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Modular properties of two-loop maximal supergravity and connections with string theory

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Abstract: The low-momentum expansion of the two-loop four-graviton scattering amplitude in eleven-dimensional supergravity compactified on a circle and a two-torus is considered up to terms of order $S^6\,\mathcal{R}^4$ (where S is a Mandelstam invariant and $\mathcal{R}$ is the linearized Weyl curvature). In the case of the toroidal compactification the coefficient of each term in the low energy expansion is generically a sum of a number of $SL(2,\mathbb{Z})$-invariant functions of the complex structure of the torus. Each such function satisfies a separate Poisson equation on moduli space with particular source terms that are bilinear in coefficients of lower order terms, consistent with qualitative arguments based on supersymmetry. Comparison is made with the low-energy expansion of type II string theories in ten and nine dimensions. Although the detailed behaviour of the string amplitude is not expected to be reproduced by supergravity perturbation theory beyond certain protected terms, at the orders considered here we find a high degree of agreement with direct results from string perturbation theory. These results point to a fascinating pattern of interrelated Poisson equations for the IIB coefficients at higher orders in the momentum expansion which may have a significance beyond the particular methods by which they were motivated.

Keywords: Supergravity, Superstring.

Contents

1. Introduction

The rich network of string theory dualities provides powerful constraints on the structure of M-theory. These are particularly restrictive for maximally supersymmetric backgrounds although the full power of maximal supersymmetry has proved difficult to exploit. The purpose of this paper is to further investigate the small corner of string theory associated with the low-energy expansion of the four-graviton scattering in nine or ten dimensions and its connection to eleven-dimensional supergravity. In terms of an effective action, this corresponds to an investigation of terms involving derivatives acting on four powers of the linearized Riemann curvature.

More precisely, our aim is to further develop the connections between multi-loop eleven-dimensional supergravity compactified on $\mathcal{S}^1$ and $\mathcal{T}^2$ and the type II superstring theories, making use of the conjectured relationships between M-theory and type IIA and IIB superstring theories [1, 2, 3, 4]. In earlier work, a number of terms in the low-energy expansion of the type II string theory amplitudes were determined from the compactified one-loop and two-loop supergravity amplitudes [5, 6, 7, 8]. The fact that the full, nonperturbative moduli dependence of the string amplitudes was reproduced is presumably a consequence of the constraints of maximal supersymmetry on 'protected' terms. In the absence of a complete understanding of which terms are protected it is of interest to pursue the connections with quantum supergravity further. Here we will develop the low-energy expansion of the two-loop supergravity amplitude in a more systematic fashion and determine several orders beyond those considered previously. We will, furthermore, investigate the extent to which this makes contact with the type II string theories in nine and ten dimensions. We will find that the scalar-field dependent coefficients of the higher-derivative terms in the expansion satisfy a suggestive pattern of differential equations on moduli space. Comparing these coefficients with known 'data' from the low-energy expansion of tree-level and genus-one perturbative string theory in nine and ten dimensions [9, 10] shows a surprising degree of agreement. Although it is obvious that there is far more to M-theory than perturbative supergravity, these results suggest patterns that could persist to all orders in the low-energy expansion.

1.1 Overview of low orders in the momentum expansion

In ten dimensions there is a clear distinction between type IIA and type IIB superstring theories even though it is known that they have identical four-graviton amplitudes at least up to, and including, genus-four in string perturbation theory [11]. The IIA theory has a single real modulus, and at strong coupling this is identified with the radius of a single compact dimension in eleven-dimensional supergravity [2]. The ten-dimensional IIB theory has a complex modulus (a complex scalar coupling constant) that is identified with the complex structure of the torus in the $\mathcal{T}^2$ compactification of eleven-dimensional supergravity in the limit in which the torus volume vanishes [12, 4]. Invariance of M-theory under large diffeomorphisms of $\mathcal{T}^2$ implies that the IIB theory possesses a $SL(2,\mathbb{Z})$ duality symmetry [1] that relates strong and weak coupling in a manner that involves both the perturbative and non-perturbative (D-instanton) interactions. After compactification to nine dimensions on a circle the two string theories are identified by the action of T-duality, which inverts the radius of the compact dimension and transforms the dilaton appropriately. The nine-dimensional duality group is $SL(2,\mathbb{Z})\otimes\mathbb{R}^+$.

Although the explicit calculations in this paper concern the four-graviton amplitude, maximal supersymmetry ensures that the conclusions apply equally to the scattering of any four states in the supermultiplet. In fact, maximal supersymmetry guarantees that the general type IIA or IIB amplitude has the structure[1]

$$\mathbf{A}_{\zeta_1,\zeta_2,\zeta_3,\zeta_4} = F(s,t,u)\,\mathbf{R}^4_{\zeta_1,\zeta_2,\zeta_3,\zeta_4}\,, \tag{1.1}$$

where we have labeled each external massless particle by its superhelicity ζ_r, which takes 256 values (the dimensionality of the maximal supergravity multiplet) and its momentum p_r ($r = 1,2,3,4$), where $p_r^2 = 0$. $F(s,t,u)$ is a function of the Mandelstam invariants[2] s, t, u. The kinematical factor in (1.1) is given by (see (7.4.57) of [13])

$$\mathbf{R}^4_{\zeta_1,\zeta_2,\zeta_3,\zeta_4}(p_1,p_2,p_3,p_4) = \zeta_1^{AA'}\zeta_2^{BB'}\zeta_3^{CC'}\zeta_4^{DD'}\,K_{ABCD}\,\tilde{K}_{A'B'C'D'}\,, \tag{1.2}$$

where the indices A,B on the polarization tensors ζ_r^{AB} run over both vector and spinor values (for example, the graviton polarization is $\zeta^{\mu\nu}$, where $\mu,\nu = 0,1,\dots,9$) and the tensor $K\tilde{K}$ is

[1] We are grateful to Nathan Berkovits for emphasizing the generality of this structure.

[2] The (dimensionless) Mandelstam invariants, $s = -\alpha'(p_1+p_2)^2$, $t = -\alpha'(p_1+p_4)^2$ and $u = -\alpha'(p_1+p_3)^2$, are subject to the mass-shell condition $s+t+u=0$ and $\sqrt{\alpha'} = l_s$ is the string length scale.

defined in [13]. For the purposes of this paper we will consider the case of external gravitons for which $\mathbf{R}$ reduces to the the momentum-space form of the linearized Weyl tensor,

$$\mathcal{R}_{\mu\nu\rho\sigma} = -4\, p_{[\mu}\zeta_{\nu][\sigma}p_{\rho]} \ , \tag{1.3}$$

where the symmetric traceless polarization tensor satisfies $p^\mu\zeta_{\mu\nu} = 0$. The kinematic factor $\mathbf{R}^4_{\zeta_1,\zeta_2,\zeta_3,\zeta_4}$ in (1.1) becomes $\mathcal{R}^4$, which denotes the product of four Weyl curvatures contracted into each other by a well-known sixteen-index tensor (often denoted t_8t_8).

The low-energy expansion of the four-particle amplitude requires the expansion of the function $F(s,t,u)$ in (1.1) for small s,t,u. This can be expressed as a complicated mixture of terms that are analytic and nonanalytic functions of the Mandelstam invariants. The analytic terms may be expanded as power series' in integer powers of s, t and u in a straightforward manner. The lowest-order terms contain the poles and contact terms characteristic of the supergravity tree diagrams. A great deal is also known about higher-order analytic terms up to order α'^6. The nonanalytic terms contain massless threshold singularities whose form is determined by unitarity and depends on the number of noncompact space-time dimensions. Generically, there are fractional powers or logarithmic branch points, giving rise to non-integer powers of s or $\log s$ factors.

In what follows we shall separate the low-energy expansion of the ten-dimensional amplitude in either type II theory into the sum of an analytic part and a non-analytic part,

$$A_{II} = i\alpha'^4\,(A^{an}_{II} + A^{nonan}_{II})\,, \tag{1.4}$$

where A_{II} has been normalized to be dimensionless. In the IIB theory the coefficients in the series in the analytic term A^{an}_{II} in (1.4) are $SL(2,\mathbb{Z})$-invariant functions of the complex coupling and the series has the form

$$A^{an}_{IIB} = \sum_{p\ge 0, q\ge -1} g_B^{p+\frac{3}{2}q-\frac{1}{2}}\,\mathcal{E}_{(p,q)}(\Omega)\,\hat\sigma_2^p\hat\sigma_3^q\,\mathcal{R}^4\,, \tag{1.5}$$

where

$$\hat\sigma_n = \frac{s^n+t^n+u^n}{4^n}\,. \tag{1.6}$$

The factors $\hat\sigma_2^p\,\hat\sigma_3^q$ are the most general scalars that are symmetric monomials in s, t, u of order $2p+3q$. The functions $\mathcal{E}_{(p,q)}$'s are modular functions of the complex scalar, $\Omega = \Omega_1 + i\Omega_2$, where

$$\Omega_1 = C^{(0)}\,, \qquad \Omega_2 = e^{-\phi_B} = g_B^{-1}\,, \tag{1.7}$$

and $C^{(0)}$ is the Ramond–Ramond scalar, ϕ_B is the type IIB dilaton and g_B is the type IIB coupling constant. The expression (1.5) includes the Born term with its poles and the coefficient $\mathcal{E}_{(0,-1)} = 1$. The nonanalytic contribution is a series that contains multi-particle thresholds of symbolic form

$$A^{nonan}_{IIB} = \left(s\log(-s) + g_B^{\frac{3}{2}}\,\mathcal{F}_4(\Omega)\,s^4\log(-s) + g_B^2\,\mathcal{F}_5(\Omega)\,s^5\log(-s) + \dots\right)\mathcal{R}^4\,, \tag{1.8}$$

where the $\mathcal{F}_r(\Omega)$'s are modular functions of Ω, which begin with terms that are genus-one or higher.

The coefficients in the expansion are known up to terms of order $\hat\sigma_3\,\mathcal{R}^4$,

$$
\begin{aligned}
A_{\rm IIB} \;=\; & e^{-2\phi}\frac{1}{3\hat\sigma_3}\,\mathcal{R}^4 + e^{-\phi_B/2}\,E_{\frac32}(\Omega)\,\mathcal{R}^4 + e^{\phi_B/2}\frac{1}{2}\,E_{\frac52}(\Omega)\,\hat\sigma_2\,\mathcal{R}^4 \\
& + e^{\phi_B}\frac{1}{6}\,\mathcal{E}_{(3/2,3/2)}(\Omega)\,\hat\sigma_3\,\mathcal{R}^4 + \cdots\,, \qquad (1.9)
\end{aligned}
$$

The terms in (1.9) are analytic in s, t and u and translate into local higher-derivative interactions in a $SL(2,\mathbb{Z})$-invariant effective action,

$$
\begin{aligned}
S_{\rm IIB} \;=\; & \frac{1}{\alpha'^4\,2^7\pi^6}\int d^{10}x\sqrt{-g}\,\Big[e^{-2\phi}\,R^{(10)} + \alpha'^3\,e^{-\phi_B/2}\,E_{\frac32}(\Omega)\,\mathcal{R}^4 + \frac{\alpha'^5}{2}e^{\phi_B/2}\,E_{\frac52}(\Omega)\,D^4\mathcal{R}^4 \\
& + \frac{\alpha'^6}{6}e^{\phi_B}\,\mathcal{E}_{(\frac32,\frac32)}(\Omega)\,D^6\mathcal{R}^4\Big] + \cdots\,, \qquad (1.10)
\end{aligned}
$$

where $R^{(10)}$ is the curvature scalar, g the ten-dimensional type IIB string metric and the coefficients $E_s(\Omega)$ and $E_{(\frac32,\frac32)}(\Omega)$ will be described below. The derivatives in (1.10) are contracted so that the four-point amplitude contributions arise in a manner that is defined by the pattern of Mandelstam invariants in (1.9). From (1.9) it follows that the coefficients in (1.5) are given by

$$
\mathcal{E}_{(0,0)}(\Omega) = E_{\frac32}(\Omega), \qquad \mathcal{E}_{(1,0)}(\Omega) = \frac12\,E_{\frac52}(\Omega)\,, \qquad \mathcal{E}_{(0,1)}(\Omega) = \frac16\mathcal{E}_{(\frac32,\frac32)}(\Omega)\,. \qquad (1.11)
$$

The quantities E_s in (1.9) and (1.10) are Eisenstein series that solve the Laplace eigenvalue equations on the fundamental domain of $SL(2,\mathbb{Z})$,

$$
\Delta_\Omega E_s \equiv \Omega_2^2\left(\frac{\partial^2}{\partial\Omega_1^2} + \frac{\partial^2}{\partial\Omega_2^2}\right)\,E_s = s(s-1)E_s\,. \qquad (1.12)
$$

Given the fact that the E_s is a $SL(2,\mathbb{Z})$ function that can have no worse than power growth as $\Omega_2\to\infty$ (which is required for consistency with string perturbation theory at weak coupling) the solution of this equation is uniquely given by

$$
E_s = \sum_{(m,n)\neq(0,0)}\frac{\Omega_2^s}{|m+n\Omega|^{2s}}\,, \qquad (1.13)
$$

which can be expanded at weak coupling in the form

$$
\begin{aligned}
E_s(\Omega) \;=\; & 2\zeta(2s)\Omega_2^s + 2\sqrt{\pi}\Omega_2^{1-s}\frac{\Gamma(s-\frac12)\zeta(2s-1)}{\Gamma(s)} \\
& + \frac{2\pi^s}{\Gamma(s)}\sum_{k\neq0}\mu(k,s)e^{-2\pi(|k|\Omega_2 - ik\Omega_1)}|k|^{s-1}\left(1+\frac{s(s-1)}{4\pi|k|\Omega_2}+\dots\right)\,. \qquad (1.14)
\end{aligned}
$$

The two power-behaved terms in this expansion correspond to the tree-level and genus-$(s-1/2)$ contributions in string theory[3], as can be seen by taking into account the powers of e^{ϕ_B} in (1.9)

[3] In order to avoid confusion, we will refer to the number of 'loops' (denoted by L) in the context of the supergravity Feynman rules, and the 'genus' (denoted by h) in the context of the string theory perturbative expansion.

and identifying Ω_2^{-1} with the IIB string coupling, g_B. The exponential terms correspond to the infinite set of D-instanton contributions.

The fact that $\mathcal{E}_{(0,0)}(\Omega) = E_{3/2}(\Omega)$ is the coefficient associated with the $\mathcal{R}^4$ term in (1.9) was initially deduced via indirect arguments [14, 5]. One of these made use of properties of loop amplitudes of eleven-dimensional supergravity compactified on a circle or on a two-torus, combined with dualities that relate M-theory to type II string theory in nine dimensions. In this way the function $E_{3/2}(\Omega)$ describes the dependence of the low-energy limit of the one-loop ($L=1$) four-graviton scattering amplitude on the modulus of the compactification torus [5]. The ultraviolet divergence, which behaves as $\Lambda^3\mathcal{R}^4$, where Λ is a momentum cutoff, is independent of Ω and can be subtracted by a local counterterm. The coefficient of this counterterm is fixed by requiring the IIA and IIB amplitudes to be equal, as they are known to be. The modular function $E_{3/2}$ can also be derived as a consequence of supersymmetry combined with $SL(2,\mathbb{Z})$-duality [15]. Although it is suspected that the other modular functions appearing in higher derivative terms (at least up to the order shown in (1.9)) should also be determined by supersymmetry combined with non-perturbative dualities, there is no systematic procedure for doing this (a sketchy outline is given in section 5 of this paper).

Expanding the $L=1$ supergravity amplitude in powers[4] of S, T and U leads to higher-order terms in the derivative expansion of the form [6, 7]. This results in an infinite set of analytic terms that are interpreted in IIB string coordinates as modular invariant coefficients multiplying powers of order $r_B^{1-2k}\, s^k$,

$$A_{L=1} = r_B \left(g_B^{-\frac{1}{2}} E_{\frac{3}{2}}(\Omega)\mathcal{R}^4 + \sum_{k=2}^{\infty} h_k\, r_B^{-2k} g_B^{k-\frac{1}{2}} E_{k-\frac{1}{2}}(\Omega)\, \mathcal{S}^{(k)}\, \mathcal{R}^4\right) + \cdots\,, \qquad (1.15)$$

where the ellipsis stand for the non-analytic contributions [7] and h_k are simple constants and $\mathcal{S}^{(k)}$ is a polynomial in $\hat\sigma_2$ and $\hat\sigma_3$ of order $k = 2p+3q$ in the Mandelstam invariants. All contributions with $k \geq 2$ vanish in the ten-dimensional type IIB limit where the two-torus volume, $\mathcal{V}_2$, vanishes. So we see that in the ten-dimensional limit the compactified one-loop ($L=1$) eleven-dimensional supergravity amplitude contributes only at order $\mathcal{R}^4$. In order to obtain higher-derivative interactions one has to consider eleven-dimensional supergravity at higher loops ($L>1$). The coefficient $\mathcal{E}_{(1,0)} = E_{5/2}(\Omega)/2$ of the ten-dimensional IIB theory indeed arises from a one-loop subdivergence of the low-energy limit of the two-loop amplitude of eleven-dimensional supergravity compactified on a two-torus in the limit in the limit $\mathcal{V}_2 \to 0$ [7, 8].

The function $\mathcal{E}_{(0,1)}\,(\Omega)$ in (1.5) is obtained by expanding the two-loop supergravity amplitude to the next order in S, T, U and compactifying on a two-torus [8]. It satisfies the Poisson equation

$$\Delta_\Omega \mathcal{E}_{(0,1)} = 12\mathcal{E}_{(0,1)} - E_{\frac{3}{2}}\, E_{\frac{3}{2}}\,, \qquad (1.16)$$

in which the source term on the right-hand side is quadratic in the $O(\alpha'^3)$ modular function $E_{3/2}$. We will denote the solution to this equation by $\mathcal{E}_{(0,1)} = \mathcal{E}_{(3/2,3/2)}/6$, as in [8]. This source term

[4] The dimensionless Mandelstam invariants of eleven-dimensional supergravity are denoted by upper case letters $S = -l_{11}^2\,(p_1+p_2)^2$, $T = -l_{11}^2\,(p_1+p_4)^2$, $U = -l_{11}^2\,(p_1+p_3)^2$, where l_{11} is the eleven-dimensional Planck length, and related to the invariants in the ten-dimensional string frame by $S = R_{11}\, s\, \ldots$, where R_{11} is the radius of the eleventh dimension.

makes the equation quite different from the Laplace eigenfunction equation (1.12). Its structure was argued in [8] to follow, at least qualitatively, from the constraints of supersymmetry. The solution of (1.16) is complicated, but the zero-mode of $\mathcal{E}_{(0,1)}$, which contains the perturbative terms, is found to have the form

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \Omega_2^{-1}\, \mathcal{E}_{(0,1)}\, d\Omega_1 = \frac{2}{3}\zeta(3)^2\Omega_2^2 + \frac{4}{3}\zeta(2)\zeta(3) + \frac{8}{5}\zeta(2)^2\Omega_2^{-2} + \frac{4}{27}\zeta(6)\Omega_2^{-4} + O(\exp(-4\pi\Omega_2))\,, \quad (1.17)$$

so it contains tree-level, genus-one, genus-two and genus-three perturbative string theory terms as well an infinite series of D-instanton – anti-D-instanton pairs. The tree-level and genus-one terms agree precisely with direct string theory calculations, while the genus-two term has not yet been extracted directly from string theory. The genus-three term cannot yet be computed in string perturbation theory but it is gratifying that the value of its coefficient agrees, as it should, with that of the genus-three term in the IIA theory that is predicted from one loop in eleven-dimensional supergravity compactified on $\mathcal{S}^1$. This agreement is striking since extracting the coefficient in the IIB theory from the $L = 2$ amplitude involves the use of a Ramanujan identity (see appendix D.1), whereas the coefficient in the IIA theory obtained from the $L = 1$ amplitude arises from a simple integral. Although there is no proof that $\mathcal{E}_{(0,1)}(\Omega)$ is the exact modular function, these agreements strongly suggest that it is. It is notable that the terms in the expression (1.17) are not of uniform transcendental weight. Whereas, there is a correlation of the power of Ω_2 and the weight of the ζ values for the first three terms, this breaks down for the genus-three term. We will see an analogous lack of transcendentality in many of the examples to be described later in this paper.

The first nonanalytic term beyond the Born (pole) term arises at order α'^4 and comes from the ten-dimensional supergravity one-loop diagrams. It has the symbolic form given by the first term on the right-hand side of (1.8). Its precise expression, reviewed in [10], has a much more complicated threshold structure but it has the notable property that the scale of the logarithm cancels, using $s + t + u = 0$.

Obviously the analysis of Feynman diagrams of eleven-dimensional supergravity has limited use since it does not capture the full content of quantum string theory, or M-theory. To begin with, eleven-dimensional supergravity is not renormalizable. Our procedure is to regulate the ultraviolet divergences by introducing a momentum cutoff and subtracting the divergences with counterterms. The result is finite but the counterterms contribute arbitrary coefficients that parameterize our ignorance of the short-distance physics. However, at low orders the values of some of these coefficients are known to be determined by supersymmetry if we also assume the result should be in accord with string dualities. One of the aims here is to investigate the extent to which this continues at higher orders.

A related issue is that the Feynman diagrams describe a semi-classical approximation to the theory in a particular classical background space-time. This can only be motivated in the limit in which the radii of the compact dimensions are much larger than the eleven-dimensional Planck length. This means $R_{11} \gg 1$ for the $\mathcal{S}^1$ compactification (where R_{11} is the dimensionless radius of the eleventh dimension in Planck units). This is the limit of large IIA string coupling, $g_A = R_{11}^{3/2} \gg 1$. Bearing in mind that this is far from the regime of string perturbation theory, we will see to what extent there is agreement between the compactified two-loop Feynman diagrams and corresponding perturbative string theory results. For the $\mathcal{T}^2$ compactification the analogous condition is $\mathcal{V}_2 = R_{10}R_{11} \gg 1$ (where $\mathcal{V}_2$ is the dimensionless volume of $\mathcal{T}^2$ in Planck units). In IIA string theory compactified to nine dimensions this is the limit in which $r_A = 1/r_B \gg g_A^{-\frac{1}{3}}$, where r_A is the radius of the compact dimension (in string units). Nevertheless, the coefficients of the $\mathcal{R}^4$, $\mathcal{D}^4\,\mathcal{R}^4$ and $\mathcal{D}^6\,\mathcal{R}^4$ terms reviewed above give the correct values in the $r_B \to \infty$ limit – presumably the extrapolation from small r_B to large r_B works because these terms are protected by supersymmetry. The low energy limit we are considering is one in which $S\,R_{11}^2 \ll 1$. Since the supergravity loop diagrams are ultraviolet divergent we will also introduce a dimensionless momentum cutoff $\Lambda \gg R_{11}^{-1} \gg 1$ measured in units of l_{11} the eleven-dimensional Planck length. We will see that the low energy expansion of the Feynman diagrams possesses a very rich structure. In particular, the coefficients that depend on the scalar fields satisfy a series of mathematically intriguing Poisson equations that are nontrivial extensions of (1.16) satisfied by $\mathcal{E}_{(0,1)}$, as we will see.

1.2 Outline of paper

In this paper we will consider the higher-order terms in the low-energy expansion of the four-graviton amplitude that are obtained by expanding the two-loop amplitude of eleven-dimensional supergravity, compactified to ten dimensions on $\mathcal{S}^1$ and nine dimensions on $\mathcal{T}^2$ to several higher orders in the Mandelstam invariants.

The four-graviton amplitude (1.4) at two loops ($L = 2$) in maximal supergravity has the form [16]

$$A^{an}_{sugra} + A^{nonan}_{sugra} = i\frac{\kappa_{11}^6}{2\,(2\pi)^{22}\,l_{11}^{12}}\,\mathcal{R}^4\,\mathcal{I}(S,T,U)\,, \tag{1.18}$$

where the scalar function $\mathcal{I}(S,T,U)$ has the structure

$$\mathcal{I}(S,T,U) = S^2 I^{(S)}(S;T,U) + T^2 I^{(T)}(T;U,S) + U^2 I^{(U)}(U;S,T)\,. \tag{1.19}$$

The terms in brackets are sums of φ^3 scalar field theory two-loop planar and non-planar ladder diagrams,

$$I^{(S)}(S;T,U) = I^P(S;T,U) + I^P(S;U,T) + I^{NP}(S;T,U) + I^{NP}(S;U,T)\,, \tag{1.20}$$

with analogous expressions for $I^{(T)}$ and $I^{(U)}$. The expression (1.18) has an overall prefactor of $\mathcal{R}^4$, which has eight powers of the external momenta, together with four more powers from the factors of S^2, T^2 or U^2. This means that the loop integrals, $I^{(P)}$ and $I^{(NP)}$, are much less divergent than they would naively appear. We will be interested in the compactified amplitude, so that $\mathcal{I}(S,T,U)$ is a function of the moduli of the compact space. Ignoring for the moment the nonanalytic pieces, we shall expand the analytic part of $\mathcal{I}(S,T,U)$ in a power series,

$$\mathcal{I}^{an}(S,T,U) = \sum_{p,q\geq 0} n_{(p,q)}\sigma_2^p\,\sigma_3^q\,I_{(p,q)}\,, \tag{1.21}$$

where $I_{(p,q)}$ is a function of the moduli that will be defined by the integral (2.56) and the constant coefficients $n_{(p,q)}$ can be read off from (2.55). Note that $I_{(0,0)} = 0$ since the well-known $\mathcal{R}^4$ term only arises at one loop. The dependence on the Mandelstam invariants in (1.21) is contained in the σ_2 and σ_3, which are defined by

$$\sigma_n = S^n + T^n + U^n \tag{1.22}$$

(whereas in the string variables we used the symbol $\hat{\sigma}_n$ in (1.6)). The coefficients $I_{(p,q)}$ in (1.21) depend on the $\mathcal{T}^n$ moduli in a manner to be determined. The infrared massless threshold effects give rise to nonanalytic terms that we will also need to discuss.

In section 2 we will show how the expression for $\mathcal{I}(S,T,U)$ can be reexpressed in a useful form that would also arise naturally in a world-line functional integral describing the two-loop process. This involves attaching vertex operators for external states of momentum p_r ($r = 1,2,3,4$) to points t_r on the three world-lines, of length L_k ($k = 1,2,3$), of the two-loop vacuum diagram. The amplitude involves the the usual factor of $\exp(-\sum_{r,s} p_r \cdot p_s\, G_{rs})$, where G_{rs} is the Green function connecting pairs of points on these world-lines, as discussed in [17]. This provides a very compact expression for the sum of all diagrams as an integral over all insertion points t_r and over the lengths L_k of the three world-lines, with an appropriate measure. The low energy expansion is obtained, formally, by expanding the integrand in powers of the Green function

$$\exp(-\sum_{r,s=1}^{4} p_r \cdot p_s\, G_{rs}) = \sum_{N=0}^{\infty} \frac{1}{N!} (-\sum_{r,s=1}^{4} p_r \cdot p_s\, G_{rs})^N\,, \tag{1.23}$$

which are to be integrated over the positions t_r with a specific measure.

We will discuss a 'hidden' modular invariance that acts on the three Schwinger parameters, L_k. This symmetry is particularly useful in evaluating the compactification of the amplitude on a spatial n-torus and was used in [7, 8] in evaluating terms of order $\mathcal{D}^4\mathcal{R}^4$ and $\mathcal{D}^6\mathcal{R}^4$. This becomes more explicit after a change of variables from the Schwinger parameters, L_k, to variables τ_1, τ_2 and V. The quantity $\tau = \tau_1 + i\tau_2$ enters in a manner analogous to the modulus of a world-sheet torus embedded in the target space in genus-one string theory. After the above redefinition of variables we will see that the coefficient $I_{(p,q)}$ in (1.21) has the schematic form (the precise coefficients will be included later)

$$I_{(p,q)} = \int dV V^{5-2p-3q} \int \frac{d^2\tau}{\tau_2^2}\, B_{(p,q)}(\tau)\, \Gamma_{(n,n)}(G_{IJ}; V, \tau)\,, \tag{1.24}$$

where $\Gamma_{(n,n)}$ is a lattice factor that contains the information about the compactified target space with metric G_{IJ} ($I, J = 1, \cdots, n$). It will be important that the integrand is invariant under $SL(2,\mathbb{Z})$, when suitably extended outside the fundamental domain. This integral has ultraviolet and infrared divergences, depending on the values of p and q. These will require a careful treatment of the integration limits, which will be discussed in detail in section 2.3.

An important property of the coefficients, $B_{(p,q)}(\tau)$ in the integrand is that they can be written as sums of components $b^r_{(p,q)}(\tau)$,

$$B_{(p,q)}(\tau) = \sum_{i=0}^{\lceil 3N/2 \rceil} b^{3N-2i}_{(p,q)}(\tau) \tag{1.25}$$

where $N = 2p + 3q - 2$ and the components satisfy Green function equations in τ of the form

$$(\Delta_\tau - r(r+1))\, b^r_{(p,q)} = \tau_2\, c^r_{(p,q)}(\tau_2)\, \delta(\tau_1)\,, \tag{1.26}$$

where $\Delta_\tau = \tau_2^2\,(\partial^2_{\tau_1} + \partial^2_{\tau_1})$, $c^r_{(p,q)}$ is a polynomial in $\tau_2 + \tau_2^{-1}$ of degree $N-1$ (see appendix A for details)[5]. This property will be used extensively to determine $I_{(p,q)}$.

The $\mathcal{S}^1$ compactification to ten dimensions will be described in section 3, together with appendix B. This will lead to coefficients for higher-momentum terms in the type IIA theory up to order $S^6\,\mathcal{R}^4$. Although this reproduces the terms considered in earlier work, important new issues are encountered at order $S^4\,\mathcal{R}^4$ ($k = 4$) where further non-analytic terms arise. Such nonanalytic behaviour arises from infrared threshold effects that are not captured by the power series expansion (1.23), so we will be careful to regulate the infrared limit of the integrals. In ten dimensions unitarity implies that such thresholds are logarithmic and arise at this order in α' at genus-one and genus-two. Further logarithmic singularities arise at genus-two at order $s^5\,\mathcal{R}^4$, and at genus-one and genus-three at order $s^6\,\mathcal{R}^4$, with a complicated pattern of thresholds at all orders in α' thereafter. Unlike in the case of the lowest-order nonanalytic term (1.8), the scales of the logarithms, which we will not evaluate, do not cancel. The translation of these supergravity results into the language of type IIA superstring theory is summarized in section 3.3.

Compactification to nine dimensions on a two-torus will be considered in section 4. The coefficients in the expansion now have a richer structure since they depend on the three moduli of $\mathcal{T}^2$, or the complex coupling, Ω, and the radius of the compact dimension, r_B, in the type IIB string theory language. Each term with a distinct kinematic structure must have a coefficient that is an independent function that is invariant under the nine-dimensional duality group, $SL(2,\mathbb{Z}) \times \mathbb{R}^+$. We will determine certain analytic terms in the double expansion of the amplitude up to order $S^6\,\mathcal{R}^4$ that are associated with particular inverse powers of r_B. In order for the Feynman diagram approximation to have a chance of being a sensible approximation it is necessary that $r_B \ll 1$, or $r_A \gg 1$. The coefficients will be modular functions of Ω. In fact, we will see that each coefficient is generally a sum of a number of modular functions that satisfy independent Poisson equations analogous to (1.16). The structure of these equations, which generalizes (1.12), is summarized by (4.15), which is one of the most intriguing results of this paper.

In nine dimensions almost all the low-order nonanalytic terms have branch points that are non-integer powers of the Mandelstam invariants rather than logarithms, and so they can be separated from the analytic part unambiguously – the exception is the term of order $S^5\,\log(-S)$, which is the contribution from nine-dimensional supergravity and can be obtained by dimensional regularization, as summarized in appendix E.3. However, there are terms that are power-behaved in r as well as terms containing, factors such as $\log r^2$, which is nonanalytic in r, and exponentially suppressed terms of the form e^{-cr}. A series of terms that are power behaved in r_B was seen to arise from the expansion of the $L = 1$ supergravity amplitude in (1.15). Similarly, we will find that the momentum expansion of the $L = 2$ amplitude gives a sum of higher-momentum

[5] We would like to thank Don Zagier for explaining the mathematical significance of this decomposition

modular invariant terms,

$$A_{L=2}^{an} = \sum_{q\geq 1}\sum_{p\geq 0}\sum_{l} r_B^{1-l} g_B^{\frac{1}{2}N+\frac{1}{2}+\frac{l}{4}} \mathcal{E}_{(p,q)}^{(l)}(\Omega)\, \hat\sigma_2^p \hat\sigma_3^q\, \mathcal{R}^4\,, \tag{1.27}$$

for various values of l that will be specified later. Terms proportional to r reproduce the $d=10$ expansion, so that $\mathcal{E}_{(p,q)}^{(0)} \equiv \mathcal{E}_{(p,q)}$. All contributions with $l \geq 0$ vanish in the ten-dimensional type IIB limit, but they give rise to well defined modular functions in nine dimensions. In addition to terms that are power-behaved in the radius r_A or r_B, there are also terms proportional to $\log r_A$ or $\log r_B$. Such terms arise explicitly at genus-one in nine-dimensional string theory [10]. For example, there is a term of the form $r \log r \times s^4 \mathcal{R}^4$, which is intimately related to the presence of the genus-one $s^4 \log s\, \mathcal{R}^4$ term in ten dimensions determined in [10]. We will see in the following that this dependence on r can also be seen from the $\mathcal{T}^2$ reduction of two-loop ($L=2$) eleven-dimensional supergravity. Terms of the form e^{-cr_B} that arise in string theory when $2p+2q \geq 4$ are not reproduced by Feynman diagrams at any number of loops.

