{(g)}(z, z_1, \dots, z_n)$ could have at most a simple pole at such double points. But because of equations (5.4) and (5.3), the residue must vanish, and thus there is no pole.

Therefore, we obtain the following structure for $\omega_n^{(g)}$'s.

Lemma 5.1. *$\omega_n^{(g)}$ can have poles only at branch points when $2g + n \geqslant 3$.*

Moreover, we see from equations (5.4) and (5.3) that if $2g + n \geqslant 3$, we have

$$\oint_{x_1 \in \mathcal{A}_i} \omega_n^{(g)}(x_1, \dots, x_n) = 0. \tag{5.11}$$

5.2.5. Symplectic invariants. Let

$$\mathrm{d}S_{z_1,z_2}(z) = \int_{z_2}^{z_1} B(z, z')$$

be the third kind differential in z, having a simple pole at $z = z_1$ with residue $+1$, a simple pole of residue -1 at $z = z_2$ and no other poles, and normalized on $\mathcal{A}$-cycles:

$$\oint_{\mathcal{A}_i} \mathrm{d}S_{z_1,z_2} = 0. \tag{5.12}$$

The fact that it has only simple poles with residue $+1$ at z_1 allows one to write the Cauchy formula on the spectral curve (o being an arbitrary base point on $\mathcal{L}_{1\mathrm{MM}}$):

$$\omega_{n+1}^{(g)}(z, z_1, \dots, z_n) = \operatorname*{Res}_{z' \to z} \mathrm{d}S_{z',o}(z)\omega_{n+1}^{(g)}(z', z_1, \dots, z_n). \tag{5.13}$$

On the other hand, the differential form $\omega_{n+1}^{(g)}(z, z_1, \dots, z_n)$ has poles only at the branch points $z \to a_i$, and thus the Riemann bilinear identity tells us that

$$\begin{aligned}
&\operatorname*{Res}_{z' \to z} \mathrm{d}S_{z',o}(z)\omega_{n+1}^{(g)}(z', z_1, \dots, z_n) + \sum_i \operatorname*{Res}_{z' \to a_i} \mathrm{d}S_{z',o}(z)\omega_{n+1}^{(g)}(z', z_1, \dots, z_n) \\
&\quad = \sum_i \oint_{z' \in \mathcal{A}_i} B(z, z') \oint_{z' \in \mathcal{B}_i} \omega_{n+1}^{(g)}(z', z_1, \dots, z_n) \\
&\quad\quad - \sum_i \oint_{z' \in \mathcal{B}_i} B(z, z') \oint_{z' \in \mathcal{A}_i} \omega_{n+1}^{(g)}(z', z_1, \dots, z_n).
\end{aligned} \tag{5.14}$$

Due to equations (5.12) and (5.11), the right-hand side vanishes and thus

$$\operatorname*{Res}_{z'\to z} \mathrm{d}S_{z',o}(z)\omega_{n+1}^{(g)}(z', z_1, \ldots, z_n) = -\sum_i \operatorname*{Res}_{z'\to a_i} \mathrm{d}S_{z',o}(z)\omega_{n+1}^{(g)}(z', z_1, \ldots, z_n).$$

Finally, one can plug in the loop equation (5.6) and note that the polynomial $P_1^{(g)}$ has no pole at the branch points, and thus we find

$$\omega_{n+1}^{(g)}(z, z_1, \ldots, z_n) = \sum_i \operatorname*{Res}_{z'\to a_i} K(z, z')\Bigg[\omega_{n+2}^{(g-1)}(z', \bar{z'}, z1, \ldots, z_n) + \sum_{h=0}^{g} {\sum_{I\subset\{z_1,\ldots,z_n\}}}' \omega_{1+|I|}^{(h)}(z', I)\omega_{1+n-|I|}^{(g-h)}(\bar{z'}, \{z_1, \ldots, z_n\}\setminus I)\Bigg],$$

where $K(z, z')$ is the kernel:

$$K(z, z') = \frac{\mathrm{d}S_{\bar{z}',z'}(z)}{2(y(z') - y(\bar{z}'))\,\mathrm{d}x(z')}.$$

In other words,

Theorem 5.3. *$\omega_n^{(g)}(z_1, \ldots, z_n)$ and $F^{(g)}$ are, respectively, the correlators and symplectic invariants of the spectral curve $\mathcal{E}_{1\mathrm{MM}}$ of equation $H_{1\mathrm{MM}}(x, y) = 0$.*

To get this theorem, we also recover F_g's from theorem 4.3 or just by homogeneity; see [40] for details. For the 1-matrix model, F_0 has been known from the origin of random matrices; F_1 was first found in [12] for the 1-cut case and in [39] for the multicut case. The other F_g's were first found in [40].

5.2.6. Symplectic invariant's properties in the matrix model's language. Since the context of random matrices might be more familiar to some of the readers, let us translate some of the properties of the symplectic invariants in this language.

- *Moduli of the model.* A formal 1-matrix integral is defined by two types of data: the coefficients of the potential $V(x)$ and the filling fractions ϵ_i. These moduli correspond to the moduli of the spectral curve of the model as described in 4.3 since the coefficients of $\frac{V'(x)}{2} = \sum_{k=0}^{d-1} t_k x^k$ give the Taylor expansion of $y\,\mathrm{d}x$ around infinity and the filling fraction is the cycle integrals $\epsilon_i = \oint_{\mathcal{A}_i} y\,\mathrm{d}x$ entering the decomposition formula (4.5).
- *Loop operator.* One can interpret the definition of the one-point correlation function as the action of a differential operator on the formal matrix integral:

$$W_1(x) = \sum_{i=1}^{d} \epsilon_i \left[\sum_{k=0}^{\infty} \frac{1}{(x-\xi_i)^{k+1}} \frac{\partial}{\partial t_{k,i}}\right] \ln Z_{1\mathrm{MM}},$$

  where $t_{k,i} := \frac{V^{(k)}(\xi_i)}{k!}$ is the kth coefficient of the Taylor expansion of the potential near ξ_i:

$$\forall\, i = 1, \ldots, d, \qquad V(x) = \sum_{k=0}^{\infty} t_{k,i}(x-\xi_i)^k = \sum_{k=0}^{d} t_k x^k.$$

  This operator thus studies the derivatives with respect to all the moduli of the potential at once, by looking at its expansion around $x \to \infty$. It is usually referred to as the loop operator in the matrix model community and denoted by $\frac{\partial}{\partial V(x)}$, hence the notation of section 4.3. It should be remembered that the action of this differential operator was the basis of a first recursive method allowing one to compute all the correlation functions of the 1-Hermitian matrix model [12].

- *Virasoro constraints.* Using this loop operator, acting as a derivation with respect to the coefficients of the potential, one can rewrite the loop equation (5.6) as a set of differential equations for the partition function when $x \to \infty$. For example, for $n = 1$, after summing over all $g \geqslant 0$, one gets the equations

$$\forall n \geqslant -1, \qquad \left[\sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{k+n}} + \frac{1}{N^2} \sum_{a+b=n} \frac{\partial^2}{\partial t_a \partial t_b}\right],$$

which are the local Virasoro constraints at the poles studied in section 4.7.

5.3. 2-matrix model

The method of loop equations can also be used to solve the formal 2-matrix model. One of the main applications and reasons for introducing the 2-matrix model was the problem of counting Ising model configurations on random discrete surfaces or, in other words, bi-colored maps (see section 7); it was first introduced and solved by Kazakov [98]. It corresponds to a formal 2-matrix integral. It can be rephrased in terms of symplectic invariants too.

For this purpose, one generalizes the notion of formal matrix integral to integrals over two normal matrices.

Definition 5.3. *Let N be an integer and V_1 and V_2 two polynomial potentials:*

$$V_1(x) = -\sum_{k=2}^{d_1+1} \frac{t_k}{k} x^k, \qquad V_2(y) = -\sum_{k=2}^{d_2+1} \frac{\tilde{t}_k}{k} y^k.$$

Let $d = d_1 d_2$ and $\vec{n}$ be a d-partition of N:

$$\vec{n} := \{n_1, n_2, \ldots, n_d\} \qquad \text{such that} \quad \sum_{i=1}^{d} n_i = N.$$

Let $\{(\xi_i, \eta_i)\}_{i=1}^{d_1 d_2}$ be the $d = d_1 d_2$ solutions of the system of equations

$$\begin{cases} V_1'(\xi_i) = \eta_i \\ V_2'(\eta_i) = \xi_i. \end{cases}$$

One defines the non-quadratic part of the Taylor expansions of the potentials around these saddle points:

$$\delta V_{1,i}(x) = V_1(x) - V_1(\xi_i) - \frac{V_1''(\xi_i)}{2}(x - \xi_i)^2$$

and

$$\delta V_{2,i}(y) = V_2(y) - V_2(\eta_i) - \frac{V_2''(\eta_i)}{2}(y - \eta_i)^2.$$

For all l, one defines the polynomial in t:

$$\begin{aligned} \sum_{k=l/2}^{ld} A_{k,l} t^k = \frac{(-1)^l N^l}{l! t^l} \int \mathrm{d}M_1 \ldots \mathrm{d}M_d \, \mathrm{d}\tilde{M}_1 \ldots \mathrm{d}\tilde{M}_d \left(\sum_i \operatorname{Tr} \delta V_{1,i}(M_i) + \delta V_{2,i}(\tilde{M}_i)\right)^l \\ \times \prod_{i=1}^{d} \mathrm{e}^{-\frac{N}{t}\left(\operatorname{Tr}\frac{V_1''(\xi_i)}{2}(M_i - \xi_i \mathbf{1}_{n_i})^2 + \frac{V_2''(\eta_i)}{2}(\tilde{M}_i - \eta_i \mathbf{1}_{n_i})^2 - (M_i - \xi_i \mathbf{1}_{n_i})(\tilde{M}_i - \eta_i \mathbf{1}_{n_i})\right)} \\ \times \prod_{i>j} \det(M_i \otimes \mathbf{1}_{n_j} - \mathbf{1}_{n_i} \otimes M_j) \prod_{i>j} \det\left(\tilde{M}_i \otimes \mathbf{1}_{n_j} - \mathbf{1}_{n_i} \otimes \tilde{M}_j\right) \end{aligned} \tag{5.15}$$

as a Gaussian integral over Hermitian matrices M_i and $\tilde{M}_i$ of size $n_i \times n_i$.

The formal 2-matrix model partition function is then defined as a formal power series in t (cf [64]):

$$Z_{2\mathrm{MM}} := \sum_{k=0}^{\infty} t^k \left(\sum_{j=0}^{2k} A_{k,j} \right). \tag{5.16}$$

As in the 1-matrix model, one uses the notation

$$Z_{2\mathrm{MM}} = \int_{\text{formal}} \mathrm{e}^{-\frac{N}{t}\mathrm{Tr}\, V_1(M_1)+V_2(M_2)-M_1M_2}\, \mathrm{d}M_1\, \mathrm{d}M_2.$$

One is also interested in the formal logarithm of the partition function: the free energy

$$F_{2\mathrm{MM}} := \ln Z_{2\mathrm{MM}},$$

which has a topological expansion (again due to 't Hooft's observation [133])

$$F_{2\mathrm{MM}} = \sum_{g\geqslant 0} \left(\frac{N}{t}\right)^{2-2g} F_g,$$

where each F_g is a formal power series of t.

5.3.1. Loop equations and spectral curve. As in the 1-matrix model (cf equation (5.2), one also defines correlation functions by

$$W_{k,l}(x_1,\ldots,x_k,y_1,\ldots,y_l) := \left\langle \prod_{i=1}^{k} \mathrm{Tr}\frac{1}{x_i - M_1} \prod_{i=1}^{l} \mathrm{Tr}\frac{1}{y_i - M_2} \right\rangle_c$$

denoted as the non-mixed correlation functions[16]. These correlation functions also admit a topological expansion

$$W_{k,l}(x_1,\ldots,x_k,y_1,\ldots,y_l) = \sum_{g\geqslant 0} \left(\frac{N}{t}\right)^{2-2g-k} W_{k,l}^{(g)}(x_1,\ldots,x_k,y_1,\ldots,y_l).$$

One also needs the polynomials in x_1 and y:

$$P_n(x_1,y;x_2,\ldots,x_n) = \left\langle \mathrm{Tr}\left(\frac{V_1'(x_1)-V_1'(M_1)}{x_1-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\right)\prod_{i=2}^{n}\mathrm{Tr}\frac{1}{x_i-M_1}\right\rangle_c$$

as well as

$$U_n(x_1,y;x_2,\ldots,x_n) = \left\langle \mathrm{Tr}\left(\frac{1}{x_1-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\right)\prod_{i=2}^{n}\mathrm{Tr}\frac{1}{x_i-M_1}\right\rangle_c,$$

which are polynomials in y only.

[16] There exist more general correlation functions mixing the two types of matrices M_1 and M_2 inside the same trace, for example $\langle \mathrm{Tr}(M_1^k M_2^l)\rangle$, but their study is too far from the main topic of this review to be treated here. It is studied in [68, 69].

5.3.2. Loop equations. Loop equations proceed from integration by parts in the formal matrix integral (i.e. integration by parts in each Gaussian integral for each power of t) or by writing invariance under changes of variables, i.e. they are Schwinger–Dyson equations.

The loop equations for the 2-matrix model were first studied by Staudacher [131], and then written in a more concise form in [59, 62]. The loop equations for the 2-matrix model are (where $J = \{x_1, \dots, x_n\}$)

$$\begin{aligned}\frac{N}{t}(y - V_1'(x))U_{n+1}(x, y; J) + U_{n+2}(x, y; x, J) + \sum_{I\subset J} W_{1+|I|,0}(x, I)U_{1+n-|I|}(x, y; J/I)\\ + \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\frac{U_n(x, y; J/\{x_j\}) - U_n(x_j, y; J/\{x_j\})}{x - x_j}\\ = -\frac{N}{t}P_{n+1}(x, y; J) + \frac{N^2}{t^2}. \end{aligned} \tag{5.17}$$

And then, identifying the coefficients of polynomials of $1/N^2$ for each power of t, we have

$$\big(y - V_1'(x) + W_{1,0}^{(0)}(x)\big)U_1^{(0)}(x, y) = (V_2'(y) - x)W_{1,0}^{(0)}(x) - P_1^{(0)}(x, y) + 1$$

and for $(g, n) \neq (0, 0)$,

$$\begin{aligned}\big(y - V_1'(x) + W_{1,0}^{(0)}(x)\big)U_{n+1}^{(g)}(x, y; J) + W_{n+1,0}^{(g)}(x, J)U_1^{(0)}(x, y)\\ + U_{n+2}^{(g-1)}(x, y; x, J) + \sum_{h=0}^{g}{\sum_{I\subset J}}' W_{1+|I|,0}^{(h)}(x, I)U_{1+n-|I|}^{(g-h)}(x, y; J/I)\\ + \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\frac{U_n^{(g)}(x, y; J/\{x_j\}) - U_n^{(g)}(x_j, y; J/\{x_j\})}{x - x_j}\\ = -P_{n+1}^{(g)}(x, y; J), \end{aligned} \tag{5.18}$$

where $\sum_h \sum_I'$ means that we exclude the terms $(h, I) = (0, \emptyset)$ and (g, J).

5.3.3. Spectral curve. Consider the first loop equation:

$$\big(y - V_1'(x) + W_{1,0}^{(0)}(x)\big)U_1^{(0)}(x, y) = (V_2'(y) - x)W_{1,0}^{(0)}(x) - P_1^{(0)}(x, y) + 1.$$

It is valid for any x and y, and in particular we may choose

$$y = y(x) = V_1'(x) - W_{1,0}^{(0)}(x).$$

Since $U_1^{(0)}(x, y)$ is a polynomial in y, it cannot have a pole at this value of $y = y(x)$, and thus, for this value of $y = V_1'(x) - W_{1,0}^{(0)}(x)$, we have the algebraic equation

$$H_{2\mathrm{MM}}(x, y(x)) = (V_2'(y(x)) - x)(V_1'(x) - y(x)) - P_1^{(0)}(x, y(x)) + 1 = 0$$

known as the 2-matrix model spectral curve.

Let us study the specificities of the 2-matrix model spectral curve $(\mathcal{L}_{2\mathrm{MM}}, x, y)$ (see [100]).

Genus of $\mathcal{L}_{2\mathrm{MM}}$. The Riemann surface $\mathcal{L}_{2\mathrm{MM}}$ has genus $\bar{g}$ lower than $d_1d_2 - 1$:

$$\bar{g} \leqslant d_1d_2 - 1.$$

Sheeted structure. The polynomial $H_{2\mathrm{MM}}(x, y)$ has degree $d_2 + 1$ (respectively $d_1 + 1$) in y (respectively x). This means that the embedding of $\mathcal{L}_{2\mathrm{MM}}$ as a branch-covering of the $\mathbb{P}^1x$-plane (respectively y-plane) is composed of $d_2 + 1$ (respectively $d_1 + 1$) copies of the Riemann

sphere $\mathbb{P}^1$, called x-sheets (respectively y-sheets), glued by cuts, so that the resulting Riemann surface $\mathcal{L}_{2\mathrm{MM}}$ has genus $\bar{g}$. Each copy of the Riemann sphere corresponds to one particular branch of the solutions of the equation $H_{2\mathrm{MM}}(x, y)$ in y (respectively x).

Since there are d_2+1 x-sheets, this means that there are d_2+1 points in $\mathcal{L}_{2\mathrm{MM}}$ corresponding to the same value of x. We write

$$x(z^i) = x(z), \qquad i = 0, \dots, d_2. \tag{5.19}$$

We will take the convention that $z^0 = z$.

Branch points are zeros of $\mathrm{d}x$, and they are also places where two sheets merge $z \to z^i$ for some i. By convention, we call this other point z^i, and thus $\bar{z}$ is one of z^i's. In general, $\bar{z}$ is not globally defined but only defined locally around the branch points.

Pole structure of the functions $x(z)$ and $y(z)$. The function $x(z)$ (respectively $y(z)$) on the Riemann surface $\mathcal{L}_{2\mathrm{MM}}$ has two poles: one of degree 1 (respectively degree d_1) at a pole called ∞_x and one of degree d_2 (respectively degree 1) at a pole called ∞_y. It means that $d_2 x$-sheets merge at ∞_y, and only one x-sheet contains ∞_x alone.

Near ∞_x, a local parameter is $1/x$, and we have

$$y(z) \underset{z\to\infty_x}{\sim} V_1'(x(z)) - \frac{t}{x(z)} + O(1/x(z)^2).$$

Near ∞_y, a local parameter is $1/y$, and we have

$$x(z) \underset{z\to\infty_y}{\sim} V_2'(y(z)) - \frac{t}{y(z)} + O(1/y(z)^2).$$

According to section 4.7, the fact that we have two poles means that the τ function built from the symplectic invariants of this curve is the τ function of the (1 + 1) KP hierarchy[17].

5.3.4. Preliminaries to the solution of loop equations. As in the preceding section, we promote the correlation functions to differential forms on the spectral curve to make them monovalued.

Definition 5.4.

$$\omega_n^{(g)}(z_1, \dots, z_n) = W_{n,0}^{(g)}(x(z_1), \dots, x(z_n))\, \mathrm{d}x(z_1) \dots \mathrm{d}x(z_n) + \delta_{n,2}\delta_{g,0} \frac{\mathrm{d}x(z_1)\, \mathrm{d}x(z_2)}{(x(z_1) - x(z_2))^2}.$$

Exactly like in the 1-matrix formal model, the very definition of the model as a formal series in t implies that each $\omega_n^{(g)}$ with $2g+n \geqslant 3$ is a meromorphic n-form with poles only at the branch points (i.e. the zeros of $\mathrm{d}x$) and with vanishing $\mathcal{A}$-cycle integrals. The only exception is $\omega_2^{(0)}(z_1, z_2)$ which can have a pole only at $z_1 = z_2$ and is found to be the Bergmann kernel (see [42]):

$$\omega_2^{(0)}(z_1, z_2) = B(z_1, z_2).$$

Before proving that the solution of loop equations for $\omega_n^{(g)}$'s is the symplectic invariant's correlators, we need a small lemma. Consider the 'full spectral curve':

$$E(x, y) = (V_1'(x) - y)(V_2'(y) - x) - \frac{t}{N} P_1(x, y) + 1,$$

where $P_1(x, y) = \sum_g (N/t)^{1-2g} P_1^{(g)}(x, y)$. We also consider its descendants:

$$E_{n+1}(x, y; z_1, \dots, z_n) = \delta_{n,0}(V_1'(x) - y)(V_2'(y) - x) - \frac{t}{N} P_{n+1}(x, y; z_1, \dots, z_n) + \delta_{n,0}.$$

[17] Actually, one should take d_1 and d_2 arbitrary large to obtain the (1+1) KP τ function, the times of the hierarchy being given by the coefficients of the Laurent expansions of $y\,\mathrm{d}x$ and $x\,\mathrm{d}y$ around ∞_x and ∞_y respectively.

We have

Lemma 5.2.

$$E_{n+1}(x(z), y; z_1, \dots, z_n) = -\tilde{t}_{d_2+1} \text{“}\left\langle \prod_{i=0}^{d_2} \left(y - V_1'(x(z^i)) + \frac{t}{N} \mathrm{Tr} \frac{1}{x(z^i) - M_1} \right) \times \prod_{j=1}^{n} \mathrm{Tr} \frac{1}{x(z_j) - M_1} \right\rangle_{c,\{x_1,\dots,x_n\}} \text{”} \tag{5.20}$$

and

$$\frac{t}{N} U_{n+1}(x(z), y) - \delta_{n,0}(V_2'(y) - x(z)) = -\tilde{t}_{d_2+1} \text{“}\left\langle \prod_{i=1}^{d_2} \left(y - V_1'(x(z^i)) + \frac{t}{N} \mathrm{Tr} \frac{1}{x(z^i) - M_1} \right) \times \prod_{j=1}^{n} \mathrm{Tr} \frac{1}{x(z_j) - M_1} \right\rangle_{c,\{x_1,\dots,x_n\}} \text{”}, \tag{5.21}$$

where $\tilde{t}_{d_2+1}$ is the leading coefficient of $V_2'(y)$, z^i are the preimages of $x(z)$ (see equation (5.19)), the subscript $_{c,\{x_1,\dots,x_n\}}$ means that we take the connected part with respect to the $\mathrm{Tr}\frac{1}{x(z_j)-M_1}$ terms, but not the $\mathrm{Tr}\frac{1}{x(z^i)-M_1}$ terms, and the inverted comas “$\langle . \rangle$” mean that every time one encounters a two-point function in the cumulant expansion, one replaces it by[18]

$$W_{2,0}(x, x') := \left\langle \mathrm{Tr} \frac{1}{x - M_1} \mathrm{Tr} \frac{1}{x' - M_1} \right\rangle + \frac{1}{(x - x')^2}.$$

For example, formula (5.20), for $n = 1$, reads to the first subleading order in t/N:

$$\begin{aligned} P_1^{(1)}(x, y; z_1) = {} & \tilde{t}_{d_2+1} \sum_{j=0}^{d_2} W_{2,0}^{(1)}(z^j, z_1) \prod_{i \neq j, i=0}^{d_2} (y - y(z^j)) \\ & + \tilde{t}_{d_2+1} \sum_{j \neq k=0}^{d_2} W_{2,0}^{(0)}(z^j, z_1) W_{1,0}^{(1)}(z^k) \prod_{i \neq j,k,i=0}^{d_2} (y - y(z^j)) \\ & + \tilde{t}_{d_2+1} \sum_{j \neq k=0}^{d_2} W_{3,0}^{(0)}(z^j, z^k, z_1) \prod_{i \neq j,k,i=0}^{d_2} (y - y(z^j)) \\ & + \tilde{t}_{d_2+1} \sum_{j \neq k \neq l=0}^{d_2} W_{2,0}^{(0)}(z^j, z_1) W_{2,0}^{(0)}(z^k, z^l) \prod_{i \neq j,k,l,i=0}^{d_2} (y - y(z^j)). \end{aligned}$$

Formula (5.21) would be almost the same, but with the indices $i, j, k, l \geqslant 1$ instead of $\geqslant 0$.

This lemma was proved in [41] and relies on the fact that the loop equation (5.18) has a unique solution admitting a topological expansion. The way to prove this lemma mostly follows from the Lagrange interpolation formula for polynomials as well as the Cauchy residue formula on $\mathcal{L}_{2\mathrm{MM}}$.

[18] Note that this notation recalls the notation ‘det’ in theorem 4.1.

Since $E_{n+1}(x, y; z_1, \dots, z_n)$ is a polynomial of y of degree $\geqslant d_2 + 1$, by expanding $E_{n+1}(x, y; z_1, \dots, z_n)$ in powers of y, this lemma gives $d_2 + 1$ equations. In particular, the term in y^{d_2} gives (if $2g + n \geqslant 2$)

$$\sum_{i=0}^{d_2} \omega_{n+1,0}^{(g)}(z^i, z_1, \dots, z_n) = 0. \tag{5.22}$$

The term in y^{d_2-1} gives a bilinear equation in the correlation functions:

$$\sum_{i\neq j}^{d_2} \omega_{n+2,0}^{(g-1)}(z^i, z^j, J) + \sum_{h=0}^{g} {\sum_{I\subset J}}' \omega_{1+|I|,0}^{(h)}(z^i, I)\omega_{1+n-|I|,0}^{(g-h)}(z^j, J/I)$$
$$= \sum_{i\neq j} y(z^i)\omega_{n+1,0}^{(g)}(z^j, J) + y(z^j)\omega_{n+1,0}^{(g)}(z^i, J) - f_n^{(g)}(x(z), J)\,\mathrm{d}x(z)^2, \tag{5.23}$$

where $f_n^{(g)}(x(z), J)$ is a rational function of $x(z)$ with no pole when z approaches a branch point.

5.3.5. Solution of loop equations and symplectic invariants. Let us write the Cauchy formula, exactly like for the 1-matrix equation (5.13),

$$\omega_{n+1}^{(g)}(z_0, J) = -\operatorname*{Res}_{z\to z_0} \mathrm{d}S_{z,0}(z_0)\omega_{n+1}^{(g)}(z, J)$$
$$= \sum_i \operatorname*{Res}_{z\to a_i} \mathrm{d}S_{z,0}(z_0)\omega_{n+1}^{(g)}(z, J), \tag{5.24}$$

where we have moved the integration contours using Riemann bilinear identity like for the 1-matrix model (5.14).

Now, note that near a branch point a_i, $\omega_{n+1,0}^{(g)}(z, J)$ and $\omega_{n+1,0}^{(g)}(\bar{z}, J)$ have a pole at $z = a_i$ and all $\omega_{n+1,0}^{(g)}(z^i, J)$, such that $z^i \neq z, \bar{z}$, have no pole at $z \to a_i$. According to equation (5.22), we have

$$\omega_{n+1,0}^{(g)}(z, J) + \omega_{n+1,0}^{(g)}(\bar{z}, J) = \text{regular}$$

and from equation (5.23), we have near a_i

$$(y(z) - y(\bar{z}))\omega_{n+1,0}^{(g)}(z, J) = \omega_{n+2,0}^{(g-1)}(z, \bar{z}, J) + \sum_{h=0}^{g} {\sum_{I\subset J}}' \omega_{1+|I|,0}^{(h)}(z, I)\omega_{1+n-|I|,0}^{(g-h)}(\bar{z}, J/I)$$
$$+ \text{regular}. \tag{5.25}$$

Inserting this last equation into the Cauchy formula (5.24), we find

$$\omega_{1+n}^{(g)}(z_0, J) = \sum_i \operatorname*{Res}_{z\to a_i} K(z_0, z)\left[\omega_{n+2,0}^{(g-1)}(z, \bar{z}, J) + \sum_{h=0}^{g} {\sum_{I\subset J}}' \omega_{1+|I|,0}^{(h)}(z, I)\omega_{1+n-|I|,0}^{(g-h)}(\bar{z}, J/I)\right],$$

where

$$K(z_0, z) = \frac{\mathrm{d}S_{\bar{z},z}(z_0)}{2(y(z) - y(\bar{z}))\,\mathrm{d}x(z)}.$$

This gives the theorem

Theorem 5.4. *$\omega_n^{(g)}(z_1, \dots, z_n)$ and F_g are the correlators and symplectic invariants of the spectral curve $(\mathcal{L}_{2\mathrm{MM}}, x, y)$ respectively.*

Here we have only briefly sketched the proof of [41], and we refer the reader to details there, in particular for finding F_g's.

F_0 was found for example in [23, 107], F_1 was found in [59–61] and the other F_g's in [41].

Remark 5.3. The correlation functions $\omega_n^{(g)}$ which we consider here are expectation values of traces of only the matrix M_1 (in terms of combinatorics of maps of section 7, they are generating functions for bicolored maps, whose boundaries are of color 1 only). There exists a generalization of symplectic invariants for all other possible expectation values, i.e. all possible boundary conditions, but this is largely outside of the scope of this review. We refer the reader to [69] for further details.

An obvious remark is that colors 1 and 2, i.e. functions x and y, play similar roles. We can obtain generating functions for bicolored maps, whose boundaries are of color 2 only, by just exchanging the roles of x and y, i.e. by computing residues at the zeros of $\mathrm{d}y$.

In particular we may compute generating functions for bicolored maps with no boundaries (i.e. F_g's), with either x or y. In other words, F_g's are unchanged if we exchange x and y. This is a special case of the symplectic invariance property $F_g(\mathcal{L}, x, y) = F_g(\mathcal{L}, y, x)$.

In fact, the general proof of symplectic invariance consists in defining some mixed generating functions for bicolored maps, whose boundaries are bicolored; it was done in [71].

5.4. Chain of matrices in an external field

Another matrix model which can be solved with the same techniques is the chain of matrices matrix model.

Consider the model of an arbitrary long open chain of matrices in an external field, which includes the 1- and 2-matrix models as particular cases.

Consider m potentials $V_k(x) = -\sum_{j=2}^{d_k+1} \frac{t_{k,j}}{j} x^j, k = 1, \dots, m$. The formal chain of matrices matrix integral is

$$Z_{\text{chain}} = \int_{\text{formal}} \mathrm{e}^{-\frac{N}{t}\mathrm{Tr}\left(\sum_{k=1}^m V_k(M_k) - \sum_{k=1}^m c_{k,k+1} M_k M_{k+1}\right)} \mathrm{d}M_1 \dots \mathrm{d}M_m = \mathrm{e}^{\sum_g (N/t)^{2-2g} F_g},$$

where the integral is a formal integral in the sense of the preceding sections and M_{m+1} is a constant given diagonal matrix $M_{m+1} = \Lambda$ with s distinct eigenvalues λ_i with multiplicities l_i:

$$M_{m+1} = \Lambda = \mathrm{diag}\left(\overbrace{\lambda_1, \dots, \lambda_1}^{l_1}, \dots, \overbrace{\lambda_i, \dots, \lambda_i}^{l_i}, \dots, \overbrace{\lambda_s, \dots, \lambda_s}^{l_s}\right)$$

with $\sum_i l_i = N$.