Perturbative contributions to the string amplitude are obtained from the weak-coupling expansion of these modular functions (making use of the methods described in appendix D). Each term in the momentum expansion derived in this manner is accompanied by a particular inverse power of the radius r_B and the new terms do not contribute in the large-r_B limit. However, after T-duality to the IIA theory, we are able to compare a number of coefficients with those derived explicitly from genus-one in string theory compactified on a circle [10] and find precise agreement. Special issues concerning the terms that contain $\log r$ factors will also be discussed. The issue of the pattern of logarithms is intimately related to the threshold behaviour in maximal supergravity in various dimensions. In appendix E we will evaluate the supergravity amplitude in nine, ten and eleven dimensions, making use of dimensional regularization. These expressions are of relevance to various pieces of the argument in the body of the paper. For example, in ten dimensions the pole term gives rise to a term of order $S^5 \log S\, \mathcal{R}^4$ that is identified with a genus-two contribution to $s^5 \log s\, \mathcal{R}^4$ in ten-dimensional string theory. In section 5 we will sketch the way in which supersymmetry constrains higher derivative terms and argue that the structure of the Poisson equations satisfied by the coefficients of the terms in the derivative expansion of the nine-dimensional IIB theory can be motivated by supersymmetry.

2. Properties of the two-loop supergravity amplitude

It has been known for a long time that the sum of one-loop Feynman diagrams that contribute to four-graviton scattering in maximal supergravity in any dimension has the form of a box diagram of φ^3 scalar field theory multiplying $\mathcal{R}^4$, where $\mathcal{R}$ is the linearized Weyl curvature, as discussed in the introduction. Similarly, the sum of all two-loop diagrams, $A(S,T,U)$, is very economically expressed in terms of two particular diagrams of φ^3 scalar field theory [16]. These are the planar double-box diagram, $I^P(S,T)$ of figure 1(a), and the non-planar double box diagram, $I^{NP}(S,T)$ of figure 1(b), together with the other diagrams obtained by permuting the external particles. In addition, one must include the one-loop triangle diagram of figure 1(c) containing a one-loop counterterm at one vertex (indicated by the blob), which subtracts the

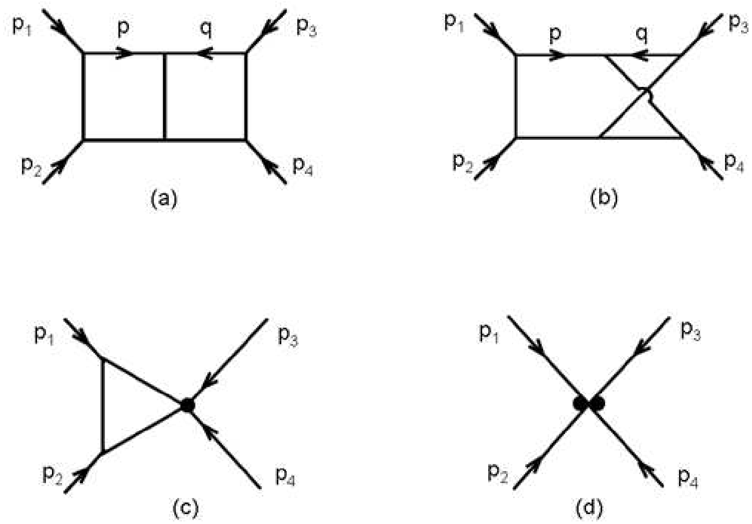

Figure 1: The two-loop four-graviton amplitude in eleven dimensions. (a) The S-channel planar diagram reduces to $S^2\,\mathcal{R}^4$ multiplying a scalar field theory double-box diagram. (b) The S-channel nonplanar diagram reduces to $S^2\,\mathcal{R}^4$ multiplying a nonplanar scalar field theory two-loop diagram. (c) The triangle diagram with a one-loop counterterm at one vertex that subtracts a sub-divergence. (d) A new two-loop primitive divergence.

one-loop sub-divergences from the two-loop diagrams. In addition there are two-loop primitive divergences (that are indicated by the double-blob in figure 1(d)).

The two-loop integrals appearing in the amplitude are sums of planar and non-planar pieces, (1.20). We are interested in compactifying these expressions on the n-torus $\mathcal{T}^n$ with $n=1$ or 2. After manipulations that are given in [7] the loop integrals can be expressed as integrals over seven Schwinger parameters, one for each propagator. The integrations over loop momenta in the compact directions are replaced by sums over the Kaluza–Klein integers in each loop m_I and n_I, where $I=1,\ldots,n$. After performing the integration over the continuous $(11-n)$-dimensional loop momenta, the planar and non-planar diagrams reduce to

$$I^P(S;T,U) = \frac{\pi^{11-n}}{\mathcal{V}_n^2}\int_0^\infty dL_1 dL_2 dL_3\,\Gamma_{(n,n)}\int_0^{L_3} dt_4\int_0^{t_4} dt_3\int_0^{L_1} dt_2\int_0^{t_2} dt_1\,\Delta^{\frac{n-11}{2}}\,e^{h_P}\,, \tag{2.1}$$

and[6]

$$I^{NP}(S;T,U) = \frac{\pi^{11-n}}{\mathcal{V}_n^2}\int_0^\infty dL_1 dL_2 dL_3\,\Gamma_{(n,n)}\int_0^{L_3} dt_3\int_0^{L_2} dt_4\int_0^{L_1} dt_2\int_0^{t_2} dt_1 \Delta^{\frac{n-11}{2}}\,e^{h_{NP}}\,, \tag{2.2}$$

where

$$\Delta = L_1L_2 + L_3L_1 + L_2L_3\,. \tag{2.3}$$

The lattice factor $\Gamma_{(n,n)}$ is defined by

$$\Gamma_{(n,n)}(G^{IJ};\{L_k\}) = \sum_{(m_I,n_I)\in\mathbb{Z}^{2n}} e^{-\pi G^{IJ}(L_1 m_I m_J + L_3 n_I n_J + L_2 (m+n)_I (m+n)_J)}\,. \tag{2.4}$$

[6]In this section we ignore the ultraviolet and infrared divergences. A treatment of these divergences and a proper definition of the integration limits of the integrals will be discussed in section 2.3.

where G^{IJ} is the inverse metric on $\mathcal{T}^n$ and $\mathcal{V}_n = \sqrt{\det G_{IJ}}$ is its volume. The quantities h_P and h_{NP} are given by[7]

$$\begin{aligned} h_P &= T\frac{L_2}{\Delta}(t_4-t_3)(t_2-t_1) \\ &+ S\left[\frac{L_2}{L_1L_3\Delta}(L_1t_3-L_3t_1)(L_1t_4-L_3t_2)+\frac{1}{L_3}t_3(L_3-t_4)+\frac{1}{L_1}t_1(L_1-t_2)\right] \\ &= \frac{1}{\Delta}\left(-S(t_1t_2(L_2+L_3)+t_3t_4(L_2+L_1))+T(t_2t_4+t_1t_3)L_2+U(t_1t_4+t_2t_3)L_2\right) \\ &\quad +S\,t_3+S\,t_1\,, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} h_{NP} &= T\frac{1}{\Delta}(L_2t_3-L_3t_4)(t_2-t_1) \\ &+ S\left(\frac{1}{L_1\Delta}(L_1t_4-L_2t_1)(L_1t_3-L_3t_2)+\frac{1}{L_1}t_1(L_1-t_2)\right) \\ &= \frac{1}{\Delta}\left(S(-t_1t_2(L_2+L_3)+t_3t_4L_1)+T(t_1t_4L_3+t_2t_3L_2)+U(t_1t_3L_2+t_2t_4L_3)\right)+S\,t_1\,. \end{aligned} \tag{2.6}$$

In writing these expressions we have ignored the ultraviolet divergences, which are manifested as divergences at the $L_k = 0$ endpoints ($k = 1,2,3$) that will be regulated by a cutoff in subsection 2.3.1 (as in [7]). The complete expression, $\mathcal{I}(S,T,U)$ in (1.19) is obtained by summing the S-channel, T-channel and U-channel diagrams.

2.1 World-line presentation of the two-loop amplitude

The above structure of the two-loop amplitude can, in principle, be deduced by considering the quantum mechanics functional integral associated with the world-lines for the internal propagators in the two-loop diagrams. This has a structure that bears a close resemblance to the world-sheet description of the genus-two string theory amplitude (although that is formulated in ten-dimensional space-time). We will here rewrite the expressions for the two-loop Feynman diagrams of the previous subsection in order to make this explicit. The advantage of this description is that it naturally packages together the planar and nonplanar diagrams of the S, T and U channels.

The 'skeleton', or vacuum diagram, has three scalar propagators joining the junction A to junction B in figure 2. The lengths of these lines, L_k ($k = 1,2,3$), are moduli that are to be integrated between 0 and ∞. The scattering particles with momenta p_r^μ ($r = 1,2,3,4$) are associated with plane-wave vertex operators that are inserted at positions t_r on any of the three lines of the skeleton, as shown in figure 2. These positions are then to be integrated over the whole network. Since there are four vertex operators and only three lines, at any point in the integration domain one pair of vertex operators is attached to one line, say line 1, while the other two may both be attached to one of the other two lines (line 2 or 3), which is the planar situation, or else the other two lines may have only one vertex operator attached, which is the

[7]The variables in this section are related to those of [7] by $L_1 = \lambda$, $L_2 = \rho$, $L_3 = \sigma$, $t_1 = L_1w_1$, $t_2 = L_1w_2$, $t_3 = L_3v_1$ and in the planar case, $t_4 = L_3v_2$, while in the non-planar case, $t_4 = L_2u_1$.

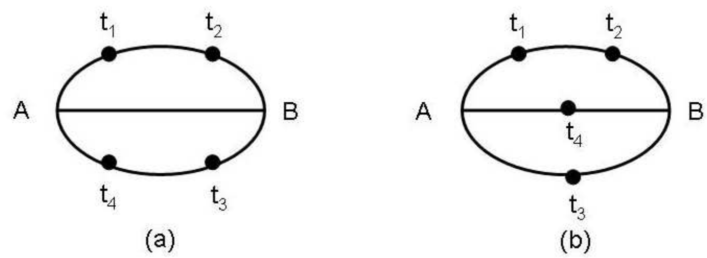

Figure 2: (a) A planar diagram is represented by the skeleton with a pair of external states connected to each of two internal lines. (b) The nonplanar configuration in which one pair of external states is attached to a single line and the other states are each attached to separate lines. Integrating the positions of the four states over the whole network generates the sum of all Feynman diagrams.

non-planar situation. The labelling of the positions t_r of the vertex operators is arbitrary, but it is convenient to choose coordinates $t_r^{(k_r)}$ for particle r on line k_r such that

$$t_r = t_r^{(k_r)}\,, \tag{2.7}$$

where $0 \le t_r^{(k_r)} \le L_{k_r}$, and $t_1^{(k_1)} = 0$, $t_2^{(k_2)} = 0$, $t_3^{(k_3)} = 0$ and $t_4^{(k_4)} = 0$ coincide at the junction A. In other words, the integral over the whole network decomposes into sectors labeled by $\{k_r\}$,

$$\oint \prod_{r=1}^{4} dt_r \equiv \sum_{\{k_r\}} \int_0^{L_{k_1}} dt_1^{(k_1)} \cdots \int_0^{L_{k_4}} dt_4^{(k_4)} \tag{2.8}$$

The expression for the Feynman diagrams can be written in a compact form in terms of the Green function, G_{rs}, between two vertices at points t_r and t_s on the skeleton diagram. Following [17] this is written in terms of two-vectors

$$\mathbf{v}^{(\mathbf{k_r})} = t_r^{(k_r)}\,\mathbf{u}^{(k_r)} \quad \text{or, in components,} \quad v_I^{(k_r)} = t_r^{(k_r)}\,u_I^{(k_r)}\,, \tag{2.9}$$

where $I = 1, 2$ labels the loop and $u_I^{(k)}$ are constant vectors

$$\mathbf{u}^{(\mathbf{1})} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{u}^{(\mathbf{2})} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}\,, \qquad \mathbf{u}^{(\mathbf{3})} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{2.10}$$

With this notation the sum of all two-loop contributions to the amplitude defined in (1.18) and (1.19) is given by

$$\mathcal{I}(S,T,U) = \frac{\pi^{11-n}}{\mathcal{V}_n^2} \int_0^\infty dL_1\, dL_2\, dL_3\; \Gamma_{(n,n)} \oint \prod_{r=1}^{4} dt_r\, W^2\, \Delta^{\frac{n-11}{2}}\; e^{-\sum_{r,s=1}^{4} p_r\cdot p_s\, G_{rs}}\,, \tag{2.11}$$

where G_{rs} is the one-dimensional Green function for the Laplace operator evaluated between the points t_r and t_s on the skeleton diagram, to be discussed below. The lattice factor is defined in (2.4).

The function W appearing in the measure in (2.11) is defined by[8]

$$
\begin{aligned}
3W &= (T-U)\,\Delta_{12}\Delta_{34} + (S-T)\,\Delta_{13}\Delta_{24} + (U-S)\,\Delta_{14}\Delta_{32} \\
&= \quad S(u_1^{(k_1)}u_1^{(k_2)}u_2^{(k_3)}u_2^{(k_4)} + u_2^{(k_1)}u_2^{(k_2)}u_1^{(k_3)}u_1^{(k_4)}) \\
&\quad +T(u_1^{(k_1)}u_2^{(k_2)}u_2^{(k_3)}u_1^{(k_4)} + u_2^{(k_1)}u_1^{(k_2)}u_1^{(k_3)}u_2^{(k_4)}) \\
&\quad +U(u_1^{(k_1)}u_2^{(k_2)}u_1^{(k_3)}u_2^{(k_4)} + u_2^{(k_1)}u_1^{(k_2)}u_2^{(k_3)}u_1^{(k_4)})
\end{aligned}
\tag{2.12}
$$

where

$$
\Delta_{rs} = \epsilon^{IJ}\, u_I^{(k_r)}\, u_J^{(k_s)}\,. \tag{2.13}
$$

Note, in particular, that $\Delta_{rs} = 0$ if $k_r = k_s$ (i.e., t_r and t_s are on the same line). Furthermore $W = 0$ if three of the vertices on the same line (using $S+T+U=0$), so that the only non-zero contributions come from the planar and non-planar diagrams of figure 1. It is easy to see that in any region in which t_r and t_s are on the same line, $W = -6k_r \cdot k_s$, so that

- $W^2 = S^2$ if $k_1 = k_2$ and/or $k_3 = k_4$;
- $W^2 = T^2$ if $k_1 = k_4$ and/or $k_2 = k_3$;
- $W^2 = U^2$ if $k_1 = k_3$ and/or $k_2 = k_4$.

This setup makes contact with the discussion in [17], where the Green function for an arbitrary Feynman diagram of φ^3 scalar field theory was described. Our case differs only due to the presence of a measure factor W^2 in (2.11) which encodes the fact that we are discussing maximal supergravity. However, the exponential factor involves the same Green function as in [17], which has the form

$$
G_{rs} = -\frac{1}{2} d_{t_r^{(k_r)} t_s^{(k_s)}} + \frac{1}{2}(\mathbf{v}^{(\mathbf{k_r})\,\mathbf{T}} - \mathbf{v}^{(\mathbf{k_s})\,\mathbf{T}})\, K^{-1}\, (\mathbf{v}^{(\mathbf{k_r})} - \mathbf{v}^{(\mathbf{k_s})})\,, \tag{2.14}
$$

where $d_{t_r^{(k_r)} t_s^{(k_s)}}$ is the modulus of the distance between t_r on line k_r and t_s on line k_s. If $k_r = k_s$ then $d_{t_r^{(k_r)} t_s^{(k_r)}} = |t_r^{(k_r)} - t_s^{(k_r)}|$, if $k_r \neq k_s$ then $d_{t_r^{(k_r)} t_s^{(k_s)}} = (t_r^{(k_r)} + t_s^{(k_s)})$. The matrix K^{-1} (analogous to the inverse of the imaginary part of the period matrix in the genus-two string calculation) is defined by

$$
K^{-1} = \frac{1}{\Delta} \begin{pmatrix} L_3 + L_2 & L_2 \\ L_2 & L_1 + L_2 \end{pmatrix}\,, \tag{2.15}
$$

where Δ is defined in (2.3). The function G_{rs} is constructed to be the Green function of the one-dimensional Laplace operator that satisfies

$$
\frac{d^2}{dt_r^2}\, G_{rs} = -\delta(t_r - t_s) + \rho\,, \tag{2.16}
$$

and $G_{r,r} = 0$, where

$$
\oint \rho dt \equiv \sum_{k=1}^{3} \int_0^{L_k} \rho^{(k)}\, dt^{(k)} = 1\,, \tag{2.17}
$$

[8]The world-line formulation of the two-loop four-graviton amplitude in ten dimensions would arise from a field theory limit of four-graviton genus two amplitude in type II superstring theory. The function W, used here, is the field theory limit of the function $\mathcal{Y}_S$ that enters the string amplitude derived in [18].

and $\rho^{(k)}$ is a constant on line k. The presence of ρ in (2.16) ensures $\oint \ddot{G}_{rs}\, dt_s = 0$, which is required by Gauss' law on the compact one-dimensional network.

The Green function (2.14) satisfying the conditions (2.16) and (2.17) has a functional form that depends on whether the points t_r and t_s are on the same line or on different lines. If t_r and t_s are on the same line ($k_r = k_s$)

$$G_{rs} = -\frac{1}{2}\,|t_r^{(k_r)} - t_s^{(k_r)}| + \frac{1}{2\Delta}\,(L_l + L_m)\,(t_r^{(k_r)} - t_s^{(k_r)})^2\,, \tag{2.18}$$

where $l \neq m \neq k_r = k_s$. If they are on different lines ($k_r \neq k_s$) the Green function is given by

$$G_{rs} = -\frac{1}{2}\,(t_r^{(k_r)} + t_s^{(k_s)}) + \frac{\Big((L_l + L_{k_s})\,(t_r^{(k_r)})^2 + (L_l + L_{k_r})\,(t_s^{(k_s)})^2 + 2t_r^{(k_r)}\,t_s^{(k_s)}\,L_l\Big)}{2\Delta}\,, \tag{2.19}$$

where $l \neq k_r \neq k_s$. In verifying the conditions (2.16) and (2.17) we find that $\rho^{(k)} = (L_l + L_m)/\Delta$, where $k \neq l \neq m$. The terms quadratic in a single t_r or t_s in G_{rs} do not contribute to the exponent in (2.11) due to the condition $S + T + U = 0$ so the exponential factor ends up being extremely simple.

The integral over the vertex operator positions, $t_r^{(k_r)}$, separates into the two distinct classes described above, namely: (a) Planar configurations in which one pair is attached to one of the three internal lines, and the other pair is attached to one of the other lines; (b) Non-planar configurations in which one pair is attached to one of the internal lines while the other vertices are each attached to the other two internal lines. It is straightforward to see that these contributions are identical to those given by the integrals (2.1) and (2.2).

The complete integral over the t_r's in (2.11) automatically adds contributions that permute the lines and the positions of the four states attached to them. Using these expressions for G_{rs} and W the expression (2.11) reproduces the sum of terms inside the square bracket in the last line of (1.18), which is the sum of planar and nonplanar diagrams in the S, T and U channels. The expression (2.11) has an obvious discrete symmetry under the shift $k_r \to k_r + 1$, fixing all k_s with $s \neq r$ (and with the identification $u_I^{(4)} \equiv u_I^{(1)}$), which moves a vertex operator from one line of the skeleton to the next. This can be thought of as a discrete remnant of the reparametrization invariance of the world-line functional integral that corresponds to cutting two of the lines of the skeleton to produce four endpoints, and regluing the endpoints in a different order. One important insight one gains from this symmetry is that the planar and nonplanar diagrams in all channels are required and their relative normalizations are fixed.

It is important to exploit the symmetries of the complete integral (2.11), which automatically combines the planar and nonplanar diagrams and symmetrizes (2.1) and (2.2) over permutations of L_1, L_2 and L_3.

Formally, if we ignore divergences, the low energy expansion of (2.11) can be written as a power series in symmetric monomials of the Mandelstam invariants, as in (1.21), by expanding the factor of $e^{-\sum p_r \cdot p_s G_{rs}}$. The resulting coefficients in (1.21) may be written as

$$\tilde{I}_{(p,q)} = \frac{\pi^{11-n}}{N!} \int \prod_{k=1}^{3} dL_k\, \Delta^{-\frac{1}{2}+p+\frac{3}{2}q}\, \tilde{B}_{(p,q)}(L_2/L_1, L_3/L_1)\, \Gamma_{(n,n)}\,, \tag{2.20}$$

where $\tilde{B}_{(p,q)}$ is the coefficient of $\sigma_2^p\sigma_3^q$ (which is of order $N+2$ in the Mandelstam invariants, with $N = 2p+3q-2$) in the expansion of the exponential in the integrand of (2.11) and is given by

$$\sum_{2p+3q=N+2} \sigma_2^p\sigma_3^q\,\tilde{B}_{(p,q)}(L_2/L_1, L_3/L_1) = \Delta^{-2-\frac{1}{2}N} \oint \prod_{r=1}^{4} dt_r\, W^2 \left(-\sum_{r,s=1}^{4} p_r\cdot p_s\, G_{rs}\right)^N . \quad (2.21)$$

We will need to evaluate the coefficients $\tilde{B}_{(p,q)}$ in order to evaluate $\mathcal{I}(S,T,U)$ in (1.21). The first two cases are known. The zeroth order term has $N = 0$ ($p = 1, q = 0$) and is given by

$$\tilde{B}_{(1,0)} = \Delta^{-2} \sum_{\{k_r\}\in\{L_r\}} \int_0^{L_{k_1}} dt_1^{(k_1)} \ldots \int_0^{L_{k_4}} dt_4^{(k_4)} = 1\,, \quad (2.22)$$

which agrees with [7]. For $N = 1$ ($p = 0$ and $q = 1$), substituting the expression for G_{rs} leads to

$$\begin{aligned}\sigma_3\,\tilde{B}_{(0,1)} &= \frac{\Delta^{-\frac{5}{2}}}{3} \sum_{\{k_r\}\in\{L_r\}} \left(\prod_{r=1}^{4}\int_0^{L_{k_r}} dt_r^{(k_r)}\right) W^2\, \left(S(G_{12}+G_{34}) + T(G_{14}+G_{23}) + U(G_{13}+G_{24})\right)\\ &= \frac{\sigma_3}{12}\left(\frac{L_1+L_2+L_3}{\Delta^{\frac{1}{2}}} + 5\frac{L_1L_2L_3}{\Delta^{\frac{3}{2}}}\right), \end{aligned} \quad (2.23)$$

in agreement with [8] (allowing for the extra normalization factor of 1/12). In evaluating the integrals over L_k in (2.20) care must be taken to subtract the ultraviolet divergent parts, as we will review later. There are also singular infrared effects associated with the occurrence of massless particle thresholds, giving rise to nonanalytic behaviour that is not captured by the expansion (1.21), as will also be seen later.

First we will describe a change of integration variables that is very useful for evaluating the integral.

2.2 Redefinition of the integration parameters

As in [8] it is very useful to redefine the Schwinger parameters by replacing L_1, L_2, L_3, by the variables V and $\tau = \tau_1 + i\tau_2$, defined by

$$\tau_1 = \frac{L_1}{L_2+L_1}\,, \qquad \tau_2 = \frac{\sqrt{\Delta}}{L_2+L_1}\,, \qquad V = \Delta^{-\frac{1}{2}}\,. \quad (2.24)$$

The integration measure transforms as

$$dL_1\,dL_2\,dL_3 = 2\frac{dV}{V^4}\,\frac{d^2\tau}{\tau_2^2}\,. \quad (2.25)$$

The ranges of the new variables are

$$0 \le \tau_1 \le 1\,, \qquad |\tau - \tfrac{1}{2}| \ge \tfrac{1}{2}\,, \qquad 0 \le V \le \infty\,, \quad (2.26)$$

which is the shaded region shown in figure 3(a). This is a fundamental domain of the group $\Gamma_0(2)$. The three segments of the boundary of this region are: $\tau_1 = 0$ that comes from $L_1 \to 0$

with L_2, L_3 fixed; $\tau_1 = 1$ that comes from $L_2 \to 0$ with L_1, L_3 fixed and $|\tau|^2 - \tau_1 = 0$ that comes from $L_3 \to 0$. with L_2, L_3 fixed. It follows that that the ultraviolet divergences, arise at the boundary of this region. From its construction it is evident that the sum of two-loop integrals is invariant under the action of the symmetric group, $\mathfrak{S}^3$ on the three parameters L_1, L_2 and L_3, which maps the six regions in the shaded domain in figure 3(a) into each other. The action of the two-cycles in $\mathfrak{S}^3$ on the Schwinger parameters is given by the following actions on τ,

$$L_1 \leftrightarrow L_2 \;:\quad \tau \to 1 - \tau^*\,, \quad L_1 \leftrightarrow L_3 \;:\quad \tau \to \frac{1}{\tau^*}\,, \quad L_2 \leftrightarrow L_3 \;:\quad \tau \to \frac{\tau^*}{\tau^* - 1}\,. \tag{2.27}$$

where $\tau^* = \tau_1 - i\tau_2$ is the complex conjugate of τ

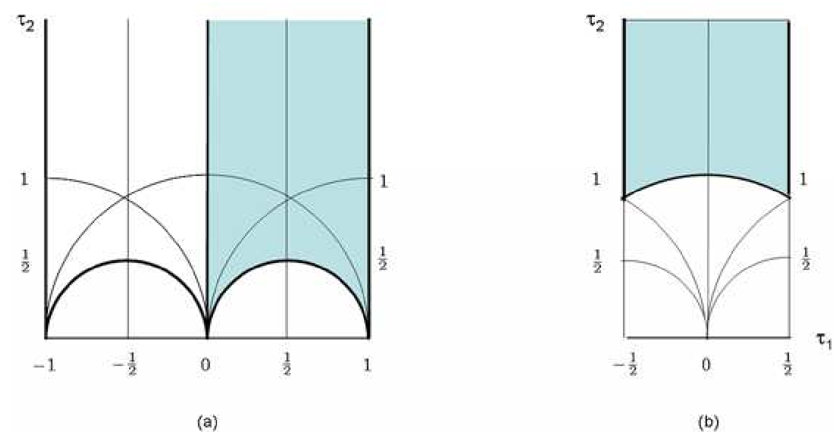

Figure 3: (a) The region of integration of $\tau = \tau_1 + i\tau_2$ is equivalent to a fundamental domain of $\Gamma_0(2)$. Ultraviolet divergences arise on the boundary of this region. (b) The integrand can be mapped into a threefold cover of the fundamental domain, $\mathcal{F}$, of $SL(2,\mathbb{Z})$. The ultraviolet divergences arise from the $\tau_1 = 0$ axis.

It is also easy to see that the integrand is invariant under the T and S transformations defined by

$$T\;: \qquad L_1 \to 2L_1 + L_2\,, \qquad L_2 \to -L_1\,, \qquad L_3 \to 2L_1 + L_3\,, \tag{2.28}$$

and

$$S\;: \qquad L_1 \to -L_1\,, \qquad L_2 \to 2L_1 + L_3\,, \qquad L_3 \to 2L_1 + L_2\,. \tag{2.29}$$

Invariance under the these transformations depends on the invariance of the lattice factor $\Gamma_{(n,n)}$. For example, consider the $\mathcal{S}^1$ compactification, where the lattice factor $\Gamma_{(1,1)}(\{L_r\})$ is given as a sum over m and n in (2.4). The T transformation is an invariance when accompanied by the shift $m \to m+n$. Likewise, the S transformation is an invariance when accompanied by the transformation $m \to -n$, $n \to m$. In terms of the new variables these transformations become

$$T\;: \quad \tau \to \tau + 1\,, \qquad S\;: \quad \tau \to -\frac{1}{\tau}\,. \tag{2.30}$$

Note that the cyclic permutation of the Schwinger parameters is generated by

$$T\,S\;: \qquad L_1 \to L_3\,, \qquad L_2 \to L_1\,, \qquad L_3 \to L_2\,, \tag{2.31}$$

We may now use the S and T transformations to map the shaded domain in figure 3(a) into a three-fold cover of the shaded area in figure 3(b). This is $\mathcal{F}$, the fundamental domain of $SL(2,\mathbb{Z})$, which is defined by $\{-\frac{1}{2} \le \tau_1 \le \frac{1}{2}, |\tau > 1\}$. The boundaries at $\tau_1 = 0$, $\tau_1 = 1$ and $|\tau| = \frac{1}{2}$ in figure 3(a) map into the line $\tau_1 = 0$ in $\mathcal{F}$. In other words the integral over the domain (2.26) is three times the integral over $\mathcal{F}$.

In terms of τ and V the matrix K^{-1} in (2.15) takes the $SL(2,\mathbb{Z})$-covariant form

$$K^{-1} = \frac{V}{\tau_2}\begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix}, \tag{2.32}$$

while (2.14) becomes

$$G_{rs} = -\frac{1}{2}d_{t_r t_s} + \frac{V}{2\tau_2}|v_2^{(k_r)} - v_2^{(k_s)} + \tau(v_1^{(k_r)} - v_1^{(k_s)})|^2 \,. \tag{2.33}$$

For much of what follows it will be useful to perform Poisson resummations on the Kaluza–Klein modes (m_I, n_I) to express the lattice factor in terms of winding numbers $(\hat{m}^I, \hat{n}^I)$, just as in [7][9],

$$\Gamma_{(n,n)}(G_{IJ}; V, \tau) = \mathcal{V}_n^2\, \Delta^{-\frac{n}{2}}\, \hat{\Gamma}_{(n,n)}(G_{IJ}; V, \tau)\,, \tag{2.34}$$

so that the n-dimensional lattice factor in (2.1), (2.2) and (2.4) is given in terms of sums over winding numbers by

$$\hat{\Gamma}_{(n,n)}(G_{IJ}; V, \tau) = \sum_{(\hat{m}^I, \hat{n}^I)\in\mathbb{Z}^{2n}} e^{-\pi \frac{G_{IJ}}{\tau_2 V}(\hat{m}^I + \hat{n}^I\tau)(\hat{m}^J + \hat{n}^J\bar{\tau})} = \sum_{(\hat{m}^I, \hat{n}^I)\in\mathbb{Z}^{2n}} e^{-\pi \hat{E}}\,, \tag{2.35}$$

where we have defined

$$\hat{E}(G_{IJ}; V, \tau) = \frac{G_{IJ}}{\tau_2 V}(\hat{m}^I + \hat{n}^I\tau)(\hat{m}^J + \hat{n}^J\bar{\tau})\,. \tag{2.36}$$

This expression is familiar in string theory as the partition function for the mapping of a world-sheet torus with complex structure τ into a target space torus with metric G_{IJ}. This will prove to be important in evaluating the integrals (as it was in [7, 8]). Note that the factor $\mathcal{V}_n^{-2}\Delta^{n/2}$ in the measure of I^P and I^{NP} cancels in the winding number basis. In terms of the new variables the expression (2.11) has the form

$$\mathcal{I}(S,T,U) = \int_0^\infty dV V^3 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \oint \prod_{r=1}^4 dt_r\, W^2\, e^{-p_r\cdot p_s\, G_{rs}}\, \hat{\Gamma}_{(n,n)}\,. \tag{2.37}$$

In section 2.3.1 we will discuss how the ultraviolet cutoff on the loop momentum can be imposed by suitable choice of limits on these integrals.

In order to evaluate the low energy expansion of the amplitude we need to evaluate the coefficients $I_{(p,q)}$ in (1.21), which arise as the coefficients of the terms in the expansion of the integrand in (2.37). Since we want to express the integral in terms of the new variables τ and

[9] Recall that a Poisson resummation that replaces a sum over a Kaluza–Klein charge m by a sum over a winding number $\hat{m}$ is expressed by the identity $\sum_{m=-\infty}^{\infty} e^{-\pi A m^2 + 2m\pi A s} = A^{-\frac{1}{2}} e^{\pi A s^2} \sum_{\hat{m}=-\infty}^{\infty} e^{-\pi A^{-1}\hat{m}^2 - 2i\pi\hat{m}s}$.

V, we need to change variables in the functions $\tilde{B}_{(p,q)}(L_2/L_1, L_3/L_1)$, defined in (2.20). In the process of changing variables from (L_1, L_2, L_3) to (τ_1, τ_2, V) it will turn out to be convenient to redefine the normalization of $\tilde{B}_{(p,q)}$ by a multiplicative constant factor in order to arrive at final equations with simple coefficients. We therefore define the rescaled coefficients $B_{(p,q)}(\tau)$ by

$$B_{(p,q)}(\tau) = d_{(p,q)}\, \tilde{B}_{(p,q)}(L_2/L_1, L_3/L_1)\,, \tag{2.38}$$

where, up to the order considered in this paper, the integer coefficients $d_{(p,q)}$ are arbitrarily chosen to be

$$\begin{aligned} d_{(1,0)} &= 1\,, \quad d_{(0,1)} = 12\,, \quad d_{(2,0)} = 144\,, \\ d_{(1,1)} &= 15120\,, \quad d_{(3,0)} = 302400\,, \quad d_{(0,2)} = \frac{3}{4}\,302400\,. \end{aligned} \tag{2.39}$$

The simplest examples of the $B_{(p,q)}$'s are $B_{(1,0)} = 1$, obtained in (2.22) and [7], and $B_{(0,1)}$ obtained in (2.23) and [8], which is given in terms of τ_1 and τ_2 by

$$B_{(0,1)}(\tau) = \frac{1}{\tau_2}(|\tau|^2 - |\tau_1| + 1) + \frac{5}{\tau_2^3}(\tau_1^2 - |\tau_1|)(|\tau|^2 - |\tau_1|)\,. \tag{2.40}$$

A most important feature of $B_{(0,1)}$ is that it satisfies the Poisson equation

$$(\Delta_\tau - 12) B_{(0,1)}(\tau) = -12\tau_2 \delta(\tau_1)\,, \tag{2.41}$$

where the laplacian can be written in terms of the original Schwinger parameters as[10]

$$\Delta_\tau \equiv \tau_2^2(\partial_{\tau_1}^2 + \partial_{\tau_2}^2) = \Delta\, \partial_{L_k} \partial_{L_k} - 2L_k\, \partial_{L_k}\,. \tag{2.42}$$

This property is very useful for determining properties of the coefficients $I_{(p,q)}(S,T,U)$, in the low energy expansion in (2.20).

As we will show in appendix A, the higher-order coefficients satisfy generalizations of this Poisson equation. In general these coefficients are sums of the form $B_{(p,q)}(\tau) = \sum_i b^i_{(p,q)}(\tau)$, where the components $b^i_{(p,q)}$ individually satisfy Poisson equations that generalize (2.41). The functions $B_{(2,0)}\,, B_{(1,1)}\,, B_{(3,0)}\,, B_{(0,2)}$ are described in detail in appendix A together with the detailed Poisson equations of the form (A.5) satisfied by the $b^i_{(p,q)}$'s. These Poisson equations are again the key to understanding the structure of the coefficients $I_{(p,q)}$.

2.3 Integration limits

Up to now we have ignored the fact that the Feynman integrals are ultraviolet divergent and therefore need to be regulated in a systematic manner, such as by introducing a momentum cutoff. We will implement this by introducing a lower cutoff on the Schwinger parameters, L_k, that are conjugate to the loop momentum [5], as will be discussed in the nest two subsections. Although this is feasible at one and two loops ($L = 1$ and $L = 2$) it is unlikely to be convenient, and may not even be consistent, for $L > 2$. In addition, although the loops are not infrared divergent, they contain infrared singularities due to the presence of massless intermediate states. Such nonanalytic terms cannot be expanded in a power series in S, T and U and need separate consideration, as described in subsection 2.3.3.

[10] The symbol $\Delta = L_1L_2 + L_2L_3 + L_1L_2$ should not be confused with Δ_τ.

2.3.1 Ultraviolet divergences and counter-terms

The nonrenormalizable divergences of eleven-dimensional supergravity may be subtracted by the addition of local counterterms but this results in an increasing number of apparently arbitrary coefficients as the number of loops is increased. However, some of these coefficients are fixed by imposing extra conditions implied by the correspondence of the compactified theory with string theory. Investigating the extent to which coefficients can be determined in this manner is one of the main motivations of this work[11].

The most basic example arises at one loop. The sum of Feynman diagrams in that case has the form of a prefactor of $\mathcal{R}^4$ multiplying a φ^3 scalar field theory box diagram with eleven-dimensional loop momentum p. The presence of eight factors of momentum in the prefactor implies that the loop amplitude has a Λ^3 ultraviolet divergence in the presence of a cutoff at $|p| = \Lambda$. Since the loop momentum enters the integral over with a factor $e^{-p^2 L}$, where L is the Schwinger parameter, the momentum cutoff may implemented in a gaussian manner by introducing the cutoff $L \geq \Lambda^{-2}$, instead of a step function momentum cutoff. Upon compactification, this short-distance divergence arises entirely in the sector with zero winding number. This dependence on Λ can be subtracted by introducing a local counterterm, $\delta_1 A = c_1 \mathcal{R}^4$, which replaces Λ^3 by a specific finite value. The value of c_1 was precisely determined in [5, 7] to be

$$c_1 = \frac{2\pi^2}{3} - \frac{4\pi}{3}\Lambda^3 \,, \tag{2.43}$$

which followed from the fact that the one-loop terms in the four-graviton scattering amplitude in the type IIA and type IIB theories are equal (and equal to the genus-one term in the perturbative expansion of $E_{3/2}$).

At two loops ($L = 2$) there are new issues. Firstly, there are new primitive divergences. The naive degree of divergence is Λ^{20} but there is now a prefactor of order $S^2\,\mathcal{R}^4$ [16], which has twelve powers of momentum. This means that the naive primitive divergence is reduced from Λ^{20} to Λ^8. But upon compactification on $\mathcal{T}^n$ powers of Λ may be traded in for inverse powers of the radii of the compact dimensions. In addition, there are one-loop sub-divergences behaving as Λ^3. We would like to impose a cutoff on the two-loop Schwinger parameters, L_k, in a manner that is consistent with that imposed at one loop. In particular, the two-loop amplitude has massless intermediate two-particle thresholds arising in the S-channel from intermediate states with $m_I = 0$ (or $n_I = 0$) in (2.1) and (2.2). The discontinuity across this threshold is obtained by setting $m_I = 0$ and is proportional to the one-loop four-point amplitude multiplied by the tree-level amplitude. It is evident from (2.1) and (2.2) that the Schwinger parameter for the one-loop sub-amplitude is $L_1 + L_2$. Therefore, in order to reproduce the same Λ-dependence as in the one-loop calculation we must introduce a cutoff $L_1 + L_2 \geq \Lambda^{-2}$. By symmetry the individual Schwinger parameters satisfy $L_k \geq \Lambda^{-2}/2$, which means, using (2.24), that V is integrated over the range

$$0 \leq V \leq V^{\Lambda} \equiv \frac{2}{\sqrt{3}}\Lambda^2 \,, \tag{2.44}$$

[11] We cannot make use of dimensional regularization since this cannot be consistently applied to a nonrenormalizable theory. In particular, standard dimensional regularization would set to zero all the dimensional terms associated with power divergences that we will need to keep. Nevertheless it is extremely useful for evaluating a subset of the nonanalytic terms, as we will see in appendix E

Only the fact that the upper cutoff is linear in Λ^2 will prove to be relevant in this paper, whereas the precise coefficient of $2/\sqrt{3}$ will be irrelevant. The cutoff on $L_1 + L_2$ implies

$$V\tau_2 \le \Lambda^2\,, \tag{2.45}$$

which imposes the same ultraviolet cutoff on τ_2 as in [7], so that $\tau = \tau_1 + i\tau_2$ is integrated over the cutoff fundamental domain,

$$\mathcal{F}_\Lambda: \qquad \{1 \le |\tau|\,, \qquad \tau_2 \le \tau_2^\Lambda = \Lambda^2/V\,, \qquad -1/2 \le \tau_1 \le 1/2\}\,. \tag{2.46}$$

Primitive divergences and sub-divergences will be subtracted by diagrams containing counterterms, as illustrated in figure 1. In order to subtract the Λ^3 sub-divergences we need to include the one-loop 'triangle' diagram in which there are two supergravity vertices and one $\mathcal{R}^4$ contact interaction, which is the counterterm that cancels the Λ^3 one-loop $\mathcal{R}^4$ divergence and replaces it with a specific finite constant (see figure 1(c)). The contribution of the diagram (denoted by δA in section 4.3 of reference [7]) is given by

$$A_{sugra\,\triangleright}(S,T,U) = ic_1 \frac{\kappa_{11}^6}{(2\pi)^{22}}\,\frac{\pi^3}{2l_{11}^{12}}\,\left(S^2\, I_\triangleright(S) + T^2\, I_\triangleright(T) + U^2\, I_\triangleright(U)\right)\,\mathcal{R}^4\,, \tag{2.47}$$

where

$$I_\triangleright(S) = \pi^{11-n}\int_{\Lambda^{-2}}^\infty dL\, L^{\frac{n-7}{2}}\int_0^1 du_2\int_0^{u_2} du_1\, e^{-Lu_1(1-u_2)\,S}\sum_{m\in\mathbb{Z}^n} e^{-\pi\, L\, G^{IJ} m_I n_J} \tag{2.48}$$

In addition to the sub-divergences there are local 'primitive' divergences of the form $\Lambda^8\, S^2\, \mathcal{R}^4$ and $\Lambda^6\, S^3\, \mathcal{R}^4$, as well as 'overlapping' divergences that lead to an extra $\Lambda^6\, S^3\, \mathcal{R}^4$ term. All of these need to be subtracted by additional local counterterms.

$$\delta_2 A_{sugra} = -\frac{\kappa_{11}^6}{(2\pi)^{22}\, l_{11}^{12}}\,(a\,\Lambda^8\, S^2\,\mathcal{R}^4 + (b\Lambda^6 - c\,\Lambda^3 - d)\, S^3\,\mathcal{R}^4 + \cdots) \tag{2.49}$$

where a, b, c and d are constants. Obviously, a local counterterm has to be independent of the moduli of the compact space.

2.3.2 Examples of renormalized interactions

One can anticipate the kinds of cutoff-dependent terms that arise upon compactification on $\mathcal{T}^n$ by simple dimensional considerations. Such terms translate into particular perturbative terms in type IIA string theory by use of the dictionary that translates between M-theory parameters and string theory parameters. Such terms will be analyzed in detail later in this paper, but here we will sketch some features that arise.

The following are examples of terms that will arise after compactification on a circle of radius R_{11}. In this case the translation of the supergravity results to IIA string theory makes the identifications $S = R_{11}\, s$, $R_{11}^3 = g_A^2$, and the implicit momentum conservation delta function, $\delta^{(10)}(\sum_r p_r^\mu)$, transforms in a manner that cancels the transformation of $\mathcal{R}^4$.

- The lowest-order term with a one-loop sub-divergence has the form $S^2\,\mathcal{R}^4\,R_{11}^{-5}\,\Lambda^3$ and corresponds to tree-level IIA. After renormalizing this by adding the contribution of the triangle diagram containing the one-loop counterterm with coefficient c_1 this has a value equal to that of the tree-level contribution contained in the modular function $E_{5/2}$ that multiplies $S^2\,\mathcal{R}^4$ in the IIB theory – as it should since the IIA and IIB theories have identical perturbative expansions up to at least genus four [11].

- Expanding the one-loop sub-divergence to next order in S, T, U results in a term of the form $S^3\,\mathcal{R}^4\,R_{11}^{-3}\Lambda^3$ that contributes to a genus-one term in IIA string theory and, after renormalization, equals the genus-one part of the the modular function $\mathcal{E}_{(0,1)}$ (or $\mathcal{E}_{(3/2,3/2)}$ in the notation of [8]).

- A term of the form $S^4\,\mathcal{R}^4\,R_{11}^{-1}\Lambda^3$, is associated with a genus-two IIA string contribution that contributes a term of the form $g_A^2\,s^4\,\log(-s)\,\mathcal{R}^4$. [A similar finite term of the form $R_{11}^{-4}\,S^4\,\mathcal{R}^4$ corresponds to a genus-one IIA string theory contribution.]

- A further term arising from one-loop sub-divergences is $S^5\,\mathcal{R}^4\,R_{11}\Lambda^3$, which contributes to a genus-three term of order $g_A^4\,S^5\,\mathcal{R}^4$ in IIA string theory.

- In addition to these contributions from sub-divergences, there is a primitive divergence of the form $S^2\,\mathcal{R}^4\,\Lambda^8$ which has to be canceled by a new local counterterm. Since this translates into a type IIA string-theory term proportional to $g_A^{8/3}$, which is not a consistent power, this must have a vanishing renormalized value.

- Similarly, a possible term $S^4\,\mathcal{R}^4\,\Lambda^4$ does not have a sensible perturbative string theory interpretation and so we will set its renormalized value to zero.

- By contrast, a term of the form $S^3\,\mathcal{R}^4\,\Lambda^6$ is to be canceled by another new two-loop local counterterm, but the finite renormalized value must take a specific value (just as we saw in the cancellation of the one-loop $\mathcal{R}^4\,\Lambda^3$ term) equal to the genus-two term in the IIB modular function, $\mathcal{E}_{(0,1)}$.

- A new phenomenon that arises at order $S^6\,\mathcal{R}^4$ is the occurrence of a primitive logarithmic divergence, $\log\Lambda$, which is the divergence manifested as a pole in ϵ in dimensional regularization of the eleven-dimensional theory. This should again be subtracted by a local counterterm.

An important feature of the two-loop divergences is that they describe local terms that are independent of R_{11} and can indeed be subtracted by the addition of new local counterterms. Whether this continues to be the case at higher loops ($L > 2$) is an interesting question that is not addressed here.

Corresponding terms arise in the $\mathcal{T}^2$ compactification to nine dimensional string theory, in which there is also dependence on the radii r_B or r_A.

2.3.3 Dealing with infrared threshold effects

Although the four-graviton amplitude has no infrared divergences in nine or ten dimensions, there are subtleties in extracting the nonanalytic threshold terms, which are infrared consequences of intermediate massless multi-particle states. These states are zero Kaluza–Klein modes.

At one loop ($L = 1$) there is a complicated S, T and U-channel discontinuity structure (reviewed in [10]) with the property that the scales of various logarithmic factors cancel out and the form of the nonanalytic terms is invariant under rescaling the Mandelstam invariants.

The massless intermediate states originate in the two-loop case ($L = 2$) from zero Kaluza–Klein modes in the factor $\Gamma_{(n,n)}$ in (2.4). The nonanalytic terms have discontinuities that arise from the long-time propagation of these states. For example, the S-channel configuration shown in figure 1 has two-particle thresholds associated with $m_I = 0$ or $n_I = 0$, or both, arising from the integration limits $L_1 \to \infty$, $L_3 \to \infty$ (or both simultaneously). In the low energy expansion this generates terms of the form $A_k\, S^k \log(-S/C_k)$ (where A_k and C_k are constants) that are required by unitarity. However, if we were to simply expand the integrand in integer powers of S the signature of such thresholds would be the occurrence of terms of order S^k with divergent coefficients, $\lim_{\mu\to 0} C_k\, S^k \log \mu$, where μ is an infrared cutoff.

In the following we will not be interested in the details of the nonanalytic thresholds[12], but will simply concentrate on the dependence of the nonanalytic terms in the amplitude on the scale χ ($\chi > 0$) of the Mandelstam invariants, defined by

$$S = \chi\, S_0\,, \qquad T = \chi\, T_0\,, \qquad U = -\chi(S_0 + T_0)\,, \tag{2.50}$$

where S_0 and T_0 are arbitrary constants. Since the limit $L_k \to \infty$ (for any k) translates into the limit $V \to 0$, the infrared nonanalytic effects arises when there are sufficient inverse powers of V. The signature of these contributions is the presence of divergent coefficients, $I_{(p,q)}$, in the power series expansion (1.21).

A simple model for the parts of our expressions that give rise to nonanalytic thresholds is given by considering the convergent integral, $H = \int_0^{\Lambda^2} dV\, V^a e^{-\chi/V}$, where $a > -1$. On the one hand this can be expanded for $\chi/\Lambda^2 < 1$ as

$$\begin{aligned} H = {} & \frac{\Lambda^{2(a+1)}}{a+1} + \frac{\Lambda^{2a}\,(-\chi)}{a} + \cdots + \frac{\Lambda^{2a-2r+2}\,(-\chi)^r}{(a-r+1)r!} + \ldots \\ & + \frac{(-\chi)^{a+1}}{(a+1)!}\,(\log(\Lambda^2/\chi) - \Gamma'(a+2)/\Gamma(a+2)) + O(\chi^{a+2}/\Lambda^2)\,. \end{aligned} \tag{2.51}$$

On the other hand, the analogue of the procedure adopted in this paper is to consider the formal series obtained by expanding the integrand of H,

$$H = \sum_{r=0}^{\infty} \int_0^{\Lambda^2} \frac{1}{r!} V^{a-r}\,(-\chi)^r dV. \tag{2.52}$$

[12] The detailed threshold structure is obtained much more simply using dimensional regularization than with the cutoff procedure we are adopting.

Clearly, terms with $r > a$ are divergent at the small V endpoint, despite the convergence of the original integral. However, we are only interested in the terms with non-negative powers of Λ (i.e., terms of order χ^r with $r \le a+1$) in the exact expansion given by (2.51). So the correct result is obtained by simply ignoring all the divergent terms in the formal expansion, with the exception of the term with $r = a+1$, which has a logarithmic divergence. This term is to be interpreted as $\log(\Lambda^2/\chi)$. An efficient way of describing this is to replace the formal expression (2.52) by the regulated expression

$$\begin{aligned} H_{reg} &= \int_0^{\Lambda^2} \sum_{r=0}^{a+1} \frac{1}{r!} V^{a-r} \, (-\chi)^r \, e^{-\chi f/V} dV \\ &\equiv \int_{0_\chi}^{\Lambda^2} \sum_{r=0}^{\infty} \frac{1}{r!} V^{a-r} \, (-\chi)^r dV \end{aligned} \tag{2.53}$$

with $f \ll 1$. The second line defines the notation to be used in discussing the analogous integrals that we will meet later in this paper. The expression (2.53) reproduces the exact expansion (2.51), apart from the scale inside the log factor, which is now proportional to $\chi^{a+1} \log(\Lambda^2/\chi f)$. The above simple example illustrates how the terms in the expansion (2.20) of the integrand in (2.11) give the correct series expansion of the amplitude, including the parts with logarithmic nonanalytic behaviour. However, the scale of the logarithm is not determined.

As an example of such a nonanalytic term let us consider the explicit ten-dimensional string loop calculation [10] that determines a genus-one logarithmic threshold at order s^4. This can be written as $\mathcal{R}^4$ multiplied by

$$S^4 \log(-SR_{11}^2/\mu_2) + T^4 \log(-TR_{11}^2/\mu_2) + U^4 \log(-UR_{11}^2/\mu_2) = \sigma_2^2 \, \log(\chi R_{11}^2/C_{(2,0)}), \tag{2.54}$$

where μ_2 is a specific constant and $C_{(2,0)}(S_0, T_0)$, has a specific dependence on S_0, and T_0 (but not on χ). This scale encodes the precise details of the multiparticle thresholds. In this paper we will reproduce the expression on the second line of (2.54) from the $\mathcal{S}^1$ compactification of $L = 2$ eleven-dimensional supergravity, but the scale $C_{(2,0)}$ multiplying χ inside the log factor will not be determined. More generally, in the compactification to ten dimensions we will obtain nonanalytic contributions of the form $K_{(p,q)} \, \sigma_2^p \, \sigma_3^q \, \log(\chi R_{11}^2/C_{(p,q)})$, where the constant coefficients, $K_{(p,q)}$, will be evaluated, but not $C_{(p,q)}$. In principle, $C_{(p,q)}$ can be reconstructed, apart from a multiplicative constant, from two-particle and three-particle unitarity.

In the compactification on $\mathcal{T}^2$ to nine dimensions most of the nonanalytic contributions are characterized by half-integral powers of S, T and U. These are easily separated from the analytic parts and we will not consider them here. However, a logarithmic term will also arise at order $l_{11}^8 \, \mathcal{D}^8 \, \mathcal{R}^4 \sim \sigma_2^2 \mathcal{R}^4$, which is known in detail using dimensional regularization [16] (and is reviewed in appendix E) and will also be considered in the Λ cutoff procedure.

2.4 The general form of the expansion of two loops on $\mathcal{T}^n$

After taking care of various normalization constants, the low-energy expansion of the two-loop supergravity amplitude (1.18) compactified on a n-torus can be written as

$$A_{sugra} = i\frac{\kappa_{11}^6}{(2\pi)^{22}}\,\mathcal{R}^4\,\pi^6\Big[\sigma_2 I_{(1,0)} + \frac{\sigma_3}{12}\,I_{(0,1)} + \frac{\sigma_2^2}{2!\cdot 144}\,I_{(2,0)} + \frac{\sigma_2\sigma_3}{3!\cdot 15120}I_{(1,1)} + \frac{1}{4!\,302400}\big(\sigma_2^3 I_{(3,0)} + \frac{4}{3}\sigma_3^2 I_{(0,2)}\big) + \dots\Big]\,. \tag{2.55}$$

where the coefficients are functions of the moduli that are given by the integrals

$$I_{(p,q)} = \pi^{N+1}\int_{0_\chi}^{V^\Lambda} dV V^{3-N}\int_{\mathcal{F}_\Lambda}\frac{d^2\tau}{\tau_2^2}B_{(p,q)}(\tau)\,\hat\Gamma_{(n,n)}(G_{IJ}V,\tau)\,, \tag{2.56}$$

where $N = 2p + 3q - 2$ and the functions $B_{(p,q)}(\tau)$ are defined via (2.21) and (2.38). These can be evaluated by symbolic computer methods and the resultant expressions are given up to order $2p + 3q = 6$ in appendix A, together with their decomposition into functions satisfying Poisson eigenvalue equations. Recall that the notation for the integration limits in (2.56) builds in the S, T, U-dependence of the nonanalytic thresholds, so that factors of $\log\chi$ are present in certain terms in (2.55). The lattice factor $\hat\Gamma_{(n,n)}$ defined in (2.35) contains the information about the spatial torus.

We will now turn to the evaluation of the $I_{(p,q)}$'s explicitly for the cases $n = 1$ (compactification on $\mathcal{S}^1$), and $n = 2$ (compactification on $\mathcal{T}^2$).

3. Circle compactification to ten dimensions

We will now consider the value of the two-loop amplitude after compactification on a circle of radius $l_{11}R_{11}$ in the special case in which the external momenta are zero in the compact direction. The metric of the eleven-dimensional theory is related to the string-frame ten-dimensional type IIA metric in the usual manner by

$$ds^2 = G^{(11)}_{MN}dx^M dx^N = \frac{l_{11}^2}{l_s^2 R_{11}}\,g_{\mu\nu}dx^\mu dx^\nu + R_{11}^2 l_{11}^2(dx^{11} - C_\mu dx^\mu)^2, \tag{3.1}$$

where $g_{\mu\nu}$ is the string frame metric and $R_{11}\,l_{11}$ is the radius of the eleventh dimension (and we will not be interested in the one-form C_μ here). The dictionary for translating between M-theory and type IIA string theory relates the string coupling and string-frame Mandelstam invariants to R_{11} and the eleven-dimensional invariants by

$$l_{11} = R_{11}^{\frac{1}{2}}\,l_s \qquad g_A^2 = R_{11}^3\,, \qquad S = R_{11}\,s\,. \tag{3.2}$$

In order to evaluate the ten-dimensional amplitude, $A^{(d=10)}$, we will need to evaluate the integral $I_{(p,q)}$ (2.56) in the case $n = 1$, which we will call $I^{(d=10)}_{(p,q)}$.

3.1 Evaluation of $I^{(d=10)}_{(p,q)}$

In this case the metric of the compact dimension is simply $G_{IJ} = R_{11}^2$ so from (2.36) we have

$$\hat E = V\,v\,\frac{|\hat m + \tau\hat n|^2}{\tau_2}\,, \tag{3.3}$$

which is to be used in (2.56), and where we have set

$$v = R_{11}^2 \,. \tag{3.4}$$

The integral $I^{(d=10)}_{(p,q)}$ could be evaluated directly by use of the 'unfolding trick', but it is more straightforward to use the method devised in [8] for studying $I^{(d=10)}_{(0,1)}$. This begins by noting that

$$\left(v^2 \frac{\partial^2}{\partial v^2} + 2v \frac{\partial}{\partial v}\right) e^{-\pi \hat{E}} = \Delta_\tau \, e^{-\pi \hat{E}} \equiv \tau_2^2 \left(\frac{\partial^2}{\partial \tau_1^2} + \frac{\partial^2}{\partial \tau_2^2}\right) e^{-\pi \hat{E}} \,. \tag{3.5}$$

This means that $I^{(d=10)}_{(p,q)}$ satisfies

$$\begin{aligned} \left(v^2 \frac{\partial^2}{\partial v^2} + 2v \frac{\partial}{\partial v}\right) \quad I^{(d=10)}_{(p,q)} &= \pi^{N+1} \int_{0_\chi}^{V^\Lambda} dV V^{3-N} \int_{\mathcal{F}_\Lambda} \frac{d^2\tau}{\tau_2^2} B_{(p,q)}(\tau) \sum_{(\hat{m},\hat{n}) \in \mathbb{Z}^2} \Delta_\tau \, e^{-\pi \hat{E}} \\ &= \pi^{N+1} \sum_{(\hat{m},\hat{n}) \neq (0,0)} \int_{0_\chi}^{V^\Lambda} dV V^{3-N} \int_{\mathcal{F}_\Lambda} \frac{d^2\tau}{\tau_2^2} \Delta_\tau \, B_{(p,q)}(\tau) \, e^{-\pi \hat{E}} - \partial I^{(d=10)}_{(p,q)} \,, \end{aligned} \tag{3.6}$$

where we have integrated by parts and the boundary term is given by

$$\partial I^{(d=10)}_{(p,q)} = \pi^{N+1} \sum_{\hat{m},\hat{n}} \int_{0_\chi}^{V^\Lambda} dV V^{3-N} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \, \left(B_{(p,q)}(\tau) \, \partial_{\tau_2} \, e^{-\pi \hat{E}} - \partial_{\tau_2} \, B_{(p,q)}(\tau) \, e^{-\pi \hat{E}}\right) \Bigg|_{\tau_2 = \tau_2^\Lambda} \tag{3.7}$$

(where we recall that $V \, \tau_2^\Lambda = \Lambda^2$). After substituting $B_{(p,q)} = \sum_{i=1}^{\lceil 3N/2 \rceil} b^i_{(p,q)}$ (as in (A.4)) and writing

$$I^{(d=10)}_{(p,q)} = \sum_{i=0}^{\lceil 3N/2 \rceil} h^i_{(p,q)}(v) \,, \tag{3.8}$$

equation (3.6) is replaced by a set of component equations of the form

$$\left(v^2 \frac{\partial^2}{\partial v^2} + 2v \frac{\partial}{\partial v}\right) \, h^i_{(p,q)} = j^i_{(p,q)} - \partial b^i_{(p,q)} \equiv J^i_{(p,q)} \,, \tag{3.9}$$

where the bulk term is given by

$$j^i_{(p,q)} = \pi^{N+1} \int_{0_\chi}^{V^\Lambda} dV V^{3-N} \int_{\mathcal{F}_\Lambda} \frac{d^2\tau}{\tau_2^2} \, \Delta_\tau \, b^i_{(p,q)}(\tau) \sum_{(\hat{m},\hat{n}) \in \mathbb{Z}^2} e^{-\pi \hat{E}} \,, \tag{3.10}$$

and $\partial b^i_{(p,q)}$ is the component form of the last term in (3.7). After using the Poisson equation (A.5) satisfied by $b^i_{(p,q)}$, equation (3.9) reduces to a set of simple second order differential equations.

As we will see later, for terms that are analytic in the Mandelstam invariants the right-hand side of (3.9), $J^i_{(p,q)}$, has a dependence on v of the form

$$J^i_{(p,q)} = v^{N-4} \, J^{(4-N)\, i}_{(p,q)} + \Lambda^3 \, v^{N-\frac{5}{2}} \, J^{(\frac{5}{2}-N)\, i}_{(p,q)} + \Lambda^{8-2N} \, J^{(0)\, i}_{(p,q)} \,, \tag{3.11}$$

where $J^{(\alpha)\,i}_{(p,q)}$ are constants, the superscript (α) indicates the power of v, and $N = 2p+3q-2$. The first term in (3.11) is the finite contribution from non-zero winding numbers $\hat m \neq 0$, $\hat n \neq 0$, and the power v^{N-4} follows by a simple dimensional argument. The second term, proportional to Λ^3, comes from the one-loop sub-divergent contributions in the sectors with $\hat m = 0$, $\hat n \neq 0$ and $\hat m \neq 0$, $\hat n = 0$ and the power $v^{N-5/2}$ is again determined by a simple dimensional argument. The third term with the power Λ^{8-2N} comes from the sector with zero winding number, $\hat m = 0, \hat n = 0$. In the $N = 4$ cases, $(p,q) = (3,0)$ and $(p,q) = (0,2)$, the powers of v^{4-N} and Λ^{8-2N} include pieces that should be interpreted as $\log v$, $\log \Lambda$, as we will see in the explicit evaluation later in this section. After substituting the structure (3.11) into (3.10) and (3.9) each term in (3.8) is seen to decompose in the same manner into $v^{N-4}\, h^{(4-N)\,i}_{(p,q)} + \Lambda^3\, v^{N-\frac{5}{2}}\, h^{(\frac{5}{2}-N)\,i}_{(p,q)} + \Lambda^{8-2N}\, h^{(0)\,i}_{(p,q)}$ and (3.6) is solved by substituting

$$\left(v^2 \frac{\partial^2}{\partial v^2} + 2v \frac{\partial}{\partial v}\right) v^{-\alpha}\, h^{(\alpha)\,i}_{(p,q)} = \alpha(\alpha-1)\, v^{-\alpha}\, h^{(\alpha)\,i}_{(p,q)}\,, \tag{3.12}$$

so that

$$h^{(\alpha)\,i}_{(p,q)} = \frac{1}{\alpha(\alpha-1)} J^{(\alpha)\,i}_{(p,q)}\,. \tag{3.13}$$

Therefore, $I_{(p,q)}$ (2.56) decomposes into the sum of three terms with distinct powers of v,

$$I^{(d=10)}_{(p,q)} = v^{N-4}\, I^{(4-N)}_{(p,q)} + \Lambda^3\, v^{N-\frac{5}{2}}\, I^{(\frac{5}{2}-N)}_{(p,q)} + \Lambda^{8-2N}\, I^{(0)}_{(p,q)}\,. \tag{3.14}$$

The one-loop sub-divergences proportional to Λ^3 are canceled by adding the triangle diagram, $I^{(d=10)}_{\triangleright\,(p,q)}$, with the one-loop $\mathcal{R}^4$ counterterm at one vertex (figure 1 (c)). The term proportional to Λ^{8-2N} is a primitive divergence that has to be subtracted by a new two-loop local counterterm, whose contribution to $I^{(d=10)}_{(p,q)}$ will be denoted $\delta_2 I^{(d=10)}_{(p,q)}$.

We will also be concerned, when $N \geq 2$, with the situation in which there are logarithmic terms on the right-hand side of (3.11) proportional to $v^\alpha \log v$ and $v^\alpha (\log v)^2$, with various values of α. In the presence of such source terms the solutions of (3.6) also have logarithms. To be specific, the general form of the equations in the examples that follow is

$$(v^2\partial_v^2 + 2v\partial_v - \lambda)\, f(v) = a\, \log v + b\, \frac{\log v}{v^{\frac{1}{2}}} + c\, \frac{\log v}{v^2} + d\, \frac{(\log v)^2}{v}\,. \tag{3.15}$$

It is easy to verify that the solution of this equation is

$$\begin{aligned} f(v) \,=\, & -a\, \frac{1}{\lambda^2}(1+\lambda \log v) - b\, \frac{4}{1+4\lambda}\, \frac{\log v}{v^{\frac{1}{2}}} + c\, \frac{1}{v^2(2-\lambda)} \left(\log v + \frac{3}{2-\lambda}\right) \\ & - d\, \frac{1}{v\lambda^3}(2+2\lambda - 2\lambda \log v + \lambda^2 (\log v)^2) + f_0(v)\,, \end{aligned} \tag{3.16}$$

where f_0 is the solution of the homogeneous equation that will be irrelevant to us. We will see that these non-analytic terms are logarithmic thresholds expected from unitarity in ten dimensions.

This procedure is implemented in detail in appendix B in the order to evaluate $I_{(p,q)}$ with $N = 1, 2, 3, 4$, leading to terms in the four-graviton amplitude that we will now review.