Also note that we may choose $c_{m,m+1} = 1$ since it can be reabsorbed as a rescaling of Λ.

Once again, in the definition of the formal integral, one has to choose around which saddle point one expands. Saddle points are solutions of

$$\forall k = 1, \dots, m, \qquad V_k'(\xi_k) = c_{k-1,k}\xi_{k-1} + c_{k,k+1}\xi_{k+1}, \qquad \exists j, \xi_{m+1} = \lambda_j.$$

This system is an algebraic equation with $D = s d_1 d_2 \dots d_m$ solutions.

Therefore, the choice of a saddle point is encoded in the choice of a set of filling fractions:

$$\epsilon_i = T\frac{n_i}{N},$$

for $i = 1, \dots, D$ with $D = d_1 d_2 \dots d_m s$ and n_i arbitrary integers satisfying

$$\sum_i n_i = N.$$

5.4.1. Definition of the correlation functions. The loop equations of the chain of matrices were derived in [62, 63] and they require the definition of several quantities, as follows:

For convenience, we introduce in the sense of equation (5.2)

$$G_i(x_i) := \frac{1}{x_i - M_i}.$$

We also consider the minimal polynomial of Λ, such that $S(\Lambda) = 0$, i.e.

$$S(z) = \prod_{i=1}^{s}(z - \lambda_i),$$

and we introduce the following polynomial in z:

$$Q(z) = \frac{1}{c_{n,n+1}} \frac{S(z) - S(\Lambda)}{z - \Lambda}.$$

We also define the polynomials $f_{i,j}(x_i, \ldots, x_j)$ by $f_{i,j} = 0$ if $j < i - 1$, $f_{i,i-1} = 1$, and

$$f_{i,j}(x_i, \ldots, x_j) = \det \begin{pmatrix} V_i'(x_i) & -c_{i,i+1}x_{i+1} & & 0 \\ -c_{i,i+1}x_i & V_{i+1}'(x_{i+1}) & \ddots & \\ & \ddots & \ddots & -c_{j-1,j}x_j \\ 0 & & -c_{j-1,j}x_{j-1} & V_j'(x_j) \end{pmatrix}$$

if $j \geqslant i$. They satisfy the recursion

$$c_{i-1,i} f_{i,j}(x_i, \ldots, x_j) = V_i'(x_i) f_{i+1,j}(x_{i+1}, \ldots, x_j) - c_{i,i+1}x_i x_{i+1} f_{i+2}(x_{i+2}, \ldots, x_j).$$

Let us then define the correlation functions and auxiliary functions:

$$W_0(x) = \langle \operatorname{Tr} G_1(x) \rangle.$$

For $i = 2, \ldots, m$, we define

$$W_i(x_1, x_i, \ldots, x_m, z) = \operatorname{Pol}_{x_i,\ldots,x_m} f_{i,m}(x_i, \ldots, x_m) \langle \operatorname{Tr}(G_1(x_1)G_i(x_i)\ldots G_m(x_m)Q(z)) \rangle,$$

which is a polynomial in variables $x_i, \ldots, x_m, z$, but not in x_1. For $i = 1$, we define

$$W_1(x_1, x_2, \ldots, x_m, z) = \operatorname{Pol}_{x_1,\ldots,x_m} f_{1,m}(x_1, \ldots, x_m) \langle \operatorname{Tr}(G_1(x_1)G_2(x_2)\ldots G_m(x_m)Q(z)) \rangle,$$

which is a polynomial in all variables.

We also define

$$W_{i;1}(x_1, x_i, \ldots, x_m, z; x_1') = \operatorname{Pol}_{x_i,\ldots,x_m} f_{i,m}(x_i, \ldots, x_m) \langle \operatorname{Tr}(G_1(x_1')) \operatorname{Tr}(G_1(x_1)G_i(x_i)\ldots G_m(x_m)Q(z)) \rangle_c.$$

All these functions admit a topological expansion, for example

$$W_0 = \sum_g (N/t)^{1-2g} W_0^{(g)}, \qquad W_1 = \sum_g (N/t)^{1-2g} W_1^{(g)},$$
$$W_i = \sum_g (N/t)^{1-2g} W_i^{(g)} \qquad \text{and} \qquad W_{i;1} = \sum_g (N/t)^{-2g} W_{i;1}^{(g)}.$$

5.4.2. Loop equations and spectral curve. In this model, the master loop equation reads as [62, 63]

$$W_{2;1}(x_1,\dots,x_{m+1};x_1)+(c_{1,2}x_2-V_1'(x_1)+\frac{t}{N}W_0(x_1))\left(\frac{t}{N}W_2(x_1,\dots,x_{m+1})-S(x_{m+1})\right)$$
$$=-\frac{t}{N}W_1(x_1,\dots,x_{m+1})+(V_1'(x_1)-c_{1,2}x_2)S(x_{m+1})$$
$$+\frac{t}{N}\sum_{i=2}^{m}(V_i'(x_i)-c_{i-1,i}x_{i-1}-c_{i,i+1}x_{i+1})W_{i+1}(x_1,x_i,\dots,x_{m+1}). \tag{5.26}$$

This equation is valid for any set of variables $x_1, x_2, \dots, x_{m+1}$; however, it can be simplified by choosing special values for these variables, in particular values for which the last terms on the rhs vanish. For this purpose, one defines some $\hat{x}_i(x_1, x_2)$ as functions of the first two variables x_1 and x_2, as follows:

$$\hat{x}_1(x_1,x_2)=x_1, \qquad \hat{x}_2(x_1,x_2)=x_2,$$

and for $i = 2, \dots, m$,

$$c_{i,i+1}\hat{x}_{i+1}(x_1,x_2)=V_i'(\hat{x}_i(x_1,x_2))-c_{i-1,i}\hat{x}_{i-1}(x_1,x_2). \tag{5.27}$$

Choosing $x_i = \hat{x}_i(x_1, x_2)$ reduces the master loop equation to an equation in x_1 and x_2:

$$\widehat{W}_{2;1}(x_1,x_2;x_1)+\frac{t}{N}\left(c_{1,2}x_2-Y(x_1)\right)\widehat{U}(x_1,x_2)=\widehat{E}(x_1,x_2),$$

where

$$Y(x)=V_1'(x)-\frac{t}{N}W_0(x), \qquad \widehat{U}(x_1,x_2)=W_2(x_1,x_2,\hat{x}_3,\dots,\hat{x}_{m+1})-\frac{N}{t}S(\hat{x}_{m+1}),$$

$$\widehat{W}_{2;1}(x_1,x_2;x_1)=W_{2;1}(x_1,x_2,\hat{x}_3,\dots,\hat{x}_{m+1};x_1)$$

and

$$\widehat{E}(x_1,x_2)=-\frac{t}{N}\widehat{W}_1(x_1,x_2)+(V_1'(x_1)-c_{1,2}x_2)\widehat{S}(x_1,x_2)$$

with

$$\widehat{S}(x_1,x_2)=S(\hat{x}_{m+1}), \qquad \widehat{W}_1(x_1,x_2)=W_1(x_1,x_2,\hat{x}_3,\dots,\hat{x}_{m+1}).$$

Note that $\widehat{W}_1(x_1, x_2)$ and thus $\widehat{E}(x_1, x_2)$ are polynomials in both x_1 and x_2.

Finally, the leading order in the topological expansion gives

$$\widehat{E}^{(0)}(x_1,x_2)=(c_{1,2}x_2-Y^{(0)}(x_1))\widehat{U}^{(0)}(x_1,x_2). \tag{5.28}$$

We may note that this equation is more or less the same as in the 2-matrix model, and it is solved in the same way.

Again, this equation is valid for any x_1 and x_2, and if we choose x_2 such that $c_{1,2}x_2 = Y^{(0)}(x_1)$, we get

$$H_{\text{chain}}(x_1,x_2):=\widehat{E}^{(0)}(x_1,x_2)=0. \tag{5.29}$$

This algebraic equation is the spectral curve of our model.

Study of the spectral curve. The algebraic plane curve $H_{\text{chain}}(x_1, x_2) = 0$ can be parametrized by a variable z living on a compact Riemann surface $\mathcal{L}_{\text{chain}}$ of some genus $\bar{g}$ and two meromorphic functions $x_1(z)$ and $x_2(z)$ on it. Let us study it in greater details.

Genus of $\mathcal{L}_{\text{chain}}$. The Riemann surface $\mathcal{L}_{\text{chain}}$ has genus $\bar{g}$ lower than $D - s$:

$$\bar{g} \leqslant D - s,$$

where $D = sd_1 \dots d_m$.

Sheeted structure. The polynomial $H_{2\text{MM}}(x_1, x_2)$ has degree $1 + \frac{D}{d_1}$ (respectively $d_1 + \frac{D}{d_1 d_2}$) in x_2 (respectively x_1). This means that the embedding of $\mathcal{L}_{\text{chain}}$ is composed of $1 + \frac{D}{d_1}$ (respectively $d_1 + \frac{D}{d_1 d_2}$) copies of the Riemann sphere, called x_1-sheets (respectively x_2-sheets), glued by cuts so that the resulting Riemann surface $\mathcal{L}_{\text{chain}}$ has genus $\bar{g}$. Each copy of the Riemann sphere corresponds to one particular branch of the solutions of the equation $H_{\text{chain}}(x_1, x_2) = 0$ in x_2 (respectively x_1).

Pole structure. In the preceding cases (1- and 2-matrix models), one was interested in the pole structure of only two functions x and y on the spectral curve. In the case of the chain of matrices, the problem is slightly richer since one can consider not only the meromorphic functions x_1 and x_2, but also all $x_i(p) := \hat{x}_i(x_1(p), x_2(p))$ as meromorphic functions on $\mathcal{L}_{\text{chain}}$. Their negative divisors are given by

$$[x_k(p)]_- = -r_k \infty - s_k \sum_{i=1}^{s} \hat{\lambda}_i,$$

where ∞ is the only point of $\mathcal{L}_{\text{chain}}$ where x_1 has a simple pole, $\hat{\lambda}_i$ are the preimages of λ_i under the map $x_{m+1}(p)$:

$$x_{m+1}(\hat{\lambda}_i) = \lambda_i,$$

and the degrees r_k and s_k are integers given by

$$r_1 := 1, \quad r_k := d_1 d_2 \dots d_{k-1}, \quad s_{m+1} := 0, \quad s_m := 1 \quad \text{and} \quad s_k := d_{k+1} d_{k+2} \dots d_m s.$$

Note that the presence of an external matrix creates as many poles as the number of distinct eigenvalues of this external matrix $M_{m+1} = \Lambda$.[19]

Remark 5.4. This matrix model also has a combinatorics interpretation in terms of counting colored surfaces. This interpretation is discussed in section 7.

5.4.3. Solution of the loop equations. The loop equations have been solved in [63] by the same method as the 2-matrix model. It proceeds in three steps. One first shows that the loop equations (5.26) have a unique solution admitting a topological expansion. One then proposes an ansatz of solution and proves that it is indeed right. This gives

Theorem 5.5.

$$E(x(z), y) = -\tilde{t}_{d_2+1} \text{“} \left\langle \prod_{i=0}^{d_2} \left(y - V_1'(x(z^i)) + \frac{t}{N} \text{Tr} \frac{1}{x(z^i) - M_1} \right) \right\rangle \text{”}.$$

One finally develops this expression as a polynomial in y to get the bilinear relation

$$\omega_1^{(g)}(z_0) = \sum_i \operatorname*{Res}_{z \to a_i} K(z_0, z) \left[\omega_2^{(g+1)}(z, \bar{z}) + \sum_{h=0}^{g} \omega_1^{(h)}(z) \omega_1^{(g-h)}(\bar{z}) \right],$$

[19] The cases of matrix models without an external field correspond to a totally degenerate external matrix $\Lambda = c\text{Id}$ with only 1-eigenvalue. There are thus two poles as in the 1- or 2-matrix models studied earlier.

where, as in the preceding section,

$$\omega_n^{(g)}(z_1,\dots,z_n) = W_n^{(g)}(x(z_1),\dots,x(z_n))\,\mathrm{d}x(z_1)\dots\mathrm{d}x(z_n) + \delta_{n,2}\delta_{g,0}\frac{\mathrm{d}x(z_1)\,\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}.$$

This allows one to obtain the theorem:

Theorem 5.6. *$\omega_n^{(g)}(z_1,\dots,z_n)$ and F_g are the correlators and symplectic invariants of the spectral curve $(\mathcal{L}_{\text{chain}}, c_{1,2}x_1, x_2)$.*

Since $c_{1,2}x_1 + c_{2,3}x_3 = V_2'(x_2)$, we may use the symplectic invariance theorem 4.1, and homogeneity theorem equation (4.1), which also allow one to write

$$\begin{aligned} F_g &= F_g(\mathcal{L}_{\text{chain}}, c_{1,2}x_1, x_2)\\ &= F_g(\mathcal{L}_{\text{chain}}, -c_{2,3}x_3, x_2)\\ &= F_g(\mathcal{L}_{\text{chain}}, c_{2,3}x_3, x_2)\\ &= F_g(\mathcal{L}_{\text{chain}}, x_2, c_{2,3}x_3)\\ &= F_g(\mathcal{L}_{\text{chain}}, c_{2,3}x_2, x_3). \end{aligned}$$

By an easy recursion, for any $k = 1,\dots,m$,

$$F_g = F_g(\mathcal{L}_{\text{chain}}, c_{k,k+1}x_k, x_{k+1}) = F_g(\mathcal{L}_{\text{chain}}, x_k, c_{k,k+1}x_{k+1}). \tag{5.30}$$

In other words, F_g's can be computed by choosing the spectral curve of any two consecutive x_k's. It means that it does not depend on k, i.e. it does not depend on where we are in the chain.

5.4.4. Matrix quantum mechanics. Matrix quantum mechanics is the limit of an infinitely long chain of matrices $m \to \infty$. This model is very useful in string theory [11].

In this limit, the index k of the matrix M_k becomes a continuous time variable t. The coefficients are scaled in a way such that the coupling term $\mathrm{Tr}(M_k - M_{k+1})^2$ becomes a kinetic energy $\mathrm{Tr}(\mathrm{d}M/\mathrm{d}t)^2$.

More explicitly, consider the chain of m matrices:

$$Z = \int \mathrm{d}M_1 \dots \mathrm{d}M_m\, \mathrm{e}^{-N\,\mathrm{Tr}[\sum_{k=1}^m \eta V_k(M_k) + \frac{\mu}{2\eta}\sum_{k=1}^{m-1}(M_k - M_{k+1})^2]}$$

and take the $\eta \to 0$ and $m \to \infty$ limits such that $T = m\eta$ is of order 1. The index k becomes a time $t = k\eta$, and in the $\eta \to 0$ limit we have

$$Z = \int D[M(t)]\, \mathrm{e}^{-N\,\mathrm{Tr}\int_0^T [V(M(t),t) + \frac{\mu}{2}(\mathrm{d}M/\mathrm{d}t)^2]\mathrm{d}t}.$$

The spectral curve is characterized as before:

find a compact Riemann surface $\mathcal{L}$ and some time-dependent function $x(z,t)$ analytical in the variable z on some domain of $\mathcal{L}$, which satisfy equation (5.27) which become Newton's equations of motion:

$$\mu\ddot{x}(z,t) = -V'(x(z,t),t) \tag{5.31}$$

and such that the initial and final impulsions

$$p(x(z,0),0) = \mu\dot{x}(z,0), \qquad p(x(z,T),T) = \mu\dot{x}(z,T)$$

are analytical outside some cuts.

Therefore, theorem 5.6 gives $\forall\, t \in [0,T]$

$$\ln Z = \sum_g N^{2-2g} F_g(\mathcal{L}, x(z,t), \mu\dot{x}(z,t)).$$

In other words, the spectral curve here is the relationship between impulsion and position for the classical equation of motion. Although the spectral curve depends on time t, F_g's are independent of t.

In the case where the potential $V(x,t) = V(x)$ does not depend on time t, the equations of motion (5.31) can be integrated and give the energy conservation:

$$E(z) = \frac{\mu}{2}\dot{x}^2(z,t) + V(x(z,t)),$$

i.e.

$$\mu\dot{x}(z,t) = \sqrt{2\mu(E(z) - V(x(z,t))},$$

and we have

$$\ln Z = \sum_g N^{2-2g} F_g(\mathcal{L}, x(z,t), \sqrt{2\mu(E(z) - V(x(z,t)))}).$$

5.5. 1-matrix model in an external field

As a special example of the chain of matrices, let us consider the special case $m = 1$, i.e. 1-matrix model with an external field.

The formal 1-matrix model in an external field $\hat{\Lambda}$ is defined as [144]

$$Z_{M\text{ext}}(\hat{\Lambda}) = \int_{\text{formal}} \mathrm{e}^{-\frac{N}{t}\mathrm{Tr}(V(M)-\hat{\Lambda}M)}\,\mathrm{d}M, \qquad \hat{\Lambda} = \mathrm{diag}\left(\overbrace{\hat{\Lambda}_1,\ldots,\hat{\Lambda}_1}^{m_1},\ldots,\overbrace{\hat{\Lambda}_s,\ldots,\hat{\Lambda}_s}^{m_s}\right),$$

where *formal* as usual means that we Taylor expand near a critical value and then exchange the order of Taylor expansion and Gaussian integral. A critical point is a matrix M_0 solution of $V'(M_0) = \hat{\Lambda}$. Let us assume that $\hat{\Lambda}$ has s distinct eigenvalues $\hat{\Lambda}_i$ of multiplicities m_i. Its minimal polynomial is

$$S(y) = \prod_{i=1}^{s}(y - \hat{\Lambda}_i).$$

For each $\hat{\Lambda}_i$, let $\xi_{i,j}$, $j = 1,\ldots,\deg V'$ be the $d = \deg V'$ solutions of $V'(\xi_{i,j}) = \hat{\Lambda}_i$. A critical point M_0 is characterized by s partitions of m_i's into at most $d = \deg V'$ parts:

$$m_i = \sum_{j=1}^{d} n_{i,j}.$$

It is of the form

$$M_0 = \mathrm{diag}\left(\overbrace{\xi_{i,j},\ldots,\xi_{i,j}}^{n_{i,j}}\right).$$

The parameters

$$\epsilon_{i,j} = \frac{t n_{i,j}}{N}$$

are called the filling fractions, and they parametrize which formal integral we consider.

The 1-matrix model in an external field is of course a special case of the chain of matrices described in section 5.4, and therefore one finds that it has a topological expansion given by the symplectic invariants of a spectral curve:

$$\ln(Z_{M\text{ext}}(\hat{\Lambda})) = \sum_{g=0}^{\infty}(N/T)^{2-2g} F_g(\mathcal{E}_{M\text{ext}})$$

and

$$\left\langle \mathrm{Tr}\frac{\mathrm{d}x(z_1)}{x(z_1)-M}\dots\mathrm{Tr}\frac{\mathrm{d}x(z_n)}{x(z_n)-M}\right\rangle_c = \sum_g (N/t)^{2-2g-n}\omega_n^{(g)}(z_1,\dots,z_n).$$

However, let us make the results of section 5.4 a little bit more explicit in that case.

5.5.1. Spectral curve. The spectral curve $\mathcal{E}_{M\mathrm{ext}}$ obeys the equation

$$0 = \mathcal{E}_{M\mathrm{ext}}(x,y) = (V'(x)-y) - \frac{t}{N}\sum_{j=1}^{s}\frac{P_j(x)}{y-\hat{\Lambda}_j}, \tag{5.32}$$

where $P_j(x)$ is a polynomial of degree at most $d-1$. Such a spectral curve is typically of genus $\bar{g} \leqslant sd-s$.

All the coefficients of all P_j's are fixed by the filling fractions requirement that (order by order in t)

$$\frac{1}{2\mathrm{i}\pi}\oint_{\mathcal{A}_I} y\,\mathrm{d}x = \epsilon_I,$$

where $I=(i,j)$ runs through the values $i=1,\dots,s$, $j=1,\dots,d$ and $\mathcal{A}_I$ is a small circle around $\xi_{i,j}$. We recall that $\epsilon_{i,j}$ are not independent; only $sd-s$ of them are independent:

$$\sum_{j=1}^{d}\epsilon_{i,j} = \frac{tm_i}{N}.$$

The two functions $x(z)$ and $y(z)$ are characterized by the fact that $x(z)$ has a simple pole at ∞, simple poles at some λ_i such that $y(\lambda_i)=\hat{\Lambda}_i$ and $y(z)$ has a pole of degree d at ∞. And, we have

$$V'(x(z))-y(z) \underset{z\to\infty}{\sim} \frac{1}{x(z)} + O(1/x(z)^2)$$

and near λ_i, the residues of x are such that

$$\operatorname*{Res}_{z\to\lambda_i} x\,\mathrm{d}y = -\frac{tm_i}{N}$$

and the cycle integrals

$$\frac{1}{2\mathrm{i}\pi}\oint_{\mathcal{A}_{i,j}} y\,\mathrm{d}x = \epsilon_{i,j} = \frac{tn_{i,j}}{N}.$$

5.5.2. Rational case. It is interesting to study the case of a rational spectral curve. The two rational functions $x(z)$ and $y(z)$ are of the form

$$\mathcal{E}_{M\mathrm{ext}} = \begin{cases} x(z) = z - \dfrac{t}{N}\,\mathrm{Tr}\,\dfrac{1}{Q'(\Lambda)(z-\Lambda)} \\ y(z) = Q(z), \end{cases}$$

where Λ is a diagonal matrix determined by

$$Q(\Lambda) = \hat{\Lambda}$$

and Q is a polynomial of degree $d = \deg V'$ determined by

$$V'(x(z)) = Q(z) + \frac{t}{z} + O(z^{-2}).$$

This equation is sufficient to determine $Q(z)$ and therefore to determine the spectral curve. We shall see an application of this in section 10.1 about the Kontsevich integral.

5.6. Convergent matrix integrals

So far, we have been discussing formal matrix integrals, which consist in exchanging integration and the small t Taylor series of $\mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(M)}$. Formal matrix integrals always have a 'topological expansion' of the type

$$\ln Z = \sum_g (N/t)^{2-2g} F_g(t)$$

(remember that this equality holds order by order in powers of t and this is not a large N expansion).

Now, let us consider a 'convergent' matrix integral:

$$Z = \int_{H_N(\gamma)} \mathrm{d}M\, \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(M)}, \tag{5.33}$$

where the integration domain $H_N(\gamma)$ is the set of normal matrices with eigenvalues on a path γ:

$$H_N(\gamma) = \{M/M = U\Lambda U^\dagger, U \in U(N), \Lambda = \operatorname{diag}(\Lambda_1, \ldots, \Lambda_N), \Lambda_i \in \gamma\},$$

equipped with the complex $U(N)$ invariant measure

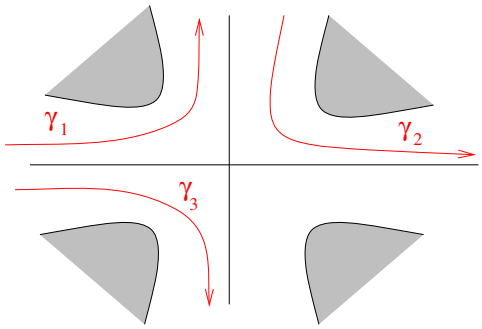
$$\mathrm{d}M = \prod_{i>j} (\Lambda_i - \Lambda_j)^2\, \mathrm{d}U \prod_i \mathrm{d}\Lambda_i,$$

where $\mathrm{d}U$ is the Haar measure on $U(N)$ and $\mathrm{d}\Lambda_i$ is the curviline measure along γ.

- For example, $H_N(\mathbb{R}) = H_N$ is the set of Hermitian matrices.
- For example, $H_N(S_1) = U(N)$ is the set of unitary matrices (S_1 is the unit circle).
  After integrating on $U \in U(N)$, the convergent matrix integral can be rewritten as an integral over the eigenvalues:

$$Z(\gamma) = \frac{1}{N!} \int_{\gamma^N} \mathrm{d}x_1 \ldots \mathrm{d}x_N \prod_{i>j} (x_i - x_j)^2 \prod_{i=1}^N \mathrm{e}^{-\frac{N}{t} V(x_i)}. \tag{5.34}$$

Imagine that V is a polynomial of degree $d+1$ (the present discussion can be easily extended to a 2-matrix model or a chain of matrices, and to all cases where V' is a rational fraction [21]). There are d homologically independent paths on which the integral $\int \mathrm{e}^{-V(x)}\, \mathrm{d}x$ is convergent; let us call $\gamma_1, \ldots, \gamma_d$ a basis of such paths (a choice of basis is not unique). See the figure for the example of a quartic potential ($d = 3$):



The path γ in the matrix integral (5.33) is a linear combination of such paths:

$$\gamma = \sum_{i=1}^d c_i \gamma_i,$$

and the convergent matrix integral (5.34) can be written as

$$Z(\gamma) = \sum_{n_1+\cdots+n_d=N} \frac{c_1^{n_1}\ldots c_d^{n_d}}{n_1!\ldots n_d!} \int_{\gamma_1^{n_1}\times\cdots\times\gamma_d^{n_d}} \mathrm{d}x_1\ldots\mathrm{d}x_N \prod_{i>j}(x_i-x_j)^2 \prod_{i=1}^{N} \mathrm{e}^{-\frac{N}{t}V(x_i)}.$$

This leads us to define the convergent matrix integral with *fixed filling fractions* n_i, as

$$\hat{Z}_{n_1,\ldots,n_d} \stackrel{\text{def}}{=} \frac{1}{n_1!\ldots n_d!} \int_{\gamma_1^{n_1}\times\cdots\times\gamma_d^{n_d}} \mathrm{d}x_1\ldots\mathrm{d}x_N \prod_{i>j}(x_i-x_j)^2 \prod_{i=1}^{N} \mathrm{e}^{-\frac{N}{t}V(x_i)},$$

and thus we have

$$Z(\gamma) = \sum_{n_1+\cdots+n_d=N} c_1^{n_1}\ldots c_d^{n_d}\hat{Z}_{n_1,\ldots,n_d}. \tag{5.35}$$

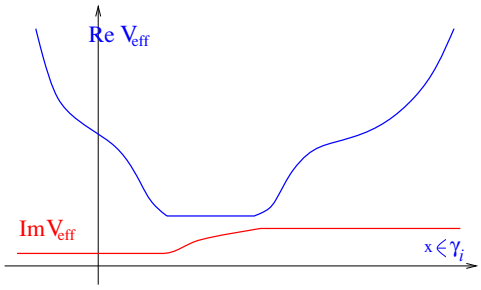
This holds for any choice of basis $\gamma_1,\ldots,\gamma_d$.

There is a conjecture[20] that there exists a 'good' basis of paths $\gamma_1,\ldots,\gamma_d$ [45], such that $\hat{Z}_{n_1,\ldots,n_d}$ is a formal matrix integral (and thus it has a topological expansion in powers of t/N)!

The 'good' paths $\gamma_1,\ldots,\gamma_d$ can be seen as 'steepest descent' paths; they should be such that the effective potential

$$V_{\text{eff}}(x) = V(x) - \frac{t}{N}\langle \ln(\det(x-M))\rangle$$

is such that along each γ_i, the real part of the large N leading order of V_{eff} decreases, then constant and finally increases, and at the same time, the imaginary part of V_{eff} is constant, then increases and is then again constant:



If such paths exist, then we can write

$$\hat{Z}_{n_1,\ldots,n_d} = \mathrm{e}^{\sum_g (N/t)^{2-2g}F_g(\epsilon_i)} \tag{5.36}$$

where ϵ_i are the filling fractions:

$$\epsilon_i = \frac{tn_i}{N}$$

and the coefficients F_g in the expansion are the symplectic invariants F_g of the corresponding formal matrix integral. Since F_g's are analytical functions of the filling fractions, we have, for any 'background' filling fraction $\eta = (\eta_1,\ldots,\eta_{d-1})$,

$$F_g(\epsilon_i) = \sum_{k=0}^{\infty} \frac{(\epsilon-\eta)^k}{k!}\partial_\eta^k F_g(\eta),$$

[20] This conjecture is proved in some cases and in particular proved for the 1-matrix model with arbitrary γ and arbitrary polynomial V (the proof follows from Bertola's work [22]), but, at the time this review was written, it was not proved in more general cases, for instance not proved for the general 2-matrix model.

where we assume tensorial notations and sums over indices. For simplicity, the unfamiliar reader may assume $d = 2$, i.e. η and ϵ are scalar.

Equation (5.36) thus becomes

$$\begin{aligned}\hat{Z}_{n_1,\dots,n_d} &= \mathrm{e}^{\sum_g (N/t)^{2-2g} F_g(\eta)}\, \mathrm{e}^{(n-N\eta/t)NF_0'/t}\mathrm{e}^{\frac{1}{2}(n-N\eta/t)^2 F_0''} \\ &\quad\times \mathrm{e}^{\sum_g \sum_{k\geqslant 2-2g} \frac{(N/t)^{2-2g-k}}{k!}(n-N\eta/t)^k F_g^{(k)}(\eta)} \\ &= \mathrm{e}^{\sum_g (N/t)^{2-2g} F_g(\eta)}\, \mathrm{e}^{(n-N\eta/t)NF_0'/t}\mathrm{e}^{\frac{1}{2}(n-N\eta/t)^2 F_0''} \\ &\quad\times \sum_l \frac{1}{l!} \sum_{g_1,\dots,g_l} {\sum_{k_1,\dots,k_l}}' \frac{(N/t)^{\sum_i (2-2g_i-k_i)}}{k_1!\dots k_l!} \prod_i F_{g_i}^{(k_i)}(\eta)(n-N\eta/t)^{\sum k_i},\end{aligned}$$

where we have separated the terms with a positive power of N from those with a negative power of N ($\sum'$ means that we consider only terms with $k_i > 0$ and $2g_i + k_i - 2 > 0$). Then, writing

$$c_i = \mathrm{e}^{2\mathrm{i}\pi v_i},$$

we perform the sum over filling fractions in equation (5.35) and get [27, 65]

$$\begin{aligned}Z(\gamma) &\sim \mathrm{e}^{\sum_g (N/t)^{2-2g} F_g(\eta)} \sum_{n\in\mathbb{Z}^{d-1}} \mathrm{e}^{2\mathrm{i}\pi n v}\, \mathrm{e}^{(n-N\eta/t)NF_0'/t}\mathrm{e}^{\frac{1}{2}(n-N\eta/t)^2 F_0''} \\ &\quad\times \sum_l \frac{1}{l!} \sum_{g_1,\dots,g_l} \sum_{k_1,\dots,k_l}{}' \frac{(N/t)^{\sum_i (2-2g_i-k_i)}}{k_1!\dots k_l!} \prod_i F_{g_i}^{(k_i)}(\eta)(n-N\eta/t)^{\sum k_i}.\end{aligned}$$

That is, we find the nonperturbative partition function of section 4.5 (see [65, 74]):

$$\begin{aligned}Z(\gamma) &\sim \mathrm{e}^{\sum_g (N/t)^{2-2g} F_g(\eta)} \qquad (5.37)\\ &\quad\times \sum_l \frac{1}{l!} \sum_{g_1,\dots,g_l} \sum_{k_1,\dots,k_l}{}' \frac{(N/t)^{\sum_i (2-2g_i-k_i)}}{k_1!\dots k_l!} \prod_i F_{g_i}^{(k_i)}(\eta)\Theta_{(0,v)}^{(\sum k_i)}(NF_0'/t, F_0''), \qquad (5.37)\end{aligned}$$

where

$$\Theta_{(\mu,v)}(u, F_0'') = \sum_{n\in\mathbb{Z}^{d-1}} \mathrm{e}^{2\mathrm{i}\pi n v}\, \mathrm{e}^{2\mathrm{i}\pi(n-N\eta/t+\mu)u}\, \mathrm{e}^{\frac{1}{2}(n-N\eta/t)^2 F_0''}.$$

This formula is expected to give the large N expansion of convergent matrix models. It was proved in several cases, but a general proof is still missing.

To the first orders, equation (5.37) reads as

$$Z(\gamma) \sim \mathrm{e}^{\frac{N^2}{t^2}F_0}\, \mathrm{e}^{F_1}\left[\Theta + \frac{t}{N}\left(\Theta' F_1' + \frac{\Theta''' F_0'''}{6}\right) + \cdots\right].$$

We have to make several remarks as follows.

- *Background independence.* Formula (5.37) is independent of the background η. This was discussed in section 4.5 and is related to the fact that the nonperturbative partition function is modular.

  This also implies that equation (5.37) cannot be a good large N asymptotic expansion for all values of η. The conjecture is that one should choose η as a real minimum of $\mathrm{Re}\, F_0$. Note that if η is real, then

  $$\mathrm{Re}\, F_0'' = 2\pi\, \mathrm{Im}\, \tau > 0,$$

  where τ is the Riemann matrix of periods of the spectral curve, and thus $\mathrm{Re}\, F_0$ is a convex function of η, and one could think that it must have a unique minimum, and we should choose η as the unique minimum of $\mathrm{Re}\, F_0$. But in fact, the space of possible

filling fractions η is the moduli space of spectral curves corresponding to the potential V. This space is rather complicated; it is made of several cells and Re F_0 is convex only within each cell. Unfortunately, this moduli space is not very well known; in particular its convexity properties are not well known and it is not known how to find the minimum of Re F_0 in general.

If such η can be found we have $\frac{1}{2i\pi}\oint_{\mathcal{A}} y\,dx = \eta \in \mathbb{R}$, and since $F_0' = \oint_{\mathcal{B}} y\,dx$, if η is a minimum of Re F_0, we should also have Re $\oint_{\mathcal{A}} y\,dx = 0$. Since $\oint_{\mathcal{C}} y\,dx = 0$ for all reducible contour $\mathcal{C}$, and since all irreducible contours are combinations of $\mathcal{A}$ and $\mathcal{B}$, we find that

$$\forall \mathcal{C}, \qquad \mathrm{Re}\oint_{\mathcal{C}} y\,dx = 0.$$

A spectral curve with this property is called a 'Boutroux' curve (see [22]).

- *Characteristics* (μ, ν). We see here that the characteristic (μ, ν) of the Θ function is of the form $\mu = 0$ and $c_i = e^{2i\pi\nu_i}$ encode the path $\gamma = \sum_i c_i\gamma_i$ chosen at the beginning to define the convergent integral on $H_N(\gamma)$. It is not fully understood how to associate a path with a characteristic and vice versa.

6. Non-intersecting Brownian motions

6.1. Dyson motions and integrability: introduction

Let us consider N Brownian motions on the real line whose positions at time t are denoted by $x_i(t)$ for $i = 1, \ldots, N$. Let us constrain them not to intersect and fix their starting and ending points: the particle i goes from a_i at $t = 0$ to b_i at $t = 1$:

$$x_i(0) = a_i, \qquad x_i(1) = b_i$$

with

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_N \qquad \text{and} \qquad b_1 \leqslant b_2 \leqslant \cdots \leqslant b_N.$$

Once these parameters are fixed, one is interested in the statistic of these Brownian movers at a given time $t \in (0, 1)$. They are given by the correlation functions

$$R_k(\lambda_1, \lambda_2, \ldots, \lambda_k|t) = \frac{1}{N^k}\left\langle \prod_{i=1}^{k} \mathrm{Tr}\,\delta(\lambda_i - M)\right\rangle_t,$$

where we denote $M = \mathrm{diag}(x_1, \ldots, x_N)$.

Using the Karlin–McGregor formula [94], these correlation functions can be written under a determinantal form [55, 116]

$$R_k(x_1, \ldots, x_k|t) = \det[H_{N,t}(x_i, x_j)]_{i,j=1}^{k}$$

for some kernel $H_{N,t}(x_i, x_j)$ depending on time t. In particular, the density of Brownian movers at time t is given by

$$R_1(x) = H_{N,t}(x, x).$$

Let us now consider a particular case where some of the starting and ending points merge in groups:

$$(a_1, \ldots, a_N) = (\overbrace{\alpha_1, \ldots, \alpha_1}^{n_1}, \overbrace{\alpha_2, \ldots, \alpha_2}^{n_2}, \ldots, \overbrace{\alpha_2, \ldots, \alpha_2}^{n_2}, \ldots, \overbrace{\alpha_p, \ldots, \alpha_p}^{n_p})$$

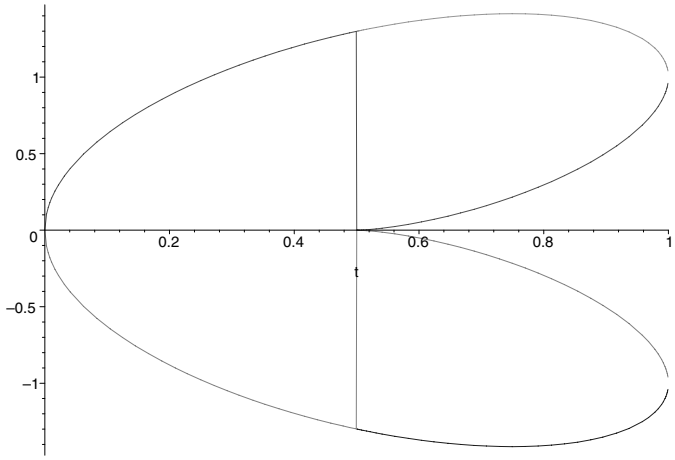


Figure 1. Example of one starting point at 0 and two ending points at +1 and − 1 respectively, half of the movers going to +1 and the other half to − 1 (the particles go from left to right). In the large N limit, the Brownian movers fill the sector of the spacetime delimited by the figure. One sees that before $t = \frac{1}{2}$, they are distributed along a unique segment which splits into two disjoint segments for $t > \frac{1}{2}$.

and

$$(b_1, \ldots, b_N) = (\overbrace{\beta_1, \ldots, \beta_1}^{\tilde{n}_1}, \overbrace{\beta_2, \ldots, \beta_2}^{\tilde{n}_2}, \ldots, \overbrace{\beta_2, \ldots, \beta_2}^{\tilde{n}_2}, \ldots, \overbrace{\beta_q, \ldots, \beta_q}^{\tilde{n}_q})$$

with

$$\sum_{i=1}^{p} n_i = \sum_{i=1}^{q} \tilde{n}_i = N.$$

It means that one considers p groups of n_i particles starting from distinct points α_i at $t = 0$, merging for intermediate times and splitting into q groups of $\tilde{n}_i$ movers reaching β_i at $t = 1$. Note [1] that the kernel reduces to the kernel of the $p + q$ multi-component KP integrable hierarchy:

$$H_{N,t}(x, x') = H_{N,t}^{(p,q)}(x, x')$$

since it satisfies the corresponding Hirota equation.

In the following, one studies the behavior of this phenomenon as the number of particles goes to infinity while the ratios

$$\epsilon_i = \frac{n_i}{N} \qquad \text{and} \qquad \tilde{\epsilon}_i = \frac{\tilde{n}_i}{N}$$

are kept fixed and finite. In this case, the Brownian movers form clouds which fill a connected region of the complex plane describing the spacetime. For example, for one starting point and two ending points one typically gets a configuration of the type depicted in figure 1: all movers leave the origin and begin to flee from one another. It creates a larger and larger segment of the space filled by the Brownian movers. As the time grows, because the movers want to reach different points, they split into two groups heading toward these two endpoints.

As often in the study of such an integrable system, the kernel exhibits universal behaviors: in any point of the spacetime, one can rescale the kernel so that one obtains a universal kernel

independent of the position of the point considered; typically, one recovers the Sine, Airy and Pearcy kernels. The purpose of this part is to emphasize the role played by the spectral curve and algebraic geometry in the study of these universality properties.

6.2. Particles starting from one point: a matrix model representation

Let us consider the particular case of one starting point $p = 1$. This means that all the Brownian movers start from the same point which we may assume to be located at the origin 0 of the real axis. It was proved [56] that this case is equivalent to a matrix model. More precisely, the statistic of the Brownian movers at a given time t is the same as the statistic of the rescaled eigenvalues of Hermitian random matrix M of size $N \times N$ submitted to an external field $A(t)$. This matrix model is given by the partition function

$$Z(A(t)) = \int \mathrm{d}M \, \mathrm{e}^{-N \operatorname{Tr}\left(M^2 + A(t) M\right)}$$

and

$$A(t) = \operatorname{diag}\left(\overbrace{A_1(t), \ldots, A_1(t)}^{\tilde{n}_1}, \overbrace{A_2(t), \ldots, A_2(t)}^{\tilde{n}_2}, \ldots, \overbrace{A_q(t), \ldots, A_q(t)}^{\tilde{n}_q} \right)$$

with the time-dependent elements

$$A_i(t) = \sqrt{\frac{2t}{t(t-1)}} \beta_i.$$

The limit of a large number of particles corresponds to the large matrix limit.

This model can also be expressed in terms of the eigenvalues $(x_1, \ldots, x_N)$ of the random matrix M using the well known HCIZ integral formula [83, 89]. They are submitted to a probability measure

$$\mathrm{d}\mu(x_1, \ldots, x_N) = \prod_{i=1}^{N} \mathrm{d}x_i \, \Delta(x)^2 \, \mathrm{e}^{-N \sum_i \left(\frac{x_i^2}{2} - x_i a_i\right)}.$$

After the rescaling

$$x_i \to x_i \sqrt{t(1-t)},$$

they have the same statistic as N Brownian movers starting from the origin.

The correlation functions

$$R_k(x_1, x_2, \ldots, x_k) = \frac{1}{N^k} \left\langle \prod_{i=1}^{k} \operatorname{Tr} \delta(x_i - M) \right\rangle$$

have also a determinantal expression

$$R_k(x_1, \ldots, x_k) = \det[H_{N,t}(x_i, x_j)]_{i,j=1}^{k},$$

in terms of a kernel $H_{N,t}$. As N goes to infinity, these eigenvalues merge into dense intervals of the real axis and the density $R_1(x)$ is supported by a finite number of segments $[z_{2i-1}, z_{2i}]$. A classical result of the study of random matrices states that these segments correspond to the discontinuity of the resolvent $W(x) = \left\langle \sum_i \frac{1}{x - x_i} \right\rangle$:

$$W_+(x) - W_-(x) = R_1(x) \tag{6.1}$$

for $x \in [z_{2i-1}, z_{2i}]$.

6.2.1. Gaussian matrix model in an external field. Let us now quickly recall this matrix model's loop equations and large N solution. This model is a special case of the matrix model in an external field studied in section 5.5.

The resolvent $W(x)$ is thus the solution of the algebraic equation (5.32) with $V'(x) = x$, i.e.

$$E(x, Y(x)) = Y(x) - x + \sum_{i=1}^{q} \frac{\epsilon_i}{Y(x) - a_i(t)} = 0, \tag{6.2}$$

where

$$Y(x) = W(x) - x.$$

This equation can be seen as the embedding of a Riemann surface $\mathcal{L}$ into $\mathbb{CP}^1 \times \mathbb{CP}^1$.

First of all, one can see that this spectral curve always has genus 0: it admits a simple rational parametrization

$$\begin{cases} x(z) = z + \sum_{i=1}^{q} \frac{n_i}{N(z-a_i(t))} \\ y(z) = z. \end{cases}$$

It is composed of $q+1$ sheets, one of which contains the pole $z = \infty$. This sheet is called the physical sheet.

For a fixed number q of distinct eigenvalues of the external matrix, it may have up to q cuts linking the physical sheet to the q others. These cuts correspond to the discontinuity of the resolvent giving rise to the density of eigenvalues equation (6.1). The cuts $[z_{2i-1}, z_{2i}]$ are thus the support of the eigenvalues. Each z_i is solution of $x'(z_i) = 0$. To be precise, using the notations of section 2, the density of eigenvalues on the cut i is given by

$$\rho(x(p)) = y(p^{(0)}) - y(p^{(i)}), \qquad \text{for} \quad x(p) \in [z_{2i-1}, z_{2i}]$$

where one labels the physical sheet by 0 and the label i corresponds to the sheet linked to the physical one by the cut $[z_{2i-1}, z_{2i}]$.

Let us now study the evolution of the structure of this spectral curve as the time evolves from 0 to 1: one is particularly interested in the time evolution of the position of the branch points $z_i(t)$.

For a given time t, these branch points are the simple real roots of the equation $\partial_y E(x, y)|_{y=y(x)} = 0$:

$$x'(z_j) = 1 - \sum_{i=1}^{k} \frac{n_i}{N(z_j - a_i(t))^2} = 0.$$

Let us first consider large times close to 1. In this case, the eigenvalues $a_i(t)$ become large and are far apart from one another. Thus, the equation has $2k$ distinct real roots in z: the spectral curve has k distinct cuts $[z_{2i-1}, z_{2i}]$ (see figure 2).

Then, as the time decreases, the branch points come closer from one another and merge for some critical time before becoming complex conjugated with an increasing imaginary part. It means that two real cuts merge into one and an imaginary cut linking two non-physical sheets appear (see figure 3).

Finally, as the time becomes small and approaches zero, all the branch points are coupled complex conjugated numbers except two of them: those with the smallest and the largest real parts; there is only 1 real cut left.

This time evolution of the spectral curve has a simple interpretation in terms of the statistic of the eigenvalues. Indeed, thanks to equation (6.1), the real cuts are the support

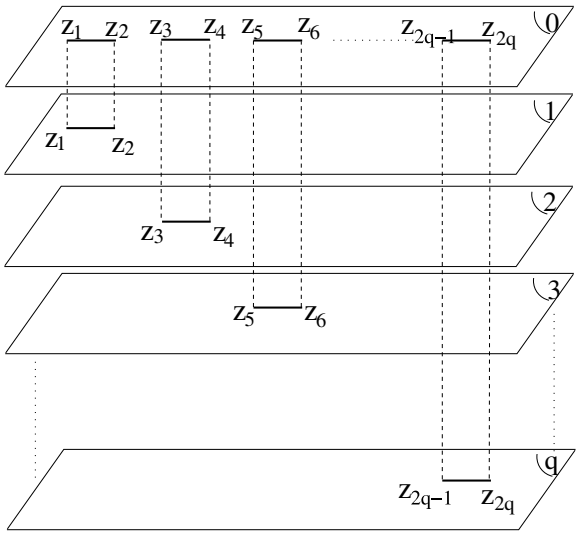


Figure 2. For large times, the spectral curve is composed of $q + 1$ sheets linked by q real cuts $[z_{2i-1}, z_{2i}]$. These cuts can be seen as the section of the support of the Brownian movers at fixed time. In the example depicted in figure 1, it corresponds to times greater than $\frac{1}{2}$.

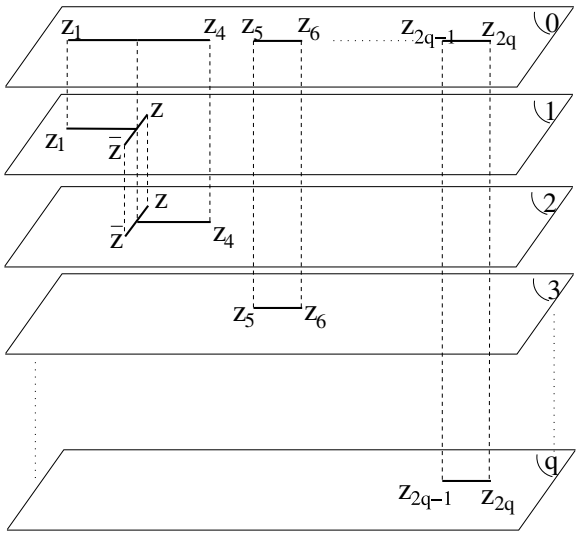


Figure 3. For intermediate times, some of the real cuts have merged and created imaginary cuts linking two non-physical sheets. In this example, z_2 and z_3 have collapsed and given rise to the imaginary cut $[z, \bar{z}]$.

of the random matrix eigenvalues (whereas imaginary cuts just follow from the interaction between the different groups of eigenvalues). In terms of non-intersecting Brownian motions, these real cuts are the segments filled by the Brownian movers at a given time t; it is the constant time section of the region of spacetime filled by the Brownian movers.

The time evolution of the spectral curve can thus be interpreted as follows. For times close to 1, the movers form q groups lying on segments centered around q end points and whose extremities are the branch points of the spectral curve. As the time decreases, the branch points come closer to each other and finally some of them merge, i.e. two of the disjoint segments supporting the Brownian movers merge into one and the sector filled by the Brownian movers exhibits a cusp. Then the different segments keep on merging as time decreases until they give a simply connected support for $t \to 0$. This follows the intuition that all the particles leave 0 in one group which step by step splits into smaller groups to end up with q groups reaching the end points at $t = 1$ (see figure 1 for $q = 2$).

Remark 6.1. For the following, it is interesting to note that the critical times when two disjoint segments merge correspond to singular spectral curves in the sense of definition 2.3. Indeed, at this time, the spectral curve has a double branch point at the location where the two simple branch points merge.

6.2.2. Replica formula and spectral curve. Let us now follow another approach, exact for finite N, exhibiting the role played by the spectral curve directly in the formulation of the kernel H_N. For this purpose, we sketch the derivation of a double integral representation of the kernel using the replica method developed in this context by Brézin and Hikami [32–35].

Let us first consider the 'Fourier' transforms of the correlation functions:

$$U_l(t_1, t_2, \ldots, t_l) = \left\langle \prod_{i=1}^{l} \operatorname{Tr} \mathrm{e}^{\mathrm{i}Nt_i M} \right\rangle$$

and, in particular, one gets the Fourier transform of the two-point correlation function:

$$U_2(t_1, t_2) = \frac{1}{Z(A)N^2} \sum_{\alpha_1,\alpha_2=1}^{N} \int \left(\prod_{j=1}^{N} \mathrm{d}x_j \right) \frac{\Delta(x)}{\Delta(a)} \mathrm{e}^{-N\sum_{j=1}^{N}[\frac{x_j^2}{2} - x_j(a_j + \mathrm{i}t_1\delta_{j,\alpha_1} + \mathrm{i}t_2\delta_{j,\alpha_2})]}.$$

One can now integrate the variables x_j by noting that

$$\int \left(\prod_{j=1}^{N} \mathrm{d}x_j \right) \Delta(x)\, \mathrm{e}^{-N\sum_{j=1}^{N}[\frac{x_j^2}{2} + x_j b_j]} = \Delta(b)\, \mathrm{e}^{\frac{N}{2}\sum_{j=1}^{N} b_j^2}$$

and using the expansion $\Delta(x) = \prod_{i\neq j}(x_i - x_j)$:

$$U_2(t_1, t_2) = \sum_{\alpha_1,\alpha_2=1}^{N} \mathrm{e}^{N(\mathrm{i}t_1 a_{\alpha_1} + \mathrm{i}t_2 a_{\alpha_2} - \frac{t_1^2+t_2^2}{2} - t_1 t_2 \delta_{\alpha_1,\alpha_2})} \times \frac{\prod_{1\leqslant l<m\leqslant N}(a_l - a_m + \mathrm{i}t_1(\delta_{l,\alpha_1} - \delta_{m,\alpha_1}) + \mathrm{i}t_2(\delta_{l,\alpha_2} - \delta_{m,\alpha_2}))}{\prod_{1\leqslant l<m\leqslant N}(a_l - a_m)}.$$

One can see that this can be written as a double contour integral:

$$U_2(t_1, t_2) = \frac{\mathrm{e}^{-N\frac{t_1^2+t_2^2}{2}}}{t_1 t_2} \oint\oint \frac{\mathrm{d}u\, \mathrm{d}v}{(2\mathrm{i}\pi)^2} \mathrm{e}^{N\mathrm{i}(t_1 u + t_2 v)} \frac{(u - v + \mathrm{i}t_1 - \mathrm{i}t_2)(u - v)}{(u - v + \mathrm{i}t_1)(u - v - \mathrm{i}t_2)} \times \prod_k \left(1 + \frac{\mathrm{i}t_1}{u - a_k}\right)\left(1 + \frac{\mathrm{i}t_2}{v - a_k}\right)$$

or

$$U_2(t_1,t_2) = \frac{\mathrm{e}^{-N\frac{t_1^2+t_2^2}{2}}}{t_1t_2}\oint\oint\frac{\mathrm{d}u\,\mathrm{d}v}{(2\mathrm{i}\pi)^2}\,\mathrm{e}^{N\mathrm{i}(t_1u+t_2v)}\left(1-\frac{t_1t_2}{(u-v+\mathrm{i}t_1)(u-v-\mathrm{i}t_2)}\right)$$
$$\times\prod_k\left(1+\frac{\mathrm{i}t_1}{u-a_k}\right)\left(1+\frac{\mathrm{i}t_2}{v-a_k}\right),$$

where the integration contours encircle all the eigenvalues a_k and the pole $v = u - \mathrm{i}t_1$.[21]

We can now go back to the correlation function

$$R_2(\lambda,\mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}t_1\,\mathrm{d}t_2}{4\pi^2}\,\mathrm{e}^{-\mathrm{i}N(t_1\lambda+t_2\mu)}U(t_1,t_2).$$

By first integrating on t_1 and t_2 with the shifts $t_1 \to t_1 - \mathrm{i}u$ and $t_2 \to t_2 - \mathrm{i}u$, we can show that

$$R_2(\lambda,\mu) = K_N(\lambda,\lambda)K_N(\mu,\mu) - K_N(\mu,\lambda)K_N(\lambda,\mu),$$

where the kernel is defined by

$$K_N(\lambda,\mu) = \int\frac{\mathrm{d}t}{2\pi}\oint\frac{\mathrm{d}v}{2\mathrm{i}\pi}\prod_{k=1}^{N}\left(\frac{\mathrm{i}t-a_k}{v-a_k}\right)\frac{1}{v-\mathrm{i}t}\,\mathrm{e}^{-N(\frac{v^2+t^2}{2}+\mathrm{i}t\lambda-v\mu)},$$

where the integration contour for v goes around all the points a_k and the integration for t is parallel to the real axis and avoids the v contour. Moreover, it is straightforwardly proven that any k-point function can be written as the Fredholm determinant:

$$R_k(x_1,\ldots,x_k) = \det[K_N(x_i,x_j)]_{i,j=1}^{k}.$$

By Wick rotating the integration variable $t \to \mathrm{i}t$, one gets

$$K_N(\lambda,\mu) = \int\frac{\mathrm{d}t}{2\mathrm{i}\pi}\oint\frac{\mathrm{d}v}{2\mathrm{i}\pi}\prod_{k=1}^{N}\left(\frac{t-a_k}{v-a_k}\right)\frac{1}{v-t}\,\mathrm{e}^{-N(\frac{v^2-t^2}{2}+t\lambda-v\mu)},$$

where the integration contour for t is now parallel to the imaginary axis. One can then rewrite it under a more factorized form:

$$K_N(\lambda,\mu) = \int\frac{\mathrm{d}t}{2\mathrm{i}\pi}\oint\frac{\mathrm{d}v}{2\mathrm{i}\pi}\mathrm{e}^{-N(S(\mu,v)-S(\lambda,t))}\frac{1}{v-t},$$

where

$$S(x,y) = \frac{y^2}{2} - xy + \sum_{i=1}^{q}\epsilon_i\ln(y-a_i)$$

with the filling fractions given by

$$\epsilon_i := \frac{n_i}{N}.$$

How to compute such an integral? Let us use the saddle-point method for both integrals. The saddle points of the first exponential are given by the y solutions of

$$\partial_y S(x,y) = y - x + \sum_{i=1}^{k}\frac{\epsilon_i}{y-a_i} = 0,$$

which is nothing but the equation of the spectral curve equation (6.2)!

In this setup, the spectral curve can thus be seen as the location of the saddle points of the action $S(x,y)$ in this formulation of the kernel. Also, note that this formulation of the kernel is very similar to the formulation of theorem 4.4 in terms of symplectic invariants.

[21] See, for instance, [32] for more details around equations (2)–(20) and equations (4)–(40).

6.2.3. From the replica formula to the symplectic invariants formalism. Let us now start from the expression of theorem 4.4 for the kernel. For this purpose, one has to compute the 1-form $y\,\mathrm{d}x$. Using the global parametrization

$$\begin{cases} x(z) = z + \sum_i \dfrac{\epsilon_i}{z - a_i(t)} \\ y(z) = z, \end{cases}$$

one gets

$$y\,\mathrm{d}x(z) = z\,\mathrm{d}z + \sum_i \frac{\epsilon_i\,\mathrm{d}z}{z - a_i(t)}.$$

Integrating by parts, one can see that

$$\int_{z_2}^{z_1} y\,\mathrm{d}x = -\int_{z_2}^{z_1} x(z)\,\mathrm{d}y(z) + [x(z)y(z)]_{z_2}^{z_1}.$$

Moreover, the function $y(z)$ can also be considered as a global coordinate on the spectral curve since it coincides with the z coordinate. The previous equation can hence be written as

$$\int_{y_2}^{y_1} y\,\mathrm{d}x(y) = -\int_{y_2}^{y_1} x(y)\,\mathrm{d}y + [x(y)y]_{y_2}^{y_1}.$$

On the other hand, the spectral curve has a particular form: it has only one x-sheet and takes the form

$$H(x, y) = x - x(y) = 0$$

with the function

$$x(y) = y + \sum_i \frac{\epsilon_i}{y - a_i(t)}.$$

Thus the action, i.e. the integral of the spectral curve w.r.t. y, reads as

$$S(x, y) = xy - \int x(y)\,\mathrm{d}y$$

and further

$$\begin{aligned} S(x_1, y_1) - S(x_2, y_2) &= x_1y_1 - x_2y_2 - \int_{y_1}^{y_2} x(y)\,\mathrm{d}y \\ &= \int_{y_2}^{y_1} y\,\mathrm{d}x(y) + y_1[x_1 - x(y_1)] + y_2[x_2 - x(y_2)]. \end{aligned}$$

It is now possible to compare the kernel built from the symplectic invariants and the spectral curve:

$$H_N(x_1, x_2) = \frac{1}{x_1 - x_2}\,\mathrm{e}^{N\int_{y(x_2)}^{y(x_1)} y\,\mathrm{d}x(y)}\left[1 + O\left(\frac{1}{N}\right)\right]$$

and the kernel following the replica formula:

$$\widetilde{H}_N(x_1, x_2) = \oint\oint \mathrm{d}y_1\,\mathrm{d}y_2 \frac{1}{y_1 - y_2}\,\mathrm{e}^{N[S(x_1,y_1) - S(x_2,y_2)]}.$$

Indeed, the latter reads as

$$\widetilde{H}_N(x_1, x_2) = \oint\oint \mathrm{d}y_1\,\mathrm{d}y_2\, H_N(x(y_1), x(y_2))\,\mathrm{e}^{N\{y_1[x_1 - x(y_1)] + y_2[x_2 - x(y_2)]\}}.$$

The saddle-point equation then directly states the equality of both kernels in the large N limit if the integration contours are good steepest descent contours.

Remark 6.2. Note that this particular representation of the kernel under the form of a double integral on the complex plane (or more precisely the Riemann sphere) is a direct consequence of the specific parametrization of the spectral curve $x = x(y)$.

6.2.4. Critical behavior and singular spectral curves. Since both kernels coincide, one can use the singular limits derived in the symplectic invariant setup to obtain some critical universal behavior of the exclusion process.

- *Universality on the edge: the Airy kernel.* Let us study the statistic of Brownian movers around the edge of their limiting support. It means that one is interested in a point (x, t) of the spacetime such that $(x, y(x))$ is close to a branch point $(x_c, y_c) = (x(z_c), y(z_c))$ of the spectral curve at time t_c. Following the study of section 4.8, let us rescale the global coordinate $z = z_c + \frac{1}{N^{\frac{1}{3}}} Z$ and one gets the blown-up spectral curve

$$\begin{cases} x(z) = x(z_c) + \dfrac{Z}{N^{\frac{1}{3}}} x'(z_c) + \dfrac{Z^2}{2N^{\frac{2}{3}}} x''(z_c) + O\left(\dfrac{1}{N}\right) \\ y(z) = y(z_c) + \dfrac{Z}{N^{\frac{1}{3}}} y'(z_c) + O\left(\dfrac{1}{N^{\frac{2}{3}}}\right). \end{cases}$$

  Since the point z_c is a branch point, one has $x'(z_c) = 0$ and the blown-up curve reduces to

$$\begin{cases} x_{\mathrm{Airy}}(Z) = \dfrac{1}{N^{\frac{2}{3}}} \dfrac{x''(z_c)}{2} Z^2 \\ y_{\mathrm{Airy}}(Z) = \dfrac{1}{N^{\frac{1}{3}}} y'(z_c) Z \end{cases}$$

  to leading order as $N \to \infty$. Note that the scaling is such that $y\,\mathrm{d}x(Z) = O\left(\frac{1}{N}\right)$. This curve is the Airy curve described in the second example of section 2.1.1. Theorem 4.22 states that, in terms of the rescaled variable Z, i.e. the distance from the critical point considered, the kernel and the correlation functions reduce to that of the Airy curve, independent of the position of the critical point.