3.2 The ten-dimensional type IIA low energy string scattering amplitude

We will now summarize the expressions deduced in detail in appendix B from the analysis of two-loop eleven-dimensional supergravity on $\mathcal{S}^1$. The complete expressions will also include the contribution $I^{(d=10)}_{\triangleright\,(p,q)}$ that comes from the triangle diagram where one vertex is the $\mathcal{R}^4$ one-loop counterterm and the primitive divergence, $\delta_2 I^{(d=10)}_{(p,q)}$. We will begin by reviewing the $(1,0)$ and $(0,1)$ cases before considering the higher-order terms.

3.2.1 $(p,q) = (1,0)$

The contribution to the coefficient of the $\sigma_2\,\mathcal{R}^4 \sim \mathcal{D}^4\mathcal{R}^4$ term was shown in [7] to have the form

$$I^{(d=10)}_{(1,0)} + I^{(d=10)}_{\triangleright\,(1,0)} + \delta_2 I^{(d=10)}_{(1,0)} \;=\; \frac{\zeta(5)}{4\,R_{11}^5}\,, \tag{3.17}$$

which corresponds to the tree-level type IIA contribution $\zeta(5)\,\hat\sigma_2\mathcal{R}^4/g_A^2$. In this case the finite piece from the $\hat m \neq 0$, $\hat n \neq 0$ sector vanishes and the right-hand side arises entirely from the Λ^3 sub-divergence of $I^{(d=10)}_{(1,0)}$, together with $I^{(d=10)}_{\triangleright\,(1,0)}$, which cancels that divergence and replaces it with a specific finite expression. The power of R_{11} corresponds to a tree-level contribution in type IIA string theory, using the identifications in (3.2) (recalling that $S = R_{11}\,s$). The Λ^8 contribution to $I^{(d=10)}_{(1,0)}$ coming from the $\hat m = 0, \hat n = 0$ sector has been set to zero by subtracting it with a two-loop counterterm, $\delta_2 I^{(d=10)}_{(1,0)}$. There can be no finite remainder since $S^2\,\mathcal{R}^4$ has no dependence on R_{11} and translates into a IIA string contribution $g_A^{2/3}\,S^2\,\mathcal{R}^4$, which would not make sense in string perturbation theory. For completeness, recall that there is also a genus-two contribution to $R_{11}\,\sigma_2\mathcal{R}^4 \sim g_A^2\,\hat\sigma_2\,\mathcal{R}^4$ that is obtained from the $\mathcal{S}^1$ compactification of *one-loop* ($L = 1$) eleven-dimensional supergravity. These tree-level and two-loop type IIA string theory contributions are precisely the same as those contained in the modular function $E_{5/2}$ that arises in the type IIB theory reviewed in the introduction. It is also notable that the two-loop supergravity calculation does not generate a genus-one contribution to $S^2\,\mathcal{R}^4$. Such a term is known to be absent in string perturbation theory [5].

The perfect agreement of the predictions from two-loop eleven-dimensional supergravity with string theory found in [7] strongly indicated that higher-loop supergravity does not contribute further terms at order $\mathcal{D}^4\mathcal{R}^4$. This suggested [7] that the three-loop amplitude should be of order $\mathcal{D}^6\mathcal{R}^4$ (or higher), as has recently been shown explicitly [19].

3.2.2 $(p,q) = (0,1)$

The expression $I^{(d=10)}_{(0,1)}$ is the coefficient of the $\sigma_3\,\mathcal{R}^4$ (or $\mathcal{D}^6\mathcal{R}^4$) term. The finite part of $I^{(d=10)}_{(0,1)}$ (the non-zero winding number sector) was considered in [8]. In addition there is a Λ^3 sub-divergence (from the sector in which one winding number vanishes) that needs to be subtracted by $I^{(d=10)}_{\triangleright\,(0,1)}$, as well as a $\Lambda^6\,\mathcal{D}^6\mathcal{R}^4$ primitive divergence (from the sector in which both winding numbers vanish) that is subtracted by $\delta_2 I^{(d=10)}_{(0,1)}$. This divergent term translates into a possible IIA string term, proportional to $g_A^2\,\mathcal{D}^6\mathcal{R}^4$ term. This is not only a possible contribution, but is known to be present since it is present in the type IIB theory with a coefficient that is determined by $\mathcal{E}_{(0,1)}$ and was reviewed in the introduction. The net result is

$$I^{(d=10)}_{(0,1)} + I^{(d=10)}_{\triangleright\,(0,1)} + \delta_2 I^{(d=10)}_{(0,1)} = \frac{\zeta(3)^2}{2\,R_{11}^6} + \frac{\zeta(3)\zeta(2)}{R_{11}^3} + \frac{6\zeta(2)^2}{5}\,, \tag{3.18}$$

In terms of type IIA string theory, these terms correspond to tree-level, genus-one and genus-two contributions with coefficients g_A^{-2}, g_A^0 and g_A^2, respectively Whereas, the coefficients of the first two terms are derived from the $\mathcal{S}^1$ compactification, the coefficient of the last term has been fixed by choosing $\delta_2 I^{(d=10)}_{(0,1)}$ so that it coincides with the coefficient of the two-loop term in the type IIB theory that came from the $\mathcal{T}^2$ compactification. For completeness, recall that there is a genus-three contribution to $\mathcal{D}^6\mathcal{R}^4$ that is again obtained as a finite contribution from the $\mathcal{S}^1$ compactification of one-loop ($L=1$) eleven-dimensional supergravity and which is also contained in the IIB expression $\mathcal{E}_{(0,1)}$, with precisely the same coefficient.

3.2.3 $(p,q)=(2,0)$

When $N \geq 4$ the $L=2$ amplitude develops logarithmic singularities corresponding to string theory threshold contributions. These contributions require careful treatments which is detailed in the appendices B and in C.

The expression $I^{(d=10)}_{(2,0)}$ is the coefficient of $\sigma_2^2\,\mathcal{R}^4 \sim \mathcal{D}^8\,\mathcal{R}^4$. At this order there is a second logarithmic singularity (after the one-loop ($L=1$) supergravity threshold of ten-dimensional supergravity, which is of order $S\log(-S)$) corresponding to a threshold of string perturbation theory. The contribution at order $\sigma_2^2\,\mathcal{R}^4$, derived in the appendix B.3, is

$$I^{(d=10)}_{(2,0)} + I^{(d=10)}_{\triangleright\,(2,0)} + \delta_2 I^{(d=10)}_{(2,0)} = -\,\frac{12}{5}\,\zeta(2)\,\left[\frac{\zeta(3)}{R_{11}^4} + \frac{2\zeta(2)}{R_{11}}\right]\,\log(\chi R_{11}^2/C_{(2,0)})\,. \tag{3.19}$$

When converting these expressions to the string frame $\log(\chi R_{11}^2)$ becomes $\log(\chi\, g_A^2)$, where we have used the relation $S=\chi S_0 = sR_{11} = \chi s_0 R_{11}$ (in other words, we have rescaled the Mandelstam invariants in the string frame by the same factor χ as in the eleven-dimensional frame). This non-analytic contribution is seen to correspond to the genus-one and genus-two normal massless thresholds of the string amplitude with coefficients $g_A^0\,\log(\chi)$ and $g_A^2\,\log(\chi)$, respectively The coefficient of the genus-two threshold contribution arises from a Λ^3 sub-divergence, which is regulated by the counter-term $I^{(d=10)}_{\triangleright\,(2,0)}$. The scale of the logarithms indicated by $C_{(2,0)}$ has not been determined by this computation and hides the details of the T-channel and U-channel thresholds, although in this case the complete calculation is straightforward and leads to (2.54).

3.2.4 $(p,q)=(1,1)$

The coefficient of the $\sigma_2\,\sigma_3\,\mathcal{R}^4 \sim D^{10}\,\mathcal{R}^4$ term is determined by the integral $I^{(d=10)}_{(1,1)}$, together with the triangle diagram containing the one-loop sub-divergence and the two-loop counterterm. These give

$$I^{(d=10)}_{(1,1)} + \hat{I}^{(d=10)}_{\triangleright\,(1,1)} + \delta_2 \hat{I}^{(d=10)}_{(1,1)} = \frac{448}{R_{11}^2}\zeta(4)\zeta(3) + \frac{675}{2}\zeta(2)\zeta(4)R_{11} + 182\,\zeta(2)^2\,R_{11}^2\,\log(\frac{\chi R_{11}^2}{C_{(1,1)}}) \tag{3.20}$$

The first two terms will translate into genus-two and genus-three contributions to $S^5\,\mathcal{R}^4$ in string theory, while the last term is a genus-two threshold contribution In fact, the coefficient of $\log(\chi)$ is precisely the same as in the ten-dimensional supergravity calculation in appendix E.1, which contains the detailed threshold dependence.

3.2.5 $(p,q)=(3,0)$ and $(p,q)=(0,2)$

In the case of $\sigma_2^3\,\mathcal{R}^4$ and $\sigma_3^2\,\mathcal{R}^4$ the coefficients are determined by the integrals $I^{(d=10)}_{(3,0)}$ and $I^{(d=10)}_{(0,2)}$, together with the contributions from the one-loop and two-loop counterterms, that are evaluated in the appendix B.5,

$$I^{(d=10)}_{(3,0)}+I^{(d=10)}_{\triangleright\,(3,0)}+\delta_2 I^{(d=10)}_{(3,0)}=-3465\,\zeta(6)\,\log(\chi R_{11}^2/C_{(3,0)})+\frac{100647}{715}\,\zeta(3)\,\zeta(6)+210\,\zeta(8)\,R_{11}^3\,, \tag{3.21}$$

and

$$I^{(d=10)}_{(0,2)}+I^{(d=10)}_{\triangleright\,(0,2)}+\delta_2 I^{(d=10)}_{(0,2)}=-\frac{6615}{2}\,\zeta(6)\,\log(\chi R_{11}^2/C_{(3,0)})+\frac{15827}{110}\,\zeta(3)\,\zeta(6)+210\,\zeta(8)\,R_{11}^3\,, \tag{3.22}$$

corresponding to type-IIA genus-three, genus-four and genus-six contributions, respectively.

3.3 Connections with string perturbation theory in ten dimensions.

We will now summarize these results and translate them into perturbative terms in IIA string theory. We will be interested in comparing these terms with direct calculations in string perturbation theory at genus-one and genus-two. The fact that the perturbative terms in the type IIA and type IIB theories are equal up to at least genus-four will provide additional data for determining the $SL(2,\mathbb{Z})$-invariant coefficients of the ten-dimensional IIB theory.

3.3.1 Analytic terms

First recall the analytic terms in the derivative expansion of one-loop supergravity ($L=1$) compactified on $\mathcal{S}^1$ up to the order of interest in this paper are [5, 6]

$$\begin{aligned}A^{an}_{L=1}=&i\frac{\kappa_{11}^4}{(2\pi)^{11}\,l_{11}^3}\,4\pi^4\,\mathcal{R}^4\,\Big[\frac{2\zeta(3)}{R_{11}^3}+4\zeta(2)+\frac{4\zeta(4)}{3}R_{11}\,\frac{\sigma_2}{4^2}+\frac{4\zeta(6)}{27}\,R_{11}^3\,\frac{\sigma_3}{4^3}\\&+\frac{64\zeta(8)}{2835}\,R_{11}^5\,\frac{\sigma_2^2}{4^4}+\frac{16\zeta(10)}{1125}\,R_{11}^7\,\frac{\sigma_2\sigma_3}{4^5}+\frac{64\zeta(12)}{31185\cdot 691}\,R_{11}^9\,\frac{(675\sigma_2^3+872\sigma_3^2)}{4^6}\cdots\Big]\,.\end{aligned} \tag{3.23}$$

The analytic part obtained from two-loop ($L=2$) supergravity on $\mathcal{S}^1$, obtained earlier in this section, together with the known two-loop results of [7] and [8], are given by

$$\begin{aligned}A^{an}_{L=2}=&i\frac{\kappa_{11}^6}{(2\pi)^{22}\,l_{11}^{12}}\;4\pi^6\mathcal{R}^4\,\Big[\frac{\zeta(5)}{R_{11}^5}\,\frac{\sigma_2}{4^2}+\frac{4}{3}\left(\frac{\zeta(3)^2}{2R_{11}^6}+\frac{\zeta(2)\zeta(3)}{R_{11}^3}+\frac{6\zeta(2)^2}{5}\right)\frac{\sigma_3}{4^3}\\&+\frac{8}{2835}\left(\frac{675}{2}\zeta(2)\zeta(4)\,R_{11}+448\,\frac{\zeta(4)\zeta(3)}{R_{11}^2}\right)\frac{\sigma_3\sigma_2}{4^5}\\&+\frac{2}{14175}\left(70\,\zeta(8)\,R_{11}^3\,\frac{3\sigma_2^3+4\sigma_3^2}{4^6}+\zeta(6)\zeta(3)\,\left(\frac{100647}{715}\frac{\sigma_2^3}{4^6}+\frac{4}{3}\,\frac{15827}{110}\,\frac{\sigma_3^2}{4^6}\right)\right)+\cdots\Big]\,.\end{aligned} \tag{3.24}$$

After conversion to the string frame the $L=1$ and $L=2$ analytic contributions combine to give the following terms in type IIA string theory coordinates,

$$
\begin{aligned}
A^{an}_{L=1} + A^{an}_{L=2} = i\kappa_{10}^2\,\mathcal{R}^4 \Big[& \frac{2\zeta(3)}{g_A^2} + 4\zeta(2) + \left(\frac{\zeta(5)}{g_A^2} + \frac{4\zeta(4)}{3}\,g_A^2\right)\hat\sigma_2 \\
& + \frac{2}{3}\left(\frac{\zeta(3)^2}{g_A^2} + 2\zeta(2)\zeta(3) + \frac{12\zeta(2)^2}{5}\,g_A^2 + \frac{2\zeta(6)}{9}\,g_A^4\right)\hat\sigma_3 \\
& + \frac{64\zeta(8)}{2835}\,g_A^6\sigma_2^2 + \left(\frac{512}{405}\,\zeta(4)\zeta(3)\,g_A^2 + \frac{20}{21}\zeta(2)\zeta(4)\,g_A^4 + \frac{16\zeta(10)}{1125}\,g_A^8\right)\hat\sigma_2\hat\sigma_3 \\
& + \left(\zeta(3)\zeta(6)\,\frac{22366}{1126125}\,g_A^4 + \frac{4}{135}\zeta(8)g_A^6 + \frac{320\zeta(12)}{231\cdot 691}\,g_A^{10}\right)\hat\sigma_2^3 \\
& + \left(\zeta(3)\zeta(6)\,\frac{9044}{334125}\,g_A^4 + \frac{16}{405}\zeta(8)g_A^6 + \frac{64\cdot 872\zeta(12)}{31185\cdot 691}\,g_A^{10}\right)\hat\sigma_3^2 + \cdots \Big]\,.
\end{aligned}
\tag{3.25}
$$

These are some of the terms that could, in principle, be obtained from string perturbation theory. Other perturbative string terms should emerge from from higher-loop ($L>2$) supergravity. For example, tree-level terms beyond order $\mathcal{D}^6\mathcal{R}^4$ are not obtained from supergravity Feynman diagrams of loop number $L \le 2$, but are obtained from higher-loop ($L>2$) contributions. Thus, the tree-level $\mathcal{D}^4\mathcal{R}^4$ term can be deduced from a two-loop subdivergent contribution to the three-loop ($L=3$) Feynman diagrams of eleven-dimensional supergravity compactified on a circle as shown in figure 4.

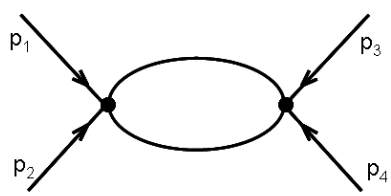

Figure 4: The double subdivergence of the three-loop diagrams that contributes at order $E_{7/2}\,S^4\,\mathcal{R}^4$ in ten-dimensional type IIB.

Although the momentum expansion of string theory at genus greater than one has not been explicitly considered, several of the low-lying terms in (3.25) are known to be in precise agreement with the expansion of tree-level and genus-one string theory amplitudes. So, for completeness, we now list the analytic terms that have been extracted from string theory at tree-level ($h=0$) and genus-one ($h=1$) [9, 10],

$$
\begin{aligned}
A^{an}_{h=0} + A^{an}_{h=1} = i\kappa_{10}^2\,\mathcal{R}^4 \Big[& \left(\frac{2\zeta(3)}{g_A^2} + 4\zeta(2)\right) + \frac{\zeta(5)}{g_A^2}\,\hat\sigma_2 + \frac{2}{3}\left(\frac{\zeta(3)^2}{g_A^2} + 2\zeta(2)\zeta(3)\right)\hat\sigma_3 \\
& + \frac{\zeta(7)}{2g_A^2}\,\hat\sigma_2^2 + \left(\frac{2}{3g_A^2}\zeta(3)\zeta(5) + \frac{97}{270}\,\zeta(2)\zeta(5)\right)\hat\sigma_2\hat\sigma_3 \\
& + \left(\frac{\zeta(9)}{4g_A^2} + \frac{2\zeta(2)\zeta(3)^3}{15}\right)\hat\sigma_2^3 + \left(\frac{2}{27g_A^2}(2\zeta(3)^2+\zeta(9)) + \frac{61}{270}\zeta(2)\zeta(3)^2\right)\hat\sigma_3^2 + \cdots \Big]\,.
\end{aligned}
\tag{3.26}
$$

We see that the terms that overlap with those of (3.25) have precisely the same coefficients. However, there are terms that occur in either (3.25) or (3.26) that do not occur in the other.

For example, the genus-zero term of order $S^5\,\mathcal{R}^4$ is not obtained from the $L=2$ supergravity diagrams described in this paper. However, it is expected to arise from a one-loop sub-divergence (proportional to Λ^3) of the three-loop diagrams ($L=3$). Similarly, its genus-one partner should arise from a double sub-divergence of the $L=3$ diagrams.

The above expressions have been obtained in the limit appropriate for comparison with perturbative ten-dimensional type IIA string theory. However, the IIA and IIB theories are known to have identical four-graviton amplitudes up to at least genus-four [11] so that, to the extent that the results match the string theory results, they should also apply to the ten-dimensional type IIB theory up to genus-four, at least.

3.3.2 Nonanalytic terms

The nonanalytic part of the one-loop ($L=1$) supergravity amplitude in ten dimensions is just the ten-dimensional maximal supergravity loop amplitude, which is well known [20] (reviewed in [10]). In this case the thresholds obtained by dimensional regularization give rise to terms of order $S\log(-S)$. The detailed structure of these terms is not relevant here, but it is notable that the scales of the Mandelstam invariants inside the logarithms cancel (the result is proportional to $(S+T+U)\log\chi$). This means that the result does not depend on details of the regularization scheme.

The nonanalytic terms obtained from two-loop ($L=2$) supergravity compactified on $\mathcal{S}^1$ in appendix B.3 are

$$\begin{aligned}A^{nonan}_{L=2} =& i\frac{\kappa_{11}^6}{(2\pi)^{22}\, l_{11}^{12}}\, \mathcal{R}^4\, 4\pi^6 \Big[-\frac{8}{15\, R_{11}^4}\,\zeta(2)\left(\frac{\zeta(3)}{R_{11}^4}+\frac{2\zeta(2)}{R_{11}}\right)\log(\chi R_{11}^2/C_{(2,0)})\frac{\sigma_2^2}{4^4}\\ &-\frac{832}{1296\, R_{11}^2}\,\zeta(4)\,\log(\chi R_{11}^2/C_{(1,1)})\frac{\sigma_2\sigma_3}{4^5}\\ &-\frac{1}{45}\zeta(6)\log(\chi R_{11}^2/C_{(3,0)})\left(11\frac{\sigma_2^3}{4^6}+14\,\frac{\sigma_3^2}{4^6}\right)+\cdots\Big].\end{aligned}\tag{3.27}$$

As before, the scales of the logarithms, $C_{(p,q)}$, are complicated functions of S_0 and T_0, and contain the information about the nonanalytic multiple logarithm terms (for example, the term of order $S^2\,\log(\chi)\,\mathcal{R}^4$ is given in detail in (2.54)). This expression involves factors with logarithms of the Mandelstam invariants, of the form $S^k\,\log(-S)\,\mathcal{R}^4$, which are correlated with discontinuities of the amplitude that are determined by unitarity. For example, at order $\sigma_2^2\,\log(-S)\,\mathcal{R}^4$ there are two terms with coefficients that differ by a power of R_{11}^3. The first of these arose from a finite contribution to the two-loop amplitude, while the second arose from the triangle diagram containing the one-loop counterterm that cancels a Λ^3 sub-divergence.

Transforming (3.27) to IIA string coordinates leads to

$$\begin{aligned}A^{nonan}_{L=2} =& i\kappa_{10}^2\,\mathcal{R}^4\Big[-\frac{\zeta(4)}{(4\pi)^3}\,\left(\zeta(3)+2\zeta(2)\,g_A^2\right)\log(\chi g_A^2/C_{(2,0)})\,\hat\sigma_2^2\\ &-\frac{13}{5184\pi}\,\zeta(4)\,g_A^2\,\log(\chi g_A^2/C_{(1,1)})\,\hat\sigma_2\hat\sigma_3\\ &-\frac{1}{11520\pi}\zeta(6)\,g_A^4\,\log(\chi g_A^2/C_{3,0})\left(11\,\hat\sigma_2^3+14\,\hat\sigma_3^2\right)+\cdots\Big],\end{aligned}\tag{3.28}$$

The first of these thresholds, of order $\hat{\sigma}_2^2 \log(\chi)\,\mathcal{R}^4$ has both a genus-one and genus-two contribution. The coefficient of the genus-one part matches the value obtained in (3.47) of [10] from genus-one string amplitude. Also present is the expected genus-two threshold of order $\hat{\sigma}_2\hat{\sigma}_3 \log(\chi)\,\mathcal{R}^4$, as well as the genus-three thresholds of order $\sigma_2^3 \log(\chi)\,\mathcal{R}^4$ and $\sigma_3^2 \log(\chi)\,\mathcal{R}^4$. The scales $C_{(p,q)}$ are undetermined by our procedure, whereas they are uniquely fixed in string perturbation theory. Such scales are also expected to be fixed by the $SL(2,\mathbb{Z})$ duality of the IIB theory.

Reinterpreting the nonanalytic terms as contributions to ten-dimensional type IIB and requiring $SL(2,\mathbb{Z})$ duality strongly suggests how certain perturbative terms combine into non-perturbative modular invariant coefficients. Thus, in the type IIB case the two terms in the coefficient of $\hat{\sigma}_2^2 \log(\chi)\,\mathcal{R}^4$ form the two perturbative terms in the expansion of the modular function $E_{3/2}$, as expected by general arguments based on unitarity of string perturbation theory [21, 10]. In similar manner, unitarity requires that the genus-three coefficients of $\hat{\sigma}_2^3 \log(\chi)\,\mathcal{R}^4$ and $\hat{\sigma}_3^2 \log(\chi)\,\mathcal{R}^4$ are pair up with genus-one threshold contributions in the ratio contained in the modular funtion $E_{5/2}$. Although these genus-one terms do not appear in the $L = 2$ supergravity calculation (dimensional analysis implies that they should arise from a one-loop sub-deivergence of three-loop $L = 3$ supergravity) their value is again known from the direct string theory calculations in [10]. Once again the coefficient in (3.28) is in accord with expectations.

4. Torus compactification to nine dimensions

Consider now the eleven-dimensional two-loop amplitude compactified on a two-torus of volume $\mathcal{V}_2$ and complex structure Ω, where the external momenta do not have any components in the compact toroidal dimensions. As before, we want to evaluate

$$I^{(d=9)}_{(p,q)} = \pi^{N+1} \int_{0_\chi}^{V^\Lambda} dV V^{3-N} \int_{\mathcal{F}_\Lambda} \frac{d^2\tau}{\tau_2^2} B_{(p,q)}(\tau) \sum_{(\hat{m}^I,\hat{n}^I)\in\mathbb{Z}^4} e^{-\pi\hat{E}}\,. \tag{4.1}$$

The metric on the torus that enters into the definition of $\hat{E}$ in (2.36) is

$$G_{IJ} = \frac{\mathcal{V}_2}{\Omega_2}\begin{pmatrix} 1 & \Omega_1 \\ \Omega_1 & |\Omega|^2 \end{pmatrix}, \tag{4.2}$$

which leads to

$$\hat{E} = \frac{\mathcal{V}_2\, V}{\Omega_2\tau_2}\left|(1\ \Omega) M \begin{pmatrix} \tau \\ 1 \end{pmatrix}\right|^2 - 2\mathcal{V}_2\, V \det M\,, \tag{4.3}$$

where

$$M = \begin{pmatrix} \hat{m}^1 & \hat{m}^2 \\ \hat{n}^1 & \hat{n}^2 \end{pmatrix}. \tag{4.4}$$

The dependence on $\mathcal{V}_2$ can be factored out by rescaling V (with a corresponding rescaling of V^Λ), which leads to

$$I^{(d=9)}_{(p,q)}(\Omega,\mathcal{V}_2) = \frac{1}{4}\pi^{3/2}\Gamma(N-3/2)\,\mathcal{V}_2^{N-4}\,\mathcal{E}^{(N+2)}_{(p,q)}(\Omega)\,, \tag{4.5}$$

with $N = 2p + 3q - 2$. The normalization has been chosen so that the functions $\mathcal{E}^{(N+2)}_{(p,q)}(\Omega)$ correspond to those defined in (1.27). The dictionary relating M-theory to type IIB string theory includes the identifications

$$r_B = \mathcal{V}_2^{-3/4}\,\Omega_2^{-1/4}\,, \qquad e^{\phi_B} = g_B = \Omega_2^{-1}\,. \tag{4.6}$$

Almost all the massless thresholds in nine dimensions for the terms up to the order we are considering here involve half-integral powers of the Mandelstam invariants rather than being logarithmic. In contrast to the ten-dimensional case in the previous section, these nonanalytic terms are easily distinguished from the analytic terms and we will ignore them in the following. The one exception is the logarithmic threshold that arises in the zero Kaluza–Klein ($m_I = n_I = 0$) term, which is the genus-two massless supergravity sector discussed in appendices C and E.

4.1 Evaluation of $\Delta_\Omega I^{(d=9)}_{(p,q)}$

Following the method used in [8] we apply the Laplace operator $\Delta_\Omega \equiv 4\Omega_2^2\,\partial_\Omega\partial_{\bar\Omega}$ to $I^{(d=9)}_{(p,q)}$ and use

$$\Delta_\Omega\, e^{-\pi\hat E} = \Delta_\tau\, e^{-\pi\hat E} \tag{4.7}$$

to give

$$\Delta_\Omega I^{(d=9)}_{(p,q)} = \pi^{N+1}\int_{0_\chi}^{V^\Lambda} dV V^{3-N}\int_{\mathcal{F}_\Lambda}\frac{d^2\tau}{\tau_2^2}\,B_{(p,q)}(\tau)\sum_{(\hat m,\hat n)\in\mathbb{Z}^4}\Delta_\tau\, e^{-\pi\hat E}\,, \tag{4.8}$$

where $N = 2p + 3q - 2$. Integrating by parts gives the Laplace operator Δ_τ acting on $B_{(p,q)}(\tau)$, while the boundary term vanishes in the sector with $(\hat m^1, \hat m^2) \neq (0,0)$ and $(\hat n^1, \hat n^2) \neq (0,0)$. For the moment we will restrict our considerations to this sector, which turns out to give terms independent of the cutoff Λ, which can therefore be set equal to ∞. We will also (in appendix C) need to consider the sector with $(\hat m^1, \hat m^2) = (0,0)$ and $(\hat n^1, \hat n^2) \neq (0,0)$.

After using (A.4) and (A.5) we see, as in the $\mathcal{S}^1$ compactification, that $I^{(d=9)}_{(p,q)}$ is itself a sum of components,

$$I^{(d=9)}_{(p,q)} = \sum_{i=0}^{\lceil 3N/2\rceil} h^i_{(p,q)}(\Omega, \mathcal{V}_2)\,, \tag{4.9}$$

and $N = 2p + 3q - 2$ as before. The function $h^i_{(p,q)}$ satisfies a Poisson equation

$$(\Delta_\Omega - \lambda^i_{(p,q)})\, h^i_{(p,q)} = \pi^{N+1}\int_{0_\chi}^{V^\Lambda} dV V^{3-N}\int_1^{\Lambda^2/V}\frac{d\tau_2}{\tau_2}\,c^i_{(p,q)}(\tau_2)\sum_{(\hat m,\hat n)\neq(0,0)} e^{-\pi\hat E}\Big|_{\tau_1=0}\,, \tag{4.10}$$

where $c^i_{(p,q)}$ is the coefficient of $\delta(\tau_1)$ in (1.26) and (A.5) and

$$\hat E\Big|_{\tau_1=0} = \mathcal{V}_2\, V\,\frac{|\hat m^1 + \hat m^2\Omega|^2 + \tau_2^2|\hat n^1 + \hat n^2\Omega|^2}{\Omega_2\tau_2} \tag{4.11}$$

In order to determine the right-hand side of (4.10) we will use the fact that $c^i_{(p,q)}(\tau_2)$ is a polynomial in $\tau_2 + \tau_2^{-1}$ of degree $N - 1 = 2p + 3q - 3$

$$c^i_{(p,q)}(\tau_2) = \sum_{r=0}^{N-1} c_r(\tau_2 + \tau_2^{-1})^r\ . \tag{4.12}$$

Substituting (4.12), using the symmetry of the integrand under $\tau_2 \to 1/\tau_2$ to extend the range of integration to $0 \le \tau_2 \le \infty$, and changing integration variables to $x = V/\tau_2$, $y = V\tau_2$ the right-hand side of (4.10) becomes

$$\frac{\pi^{N+1}}{4} \times \sum_{r=0}^{\lceil 3N/2 \rceil} c_r \int_0^\infty \frac{dx}{x} \int_0^\infty \frac{dy}{y} x^{2-\frac{N}{2}+\frac{n_r}{2}} y^{2-\frac{N}{2}-\frac{n_r}{2}} \sum_{(\hat m,\hat n)'} e^{-\pi \mathcal{V}_2 \left(y \frac{|\hat m^1+\hat m^2\Omega|^2}{\Omega_2} + x \frac{|\hat n^1+\hat n^2\Omega|^2}{\Omega_2}\right)}$$
$$= \frac{\pi^{N+1}}{4\mathcal{V}_2^{4-N}} \sum_{r=0}^{\lceil 3N/2 \rceil} c_r\, E^*_{2-\frac{N}{2}+\frac{n_r}{2}}(\Omega)\ E^*_{2-\frac{N}{2}-\frac{n_r}{2}}(\Omega)\,, \tag{4.13}$$

for $-N+1 \le n_r \le N-1$. The Eisenstein series E^*_s is defined in terms of E_s by

$$E^*_s(\tau) \equiv \frac{\Gamma(s)}{\pi^s} E_s(\tau) = 2\zeta^*(2s)\,\tau_2^s + 2\zeta^*(2-2s)\,\tau_2^{1-s} + O(\exp(-\tau_2)) \tag{4.14}$$

and satisfies the symmetry relation $E^*_s = E^*_{1-s}$, where $\zeta^*(2s) = \pi^{-s}\Gamma(s)\zeta(2s)$. The cutoffs on the integration limits have been removed in (4.13) since the result is finite (if the E_s functions for $s \le 1/2$ are defined by analytic continuation from $s > 1/2$).

So we finally obtain the Poisson equations for the components of the (p,q) term,

$$(\Delta_\Omega - \lambda^i_{(p,q)})\, h^i_{(p,q)} = \frac{\pi^{N+1}}{4\mathcal{V}_2^{4-N}} \sum_{r=0}^{N-1} c_r\, E^*_{2-\frac{N}{2}+\frac{n_r}{2}}(\Omega)\ E^*_{2-\frac{N}{2}-\frac{n_r}{2}}(\Omega)\,, \tag{4.15}$$

where $N = 2p+3q-2$ and $1 \le i \le \lceil 3N/2 \rceil$. The right-hand side of this equation is a sum with a finite number of terms that depends on the value of N. The solutions of this equation for given values of (p,q) and the corresponding values of the index i determine $I^{(d=9)}_{(p,q)}$ and hence, the coupling constant dependence of the coefficient of the term in A_{IIB} of order $S^{N+2}\,\mathcal{R}^4$. It is notable that the right-hand side of (4.15) is quadratic in the Eisenstein series, each of which will later (in section 5) be identified with a coefficient of a lower-order term in the action.

The dependence of $I^{(d=9)}_{(p,q)}$ on the volume, $\mathcal{V}_2^{N-4}$, in (4.13) translates into the IIB string theory description as

$$g_B^{N-1}\, r_B^{3-2N}\,, \tag{4.16}$$

using the correspondence between the supergravity and IIB string parameters given in (4.6), together with the identification $S = R_{11}\, s$.

In the above analysis we only considered terms with $(\hat m^1, \hat m^2) \neq (0,0)$ and $(\hat n^1, \hat n^2) \neq (0,0)$, which are independent of Λ. Certain terms with $\log \Lambda$ dependence also entered into the zero eigenvalue parts of the modular functions $\mathcal{E}^{(2)}_{(2,0)}$, $\mathcal{E}^{(6)}_{(3,0)}$ and $\mathcal{E}^{(6)}_{(0,2)}$ in appendix C. However, for economy of space we have not considered the terms that arise from $(\hat m^1, \hat m^2) = (0,0)$ with $(\hat n^1, \hat n^2) \neq (0,0)$, which correspond to subdivergences and have a power dependence on Λ that needs to be subtracted by counterterms.

In the $N = 0$ case ($(p,q) = (1,0)$), which corresponds to the $\mathcal{D}^4\mathcal{R}^4$ term, the source on the right-hand side of (4.15) vanishes and the equation reduces to the Laplace eigenvalue equation (1.12) for the value $s = 5/2$, as in [7]. In the $N = 1$ case ($(p,q) = (0,1)$), which corresponds to the $D^6\mathcal{R}^4$ term, the source on the right-hand side of (4.15) is quadratic in $E_{3/2}$ and there is a single eigenvalue $\lambda^1_{(0,1)} = 12$, reproducing (1.16), as obtained in [8]. We will now analyze these solutions for the cases $(2,0)$, $(1,1)$, $(3,0)$ and $(0,2)$, which raise a number of new issues.

4.1.1 $(p, q) = (2, 0)$

The expression for $B_{(2,0)}(\tau)$ given in (A.14) is written as the sum of the $b^i_{(2,0)}$'s in (A.18). Applying the method described in the previous subsection, using the explicit Poisson equation (A.19) satisfied by each $b^i_{(2,0)}$, we determine that the modular function $\mathcal{E}^{(2)}_{(2,0)}$ associated with the $\sigma_2^4\,\mathcal{R}^4$ term has the form

$$\frac{4\mathcal{V}_2^2}{\pi^2}\, I_{(2,0)}(\Omega,\mathcal{V}_2) \equiv \mathcal{E}^{(2)}_{(2,0)}(\Omega) = \sum_{i=0}^{3} \mathcal{E}^{(2)2i}_{(2,0)}(\Omega)\;, \tag{4.17}$$

where $\mathcal{E}^{(2)i}_{(2,0)}(\Omega)$ are modular functions satisfying the Poisson equations

$$(\Delta_\Omega - r(r+1))\mathcal{E}^{(2)r}_{(2,0)}(\Omega) = -2u_r E_{\frac{3}{2}} E_{\frac{1}{2}}\;, \qquad \text{for } r = 2i = 2,4,6 \tag{4.18}$$

and u_r are constants given in appendix A.1.

The function $E_{1/2}$ is defined as the limit

$$\lim_{s\to 1/2} E_s = 2\Omega_2^{\frac{1}{2}}\,\log\frac{\Omega_2}{4\pi c_e} + 4\Omega_2^{\frac{1}{2}}\sum_{n\neq 0} d_{|n|}\,K_0(2\pi|n|\Omega_2)\,e^{2i\pi\,n\,\Omega_1}\;, \qquad c_e = e^{-\gamma}\,, \tag{4.19}$$

where γ is Euler's constant and $d_{|n|}$ is the number of divisors of n. This is a very special Eisenstein series which has a large-Ω_2 expansion that has no purely power-behaved terms, but starts with $\Omega_2^{1/2}\,\log\Omega_2$. The interpretation of the $\log\Omega_2$ factor will be given in the section 4.3 where it will shown to be associated with the presence of massless thresholds.

The $i = 0$ term, $\mathcal{E}^{(2)0}_{(2,0)}$ associated with the constant $b^0_{(2,0)} = -13/21$, is shown in appendix C to be equal to

$$\mathcal{E}^{(2)0}_{(2,0)} = -\frac{104}{21}\zeta(2)\log\left(-S\,\mathcal{V}_2/\Omega_2\,C_{(2,0)}\right), \tag{4.20}$$

where $C_{(2,0)}$ is, as before, an undetermined function of S_0 and T_0, but is independent of $\mathcal{V}_2$ and Ω_2. The logarithm comes from the contribution of the zero Kaluza–Klein modes, $m_I = n_I = 0$ in the $\mathcal{T}^2$ reduction from eleven dimensions and should coincide with the supergravity calculation in nine dimensions discussed in section E. It is notable that the coefficient of $\log(\chi)$ in (4.20) does indeed coincide with the coefficient of the ϵ pole in dimensional regularization of two-loop maximal supergravity around nine dimensions. However, in our case the scale depends on the compactification moduli rather than an arbitrary cutoff.

When translated to IIB coordinates the $\mathcal{E}_{(2,0)}$ contribution has the form $r_B^{-1}\,\mathcal{E}^{(2)}_{(2,0)}(\Omega)\,S^4\,\mathcal{R}^4$.

4.1.2 $(p, q) = (1, 1)$

With some effort one can use (A.22) to write $B_{(1,1)} = \sum_{j=0}^4 b^j_{(1,1)}$ where the $b^j_{(1,1)}$'s satisfy the Poisson equations (A.28). It is straightforward to extend the general method described in subsection 4.1 to determine the coefficient, $\mathcal{E}^{(4)}_{(1,1)}$, of the $\sigma_2\sigma_3\,\mathcal{R}^4$ term in the amplitude. This is given by

$$\frac{8\mathcal{V}_2}{\pi^2}\, I_{(1,1)} \equiv \mathcal{E}^{(4)}_{(1,1)}(\Omega) = \sum_{j=0}^{4} \mathcal{E}^{(4)2j+1}_{(1,1)}(\Omega)\;, \tag{4.21}$$

where $\mathcal{E}^{(4)j}_{(1,1)}(\Omega)$ are modular functions satisfying

$$(\Delta_\Omega - r(r+1))\mathcal{E}^{(4)r}_{(1,1)}(\Omega) = -2v_r E_{\frac{3}{2}} E_{\frac{3}{2}} - 4\pi^2 w_r E_{\frac{1}{2}} E_{\frac{1}{2}}\,, \qquad r = 2j+1 = 1,3,5,7,9\,, \quad (4.22)$$

The coefficients $v_j,\ w_j$ are given in appendix A.2.

When translated to IIB coordinates this contribution has the form $r_B^{-3}\,\mathcal{E}^{(4)}_{(1,1)}(\Omega)\,S^5\,\mathcal{R}^4$.

4.1.3 The cases $(p,q) = (3,0)$ and $(p,q) = (0,2)$

In this case there are two modular functions, $\mathcal{E}^{(6)}_{(3,0)}$ and $\mathcal{E}^{(6)}_{(0,2)}$, multiplying the independent kinematical structures σ_2^3 and σ_3^2. Equations in (A.31) determine that $B^k_{(3,0)} = \sum_{k=0}^6 b^{2k}_{(3,0)}$ and $B^k_{(0,2)} = \sum_{k=0}^6 b^{2k}_{(0,2)}$ where $b^k_{(3,0)}$ and $b^k_{(0,2)}$ satisfy the Poisson equations (A.46). This leads to the expressions for the coefficients of the two kinematic structures at order $S^6\,\mathcal{R}^4$,

$$\begin{aligned}\frac{16}{3\pi^2} I_{(3,0)} \equiv \mathcal{E}^{(6)}_{(3,0)}(\Omega) &= \sum_{k=0}^{6} \mathcal{E}^{(6)2k}_{(3,0)}(\Omega)\,,\\ \frac{16}{3\pi^2} I_{(0,2)} \equiv \mathcal{E}^{(6)}_{(0,2)}(\Omega) &= \sum_{k=0}^{6} \mathcal{E}^{(6)2k}_{(0,2)}(\Omega)\,,\end{aligned} \quad (4.23)$$

where $\mathcal{E}^{(6)k}_{(3,0)}(\Omega)$ and $\mathcal{E}^{(6)k}_{(0,2)}(\Omega)$ are modular functions satisfying,

$$(\Delta_\Omega - r(r+1))\mathcal{E}^{(6)r}_{(p,q)}(\Omega) \;=\; -2f^r_{(p,q)} E_{\frac{3}{2}} E_{\frac{5}{2}} - 16\zeta(2)\,(f^r_{(p,q)} + g^r_{(p,q)})\,E_{\frac{1}{2}} E_{\frac{3}{2}}\,, \quad (4.24)$$

where $r = 2k = 2,4,6,8,10,12$ and $(p,q) = (3,0)$ or $(p,q) = (0,2)$ and the coefficients $f^r_{(p,q)}$ and $g^r_{(p,q)}$ are given in appendix A.3. The expressions for the functions $\mathcal{E}^{(6)\,0}_{3,0}$ and $\mathcal{E}^{(6)\,0}_{0,2}$ associated with the constant function $b^0_{(3,0)} = 12264/715$ and $b^0_{(0,2)} = 2716/165$ can be obtained by direct evaluation of the integrals as in (C.2) and (C.25).

When translated to IIB coordinates these $\mathcal{D}^{12}\mathcal{R}^4$ contributions have the form $r_B^{-5}\,(\mathcal{E}^{(6)}_{(3,0)}\,\hat\sigma_2^3 + \mathcal{E}^{(6)}_{(0,2)}\,\hat\sigma_3^2)\,\mathcal{R}^4$.

4.2 The nine-dimensional type IIB low energy string scattering amplitude

To summarize, we have determined a number of terms in the expansion of the the $\mathcal{T}^2$ compactification of two-loop ($L=2$) eleven-dimensional supergravity up to order $S^6\,\mathcal{R}^4$. Adding these to the terms found previously, gives the following expression in terms of the type IIB string theory parametrization:

$$\begin{aligned}A^{(d=9)}_{L=1} + A^{(d=9)}_{L=2} \;&=\; r_B(g_B^{-\frac{1}{2}}\,\mathcal{E}_{(0,0)}(\Omega)\,\mathcal{R}^4 + g_B^{\frac{1}{2}}\,\mathcal{E}_{(1,0)}(\Omega)\,\hat\sigma_2\mathcal{R}^4 + g_B\,\mathcal{E}_{(0,1)}(\Omega)\,\hat\sigma_3\mathcal{R}^4)\\ &+\;(4\pi)^2\frac{g_B^2}{r_B}\,\mathcal{E}^{(2)}_{(2,0)}(\Omega)\,\frac{\hat\sigma_2^2}{288}\mathcal{R}^4 + (4\pi)^2\,\frac{g_B^3}{r_B^3}\,\mathcal{E}^{(4)}_{(1,1)}(\Omega)\,\frac{\hat\sigma_2\hat\sigma_3}{3!\,15120}\mathcal{R}^4\\ &+\;\frac{(4\pi)^2}{4!\,302400}\,\frac{12g_B^4}{r_B^5}\left(\mathcal{E}^{(6)}_{(3,0)}(\Omega)\,\hat\sigma_2^3\mathcal{R}^4 + \frac{4}{3}\mathcal{E}^{(6)}_{(0,2)}(\Omega)\,\hat\sigma_3^2\mathcal{R}^4\right)\end{aligned} \quad (4.25)$$

Here we have included the coefficients given in (2.55) and the powers of r_B and g_B in (4.16). The terms in the first line are the ones found in previous work, namely, the function $\mathcal{E}_{(0,0)}$, which was derived from the $L=1$ amplitude [5], and the functions $\mathcal{E}_{(1,0)}$ and $\mathcal{E}_{(0,1)}$ that were derived from the $L=2$ amplitude in [7, 8]. We have not included terms that arise from renormalised subdivergent contributions (apart from the $\hat{\sigma}_2\mathcal{R}^4$ term), although these are easy to evaluate.

We emphasize again that the Feynman diagrams of supergravity are only expected to be an approximation to low energy string theory in a limited range of moduli space, although some some very special processes are presumably protected by supersymmetry. This requires, in particular, that $r_B \ll 1$ or $r_A \gg 1$ with $\alpha' s\, r_A^2 \ll 1$. The type IIA expression follows by use of the usual T-duality relations

$$r_A = r_B^{-1}\,, \qquad g_A = r_B^{-1}\, g_B\,. \tag{4.26}$$

The functions $\mathcal{E}^{(2)}_{(2,0)}$, $\mathcal{E}^{(4)}_{(1,1)}$, $\mathcal{E}^{(6)}_{(3,0)}$ and $\mathcal{E}^{(6)}_{(0,2)}$ are the unique $SL(2,\mathbb{Z})$-invariant solutions of the Poisson equations obtained earlier, subject to the condition that they are no worse than power-behaved in g_B as $g_B \to 0$. Our interest here is in obtaining the terms in these functions that are power-behaved in the string coupling $g_B = \Omega_2^{-1}$, which is the subject of the following sub-section.

4.2.1 The perturbative expansion of $A^{(d=9)}_{IIB}$

In analyzing the perturbative parts of the solutions to the preceding Poisson equations we may replace Δ_Ω by $\Omega_2^2\,\partial^2_{\Omega_2}$ since the perturbative terms are independent of Ω_1. The cases $(1,0)$ and $(0,1)$ were discussed in [7, 8] and reviewed in the introduction, so we will begin with the next term in the expansion.

$(p,q) = (2,0)$

We start with the coefficient, $\mathcal{E}^{(2)}_{(2,0)} = \sum_{i=0}^{3} \mathcal{E}^{(2)\,2i}_{(2,0)}$, of the $\sigma_2\mathcal{R}^4 \sim \mathcal{D}^4\mathcal{R}^4$ terms. In this case, alone among the nine-dimensional terms that we are considering, there is a logarithmic singularity of order $S^4\,\log(\chi)\,\mathcal{R}^4$, which arises from the sector with zero Kaluza–Klein modes, $m_I = n_I = 0$, which enters into the function $\mathcal{E}^{(2)\,0}_{(2,0)}$. In addition to the analytic part, proportional to $\mathcal{E}^{(2)}_{(2,0)}\,\sigma_2^2\,\mathcal{R}^4$, the amplitude therefore contains a nonanalytic part,

$$A^{nonan}_{(2,0)} = \mathcal{E}^{(2)\,0}_{(2,0)}\, S^4\,\log(\chi/C_{(2,0)})\,\mathcal{R}^4\,, \tag{4.27}$$

For $r = 2,4,6$, we see from (4.18) that the perturbative parts of $\mathcal{E}^{(2)r}_{(2,0)}$ satisfy

$$(\Omega_2^2\partial^2_{\Omega_2} - r(r+1))\mathcal{E}^{(2)r\,\mathrm{pert}}_{(2,0)} = -2u_r\big(4\zeta(3)\Omega_2^2 + 8\zeta(2)\big)\log\frac{\Omega_2}{4\pi c_e}\,, \tag{4.28}$$

Hence

$$\begin{aligned}\Omega_2^{-2}\mathcal{E}^{(2)r\,\mathrm{pert}}_{(2,0)} &= \alpha^{(r)}_{(2,0)}\Omega_2^{r-1} + \beta^{(r)}_{(2,0)}\Omega_2^{-r-2} + \frac{8u_r\zeta(3)}{(r(r+1)-2)^2}\left(3 + (r(r+1)-2)\log\frac{\Omega_2}{4\pi c_e}\right)\\ &- \frac{16\zeta(2)\,u_r}{(r(r+1))^2}\Omega_2^{-2}\left(1 - r(r+1)\,\log\frac{\Omega_2}{4\pi c_e}\right)\,.\end{aligned} \tag{4.29}$$

where $u_2 = 20/21$, $u_4 = 90/77$, $u_6 = 640/165$ and $\alpha^{(r)}_{(2,0)}$ and $\beta^{(r)}_{(2,0)}$ are integration constants and must be fixed by boundary conditions. Since the term proportional to $\alpha^{(r)}_{(2,0)}$ is an odd power of the string coupling, which does not appear in string perturbation theory, we deduce that $\alpha^{(r)}_{(2,0)}$ must be zero. As shown in [8] and in appendix D, this uniquely determines the value of $\beta^{(r)}_{(2,0)}$.

Summing all contributions and using the values of u_1, u_2 and u_3 given above leads to the complete contribution (including the $i=0$ term in (4.20))

$$\Omega_2^{-2}\mathcal{E}^{(2)\text{pert}}_{(2,0)} = \frac{16}{5}\zeta(3)\log\frac{\Omega_2}{4\pi c_e} + \frac{4}{9}\zeta(4)\Omega_2^{-4} + \frac{4}{945}\zeta(6)\Omega_2^{-6} + \frac{512}{496125}\zeta(8)\Omega_2^{-8}\,. \tag{4.30}$$

Notice that the sum of the $\zeta(2)\,\Omega_2^{-2}\log\Omega_2$ terms appearing in each $\mathcal{E}^{(2)i\,\text{pert}}_{(2,0)}$ in (4.29) have canceled with the $\log\Omega_2$ factor in $\mathcal{E}^{(2)0}_{(2,0)}$ (4.20) and the only remaining genus-two term is the non-analytic term (4.27), which is proportional to $g_B^2\, s^4\, \log(\chi/r_B^2 C_{(2,0)})$ in the IIB string parametrization).

Furthermore, we see that in the language of type IIB string theory, where $\Omega_2^{-1} = g_B$, the $S^4\,\mathcal{R}^4$ coefficient contains perturbative string contributions from genus-one to genus-five. The genus-one and genus-two terms (proportional to Ω_2^0 and Ω_2^{-2}, respectively), are simply obtained by equating the corresponding terms on the left-hand and right-hand sides of (4.28). The power-behaved terms at order Ω_2^{-4}, Ω_2^{-6} and Ω_2^{-8} have been evaluated using the method described in appendix D. We recognize part of the genus-one contribution to $\hat\sigma_2^2\,\mathcal{R}^4$ in nine dimensions derived in [10]. Indeed, the $\log\Omega_2$ term originates from the stringy corrections to the massless threshold and is associated with the $\log r$ term found in [10]. It is notable that the scale of the logarithm is absolutely determined in this expression.

$(p,q) = (1,1)$

Now we turn to the power-behaved terms in $\mathcal{E}^{(4)\text{pert}}_{(1,1)}$, the coefficient of the $\sigma_2\sigma_3\,\mathcal{R}^4 \sim \mathcal{D}^{10}\mathcal{R}^4$ contribution, which are determined by (4.22). In this case the source term (the right-hand side of (4.22)) contains the powers $\Omega_2^0,\ldots,\Omega_2^{-4}$ which lead to terms with the same powers in $\mathcal{E}^{(2)\text{pert}}_{(1,1)}$. In addition there are β terms that are again deduced from the expressions in appendix D. The result is

$$\begin{aligned}\Omega_2^{-3}\mathcal{E}^{(4)\text{pert}}_{(1,1)} &= 180\zeta(3)^2 + \frac{7168}{15}\,\zeta(2)\zeta(3)\Omega_2^{-2} - \frac{1456}{3}\,\zeta(2)\Omega_2^{-2}\log\frac{\Omega_2}{4\pi c_e} + \frac{3248}{3}\zeta(4)\Omega_2^{-4}\\ &+ \frac{98}{9}\zeta(6)\Omega_2^{-6} + \frac{896}{405}\zeta(8)\Omega_2^{-8} + \frac{304}{1875}\zeta(10)\Omega_2^{-10} + \frac{185600}{15802479}\zeta(12)\Omega_2^{-12}\,. \end{aligned}\tag{4.31}$$

Note that the scale of the $\log\Omega_2$ term is determined in this expression. Thus $\Omega_2^{-3}\mathcal{E}^{(4)\text{pert}}_{(1,1)}$ contains terms that are interpreted as perturbative string theory contributions from genus-one up to genus-seven. Note, in particular, the presence of the genus-two $\log\Omega_2$ term. This is directly related to the presence of a $S^5\,\log(-S)$ term at genus-two in ten-dimensional supergravity, as we saw in (3.28) and which is required by unitarity. This will be discussed in the analysis of ten-dimensional $L=2$ supergravity in appendix E.1. Importantly, the log square terms – present in each $\mathcal{E}^{(4)j}_{(1,1)}$ contribution – have canceled out in the sum. This corresponds to the cancelation of the leading $1/\epsilon^2$ pole also described in appendix E.1.

$(p,q) = (3,0)$ *and* $(p,q) = (0,2)$

Finally, we turn to the coefficients of the two order $S^6\,\mathcal{R}^4$ terms, $\mathcal{E}^{(6)\text{pert}}_{(3,0)}$ and $\mathcal{E}^{(6)\text{pert}}_{(0,2)}$. These are determined by (4.24). In this case the source term on the right-hand side of each of these equations contains the powers $\Omega_2^0 \ldots, \Omega_2^{-6}$, which determine the corresponding powers of Ω_2 in the solutions. In addition, there are β terms with powers $\Omega_2^{-8} \ldots, \Omega_2^{-16}$ that are determined by the expressions in appendix D. The resulting perturbative terms in the solutions are

$$\begin{aligned}\Omega_2^{-4}\mathcal{E}_{(3,0)}^{(6)\text{pert}} &= 96\zeta(3)\zeta(5) + \frac{8828}{77}\zeta(2)\zeta(5)\Omega_2^{-2} - \frac{28096}{77}\zeta(3)\zeta(2)\Omega_2^{-2} \\ &+ 1760\,\zeta(4)\,\Omega_2^{-4}\log\frac{\Omega_2}{C_{(3,0)}} + \frac{280}{3}\zeta(6)\Omega_2^{-6} + \frac{1792}{135}\zeta(8)\Omega_2^{-8} + \frac{32}{45}\zeta(10)\Omega_2^{-10} \\ &+ \frac{30720}{53207}\zeta(12)\Omega_2^{-12} - \frac{707584}{7432425}\zeta(14)\Omega_2^{-14} + \frac{973635584}{41937606711}\zeta(16)\Omega_2^{-16}\end{aligned} \tag{4.32}$$

and

$$\begin{aligned}\Omega_2^{-4}\mathcal{E}_{(0,2)}^{(6)\text{pert}} &= 96\zeta(3)\zeta(5) + \frac{81148}{693}\zeta(2)\zeta(5)\Omega_2^{-2} - \frac{233512}{693}\zeta(3)\zeta(2)\Omega_2^{-2} \\ &+ 1680\zeta(4)\,\Omega_2^{-4}\,\log\frac{\Omega_2}{C_{(0,2)}} + \frac{1120}{9}\zeta(6)\Omega_2^{-6} + \frac{3584}{405}\zeta(8)\Omega_2^{-8} + \frac{32}{15}\zeta(10)\Omega_2^{-10} \\ &+ \frac{2432}{159621}\zeta(12)\Omega_2^{-12} + \frac{356268544}{3277699425}\zeta(14)\Omega_2^{-14} + \frac{215105536}{9677909241}\zeta(16)\Omega_2^{-16}\,.\end{aligned} \tag{4.33}$$

Thus, these modular functions contain perturbative string contributions from genus-one up to genus-nine. Note, in particular, that the $\Omega_2^{-2}\log\Omega_2$ terms which are present for each individual eigenvalue have canceled in the sum. There remain two genus-two terms in (4.32) and (4.33), which have coefficients proportional to $\zeta(2)\zeta(3)$ and $\zeta(2)\zeta(5)$. Here we see another example of the lack of transcendentality. The example described in the introduction arose in comparing contributions of different genera whereas here it arises purely at genus-two. The only log terms in (4.32) and (4.33) are the ones associated with the power Ω_2^{-4}, which correspond to genus-three terms in string theory. As we will see, these have the numerical values expected on the basis of string unitarity. The undetermined constants $C_{(3,0)}$ and $C_{(0,2)}$ are once again associated with the scale of these log terms.

In addition to the terms in (4.32) and (4.33) there are Ω-independent terms arising from $\mathcal{E}_{(3,0)}^{(6)\,0}$ and $\mathcal{E}_{(0,2)}^{(6)\,0}$ of the form $\zeta(2)^2\,\log(\mathcal{V}_2\Lambda)$, as given in (C.26). This is the same $\log\Lambda$ divergence that we found in the case of the $\mathcal{S}^1$ compactification to ten dimensions in (B.80) and (B.81). This new Λ-dependent term should be canceled by a local counterterm. The values of the earlier local counterterms, such as the one that cancels the Λ^3 behaviour of the one-loop amplitude, were determined by enforcing T-duality and the equality of perturbative type IIA and IIB four-graviton scattering at low genus. Whether this argument can be extended to the case of the $\log\Lambda$ terms is not clear.

4.3 Connections with string perturbation theory in nine dimensions

We can now compare the perturbative terms in the modular functions with known features of string perturbation theory. Contributions to terms that contribute in the ten-dimensional limit $r_B \to \infty$ up to order $\mathcal{D}^6\mathcal{R}^4$ arise from one-loop and two-loop eleven-dimensional supergravity compactified on $\mathcal{T}^2$, as discussed in [5, 6, 7, 8]. No further ten-dimensional terms arise from the expansion of $L = 2$ supergravity to higher orders in momenta as considered in this paper. A dimensional argument shows that in order to generate higher-order ten-dimensional string theory terms one needs to consider higher-loop supergravity amplitudes with $L > 2$, together with corresponding counterterm diagrams that cancel divergences. One example that is easy to extract explicitly is a contribution $E_{7/2}\,\mathcal{D}^4\mathcal{R}^4$ that emerges from a diagram involving two $\mathcal{R}^4$ counterterms that cancels the contribution of a pair of one-loop sub-divergences in three-loop supergravity diagrams, as shown in figure 4. The emergence of this term follows from a simple dimensional argument that takes into account the fact that the double-divergence behaves as Λ^6 (i.e., Λ^3 for each loop). However, there is no reason to expect this to be the complete $(\alpha' s)^4 \log(\chi)\,\mathcal{R}^4$ contribution.

As we saw in the last subsection, the perturbative expansions of the modular functions considered in this paper all begin with genus-one terms followed by a finite series of higher-genus corrections. Some of these terms may be compared with the known string theory results, which mostly come from the low-energy expansion of the genus-one amplitude [10].

4.3.1 Comparison with genus-one string theory

The terms in (4.25), apart from $\hat\sigma_2\,\mathcal{R}^4$ and $\hat\sigma_3\,\mathcal{R}^4$, disappear in the ten-dimensional IIB limit, $r_B \to \infty$. However, as discussed earlier, if the supergravity approximation does make contact with string theory this would happen for large values of r_A, which is described by T-duality from the IIB expression in the small-r_B limit. In this limit there are terms with both negative and positive powers of r_A. Those proportional to r_A give rise to perturbative contributions of type IIA theory in ten dimensions. These comprise a genus-one contribution to $\hat\sigma_2^2\mathcal{R}^4 \sim \mathcal{D}^8\mathcal{R}^4$, a genus-two contribution to $\hat\sigma_2\hat\sigma_3\,\mathcal{R}^4 \sim \mathcal{D}^{10}\mathcal{R}^4$ and a genus-three contribution to the $\mathcal{D}^{12}\mathcal{R}^4$ terms $\hat\sigma_2^3\,\mathcal{R}^4$ and $\hat\sigma_3^2\,\mathcal{R}^4$, which will be discussed in the following subsection. There are also terms which behave as r_A^{1+k}, which diverge in the decompactification limit $r_A \to \infty$ and must be resummed in order to reconstruct the string thresholds in ten dimensions, as explained in [21]. In addition to terms that are power-behaved in r_A, in type IIA string perturbation theory there are exponentially suppressed terms of the form e^{-r_A}. Such highly-suppressed terms do not appear in the compactification of the perturbative supergravity amplitude, which is not sensitive to terms of the form $e^{-\mathcal{V}_n}$ that decrease exponentially with the compactification volume.

- Consider the type IIA interpretation of the analytic contributions obtained in the previous subsection. The genus-one terms of the modular functions in (4.25) have the following form,

$$A^{an}_{h=1} = 2\zeta(2)\left(\frac{1}{r_A} + \frac{1}{3r_A}\,\zeta(3)\hat\sigma_3 + \frac{r_A^3}{21}\zeta(3)^2\hat\sigma_2\hat\sigma_3 + \frac{2r_A^5}{525}\zeta(3)\zeta(5)\big(\hat\sigma_2^3 + \frac{4}{3}\hat\sigma_3^2\big)\right.$$
$$\left. - \frac{4}{15}\,r_A\zeta(3)\log(g_A/r_A)\,\hat\sigma_2^2\right). \tag{4.34}$$

  The analytic terms in this expression are exactly the same as those obtained from the genus-one string theory calculation in [10], so the $L = 2$ eleven-dimensional supergravity on $\mathcal{T}^2$ precisely reproduces these genus-one terms in string theory.

  The $\log r_A$ contribution in the last line of (4.34) comes from the function $\mathcal{E}^{(2)}_{(2,0)}$ multiplying the $\hat\sigma_2^2\,\mathcal{R}^4$ contribution, which contains $\log\Omega_2 = -\log(g_s^B)$ factors of the form

$$\frac{1}{r_B}\,\frac{16}{5}\,\zeta(3)\,\log(g_s^B) = r_A\,\frac{16}{5}\,\zeta(3)\,\log(g_s^A/r_A)\,. \tag{4.35}$$

The coefficient of the $\log r_A$ piece agrees with that calculated in genus-one string perturbation theory on a circle of finite radius $r_A = r_B^{-1}$ in [10]. This leaves a term proportional to $\log g_A$.

- The genus-two term at order $\hat\sigma_2\hat\sigma_3\,\mathcal{R}^4 \sim \mathcal{D}^{10}\mathcal{R}^4$ contained in $\mathcal{E}^{(4)}_{(1,1)}$ in equation (4.31) has a $\log \Omega_2$ factor, whereas the terms proportional to $\log^2 \Omega_2$ in each $\mathcal{E}^{(4)\,j}_{(1,1)}$ cancelled after summing up all contributions. This is consistent with the absence of the $1/\epsilon^2$ pole in the total two-loop supergravity calculation detailed in appendix E. After T-duality, this $h = 2$ term transforms into a term in type IIA that is proportional to r_A and therefore survives the $r_A \to \infty$ limit. It is therefore gratifying that its coefficient agrees with that evaluated by dimensional regularization in ten dimensions, as described earlier.

- The functions $\mathcal{E}^{(6)}_{(3,0)}$ and $\mathcal{E}^{(6)}_{(0,2)}$ of equations (4.32) and (4.33) exhibit a genus-three logarithmic term of order $S^6\,\mathcal{R}^4$ of the form

$$r_B^{-5}\zeta(2)^2 g_B^4 \log g_B = r_A \zeta(2)^2 g_A^4 \log \frac{g_A}{r_A}\,, \tag{4.36}$$

  which shows that these terms are again proportional to terms nonanalytic in r_A in the IIA theory that are proportional to r_A. Therefore these terms survive the ten-dimensional IIA limit, which was obtained in the $\mathcal{S}^1$ compactification in (3.27). Since the IIA and IIB amplitudes are equal up to at least genus four, it follows that these terms also arise in ten-dimensional IIB with the same coefficients. This, in turn, is consistent with two-particle unitarity [21], which relates the order $S^6\,\mathcal{R}^4$ threshold contributions at genus-one and genus-three in string perturbation theory. The precise coefficients of the genus-one ($h = 1$) massless thresholds at order $S^6\,\mathcal{R}^4$ have been evaluated in string theory [10]. The coefficients of the genus-three terms deduced above imply that the $h = 1$ and $h = 3$ terms combine into the nonanalytic term proportional to

$$g_B^{5/2}\, E_{5/2}\, (\frac{11}{210}\hat\sigma_2^3 + \frac{1}{15}\hat\sigma_3^2) \log(\chi)\, \mathcal{R}^4\,, \tag{4.37}$$

  which is precisely the anticipated non-perturbative threshold term [21].

5. Supersymmetry and higher-derivative couplings – a schematic discussion.

In this paper we have analyzed the momentum expansion of the two-loop four-graviton amplitude in eleven-dimensional supergravity up to order $S^6\,\mathcal{R}^4$. We considered the compactification on $\mathcal{S}^1$ to make contact with the ten-dimensional IIA theory, and on $\mathcal{T}^2$ to make contact with the nine-dimensional IIB theory. In the $\mathcal{S}^1$ case we obtained a number of higher-momentum terms that correspond to terms of particular genus in string perturbation theory. In the $\mathcal{T}^2$ case we obtained a number of higher-momentum terms with coefficients that are specific $SL(2,\mathbb{Z})$-invariant functions of the complex scalar coupling multiplying particular powers of r_B. We have found some impressive matches with perturbative string-theory results at different genera that are obtained from direct calculations in string perturbation theory [10] combined with unitarity constraints [21]. However, it is clear that there would be immense problems in going further in this manner. To begin with, the pattern of ultraviolet divergences of Feynman diagrams becomes much more complicated at higher values of L, which raises questions about how to implement the cutoff on the Schwinger parameters at higher loops. Furthermore, it is unclear whether this procedure of computing supergravity amplitudes with an ultraviolet cutoff and determining the finite part by using string dualities, can account for the details of intrinsically M-theory quantum effects, such as quantum effects of membranes, to all orders in the low energy expansion of string theory. Interestingly, according to the argument in [11] that uses the pure spinor formalism, the terms of order $S^6 R^4$ are the first terms for which one does not expect a non-renormalization theorem to hold just on the basis of supersymmetry. It is therefore of interest that the genus-one pieces and the threshold pieces of the genus-three terms in the $S^6\mathcal{R}^4$ coefficient functions match the string theory results.