  Note that this Airy curve has also the form $x = x(y)$ where the function $x(y) = \frac{x''(z_c)}{2y'(z_c)^2} y^2$. The kernel can thus also be written under a double integral form:

$$H_{\mathrm{Airy}}(x_1, x_2) = \oint \oint \frac{\mathrm{d}y_1\,\mathrm{d}y_2}{y_1 - y_2}\, \mathrm{e}^{S_{\mathrm{Airy}}(x_1,y_1) - S_{\mathrm{Airy}}(x_1,y_1)}$$

  with $S_{\mathrm{Airy}}(x, y) = xy - \frac{x''(z_c)}{y'(z_c)^2} \frac{y^3}{6}$, i.e. the Airy kernel [134].
- *Universality at the cusp: the Pearcy kernel.* Let us finally consider a point where two groups of Brownian movers merge. This is obtained when two edges merge or, in the spectral curve formalism, when two branch points merge as time decreases. At this critical time t_c, the corresponding spectral curve is singular since the merging of two simple branch points gives rise to a double branch point. Let us be more specific: one considers a cusp at the position (x_c, t_c) in the spacetime. Let us blow up the spacetime around this point using the rescaling

$$\begin{cases} t := t_c + \dfrac{\alpha_t T}{N^{\frac{1}{2}}} \\ z := z_c + \dfrac{\alpha_y Z}{N^{\frac{1}{4}}}, \end{cases}$$

  where z is the global parameter of the spectral curve. The rational parametrization of the spectral curve thus reads as

$$\begin{cases} x(z,t) = x(z_c,t_c) + \dfrac{\alpha_y Z}{N^{\frac{1}{4}}} \partial_z x(z_c,t_c) + \dfrac{\alpha_y^2 Z^2}{2N^{\frac{1}{2}}} (\partial_z)^2 x(z_c,t_c) \\ \qquad + \dfrac{\alpha_y^3 Z^3}{6N^{\frac{3}{4}}} (\partial_z)^3 x(z_c,t_c) + \dfrac{\alpha_y \alpha_t Z T}{N^{\frac{3}{4}}} \partial_z \partial_t x(z_c,t_c) + O\left(\dfrac{1}{N}\right) \\ y(z) = y(z_c) + \dfrac{\alpha_y Z}{N^{\frac{1}{4}}} y'(z_c) + O\left(\dfrac{1}{N^{\frac{2}{3}}}\right). \end{cases}$$

Since the critical point is a double branch point, the blown-up curve reduces to

$$\begin{cases} x(Z,T) = \dfrac{1}{N^{\frac{3}{4}}}\left[\dfrac{\alpha_y^3 Z^3}{6}\,(\partial_z)^3\,x(z_c,t_c) + \alpha_y\alpha_t Z T\,\partial_z\partial_t x(z_c,t_c)\right] \\ y(Z) = \dfrac{\alpha_y Z}{N^{\frac{1}{4}}}\partial_z y(z_c), \end{cases}$$

which could be called the Pearcy curve. Indeed, since this curve also has the form $x = x(y)$, the associated kernel has the double contour integral representation

$$H_{\text{Pearcy}}(x_1,x_2) = \oint\oint \frac{\mathrm{d}y_1\,\mathrm{d}y_2}{y_1-y_2}\,\mathrm{e}^{S_{\text{Pearcy}}(x_1,y_1)-S_{\text{Pearcy}}(x_1,y_1)}$$

with $S_{\text{Pearcy}}(x,y) = xy - \left(\frac{\alpha_y^2(\partial_z)^3 x(z_c,t_c)}{(\partial_z y(z_c))^3}\right)\frac{y^4}{24} - \left(\frac{\alpha_t\partial_z\partial_t x(z_c,t_c)}{2\partial_z y(z_c)}\right)y^2 T$. This is nothing but the Pearcy kernel once the right integration contour is found and the rescaling coefficients are fixed by

$$\frac{\alpha_y^2\,(\partial_z)^3\,x(z_c,t_c)}{(\partial_z y(z_c))^3} = 6$$

and

$$\frac{\alpha_t\partial_z\partial_t x(z_c,t_c)}{2\partial_z y(z_c)} = 1.$$

7. Enumeration of discrete surfaces or maps

The symplectic invariants provide a solution to Tutte's equations for counting discrete surfaces (also called maps), for arbitrary topologies. Indeed, it was found by Brezin *et al* [36], and further developed, among others, by Ambjorn *et al* [13, 43, 96] that such generating functions for discrete surfaces can be written as formal matrix models. Let us review how to enumerate various ensembles of discrete surfaces.

7.1. Introduction

Definition 7.1. *Let $\mathbb{M}_n^{(g)}$ be the set of connected orientable discrete surfaces of genus g obtained by gluing together polygonal faces, namely n_3 triangles, n_4 quadrangles, n_k k-angles, as well as n marked polygonal faces of perimeters $l_1,\dots,l_n$, each of the marked faces having one marked edge on its boundary. Let us call v the number of vertices of a discrete surface and $\mathbb{M}_n^{(g)}(v)$ be the set of discrete surfaces in $\mathbb{M}_n^{(g)}$, with v vertices.*

We require that unmarked faces have perimeter $\geqslant$3, whereas marked faces are only required to have perimeter $l_i \geqslant 1$.

Note that nothing in our definition prevents from gluing a side of a polygon to another side of the same polygon.

Marked faces are also called 'boundaries' (this is a slight abuse of language, since the boundaries of marked faces do not necessarily have the topology of circle; indeed, the marked face can be glued to itself along some of its sides).

Theorem 7.1. *$\mathbb{M}_n^{(g)}(v)$ is a finite set.*

Proof. Let e be the number of edges of a discrete surface in $\mathbb{M}_n^{(g)}(v)$. The total number of half-edges is

$$2e = \sum_{j\geqslant 3} jn_j + \sum_{i=1}^{n} l_i.$$

The Euler characteristic is

$$\chi = 2 - 2g = v - e + n + \sum_{j\geqslant 3} n_j = v + n - \frac{1}{2}\sum_{j\geqslant 3}(j-2)n_j - \frac{1}{2}\sum_{i=1}^{n} l_i.$$

This implies

$$\frac{1}{2}\sum_{j\geqslant 3}(j-2)n_j + \frac{1}{2}\sum_{i=1}^{n} l_i = 2g - 2 + n + v, \tag{7.1}$$

and therefore n_j's are bounded and l_i's are bounded. There is then a finite number of possible discrete surfaces having a finite number of faces, edges and vertices. □

In order to enumerate discrete surfaces, we define the generating functions.

Definition 7.2. *The generating function is the formal power series in t:*

$$W_n^{(g)}(x_1, \ldots, x_n; t_3, \ldots, t_d; t) = \frac{t}{x_1}\delta_{n,1}\delta_{g,0} + \sum_{v=1}^{\infty} t^v \sum_{S\in\mathbb{M}_n^{(g)}(v)} \frac{1}{\#\mathrm{Aut}(S)} \frac{t_3^{n_3(S)}\ldots t_d^{n_d(S)}}{x_1^{l_1(S)}\ldots x_n^{l_n(S)}} \prod_{i=1}^{n}\frac{1}{x_i}.$$

Most often, we will write only the dependence in x_i's explicitly and write

$$W_n^{(g)}(x_1, \ldots, x_n; t_3, \ldots, t_d; t) = W_n^{(g)}(x_1, \ldots, x_n).$$

The generating functions counting surfaces with marked faces of given perimeters $l_1, \ldots, l_n$ are, by definition,

$$T^{(g)}_{l_1,\ldots,l_n} = (-1)^n \operatorname*{Res}_{x_1\to\infty} \ldots \operatorname*{Res}_{x_n\to\infty} x_1^{l_1}\ldots x_n^{l_n} W_n^{(g)}(x_1, \ldots, x_n)\,\mathrm{d}x_1\ldots\mathrm{d}x_n. \tag{7.2}$$

Note that rooted discrete surfaces, i.e. surfaces with only one marked edge, have no non-trivial automorphisms, and thus #Aut = 1 when $n = 1$.

7.2. Tutte's recursion equations

Tutte's equations are recursions on the number of edges [135, 136]. If one erases the marked edge on the first marked face whose perimeter is $l_1 + 1$, several mutually exclusive possibilities may occur as follows.

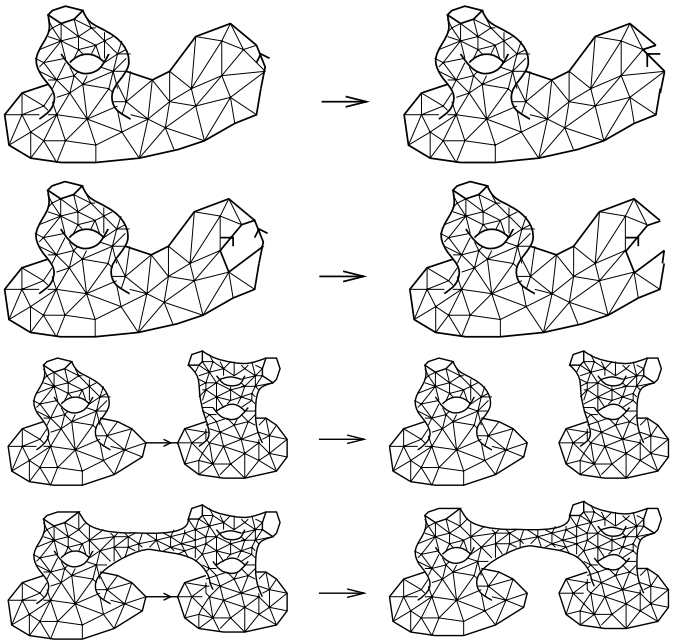
- The marked edge separates the marked face with some unmarked face (let us say a j-gon with $j \geqslant 3$), and removing that edge is equivalent to removing a j-gon (with weight t_j). We thus get a discrete surface of genus g with the same number of boundaries, and the length of the first boundary is now $l_1 + j - 1$.
- The marked edge separates two distinct marked faces (face 1 and face m with $2 \leqslant m \leqslant n$); thus the marked edge of the first boundary is one of the l_m edges of the mth boundary. We thus get a discrete surface of genus g with $n - 1$ boundaries. The other $n - 2$ boundaries remain unchanged, and there is now one boundary of length $l_1 + l_m - 1$.

- The same marked face lies on both sides of the marked edge; therefore by removing it, we disconnect the boundary. Two cases can occur: either the discrete surface itself gets disconnected into two discrete surfaces of genus h and $g-h$, one having $|J|+1$ boundaries of lengths j, J, where J is a subset of $K = \{l_2, \ldots, l_n\}$, and the other discrete surface having $k - |J|$ boundaries of lengths $l_1 - 1 - j$, K/J, or the discrete surface remains connected because there was a handle connecting the two sides, and thus by removing the marked edge, we get a discrete surface of genus $g-1$, with $n+1$ boundaries of lengths $j, l_1 - j - 1, K$.

This procedure is (up to the symmetry factors) bijective, and all these possibilities correspond to the following recursive equation:

$$\sum_{j=0}^{l_1-1}\left[\sum_{h=0}^{g}\sum_{J\subset K}\mathcal{T}^{(h)}_{j,J}\mathcal{T}^{(g-h)}_{l_1-1-j,K/J}+\mathcal{T}^{(g-1)}_{j,l_1-1-j,K}\right]+\sum_{m=2}^{n}l_m\mathcal{T}^{(g)}_{l_m+l_1-1,K/\{l_m\}}$$
$$=\mathcal{T}^{(g)}_{l_1+1,K}-\sum_{j=3}^{d}t_j\mathcal{T}^{(g)}_{l_1+j-1,K}. \tag{7.3}$$

This equation is illustrated as follows (where the first marked face is the 'exterior face'):



7.3. Loop equations

Rewritten in terms of $W^{(g)}_n$'s, Tutte's equations (7.3) read as

Theorem 7.2. *Loop equations. For any n and g, and* $L = \{x_2, \ldots, x_n\}$*, we have*

$$\sum_{h=0}^{g}\sum_{J\subset L}W^{(h)}_{1+|J|}(x_1, J)W^{(g-h)}_{n-|J|}(x_1, L/J)+W^{(g-1)}_{n+1}(x_1, x_1, L)$$

$$+\sum_{j=2}^{n}\frac{\partial}{\partial x_j}\frac{W_{n-1}^{(g)}(x_1,L/\{j\})-W_{n-1}^{(g)}(L)}{x_1-x_j}$$
$$=V'(x_1)W_n^{(g)}(x_1,L)-P_n^{(g)}(x_1,L), \tag{7.4}$$

where

$$V'(x)=x-\sum_{j\geqslant 3}t_j x^{j-1},$$

where $P_n^{(g)}(x_1,L)$ is a polynomial in x_1 of degree $d-3$ (except $P_1^{(0)}$ which is of degree $d-2$):

$$P_n^{(g)}(x_1,x_2,\dots,x_n)=-\sum_{j=2}^{d-1}t_{j+1}\sum_{i=0}^{j-1}x_1^i\sum_{l_2,\dots,l_n=1}^{\infty}\frac{\mathcal{T}^{(g)}_{j-1-i,l_2,\dots,l_n}}{x_2^{l_2+1}\dots x_n^{l_n+1}}+t\delta_{g,0}\delta_{n,1}.$$

Proof. Indeed, if we expand both sides of equation (7.4) in powers of $x_1\to\infty$ and identify the coefficients on both sides, we find that the negative powers of x_i's give precisely the loop equation (7.3), whereas the coefficients of positive powers of x_1 cancel due to the definition of $P_n^{(g)}$, which is exactly the positive part of $V'(x_1)W_n^{(g)}$:

$$P_n^{(g)}(x_1,x_2,\dots,x_n)=\operatorname*{Pol}_{x_1\to\infty}\left(V'(x_1)W_n^{(g)}(x_1,x_2,\dots,x_n)\right),$$

where Pol means that we keep only the polynomial part, i.e. the positive part of the Laurent series at $x_1\to\infty$. □

We see that Tutte's equations (7.4) are identical to the matrix model loop equation (5.6). The solution is thus the same, and it is expressed in terms of symplectic invariants. One only has to find the corresponding spectral curve.

7.3.1. Spectral curve and disc amplitude. The spectral curve is given by the function $W_1^{(0)}(x)$.

With $n=1$ and $g=0$, the loop equation (7.4) reads as

$$\left(W_1^{(0)}(x)\right)^2=V'(x)W_1^{(0)}(x)-P_1^{(0)}(x),$$

which implies

$$W_1^{(0)}(x)=\tfrac{1}{2}\left(V'(x)-\sqrt{V'(x)^2-4P_1^{(0)}(x)}\right),$$

where $P_1^{(0)}(x)$ is a polynomial of degree $\deg V'-1$ in x, namely

$$P_1^{(0)}(x)=t-\sum_{j=2}^{d-1}t_{j+1}\sum_{i=0}^{j-1}x^i\mathcal{T}^{(0)}_{j-1-i}.$$

Note that equation (7.1) implies that discrete surfaces with $g=0$ and $n=1$ must have $v\geqslant 2$, and thus

$$\mathcal{T}_j^{(0)}=t\delta_{j,0}+O(t^2).$$

Therefore, $P_1^{(0)}(x)$ is a formal series in t such that

$$P_1^{(0)}(x)=t\frac{V'(x)}{x}+O(t^2). \tag{7.5}$$

A general spectral curve with filling fractions $\epsilon_i=\frac{-1}{2i\pi}\oint_{\mathcal{A}_i}W_1^{(0)}(x)\,dx$ would correspond to (see equation (5.8)

$$P_1^{(0)}(x)=\sum_{i=1}^{d-1}\epsilon_i\frac{V'(x)}{x-X_i}+O(t^2),$$

where X_i, $i=1,\dots,d-1$, are the zeros of $V'(x)$.

In other words, the spectral curve counting discrete surfaces has only one non-vanishing filling fraction; it is a 1-cut spectral curve or, equivalently, it is a genus $\bar{g} = 0$ spectral curve.

More precisely, equation (7.5) implies that the zeros of $V'(x)^2 - 4P_1^{(0)}(x)$ have the following small t behavior:

- two zeros a, b are of the form
$$a \sim 2\sqrt{t} + O(t), \qquad b \sim -2\sqrt{t} + O(t);$$
- there are $d-2$ double zeros of the form $X_i \pm O(t)$, where $V'(X_i) = 0$ and $X_i \neq 0$.

Therefore, there exist a, b and a polynomial $M(x)$ such that
$$V'(x)^2 - 4P_1^{(0)}(x) = (x-a)(x-b)M(x)^2,$$
and thus
$$W_1^{(0)}(x) = \tfrac{1}{2}(V'(x) - M(x)\sqrt{(x-a)(x-b)}), \tag{7.6}$$
where $a = 2\sqrt{t} + O(t)$, $b = -2\sqrt{t} + O(t)$, $M(x) = \frac{V'(x)}{x} + O(t)$.

7.3.2. Rational parametrization. Since the spectral curve has only 1-cut $[a, b]$, it has genus $\bar{g} = 0$, and thus it is a rational spectral curve, and it can be parametrized by rational functions of a complex variable. Here, this can be done very explicitly.

We parametrize x as
$$x(z) = \frac{a+b}{2} + \frac{a-b}{4}\left(z + \frac{1}{z}\right) = \alpha + \gamma\left(z + \frac{1}{z}\right), \qquad \alpha = \frac{a+b}{2}, \qquad \gamma = \frac{a-b}{4}.$$
This parametrization is convenient because we have
$$\sqrt{(x-a)(x-b)} = \gamma\left(z - \frac{1}{z}\right)$$
and therefore, from equation (7.6), we see that $W_1^{(0)}(x(z))$ is a rational fraction of z.

Since $x(z) = x(1/z)$, we can find some complex numbers $u_0, u_1, \ldots, u_{d-1}$ such that
$$V'(x(z)) = \sum_{k=0}^{d-1} u_k(z^k + z^{-k}),$$
and similarly
$$M(x(z))\sqrt{(x(z)-a)(x(z)-b)} = \sum_{k=1}^{d-1} \tilde{u}_k(z^k - z^{-k}).$$
Thus, we have
$$W_1^{(0)}(x(z)) = u_0 + \frac{1}{2}\sum_{k\geqslant 1}(u_k - \tilde{u}_k)z^k + \frac{1}{2}\sum_{k\geqslant 1}(u_k + \tilde{u}_k)z^{-k}.$$
Since, by definition, $W_1^{(0)}(x(z))$ contains only negative powers of x, $W_1^{(0)}(x) \sim \frac{t}{x} + O(1/x^2)$, it must contain only negative powers of z, and therefore we must have $u_0 = 0$ and $\tilde{u}_k = u_k$, i.e. $W_1^{(0)}(x(z))$ is a polynomial in $1/z$:
$$W_1^{(0)}(x(z)) = \sum_{k=1}^{d-1} u_k z^{-k}.$$

Since $W_1^{(0)}(x) \sim \frac{t}{x}$ at large x, the coefficient of $1/z$ must be $u_1 = t/\gamma$. Therefore, a and b are determined by the two equations:

$$u_0 = 0, \qquad u_1 = \frac{t}{\gamma}.$$

All this can be summarized by the theorem

Theorem 7.3. *Let $V'(x) = x - \sum_{k=2}^{d-1} t_{k+1} x^k$, where t_k is the Boltzmann weight for k-gons. For an arbitrary α and γ, we write*

$$V'(\alpha + \gamma(z + 1/z)) = \sum_{k=0}^{d-1} u_k (z^k + z^{-k}).$$

The coefficients u_k are thus polynomials of α and γ. We determine α and γ by

$$u_0 = 0, \qquad u_1 = \frac{t}{\gamma}$$

and by the conditions that $\alpha = O(t)$ and $\gamma^2 = t + O(t^2)$ at small t.

Then the spectral curve $\mathcal{E} = (\mathbb{P}^1, x, y)$ is

$$x(z) = \alpha + \gamma(z + 1/z), \qquad y(z) = -W_1^{(0)}(x(z)) = -\sum_k u_k z^{-k}.$$

An example of the application of this theorem is given in section 7.4 for counting quadrangulations.

7.3.3. Generating function of the cylinder, annulus. For $n = 2$ and $g = 0$, Tutte's equations give

$$2W_1^{(0)}(x_1)W_2^{(0)}(x_1, x_2) + \frac{\partial}{\partial x_2}\frac{W_1^{(0)}(x_1) - W_1^{(0)}(x_2)}{x_1 - x_2} = V'(x_1)W_2^{(0)}(x_1, x_2) - P_2^{(0)}(x_1, x_2),$$

and one can prove that (see section 5.2.3)

$$W_2^{(0)}(x(z_1), x(z_2)) = \frac{1}{(z_1 - z_2)^2 x'(z_1)x'(z_2)} - \frac{1}{(x(z_1) - x(z_2))^2},$$

i.e.

$$W_2^{(0)}(x(z_1), x(z_2))\,\mathrm{d}x(z_1)\,\mathrm{d}x(z_2) + \frac{\mathrm{d}x(z_1)\,\mathrm{d}x(z_2)}{(x(z_1) - x(z_2))^2} = B(z_1, z_2) = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1 - z_2)^2},$$

which is the Bergmann kernel on $\mathbb{P}^1$, i.e. the Bergmann kernel of the spectral curve $\mathcal{E} = (\mathbb{P}^1, x, y)$.

7.3.4. Generating functions of discrete surfaces of higher topologies. For arbitrary n and g, the generating functions $W_n^{(g)}$ counting discrete surfaces of genus g with n boundaries are obtained from $\omega_n^{(g)}$'s of the spectral curve $\mathcal{E} = (\mathbb{P}^1, x, y)$, by

$$\omega_n^{(g)}(z_1, \ldots, z_n) = W_n^{(g)}(x(z_1), \ldots, x(z_n))\,\mathrm{d}x(z_1)\ldots\mathrm{d}x(z_n) + \delta_{n,2}\delta_{g,0}\frac{\mathrm{d}x(z_1)\,\mathrm{d}x(z_2)}{(x(z_1) - x(z_2))^2}.$$

In particular, with $n = 0$, the generating function for counting surfaces of genus g and no boundary is given by the symplectic invariants of $\mathcal{E}$:

$$\sum_{v=1}^{\infty} t^v \sum_{S \in \mathbb{M}_0^{(g)}(v)} \frac{t_3^{n_3(S)} \ldots t_d^{n_d(S)}}{\#\mathrm{Aut}(S)} = F_g(\mathcal{E}).$$

7.4. Example quadrangulations

If we count only quadrangulations, we choose $t_4 \neq 0$, and all other $t_k = 0$, i.e.

$$V'(x) = x - t_4 x^3.$$

Let us see how to apply theorem 7.3 in this case. We write

$$x(z) = \alpha + \gamma\left(z + \frac{1}{z}\right)$$

and

$$\begin{aligned} V'(x(z)) &= x(z) - t_4 x^3(z) \\ &= \alpha + \gamma(z + z^{-1}) - t_4(\alpha^3 + 3\alpha^2\gamma(z + z^{-1}) \\ &\quad + 3\alpha\gamma^2(z^2 + 2 + z^{-2}) + \gamma^3(z^3 + 3z + 3z^{-1} + z^{-3})). \end{aligned}$$

In other words, $V'(x(z)) = \sum_k u_k(z^k + z^{-k})$ corresponds to

$$\begin{aligned} &2u_0 = \alpha - t_4(\alpha^3 + 6\alpha\gamma^2), \qquad u_1 = \gamma - 3t_4(\alpha^2\gamma + \gamma^3) \\ &u_2 = -3t_4\alpha\gamma^2, \qquad u_3 = -t_4\gamma^3. \end{aligned}$$

The condition $u_0 = 0$ implies

$$0 = \alpha(1 - t_4(\alpha^2 + 6\gamma^2)).$$

Since we must choose a solution where $\alpha = O(t)$ and $\gamma^2 = O(t)$ at small t, we must choose $\alpha = 0$. Then the condition $u_1 = t/\gamma$ gives

$$t = \gamma^2 - 3t_4\gamma^4,$$

i.e.

$$\gamma^2 = \frac{1 - \sqrt{1 - 12tt_4}}{6t_4}.$$

Then the spectral curve is

$$x(z) = \gamma\left(z + \frac{1}{z}\right), \qquad y(z) = -\frac{t}{\gamma z} + \frac{t_4\gamma^3}{z^3}.$$

7.4.1. Rooted planar quadrangulations. The number of planar ($g = 0$) quadrangulations with only one boundary of length $2l$ is given by equation (7.2), i.e. $\mathcal{T}_{2l}^{(0)} = \mathrm{Res}_\infty x(z)^{2l} y(z)\,\mathrm{d}x(z)$, and thus, by computing the residue in the z variable, one easily finds

$$\mathcal{T}_{2l}^{(0)} = \gamma^{2l}\frac{(2l)!}{l!(l+2)!}(2(l+1)t - l\gamma^2).$$

In particular, if we require all faces, including the marked face of the boundary, to be quadrangles, we choose $2l = 4$, and we find the generating function counting planar quadrangulations with one marked edge (rooted quadrangulations)[22]:

$$\begin{aligned} \mathcal{T}_4^{(0)} &= \sum_v t^v \sum_{S \in \mathbb{M}_1^{(0)}(v), l(S)=4} t_4^{n_4(S)} \\ &= \gamma^4(3t - \gamma^2) \\ &= t^3 \sum_n \frac{23^n(2n)!}{n!(n+2)!}(tt_4)^{n-1}. \end{aligned}$$

[22] Note that discrete surfaces with $n = 1$ marked face and marked edge can have no non-trivial symmetry conserving the marked edge, and thus #Aut = 1.

Thus, we recover the famous result of Tutte [136] that the number of rooted planar quadrangulations with $n = n_4 + 1$ faces is

$$\frac{2 3^n (2n)!}{n!(n+2)!}.$$

7.4.2. Quadrangulations of the annulus. The generating function counting quadrangulations of the annulus $g = 0, n = 2$ is given by the Bergmann kernel:

$$W_2^{(0)}(x_1, x_2) = \sum_v t^v \sum_{S \in \mathbb{M}_2^{(0)}(v)} \frac{t_4^{n_4(S)}}{x_1^{l_1(s)} x_2^{l_2(s)} \#\mathrm{Aut}(S)} = \frac{1}{(z_1 - z_2)^2 x'(z_1) x'(z_2)} - \frac{1}{(x_1 - x_2)^2}$$

and thus, if we fix the perimeter lengths of the two boundaries as $2l_1$ and $2l_2$, we have from equation (7.2)

$$\begin{aligned} \mathcal{T}_{2l_1,2l_2}^{(0)} &= \sum_v t^v \sum_{S \in \mathbb{M}_2^{(0)}(v), l_1(S)=2l_1, l_2(S)=2l_2} \frac{t_4^{n_4(S)}}{\#\mathrm{Aut}(S)} \\ &= \operatorname*{Res}_{z_1 \to \infty} \operatorname*{Res}_{z_2 \to \infty} \frac{x(z_1)^{2l_1} x(z_2)^{2l_2}}{(z_1 - z_2)^2} \, \mathrm{d}z_1 \, \mathrm{d}z_2 \\ &= \gamma^{2(l_1+l_2)} \sum_{j=1}^{\min(l_1,l_2)} 2j \frac{(2l_1)!(2l_2)!}{(l_1+j)!(l_1-j)!(l_2+j)!(l_2-j)!}. \end{aligned}$$

If we require the marked boundary faces to be quadrangles, we choose $2l_1 = 2l_2 = 4$ and thus

$$\begin{aligned} \mathcal{T}_{4,4}^{(0)} &= \sum_v t^v \sum_{S \in \mathbb{M}_2^{(0)}(v), l_1(S)=4, l_2(S)=4} \frac{t_4^{n_4(S)}}{\#\mathrm{Aut}(S)} \\ &= 36\gamma^8 \\ &= 36t^4 \sum_{n \geqslant 2} \frac{(2n+2)!}{n!(n+2)!} (3tt_4)^{n-2}, \end{aligned}$$

i.e. the number of annulus quadrangulations with $n = n_4 + 2$ faces, where all faces including the two marked faces are quadrangles, is $4 \times 3^n \frac{(2n+2)!}{n!(n+2)!}$.

7.4.3. Quadrangulations on a pair of pants. The generating function for quadrangulations on the pair of pants $g = 0, n = 3$ is given by $\omega_3^{(0)}$:

$$\omega_3^{(0)}(z_1, z_2, z_3) = \frac{1}{2\gamma y'(1)} \left(\frac{\mathrm{d}z_1}{(z_1-1)^2} \frac{\mathrm{d}z_2}{(z_2-1)^2} \frac{\mathrm{d}3}{(z_3-1)^2} - \frac{\mathrm{d}z_1}{(z_1+1)^2} \frac{\mathrm{d}z_2}{(z_2+1)^2} \frac{\mathrm{d}z_3}{(z_3+1)^2} \right)$$

and, for instance, we find

$$T_{4,4,4}^{(0)} = (12)^3 \frac{\gamma^{12}}{2t} \left(1 + \frac{1}{\sqrt{1 - 12tt_4}} \right) = t^5 \sum_n 2^6 3^n \frac{(2n+1)!}{(n+2)!(n-1)!} (tt_4)^{n-3},$$

i.e. the number of quadrangulations on a pair of pants, where all n faces, including the three marked faces, are quadrangles, is $2^6 3^n \frac{(2n+1)!}{(n+2)!(n-1)!}$.

7.4.4. Quadrangulations on a genus 1 disc. The generating function for quadrangulations on the genus 1 disc $g = 1, n = 1$ is given by $\omega_1^{(1)}$:

$$\omega_1^{(1)}(z) = \frac{-z + 8z^3 - z^5 + \gamma^2(z - 5z^3 + z^5)}{3(\gamma^2 - 2)^2(z^2 - 1)^4}$$

and

$$\mathcal{T}_4^{(1)} = \frac{\gamma^6}{(\gamma^2 - 2)^2} = \frac{1}{6t_4}\left(\frac{1}{1 - 12t_4} - \frac{1}{\sqrt{1 - 12t_4}}\right)$$
$$\mathcal{T}_4^{(1)} = 2\left(1 + \binom{-1/2}{n}(-1)^{n-1}\right)(12t_4)^{n-1} = 2\left(1 - \frac{(2n-1)!!}{n!2^n}\right)(12t_4)^{n-1},$$

i.e. the number of rooted quadrangulations of genus 1 with n faces is

$$\frac{1}{6}\left(1 - \frac{(2n-1)!!}{n!2^n}\right)(12)^n = \frac{3^n}{6}\left(2^{2n} - \frac{(2n)!}{n!n!}\right).$$

7.5. Colored surfaces

Exactly like the way the 1-matrix integral is related to the enumeration of discrete surfaces, the 2-matrix integral and the chain of matrices are also related to the enumeration of discrete surfaces carrying colors (one color per matrix).

7.5.1. The Ising model on a discrete surface. The Ising model on a random lattice is a problem of the enumeration of bicolored discrete surfaces, and it is related to the 2-matrix model. In the random matrix context, it was introduced by Kazakov [98].

Consider $\mathbb{M}_n^{(g)}$ = the set of connected orientable discrete surfaces of genus g obtained by gluing together polygonal faces of two possible colors (or spin) $\pm$, namely n_{3+} triangles of color +, n_{4+} quadrangles of color +, ... $n_{k+}k$-angles of color +, and also n_{3-} triangles of color $-$, n_{4-} quadrangles of color $-$, ... $n_{k-}k$-angles of color $-$, as well as n marked polygonal faces of color + of perimeters $l_1, \ldots, l_n$, each of the marked faces having one marked edge on its boundary. Let us call v the number of vertices of a discrete surface and $\mathbb{M}_n^{(g)}(v)$ be the set of discrete surfaces in $\mathbb{M}_n^{(g)}$, with v vertices. We call n_{++} the number of edges with + on both sides, n_{--} the number of edges with $-$ on both sides and n_{+-} the number of edges separating faces of different colors.

We require that unmarked faces have perimeter $\geqslant 3$, whereas marked faces are only required to have perimeter $l_i \geqslant 1$. Moreover, we recall that marked faces are required to have color +.

We define their generating functions as follows:

the generating function is the formal power series in t:

$$W_n^{(g)}(x_1, \ldots, x_n; t_2, t_3, \ldots, t_d; \tilde{t}_2, \tilde{t}_3, \ldots, \tilde{t}_{\tilde{d}}; t)$$
$$= \frac{t}{x_1}\delta_{n,1}\delta_{g,0} + \sum_{v=1}^{\infty} t^v \sum_{S \in \mathbb{M}_n^{(g)}(v)} \frac{1}{\#\mathrm{Aut}(S)}$$
$$\times \frac{t_3^{n_{3+}(S)} \ldots t_d^{n_{d+}(S)} \tilde{t}_3^{n_{3-}(S)} \ldots \tilde{t}_{\tilde{d}}^{n_{\tilde{d}-}(S)}}{x_1^{1+l_1(S)} \ldots x_n^{1+l_n(S)}} \frac{t_2^{n_{--}(S)} \tilde{t}_2^{n_{++}(S)}}{(t_2\tilde{t}_2 - 1)^{n_{++}(S)+n_{++}(S)+n_{+-}(S)}}.$$

For instance, if we want to consider the Ising model on a random triangulation, we choose $t_k = 0$ for $k \geqslant 4$ and $\tilde{t}_k = 0$ for $k \geqslant 4$, i.e. only $t_3 \neq 0$ and $\tilde{t}_3 \neq 0$.

One may write Tutte-like recursion equations on the number of edges, which are identical to the loop equations of the 2-matrix model [59, 131] and for which we may use the results of section 5.3. Like in the 1-matrix model, the spectral curve is found by the requirement that $W_1^{(0)}(x)$ be a 1-cut solution of some algebraic equation and have a good small t expansion. The recipe for finding the correct spectral curve is summarized in the following theorem.

Theorem 7.4. *We define the potentials*

$$V_1'(x) = t_2 x - \sum_{j=2}^{d-1} t_{j+1} x^j, \qquad V_2'(y) = \tilde{t}_2 y - \sum_{j=2}^{\tilde{d}-1} \tilde{t}_{j+1} y^j.$$

For arbitrary coefficients $\gamma, \alpha_i, \beta_i$, we define the two rational functions

$$x(z) = \gamma z + \sum_{i=0}^{\tilde{d}-1} \alpha_i z^{-i}, \qquad y(z) = \gamma z^{-1} + \sum_{i=0}^{d-1} \beta_i z^i.$$

The coefficients $\gamma, \alpha_i, \beta_i$ are uniquely determined by the conditions

$$V_1'(x(z)) - y(z) \sim \frac{t}{\gamma z} + O(z^{-2}), \qquad V_2'(y(z)) - x(z) \sim \frac{tz}{\gamma} + O(z^2)$$

such that γ^2 as well as α_i, β_i are power series in t which behave like $O(t)$ for $t \to 0$.

Then, the spectral curve is

$$\mathcal{E}_{\text{Ising}} = (\mathbb{P}^1, x, y),$$

the function $W_1^{(0)}(x)$ is given by

$$W_1^{(0)}(x(z)) = V_1'(x(z)) - y(z),$$

the function $W_2^{(0)}(x_1, x_2)$ is given by

$$W_2^{(0)}(x(z_1), x(z_2))x'(z_1)x'(z_2) = \frac{1}{(z_1 - z_2)^2} - \frac{x'(z_1)x'(z_2)}{(x(z_1) - x(z_2))^2}$$

and all the stable $W_n^{(g)}$'s are given by $\omega_n^{(g)}$'s of the spectral curve $\mathcal{E}_{\text{Ising}}$ by

$$W_n^{(g)}(x(z_1), \ldots, x(z_n))\, \mathrm{d}x(z_1) \ldots \mathrm{d}x(z_n) = \omega_n^{(g)}(z_1, \ldots, z_n).$$

In particular, the generating function for counting bicolored maps with no boundaries are the symplectic invariants $F_g(\mathcal{E}_{\text{Ising}})$.

Example. *Ising model on quadrangulations.* For quadrangulations, we choose only $t_4 \neq 0$ and $\tilde{t}_4 \neq 0$; we thus have

$$V_1'(x) = t_2 x - t_4 x^3, \qquad V_2'(y) = \tilde{t}_2 y - \tilde{t}_4 y^3.$$

Applying theorem 7.4, we find

$$x(z) = \gamma z + \alpha_1 z^{-1} + \alpha_3 z^{-3}, \qquad y(z) = \gamma z^{-1} + \beta_1 z + \beta_3 z^3,$$

where the coefficients $\gamma, \alpha_i, \beta_i$ must obey the equations

$$\beta_3 = -t_4 \gamma^3, \qquad \beta_1 = t_2 \gamma - 3t_4 \gamma^2 \alpha_1, \qquad \alpha_3 = -\tilde{t}_4 \gamma^3,$$
$$\alpha_1 = \tilde{t}_2 \gamma - 3\tilde{t}_4 \gamma^2 \beta_1, \qquad \alpha_1 \beta_1 + 3\alpha_3 \beta_3 = \gamma^2 + t.$$

This gives an algebraic equation for γ^2:

$$3t_4 \tilde{t}_4 \gamma^4 + \frac{(t_2 - 3\tilde{t}_2 t_4 \gamma^2)(\tilde{t}_2 - 3t_2 \tilde{t}_4 \gamma^2)}{(1 - 9\tilde{t}_4 t_4 \gamma^4)^2} = 1 + \frac{t}{\gamma^2},$$

and we choose the unique solution such that

$$\gamma^2 \sim \frac{t}{t_2 \tilde{t}_2 - 1} + O(t^2).$$

7.5.2. The chain of matrix discrete surfaces. A chain of matrices (see section 5.4), with m matrices $M_1, \dots, M_m$ can also be interpreted as a generating function for enumerating discrete surfaces with m possible colors. The 'colors' are labeled $1, \dots, m$. This is the generalization of maps ($m = 1$) and the Ising model ($m = 2$).

We are going to consider discrete surfaces, whose unmarked faces can have any color, and are at least triangles, and marked faces have color 1.

Consider $\mathbb{M}_n^{(g)}$ to be the set of connected orientable discrete surfaces of genus g obtained by gluing together polygonal faces of m possible colors $k = 1, \dots, m$ (let $n_{j,k}$ be the number of faces of size j and color k; we assume $j \geqslant 3$) and n marked faces of color 1, with a marked edge, and of size $l_i \geqslant 1, i = 1, \dots, n$.

Let $n_{\langle i,j \rangle}$ be the number of edges such that the two sides are faces of colors i and j.

Let C be the following Toeplitz matrix of color couplings:

$$C^{-1} = \begin{pmatrix} t_{2,1} & -1 & 0 & \dots & 0 \\ -1 & t_{2,2} & -1 & & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & & -1 & t_{2,m-1} & -1 \\ 0 & & 0 & -1 & t_{2,m} \end{pmatrix}.$$

We define their generating functions as follows. The generating function is the formal power series in t:

$$W_n^{(g)}(x_1, \dots, x_n; t_{i,j}; t) = \frac{t}{x_1}\delta_{n,1}\delta_{g,0} + \sum_{v=1}^{\infty} t^v \sum_{S \in \mathbb{M}_n^{(g)}(v)} \frac{\prod_{i \geqslant 3} \prod_{j=1}^{m} t_{i,j}^{n_{i,j}(S)}}{x_1^{1+l_1(S)} \dots x_n^{1+l_n(S)}} \frac{\prod_{i,j=1}^{m} C_{i,j}^{n_{\langle i,j \rangle}(S)}}{\#\mathrm{Aut}(S)}.$$

Again, one may write Tutte-like recursion equations on the number of edges, which are identical to the loop equations of the chain of matrices model [62], and we may apply the results of section 5.4. Like in the 1-matrix model, the spectral curve is found by the requirement that $W_1^{(0)}(x)$ be a 1-cut solution of some algebraic equation and have a good small t expansion. The recipe for finding the correct spectral curve is summarized in the following theorem (proved in the same way as for maps $m = 1$).

Theorem 7.5. *We define the potentials $V_1, \dots, V_m$ by*

$$V_k'(x) = t_{2,k}x - \sum_{j=2}^{d_k - 1} t_{j+1,k} x^j, \qquad k = 1, \dots, m.$$

For arbitrary coefficients $\alpha_{i,k}$, we define the rational functions

$$x_k(z) = \sum_{j=-s_k}^{r_k} \alpha_{j,k} z^j$$

with

$$r_1 = 1, \qquad r_{k+1} = r_k \cdot (d_k - 1),$$
$$s_m = 1, \qquad s_{k-1} = s_k \cdot (d_k - 1).$$

The coefficients $\gamma, \alpha_{i,k}$ are uniquely determined by the conditions

$$\begin{aligned} &\alpha_{1,1} = \alpha_{-1,m} = \gamma, \\ &\forall\, k = 2, \dots, m-1, \qquad V_k'(x_k(z)) = x_{k-1}(z) + x_{k+1}(z), \\ &\forall\, k = 1, \dots, m-1, \qquad \sum_j j \alpha_{j,k+1} \alpha_{-j,k} = t \end{aligned} \tag{7.7}$$

such that γ^2 as well as $\alpha_{i,k}$ are power series in t which behave like $O(t)$ for small t.

Then, the spectral curve is

$$\mathcal{E}_{\text{ch.mat}} = (\mathbb{P}^1, x_1, x_2),$$

the function $W_1^{(0)}(x)$ is given by

$$W_1^{(0)}(x_1(z)) = V_1'(x_1(z)) - x_2(z),$$

the function $W_2^{(0)}(x_1(z_1), x_1(z_2))$ is given by

$$W_2^{(0)}(x_1(z_1), x_1(z_2))x_1'(z_1)x_2'(z_2) = \frac{1}{(z_1 - z_2)^2} - \frac{x_1'(z_1)x_1'(z_2)}{(x_1(z_1) - x_1(z_2))^2}$$

and all the stable $W_n^{(g)}$'s are given by $\omega_n^{(g)}$'s of the spectral curve $\mathcal{E}_{\text{ch.mat}}$ by

$$W_n^{(g)}(x_1(z_1), \dots, x_1(z_n))\, \mathrm{d}x_1(z_1) \dots \mathrm{d}x_1(z_n) = \omega_n^{(g)}(z_1, \dots, z_n).$$

In particular, $F_g(\mathcal{E}_{\text{ch.mat}})$ is the generating functions counting maps of genus g with no boundaries.

This theorem allows one to compute explicitly the numbers of such surfaces of various topologies.

Remark 7.1. Because of equation (7.7), we see that the spectral curves $(\mathbb{P}^1, x_k, x_{k+1})$ are all symplectically equivalent for any $k = 1, \dots, m-1$ and, thus, we have

$$\forall k = 1, \dots, m-1, \qquad F_g = F_g((\mathbb{P}^1, x_k, x_{k+1})) = F_g((\mathbb{P}^1, x_{k+1}, x_k)),$$

i.e. F_g is independent of k.

8. Double scaling limits and large maps

8.1. Minimal models and continuous surfaces

In the preceding section, we explained how the symplectic invariants can be used to count discrete surfaces. Theorem 4.23 allows one to find various limits of symplectic invariants, and here it can be used to find the asymptotics of generating functions of large discrete surfaces [95].

The conjecture [141] was that large discrete surfaces tend toward continuous surfaces weighted with the Liouville theory action, possibly coupled to some conformal matter fields.

The idea is to count discrete surfaces made of large numbers of polygons and send the size of polygons to 0, so that the total area remains finite [37, 52, 81].

In section 7, we found the generating function for counting maps of genus g as the symplectic invariants $F_g(\mathcal{E})$ of some rational spectral curve $\mathcal{E}$. In this enumeration of maps, the expectation value of the number of k-gons in the considered maps is given by

$$\langle n_k \rangle = t_k \frac{\partial \ln F_g}{\partial t_k}$$

and the expectation value of the number of vertices is

$$\langle v \rangle = t \frac{\partial \ln F_g}{\partial t}.$$

Large discrete surfaces are obtained when these numbers diverge, i.e. when the parameters t_k (or t) approach a singularity, for which the spectral curve is singular (in the sense of equation (2.3)).

Let us now study the blow-up of the matrix models' spectral curves around these singularities.

8.2. *Minimal model* (p, q) *and KP hierarchy*

For discrete surfaces (formal 1-matrix model), bicolored discrete surfaces with an Ising model (2-matrix model) or the formal chain of matrices, the spectral curve depends on the parameter t in an algebraic way. We choose potentials V_k as well as a critical $t = t_c$, such that at $t = t_c$ the spectral curve has a cusp singularity of the type $y(z) \sim (x(z) - x(a))^{p/q}$ (see [42] for a list of critical potentials of minimal degrees for the 2-matrix model).

We expand $\mathcal{E}$ in the vicinity of the critical branch point $z \to a$ and $t \to t_c$:

$$\begin{aligned} &z = a + (t - t_c)^{\nu}\zeta \\ &\mathcal{E} \sim \begin{cases} x(z) = x(a) + (t - t_c)^{q\nu} Q(\zeta) + o((t - t_c)^{q\nu}) \\ y(z) = y(a) + (t - t_c)^{p\nu} P(\zeta) + o((t - t_c)^{p\nu}), \end{cases} \end{aligned} \tag{8.1}$$

where Q and P are polynomials of the complex variable ζ, of respective degrees q and p, and the exponent ν is a scaling exponent such that the blown-up spectral curve is regular.

We define the double scaling limit [95] spectral curve as the blow-up of the singularity:

$$\mathcal{E}_{(p,q)} = \begin{cases} x(\zeta) = Q(\zeta) \\ y(\zeta) = P(\zeta). \end{cases} \tag{8.2}$$

In order to find the exponent ν, one may note that for the chain of matrices (and thus also 1-matrix and 2-matrix models), the derivative with respect to t of the form $y\,\mathrm{d}x$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}(y\,\mathrm{d}x(z)) = \frac{\mathrm{d}z}{z} \sim (t - t_c)^{\nu}\frac{1}{a}\,\mathrm{d}\zeta.$$

Comparing this with equation (8.1), it is easy to see that this implies

$$\nu = \frac{1}{p + q - 1}$$

and

$$pP(\zeta)Q'(\zeta) - qQ(\zeta)P'(\zeta) = \frac{1}{a}.$$

Theorem 4.23 of section 4.8 implies that the symplectic invariants defined in equation (2.6) give the double scaling limit:

$$F_g(\mathcal{E}) \sim (t - t_c)^{(2-2g)\frac{p+q}{p+q-1}} F_g(\mathcal{E}_{(p,q)})(1 + o(1)).$$

The spectral curve (8.2) is the spectral curve of the (p, q) minimal model in conformal field theory [48], coupled to gravity [102]. It corresponds to a finite dimensional irreducible representation of the group of conformal transformations and has a central charge

$$c = 1 - 6\frac{(p - q)^2}{pq}.$$

The exponent $\nu = \frac{1}{p+q-1}$ or more precisely the exponent 'γ-string'

$$\gamma = 2 - 2(p + q)\nu$$

is given by the famous KPZ formula [48, 102]. Note that the symplectic invariance of F_g's under $x \leftrightarrow y$ is related to the $(p, q) \leftrightarrow (q, p)$ duality [101].

The corresponding τ function is

$$\tau(N) = \exp\left(\sum_{g=0}^{\infty} N^{2-2g} F_g\left(\mathcal{E}_{(p,q)}\right)\right).$$

It is a τ function of the (p, q) reduction of the integrable hierarchy of KP [16]. The function

$$F(t) = \sum_{g=0}^{\infty} t^{(2-2g)\frac{p+q}{p+q-1}} F_g(\mathcal{E}_{(p,q)}) = \ln\left(\tau\left(t^{\frac{p+q-1}{p+q}}\right)\right)$$

and its second derivative

$$u(t) = F''(t)$$

can be found as follows: find two differential operators $\hat{P} = d^p + pud^{p-2} + \cdots$ and $\hat{Q} = d^q + ud^{q-2} + \cdots$ (where $d = \mathrm{d}/\mathrm{d}t$) satisfying the string equation

$$[\hat{P}, \hat{Q}] = 1.$$

This equation implies the differential equation of KP for the function $u(t)$.

For example, for pure gravity $(p, q) = (3, 2)$ (with central charge $c = 0$), we have

$$\hat{Q} = d^2 - 2u, \qquad \hat{P} = d^3 - 3ud - \tfrac{3}{2}u'$$

and $[\hat{P}, \hat{Q}] = 1$ imply the Painlevé equation for $u = F''$:

$$3u^2 - \tfrac{1}{2}u'' = t.$$

8.3. Minimal model $(p, 2)$ and KdV

As a special case, we consider the 1-matrix model. In that case, the spectral curve is always hyperelliptical $y^2 = \mathrm{Pol}(x)$ and thus the only cusp singularities must be half-integers $y \sim x^{p/2}$, i.e. $q = 2$ and $p = 2k + 1$.

The operators $\hat{Q}$ and $\hat{P}$ are such that

$$\hat{Q} = d^2 - 2u(t),$$
$$\hat{P} = d^{2k+1} - (2k+1)u(t)d^{2k-1} + \sum_{j=0}^{2k-2} v_j(t)d^j.$$

The string equation $[\hat{P}, \hat{Q}] = 1$ implies a differential equation for $u(t)$:

$$R_{k+1}(u) = t$$

where R_k is the kth Gelfand–Dikii differential polynomial (see [16, 46, 48]). They obey the recursion

$$R_0 = 2, \qquad R'_{j+1} = -2uR'_j - u'R_j + \tfrac{1}{4}R'''_j.$$

The first few of them are

$$R_0 = 2, \qquad R_1 = -2u, \qquad R_2 = 3u^3 - \tfrac{1}{2}u'', \qquad R_3 = -5u^3 + \tfrac{5}{2}uu'' - \tfrac{1}{4}u'^2 - \tfrac{1}{8}u''''.$$

We have seen in the previous section that the spectral curve (8.2) of the $(2k+1, 2)$ scaling limit of the 1-matrix model is

$$\mathcal{E}_{(2k+1,2)} = \begin{cases} \tilde{x}(z) = z^2 - 2u_0 \\ \tilde{y}(z) = \sum_{j=0}^{k} \bar{t}_k u_0^{k-j} z^{2j+1}. \end{cases} \tag{8.3}$$

It is the classical limit of $(\hat{P}, \hat{Q})$, where $d = \mathrm{d}/\mathrm{d}t$ becomes a complex number z:

$$\hat{Q} = d^2 - 2u \quad \to \quad x = z^2 - 2u_0$$
$$\hat{P} = d^{2k+1} - (2k+1)ud^{2k-1} + \cdots \quad \to \quad y = z^{2k+1} - (2k+1)u_0 z^{2k-1} + \cdots.$$

The symplectic invariants defined in equation (2.6) give the double scaling limit:

$$F_g(\mathcal{E}_{1\mathrm{MM}}) \sim (t-t_c)^{\frac{(2-2g)(2k+3)}{(2k+2)}} F_g(\mathcal{E}_{(2k+1,2)}) + o\Big((t-t_c)^{\frac{(2-2g)(2k+3)}{(2k+2)}}\Big)$$

and, if $z_1, \ldots, z_n$ all lie in the vicinity of $1 + O((t-t_c)^{\frac{1}{2k+2}})$, we have

$$\omega_n^{(g)}(\mathcal{E}_{1\mathrm{MM}})(z_1, \ldots, z_n) \sim (t-t_c)^{\frac{(2-2g-n)(2k+3)}{(2k+2)}} \omega_n^{(g)}(\mathcal{E}_{(p,2)})(\zeta_1, \ldots, \zeta_n) + o((t-t_c)^{\frac{(2-2g-n)(2k+3)}{(2k+2)}}),$$

where $z_i = 1 + (t-t_c)^{\frac{1}{2k+2}}\zeta_i$.

The formal function

$$F(\xi) = \sum_g \xi^{\frac{(2-2g)(2k+3)}{(2k+2)}} F_g(\mathcal{E}_{(2k+1,2)})$$

is such that its second derivative

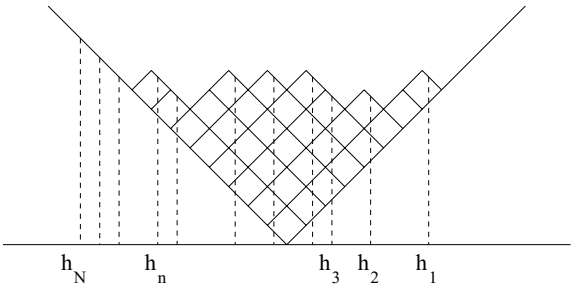
$$u(\xi) = F''(\xi)$$

satisfies the $(k+1)$st Gelfand–Dikii equation:

$$R_{k+1}(u) = \xi.$$

9. Partitions and the Plancherel measure

In many different problems of mathematics or physics, such as problems of crystal growth [92, 128, 137] and topological string theory [123, 125], one needs to count partitions with the Plancherel weight.



Given a partition $\lambda_1 \geqslant \lambda_2 \cdots \geqslant \lambda_N \geqslant 0$, we define the following.

- Its weight $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_N$.
- Its length $n(\lambda) = \#\{i, \lambda_i \neq 0\}$; we have $n(\lambda) \leqslant N$.
- Its Plancherel weight

$$\mathcal{P}(\lambda) = \left(\frac{\dim(\lambda)}{|\lambda|!}\right)^2 = \frac{\prod_{i>j}(h_i - h_j)^2}{\prod_i (h_i!)^2},$$

where $\dim(\lambda)$ is the dimension of the representation of the symmetric group indexed by λ and h_i's correspond to the down-right edges of the partition rotated by $\pi/4$:

$$h_i = \lambda_i - i + N, \qquad h_1 > h_2 > \cdots > h_N \geqslant 0.$$

- The Casimirs

$$C_k(\lambda) = \sum_{i=1}^N \left(h_i - \frac{N-1}{2}\right)^k - \sum_{i=1}^N \left(\frac{1}{2} - i\right)^k + (1-2^{-k})\zeta(-k).$$

For example,

$$C_1(\lambda) = |\lambda| - \frac{1}{24} = \sum_{i=1}^{N} h_i - \frac{N(N-1)}{2} - \frac{1}{24},$$

$$C_2(\lambda) = \sum_{i=1}^{N} \lambda_i(\lambda_i - 2i + 1) = \sum_i h_i^2 - (2N-1)\sum_i h_i + \frac{N(N+1)(N-1)}{3}.$$

9.1. The partition function

The partition function one would like to compute is

$$Z_N(Q) = \sum_{n(\lambda)\leqslant N} \left(\frac{\dim(\lambda)}{|\lambda|!}\right)^2 Q^{2|\lambda|}$$

and particularly its large Q expansion

$$\ln Z_N(Q) = \sum_g Q^{2-2g} F_g.$$

Indeed, since Q is coupled to the weight $|\lambda|$ of partitions, the large Q limit corresponds to the limit of large partitions.

More generally, one is also interested in the expectation values of moments of Casimirs and wishes to compute the partition function

$$Z_N(Q, t_k) = \sum_{n(\lambda)\leqslant N} \left(\frac{\dim(\lambda)}{|\lambda|!}\right)^2 Q^{2|\lambda|}\, \mathrm{e}^{-\sum_{k=1}^{d} \frac{t_k Q^{1-k}}{k} C_k(\lambda)}, \tag{9.1}$$

where we have taken into account the scaling of the Casimirs in the large Q limit. Again, we are interested in the large size expansion:

$$\ln Z_N(Q, t_k) \sim \sum_g Q^{2-2g} F_g(t_k, N).$$

The leading term F_0 was computed in many works [126]. The full expansion was computed in [66] and is given by the symplectic invariants of a spectral curve, which we describe in this section.

In the case where $t_k = 0$ for all k, the answer is classical, and we have the following identity [137]:

$$\sum_\lambda \left(\frac{\dim(\lambda)}{|\lambda|!}\right)^2 Q^{2|\lambda|} = \mathrm{e}^{Q^2}$$

and therefore

$$F_g = \delta_{g,0}.$$

When some of t_k's are turned on, the answer can be written in terms of a matrix model and symplectic invariants of an appropriate spectral curve. This issue is developed in the following sections.

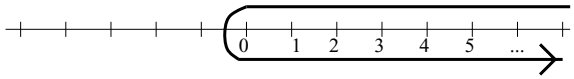
Some applications of this model include the statistics of the longest increasing subsequence of a random sequence, which is equivalent to the statistical physics of growing 2D crystals; indeed, the Plancherel measure $\mathcal{P}(\lambda)$ is precisely the probability of obtaining a shape λ by dropping square boxes from the sky at random times and random positions (see [126, 128]).

Another application concerns algebraic geometry and topological string theory. The partition function $Z_N(Q, t_k)$ is also the generating function for counting ramified branched coverings of $\mathbb{P}^1$, i.e. the Hurwitz numbers; see [112, 122, 123, 125, 126].

It has been observed in many works [15, 92, 128] that locally, in the large Q limit, the Plancherel statistics of partitions shows many similarities with universal statistical laws observed in various limits of matrix models. In fact, it was found in [66] that the similarity is much stronger, since, in fact, $Z_N(Q, t_k)$ *is a matrix model* for all Q (not only large Q). In particular, this shows that F_g's are again symplectic invariants.

9.2. *Matrix model for counting partitions*

Consider the contour $\mathcal{C}$ which surrounds all positive integer points:



The following matrix integral[23]

$$Z = \int_{H_N(\mathcal{C})} \mathrm{d}M\, \mathrm{e}^{-Q\,\mathrm{Tr}\,V(M)}$$

can be written in eigenvalues:

$$Z = \frac{1}{N!}\int_{\mathcal{C}^N} \mathrm{d}x_1 \ldots \mathrm{d}x_N \prod_{i>j}(x_i - x_j)^2 \prod_i \mathrm{e}^{-QV(x_i)},$$

where we choose

$$\mathrm{e}^{-QV(x)} = \frac{\mathrm{e}^{\mathrm{i}\pi Qx}}{\sin(\pi Qx)}\frac{Q^{2Qx}}{\Gamma(Qx+1)^2}\,\mathrm{e}^{-Q\sum_k \frac{t_k}{k}(x-(N-\frac{1}{2})/Q)^k}. \tag{9.2}$$

The integration over $\mathcal{C}$ picks up residues at the poles of $\mathrm{e}^{-QV(x)}$, i.e. at the poles of $\frac{1}{\sin \pi Qx}$, and thus forces $Qx_i \in \mathbb{N}$ and the factor $\prod_{i>j}(x_i - x_j)^2$ forces $x_i \neq x_j$. Thus we have $Qx_i = h_i \in \mathbb{N}$, and since they are dum variables, we can always reorder them so that $h_1 > h_2 > \cdots > h_N \geqslant 0$. Therefore, we have

$$\begin{aligned} Z &= \int_{H_N(\mathcal{C})} \mathrm{d}M\, \mathrm{e}^{-Q\,\mathrm{Tr}\,V(M)} \\ &= \sum_{h_1>\cdots>h_N\geqslant 0} \frac{\prod_{i>j}(h_i - h_j)^2}{\prod_i (h_i!)^2} Q^{2\sum_i h_i}\,\mathrm{e}^{-Q\sum_k \sum_i \frac{t_k}{k} Q^{-k}(h_i - N + \frac{1}{2})^k} \\ &\propto Z_N(Q, t_k), \end{aligned}$$

i.e. we recover the Plancherel generating function $Z_N(Q, t_k)$ for partitions equation (9.1). Therefore, $Z_N(Q, t_k)$ can be written as a normal matrix integral with eigenvalues supported on $\mathcal{C}$. Since the loop equations are independent of the integration contour, we have the same loop equations as for any other matrix model and thus the same solution in terms of symplectic invariants.

The spectral curve, i.e. the equilibrium density of the eigenvalues of the matrix, is computed like for any other matrix model, like in section 5. Here, the potential $V(M)$ may look quite complicated and the corresponding spectral curve also looks at first sight

[23] $H_N(\mathcal{C})$ stands for the set of normal matrices whose eigenvalues lie on a contour $\mathcal{C}$.

quite complicated. However, due to the special properties of the Γ functions, it simplifies considerably and reduces to a rather simple expression, which coincides with a 'naive' large Q limit. The naive large Q limit has been computed in many works [115, 125, 126, 137] and has been known for some time. It has the property that it is nearly independent of N. Let us just state the final result and refer the reader to [66] for details.

The spectral curve is obtained from the following recipe.

Theorem 9.1. *Define*

$$U(x) = \sum_{k=1}^{d-1} t_{k+1} x^k.$$

Then define the coefficients $u_0, u_1, \ldots, u_{d-1}$ by the equation

$$U(\mathrm{e}^{-u_0}(z + 1/z - u_1)) = \sum_{k=0}^{d-1} u_k (z^k + z^{-k}). \tag{9.3}$$

Then the spectral curve is

$$\mathcal{E}_{\text{Plancherel}} = \begin{cases} x(z) = \dfrac{N - \frac{1}{2}}{Q} + \mathrm{e}^{-u_0}(z + 1/z - u_1) \\ y(z) = \ln z + \dfrac{1}{2}\displaystyle\sum_{k=1}^{d-1} u_k (z^k - z^{-k}) \end{cases}$$

and the large Q expansion of $Z_N(Q, t_k)$ is given by

$$\ln Z_N(Q, t_k) \sim \sum_{g=0}^{\infty} Q^{2-2g} F_g(\mathcal{E}_{\text{Plancherel}}), \tag{9.4}$$

where F_g are the symplectic invariants defined in section 2.

Remark 9.1. Since $x \to x - \frac{N-\frac{1}{2}}{Q}$ is a symplectic transformation, F_g is independent of N and therefore equation (9.4) seems to be independent of N. In fact, the N dependence is smaller than any power of Q; it is exponentially small. This phenomenon is known as the arctic circle property. All partitions with size $n(\lambda) > \mathrm{e}^{-u_0}(u_1 + 2)Q$ seem to be frozen and contribute only exponentially to the partition function.