More generally, it is of interest to consider to what extent the structure of the coefficients in the momentum expansion might be determined by symmetry constraints that might generalize to higher orders. In particular, it would be of interest to determine the extent to which maximal supersymmetry controls the form of the inhomogeneous Laplace equations satisfied by the coefficients.

5.1 Supersymmetry

The structure of the Poisson equations satisfied by the coefficient functions should be highly constrained by maximal supersymmetry, although this has not been explored in detail beyond the lowest order term in the momentum expansion. In the case of the $\mathcal{R}^4$ term the supersymmetry constraints are indeed known to determine that the coefficient function is the modular function $E_{3/2}$ [15]. At general order in the momentum expansion the requirement is that the full effective action be invariant under the modified supersymmetry transformation with spinor parameter ϵ acting on any field Φ is

$$\delta\,\Phi = \left(\delta^{(0)} + \alpha'^3\,\delta^{(3)} + \alpha'^5\,\delta^{(5)} + \dots\right)\,\Phi\,, \tag{5.1}$$

where $\delta^{(0)}$ is the classical supersymmetry transformation and $\delta^{(n)}\,\Phi$ denotes the modified transformation at $O(\alpha'^n)$. Invariance of the modified action, $(\alpha')^4\,S = S^{(0)}+\alpha'\,S^{(1)}+\ldots+(\alpha')^n\,S^{(n)}+\ldots$ (where $S^{(n)}$ is the action at order α'^n) requires

$$\left(\sum_{m=0}^{r}\alpha'^m\delta^{(m)}\right)\sum_{n=o}^{r}\alpha'^n S^{(n)} = 0, \tag{5.2}$$

Furthermore, the modified supersymmetry transformations must form a closed algebra when acting on Φ, modulo terms proportional to the modified Φ equation of motion and local symmetry transformations. This means that the commutator of two supersymmetry transformations with spinorial parameters ϵ_1 and ϵ_2 is given by

$$[\delta_1\,,\delta_2]\;\Phi = -2\mathrm{Im}(\bar\epsilon_2\gamma^\mu\epsilon_1)\,\partial_\mu\Phi + \Phi\text{ eqn. of motion} + \delta_{local}\,\Phi\,, \tag{5.3}$$

where γ_μ is a Dirac Gamma matrix for the ten-dimensional theory, the second term is proportional to an equation of motion and the third term represents local symmetry transformations.

In [15] these equations at order α'^3 were used to determine that the ten-dimensional type IIB $\mathcal{R}^4$ coefficient satisfies a Laplace eigenvalue equation of the form (1.12) that has as solution the modular function $\mathcal{E}_{(0,0)} = E_{3/2}$. A similar argument at $O(\alpha'^5)$ involving $\delta^{(5)}$ determines the modular function $\mathcal{E}_{(1,0)} = E_{5/2}/2$ [22]. Similarly, the form of the Poisson equation with a quadratic source term (1.16) that determines $\mathcal{E}_{(0,1)} = \mathcal{E}_{(3/2,3/2)}/6$ is at least in qualitative accord with supersymmetry at $O(\alpha'^6)$ [8]. However, in this case, not only do the classical supersymmetries mix the $\alpha'^6\, S^{(6)}$ with the $O(\alpha'^6)$ supersymmetry transformations, $\delta^{(6)}$, but there is also mixing with the $O(\alpha'^3)$ variations, $\delta^{(3)}$, of the terms in $S^{(3)}$,

$$\delta^{(6)}\, S^{(0)} + \delta^{(3)}\, S^{(3)} + \delta^{(0)}\, S^{(6)} = 0\,, \tag{5.4}$$

as well as in the closure of the algebra, where we require (ignoring detailed coefficients)

$$[\delta_1^{(0)},\, \delta_2^{(6)}] + [\delta_1^{(6)},\, \delta_2^{(0)}] + [\delta_1^{(3)},\, \delta_2^{(3)}] = 0 + \frac{\delta S^{(6)}}{\delta \Phi^*} + \delta_{local}\, \Phi\,. \tag{5.5}$$

We may refer to terms such as $\delta^{(3)}\, S^{(3)}$ and their generalizations at higher order as 'intermediate mixing terms'. These are terms of intermediate order in α' that mix with the $\delta^{(0)}$ (i.e., classical) variation of a higher-order term. The detailed analysis of these constraints is very cumbersome and has not been carried out. However, the structure of (5.4) and (5.5) is just what is needed for the coefficient function $\mathcal{E}_{(0,1)}$ to satisfy a Poisson equation with a source term that is proportional to $E_{3/2}E_{3/2}$ arising from the presence of the contributions from intermediate mixing, $\delta^{(3)}\, S^{(3)}$ and $[\delta_1^{(3)},\, \delta_2^{(3)}]$.

More generally, at order α'^p the modified supersymmetry conditions,

$$\sum_{k=0}^{p} \delta^{(p-k)}\, S^{(k)} = 0\,, \tag{5.6}$$

mix all terms at orders $k \le p$. The Poisson equations can, in general, have a number of distinct source terms that are quadratic in different lower order terms, as we have seen. There may also be degeneracies in which several terms of the same order mix under supersymmetry.

5.2 Systematics of the nine-dimensional amplitude

These arguments suggest how the pattern might continue to higher derivatives. The general structure should involve Poisson equations with quadratic source terms that are determined by commuting two supersymmetries. Each factor that appears in the source is itself a modular function associated with a lower-order interaction or modified supersymmetry transformation. The fact that the source terms in the Poisson equations found in section 4 should be consistent with supersymmetry should therefore provide information concerning classes of terms in the nine-dimensional amplitude.

We can illustrate this in a very schematic manner by listing the subset of terms required to reproduce the Poisson equations that we earlier obtained by analyzing $L = 2$ diagrams of eleven-dimensional supergravity. In the language of the effective action, and ignoring coefficients, the effective action contains the following terms,

$$S^{(9)} = S^{(9)}_{subset} + S^{(9)}_{rest}\,, \tag{5.7}$$

where $S^{(9)}_{subset}$ is a subset of terms of the form $D^{2k}\,\mathcal{R}^4$ that will mix with each other under the intermediate supersymmetries, such as $\delta^{(3)}S^{(3)}$ in (5.4) and its higher order generalizations. The following set of terms is needed

$$
\begin{aligned}
S^{(9)}_{subset} = \int d^9x\sqrt{-G^{(9)}}\, r_B \Big(& R + \alpha'^3 E_{\frac{3}{2}}\,\mathcal{R}^4 + \alpha'^4\, r_B^{-2}\, E_{\frac{1}{2}}\, D^2\mathcal{R}^4 \\
& + \alpha'^5\,(E_{\frac{5}{2}} + r_B^{-4}\, E_{\frac{3}{2}})\, D^4\mathcal{R}^4 + \alpha'^6 (\mathcal{E}^{(0)}_{(0,1)} + r_B^{-6}\, E_{\frac{5}{2}})\, D^6\mathcal{R}^4 \\
& + \alpha'^7\,(r_B^{-2}\,\mathcal{E}^{(2)}_{(2,0)} + r_B^{-8}\, E_{\frac{7}{2}})\, D^8\mathcal{R}^4 + \alpha'^8 (r_B^{-4}\,\mathcal{E}^{(4)}_{(1,1)} + r_B^{-10}\, E_{\frac{9}{2}})\, D^{10}\mathcal{R}^4 \\
& + \alpha'^9 (r_B^{-6}\mathcal{E}^{(6)}_{(3,0)} + r_B^{-12}\, E_{\frac{11}{2}})\, D^{12}\mathcal{R}^4 \Big)\,, \qquad (5.8)
\end{aligned}
$$

where $D^{12}\mathcal{R}^4$ stands for both kinematic structures $\sigma_2^3\mathcal{R}^4$ and $\sigma_3^2\mathcal{R}^4$ (and $G^{(9)}$ is the metric in the nine-dimensional space transverse to the torus). The coefficient functions are various modular functions, including some that have been discussed in this and earlier papers. We have included the interaction $r_B^{-1}\, E_{1/2}\, D^2\mathcal{R}^4 \sim r_B^{-1}\, E_{1/2}\,\sigma_1\,\mathcal{R}^4$, where $\sigma_1 = S+T+U$, even though it vanishes on shell when the dilaton is constant, because it is important for the structure of α'–corrected supersymmetry transformations. In considering the supersymmetry variations of the fields in the action we need to consider general infinitesimal transformations (that are not on-shell). This is the $k=1$ term in the series of terms, $r_B^{1-2k}\, E_{k-1/2}\, D^{2k}\mathcal{R}^4$, that arises from $L=1$ supergravity on $\mathcal{T}^2$ [6, 7].

The remaining terms, which are contained in $S^{(9)}_{rest}$, include a host of further contributions that mix with $S^{(9)}_{subset}$ under both the classical and higher-order supersymmetry transformations. Such terms, which are not of the form $D^{2k}\,\mathcal{R}^4$ but involve the other fields in the supergravity multiplet, generally carry nonzero $U(1)$ charge, u (where $U(1)$ is the R-symmetry of the IIB theory). The moduli-dependent coefficients of terms of this type are modular forms that transform with a phase under $SL(2,\mathbb{Z})$ that compensates for the non-zero phase associated with the charge u. An example of such a term is $\mathcal{E}^{-2}_{(0,0)}\, G^2\, R^3$, where G is the complex type IIB three-form that carries unit $U(1)$ charge [23] and the modular form $\mathcal{E}^{u}_{(0,0)}$ is given by acting with a $U(1)$-covariant derivative u times on the Eisenstein series $E_{3/2}$ [15][13] Such $U(1)$-violating interactions are not present in classical IIB supergravity and are believed to arise in string theory only in n-point functions with $n>4$.

The double expansion in powers of α' and powers of r_B^{-2} in (5.8) fits in with the general structure expected from supersymmetry. Demanding supersymmetry at a given order $\alpha'^{6+p} r_B^{-2p}$ gives conditions that can schematically be argued to associate modular functions with source terms as shown in the table. In the first line the source arises from the presence of $\alpha'^3\, E_{3/2}\,\mathcal{R}^4$ and $\alpha'^4\, r_B^{-2}\, E_{1/2}\, D^2\mathcal{R}^4$ in (5.8), together with their supersymmetric partners, which we have not determined. The powers of both α' and r_B are such that these terms can mix with the $\delta^{(0)}$ transformation of the $O(\alpha'^7\, r_B^{-2})$ terms. In the second line, the first source term comes from $\alpha'^3\, E_{3/2}\,\mathcal{R}^4$ with $\alpha'^5\, r_B^{-4}\, E_{3/2}\, D^4\mathcal{R}^4$, while the second source term comes from the $\alpha'^4\, r_B^{-4}\, E_{1/2}\, D^2\mathcal{R}^4$ (more precisely, from the term $\delta^{(4)}S^{(4)}$ in the supersymmetry transformation at order r_B^{-4}). In the third line the first source term comes from $\alpha'^3\, E_{3/2}\,\mathcal{R}^4$ and $\alpha'^6\, r_B^{-6}\, E_{5/2}\, D^6\mathcal{R}^4$ while the second

[13]The superscript u was suppressed for the coefficients $\mathcal{E}_{(p,q)}$ of the $U(1)$-conserving terms considered explicitly earlier in this paper, which all have $u=0$.

ORDER	COEFFICIENT	SOURCE
$\alpha'^7 r_B^{-2}$	$\mathcal{E}^{(2)}_{(2,0)}$	$E_{\frac{1}{2}} E_{\frac{3}{2}}$
$\alpha'^8 r_B^{-4}$	$\mathcal{E}^{(4)}_{(1,1)}$	$E_{\frac{3}{2}} E_{\frac{3}{2}} + E_{\frac{1}{2}} E_{\frac{1}{2}}$
$\alpha'^9 r_B^{-6}$	$\mathcal{E}^{(6)}_{(3,0)}$	$E_{\frac{3}{2}} E_{\frac{5}{2}} + E_{\frac{1}{2}} E_{\frac{3}{2}}$

Table 1: Summary of source terms associated with the inhomogeneous Laplace equations for various coefficient functions.

source term comes from $\alpha'^4\, r_B^{-2}\, E_{1/2}\, D^2\mathcal{R}^4$ and $\alpha'^5\, r_B^{-4}\, E_{3/2}\, D^4\mathcal{R}^4$. In this manner we can see how the structure of the source terms in the Poisson equations of section 4 arise.

These very sketchy arguments do not explain why the modular invariant coefficients in (5.8) are generally *sums* of modular functions satisfying Poisson equations, as we have seen in the examples derived from $L = 2$ supergravity in this paper. This could well arise from the possible degeneracies in terms that mix with each other under supersymmetry mentioned earlier, which obviously merits further study.

Finally, even the set of $D^{2k}\,\mathcal{R}^4$ terms shown explicitly in $S^{(9)}_{subset}$ in (5.8) is not complete. In the case of the lowest derivative terms, $\mathcal{R}^4$, $\hat\sigma_2\mathcal{R}^4$ and $\hat\sigma_3\mathcal{R}^4$ the complete coefficients can be deduced by imposing T-duality on the expressions obtained by compactifying $L = 1$ and $L = 2$-loop supergravity on a circle[14]. The terms of higher order in α' have not been completed, although T-duality, together with the tree-level and one-loop perturbative string theory 'data', do lead to some very suggestive constraints on the missing terms. However, we expect significant generalizations in the structure of the Poisson equations satisfied by the coefficients of the higher order terms, and a complete determination will almost certainly need an extension of the considerations of this paper.

5.3 Concluding remarks

We have determined terms in the derivative expansion of type II superstring theory that arise via duality from compactification of two-loop ($L = 2$) eleven-dimensional supergravity on a circle and on a two-torus up to order $S^6\,\mathcal{R}^4$. In the case of the two-torus compactification these coefficients are sums of modular functions of the scalar fields, satisfying an intriguing set of Poisson equations on moduli space with source terms that are bilinear in lower-order coefficients. This is the principle message of this paper. The structure of these equations has a form that is in line with the expectations based on implementing maximal supersymmetry. Although the terms that we have determined in this manner are incomplete, there are many intriguing correspondences with results directly obtained from string perturbation theory at tree-level and genus one in nine and ten dimensions. This structure should generalize to the larger moduli spaces that become relevant upon compactification to lower dimensions. Examples of this are the $SL(3,\mathbb{Z})\otimes SL(2,\mathbb{Z})$-invariant functions relevant to the compactification on $\mathcal{T}^3$ to eight dimensions that were mentioned in the previous footnote.

[14]Furthermore, the exact form of the coefficients of these terms is known in eight dimensions, where they are $SL(3,\mathbb{Z})\otimes SL(2,\mathbb{Z})$-invariant functions [24, 25, 26]. The exact nine-dimensional expression can therefore be deduced by decompatifying these expressions.

As emphasized in the introduction, supersymmetry guarantees that this structure should also apply to the low-energy expansion of the four-particle amplitudes in which the external states are any of the 256 states in the supermultiplet. These amplitudes conserve the $U(1)$ charge, u. However, as we have discussed, the full nonlinear supersymmetry relates such processes to amplitudes with total $u \neq 0$, and should therefore provide interesting constraints on these $U(1)$ non-conserving processes. However, the analysis of the complete set of conditions implied by supersymmetry is far from complete.

All this suggests that the exact expressions for the moduli-dependent coefficients at higher orders in the low-energy expansion are given by duality-invariant functions that are solutions of generalizations of the Poisson equations obtained from two-loop ($L = 2$) supergravity (4.15). For example, the $u = 0$ coefficients $\mathcal{E}_{(p,q)}$ of the ten-dimensional $\hat\sigma_2^p \hat\sigma_3^q \mathcal{R}^4$ terms with general p and q should be sums of functions, $\mathcal{E}_{(p,q)} = \sum_i \mathcal{E}^i_{(p,q)}$, where the $\mathcal{E}^i_{(p,q)}$ satisfy equations of the general form

$$(\Delta_\Omega - \lambda^i_{(p,q)})\, \mathcal{E}^i_{(p,q)} = \sum_{p'q'p''q'';jk;u} c^{p'q'p''q'';i}_{pq;jk;u}\, \mathcal{E}^{j,u}_{(p',q')}\, \mathcal{E}^{k,-u}_{(p'',q'')} \,. \tag{5.9}$$

Here we are allowing for the source to involve the moduli-dependent coefficients, $\mathcal{E}^{j,u}_{(p',q')}$, of terms carrying nonzero as well as zero values of the $U(1)$ charge, u. Such source terms are not seen in the specific $L = 2$ supergravity calculation in the body of this paper because the intermediate states in the toroidally compactified $L = 2$ diagrams necessarily carry total charge $u = 0$. It would be interesting to understand in detail how equations of the form (5.9) follow from the constraints of supersymmetry and duality invariance.

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A. Properties of the integrands $B_{(p,q)}$

In this appendix we will describe properties of the functions $B_{(p,q)}$ that enter in the integrands of the coefficients $I_{(p,q)}$ in (2.56). The coefficients $I_{(1,0)}$ and $I_{(0,1)}$ were computed in [7, 8], respectively. The higher order coefficients of interest here are $I_{(2,0)}$, $I_{(1,1)}$, $I_{(3,0)}$, $I_{(0,2)}$. Recall that the functions $B_{(p,q)}$ are proportional to the functions $\tilde B_{(p,q)}$ that enter into the expansion of the integrand in (2.11),

$$B_{(p,q)} = d_{(p,q)}\, \tilde B_{(p,q)} \,, \tag{A.1}$$

where the coefficients $d_{(p,q)}$ are arbitrarily chosen integers that avoid the occurrence of unwieldy coefficients in the main equations. The values of $d_{(p,q)}$ of relevance to the examples in this paper were given in (2.39).

After mapping the integrand from the domain in figure 3(a) to figure 3(b) the functions $B_{(p,q)}(\tau_1, \tau_2)$ are manifestly invariant under the transformation $\tau_1 \to -\tau_1$, which is equivalent to the symmetry $\tau \to 1-\tau^*$ in the original region. This means that the dependence on τ_1 enters via the combination

$$T_1 = -\tau_1^2 + |\tau_1|\,, \tag{A.2}$$

and there is a discontinuity in ∂_{τ_1} at $\tau_1 = 0$. The coefficient of the $\sigma_2\,\mathcal{R}^4 \sim \mathcal{D}^4\mathcal{R}^4$ term is simply $B_{(1,0)}(\tau) = 1$ [7]. In this notation the coefficient of the $\sigma_3\,\mathcal{R}^4 \sim \mathcal{D}^6\mathcal{R}^4$ term [8] in (2.23) and (2.40) is given by

$$B_{(0,1)}(\tau) = \tau_2 + \frac{1-6T_1}{\tau_2} + \frac{5T_1^2}{\tau_2^3}\,. \tag{A.3}$$

We will here show that the higher order functions $B_{(p,q)}(\tau)$ are given by sums of the form[15]

$$B_{(p,q)}(\tau) = \sum_{i=0}^{\lceil \frac{3}{2}N \rceil} b_{(p,q)}^{3N-2i}(\tau)\,, \tag{A.4}$$

where $N = 2p+3q-2$ and $b^i_{(p,q)}$ satisfies a Poisson equation with delta function source of general structure

$$\Delta b^i_{(p,q)}(\tau) = i(i+1)\, b^i_{(p,q)}(\tau) - \tau_2\, c^i_{(p,q)}(\tau_2)\ \delta(\tau_1)\,, \tag{A.5}$$

where $c^i_{(p,q)}(\tau_2)$ is a polynomial of order $N-1$ in $\tau_2 + \tau_2^{-1}$. The index i takes values $\lceil 3N/2 \rceil$. The range of the summation index in (A.4) is determined by the powers of $1/\tau_2$ in the expansion of $B_{(p,q)}$ which has the general form

$$B_{(p,q)}(\tau) = \sum_{i=0}^{2N} q_{2i}(|\tau_1|)\, \tau_2^{N-2i} \tag{A.6}$$

where $q_{2i}(|\tau_1|)$ are polynomials of degree i in T_1. The highest inverse power of τ_2 in this sum is given by a constant times $T_1^{2N}\,\tau_2^{-3N}$.

An important feature for later considerations is that $q_2(|\tau_1|) = q_2^{(0)}\,(1-6T_1)/6$ where $q_2^{(0)}$ is a constant. Since

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 (1-6T_1) = 0\,, \tag{A.7}$$

it follows that the zero mode with respect to τ_1 satisfies

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1\, (B_{(p,q)} - q_0 \tau_2^{2p+3q-2}) = O(\tau_2^{2p+3q-5})\,. \tag{A.8}$$

In the following subsections we will present the rather unwieldy complete expressions for the $B_{(p,q)}$ functions up to order $N = 2p+3q-2 = 4$ of interest in this paper, which result from computer evaluations. However, it is worth noting two general features of these functions that are straightforward to derive to all orders.

Firstly, for the special value $\tau_1 = 0$ (or $L_2 = 0$ in terms of the original Schwinger parameters), only the planar diagrams contribute to the amplitude and the integrals over the vertex positions t_r can be computed explicitly. The result is

$$\tilde{B}_{(p,q)}(L_2/L_1 = 0, , L_3/L_1) = \alpha_{p,q} \sum_{k=0}^{N} c(k)c(N-k)\ \tau_2^{N-2k}\,, \tag{A.9}$$

[15] We would like to thank Don Zagier for explaining us the mathematical significance of this decomposition

where

$$\alpha_{p,q} = \frac{N!\,(N+2)(p+q-1)!}{p!q!2^p3^q}\ , \qquad c(k) = \frac{\sqrt{\pi}}{2^{2k+1}(k+1)\Gamma\left(k+\frac{3}{2}\right)}\ . \tag{A.10}$$

The coefficient $\alpha_{p,q}$ arises from the conversion of σ_{N+2} to $\sigma_2^p\sigma_3^q$ using the identity [9]

$$\sigma_n = n \sum_{2p+3q=n} \frac{(p+q-1)!}{p!q!2^p3^q}\, \sigma_2^p\sigma_3^q\,, \tag{A.11}$$

while the coefficients $c(k)$ come from further combinatorics in the expansion of the integrand (2.11).

Secondly, for arbitrary values of the Schwinger parameters, L_k, the leading terms in the expansion of $\tilde{B}_{(p,q)}$ for large $\tau_2 = \Delta^{\frac{1}{2}}/(L_1+L_2)$ are

$$\tilde{B}_{(p,q)}(L_2/L_1, L_3/L_1) = \alpha_{p,q}\left(a_N\ \tau_2^N + b_N(1-6T_1)\tau_2^{N-2} + O(\tau_2^{N-4})\right)\ , \tag{A.12}$$

with

$$a_N = \frac{\sqrt{\pi}}{2^{2N+1}(N+1)\,\Gamma\left(N+\frac{3}{2}\right)}\ , \qquad b_N = \frac{\sqrt{\pi}\,(N+1)\Gamma(N-1)}{3\cdot 2^{2N+1}\ \Gamma\left(N+\frac{1}{2}\right)}\,. \tag{A.13}$$

A.1 Properties of $B_{(2,0)}$

The modular function $B_{(2,0)}$ that enters the $\sigma_2^2\,\mathcal{R}^4 \sim \mathcal{D}^4\mathcal{R}^4$ interaction has the form

$$B_{(2,0)} = \frac{4}{5}\,\tau_2^2 + (1-6\,T_1) + \frac{2}{5}\,\frac{2-15\,T_1+40\,T_1^2}{\tau_2^2} + \frac{2}{5}\,\frac{T_1^2\,(11-43\,T_1)}{\tau_2^4} + \frac{32}{5}\,\frac{T_1^4}{\tau_2^6}\ . \tag{A.14}$$

We will now describe the iterative process for writing $B_{(2,0)} = \sum_{i=0}^6 b_{(2,0)}^{2i}$, where each of the functions $b_{(2,0)}^{2i}$ satisfies a Poisson equation with delta function source of the form given in (A.5). The procedure will be the same in the cases with $N>2$. First consider the action of the laplacian $\Delta_\tau = \tau_2^2\,(\partial_{\tau_1}^2 + \partial_{\tau_2}^2)$ on a function of the form $q_n(|\tau_1|)/\tau_2^r$ with $q_n(|\tau_1|)$ polynomials of degree n in the decomposition of the $B_{(2,0)}$ in (A.6). The action of the laplacian gives two types of contributions

$$\Delta_\tau \frac{q_n(|\tau_1|)}{\tau_2^r} = r(r+1)\frac{q_n(|\tau_1|)}{\tau_2^r} + \frac{q_n''(|\tau_1|)}{\tau_2^{r-2}}\ . \tag{A.15}$$

The first contribution is proportional to the original function times an 'eigenvalue' determined by the power of τ_2. The second contribution is of the same type as the original function but with the power of τ_2 increased by 2 and the numerator is a polynomial $q_n''(|\tau_1|)$ of degree $n-2$. The linear term $|\tau_1|$ in q_n contributes to the $\delta(\tau_1)$ source in the Poisson equation using $\partial_{\tau_1}^2|\tau_1| = 2\delta(\tau_1)$. Splitting off this contribution by writing

$$q_n''(|\tau_1|) = \hat{q}_n(|\tau_1|) + q_n^{(1)}\,\delta(\tau_1) \tag{A.16}$$

one finds that (A.15) can be rewritten as

$$\begin{aligned}\Delta_\tau\left(\frac{q_n(|\tau_1|)}{\tau_2^r} + \frac{\hat{q}_n(|\tau_1|)}{(4r-1)\,\tau_2^{r-2}}\right) =& r(r+1)\left(\frac{q_n(|\tau_1|)}{\tau_2^r} + \frac{\hat{q}_n(|\tau_1|)}{(4r-1)\,\tau_2^{r-2}}\right)\\ &+ \frac{\hat{q}_n''}{(4r-1)\,\tau_2^{r-4}} + \frac{q_n^{(1)}}{\tau_2^{r-2}}\delta(\tau_1)\,.\end{aligned} \tag{A.17}$$

By iterating this procedure until the degree of the polynomial in $|\tau_1|$ is 1 or 0, one can construct an eigenfunction of the laplacian Δ_τ together with a delta function source term. This defines the function $b^{3N}_{(p,q)}$ that contains the most negative power, τ_2^{-3N}. After subtracting this function from $B_{(2,0)}$, the most negative remaining power is τ_2^{-3N+2} and the above procedure may be repeated to determine the function $b^{3N-2}_{(p,q)}$, and so on until the complete set of functions has been determined.

Applying this procedure to $B_{(2,0)}$ leads to a sum of the following $b^i_{(2,0)}$ functions,

$$\begin{aligned}
b^0_{(2,0)}(\tau) &= -\frac{13}{21} \\
b^2_{(2,0)}(\tau) &= \frac{10}{21}\left(\tau_2^2 + 1 - 2T_1 + \frac{(1-T_1)^2}{\tau_2^2}\right) \\
b^4_{(2,0)}(\tau) &= \frac{10}{77}\Big(\tau_2^2 + \frac{3}{5}\,(4-15T_1) + \frac{1-9T_1+15T_1^2}{\tau_2^2} + 7T_1^2\,\frac{1-T_1}{\tau_2{}^4}\Big) \\
b^6_{(2,0)}(\tau) &= \frac{32}{165}\Big(\tau_2{}^2 + \frac{10}{7}(3-14T_1) + \frac{1-20T_1+70T_1^2}{\tau_2{}^2} \\
&\quad + 6T_1^2\,\frac{3-14T_1}{\tau_2{}^4} + \frac{33\,T_1^4}{\tau_2{}^6}\Big)\,.
\end{aligned} \tag{A.18}$$

These functions satisfy the inhomogeneous Laplace equations for $r = 0, 2, 4, 6$

$$\Delta b^r_{(2,0)}(\tau) = r(r+1)\, b^r_{(2,0)}(\tau) - 2u_r\, \tau_2\, (\tau_2 + \tau_2{}^{-1})\, \delta(\tau_1)\,, \tag{A.19}$$

where $u_0 = 0$ and, for $r = 2, 4, 6$,

$$u_r = \frac{1}{4} q_r\, (r(r+1) - 2) = (\frac{10}{21}, \frac{45}{77}, \frac{64}{33})\ , \qquad q_r = (\frac{10}{21}, \frac{10}{77}, \frac{32}{165}) \tag{A.20}$$

The value u_r can be computed by

$$\partial_{\tau_1} b^r_{(2,0)}\Big|_{\tau_1=0} = -2u_n(1+\tau_2{}^{-2}) \tag{A.21}$$

A.2 Properties of $B_{(1,1)}$

The modular function associated the coefficient of $\sigma_2\sigma_3\,\mathcal{R}^4 \sim \mathcal{D}^{10}\mathcal{R}^4$ is given by

$$\begin{aligned}
B_{(1,1)} =& \frac{45\,\tau_2^3}{2} + 35\,(1-6\,T_1)\,\tau_2 + \frac{7\,\left(10 - 75\,T_1 + 191\,T_1^2\right)}{2\,\tau_2} + \frac{45 - 420\,T_1 + 1372\,T_1^2 - 2086\,T_1^3}{2\,\tau_2^3} \\
&+ \frac{T_1^2\,\left(285 - 1264\,T_1 + 1761\,T_1^2\right)}{2\,\tau_2^5} + \frac{T_1^4\,(347 - 782\,T_1)}{2\,\tau_2^7} + \frac{145\,T_1^6}{2\,\tau_2^9}
\end{aligned} \tag{A.22}$$

Following the previous iterative procedure this function can straightforwardly be shown to be a sum of five modular functions $B_{(1,1)} = \sum_{j=0}^4 b^{2j+1}_{(1,1)}$ that again satisfy Poisson equations with delta-function source terms,

$$b^1_{(1,1)} = -\frac{245}{66}\left(\tau_2 + \frac{1-T_1}{\tau_2}\right) \tag{A.23}$$

$$b^3_{(1,1)} = -\frac{7}{429}\Big(- 679\,\tau_2^3 + 3\,(176 + 679\,T_1)\,\tau_2 + \frac{-679 + 2037\,T_1 + 4677\,T_1^2 + 679\,T_1^3}{\tau_2^3} - \frac{3\,\left(-176 + 1735\,T_1 + 679\,T_1^2\right)}{\tau_2}\Big) \tag{A.24}$$

$$b^5_{(1,1)} = \frac{49}{39}\Big(7\,\tau_2^3 - 12\,(-2+7\,T_1)\,\tau_2 + \frac{6\,\left(4 - 25\,T_1 + 35\,T_1^2\right)}{\tau_2} - \frac{7\,\left(-1 + 12\,T_1 - 36\,T_1^2 + 28\,T_1^3\right)}{\tau_2^3} + \frac{63\,(-1+T_1)^2\,T_1^2}{\tau_2^5}\Big) \tag{A.25}$$

$$b^7_{(1,1)} = -\frac{1862}{7293}\Big(- 9\,\tau_2^3 + 5\,(-11 + 45\,T_1)\,\tau_2 - \frac{5\,\left(11 - 98\,T_1 + 210\,T_1^2\right)}{\tau_2} + \frac{9\,\left(-1 + 25\,T_1 - 140\,T_1^2 + 210\,T_1^3\right)}{\tau_2^3} - \frac{33\,T_1^2\,\left(6 - 38\,T_1 + 45\,T_1^2\right)}{\tau_2^5} + \frac{429\,(-1+T_1)\,T_1^4}{\tau_2^7}\Big) \tag{A.26}$$

$$b^9_{(1,1)} = \frac{1}{4862}\Big(11172\,\tau_2^3 - \frac{18620\,(-11+45\,T_1)\,\tau_2}{3} + \frac{18620\,\left(11 - 98\,T_1 + 210\,T_1^2\right)}{3\,\tau_2} - \frac{11172\,\left(-1 + 25\,T_1 - 140\,T_1^2 + 210\,T_1^3\right)}{\tau_2^3} + \frac{40964\,T_1^2\,\left(6 - 38\,T_1 + 45\,T_1^2\right)}{\tau_2^5} - \frac{532532\,(-1+T_1)\,T_1^4}{\tau_2^7} + \frac{352495\,T_1^6}{\tau_2^9}\Big)\,. \tag{A.27}$$

The inhomogeneous Laplace equations satisfied by these functions are given by, for $r = 1, 3, 5, 7, 9$

$$\Delta_\tau b^r_{(1,1)} = r(r+1)\,b^r_{(1,1)} - 2\tau_2\big(v^r(\tau_2^2 + \tau_2^{-2}) + w^r\big)\;\delta(\tau_1)\,, \tag{A.28}$$

where the constants v^r and w^r are given by

$$\begin{aligned} &v^1 = 0\,, \quad v^3 = \frac{14\cdot 679}{143}\,, \quad v^5 = \frac{196\cdot 14}{13}\,, \quad v^7 = \frac{18620\cdot 45}{7293}\,, \quad v^9 = \frac{6090\cdot 11}{2431}\,,\\ &w^1 = -\frac{245}{33}\,, \quad w^3 = -\frac{14\cdot 1735}{143}\,, \quad w^5 = \frac{196\cdot 25}{13}\,,\\ &w^7 = \frac{18620\cdot 98}{7293}\,, \quad w^9 = \frac{6090\cdot 30}{2431}\,. \end{aligned} \tag{A.29}$$