9.2.1. Example no Casimir. When all $t_k = 0$, we have $U = 0$, i.e. $u_0 = 0$ and $u_1 = 0$. Thus, the spectral curve is

$$\begin{cases} x(z) = \dfrac{N - \frac{1}{2}}{Q} + z + 1/z \\ y(z) = \ln z = \operatorname{Arcosh}\left(\left(x - \dfrac{N - \frac{1}{2}}{Q}\right)/2\right). \end{cases}$$

For this spectral curve, one finds

$$F_0 = 1, \qquad F_1 = F_2 = F_3 = \cdots = 0,$$

which is in agreement with

$$Z(Q) = \mathrm{e}^{Q^2}.$$

9.2.2. Example: Plancherel measure with the 2nd Casimir. If we choose $t_2 \neq 0$ and all other $t_k = 0$, we have from equation (9.3)

$$t_2 \mathrm{e}^{-u_0}\left(z + \frac{1}{z} - u_1\right) = 2u_0 + u_1\left(z + \frac{1}{z}\right)$$

and thus

$$t_2^2 = -2u_0\,\mathrm{e}^{2u_0}, \qquad u_1 = \sqrt{-2u_0}.$$

This gives

$$2u_0 = L\big(-t_2^2\big),$$

where $L(x)$ is the Lambert function, the solution of $L\,\mathrm{e}^L = x$.

Thus, the spectral curve is

$$\begin{cases} x(z) = \dfrac{N - \frac{1}{2}}{Q} + \mathrm{e}^{-u_0}(z + 1/z - u_1) \\ y(z) = \ln z + \dfrac{1}{2}u_1\left(z - \dfrac{1}{z}\right). \end{cases}$$

For this spectral curve, one finds the first orders' free energies:

$$F_0 = \frac{\mathrm{e}^{-2u_0}}{2}(1 + u_0)(2 - u_0),$$
$$F_1 = \frac{1}{24}\ln\left(\mathrm{e}^{-2u_0}(1 + 2u_0)\right),$$
$$F_2 = \frac{\mathrm{e}^{2u_0}}{180}\frac{u_0^3(1 - 12u_0)}{(1 + 2u_0)^5}.$$

The higher order free energies can be easily computed using the recursive procedure.

9.3. q-deformed partitions

The q-deformed Plancherel weight is obtained by replacing integer numbers $h_i \in \mathbb{N}$ by the q-numbers $[h_i] = \frac{q^{\frac{h_i}{2}} - q^{\frac{-h_i}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$, and thus

$$\mathcal{P}_q(\lambda) = \left(\frac{\dim_q(\lambda)}{[|\lambda|]!}\right)^2 = \left(\frac{\prod_{i>j}[h_i - h_j]}{\prod_{i=1}^N \prod_{j=1}^{h_i}[j]}\right)^2 = \frac{\prod_{i>j}\left(q^{\frac{h_i - h_j}{2}} - q^{\frac{h_j - h_i}{2}}\right)^2}{\prod_{i=1}^N \prod_{j=1}^{h_i}\left(q^{\frac{j}{2}} - q^{\frac{-j}{2}}\right)^2},$$

which can also be written as

$$\mathcal{P}_q(\lambda) = \prod_{i>j}(q^{h_i} - q^{h_j})^2 \prod_{i=1}^N q^{(1-N)h_i} q^{\frac{h_i(h_i+1)}{2}} \left(\frac{g(q^{-h_i})}{g(1)}\right)^2,$$

where $g(x)$ is the q-product:

$$g(x) = \prod_{n=1}^{\infty}\left(1 - \frac{1}{x}q^n\right).$$

Again, our goal, for various applications in physics and mathematics, is to compute the following sum:

$$Z_N(q; t_k) = \sum_{n(\lambda) \leqslant N} \mathcal{P}_q(\lambda)\,\mathrm{e}^{\frac{1}{\ln q}\sum_k \frac{t_k}{k}(\ln q)^k C_k(\lambda)}, \tag{9.5}$$

and one would like to compute it in the $q \to 1$ limit, i.e.

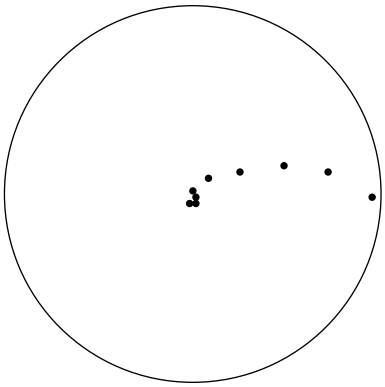
$$\ln Z_N \sim \sum_{g=0}^{\infty} (\ln q)^{2g-2} F_g.$$

Again, we will find that F_g's are the symplectic invariants of some spectral curve.

The main application concerns algebraic geometry and topological strings. $Z_N(q; t_k)$ is the partition function for the Gromov–Witten invariants of some family of Calabi–Yau threefold; see [112].

9.3.1. Matrix model. Again, the idea is to represent the sum (9.5) as a matrix integral.

Consider the contour $\mathcal{C}$ which surrounds all points of the form $1, q, q^2, q^3, \ldots$, i.e. $\mathcal{C}$ is a circle of radius $1 < r < |q^{-1}|$:



The following matrix integral

$$Z = \int_{H_N(\mathcal{C})} \mathrm{d}M \, \mathrm{e}^{\frac{1}{\ln q} \operatorname{Tr} V(M)}$$

can be written in eigenvalues:

$$Z = \frac{1}{N!} \int_{\mathcal{C}^N} \mathrm{d}x_1 \ldots \mathrm{d}x_N \prod_{i>j} (x_i - x_j)^2 \prod_i \mathrm{e}^{\frac{1}{\ln q} V(x_i)},$$

where we choose

$$\mathrm{e}^{\frac{1}{\ln q} V(x)} = \mathrm{e}^{\frac{1}{\ln q} \sum_k \frac{t_k}{k} (\ln x - (N - \frac{1}{2}) \ln q)^k} \, \mathrm{e}^{(1-N) \ln x} \frac{\sqrt{x} g(1/x)^2}{g(1)^2} \, \mathrm{e}^{\frac{(\ln x)^2}{2 \ln q}} f(x) \tag{9.6}$$

with

$$f(x) = -\frac{\mathrm{e}^{\mathrm{i}\pi \frac{\ln x}{\ln q}} g(1)^2 \, \mathrm{e}^{\frac{(\ln x)^2}{2 \ln q}}}{\sqrt{x}(1-x) g(x) g(1/x)}.$$

The integration over $\mathcal{C}$ picks up residues at the poles of $\mathrm{e}^{\frac{1}{\ln q} V(x)}$, i.e. at the poles of $f(x)$ and thus forces $\frac{\ln x_i}{\ln q} \in \mathbb{N}$ and the factor $\prod_{i>j} (x_i - x_j)^2$ forces $x_i \neq x_j$; thus we have $x_i = q^{h_i}$ where $h_i \in \mathbb{N}$, and since they are dum variables, we can always reorder them so that $h_1 > h_2 > \cdots > h_N \geqslant 0$. The residues are such that we have

$$Z_N(q; t_k) \propto \int_{H_N(\mathcal{C})} \mathrm{d}M \, \mathrm{e}^{\frac{1}{\ln q} \operatorname{Tr} V(M)}.$$

Therefore, as in the preceding case, $Z_N(q, t_k)$ can be written as a normal matrix integral with eigenvalues supported on $\mathcal{C}$. Once again, the small $\ln(q)$ expansion of the free energy is thus given by the corresponding symplectic invariants.

9.3.2. Spectral curve. It remains to compute the spectral curve, i.e. the equilibrium density of eigenvalues for the matrix potential:

$$V(x) = \sum_k \frac{t_k}{k}\left(\ln x - \left(N - \frac{1}{2}\right)\ln q\right)^k + (\ln x)^2 + \mathrm{i}\pi \ln x$$
$$+ (1-N)\ln q \ln x + \ln q \ln \frac{g(1/x)}{g(x)} - \ln q \ln(1-x).$$

Again, the spectral curve, i.e. the equilibrium density of eigenvalues of the matrix, is computed like for any other matrix model, like in section 5. Here, the potential $V(M)$ may look quite complicated and the corresponding spectral curve also looks at first sight quite complicated. However, due to the special properties of the q-product $g(x)$, it simplifies considerably and reduces to a rather simple expression, which coincides with a 'naive' small $\ln q$ limit. The naive small $\ln q$ limit has been computed in many works (see for example [112] and references therein) and has been known for some time. It has the property that it is nearly independent of N. Let us just state the final result and refer the reader to [66] for details.

In principle, the spectral curve could be found for all t_k's, but for simplicity, we compute it explicitly for $t_3 = t_4 = \cdots = 0$, i.e. for only t_1 and t_2. We write

$$t_1 = t, \qquad t_2 = p-1, \qquad t_3 = t_4 = t_5 = \cdots = 0.$$

In that case, the spectral curve is

$$\mathcal{E} = \begin{cases} x(z) = \dfrac{\left(1-\frac{z}{z_0}\right)\left(1-\frac{1}{zz_0}\right)}{\left(1+\frac{1}{z_0}\right)^2} \\ y(z) = \dfrac{1}{x(z)}\left(-\ln z + \dfrac{p}{2}\ln\left(\dfrac{1-z/z_0}{1-1/zz_0}\right)\right), \end{cases} \qquad \mathrm{e}^{-t} = \frac{1}{z_0^2}\left(1-\frac{1}{z_0^2}\right)^{p(p-2)}, \tag{9.7}$$

and thus we have

$$Z = \sum_{n(\lambda)\leqslant N}\left(\frac{\dim_q(\lambda)}{[|\lambda|]!}\right)^2 \mathrm{e}^{-t|\lambda|} q^{\frac{p-1}{2}C_2(\lambda)} \sim \mathrm{e}^{\sum_g (\ln q)^{2g-2} F_g(\mathcal{E})}.$$

The small $\ln q$ expansion of $\ln Z$ is given by the symplectic invariants of curve $\mathcal{E}$, and this expansion is independent of N, provided that $N > \overline{n}$, where

$$\overline{n} = \frac{1}{2} - \frac{1}{\ln q}\left(p\ln(1-1/z_0) + (p-2)\ln(1+1/z_0)\right).$$

In fact, this means that the N dependence is in nonperturbative terms smaller than any power of $\ln q$. This is again the arctic circle phenomenon; the partitions of size $> \overline{n}$ have an exponentially small probability and the size of the system seems to be frozen to $\overline{n}$.

9.3.3. Mirror curve. In topological strings (see [3, 28, 86, 110, 138] and section 11), it is known that Gromov–Witten invariants of some toric Calabi–Yau manifolds can be written as sums over partitions; this is called the topological vertex method.

In particular, for the family of the Hirzebruch Calabi–Yau manifold $X_p = \mathcal{O}(-p) \oplus \mathcal{O}(p-2) \to \mathbb{P}^1$, which is a rank-2 bundle over $\mathbb{P}^1$, the Gromov–Witten invariants $\mathcal{N}_{g,d}(X_p)$ which count the number of Riemann surfaces of genus g and degree d which can be embedded in X_p and go through given points [125, 127, 138] are given by [3]

$$\sum_g (\ln q)^{2g-2} \sum_d \mathrm{e}^{-td}\mathcal{N}_{g,d}(X_p) = \ln Z,$$

where

$$Z = \sum_{\lambda} \left(\frac{\dim_q(\lambda)}{[|\lambda|]!} \right)^2 \mathrm{e}^{-t|\lambda|} q^{\frac{p-1}{2} C_2(\lambda)}.$$

Therefore, we have found in the previous section that the Gromov–Witten invariants of X_p are given by the symplectic invariants of $\mathcal{E}$:

$$\sum_d \mathrm{e}^{-td} \mathcal{N}_{g,d}(X_p) = F_g(\mathcal{E}).$$

This result is interesting by itself, since it already gives a practical way of computing the Gromov–Witten invariants of X_p.

But we can go further. Note that $x(z) = u(z)$ is a rational function of z and $y(z) = \frac{1}{u(z)} \ln v(z)$ where $v(z)$ is also a rational function of z, and note that

$$\mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}\ln u \wedge \mathrm{d}\ln v;$$

thus, $\mathcal{E}$ is symplectically equivalent to the following spectral curve:

$$\tilde{\mathcal{E}} = \begin{cases} \tilde{x}(z) = \ln\left(\left(1 - \dfrac{z}{z_0}\right)\left(1 - \dfrac{1}{zz_0}\right)\right) \\ \tilde{y}(z) = \ln\left(\dfrac{1}{z}\left(\dfrac{1 - z/z_0}{1 - 1/zz_0}\right)^{\frac{p}{2}}\right), \end{cases} \qquad \mathrm{e}^{-t} = \frac{1}{z_0^2}\left(1 - \frac{1}{z_0^2}\right)^{p(p-2)}, \tag{9.8}$$

and thus we have

$$F_g(\mathcal{E}) = F_g(\tilde{\mathcal{E}}).$$

Note that $u = \mathrm{e}^{\tilde{x}}$ and $v = \mathrm{e}^{\tilde{y}}$ are both rational fractions of z, and thus, by eliminating z, there exists an algebraic relationship between them, i.e. there exists a polynomial H such that

$$H(u, v) = 0.$$

This curve is known in the context of topological strings [86]

$$H(\mathrm{e}^{\tilde{x}}, \mathrm{e}^{\tilde{y}}) = \omega_+ \omega_-$$

such that it is a Calabi–Yau three-fold $\tilde{X}_p$, which is a mirror of X_p under mirror symmetry, and $H(\mathrm{e}^x, \mathrm{e}^y) = 0$ is the singular locus of $\tilde{X}_p$. Therefore, we have obtained that, in agreement with the 'remodeling of the B model proposal' (see [28] and section 11), we have: *the Gromov–Witten invariants of X_p are the symplectic invariants of the singular curve of its mirror $\tilde{X}_p$.*

10. Intersection numbers and volumes of moduli spaces

The Kontsevich matrix integral is important because it encodes the topology of moduli spaces of Riemann surfaces. More precisely, it computes the intersection numbers of Chern classes of line bundles over the moduli spaces of Riemann surfaces [103, 141].

Let $\mathcal{M}_{g,n}$ be the moduli space of Riemann surfaces Σ of genus g, with n marked points $p_1, \ldots, p_n$. This moduli space is a complex manifold of dimension:

$$\dim \mathcal{M}_{g,n} = d_{g,n} = 3g - 3 + n.$$

It can be compactified into a compact space $\overline{\mathcal{M}}_{g,n}$ by adding all stable nodal surfaces (stable means that each component has a Euler characteristic <0).

The cotangent bundle $\mathcal{L}_i$ is the bundle over $\mathcal{M}_{g,n}$, whose fiber is the cotangent space of Σ at the point p_i. Let

$$\psi_i = c_1(\mathcal{L}_i)$$

be its first Chern class. $\mathcal{L}_i$ and ψ_i can be extended to $\overline{\mathcal{M}}_{g,n}$.

Chern classes ψ_i provide useful information on the topology of $\overline{\mathcal{M}}_{g,n}$. The intersection numbers are defined as

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle \stackrel{\text{def}}{=} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \psi_2^{d_2} \dots \psi_n^{d_n}$$

and are non-zero only if $\sum_i d_i = d_{g,n} = 3g - 3 + n$.

Another way to encode these intersection numbers is to define a generating function:

$$\begin{aligned} A_g(L_1, \dots, L_n) &= \sum_{d_1+\dots+d_n=d_{g,n}} \frac{d_{g,n}!}{d_1! \dots d_n!} L_1^{2d_1} \dots L_n^{2d_n} \langle \tau_{d_1} \dots \tau_{d_n} \rangle \\ &= \int_{\overline{\mathcal{M}}_{g,n}} \left(L_1^2 \psi_1 + L_2^2 \psi_2 + \dots + L_n^2 \psi_n \right)^{d_{g,n}}. \end{aligned}$$

Kontsevich [103] introduced a larger space:

$$\overline{\mathcal{M}}_{g,n}^{\text{comb}} = \overline{\mathcal{M}}_{g,n} \times \mathbb{R}_+^n$$

in which computations are easier.

First, since $\mathbb{R}_+^n$ is trivial, it is equivalent to computing Chern classes in $\overline{\mathcal{M}}_{g,n}^{\text{comb}}$ or in $\overline{\mathcal{M}}_{g,n}$.

Then, it was found, using the notion of Strebel differentials [103], that there is an (orbifold, i.e. divide by the symmetry group) isomorphism between the set of metric ribbon graphs and $\overline{\mathcal{M}}_{g,n}^{\text{comb}}$:

$$\overline{\mathcal{M}}_{g,n}^{\text{comb}} \sim \oplus \text{Ribbon graphs} \times \mathbb{R}_+^{\#\text{edges}},$$

where the sum is an orbifold sum (weighted by the inverse of the symmetry group) over all ribbon graphs of genus g, with n faces and with $6g - 6 + 3n$ edges. A metric graph means that we associate a length l_e with each edge $e = 1, \dots, 6g - 6 + 3n$.

This isomorphism tells that every Riemann surface of genus g with n marked points $p_1, \dots, p_n$ and n perimeters $L_1, \dots, L_n$ can be obtained in a unique way, by gluing n discs of perimeters $L_1, \dots, L_n$, along the edges of a Ribbon graph.

The lengths l_e, of the $6g - g + 3n$ edges provide a system of flat coordinates on $\overline{\mathcal{M}}_{g,n}^{\text{comb}}$, and Kontsevich noted that $\left(L_1^2 \psi_1 + L_2^2 \psi_2 + \dots + L_n^2 \psi_n \right)^{d_{g,n}} \mathrm{d}L_1 \dots \mathrm{d}L_n$ is a top dimensional volume form (indeed each ψ_i is a 2-form and thus $2d_{g,n} + n = 6g - 6 + 3n$), and it is flat in these coordinates, i.e. it is such that

$$\left(L_1^2 \psi_1 + L_2^2 \psi_2 + \dots + L_n^2 \psi_n \right)^{d_{g,n}} \mathrm{d}L_1 \dots \mathrm{d}L_n = \rho_{g,n} \prod_{e \in \text{edges}} \mathrm{d}l_e,$$

and he computed the proportionality constant

$$\rho_{g,n} = 2^{d_{g,n}} 2^{2g-2+n}.$$

Then the sum over Ribbon graphs can be realized as a matrix integral, and one may note that Ribbon graphs which are not trivalent form a subset of measure zero, so that one gets a cubic matrix integral. Also, instead of working with fixed $L_1, \dots, L_n$, it is easier to consider the Laplace transform and one gets the Kontsevich integral.

10.1. Kontsevich integral and intersection numbers

10.1.1. Matrix integral. Let Λ be a $N \times N$ diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$. Let

$$t_k = \frac{1}{N} \operatorname{tr} \Lambda^{-k}.$$

Kontsevich's integral [103] is the following formal matrix integral (defined as a formal power series in large Λ):

$$Z_{\text{Kontsevich}}(\Lambda) = \int \mathrm{d}M \, \mathrm{e}^{-N \operatorname{Tr} \frac{M^3}{3} + \Lambda M^2},$$

and we consider its topological expansion:

$$\ln Z_{\text{Kontsevich}}(\Lambda) = \sum_{g=0}^{\infty} N^{2-2g} F_g.$$

Note that, upon shifting $M \to M - \Lambda$, we have

$$Z_{\text{Kontsevich}}(\Lambda) = \mathrm{e}^{\frac{N}{3} \operatorname{Tr} \Lambda^3} \int \mathrm{d}M \, \mathrm{e}^{-N \operatorname{Tr}[\frac{M^3}{3} - \Lambda^2 M]},$$

and thus this integral is a special case of 1-matrix integral with an external field (see section 5.5), which implies that the coefficients F_g are the symplectic invariants of the corresponding spectral curve.

The Kontsevich integral is a generating function for intersection numbers:

$$\ln Z_{\text{Kontsevich}}(\Lambda) = \sum_g (N/2)^{2-2g} \sum_n \sum_{d_1 + \dots + d_n = d_{g,n}} \prod_i (2d_i - 1)!! \frac{t_{2d_i+1}}{2} \langle \tau_{d_1} \dots \tau_{d_n} \rangle. \tag{10.1}$$

For example we have (we choose $t_1 = 0$ for simplicity)

$$F_2 = \sum_k (t_3/2)^k \left(\frac{105 t_9}{2} \langle \tau_1^k \tau_4 \rangle + \frac{45 t_5 t_7}{4} \langle \tau_1^k \tau_2 \tau_3 \rangle + \frac{27 t_5^3}{8} \langle \tau_1^k \tau_2^3 \rangle \right).$$

10.1.2. Kontsevich's spectral curve. We have seen in section 5.5 that the topological expansion of a matrix integral with an external field of the type

$$Z = \int \mathrm{d}M \, \mathrm{e}^{-N \operatorname{Tr} V(M) - \Lambda^2 M}$$

is given by the symplectic invariants of its spectral curve:

$$\ln Z = \sum_g N^{2-2g} F_g(\mathcal{E}),$$

where the spectral curve $\mathcal{E}$ is characterized by the algebraic equation

$$V'(x) - y = \frac{1}{N} \sum_i \frac{P_i(x)}{y - \lambda_i^2}, \tag{10.2}$$

where $P_i(x)$ is a polynomial of x of degree at most $\deg V''$, which behaves at large x like $V'(x)/x$. Here, we have $V(x) = \frac{x^3}{3}$; hence $V'(x) = x^2$ and thus $P_i(x) = x + P_i(0)$.

The fact that Z is to be understood as a formal power series at large Λ means that we look for a rational spectral curve, and this determines all the polynomials $P_i(x)$.

Here, we find that the rational spectral curve of form (10.2) is

$$\mathcal{E}_K = \begin{cases} x(z) = z - \dfrac{1}{N} \operatorname{Tr} \dfrac{1}{2\hat{\Lambda}(\hat{\Lambda} - z)} \\ y(z) = z^2 + \hat{t}_1, \end{cases}$$

where $\hat{t}_1 = \frac{1}{N}\operatorname{Tr}\frac{1}{\hat{\Lambda}}$ and

$$\Lambda^2 = \hat{\Lambda}^2 + \hat{t}_1.$$

From now on, for simplicity, we shall assume that $\hat{t}_1 = 0$:

$$\hat{t}_1 = 0 = \frac{1}{N}\operatorname{Tr}\frac{1}{\hat{\Lambda}},$$

and therefore we have

$$\hat{\Lambda} = \Lambda$$

and the spectral curve is

$$\mathcal{E}_K = \begin{cases} x(z) = z - \dfrac{1}{N}\operatorname{Tr}\dfrac{1}{2\Lambda(\Lambda - z)} \\ y(z) = z^2. \end{cases}$$

10.1.3. Symplectic invariants. To compute F_g's of the spectral curve $\mathcal{E}_K$, we need to consider the branch points, i.e. the zeros of $x'(z)$, and they are quite complicated.

However, we may use symplectic invariance and compute F_g's after exchanging the roles of x and y and thus consider the spectral curve

$$\tilde{\mathcal{E}}_K = \begin{cases} x(z) = z^2 \\ y(z) = z - \dfrac{1}{N}\operatorname{Tr}\dfrac{1}{2\Lambda(\Lambda - z)}. \end{cases}$$

This spectral curve has now only one branch point solution of $x'(z) = 0$, which is located at $z = 0$. Since F_g's are obtained by computing residues near $z = 0$, we may Taylor expand $y(z)$ near $z = 0$, and we have

$$\tilde{\mathcal{E}}_K = \begin{cases} x(z) = z^2 \\ y(z) = z - \dfrac{1}{2}\displaystyle\sum_{k=0}^{\infty} t_{k+2} z^k. \end{cases} \tag{10.3}$$

Now, it is rather easy to compute the first few symplectic invariants:

$$\begin{aligned}
\omega_1^{(1)}(z) &= -\frac{\mathrm{d}z}{8(2-t_3)}\left(\frac{1}{z^4} + \frac{t_5}{(2-t_3)z^2}\right), \\
\omega_3^{(0)}(z_1, z_2, z_3) &= -\frac{1}{2-t_3}\frac{\mathrm{d}z_1\,\mathrm{d}z_2\mathrm{d}z_3}{z_1^2 z_2^2 z_3^2}, \\
\omega_2^{(1)}(z_1, z_2) &= \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{8(2-t_3)^4 z_1^6 z_2^6}\big[(2-t_3)^2\big(5z_1^4 + 5z_2^4 + 3z_1^2 z_2^2\big) \\
&\qquad + 6t_5^2 z_1^4 z_2^4 + (2-t_3)\big(6t_5 z_1^4 z_2^2 + 6t_5 z_1^2 z_2^4 + 5t_7 z_1^4 z_2^4\big)\big], \\
\omega_1^{(2)}(z) &= -\frac{\mathrm{d}z}{128(2-t_3)^7 z^{10}}\big[252t_5^4 z^8 + 12t_5^2 z^6(2-t_3)(50t_7 z^2 + 21t_5) \\
&\quad + z^4(2-t_3)^2\big(252t_5^2 + 348t_5 t_7 z^2 + 145t_7^2 z^4 + 308t_5 t_9 z^4\big) \\
&\quad + z^2(2-t_3)(203t_5 + 145z^2 t_7 + 105z^4 t_9 + 105z^6 t_{11}) \\
&\quad + 105(2-t_3)^4\big],
\end{aligned}$$

and so on.

For example, the first- and second-order free energies are

$$F^{(1)}_{\text{Kontsevitch}} = -\frac{1}{24}\ln\left(1-\frac{t_3}{2}\right)$$

and

$$F^{(2)}_{\text{Kontsevitch}} = \frac{1}{1920}\frac{252t_5^3 + 435t_5t_7(2-t_3) + 175t_9(2-t_3)^2}{(2-t_3)^5}. \tag{10.4}$$

Remark 10.1. The fact that the symplectic invariants depend only on odd t_k's can be understood in terms of symplectic invariance; indeed, adding to $y(z)$ any rational function of $x(z)$ (i.e. any rational function of z^2) is a symplectic transformation. It does not change F_g's, and therefore F_g's depend only on the odd part of $y(z)$, i.e. only on the odd t_k's.

10.1.4. Correlators and unmarked faces. The symplectic invariants of the spectral curve (10.3) are F_g's. They generate intersection numbers of ψ-classes, i.e. Chern classes of cotangent line bundles over marked points, i.e. they generate the intersection numbers of the type

$$\left\langle \psi_1^{d_1}\dots\psi_n^{d_n}\right\rangle.$$

The correlators $\omega_n^{(g)}$s of this spectral curve also have some interpretation in terms of integrals of some classes over moduli spaces.

Note that the Kontsevich integral (10.1) contains a summation over n, i.e. over the number of marked points. One may wish to distinguish some of these marked points, fix marked faces around them and perform the sum over the other marked points; in some sense, forget the other marked points.

The forgetful map is the map from $\overline{\mathcal{M}}_{g,n+m}$ to $\overline{\mathcal{M}}_{g,n}$, which consists in forgetting m marked points. Under this map, ψ classes project to the Mumford κ-classes:

$$\int_{\overline{\mathcal{M}}_{g,n+m}} \psi_1^{d_1}\dots\psi_n^{d_n}\prod_{k=1}^{m}\psi_{n+k}^{a_k+1} = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\dots\psi_n^{d_n}\sum_{\sigma\in\Sigma_m}\prod_{c=\text{cycles of }\sigma}\kappa_{(\sum_{i\in c}a_i)}.$$

For examples with $m=1$ and $m=2$,

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1}\dots\psi_n^{d_n}\psi_{n+1}^{a+1} = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\dots\psi_n^{d_n}\kappa_a,$$

$$\int_{\overline{\mathcal{M}}_{g,n+2}} \psi_1^{d_1}\dots\psi_n^{d_n}\psi_{n+1}^{a_1+1}\psi_{n+2}^{a_2+1} = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\dots\psi_n^{d_n}\left(\kappa_{a_1+a_2}+\kappa_{a_1}\kappa_{a_2}\right).$$

One finds [67] that the correlators $\omega_n^{(g)}(z_1,\dots,z_n)$, are the generating functions for κ classes, coupled to n ψ-classes:

$$\omega_n^{(g)}(z_1,\dots,z_n) = 2^{-d_{g,n}}(t_3-2)^{2-2g-n}\sum_{d_0+d_1+\dots+d_n=d_{g,n}}\sum_{k=1}^{d_0}\frac{1}{k!}\sum_{b_1+\dots+b_k=d_0,b_i>0}$$
$$\times\prod_{i=1}^{n}\frac{2\mathrm{d}_i+1!}{\mathrm{d}_i!}\frac{\mathrm{d}z_i}{z_i^{2d_i+2}}\prod_{l=1}^{k}\tilde{t}_{b_l}\left\langle\prod_{l=1}^{k}\kappa_{b_l}\prod_{i=1}^{n}\psi_i^{d_i}\right\rangle_{g,n},$$

where the coefficients $\tilde{t}_b$ are the Schur transform of t_k's:

$$\tilde{t}_b = \sum_{l=1}^{b}\frac{(-1)^l}{l}\sum_{a_1+\dots+a_l=b,a_i>0}\prod_j\frac{2a_j+1!}{a_j!}\frac{t_{2a_j+3}}{t_3-2}. \tag{10.5}$$

Their generating function is obtained by

$$f(z) = \sum_{a=1}^{\infty} \frac{2a+1!}{a!} \frac{t_{2a+3}}{2-t_3} z^a, \qquad -\ln(1-f(z)) = \sum_{b=1}^{\infty} \tilde{t}_b z^b. \tag{10.6}$$

Example.

$$\tilde{t}_1 = -6\frac{t_5}{t_3-2}, \qquad \tilde{t}_2 = -60\frac{t_7}{t_3-2} + 18\frac{t_5^2}{(t_3-2)^2}, \ldots.$$

$\omega_n^{(g)}$ is the Laplace transform of

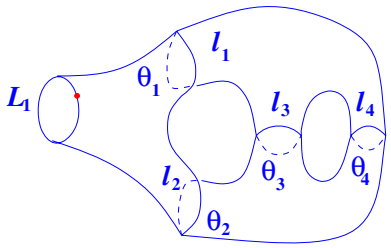
$$2^{d_{g,n}}(t_3-2)^{2g-2+n} V_{g,n}(L_1,\ldots,L_n)$$
$$= \sum_{d_0+d_1+\cdots+d_n=d_{g,n}} \prod_{j=1}^{n} \frac{L_j^{2\mathrm{d}_j}}{\mathrm{d}_j!} \sum_k \frac{1}{k!} \sum_{b_1+b_2+\cdots+b_k=d_0, b_i>0} \prod_{i=1}^{k} \tilde{t}_{b_i} \left\langle \prod_{i=1}^{k} \kappa_{b_i} \prod_j \psi_F^{d_j} \right\rangle. \tag{10.7}$$

$V_{g,n}$ can be interpreted as the generating function for counting intersection numbers of κ-classes, on the moduli-space of Riemann surfaces with n boundaries (discs removed from the surface), of perimeters $L_1,\ldots,L_n$.

10.2. Application: Weil–Petersson volumes

Consider the stable moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n boundaries (stability means $2-2g-n<0$). Every surface in $\mathcal{M}_{g,n}$ has a negative Euler characteristic and thus a negative average curvature. It can be equipped with a unique constant negative curvature metric, called the Poincaré metric, such that the boundaries are geodesics. Let $L_1,\ldots,L_n$ be the geodesic lengths of the boundaries and $\mathcal{M}_{g,n}(L_1,\ldots,L_n)$ be the moduli space of Riemann surfaces of genus g with n boundaries of geodesic lengths $L_1,\ldots,L_n$.

Every Riemann surface in $\mathcal{M}_{g,n}(L_1,\ldots,L_n)$ can be decomposed into $2g-2+n$ pairs of pants, whose boundaries are geodesics (such a decomposition is not unique). Conversely, $2g-2+n$ pairs of pants with fixed given boundary perimeters can be glued together to form a Riemann surface of $\mathcal{M}_{g,n}(L_1,\ldots,L_n)$ provided that the lengths of boundaries which are glued together match. The gluing is not unique because two circles of the same perimeter can be glued in many ways, twisted by an arbitrary angle.



The $3g-3+n$ geodesic lengths of the glued boundaries $l_1,\ldots,l_{3g-3+n}$, together with the $3g-3+n$ twisting angles $\theta_1,\ldots,\theta_{3g-3+n}$, provide a local set of coordinates parametrizing $\mathcal{M}_{g,n}(L_1,\ldots,L_n)$. It turns out that although we do not have uniqueness of the decomposition, the corresponding symplectic form, called the Weil–Petersson symplectic metric, is well defined on $\mathcal{M}_{g,n}(L_1,\ldots,L_n)$:

$$\Omega = \prod_{i=1}^{3g-3+n} \mathrm{d}l_i \wedge \mathrm{d}\theta_i,$$

and it can be extended to the compactified $\overline{\mathcal{M}}_{g,n}(L_1,\dots,L_n)$. The Weil–Petersson volume of $\overline{\mathcal{M}}_{g,n}(L_1,\dots,L_n)$ is then defined as

$$\mathrm{Vol}_{g,n}(L_1,\dots,L_n)=\int_{\overline{\mathcal{M}}_{g,n}(L_1,\dots,L_n)}\Omega,$$

where the n external boundaries are restricted to have fixed geodesic lengths L_i's and fixed angles (i.e. some marked points).

It can be proved [142] that the Weil–Petersson metrics comes from the Kähler metrics

$$2\pi^2\kappa_1,$$

i.e. we have

$$\begin{aligned}\mathrm{Vol}_{g,n}(L_1,\dots,L_n)&=\left\langle\left(2\pi^2\kappa_1+\frac{1}{2}\sum_{i=1}^{n}L_i^2\psi_i\right)^{d_{g,n}}\right\rangle\\&=2^{-d_{g,n}}\sum_{d_0+\cdots+d_n=3g-3+n}\frac{2^{d_0}}{d_0!}\prod_{i=1}^{n}\frac{L_i^{2\mathrm{d}_i}}{\mathrm{d}_i!}\left\langle(2\pi^2\kappa_1)^{d_0}\psi_1^{d_1}\dots\psi_n^{d_n}\right\rangle.\end{aligned}$$

It can be made to coincide with equation (10.7), provided that we choose $\tilde{t}_1=4\pi^2$ and $t_3=3$. Doing the reverse transform using equation (10.6), it corresponds to $-\ln(1-f(z))=4\pi^2 z$, i.e. $f(z)=1-\mathrm{e}^{-4\pi^2 z}$, and thus

$$t_{2d+3}=\frac{(2\mathrm{i}\pi)^{2d}}{(2d+1)!}+2\delta_{d,0}$$

i.e. it corresponds to the spectral curve

$$\mathcal{E}_{\mathrm{WP}}=\begin{cases}x(z)=z^2\\ y(z)=\dfrac{1}{2\pi}\sin(2\pi z).\end{cases}\tag{10.8}$$

In other words, the Laplace transforms of the Weil–Petersson volumes $\mathrm{Vol}_{g,n}(L_1,\dots,L_n)$ are $W_n^{(g)}$'s of the spectral curve $\mathcal{E}_{\mathrm{WP}}$.

Taking the inverse Laplace transform of the recursion relations (2.4) which define $W_n^{(g)}$'s for the spectral curve (10.8), we recover Mirzakhani's recursion [120, 121] relations for the Weil–Petersson volumes (see [67, 72] for the proof).

10.3. Application: Witten–Kontsevich theorem

Witten's conjecture [141] was the assertion that the limit of the generating function of large discrete surfaces was indeed the generating function of intersection numbers for continuous Riemann surfaces. In other words, the double scaling limit of F_g's of maps should coincide with F_g's of the Kontsevich integral.

We have seen that F_g's of maps are given by symplectic invariants and thus their limits as $t\to t_{\mathrm{c}}$ are given by the symplectic invariants of the blown-up curve. Thus, F_g's of the double scaling limit of maps are F_g's of the minimal $(p,2)$ model, i.e. F_g's of the spectral curve

$$F_g(\mathcal{E}_{\mathrm{maps}})\sim(t-t_{\mathrm{c}})^{(2-2g)\mu}F_g(\mathcal{E}_{(p,2)}),$$

where $\mu=\frac{p+2}{p+1}$, and

$$\mathcal{E}_{(p,2)}=\begin{cases}x(z)=z^2-2u\\ y(z)=\displaystyle\sum_{k=0}^{p}\bar{t}_k z^k.\end{cases}$$

On the other hand, F_g's of the Kontsevich integral are also given by the symplectic invariants of a spectral curve, which is the equilibrium density of eigenvalues in the Kontsevich integral's matrix Airy function. The spectral curve (see section 10.1) is (again we assume for simplicity that $t_1 = 0$)

$$\mathcal{E}_K = \begin{cases} x(z) = z - \dfrac{1}{N} \operatorname{Tr} \dfrac{1}{2\Lambda(\Lambda - z)} \\ y(z) = z^2. \end{cases}$$

We have seen (theorem 4.1) that if two spectral curves are equivalent under symplectic transformations, then they have the same F_g's. In particular, F_g's do not change if we change $x \to y$ and $y \to -x$; thus

$$\mathcal{E}_k \sim \tilde{\mathcal{E}}_K = \begin{cases} x(z) = z^2 \\ y(z) = -z + \dfrac{1}{N} \operatorname{Tr} \dfrac{1}{2\Lambda(\Lambda - z)}. \end{cases}$$

In this new formulation, there is only one branch point located at $z = 0$, and since all quantities computed are residues at this branch point, we may Taylor expand $y(z)$ in the vicinity of $z = 0$ and thus

$$\mathcal{E}_K \sim \begin{cases} x(z) = z^2 \\ y(z) = -z + \dfrac{1}{2} \displaystyle\sum_{k=0}^{\infty} z^k \frac{1}{N} \operatorname{Tr} \Lambda^{-k-2}. \end{cases}$$

If we choose the diagonal matrix Λ such that

$$\bar{t}_k = \frac{1}{2N} \operatorname{Tr} \Lambda^{-k-2} - \delta_{k,1},$$

we have

$$\mathcal{E}_K \sim \mathcal{E}_{(p,2)}$$

and therefore

$$F_g(\mathcal{E}_K) = F_g(\mathcal{E}_{(p,2)}),$$

which confirms Witten's conjecture.

In fact, this conjecture was first proved by Kontsevich [103]. Kontsevich's method consisted of two parts: first prove that the Airy matrix integral now known as the Kontsevich integral was the generating function of intersection numbers and then prove that both $F_{g,K}$ and $F_{g,(p,2)}$ obeyed the same set of differential equations, namely KdV hierarchy. Witten's conjecture has received several proofs since then, in particular Looijenga's [109], or Okounkov–Pandharipande's [127].

Here we see that the spectral curve of the Kontsevich model and the spectral curve of the $(p, 2)$ model are obtained to one another by the symplectic transformation $x \to y$, $y \to -x$. That symplectic transformation (the $\pi/2$ rotation in the (x, y) plane), leaves F_g invariant, but it changes $\omega_n^{(g)}$'s.

11. Application: topological strings

The counting of surfaces with particular weights, or enumerative geometry, has been very important in physics since the arising of string theories. Indeed, these theories consist in replacing the point-like particles of usual theories such as classical mechanics (0-dimensional objects) by strings (one-dimensional objects); a state is now given by a string state instead of

a point state. A string evolving in time describes a surface in the spacetime, the worldsheet, instead of a line for a point evolving in spacetime. Hence, the usual path integral corresponding to sum over all possible histories from one initial state to one final state is now a 'sum' over all possible surfaces in spacetime (target space), linking the initial strings to the final ones. This could explain why these theories are related to the symplectic invariants since the latter already appear in many problems of 'surface enumeration'. So far, no proof is available but there exist many hints that the symplectic invariants of some specific spectral curve should be the partition functions of some particular string theories: type IIB topological strings on some special target space. Many checks have been made that this is indeed the case [112], so that Bouchard *et al* [28] proposed to define the topological string partition function and observables as the symplectic invariants and correlation functions, considering the spectral curve as the target space of the string theory.

Even if this conjecture is not proved yet, some new clues were recently given by Dijkgraaf and Vafa [50] who already conjectured that some matrix models are dual to some particular topological string theories [49]. These new hints rely on the study of an effective field theory for the string theory: the Kodaira–Spencer theory. We review these advances in the second part of the present section.

11.1. Topological string theory

11.1.1. Introduction. Type IIB topological string theories are obtained by twisting $N = 2$ superconformal sigma model in dimensions. The precise description would lead us too far away from the main topic of this review, and the interested reader is invited to consult [86, 110, 111, 138] for details as well as [29] for the particular topic of toric geometries. There exist two ways of twisting this theory leading to two different models referred to as A and B models. We will be particularly interested in this paper in the B model and would not develop the A model too far. Nevertheless, since the special geometries of the target space of the B model we are interested in are inherited from the A model, we will say a few words about the latter and especially the possible geometries of its target space and of the objects one wants to compute in the following section.

First of all, let us mention that the topological string theories can be thought of as theories of the maps from a Riemann surface Σ (the worldsheet) to a Calabi–Yau manifold M (the target space) of arbitrary dimension. More precisely, in the A model side, the amplitudes to be computed are related to the Gromov–Witten invariants as follows. Let us consider a worldsheet $\Sigma_k^{(g)}$ of genus g with k holes (or boundaries). One wants to count maps which map the boundaries to a Lagrangian submanifold of M denoted as the brane[24] L. Such maps are characterized by two additional parameters prescribing how the boundaries are mapped to the brane L: a bulk class $\beta \in H_2(M, L)$[25] and winding numbers $\omega_i \in \mathbb{Z}$, $i = 1, \ldots, k$, telling how many times the boundaries wrap around the brane L. One sums up these 'number of maps' called Gromov–Witten invariants $N_{\beta,\omega}^{(g)}$ in generating functions:

$$F_\omega^{(g)} := \sum_{\beta \in H_2(M,L)} N_{\beta,\omega}^{(g)} \, \mathrm{e}^{-\beta t}$$

and one also considers the open string amplitudes

$$A_k^{(h)}(z_1, \ldots, z_k) := \sum_{\omega \in \mathbb{Z}^k} F_\omega^{(g)} \prod_{i=1}^{k} z_i^{\omega_i},$$

[24] The geometry of such submanifolds is studied in the following section.

[25] One assumes for simplicity that $b_1(L) = 1$.

where the open string parameters z_i are parameters of the moduli space of the brane L as well as the closed string amplitudes:

$$\mathcal{F}^{(g)} := F_0^{(g)}.$$

Let us also precisely describe the moduli spaces of both models, since they play an important role in the link between topological string theories and the symplectic invariants. In the A model side, the moduli are the Kälher parameters of the target space M whereas the B model depends on the complex structure of M.

11.1.2. Mirror symmetry, branes and toric geometries. One of the most fascinating features of topological string theories is the existence of a duality linking the A and B models: the mirror symmetry. This symmetry states the equivalence of the A model on a target space M and the B model on a mirror target space $\widetilde{M}$ obtained from M by a mirror map which exchanges the Kälher structure of M with the complex structure of $\widetilde{M}$.[26]

In the following, we will only be concerned with a special but interesting class of target spaces M: toric Calabi–Yau threefolds for the A model and their image under mirror symmetry. We now precisely describe the structure of these geometries as well as the geometry of their Lagrangian submanifolds.

Let us start with a toric Calabi–Yau on the A model side. A toric Calabi–Yau three-fold M can be built as a submanifold of $\mathbb{C}^{k+3}$ as follows. Consider $k+3$ complex scalars $X_i = |X_i|\,\mathrm{e}^{\mathrm{i}\theta_i}$, $i = 1, \ldots, k+3$, transforming under the action of $U(1)^k$ as

$$X_i \to \mathrm{e}^{\mathrm{i}Q_i^\alpha \epsilon_\alpha} X_i$$

for some integers Q_i^α, $\alpha = 1, \ldots, k$. One then considers the three-dimensional submanifold of $\mathbb{R}_+^{k+3}$ obtained by constraining $|X_i|^2$'s to satisfy

$$\sum_{i=1}^{k+3} Q_i^\alpha |X_i|^2 = r_\alpha, \qquad \alpha = 1, \ldots, k. \tag{11.1}$$

The CY three-fold M is the bundle of tori generated by θ_i's modulo the action of $U(1)^k$, over this real submanifold.

The parameters r_α are the Kähler moduli of the toric three-fold M. Then the Calabi–Yau condition is

$$\forall\, \alpha = 1, \ldots, k, \qquad \sum_{i=1}^{k+3} Q_i^\alpha = 0.$$

Note that one can see the coordinates X_i as a S_1-fibration (coordinates $\mathrm{e}^{\mathrm{i}\theta_i}$) over $\mathbb{R}^+$ (coordinates $|X_i|$) giving to M a structure of T^3 fibration over the subspace of $\mathbb{R}_+^3$ defined by the constraints (11.1). It is then interesting to note that this fibration has singular loci when one or several $|X_i|$ vanish. Indeed, the S^1 fiber defined by the corresponding θ_i shrinks. These loci will be important in the following study of the type A branes and they are encoded in the so-called toric graph of the three-fold M [28, 29, 112].

The boundaries of the worldsheet must be mapped to special Lagrangian submanifolds of X called branes. For this purpose, let us remind the notation

$$X_i = |X_i|\,\mathrm{e}^{\mathrm{i}\theta_i}.$$

[26] The works on the subject of mirror symmetry are numerous in physics and mathematics. Entering this subject would quickly lead us too far away from our main topic. For a nice review of the subject, one can refer to the excellent book [86].

The canonical symplectic form on M is then given by

$$\omega = \frac{1}{2}\sum_{i=1}^{3} \mathrm{d}|X_i|^2 \wedge \mathrm{d}\theta_i.$$

One can cancel this form and obtain a special Lagrangian submanifold L by fixing θ_i's with the equation

$$\sum_{i=1}^{3} \theta_i = 0 \quad [\pi]$$

as well as constraining the moduli of X_i by

$$\sum_{i=1}^{k+3} q_i^\alpha |X_i|^2 = c^\alpha \tag{11.2}$$

for $\alpha = 1, \ldots, r$, where q_i^α satisfy

$$\sum_{i=1}^{k+3} q_i^\alpha = 0.$$

A special submanifold in M is then given by the set of complex numbers q_i^α and c^α.

Moreover, one can consider such manifolds L passing through some singular locus of the manifold M. In this case, it splits into two submanifolds L^+ and L^-. This is one of the latter submanifolds that we consider as A brane, e.g. L^+[27].

Let us now describe the mirror geometry of this target space and of the branes in the B model.

The mirror map transforming M into $\widetilde{M}$ can be built as follows. $\widetilde{M}$ has homogenous coordinates $\widetilde{X}_i := \mathrm{e}^{x_i} \in \mathbb{C}^*$ with $i = 1, \ldots, k+3$ whose moduli are constrained by

$$|\widetilde{X}_i| = \mathrm{e}^{-|X_i|^2}.$$

The mirror geometry $\widetilde{M}$ of M is then given by

$$\omega^+\omega^- = \sum_{i=1}^{k+3} \widetilde{X}_i$$

for two complex scalars $(\omega^+, \omega^-) \in \mathbb{C}^2$ and non-vanishing complex homogenous coordinates $\widetilde{X}_i \in \mathbb{C}^*$ satisfying

$$\prod_{i=1}^{k+3} \widetilde{X}_i^{Q_i^\alpha} = \mathrm{e}^{-t_\alpha} = q_\alpha$$

for any $\alpha = 1, \ldots, k$, where

$$t_\alpha := r_\alpha + \mathrm{i}\theta_\alpha$$

are complexified Kähler parameters of the three-fold M. The equation of the mirror geometry $\widetilde{M}$ reduces to

$$\widetilde{H}(\widetilde{X}, \widetilde{Y}|t_\alpha) = \omega^+\omega^- = H(\mathrm{e}^x, \mathrm{e}^y|t_\alpha)$$

[27] One can chose L^+ or L^- as this brane without changing anything in the following.

where $\widetilde{X} = \mathrm{e}^x$ and $\widetilde{Y} = \mathrm{e}^y$ are two non-vanishing coordinates chosen among $\widetilde{X}_i$'s.[28] The holomorphic volume form on $\widetilde{M}$ is then given by

$$\Omega = \frac{\mathrm{d}\omega\,\mathrm{d}\widetilde{X}\,\mathrm{d}\widetilde{Y}}{\omega\widetilde{X}\widetilde{Y}} = \frac{\mathrm{d}\omega}{\omega}\,\mathrm{d}x\,\mathrm{d}y.$$

Under this mirror map, one can easily characterize the B model branes, i.e. the image of the special Lagrangian submanifolds of M under the mirror map [4]. The constraints (11.2) are translated into constraints on $\widetilde{X}_i$:

$$\prod_{i=1}^{k+3} \widetilde{X}_i^{q_i^\alpha} = \mathrm{e}^{-c_\alpha}$$

for $\alpha = 1, \ldots, r$. Moreover, for $r = 2$, if one considers the singular brane L^+, its image is a one-dimensional complex submanifold described by the algebraic equation

$$H(\mathrm{e}^x, \mathrm{e}^y) = 0 = \widetilde{H}(\widetilde{X}, \widetilde{Y})$$

on $\mathbb{C}^*$ and can thus be obtained by fixing $\omega^- = 0$ and considering ω^+ as a parameter of this brane. In the following, we will consider this equation as the spectral curve.

11.1.3. Embedding of the spectral curve and open string parameters. In the preceding section, we showed that the spectral curve corresponds to the moduli space of the open string boundaries, i.e. of the B branes. In particular, it was shown that the description of this moduli space depends on a choice of local coordinates. Let us make this statement more precise by studying the whole target space of the B model and its projection to local patches.

Remember that the moduli space of the B branes is a Riemann surface $\mathcal{L}$ given by a set of coordinates $\widetilde{X}_i$ and $\omega_\pm$ constrained by

$$\sum_{i=1}^{k+3} \widetilde{X}_i = \omega_- = 0 \tag{11.3}$$

as well as the constraints

$$\prod_{i=1}^{k+3} \widetilde{X}_i^{q_i^\alpha} = \mathrm{e}^{-t_\alpha}, \qquad \alpha = 1, \ldots, k \tag{11.4}$$

and

$$\prod_{i=1}^{k+3} \widetilde{X}_i^{q_i^\alpha} = \mathrm{e}^{-c_\alpha}, \qquad \alpha = 1, \ldots, r. \tag{11.5}$$

In order to describe this moduli space, one has to choose as set of coordinates characterizing it. Indeed, one can describe it as an embedding of the Riemann surface $\mathcal{L}$ into $\mathbb{C}^* \times \mathbb{C}^*$ (respectively to $\mathbb{C} \times \mathbb{C}$) through the coordinates $\widetilde{X}_i$ (respectively the coordinates x_i), e.g. the choice of two coordinates $\widetilde{X}_i$ and $\widetilde{X}_j$ (respectively x_i and x_j) among the set $\{\widetilde{X}_a\}_{a=1}^{k+3}$ (respectively $\{x_a\}_{a=1}^{k+3}$) allowing one to describe $\mathcal{L}$ by an equation

$$\widetilde{H}_{i,j}(\widetilde{X}_i, \widetilde{X}_j) = H_{i,j}(\mathrm{e}^{x_i}, \mathrm{e}^{x_j}) = 0$$

obtained by elimination of all the other coordinates $\widetilde{X}_a$ in equations (11.3)–(11.5). In the following, one generically denotes this embedding of the spectral curve by the equation

$$H(\mathrm{e}^x, \mathrm{e}^y) = 0 = \widetilde{H}(\widetilde{X}, \widetilde{Y}).$$

[28] The choice of such coordinates $\widetilde{X}_i$ and $\widetilde{X}_j$ depends on the sector of the moduli space that we are studying. We explicitly describe this choice in the following section.

If all these equations represent the embedding of the same surface, they correspond to different descriptions of the branes, i.e. different types of boundary conditions for the worldsheet. The choice of such an embedding is not random: depending on the regime we consider, i.e. the sector of the moduli space we stud, some embeddings are more appropriate (see for example the discussion in section 2.2 of [28]).

Note that the reparametrization group of the spectral curve $\mathcal{L}$ is

$$G_{\mathcal{L}} = SL(2,\mathbb{Z}) \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

in terms of the variables $(\widetilde{X}, \widetilde{Y})$ of $\widetilde{H}(\widetilde{X}, \widetilde{Y}) = 0$.

Remark 11.1. Note that these transformations preserve the symplectic form $|\mathrm{d}x \wedge \mathrm{d}y| = \left|\frac{\mathrm{d}\widetilde{X}}{\widetilde{X}} \wedge \frac{\mathrm{d}\widetilde{Y}}{\widetilde{Y}}\right|$. This reminds us of the symplectic transformations in section 4.2 which do not change the symplectic invariants. In this topological string setup, these transformations acting on the open string parameters should preserve the closed string amplitudes.

These transformations, i.e. changing the open string parameters, are important in the study of the open string amplitudes. Indeed, the whole open string moduli space, or moduli space of branes, exhibits different phases. In each of these particular regimes, one can use a specific embedding to describe the brane moduli space in appropriate coordinates (see [28, 86] for a review on the subject). The usual methods of computation allow one to know the open string amplitudes in some very particular regime. This means that one can compute these amplitudes in a specific patch of $\widetilde{M}$ and not on the others. It is thus interesting to be able to go from one patch to the others. These 'phase transitions', corresponding for example to blow-ups of $\widetilde{M}$, are elements of $G_{\mathcal{L}}$ encoding the transition from the open string parameters of one embedding to the others.

The choice of an embedding, i.e. the choice of a coordinate x, does not only fix the location of a brane but also fix the last remaining ambiguity known as the framing[29]. Roughly speaking, the framing consists in a discrete ambiguity and can be fixed by choosing an integer f. This ambiguity corresponds to the elements of $G_{\mathcal{L}}$:

$$(\widetilde{X}, \widetilde{Y}) \to (\widetilde{X}\widetilde{Y}^f, \widetilde{Y})$$

or, in x and y coordinates,

$$(x, y) \to (x + fy, y)$$

for integer f. Note that this is also a symplectic transformation considered in section 2.

We have thus shown that fixing an embedding of the spectral curve $\mathcal{L}$ in $\mathbb{C} \times \mathbb{C}$ (or $\mathbb{C}^* \times \mathbb{C}^*$), one fixes the open string parameter space which can be seen as the coordinate x. Going from one patch in the parameter space to another is obtained by changing embedding, thanks to an element of $G_{\mathcal{L}}$.

11.1.4. Open/closed flat coordinates. As was already mentioned in the preceding sections, the moduli space of the B model (respectively A model) is given by the complex (respectively Kähler) parameters of the target space $\widetilde{M}$ (respectively M). Let us precisely describe the flat coordinates describing these complex and Kähler structures which are mapped to each other by the closed string mirror map.

[29] See [28] for explanations on this phenomenon.

These flat coordinates $T^\alpha, \alpha = 1, \dots, \bar{g}$, are given by the periods of the meromorphic 1-form $y\,\mathrm{d}x = \ln \widetilde{Y} \frac{\mathrm{d}\widetilde{X}}{\widetilde{X}}$ on the spectral curve $\mathcal{L}$:

$$T^\alpha = \frac{1}{2\mathrm{i}\pi} \oint_{\mathcal{A}_\alpha} y\,\mathrm{d}x,$$

where $(\mathcal{A}_\alpha, \mathcal{B}_\alpha)$ is a canonical basis of one-cycles on $\mathcal{L}$. This also ensures the existence of a holomorphic function $F(T_\alpha)$ such that the dual periods can be expressed as

$$\frac{\partial F}{\partial T_\alpha} = \frac{1}{2\mathrm{i}\pi} \oint_{\mathcal{B}_\alpha} y\,\mathrm{d}x.$$

What about the open flat coordinates? If the closed coordinates are given by closed integrals of λ over the cycles, the open flat coordinate is expected to be given by chain integrals

$$U = \frac{1}{2\mathrm{i}\pi} \int_{\alpha_x} y\,\mathrm{d}x,$$

where α_x is an open path over which y jumps by $2\mathrm{i}\pi$.

Moreover, it is interesting to note that the open string disc amplitude $A_1^{(0)}(u)$ can be computed explicitly; it is also a chain integral of the 1-form Θ

$$A_1^{(0)}(x) = \int_{[x^*,x]} y\,\mathrm{d}x.$$

Remark 11.2. Once again, it is interesting to note the similarities between the theory of symplectic invariants and B model. Let us summarize this correspondence in a short array:

Symplectic invariants	B model
Spectral curve	Brane moduli space
Symplectic transformations	Phase transition
Filling fraction ϵ_i	Closed flat coordinates T_α
Genus zero free energy $F^{(0)}$	Prepotential F
Variations $\frac{\partial F^{(0)}}{\partial \epsilon_i} = \frac{1}{2\mathrm{i}\pi} \oint_{\mathcal{B}_i} y\,\mathrm{d}x$	Variations $\frac{\partial F}{\partial T_\alpha} = \frac{1}{2\mathrm{i}\pi} \oint_{\mathcal{B}_\alpha} y\,\mathrm{d}x$
Genus 0 one-point function $y\,\mathrm{d}x$	Disc amplitude $\int y\,\mathrm{d}x$

11.1.5. Symplectic invariants formalism: a conjecture. In [28], following some checks of [112] and the seminal paper [49], Bouchard *et al* proposed to define the open and closed string amplitudes of the A model as the symplectic invariants and correlation functions computed on the spectral curve of the mirror B model branes.

The conjecture of [28] simply states that the open string amplitudes $A_k^{(h)}(z_1, \dots, z_k)$ and closed string amplitudes $\mathcal{F}^{(h)}$ of the A model whose mirror background gives rise to the spectral curve $H(\mathrm{e}^x, \mathrm{e}^y) = 0$ are given by correlation functions and symplectic invariants built from the equation $H(\mathrm{e}^x, \mathrm{e}^y) = 0$:

$$A_k^{(h)}(z_1, \dots, z_k) = \int \omega_{k,\mathrm{string}}^{(h)}(z_1, \dots, z_k)$$

and

$$\mathcal{F}^{(h)} = F_h(t_\alpha).$$

Remark 11.3. In [28], the authors seem to slightly change the recursive rules defining the correlation functions and symplectic invariants. However, this apparent transformation results from their choice to work with the coordinates $\widetilde{X} = e^x$ and $\widetilde{Y} = e^y$ instead of x and y in order to start from an equation

$$\widetilde{H}(\widetilde{X}, \widetilde{Y}) = 0,$$

which is algebraic. It appears that the algebraicity of the spectral curve is not essential and that one can directly work with the coordinates x and y, avoiding this change of coordinates.

Let us emphasize a few important points. In order to get the A model amplitudes, one first has to compute the correlation functions and symplectic invariants from the B model spectral curve and then plug in the mirror map to obtain the result in terms of the A model parameters. It should be underlined that the choice of coordinates x and y out of x_i's, i.e. a choice of embedding of $\mathcal{L}$ in $\mathbb{C} \times \mathbb{C}$, corresponds to a choice of brane in the A model[30]. Thus, it is interesting to study the behavior of the amplitudes when one moves in the brane moduli space, which corresponds to the space of parametrizations of the spectral curve. In particular, the closed amplitudes should not depend on this choice of parametrization since only the boundaries of the worldsheet are sensitive to the definition of the branes. This independence of the closed amplitudes on the embedding of the spectral curve is indeed true and follows directly from the symplectic invariance of $F^{(g)}$'s.

11.1.6. Checks of the conjecture. There have been many checks of this conjecture before and after the definition of the symplectic invariants.

First of all, most of this conjecture was inspired by the idea of Dijkgraaf and Vafa who conjectured that the partition function of the type B topological string on some special backgrounds is given by a random matrix integral [49]. Now that the symplectic invariants extend the notion of random matrix partition function to any spectral curve, it seemed natural to conjecture that these symplectic invariants do coincide with the partition function of B model topological string with more general backgrounds.

Another further property of the symplectic invariants points in the same direction. As is reminded in section 4.4.2, using the modular properties of the symplectic invariants, it was proved that one can promote the latter to modular invariants whose non-holomorphic part is fixed by the same set of equations as the B model topological string partition function [73]: the holomorphic anomaly equations of BCOV [20]. This means that the non-holomorphic part of these functions coincide. To prove the conjecture, one thus have to prove it only for the holomorphic part.

Further studies were led by Mariño and collaborators [28, 112]: they checked for many explicit examples of possible backgrounds for the B model topological strings that the partition function and open string amplitudes indeed coincide with the correlation functions and symplectic invariants computed on the associated spectral curve. Every single check indeed works, giving more weight to this conjecture!

Another general check can be made by looking at the short summary made in the array in section 11.1.4. It is also interesting to note that the disc and cylinder amplitudes can be computed independently from this conjecture for any background in the B model: they satisfy the relation conjectured by [28].

Moreover, the computation of the sum over large partition with respect to the q-deformed Plancherel measure makes the link with the topological vertex approach and proves the

[30] That is to say, the choice of the location of the brane as well as a choice of framing. This topic is well developed in [28].

conjecture in a particular family of target spaces. The extension of this method could lead to a direct proof of this conjecture.

Finally, a last clue has been added recently by Dijkgraaf and Vafa [50], using an effective field theory conjectured to be equivalent to B model: the Kodaira–Spencer theory. This check is the subject of the following section.

11.2. Kodaira–Spencer theory

11.2.1. Introduction: an effective field theory for the B model. The six-dimensional Kodaira–Spence theory is the string field theory for the B model on Calabi–Yau threefold. Consider the case of non-compact Calabi–Yau three-fold $\widetilde{M}$ defined by

$$H(x, y) = \omega^+ \omega^-,$$

whose holomorphic volume form is

$$\Omega = \frac{\mathrm{d}\omega}{\omega}\,\mathrm{d}x\,\mathrm{d}y.$$

The Kodaira–Spencer theory is the quantization of the cohomologically trivial variations of the operator $\overline{\partial}$ on $\widetilde{M}$ with a fixed complex structure.