A.3 Properties of $B_{(3,0)}$ and $B_{(0,2)}$

The integrands that define the coefficients $I_{(3,0)}$, $I_{(0,2)}$ of the $S^6\,\mathcal{R}^4$ contributions $\sigma_2^3\,\mathcal{R}^4$ and $\sigma_3^2\,\mathcal{R}^4$, respectively, are

$$\begin{aligned} B_{(3,0)} =& 24\,\tau_2^4 + 45\,(1-6\,T_1)\,\tau_2^2 + 4\,\left(14 - 105\,T_1 + 270\,T_1^2\right) + \frac{3}{\tau_2^2}\,\left(15 - 140\,T_1 + 462\,T_1^2 - 756\,T_1^3\right)\\ &+ \frac{6}{\tau_2^4}\,\left(4 - 45\,T_1 + 190\,T_1^2 - 390\,T_1^3 + 558\,T_1^4\right) + \frac{3\,T_1^2}{\tau_2^6}\,\left(58 - 334\,T_1 + 715\,T_1^2 - 1402\,T_1^3\right)\\ &+ \frac{24\,T_1^4}{\tau_2^8}\,\left(10 - 41\,T_1 + 167\,T_1^2\right) + \frac{24\,T_1^6}{\tau_2^{10}}\,(7 - 93\,T_1) + \frac{516\,T_1^8}{\tau_2^{12}} \end{aligned} \tag{A.30}$$

and

$$B_{(0,2)} = 24\,\tau_2^4 + 45\,\left(1-6\,T_1\right)\,\tau_2^2 + 14\,\left(4-30\,T_1+75\,T_1^2\right) + \frac{45-420\,T_1+1358\,T_1^2-1904\,T_1^3}{\tau_2^2}$$
$$+\frac{6}{\tau_2^4}\,\left(4-45\,T_1+185\,T_1^2-330\,T_1^3+197\,T_1^4\right) + \frac{T_1^2\,\left(174-942\,T_1+1265\,T_1^2+1418\,T_1^3\right)}{\tau_2^6}$$
$$+\frac{2T_1^4}{\tau_2^8}\,\left(105-96\,T_1-1529\,T_1^2\right) - \frac{76\,T_1^6}{\tau_2^{10}}\,\left(1-27\,T_1\right) - \frac{494\,T_1^8}{\tau_2^{12}} \tag{A.31}$$

The functions $B_{(3,0)}$ and $B_{(0,2)}$ are each given by a sum of seven functions $b^{2k}_{(3,0)}$ and $b^{2k}_{(0,2)}$ with $k=0,\ldots,6$, which satisfy Poisson equations, The detailed form of these functions is straightforward to determine using the iterative process described earlier, giving

$$b^0_{(3,0)} = \frac{12264}{715}\,, \tag{A.32}$$

$$b^2_{(3,0)} = -\frac{2408}{143}\left(\tau_2^2+1-2\,T_1+\frac{\left(1-T_1\right)^2}{\tau_2^2}\right), \tag{A.33}$$

$$b^4_{(3,0)} = \frac{42}{12155}\Big(3915\,\tau_2^4-20\,\left(-181+783\,T_1\right)\,\tau_2^2+3\,\left(547-3030\,T_1+7830\,T_1^2\right)$$
$$-\frac{10\,\left(-362+909\,T_1-732\,T_1^2+1566\,T_1^3\right)}{\tau_2^2}$$
$$+\frac{5\,\left(-1+T_1\right)\,\left(-783+2349\,T_1+413\,T_1^2+783\,T_1^3\right)}{\tau_2^4}\Big), \tag{A.34}$$

$$b^6_{(3,0)} = -\frac{1}{3553}\Big(-20322\,\tau_2^4+5\,\left(-10889+60966\,T_1\right)\,\tau_2^2-10\,\left(-1827-7280\,T_1+101610\,T_1^2\right)$$
$$+\frac{5\,\left(-10889+14560\,T_1+91294\,T_1^2+284508\,T_1^3\right)}{\tau_2^2}$$
$$-\frac{6\,\left(3387-50805\,T_1+112530\,T_1^2+152260\,T_1^3+152415\,T_1^4\right)}{\tau_2^4}$$
$$+\frac{33\,T_1^2\,\left(-6774+20322\,T_1+13295\,T_1^2+6774\,T_1^3\right)}{\tau_2^6}\Big), \tag{A.35}$$

$$b^8_{(3,0)} = \frac{2}{2717}\Big(7920\,\tau_2^4-3600\,\left(-17+66\,T_1\right)\,\tau_2^2+25200\,\left(4-30\,T_1+55\,T_1^2\right)$$
$$-\frac{3600\,\left(-17+210\,T_1-798\,T_1^2+924\,T_1^3\right)}{\tau_2^2}+\frac{7920\,\left(1-30\,T_1+225\,T_1^2-600\,T_1^3+495\,T_1^4\right)}{\tau_2^4}$$
$$-\frac{102960\,T_1^2\,\left(-1+2\,T_1\right)\,\left(2-12\,T_1+11\,T_1^2\right)}{\tau_2^6}+\frac{514800\,\left(-1+T_1\right)^2\,T_1^4}{\tau_2^8}\Big), \tag{A.36}$$

$$b^{10}_{(3,0)} = \frac{2}{1062347}\Big(-756756\,\tau_2^4 + 407484\,\left(-22+91\,T_1\right)\,\tau_2^2 - 333396\,\left(50-451\,T_1+1001\,T_1^2\right)$$
$$+\frac{407484\,\left(-22+369\,T_1-1881\,T_1^2+3003\,T_1^3\right)}{\tau_2^2}$$
$$-\frac{756756\,\left(1-49\,T_1+567\,T_1^2-2310\,T_1^3+3003\,T_1^4\right)}{\tau_2^4}$$
$$+\frac{2270268\,T_1^2\,\left(-15+215\,T_1-880\,T_1^2+1001\,T_1^3\right)}{\tau_2^6}$$
$$-\frac{12864852\,T_1^4\,\left(15-87\,T_1+91\,T_1^2\right)}{\tau_2^8}+\frac{244432188\,\left(-1+T_1\right)\,T_1^6}{\tau_2^{10}}\Big)\,, \qquad (A.37)$$

$$b^{12}_{(3,0)} = \frac{2}{96577}\Big(16770\,\tau_2^4 + -40248\,\left(-7+30\,T_1\right)\,\tau_2^2 + 119196\,\left(5-52\,T_1+130\,T_1^2\right)$$
$$-\frac{281736\,\left(-1+22\,T_1-143\,T_1^2+286\,T_1^3\right)}{\tau_2^2}+\frac{16770\,\left(1-72\,T_1+1188\,T_1^2-6864\,T_1^3+12870\,T_1^4\right)}{\tau_2^4}$$
$$-\frac{1140360\,T_1^2\,\left(-1+22\,T_1-143\,T_1^2+286\,T_1^3\right)}{\tau_2^6}+\frac{2166684\,T_1^4\,\left(5-52\,T_1+130\,T_1^2\right)}{\tau_2^8}$$
$$-\frac{4333368\,T_1^6\,\left(-7+30\,T_1\right)}{\tau_2^{10}}+\frac{24916866\,T_1^8}{\tau_2^{12}}\Big)\,, \qquad (A.38)$$

and

$$b^0_{(0,2)} = \frac{12264}{715}\,, \qquad (A.39)$$

$$b^2_{(0,2)} = -\frac{2128}{143}\,\left(\tau_2^2 + 1 - 2\,T_1 + \frac{\left(1-T_1\right)^2}{\tau_2^2}\right)\,, \qquad (A.40)$$

$$b^4_{(0,2)} = \frac{42}{12155}\Big(3385\,\tau_2^4 - 20\,\left(-69+677\,T_1\right)\,\tau_2^2 + 3\,\left(-927+2630\,T_1+6770\,T_1^2\right)$$
$$-\frac{10\,\left(-138-789\,T_1+1992\,T_1^2+1354\,T_1^3\right)}{\tau_2^2}$$
$$+\frac{5\,\left(-1+T_1\right)\,\left(-677+2031\,T_1+2807\,T_1^2+677\,T_1^3\right)}{\tau_2^4}\Big)\,, \qquad (A.41)$$

$$b^6_{(0,2)} = -\frac{1}{561}\Big(-6090\,\tau_2^4 + 525\,\left(-43+174\,T_1\right)\,\tau_2^2 - 350\,\left(61-420\,T_1+870\,T_1^2\right)$$
$$+\frac{525\,\left(-43+280\,T_1-574\,T_1^2+812\,T_1^3\right)}{\tau_2^2}-\frac{210\,\left(29-435\,T_1+1500\,T_1^2-1200\,T_1^3+1305\,T_1^4\right)}{\tau_2^4}$$
$$+\frac{1155\,T_1^2\,\left(-58+174\,T_1-65\,T_1^2+58\,T_1^3\right)}{\tau_2^6}\Big)\,, \qquad (A.42)$$

$$b^{8}_{(0,2)} = \frac{1}{143}\Big(22\,\tau_2^4 - 10\,(-17+66\,T_1)\,\tau_2^2 + 70\,\left(4-30\,T_1+55\,T_1^2\right) - \frac{10\,\left(-17+210\,T_1-798\,T_1^2+924\,T_1^3\right)}{\tau_2^2} + \frac{22\,\left(1-30\,T_1+225\,T_1^2-600\,T_1^3+495\,T_1^4\right)}{\tau_2^4} - \frac{286\,T_1^2\,(-1+2\,T_1)\,\left(2-12\,T_1+11\,T_1^2\right)}{\tau_2^6} + \frac{1430\,(-1+T_1)^2\,T_1^4}{\tau_2^8}\Big)\,, \tag{A.43}$$

$$b^{10}_{(0,2)} = -\frac{1}{55913}\Big(-90948\,\tau_2^4 + 48972\,(-22+91\,T_1)\,\tau_2^2 - 40068\,\left(50-451\,T_1+1001\,T_1^2\right) + \frac{48972\,\left(-22+369\,T_1-1881\,T_1^2+3003\,T_1^3\right)}{\tau_2^2} - \frac{90948\,\left(1-49\,T_1+567\,T_1^2-2310\,T_1^3+3003\,T_1^4\right)}{\tau_2^4} + \frac{272844\,T_1^2\,\left(-15+215\,T_1-880\,T_1^2+1001\,T_1^3\right)}{\tau_2^6} - \frac{1546116\,T_1^4\,\left(15-87\,T_1+91\,T_1^2\right)}{\tau_2^8} + \frac{29376204\,(-1+T_1)\,T_1^6}{\tau_2^{10}}\Big)\,, \tag{A.44}$$

$$b^{12}_{(0,2)} = -\frac{1}{391}\Big(130\,\tau_2^4 - 312\,(-7+30\,T_1)\,\tau_2^2 + 924\,\left(5-52\,T_1+130\,T_1^2\right) - \frac{2184\,\left(-1+22\,T_1-143\,T_1^2+286\,T_1^3\right)}{\tau_2^2} + \frac{130\,\left(1-72\,T_1+1188\,T_1^2-6864\,T_1^3+12870\,T_1^4\right)}{\tau_2^4} - -\frac{8840\,T_1^2\,\left(-1+22\,T_1-143\,T_1^2+286\,T_1^3\right)}{\tau_2^6} + \frac{16796\,T_1^4\,\left(5-52\,T_1+130\,T_1^2\right)}{\tau_2^8} - \frac{33592\,T_1^6\,(-7+30\,T_1)}{\tau_2^{10}} + \frac{193154\,T_1^8}{\tau_2^{12}}\Big)\,. \tag{A.45}$$

The inhomogeneous Laplace equations satisfied by these functions have the form

$$\Delta_\tau b^r_{(p,q)} = r(r+1)\,b^r_{(p,q)} - 2\tau_2\left(\tau_2+\tau_2^{-1}\right)\left[f^r_{(p,q)}(\tau_2^2+\tau_2^{-2}) + g^r_{(p,q)}\right]\,\delta(\tau_1)\,, \tag{A.46}$$

where

$$f^{12}_{(3,0)} = \frac{15\cdot 12384}{7429}\,,\quad f^{10}_{(3,0)} = -\frac{91\cdot 74088}{96577}\,,\quad f^{8}_{(3,0)} = \frac{11\cdot 43200}{2717}\,,\quad f^{6}_{(3,0)} = \frac{10\cdot 30483}{3553}\,,$$
$$f^{4}_{(3,0)} = \frac{174\cdot 756}{2431}\,,\qquad f^{2}_{(3,0)} = f^{0}_{(3,0)} = 0\,,$$
$$g^{12}_{(3,0)} = \frac{62\cdot 12384}{7429}\,,\quad g^{10}_{(3,0)} = -\frac{278\cdot 74088}{96577}\,,\quad g^{8}_{(3,0)} = \frac{24\cdot 43200}{2717}\,,\quad g^{6}_{(3,0)} = -\frac{10\cdot 23203}{3553}\,,$$
$$g^{4}_{(3,0)} = -\frac{73\cdot 756}{2431}\,,\quad g^{2}_{(3,0)} = -\frac{4816}{143}\,,\quad g^{0}_{(3,0)} = 0\,, \tag{A.47}$$

and

$$
\begin{aligned}
&f^{12}_{(0,2)} = -\frac{15\cdot 624}{391}\ , \quad f^{10}_{(0,2)} = \frac{91\cdot 4452}{5083}\ , \quad f^{8}_{(0,2)} = \frac{11\cdot 60}{143}\ , \quad f^{6}_{(0,2)} = \frac{87\cdot 350}{187}\ , \\
&f^{4}_{(0,2)} = \frac{84\cdot 1354}{2431}\ , \quad f^{2}_{(0,2)} = f^{0}_{(0,2)} = 0\ , \\
&g^{12}_{(0,2)} = -\frac{62\cdot 624}{391}\ , \quad g^{10}_{(0,2)} = \frac{278\cdot 4452}{5083}\ , \quad g^{8}_{(0,2)} = \frac{24\cdot 60}{143}\ , \quad g^{6}_{(0,2)} = \frac{53\cdot 350}{187}\ , \\
&g^{4}_{(0,2)} = -\frac{84\cdot 2143}{2431}\ , \quad g^{2}_{(0,2)} = -\frac{4256}{143}\ , \quad g^{0}_{(0,2)} = 0\ .
\end{aligned} \tag{A.48}
$$

B. Interactions from circle compactification to ten dimensions

We will here evaluate the integrals $I^{(d=10)}_{(p,q)}$ for $(p,q) = (2,0),(1,1),(3,0),(0,2)$ for the circle compactification that relates eleven-dimensional supergravity to ten-dimensional type IIA string theory. The method used to evaluate these integrals is an extension of that used for the $(0,1)$ case in [8], which we will review in the appendix B.2 ($I^{(d=10)}_{(1,0)}$ was evaluated in [7]). First we will discuss some expressions that need to be evaluated in the course of the calculations.

B.1 Some basic sums

In the course of these calculations we will encounter both analytic and nonanalytic terms, as discussed in section 2.3.3. The calculation will reduce to the evaluation of expressions of the form

$$\Sigma_\alpha(v,\Lambda,\chi) = \sum_{\hat m\in\mathbb{Z}} \int_0^{\Lambda^2} dx\, x^{-\alpha}\, e^{-\pi \hat m^2 v x}\, e^{-\frac{\chi f}{x}}\ . \tag{B.1}$$

in the limit $\chi\, v \ll 1$ and $\Lambda^2 v \to \infty$ with v fixed, where f may be a function of S_0 and T_0, but its exact form will be irrelevant in the following (since, in general, we will not keep track of the scale of logarithmic thresholds). The regulating factor $e^{-\chi f/x}$ is inserted to regulate the infrared logarithmic factor as in (2.53).

First consider the case $\alpha < 1/2$. In this case we can safely set $S = 0$ in (B.1), giving

$$\Sigma_{\alpha<\frac{1}{2}} = \sum_{\hat m\in\mathbb{Z}} \int_{0_\chi}^{\Lambda^2} dx\, x^{-\alpha}\, e^{-\pi \hat m^2 v x} = \frac{\Lambda^{2-2\alpha}}{1-\alpha} + 2\Gamma(1-\alpha)\,\zeta(2-2\alpha)\,(\pi v)^{\alpha-1}\,, \tag{B.2}$$

where the Λ-dependence comes from the $\hat m = 0$ term. For later reference we note

$$
\begin{aligned}
\Sigma_{-\frac{1}{2}} = \frac{2}{3}\,\Lambda^3 + \frac{1}{\pi}\,\zeta(3)\,v^{-\frac{3}{2}}\,, \qquad \Sigma_{-\frac{3}{2}} = \frac{2}{5}\,\Lambda^5 + \frac{3}{2\pi^2}\,\zeta(5)\,v^{-\frac{5}{2}} \\
\Sigma_0 = \Lambda^2 + \frac{2}{\pi}\,\zeta(2)\,v^{-1}\,.
\end{aligned} \tag{B.3}
$$

Now we consider the case $\alpha = 1/2$,

$$\Sigma_{\frac{1}{2}} = \sum_{\hat m\in\mathbb{Z}} \int_0^{\Lambda^2} \frac{dx}{\sqrt{x}}\, e^{-\pi \hat m^2 v x}\, e^{-\frac{\chi f}{x}}\ . \tag{B.4}$$

Here we cannot simply set $\chi = 0$ since this leads to a singular sum, even though each term in the sum is finite,

$$\Sigma_{\frac{1}{2}}(\chi = 0) = \sum_{\hat{m}\in\mathbb{Z}} \int_0^{\Lambda^2} \frac{dx}{\sqrt{x}}\, e^{-\pi \hat{m}^2 v x} = \sum_{\hat{m}\neq 0} \frac{1}{\hat{m}} \frac{\Gamma(\frac{1}{2})}{\sqrt{\pi v}} + 2\Lambda \sim \frac{2}{v^{\frac{1}{2}}}\,\zeta(1) + 2\Lambda\,. \tag{B.5}$$

The presence of a divergence in the form of $\zeta(1)$ shows the importance of keeping the factor of $e^{-\chi f/x}$ in (B.4).

Separating the $\hat{m} = 0$ contribution and computing the integral we find (for $Sfv \ll 1$, or $\alpha' s g_A^2 \ll 1$)

$$\begin{aligned}
\Sigma_{\frac{1}{2}} &= 2\Lambda + \sum_{\hat{m}\neq 0} \int_0^{\infty} \frac{dx}{\sqrt{x}} e^{-\pi \hat{m}^2 v x - \chi f/x} = 2\Lambda + \sum_{\hat{m}\neq 0} \frac{1}{\sqrt{v}\hat{m}} e^{-2\sqrt{\pi\chi f v}|\hat{m}|} \\
&= 2\Lambda - \frac{2}{\sqrt{v}} \log\left(1 - e^{-2\sqrt{\pi\chi f v}}\right) \\
&\cong 2\Lambda - \frac{1}{\sqrt{v}} \log(4\pi\chi f v)
\end{aligned} \tag{B.6}$$

Notably, the scale of the logarithmic depends on $v = R_{11}^2$ but is independent of the cutoff Λ.

Another special case that will be needed later is

$$\begin{aligned}
\Sigma_1 &= \sum_{\hat{m}\in\mathbb{Z}} \int_0^{\Lambda^2} \frac{dx}{x}\, e^{-\pi \hat{m}^2 v x}\, e^{-\frac{\chi f}{x}} \\
&= \log(v\,\Lambda^2/C') - 2\pi^{-\frac{1}{2}}\,(\chi f)^{-\frac{1}{2}}\, v^{-\frac{1}{2}}
\end{aligned} \tag{B.7}$$

(where C' is independent of v and χ), as can be checked by differentiating the first expression with respect to Λ and with respect to S. The inverse power of χ will be ignored in the following as described in subsection (2.3.3).

We now turn to consider $\alpha > 1$. The integrand of Σ_α is more singular at small x so

$$\begin{aligned}
\Sigma_{\alpha>1} &= \sum_{\hat{m}\in\mathbb{Z}} \int_0^{\Lambda^2} dx\, x^{-\alpha}\, e^{-\pi \hat{m}^2 v x}\, e^{-\frac{\chi f}{x}} \\
&= \frac{1}{v^{\frac{1}{2}}} \sum_{m\in\mathbb{Z}} \int_0^{\Lambda^2} \frac{dx}{x^{\frac{1}{2}}}\, x^{-\alpha}\, e^{-\pi m^2/vx}\, e^{-\frac{\chi f}{x}} \\
&= \frac{1}{v^{\frac{1}{2}}} \sum_{m\in\mathbb{Z}} \int_{1/\Lambda^2}^{\infty} dw\, w^{\alpha-\frac{3}{2}}\, e^{-\pi m^2 w/v}\, e^{-\chi f w} \\
&= \frac{1}{v^{\frac{1}{2}}}\,(\chi f)^{\frac{1}{2}-\alpha}\,\Gamma(\alpha - \tfrac{1}{2}) + \frac{2}{v^{\frac{1}{2}}}\,\left(\frac{v}{\pi}\right)^{\alpha-\frac{1}{2}}\,\Gamma(\alpha - \tfrac{1}{2})\,\zeta(2\alpha - 1)\,,
\end{aligned} \tag{B.8}$$

where we have performed a Poisson resummation to express the sum in terms of Kaluza–Klein integers m, and separated the $m = 0$ term, which is proportional to $(-S)^{\frac{1}{2}-\alpha}$ and set $S = 0$ in the terms with $m \neq 0$. We note, in particular, that after again dropping negative powers of χ,

$$\Sigma_{\frac{3}{2}} = \frac{\pi v^{\frac{1}{2}}}{3}\,, \qquad \Sigma_{\frac{5}{2}} = \frac{\pi^2 v^{\frac{3}{2}}}{45}\,. \tag{B.9}$$

B.2 Evaluation of $I^{(d=10)}_{(0,1)}$

The integral $I^{(d=10)}_{(0,1)}$ of relevance to the $\sigma_3\,\mathcal{R}^4$ interaction decomposes into three distinct pieces of the form

$$I^{(d=10)}_{(0,1)} = \frac{I^{(3)}_{(0,1)}}{v^3} + I^{(3/2)}_{(0,1)}\,\frac{\Lambda^3}{v^{3/2}} + I^{(0)}_{(1,0)}\,\Lambda^6\,, \tag{B.10}$$

where $v = R_{11}^2$. The contribution $I^{(3)}_{(0,1)}$ is the finite part of the amplitude, which comes from non-zero winding numbers, and which was evaluated in in section 4 of [8]. This corresponds to the tree-level string contribution to the $\mathcal{S}^{(3)}\mathcal{R}^4$ term in the amplitude (or the $\mathcal{D}^6\mathcal{R}^4$ interaction). The $I^{(3/2)}_{(0,1)}$ term proportional to Λ^3 comes from a one-loop sub-divergence that needs to be subtracted by the addition of the triangle diagram where one vertex is the $\mathcal{R}^4$ one-loop counterterm. The $I^{(0)}_{(0,1)}$ term proportional to Λ^6 comes from a new two-loop divergence that also needs to be subtracted by the addition of a local counterterm.

Each of these contributions satisfies a second order differential equation of the form,

$$(v^2\frac{\partial^2}{\partial v^2} + 2v\frac{\partial}{\partial v})\frac{I^{(\alpha)}_{(0,1)}}{v^\alpha} = \alpha(\alpha-1)\,\frac{I^{(\alpha)}_{(0,1)}}{v^\alpha}\,. \tag{B.11}$$

Applying the operator on the left-hand side of this equation to the explicit integral $I_{(0,1)}$ and using the explicit form of $\hat{E}$,

$$\hat{E}(\tau,V) = v\,V\frac{|\hat{m}+\hat{n}\tau|^2}{\tau_2}\,, \tag{B.12}$$

leads to

$$(v^2\frac{\partial^2}{\partial v^2} + 2v\frac{\partial}{\partial v})I^{(d=10)}_{(0,1)} = \pi^2\sum_{(\hat{m},\hat{n})\in\mathbb{Z}}\int_{0_\chi}^{V^\Lambda} dV\,V^2\int_{\mathcal{F}_\Lambda}\frac{d^2\tau}{\tau_2^2}B_{(0,1)}(\tau)\Delta_\tau e^{-\pi\hat{E}(\tau,V)}\,. \tag{B.13}$$

After integration by parts, and using the Laplace equation (2.41) satisfied by $B_{(0,1)}$ this equation can be reexpressed as

$$(v^2\frac{\partial^2}{\partial v^2} + 2v\frac{\partial}{\partial v} - 12)I^{(d=10)}_{(0,1)} = j_{(0,1)} - \partial I^{(d=10)}_{(0,1)}\,, \tag{B.14}$$

where $j_{(0,1)}$ is the bulk term

$$j_{(0,1)} = -12\pi^2\sum_{(\hat{m},\hat{n})\in\mathbb{Z}^2}\int_0^{\Lambda^2} dV\,V^2\int_1^{\frac{\Lambda^2}{V}}\frac{d\tau_2}{\tau_2}\,e^{-\pi\hat{E}}\,, \tag{B.15}$$

and $\partial I_{(0,1)}$ is the boundary term

$$\partial I_{(0,1)} = \pi^2\sum_{(\hat{m},\hat{n})\in\mathbb{Z}^2}\int_0^{\Lambda^2} dV\,V^2\left(\partial_{\tau_2}B_{(0,1)}(\tau)\,e^{-\pi\hat{E}} - B_{(0,1)}(\tau)\partial_{\tau_2}e^{-\pi\hat{E}}\right)\Big|_{\tau=\tau_2^\Lambda}\,, \tag{B.16}$$

which receives contributions from $\tau_2 = \tau_2^\Lambda = \Lambda^2/V$. Note that the upper limit on V is equal to Λ^2 (whereas $V_\Lambda = 2\Lambda^2/\sqrt{3}$) since $\tau_2^\Lambda = \Lambda^2/V \geq 1$.

The boundary contributions with $\hat{n} \neq 0$ are exponentially suppressed as $\Lambda^2 \to \infty$ because they are proportional to

$$e^{-v\,\hat{n}^2\,\Lambda^2}\,. \tag{B.17}$$

Therefore only terms with $\hat{n} = 0$ contribute to $\partial I_{(0,1)}$. These zero winding number terms contribute to the sub-leading divergence proportional to Λ^3, which is canceled by the diagram with the one-loop counterterm of (2.47). Only the leading positive power of τ_2 in $B_{(p,q)}$ contributes in an essential way to the boundary term (B.16). More explicitly, we may write $B_{(0,1)}(\tau) = \tau_2 + \alpha_1(\tau_1)\tau_2^{-1} + o(\tau_2^{-3})$ where $\int_{-1/2}^{\frac{1}{2}} d\tau_1 \alpha_1(\tau_1) = \tilde{\alpha}_1 = 0$. In that case, after some manipulations (B.16) becomes

$$\begin{aligned}
\partial I_{(0,1)} &= \pi^2 \sum_{\hat{m}\in\mathbb{Z}} \int_0^{\Lambda^2} dV\, V^2 \big(e^{-\pi\hat{E}} - (\tau_2^{\Lambda})^{-1}\,(\pi\hat{m}^2 v V)\, e^{-\pi\hat{E}}\big) \\
&= \pi^2 \sum_{\hat{m}\in\mathbb{Z}} \int_0^{\Lambda^2} dV\, V^2 \big(1 - \frac{1}{\Lambda^2}\pi v V^2 \hat{m}^2\big) e^{-\pi\frac{vV^2\hat{m}^2}{\Lambda^2}} \\
&= \frac{\pi^2}{3}\,\Lambda^6 - \frac{3}{2\pi}\,\zeta(2)\zeta(3)\, v^{-\frac{3}{2}}\,\Lambda^3\,,
\end{aligned} \tag{B.18}$$

using (B.3) in the last step.

The contribution from the bulk term in (B.14) is

$$j_{(0,1)} = -12\pi^2 \sum_{(\hat{m},\hat{n})\in\mathbb{Z}^2} \int_0^{\Lambda^2} dV\, V^2 \int_1^{\frac{V_\Lambda}{V}} \frac{d\tau_2}{\tau_2}\, e^{-\pi v\, V\,(\frac{m^2}{\tau_2} + n^2\tau_2)}\,. \tag{B.19}$$

We now change variables to

$$x = V/\tau_2\,, \qquad y = V\tau_2\,, \tag{B.20}$$

which are integrated over the domain

$$0 < y < \Lambda^2\,, \qquad 0 < x < y\,, \tag{B.21}$$

with measure

$$dV\, d\tau_2 = \frac{1}{2y}\, dx dy\,. \tag{B.22}$$

Noting that since the integrand is symmetric we can double the region of integration and integrate over x and y independently. In these variables we have

$$\begin{aligned}
j_{(0,1)} &= -3\pi^2 \sum_{(\hat{m},\hat{n})\in\mathbb{Z}} \int_0^{\Lambda^2} dx \int_0^{\Lambda^2} dy \sqrt{x\,y}\; e^{-\pi v\,(\hat{m}^2 y + \hat{n}^2 x)} \\
&= -3\pi^2\,(\Sigma_{-\frac{1}{2}})^2 = -\frac{3}{v^3}\,\zeta(3)^2 - \frac{24}{\pi v^{\frac{3}{2}}}\,\zeta(2)\zeta(3)\,\Lambda^3 - \frac{4}{3}\,\pi^2\,\Lambda^6\,.
\end{aligned} \tag{B.23}$$

Substituting the contributions to $j_{(0,1)}$ and $\partial I_{(0,1)}^{(d=10)}$ into (B.14) gives the Poisson equation

$$(v^2\partial_v^2 + 2v\partial_v - 12) I_{(0,1)}^{(d=10)} = -\frac{3}{v^3}\,\zeta(3)^2 - \frac{45}{2\pi v^{\frac{3}{2}}}\,\zeta(2)\zeta(3)\,\Lambda^3 - 10\,\zeta(2)\,\Lambda^6\,. \tag{B.24}$$

This equation is simple to solve using the general formula (3.12), (3.13), (3.16), giving

$$I^{(d=10)}_{(0,1)} = \frac{5}{6}\zeta(2)\,\Lambda^6 + \Lambda^3\frac{2\zeta(2)\zeta(3)}{\pi v^{3/2}} + \frac{\zeta(3)^2}{2\,v^3}\,. \tag{B.25}$$

The Λ^3 divergence in $I_{(0,1)}$ is canceled by the counter term $\delta A_{\triangleright}$ of equation (2.47) which, at order $\sigma_3\,\mathcal{R}^4$, contributes

$$I^{(d=10)}_{\triangleright\,(0,1)} = \frac{\pi}{4}\,c_1\,\left(\frac{2\Lambda^3}{3} + \frac{\zeta(3)}{\pi\,v^{3/2}}\right) = \frac{\zeta(2)\zeta(3)}{v^{\frac{3}{2}}} - \Lambda^3\frac{2\zeta(2)\zeta(3)}{\pi v^{\frac{3}{2}}} - \Lambda^6\frac{4\zeta(2)}{3}\,, \tag{B.26}$$

where we have used the value of c_1 given in (2.43). The relative normalisation of the counter-term triangle diagram with respect to the double box diagram, which is fixed by unitarity, is such that the Λ^3 divergence cancels. We also need to subtract the superficial Λ^6 divergence with a new counterterm

$$\delta_2 I^{(d=10)}_{(0,1)} = \frac{4\zeta(2)}{3}\Lambda^6 + \frac{6\zeta(2)^2}{5}\,, \tag{B.27}$$

where the value of the constant last term is determined from the value of the genus-two coefficient of the $\sigma_3\,\mathcal{R}^4$ interaction in type IIB string theory, which is contained in the modular function $\mathcal{E}_{(0,1)}$ [8] (using the fact that the four-graviton amplitudes in the IIA and IIB theories are identical up to four loops). The total contribution

$$I^{(d=10)}_{(0,1)} + I^{(d=10)}_{\triangleright\,(0,1)} + \delta_2 I^{(d=10)}_{(0,1)} = \frac{\zeta(3)^2}{2\,v^3} + \frac{\zeta(2)\zeta(3)}{v^{3/2}} + \frac{6\zeta(2)^2}{5}\,, \tag{B.28}$$

Using the dictionary between M-theory and string variables the first two terms coincide with the perturbative string tree-level and genus-one results. These are also reproduced by the first two terms of the perturbative expansion of $\mathcal{E}_{(0,1)}$ (while the last term in (B.28) is the genus-two term).

B.3 Evaluation of $I^{(d=10)}_{(2,0)}$

In a similar fashion to the treatment of $I^{(d=10)}_{(0,1)}$, we may write $I^{(d=10)}_{(2,0)}$ as the sum of three terms with different powers of v (recalling that $v = R_{11}^2$)

$$I^{(d=10)}_{(2,0)} = \frac{I^{(2)}_{(2,0)}}{v^2} + I^{(1/2)}_{(2,0)}\,\frac{\Lambda^3}{v^{1/2}} + I^{(0)}_{(2,0)}\,\Lambda^4\,. \tag{B.29}$$

The contribution $I^{(2)}_{(2,0)}$ is the finite part of the amplitude that comes from non-zero windings. The piece that diverges as Λ^3 comes from the sub-divergences in which there is zero winding in one loop and non-zero in the other. The leading Λ^4 divergence does not make sense in string perturbation and is subtracted (just as the $\Lambda^8\,\mathcal{D}^4\mathcal{R}^4$ term was subtracted in [7]).