The setup dual to the preceding section corresponds to the local surface

$$H(x, y) = 0.$$

Since one can see that the periods of Ω can be reduced on this local surface to the integrals of the 1-form $y\,\mathrm{d}x$, the two-dimensional reduction of the Kodaira–Spencer theory is defined by the pair $(\overline{\partial}, y\,\mathrm{d}x)$ on the spectral curve; this means that it is the study of the deformations of $\overline{\partial}$ keeping the cohomology class of $y\,\mathrm{d}x$ fixed. For this purpose, one looks at the variations of this operator under the form

$$\overline{\partial} \to \overline{\partial} - \frac{\overline{\partial}\phi}{y\,\mathrm{d}x}\partial,$$

where ϕ is a scalar field satisfying

$$\overline{\partial}\partial\phi = 0.$$

Indeed, under this variation, the cohomology class of $y\,\mathrm{d}x$ is not changed since it transforms as

$$y\,\mathrm{d}x \to y\,\mathrm{d}x + \mathrm{d}\phi.$$

Finally, we are left with a field theory on the spectral curve $\mathcal{L}$ given by the action[31]:

$$S = \int_{\mathcal{L}} \partial\phi\,\overline{\partial}\phi + \frac{y\,\mathrm{d}x}{\lambda}\overline{\partial}\phi + \frac{\lambda}{y\,\mathrm{d}x}\overline{\partial}\phi\,(\partial\phi)^2\,, \tag{11.6}$$

where we rescale the differential $y\,\mathrm{d}x$ by the string coupling constant $\lambda = \frac{t}{N}$: $y\,\mathrm{d}x \to \frac{y\,\mathrm{d}x}{\lambda}$. Let us just explain the three different terms of this action. The first term is a simple kinetic term whereas the second term corresponds to the coupling to a holomorphic background gauge field $\frac{y\,\mathrm{d}x}{\lambda}$. The most interesting term is the third one: this cubic interaction encodes the perturbative corrections and is the fundamental ingredient of this action.

Let us now move to the observables of this theory. First of all, the partition function can be written as

$$\mathcal{Z} = \mathrm{e}^{-\mathcal{F}},$$

[31] For a very pedagogical construction of this action, see [50].

where the free energy $\mathcal{F}$ has a topological expansion in terms of the string coupling constant:

$$\mathcal{F} = \sum_{g \geqslant 0} \lambda^{2g-2} F^{(g)}.$$

One also defines the correlation functions

$$W_k(z_1, \ldots, z_k|\lambda) := \langle \partial\phi(z_1) \ldots \partial\phi(z_k) \rangle_c,$$

where the subscript c denotes the connected part. These correlation functions also have a topological expansion

$$W_k(z_1, \ldots, z_k|\lambda) = \sum_{g \geqslant 0} \lambda^{2g+k-2} W_k^{(g)}(z_1, \ldots, z_k)$$

coming from the interaction term $\mathrm{e}^{\lambda \int_{\mathcal{L}} \frac{\overline{\partial}\phi(\partial\phi)^2}{y\,\mathrm{d}x}}$.

11.2.2. Recursive relations as the Schwinger–Dyson equations. One can note that the integrant in this interaction can be written as a total derivative and does not give any contribution except at the zeros at the denominator, i.e. the zeros of $y\,\mathrm{d}x$. These zeros are the zeros of $y(z)$ and $\mathrm{d}x(z)$. However, one can show that only the zeros a_i of $\mathrm{d}x$ do give non-vanishing contributions (see [50]). Thus, the interaction term can be written as follows:

$$\int_{\mathcal{L}} \frac{\overline{\partial}\phi\,(\partial\phi)^2}{y\,\mathrm{d}x} = \sum_i \oint_{a_i} \frac{\phi\partial\phi\partial\phi}{y\,\mathrm{d}x}.$$

This means that the interaction vertex is localized at the branch points.

In order to compute the correlation functions, one thus has to compute terms of the form $\left\langle \partial\phi(z_1) \oint_{z \to a_i} \frac{\phi(z)\partial\phi(z)\partial\phi(z)}{y(z)\,\mathrm{d}x(z)} \ldots \right\rangle$. A first step consists in the computation of the two-point chiral operator $\langle \partial\phi(z)\partial\phi(z_1) \rangle$, which is known to be the Bergmann kernel. From this point, one can easily compute the contraction of $\partial\phi(z_1)$ with $\phi(z)$:

$$\langle \phi(z)\partial\phi(z_1) \rangle_{\text{twist}} = \frac{1}{2} \int_{\xi(z')=-\xi(z)}^{\xi(z)} B(z', z_1),$$

where the subscript twist refers to the fact that one constrains the scalar field ϕ to be an odd function of a local variable $\xi(z)$ as z approaches a branch point:

$$\phi(-\xi(z)) = -\phi(\xi(z)).$$

In other terms, using the notations of section 2, it implies, thanks to the De l'Hôpital rule,

$$\lim_{z \to a_i} \frac{\langle \phi(z)\partial\phi(z_1) \rangle_{\text{twist}}}{y\,\mathrm{d}x(z)} = \lim_{z \to a_i} \frac{1}{2} \frac{\mathrm{d}E_z(z_1)}{(y(z) - y(\overline{z}))\,\mathrm{d}x(z)} = -\lim_{z \to a_i} K(z_1, z).$$

Finally, taking into account the normal ordering of the cubic interaction term, the Schwinger–Dyson equations of this theory give the recursion relation for the correlation functions

$$W_{n+1}^{(g)}(z_0, J) = \sum_i \operatorname*{Res}_{z \to a_i} K(z_0, z) \left[W_{n+2}^{(g-1)}(z, \bar{z}, J) + \sum_{h=0}^{g} {\sum_{I \subset J}}' W_{1+|I|}^{(h)}(z, I) W_{1+n-|I|}^{(g-h)}(\bar{z}, J \backslash I) \right].$$

The correlators of the Kodaira–Spencer theory on $\mathcal{L}$ are thus the correlation functions computed from the latter curve.

What about the partition function? On the one hand, from the topological expansion, one easily finds that

$$\lambda \frac{\partial \mathcal{F}}{\partial \lambda} = \sum_{g \geqslant 0} (2g-2) \lambda^{2g-2} F^{(g)}.$$

On the other hand, let us re-express the lhs directly in terms of the correlators of the theory. Indeed, thanks to the expression of the action (11.6), one gets

$$\lambda\frac{\partial\mathcal{Z}}{\partial\lambda} = -\frac{1}{\lambda}\left\langle\int_{\mathcal{L}} y\,\mathrm{d}x\,\overline{\partial}\phi\right\rangle + \frac{\lambda^2}{y\,\mathrm{d}x}\left\langle\int_{\mathcal{L}}\overline{\partial}\phi\,(\partial\phi)^2\right\rangle.$$

In order to compute these terms, one proceeds as in the case of the correlation functions by localizing these expressions around the branch points. One can first note that the second term vanishes since it corresponds to the interaction operator with no field inserted. Let us thus compute the first term. Since the integrant can be written as a total derivative $y\,\mathrm{d}x\,\overline{\partial}\phi = \mathrm{d}\,(y\,\mathrm{d}x\,\phi)$, this term can be localized around the poles of the integrant which are nothing but the branch points, i.e.

$$\lambda\frac{\partial\mathcal{Z}}{\partial\lambda} = \frac{1}{\lambda}\sum_i\left\langle\oint_{a_i} y\,\mathrm{d}x\,\phi\right\rangle.$$

Now consider a primitive Φ of $y\,\mathrm{d}x$:

$$\mathrm{d}\Phi = y\,\mathrm{d}x.$$

Integrating by parts, it implies

$$\lambda\frac{\partial\mathcal{Z}}{\partial\lambda} = \frac{1}{\lambda}\sum_i\oint_{a_i}\Phi\langle\partial\phi\rangle.$$

In terms of the topological expansion, this equation coincides with the definition of the symplectic invariants:

$$F^{(g)} = \frac{1}{2-2g}\sum_i\oint_{a_i}\Phi(z)W_1^{(g)}(z).$$

This means that the partition function of the Kodaira–Spencer theory is the τ function τ_N defined from the symplectic invariants in section 2.

12. Conclusion

In this review, we have presented an overview of the recent method introduced for solving matrix models' loop equations and its further extension to a more general context. We have defined the notion of symplectic invariants of a spectral curve and have studied its main applications, as in the present state of the art. In some sense, starting from the spectral curve of a classical integrable system, we have proposed a way to reconstruct the full quantum integrable system.

The study of applications to enumerative geometry and integrable systems of those notions is probably only at its beginning, and in particular the consequences for topological string theory are still mostly to be understood.

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References

[1] Adler M, Van Moerbeke P and Vanhaecke P 2006 Moment matrices and multi-component KP, with application to random matrix theory arXiv:math-ph/0612064

[2] Aganagic M, Bouchard V and Klemm A 2006 Topological strings and (almost) modular forms arXiv: hep-th/0607100

[3] Aganagic M, Klemm A, Mariño M and Vafa C 2003 The topological vertex arXiv:hep-th/0305132

[4] Aganagic M and Vafa C 2000 Mirror symmetry, D-branes and counting holomorphic discs arXiv: hep-th/0012041

[5] Akemann G 1997 Universal correlators for multi-arc complex matrix models *Nucl. Phys.* B **507** 475 arXiv:hep-th/9702005

[6] Akemann G 1996 Higher genus correlators for the Hermitian matrix model with multiple cuts *Nucl. Phys.* B **482** 403 (arXiv:hep-th/9606004)

[7] Akemann G and Ambjørn J 1996 New universal spectral correlators *J. Phys. A: Math. Gen.* **29** L555–60 (arXiv:cond-mat/9606129)

[8] Alexandrov A, Mironov A and Morozov A 2007 M-theory of matrix models *Theor. Math. Phys.* **150** 179 (arXiv:hep-th/0605171)

[9] Alexandrov A, Mironov A and Morozov A 2007 Instantons and merons in matrix models *Physica* D **235** 126 (arXiv:hep-th/0608228)

[10] Alexandrov A, Mironov A and Morozov A 2004 Partition function of matrix models as the first special functions of string theory: I. Finite size Hermitean 1-matrix model *Int. J. Mod. Phys.* A **19** 4127
Alexandrov A, Mironov A and Morozov A 2005 Partition function of matrix models as the first special functions of string theory: I. Finite size Hermitean 1-matrix model *Theor. Math. Phys.* **142** 349 (arXiv: hep-th/0310113)

[11] Alexandrov S 2003 Matrix quantum mechanics and two-dimensional string theory in non-trivial backgrounds *PhD Thesis* (arXiv:hep-th/0311273)

[12] Ambjørn J, Chekhov L, Kristjansen C F and Makeenko Yu 1993 Matrix model calculations beyond the spherical limit *Nucl. Phys.* B **404** 127–72
Ambjørn J, Chekhov L, Kristjansen C F and Makeenko Yu 1995 Matrix model calculations beyond the spherical limit *Nucl. Phys.* B **449** 681 (arXiv:hep-th/9302014) (erratum)

[13] Ambjørn J, Durhuus B and Fröhlich J Diseases of triangulated random surface models, and possible cures *Nucl. Phys.* B **257** 433–49

[14] Baik J, Borodin A, Deift P and Suidan T 2006 A model for the bus system in Cuernavaca (Mexico) *J. Phys. A: Math. Gen.* **39** 8965 (doi: 10.1088/0305-4470/39/28/S11)

[15] Baik J and Rains E M 2001 Symmetrized random permutations *Random Matrix Models and their Applications* vol 40 (Cambridge: Cambridge University Press)

[16] Babelon O, Bernard D and Talon M 2003 *Introduction to Classical Integrable Systems* (Cambridge: Cambridge University Press)

[17] Ben Arous G and Guionnet A 1997 Large deviations for Wigner's law and Voiculescu's non-commutative entropy *Probability Theory and Related Fields* vol 108 (Berlin: Springer) pp 1432–2064 (doi: 10.1007/s004400050119) (ISSN 0178-8051)

[18] Bergère M and Eynard B Determinantal formulae and loop equations, in preparation

[19] Bergman S and Schiffer M 1953 *Kernel Functions and Elliptic Differential Equations in Mathematical Physics* (New York: Academic)

[20] Bershadsky M, Cecotti S, Ooguri H and Vafa C 1994 Kodaira–Spencer theory of gravity and exact results for quantum string amplitudes *Commun. Math. Phys.* **165** 311

[21] Bertola M 2006 Biorthogonal polynomials for 2-matrix models with semiclassical potentials arXiv: nlin.SI/0605008

[22] Bertola M 2007 Boutroux curves with external field: equilibrium measures without a minimization problem arXiv:0705.3062

[23] Bertola M 2003 Free energy of the two-matrix model/dtoda tau-function arXiv:hep-th/0306184

[24] Bleher P 2008 Lectures on random matrix models. The Riemann–Hilbert approach arXiv:0801.1858

[25] Bleher P and Its A 1999 Semiclassical asymptotics of orthogonal polynomials, Riemann–Hilbert problem, and universality in the matrix model *Ann. Math.* **150** 185

[26] Bohigas O, Giannoni M J and Schmit C 1984 *Phys. Rev. Lett.* **52** 1

[27] Bonnet G, David F and Eynard B 2000 Breakdown of universality in multi-cut matrix models *J. Phys. A: Math. Gen.* **33** 6739–68

[28] Bouchard V, Klemm A, Mariño M and Pasquetti S 2007 Remodeling the B-model arXiv:0709.1453

[29] Bouchard V 2006 Toric geometry and string theory *DPhil Thesis* Oxford University (arXiv:hep-th/0609123)
[30] Bouchard V and Marinño M 2007 Hurwitz numbers, matrix models and enumerative geometry arXiv:0709.1458
[31] Brézin E and Deo N 1999 *Phys. Rev.* E **59** 3901 (arXiv:cond-mat/9805096)
[32] Brézin E and Hikami S 1996 Correlations of nearby levels induced by a random potential *Nucl. Phys.* B **479** 697–706
[33] Brézin E and Hikami S 1997 Extension of level spacing universality *Phys. Rev.* E **56** 264–9
[34] Brézin E and Hikami S 1998 Universal singularity at the closure of a gap in a random matrix theory *Phys. Rev.* E **57** 4140–9
[35] Brézin E and Hikami S 1998 Level spacing of random matrices in an external source *Phys. Rev.* E **58** 7176–85
[36] Brézin E, Itzykson C, Parisi G and Zuber J B 1978 *Commun. Math. Phys.* **59** 35
[37] Brézin E and Kazakov V A 1990 *Phys. Lett.* B **236** 144
[38] Chekhov L 1995 Matrix models and geometry of moduli spaces arXiv:hep-th/9509001v1
[39] Chekhov L 2004 Genus one corrections to multi-cut matrix model solutions *Theor. Math. Phys.* **141** 1640–53 (arXiv:hep-th/0401089)
[40] Chekhov L and Eynard B 2006 Hermitian matrix model free energy: Feynman graph technique for all genera *J. High Energy Phys.* JHEP03(2006)014 (arXiv:hep-th/0504116)
[41] Chekhov L, Eynard B and Orantin N 2006 Free energy topological expansion for the 2-matrix model *J. High Energy Phys.* JHEP12(2006)053 (arXiv:math-ph/0603003)
[42] Daul J M, Kazakov V and Kostov I 1993 Rational theories of 2D gravity from the two-matrix model *Nucl. Phys.* B **409** 311–38 (arXiv:hep-th/9303093)
[43] David F 1985 Planar diagrams, two-dimensional lattice gravity and surface models *Nucl. Phys.* B **257** 45
[44] David F 1990 Loop equations and nonperturbative effects in two-dimensional quantum gravity *Mod. Phys. Lett.* A **5** 1019
[45] Deift P, Kriecherbauer T, McLaughlin K T R, Venakides S and Zhou Z 1999 Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory *Commun. Pure Appl. Math.* **52** 1335
[46] Di Francesco P, Ginsparg P and Zinn-Justin J 1995 2D gravity and random matrices *Phys. Rep.* **254** 1
[47] Francesco P Di 2006 2D quantum gravity, matrix models and graph combinatorics *Applications of Random Matrices in Physics (Les Houches Lecture Notes)* (*NATO Sci. Ser.* vol 221) (Dordrecht: Springer) pp 33–88 (arXiv:math-ph/0406013)
[48] Di Francesco P, Mathieu P and Sénéchal D 1999 *Conformal Field Theory* (Berlin: Springer)
[49] Dijkgraaf R and Vafa C 2002 Matrix models, topological strings, and supersymmetric gauge theories *Nucl. Phys.* B **644** 3–20 (arXiv:hep-th/0206255)
Dijkgraaf R and Vafa C 2002 On geometry and matrix models *Nucl. Phys.* B **644** 21–39 (arXiv:hep-th/0207106)
Dijkgraaf R and Vafa C 2002 A perturbative window into non-perturbative physics arXiv:hep-th/0208048
[50] Dijkgraaf R and Vafa C 2007 Two dimensional Kodaira–Spencer theory and three dimensional Chern–Simons gravity arXiv:0711.1932
[51] Dijkgraaf R, Verlinde H and Verlinde E 1991 Topological strings in d $<$ 1 *Nucl. Phys.* B **352** 59
[52] Douglas M and Shenker S 1990 *Nucl. Phys.* B **335** 635
[53] Dubrovin B 1996 Geometry of 2D topological field theories *Integrable Systems and Quantum Groups (Springer Lecture Notes in Math.* vol 1620*)* ed M Francaviglia and S Greco (Montecalini Terme) p 120–348
[54] Dubrovin B and Zhang Y 1998 Bihamiltonian hierarchies in 2D topological field theory at one-loop approximation *Commun. Math. Phys.* **198** 311–61 (arXiv:hep-th/9712232)
[55] Dyson F J 1970 Correlations between the eigenvales of a random matrix *Commun. Math. Phys.* **19** 235–50
[56] Dyson F J 1962 A Brownian-motion model for the eigenvalues of a random matrix *J. Math. Phys.* **3** 1191
[57] Efetov K 1994 Supersymmetry in quantum chaos and mesoscopic physics arXiv:cond-mat/9409024
[58] Eynard B 2004 Topological expansion for the 1-hermitian matrix model correlation functions *J. High Energy Phys.* JHEP11(2004)031 (arXiv:hep-th/0407261)
[59] Eynard B 2003 Large N expansion of the 2-matrix model *J. High Energy Phys.* JHEP01(2003)051 (arXiv:hep-th/0210047)
[60] Eynard B 2003 Large N expansion of the 2-matrix model, multicut case arXiv:math-ph/0307052
[61] Eynard B, Kokotov A and Korotkin D 2004 $1/N^2$ corrections to free energy in Hermitian two-matrix model arXiv:hep-th/0401166
[62] Eynard B 2003 Master loop equations, free energy and correlations for the chain of matrices *J. High Energy Phys.* JHEP11(2003)018 (arXiv:hep-th/0309036)
[63] Eynard B and Ferrer A P 2008 Topological expansion of the chain of matrices arXiv:0805.1368

[64] Eynard B 2006 Formal matrix integrals and combinatorics of maps arXiv:math-ph/0611087
[65] Eynard B 2008 Large N expansion of convergent matrix integrals, holomorphic anomalies, and background independence arXiv:0802.1788
[66] Eynard B 2008 All orders asymptotic expansion of large partitions arXiv:0804.0381
[67] Eynard B 2007 Recursion between Mumford volumes of moduli spaces arXiv:0706.4403
[68] Eynard B and Orantin N 2005 Mixed correlation functions in the 2-matrix model, and the Bethe ansatz *J. High Energy Phys.* JHEP08(2005)028 (arXiv:math-ph/0504058)
[69] Eynard B and Orantin N 2008 Topological expansion and boundary conditions *J. High Energy Phys.* JHEP06(2008)037 (arXiv:0710.0223)
[70] Eynard B and Orantin N 2007 Invariants of algebraic curves and topological expansion *Commun. Number Theory Phys.* **1** 347–52 (arXiv:math-ph/0702045)
[71] Eynard B and Orantin N 2008 Topological expansion of mixed correlations in the Hermitian 2 matrix model and x-y symmetry of the Fg invariants *J. Phys. A: Math. Theor.* **41** 015203 (arXiv:0705.0958)
[72] Eynard B and Orantin N 2007 Weil–Petersson volume of moduli spaces, Mirzakhani's recursion and matrix models arXiv:0705.3600
[73] Eynard B, Mariño M and Orantin N 2007 Holomorphic anomaly and matrix models *J. High Energy Phys.* JHEP06(2007)058 (arXiv:hep-th/0702110)
[74] Eynard B and Mariño M 2008 A holomorphic and background independent partition function for matrix models and topological strings arXiv:0810.4273
[75] Flume R, Grossehelweg J and Klitz A 2009 A Lagrangean formalism for Hermitean matrix models *Nucl. Phys.* B **812** 322–42
[76] Farkas H M and Kra I 1992 *Riemann surfaces* 2nd edn (Berlin: Springer)
[77] Fay J D 1973 *Theta Functions on Riemann Surfaces* (Berlin: Springer)
[78] Ginsparg P and Moore G 1993 *Lectures on 2d Gravity and 2d String Theory* (Cambridge: Cambridge University Press)
[79] Givental A 2001 Semisimple Frobenius structure at higher genus *Int. Math. Res. Not.* **23** 1265 (arXiv: math/0008067)
[80] Givental A 2003 A_{n-1}-singularities and nKdV hierarchies *Mosc. Math J.* **3** 475 (arXiv:math.AG/0209205)
[81] Gross D and Migdal A 1990 *Phys. Rev. Lett.* **64** 127
Gross D and Migdal A 1990 *Nucl. Phys.* B **340** 333
[82] Guhr T, Muller-Groeling A and Weidenmuller H A 1998 *Phys. Rep.* **299** 189
[83] Harish-Chandra 1957 *Am. J. Math.* **79** 87
[84] Harnad J and Yu Orlov A 2005 Fermionic construction of partition functions for two-matrix models and perturbative Schur function expansions arXiv:math-ph/0512056
[85] Harnad J and Yu Orlov A 2007 Fermionic construction of tau functions and random processes arXiv:0704.1157
[86] Hori K, Katz S, Klemm A, Pandharipande R, Thomas R, Vafa C, Vakil R and Zaslow E 2003 *Mirror Symmetry* (Providence, RI: American Mathematical Society)
[87] Hirota R 1971 *Phys. Rev. Lett.* **27** 1192
[88] Imayoshi Y and Taniguchi M 1992 *An Introduction to Teichmüller Spaces* (Berlin: Springer)
[89] Itzykson C and Zuber J B 1980 *J. Math. Phys.* **21** 411
[90] Jimbo M and Miwa T 1983 Solitons and infinite dimensional Lie algebras *Publ. Res. Inst. Math. Soc. Kyoto Univ.* **19** 943–1001
[91] Jimbo M, Miwa T and Ueno K 1981 Monodromy preserving deformations of linear ordinary differential equations with rational coefficients: I *Physica* **2D** 306–52
[92] Johansson K 1998 The longest increasing subsequence in a random permutation and a unitary random matrix model *Math. Res. Lett.* **5** 6382
[93] Kanzieper E and Freilikher V 1998 *Phys. Rev.* E **57** 6604 (arXiv:cond-mat/9709309)
[94] Karlin S and McGregor J 1959 Coincidence probabilities *Pac. J. Math.* **9** 1141
[95] Kazakov V A, Kostov I and Migdal A A 1985 Critical properties of randomly triangulated planar random surfaces *Phys. Lett.* B **157** 295
[96] Kazakov V A 1985 Bilocal regularization of models of random surfaces *Phys. Lett.* B **150** 282–4
[97] Kazakov V A 1989 The appearance of matter fields from quantum fluctuations of 2D-gravity *Mod. Phys. Lett.* A **4** 2125–39
[98] Kazakov V A 1986 Ising model on a dynamical planar random lattice: exact solution *Phys Lett.* A **119** 140–4
[99] Kazakov V A, Staudacher M and Wynter T 1996 Character expansion methods for matrix models of dually weighted graphs *Commun. Math. Phys.* **177** 451–68
[100] Kazakov V A and Marshakov A 2003 Complex curve of the two matrix model and its tau-function *J. Phys. A: Math. Gen.* **36** 3107–36 (arXiv:hep-th/0211236)

[101] Kharchev S and Marshakov A 1993 On $p - q$ duality and explicit solutions in $c < 1$ 2d gravity models arXiv:hep-th/9303100
[102] Knizhnik V G, Polyakov A M and Zamolodchikov A B 1988 *Mod. Phys. Lett.* A **3** 819
[103] Kontsevich M 1991 Intersection theory on the moduli space of curves and the matrix Airy function *Funk. Anal. Prilozh.* **25** 50–7
Kontsevich M 1992 Intersection theory on the moduli space of curves and the matrix Airy function *Commun. Math. Phys.* **147** 1–23
[104] Kostov I K 1999 Conformal field theory techniques in random matrix models arXiv:hep-th/9907060
[105] Kokotov A and Korotkin D 2003 Bergmann tau-function on Hurwitz spaces and its applications arXiv: math-ph/0310008
[106] Krbálek M and Šeba P 2000 Statistical properties of the city transport in Cuernavaca (Mexico) and random matrix theory *J. Phys. A: Math. Gen.* **33** 229–34
[107] Krichever I 1992 The τ-function of the universal Whitham hierarchy, matrix models and topological field theories *Commun. Pure Appl. Math.* **47** 437 (arXiv:hep-th/9205110)
[108] Kunz H 1998 Matrices aléatoires en physique *Cahiers de Physique* (Switzerland: Presses polytechniques et universitaires romandes)
[109] Looijenga E 1993 Intersection theory on Deligne–Mumford compactifications (after Witten and Kontsevich) *Séminaire Bourbaki* vol 1992/93 pp 187–212
[110] Mariño M 2005 *Chern-Simons Theory, Matrix Models, and Topological Strings (International Series of Monographs on Physics* vol 131*)* (Oxford: Oxford University Press)
[111] Mariño M 2006 Les Houches lectures on matrix models and topological strings *Applications of Random Matrices in Physics (Les Houches Lecture Notes)* (*NATO Sci. Ser.* vol 221) (Dordrecht: Springer) pp 319–78 (arXiv:hep-th/0410165)
[112] Mariño M 2006 Open string amplitudes and large order behavior in topological string theory arXiv: hep-th/0612127
[113] Gerasimov A, Marshakov A, Mironov A, Morozov A and Yu Orlov A 1991 Matrix models of 2D gravity and toda theory *Nucl. Phys.* B **357** 565–618
[114] Marshakov A, Mironov A and Morozov A 1991 From Virasoro constraints in Kontsevich's model to W constraints in asymmetric 2-matrix model *JETP Lett.* **54** 540
[115] Marshakov A and Nekrasov N 2006 Extended Seiberg–Witten theory and integrable hierarchy arXiv: hep-th/0612019
[116] Mehta M L *Random Matrices (Pure and Applied Mathematics* vol 142*)* 3rd edn (Amsterdam: Elsevier)
[117] Mehta M L 1986 *Commun. Math. Phys.* **14** 21
[118] Migdal A A 1983 *Phys. Rep.* **102** 199
[119] Mironov A and Morozov A 1990 On the origin of Virasoro constraints in matrix model: Lagrangian approach *Phys. Lett.* B **252** 47
[120] Mirzakhani M 2007 Simple geodesics and Weil–Petersson volumes of moduli space of bordered Riemann surfaces *Invent. Math.* **167** 179–222
[121] Mirzakhani M 2007 Weil–Petersson volumes and intersection theory on the moduli space of curves *J. Am. Math. Soc.* **20** 1–23
[122] Nekrasov N 2004 Seiberg–Witten prepotential from instanton counting *Adv. Theor. Math. Phys.* **7** 831 (arXiv:hep-th/0206161)
[123] Nekrasov N and Okounkov A 2003 Seiberg–Witten theory and random partitions arXiv:hep-th/0306238
[124] Neveu A and Schwarz J H 1971 *Nucl. Phys.* B **31** 86
[125] Okounkov A 2003 The uses of random partitions arXiv:math-ph/0309015
[126] Okounkov A, Borodin A and Olshanski G 2000 Asymptotics of Plancherel measures for symmetric groups *J. Am. Math. Soc.* **13** 481
[127] Okounkov A and Pandharipande R 2001 Gromov–Witten theory, Hurwitz numbers, and matrix models: I arXiv:math/0101147
[128] Prähofer M and Spohn H 2000 Universal distribution for growth processes in 1+1 dimensions and random matrices *Phys. Rev. Lett.* **84** 4882
[129] Sato M and Sato Y 1983 Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold *Nonlinear Partial Differential Equations in Applied Science (Tokyo, 1982)* (*North Holland Math. Stud.* vol 81) (Amsterdam: North-Holland) pp 259–71
[130] Seiberg N and Witten E 1994 Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang–Mills theory arXiv:hep-th/9401087
[131] Staudacher M 1993 Combinatorial solution of the 2-matrix model *Phys. Lett.* B **305** 332–8
[132] Szegö G 1967 *Orthogonal Polynomials* (Providence, RI: American Mathematical Society)

[133] 't Hooft G 1974 *Nucl. Phys.* B **72** 461
[134] Tracy C and Widom H 1994 Level-spacing distributions and the Airy kernel *Commun. Math. Phys.* **159** 151–74
[135] Tutte W T 1962 A census of planar triangulations *Can. J. Math.* **14** 21–38
[136] Tutte W T 1963 A census of planar maps *Can. J. Math.* **15** 249–71
[137] Vershik A and Kerov S 1977 Asymptotics of the Plancherel measure of the symmetric group and the limit form of Young tableaux *Sov. Methods Dokl.* **18** 527531
[138] Vonk M 2005 A mini-course in topological strings arXiv:hep-th/0504147
[139] Wiegmann P and Zabrodin A 2003 Large N expansion for normal and complex matrix ensembles arXiv: hep-th/0309253
[140] Wigner E P 1951 *Proc. Camb. Phil. Soc.* **47** 790
Porter C E 1965 *Statistical Theories of Spectra: Fluctuation* (New-York: Academic) (reprint)
[141] Witten E 1990 Two-dimensional gravity and intersection theory on moduli space *Surv. Differ. Geom.* **1** 243
[142] Wolpert S 1983 On the homology of the moduli space of stable curves *Ann. Math.* **118** 491–523
[143] Zirnbauer M 2004 The supersymmetry method of random matrix theory arXiv:org/abs/math-ph/0404057
[144] Zinn-Justin P 1998 Universality of correlation functions of Hermitian random matrices in an external field *Commun. Math. Phys.* **194** 631–50 (arXiv:cond-mat/9705044)