Each of the contributions satisfies

$$(v^2\frac{\partial^2}{\partial v^2} + 2v\frac{\partial}{\partial v})\,\frac{I^{(\alpha)}_2}{v^\alpha} = \alpha(\alpha-1)\,\frac{I^{(\alpha)}_2}{v^\alpha}\,.$$

Now we write $B_{(2,0)}$ as a sum of the four functions $b^0_{(2,0)}, b^2_{(2,0)}, b^4_{(2,0)}, b^6_{(2,0)}$ satisfying Poisson equations with delta function sources. $I^{(d=10)}_{(2,0)}$ is then naturally written as

$$I^{(d=10)}_{(2,0)} = \sum_{i=0}^{3} h^{2i}_{(2,0)} \ . \tag{B.30}$$

The integral $h^0_{(2,0)}$ needs separate treatment, because $b^0_{(2,0)} = -13/21$ is a constant and will be considered later. The integrals $h^2_{(2,0)}, h^4_{(2,0)}, h^6_{(2,0)}$ can be computed by following the analogous computation to that given in the last sub-section (and section 4 of [8]). By definition they satisfy the equations

$$\big(v^2 \frac{\partial^2}{\partial v^2} + 2v \frac{\partial}{\partial v}\big) h^i_2 = \pi^3 \sum_{(\hat{m},\hat{n})\in\mathbb{Z}^2} \int_{0_\chi}^{V\Lambda} dV\, V \int_{\mathcal{F}_\Lambda} \frac{d^2\tau}{\tau_2^2} b^i_{(2,0)}(\tau) \Delta_\tau e^{-\pi \hat{E}} \ . \tag{B.31}$$

where $\mathcal{F}_\Lambda$ is once again the cutoff fundamental domain $\tau_2 \leq \tau_2^\Lambda = \Lambda^2/V$ and $V_\Lambda = 2\Lambda^2/\sqrt{3}$. Integrating by parts gives for $i = 2,4,6$

$$\big(v^2 \frac{\partial^2}{\partial v^2} + 2v \frac{\partial}{\partial v} - i(i+1)\big) h^i_{(2,0)} = j^i_{(2,0)} - \partial h^i_{(2,0)} \,, \tag{B.32}$$

where the bulk term is

$$j^i_{(2,0)} = -2u_i \pi^3 \sum_{(\hat{m},\hat{n})\in\mathbb{Z}^2} \int_{0_\chi}^{V\Lambda} dV\, V \int_1^{\frac{\Lambda^2}{V}} \frac{d\tau_2}{\tau_2^2} \big(\tau_2^2 + 1\big) e^{-\pi E(0,\tau_2)} \tag{B.33}$$

and the boundary term is

$$\partial h^i_{(2,0)} = \pi^3 \sum_{(\hat{m},\hat{n})\in\mathbb{Z}^2} \int_{0_\chi}^{V\Lambda} dV\, V \left(\partial_{\tau_2} b^i_{(2,0)}(\tau)\, e^{-\pi \hat{E}} - b^i_{(2,0)}\,(\tau) \partial_{\tau_2} e^{-\pi \hat{E}} \right) \Bigg|_{\tau=\tau_2^\Lambda} \tag{B.34}$$

(where $\lambda^i_{(2,0)}$ and u_i are defined in the appendix A.1).

This boundary term again receives contributions from the region $\tau_2 \sim \tau_2^\Lambda = \Lambda^2/V$ and in the parametrization where $\hat{E} = v\, V|\hat{m} + \hat{n}\tau|^2/\tau_2$ the contribution with $\hat{n} \neq 0$ is again exponentially suppressed as $\Lambda^2 \to \infty$. The $\hat{n} = 0$ terms contribute to the sub-leading divergence which is regularised by the diagram with the one-loop counter-term of equation (2.47). As before, the only boundary contributions that matter are the leading ones, which in this case are given by using the expansion

$$b^i_{(2,0)}(\tau) = q_i \tau_2^2 + \alpha^i_2(\tau_1) + o(\tau_2^{-1}) \,, \qquad i = 2,4,6 \ , \tag{B.35}$$

to give

$$\partial h^i_{(2,0)} = u_i \frac{3\pi^3}{4} \Lambda^3\, \Sigma_{\frac{1}{2}} - \frac{3\pi}{4} \zeta(3)\, \tilde{\alpha}^i_2 \frac{\Lambda}{v^{\frac{3}{2}}} \,, \tag{B.36}$$

where $\tilde\alpha_2^i = \int_{-1/2}^{1/2} d\tau_1 \alpha_2^i(\tau_1)$, and $\Sigma_{\frac12}$ was defined in (B.4). The contribution from the bulk term (B.33) is

$$
\begin{aligned}
j^i_{(2,0)} &= -2u_i\,\pi^3 \sum_{(\hat m,\hat n)\in\mathbb{Z}^2} \int_{0_\chi}^{V^\Lambda} dV\,V \int_1^{\frac{\Lambda^2}{V}} \frac{d\tau_2}{\tau_2^2}\big(\tau_2^2+1\big)e^{-\pi v\,V\,(\frac{\hat m^2}{\tau_2}+\hat n^2\tau_2)} && \text{(B.37)}\\
&= -u_i\pi^3 \sum_{(\hat m,\hat n)\in\mathbb{Z}^2} \int_0^{\Lambda^2} dx \int_0^{\Lambda^2} dy \sqrt{\frac{x}{y}}\, e^{-\pi v\,(\hat m^2\,x+\hat n^2\,y)}\\
&= -u_i\,\pi^3\,\Sigma_{\frac12}\,\Sigma_{-\frac12}\\
&= -u_i\,\pi^3\left(2\Lambda - \frac{1}{v^{\frac12}}\log(\chi v/2\pi^2 c_e)\right)\left(\frac23\Lambda^3+\frac{\zeta(3)}{\pi\,v^{\frac32}}\right). && \text{(B.38)}
\end{aligned}
$$

Therefore (B.32) becomes for $i=2,4,6$

$$
\begin{aligned}
(v^2\partial_v^2+2v\partial_v - i(i+1))h^i_{(2,0)} = u_i\,\Big(&\frac{3\pi^3}{2}\,\Lambda^4 + \frac{17\pi\,\Lambda^3\ \log(\chi v/2\pi^2 c_e)\,\zeta(2)}{2\sqrt{v}}\\
&+\frac{1}{v^{\frac32}}\zeta(3)\zeta(2)\,\Lambda\left(\frac{9\,\hat\alpha_2^i}{2\pi}-12\right)+\frac{6\,\log(\chi v/2\pi^2 c_e)\,\zeta(2)\,\zeta(3)}{v^2}\Big).
\end{aligned} \qquad \text{(B.39)}
$$

The terms proportional to Λ will eventually cancel due to the relation (A.7). Furthermore, the Λ^4 terms are primitive divergences that we will cancel with a counterterm, so their precise coefficients are not of relevance (there can be no finite remainder since this term does not correspond to a sensible term in string perturbation theory). These equations are of the form (3.15) (with $a=d=0$), which have solutions (3.16). The explicit expressions will not be given here but their sum enters the complete expression for $I^{(d=10)}_{(2,0)}$.

Now consider the case of $h^0_{(2,0)}$ for which the integrand is a total derivative. Integration by parts shows that the integral only gets contributions from τ_2 boundary $\tau_2^\Lambda = \Lambda^2/V$, so that

$$
\begin{aligned}
(v^2\partial_v^2+2v\partial_v)h^0_{(2,0)} &= -\frac{13}{21}\,\pi^3 \sum_{(\hat m,\hat n)\in\mathbb{Z}^2}\int_0^{V_\Lambda} dV\,V\int_{\mathcal{F}_\Lambda}\frac{d^2\tau}{\tau_2^2}\Delta_\tau e^{-\pi\hat E}\\
&= -\frac{13}{21}\,\pi^4\,\frac{v}{\Lambda^4}\sum_{\hat m\neq 0}\hat m^2\int_0^{\Lambda^2} dV\,V^4\,e^{-\pi V^2\,v\hat m^2/\Lambda^2}\\
&= -\frac{39}{14}\,\zeta(2)\zeta(3)\,\frac{\Lambda}{v^{\frac32}}\,. && \text{(B.40)}
\end{aligned}
$$

Summing all the contributions to $I^{(d=10)}_{(2,0)}$ gives

$$
I^{(d=10)}_{(2,0)} = \frac{8}{5\sqrt{v}}\,\pi\,\Lambda^3\,\log(\chi v/\tilde C_{(2,0)})\,\zeta(2) - \frac{12}{50\,v^2}\,\log(\chi v/\tilde C_{(2,0)}))\,\zeta(2)\,\zeta(3)\,, \qquad \text{(B.41)}
$$

where $\tilde C_{(2,0)}$ is an unknown function of z. Note that the term with coefficient $1/v^2 = 1/R_{11}^4$ corresponds to a finite genus-one contribution in IIA string theory, while the term with coefficient $1/v^{\frac12} = 1/R_{11}$ corresponds to a genus-two string theory term, that comes from the sub-divergences (as indicated by the factor of Λ^3).

The Λ^3 divergence in $I^{(d=10)}_{(2,0)}$ is canceled by the counter-term $I^{(d=10)}_{\triangleright\,(2,0)}$ of equation (2.47) which contributes

$$I^{(d=10)}_{\triangleright\,(2,0)} = \frac{\pi^3\zeta(2)}{15}\, c_1\, v^{-\frac{1}{2}}\, \Big(-\log(\chi v/\tilde{\tilde{C}}_{(2,0)})\Big) \tag{B.42}$$

to the coefficient of $\sigma_2^2\,\mathcal{R}^4$, where c_1 given in (2.43) (and $\tilde{\tilde{C}}_{(2,0)}$ is another unknown function of z). The relative normalisation of the counter-diagram with respect to the double box diagram is such that the Λ^3 sub-divergence cancels. Furthermore, we need to introduce a counterterm that subtracts the primitive Λ^4 divergence,

$$I^{(d=10)}_{(2,0)} + I^{(d=10)}_{\triangleright\,(2,0)} + \delta_2 I^{(d=10)}_{(2,0)} = -\frac{12}{5}\,\zeta(2)\,\left[\frac{\zeta(3)}{v^2} + \frac{2\zeta(2)}{\sqrt{v}}\right]\,\log(-Sv/C_{(2,0)})\,. \tag{B.43}$$

The $\log(-Sv)$ terms are threshold contributions that correspond to the genus-one and genus-two string theory thresholds expected from unitarity, as described in the body of this paper.

B.4 Evaluation of $I^{(d=10)}_{(1,1)}$

We now consider $I^{(d=10)}_{(1,1)}$ for the term of order $\sigma_2\sigma_3\,\mathcal{R}^4$ in the expansion of the amplitude. We saw earlier that $B_{(1,1)} = \sum_{j=0}^4 b^{2j+1}_{(1,1)}$, where $b^j_{(1,1)}$ satisfies the Poisson equation (A.28). Extending the earlier cases, this leads to the decomposition

$$I^{(d=10)}_{(1,1)} = h^1_{(1,1)} + h^3_{(1,1)} + \cdots + h^9_{(1,1)}\ . \tag{B.44}$$

In this case we have

$$(v^2\frac{\partial^2}{\partial v^2} + 2v\frac{\partial}{\partial v})h^j_{(1,1)} = \pi^4 \sum_{(\hat m,\hat n)\in\mathbb{Z}^2}\int_0^{V\Lambda} dV \int_{\mathcal{F}_\Lambda}\frac{d^2\tau}{\tau_2^2} b^j_{(1,1)}(\tau)\Delta_\tau e^{-\pi\hat E}\ . \tag{B.45}$$

Proceeding as in the previous section we obtain, for $j = 1,3,5,7,9$

$$(v^2\partial_v^2 + 2v\,\partial_v - j(j+1))h^j_{(1,1)} = j^j_{(1,1)} - \partial h^j_{(1,1)}\,, \tag{B.46}$$

where

$$j^j_{(1,1)} = -2\pi^4 \sum_{(\hat m,\hat n)\in\mathbb{Z}^2}\int_0^{V\Lambda} dV\int_1^{\Lambda^2/V}\frac{d\tau_2}{\tau_2^2}\big(v_j(\tau_2^3 + \frac{1}{\tau_2}) + w_j\tau_2\big)\ e^{-\pi\hat E(i\tau_2)} \tag{B.47}$$

and

$$\partial h^j_{(1,1)} = -\pi^4 \sum_{(\hat m,\hat n)\in\mathbb{Z}^2}\int_0^{\Lambda^2} dV\,(\partial_{\tau_2} b^j_{(1,1)}\, e^{-\pi\hat E} - b^j_{(1,1)}\partial_{\tau_2}e^{-\pi\hat E})\big|_{\tau=\tau_2^\Lambda}\,, \tag{B.48}$$

which again only gets contributions from the $\hat n = 0$ sector. Furthermore, only the leading terms of

$$b^j_{(1,1)} \sim e^j_3\tau_2^3 + \tau_2\,\alpha^j_3(\tau_1) + o(\tau_2^{-1})\,, \tag{B.49}$$

contribute significantly to $\partial h^j_{(1,1)}$. Setting $x = V^2/\Lambda^2$ we get

$$\begin{aligned}
\partial h^j_{(1,1)} &= \pi^4\,\frac{\Lambda}{2}\sum_{(\hat m,\hat n)\in\mathbb{Z}^2}\int_{0_\chi}^{\Lambda^2}\frac{dx}{x^{\frac12}}\left(\frac{3e_3^j\Lambda^2}{x}+\tilde\alpha_3^j-\pi\,v\,\hat m^2\,e^j\,\Lambda^2-\pi\,v\,\hat m^2\,x\,\tilde\alpha_3^j\right)\,e^{-\pi\hat m^2 xv}\\
&= \frac{7\pi^4 e_3^j}{4}\Lambda^3\,\Sigma_{\frac32}-\frac{\pi^4}{2}\,\tilde\alpha_3^j\,\Sigma_{\frac12}\,\Lambda\\
&= \frac{7\pi^4 e_3^j}{4}\Lambda^3\,\left(\frac{1}{v^{\frac12}}\,(\chi f)^{-1}+\frac{\pi v^{\frac12}}{3}\right)-\frac{\pi^4}{2}\,\tilde\alpha_3^j\,\left(2\Lambda^2-\frac{1}{2v^{\frac12}}\,\Lambda\,\log(\chi v/C)\right)\,, \qquad \text{(B.50)}
\end{aligned}$$

where $\tilde\alpha_3^j \equiv \int_{-1/2}^{1/2} d\tau_1\,\alpha_3^j(\tau_1)$. As before the $\log(\chi)$ term arises from the massless threshold associated with $m = 0$ Kaluza-Klein charge in the intermediate states. Turning to the bulk term (B.47) we write $j^j_{(1,1)} = -2\pi^4\,(v_j\,K_1 + w_j\,K_2)$, where

$$v_jK_1+w_jK_2 \equiv 2\pi^4\sum_{(\hat m,\hat n)\in\mathbb{Z}^2}\int_0^{\Lambda^2}dV\int_1^{\frac{\Lambda^2}{V}}\frac{d\tau_2}{\tau_2}\big(v_j(\tau_2^2+\frac{1}{\tau_2^2})+w_j\big)\;e^{-\pi\hat E(i\tau_2)}\,, \qquad \text{(B.51)}$$

where (after introducing $x = V/\tau_2,\ y = V\tau_2$)

$$\begin{aligned}
K_1 &= \frac{\pi^4}{2}\sum_{(\hat m,\hat n)\in\mathbb{Z}^2}\int_0^{\Lambda^2}\frac{dxdy}{\sqrt{xy}}\big(\frac{x}{y}+\frac{y}{x}\big)\;e^{-\pi v\,(\hat m^2 y+\hat n^2 x)}\\
&= \pi^4\Sigma_{\frac32}\,\Sigma_{-\frac12}\\
&= \pi^4\,\left(\frac23\,\Lambda^3+\frac1\pi\,\zeta(3)\,v^{-\frac32}\right)\,\left(\frac{1}{v^{\frac12}}\,(\chi f)^{-1}+\frac{\pi v^{\frac12}}{3}\right)\\
&= \frac23\frac{\pi^4}{v^{\frac12}}\,\Lambda^3\,(\chi f)^{-1}+\frac43\pi^3\,\zeta(2)\,\Lambda^3\,v^{\frac12}+\frac{\pi^3\zeta(3)}{v^2}(\chi f)^{-1}+\frac{\pi^4}{3\,v}\zeta(3)\,, \qquad \text{(B.52)}
\end{aligned}$$

and

$$\begin{aligned}
K_2 &= \frac{\pi^4}{2}\sum_{(\hat m,\hat n)\in\mathbb{Z}^2}\int_0^{\infty}\frac{dxdy}{\sqrt{xy}}\;e^{-\pi v\,(\hat m^2 y+\hat n^2 x)}\\
&= \frac{\pi^4}{2}\,(\Sigma_{\frac12})^2=\frac{\pi^4}{2}\,\left(2\Lambda-\frac{1}{v^{\frac12}}\log(\chi v/C)\right)^2\\
&= 2\pi^4\,\Lambda^2-2\pi^4\,\Lambda\,\frac{1}{v^{\frac12}}\,\log(\chi v/C)+\frac{\pi^4}{2v}\,\log^2(\chi v/C)\,. \qquad \text{(B.53)}
\end{aligned}$$

The inverse powers of S in (B.50) and (B.52), originate, as anticipated, from the attempt to expand the nonanalytic amplitude in powers of S. We can drop these terms, which are singular in the limit $f \to 0$, following the argument in section 2.3.3. Summing all the other contributions of order S^5 and $S^5\log(-S)$ gives

$$\begin{aligned}
(v^2\partial_v^2+2v\partial_v-j(j+1))h^j_{(1,1)} = &-\frac{\pi^5}{36}\,\big(21\,e^j+8\,v_j\big)\,\sqrt{v}\,\Lambda^3-\frac{\pi^4}{2}\,(\hat e_j+8w_j)\,\Lambda^2\\
&+\frac{\pi^4}{4v^{\frac12}}\,(\hat\alpha_3^j+8w_j)\,\log(\chi v/2\pi^2 c_e)\,\Lambda \qquad \text{(B.54)}\\
&-\frac{\pi^4}{3v}\,\big(3\,w_j\,\log(\chi v/2\pi^2 c_e)^2+v_j\,\zeta(3)\big)\,.
\end{aligned}$$

These equations involve $\log v$ and $(\log v)^2$ factors are again of the form (3.15) (this time with $a = c = 0$) with solutions (3.16). Exploiting these solutions together with the following facts that follow from the explicit coefficients in appendix A

$$\sum_{j=0}^{4} \frac{w_{2j+1}}{(2j+1)(2j+2)} = 0 \,, \qquad \sum_{j=0}^{4} \frac{\hat{f}_j + 8w_j}{\frac{1}{4} + (2j+1)(2j+2)} = 0 \,, \tag{B.55}$$

and

$$\sum_{j=0}^{4} \frac{v_{2j+1}}{(2j+1)(2j+2)} = \frac{224}{15} \,, \qquad \sum_{j=0}^{4} \frac{\hat{\alpha}_3^{2j+1}}{(2j+1)(2j+2)} = -\frac{91}{18} \,,$$
$$\sum_{j=0}^{4} \frac{(21q_{2j+1} + 8v_{2j+1})}{(\frac{3}{4} - (2j+1)(2j+2))} = 180 \,, \tag{B.56}$$

and recalling that $I_{(1,1)} = \sum_{j=0}^{4} h_{(1,1)}^{2j+1}$, the total result is

$$I_{(1,1)}^{(d=10)} = -\frac{91}{36}\pi^4 \Lambda^2 + \frac{675}{2}\zeta(4)\sqrt{v}\,\frac{4\pi\Lambda^3}{3} + \frac{448\,\zeta(4)\,\zeta(3)}{v} + \frac{273\pi^4}{54\,v}\,\log(\chi v/\tilde{C}_{(1,1)})\,. \tag{B.57}$$

The cancelation of the $\log^2(-Sv)$ contributions corresponds to the fact that $1/\epsilon^2$ terms cancel in the two-loop diagrams of ten-dimensional type II supergravity, as we will see in detail in appendix E.1. The Λ^2 contribution is the leading superficial divergence at two string loop and must be subtracted with no finite residue since it is not accompanied by a power of $v = R_{11}^2 = g_A^{2/3}$ that is an integer power of g_A^2 and therefore cannot contribute in string theory. The Λ^3 contribution is a subleading divergence regulated by the counter term (2.47) leaving a finite genus-three string contribution. The total contribution at order $\sigma_2\sigma_3\,\mathcal{R}^4$ is

$$I_{(1,1)}^{(d=10)} + I_{\triangleright(1,1)}^{(d=10)} + \delta_2 I_{(1,1)}^{(d=10)} = \frac{4725}{8}\,\zeta(6)\sqrt{v} + \frac{448\,\zeta(4)\,\zeta(3)}{v} + \frac{455\zeta(4)}{v}\,\log(\chi v/C_{(1,1)})\,. \tag{B.58}$$

The first term in this expression corresponds to a genus-three IIA string contribution while the remaining terms (with the $1/v$ factor) are genus-two IIA string contributions. These contributions are distinguished by their distinct zeta function coefficients, so it would look very unnatural to associate the analytic $1/v$ term with the unknown scale of the logarithm in the nonanalytic $1/v$ term.

Substituting this result into the expansion for the amplitude gives the terms of order $\sigma_2\sigma_3\,\mathcal{R}^4$ as summarized in the text. It is worth noting, in particular, the presence of the logarithmic term

$$i\frac{\kappa_{(11)}^6}{(4\pi)^{10}}\,\frac{13}{466560}\,\sigma_2\sigma_3\,\log(\chi)\,\mathcal{R}^4\,, \tag{B.59}$$

which reproduces the result obtained for the coefficient of the $1/\epsilon$ pole obtained by dimensional regularization around nine dimensions in (E.3).

B.5 Evaluation of $I^{(d=10)}_{(3,0)}$ and $I^{(d=10)}_{(0,2)}$

In order to analyze the integrals $I_{(3,0)}$ and $I_{(0,2)}$ we write $B_{(p,q)} = \sum_{k=0}^{6} b^{2k}_{(p,q)}$ with $(p,q) = (3,0)$ and $(p,q) = (0,2)$, where $b^k_{(p,q)}$ satisfy Poisson equations (A.46). This leads to a decomposition

$$I_{(p,q)} = h^0_{(p,q)} + h^2_{(p,q)} + \cdots + h^{12}_{(p,q)}\,, \tag{B.60}$$

where, for $k = 2, 4, \ldots, 12$, the components satisfy the equations

$$\big(v^2 \frac{\partial^2}{\partial v^2} + 2v \frac{\partial}{\partial v}\big)\, h^k_{(p,q)} = \pi^5 \sum_{(\hat m,\hat n)\in\mathbb{Z}^2} \int_{0_\chi}^{\lambda} \frac{dV}{V} \int_{\mathcal{F}_\Lambda} \frac{d^2\tau}{\tau_2^2}\, b^k_{(p,q)}(\tau)\, \Delta_\tau e^{-\pi \hat E}\,. \tag{B.61}$$

The components $h^0_{(3,0)}$ and $h^0_{(0,2)}$ need separate treatment because they are associated with the constant contributions $b^0_{(3,0)} = 12264/715$ and $b^0_{(0,2)} = 2716/165$. In this case both the left-hand and right-hand sides of (B.61) vanish. We will return to these cases later.

B.5.1 $h^k_{(3,0)}$, $h^k_{(0,2)}$ with $k > 0$

Integrating (B.61) by parts and using (A.46) gives

$$\big(v^2 \frac{\partial^2}{\partial v^2} + 2v \frac{\partial}{\partial v} - k(k+1)\big) h^k_{(p,q)} = j^k_{(p,q)} - \partial h^k_{(p,q)}\,, \tag{B.62}$$

for $(p,q) = (3,0)$ and $(p,q) = (0,2)$ and $k = 2, 4, \ldots, 12$ where

$$j^i_{(p,q)} = -2\pi^5 \sum_{(\hat m,\hat n)\in\mathbb{Z}^2} \int_{0_\chi}^{V^\Lambda} \frac{dV}{V} \int_1^\infty \frac{d\tau_2}{\tau_2^2} \big(1+\tau_2^2\big) \big[f^k_{(p,q)}(\tau_2^2 + \tau_2^{-2}) + g^k_{(p,q)}\big]\; \delta(\tau_1) e^{-\pi E} \tag{B.63}$$

and

$$\partial h^k_{(p,q)} = -\pi^5 \int_{0_\chi}^{V^\Lambda} \frac{dV}{V} \big(\partial_{\tau_2} b^k_{(p,q)} e^{-\pi \hat E} - b^k_{(p,q)} \partial_{\tau_2} e^{-\pi \hat E}\big)\big|_{\tau_2 = \tau_2^\Lambda}\,. \tag{B.64}$$

with $(p,q) = (3,0)$ and $(p,q) = (0,2)$. We recall that the eigenvalues are the same for the two tensorial structures, so

$$(v^2 \partial_v^2 + 2v\partial_v - k)(k+1) h^{2k}_{(p,q)} \;=\; -2\,(f^{2k}_{(p,q)} H_1 + g^{2k}_{(p,q)} H_2) - \partial h^{2k}_{(p,q)}\,, \tag{B.65}$$

where (after introducing $x = V/\tau_2,\ y = V\tau_2$)

$$\begin{aligned}
H_1 \;&=\; \frac{\pi^5}{2} \sum_{(\hat m,\hat n)\in\mathbb{Z}^2} \int_0^\infty \frac{dxdy}{xy} \big(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}}\big)\big(\frac{x}{y} + \frac{y}{x}\big)\; e^{-\pi v\,(\hat m^2 y + \hat n^2 x)} \\
&=\; \frac{\pi^5}{2}\,(\Sigma_{-\frac12}\,\Sigma_{\frac52} + \Sigma_{\frac12}\,\Sigma_{\frac32}) \\
&=\; \frac{\pi^5}{2} \left(\frac{2}{3}\Lambda^3 + \frac{1}{\pi}\zeta(3)\, v^{-\frac32}\right) \left(v^{-\frac12}(\chi f)^{-2} + \frac{\pi^2}{45} v^{\frac32}\right) \\
&\quad\; \frac{\pi^5}{2} \left(-v^{-\frac12} \log(\chi v/C) + 2\Lambda\right) \left(v^{-\frac12}(\chi f)^{-1} + \frac{\pi}{3} v^{\frac12}\right)\,,
\end{aligned} \tag{B.66}$$

and

$$\begin{aligned} H_2 &= \frac{\pi^5}{2} \sum_{(\hat m,\hat n)\in\mathbb{Z}^2} \int_0^\infty \frac{dxdy}{xy} (\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}})\ e^{-\pi v\, (\hat m^2 y + \hat n^2 x)} \\ &= \frac{\pi^5}{2}\, \Sigma_{\frac12}\, \Sigma_{\frac32} \\ &= \frac{\pi^5}{2} \left(-v^{-\frac12}\log(\chi v/C) + 2\Lambda\right) \left(v^{-\frac12}(\chi f)^{-1} + \frac{\pi}{3} v^{\frac12}\right) , \end{aligned} \tag{B.67}$$

where the terms with inverse powers of χ will once more be dropped.

The relevant contributions to the boundary term come from the positive powers of τ_2 in the expansions

$$b^k_{(p,q)} = e^k_{(p,q)}\, \tau_2^4 + \alpha^k_{(p,q)}(\tau_1)\, \tau_2^2 + O(1)\,. \tag{B.68}$$

Substituting in (B.64) gives

$$\partial h^k_{(p,q)} = -\frac{\pi^5}{2} \left(\frac{11}{2}\Lambda^3 e^k_{(p,q)}\, \Sigma_{\frac52} + \frac{5}{2}\,\Lambda\, \tilde\alpha^k_{(p,q)}\, \Sigma_{\frac32} \right) , \tag{B.69}$$

where $\tilde\alpha^k_{(p,q)} = \int_{-1/2}^{1/2} \alpha^k_{(p,q)}(\tau_1)\, d\tau_1$. Putting the various contributions together (B.62) gives, for $k \neq 0$,

$$\begin{aligned} (v^2\partial_v^2 + 2v\partial_v - k(k+1))\, h^{2k}_{(p,q)} &= -\frac{315}{4}\sqrt{v}\,\Lambda\, \left(8\, f^{2k}_{(p,q)} + 8\, g^{2k}_{(p,q)} + 5\, \tilde\alpha^{2k}_{(p,q)}\right) \zeta(6) \\ &- \frac{7}{4}\,\pi\, v^{\frac32}\, \Lambda^3\, \left(8\, f^{2k}_{(p,q)} + 33\, e^{2k}_{(p,q)}\right) \zeta(6) \\ &- \left(-315\, (g^{2k}_{(p,q)} + f^{2k}_{(p,q)})\, \log(\chi v/2\pi^2 c_e) + 21\, f^{2k}_{(p,q)}\, \zeta(3)\right) \zeta(6)\,. \end{aligned} \tag{B.70}$$

for $(p,q) = (3,0)$ and $(p,q) = (0,2)$.

Once again these equations have $\log v$'s on the right-hand side and the solutions were obtained in (3.16). We also note the values of the sums,

$$\sum_{k=1}^{6} \frac{8\, f^{2k}_{(3,0)} + 8\, g^{2k}_{(3,0)} + 5\, \tilde\alpha^{2k}_{(3,0)}}{3/4 - 2k(2k+1)} = 0\,, \qquad \sum_{k=1}^{6} \frac{8\, f^{2k}_{(3,0)} + 33\, e^{2k}_{(3,0)}}{15/4 - 2k(2k+1)} = -96\,, \tag{B.71}$$

$$\sum_{k=1}^{6} \frac{8\, f^{2k}_{(0,2)} + 8\, g^{2k}_{(0,2)} + 5\, \tilde\alpha^{2k}_{(0,2)}}{3/4 - 2k(2k+1)} = 0\,, \qquad \sum_{k=1}^{6} \frac{8\, f^{2k}_{(0,2)} + 33\, e^{2k}_{(0,2)}}{15/4 - 2k(2k+1)} = -96\,, \tag{B.72}$$

$$\sum_{k=1}^{6} \frac{f^{2k}_{(3,0)} + e^{2k}_{(3,0)}}{2k(2k+1)} = \frac{1733}{715}\,, \qquad \sum_{k=1}^{6} \frac{f^{2k}_{(0,2)} + e^{2k}_{(0,2)}}{2k(2k+1)} = \frac{749}{330}\,, \tag{B.73}$$

$$\sum_{k=1}^{6} \frac{f^{2k}_{(3,0)}}{2k(2k+1)^2} = \frac{16000249}{75150075}\,, \qquad \sum_{k=1}^{6} \frac{f^{2k}_{(0,2)}}{2k(2k+1)^2} = \frac{25658819}{118918800}\,. \tag{B.74}$$

B.5.2 $h^0_{(3,0)}$, $h^0_{(0,2)}$

We now return to the $k = 0$ terms, which are determined by the values of the constants $b^0_{(2,0)}$ and $b^0_{(0,2)}$. In this case we can evaluate the integral

$$h^0_{(p,q)} = \pi^5\, b^0_{(p,q)} \sum_{(\hat m,\hat n)\in\mathbb{Z}^2} \int_{0_\chi}^{V^\Lambda} \frac{dV}{V} \int_{\mathcal{F}_\Lambda} \frac{d^2\tau}{\tau_2^2}\, e^{-\pi\, V\, v\, \frac{|\hat m + \tau \hat n|^2}{\tau_2}} \tag{B.75}$$

for $(p,q)=(3,0)$ and $(p,q)=(0,2)$. We will write the integral as the sum of two terms,

$$h^0_{(p,q)} = h^{0\,(1)}_{(p,q)} - h^{0\,(2)}_{(p,q)} \tag{B.76}$$

where in $h^{0\,(1)}_{(p,q)}$ the τ integral spans the full fundamental domain, $\mathcal{F}$, whereas $h^{0\,(2)}_{(p,q)}$ subtracts the integral over the range $\Lambda^2/V \le \tau_2 \le \infty$. In the first contribution we separate the $\hat{m}=\hat{n}=0$ term, for which the τ integral simply gives the volume of the fundamental domain, $\int d^2\tau/\tau_2^2 = \pi/3$. The integral over τ in the $(\hat{m},\hat{n}) \neq (0,0)$ piece can be 'unfolded' to the infinite strip as in [7], giving

$$\begin{aligned} h^{0\,(1)}_{(p,q)} &= \pi^5\, b^0_{(p,q)} \int_{0_\chi}^{V\Lambda} \frac{dV}{V} \left(\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} + \int_0^\infty \frac{dt}{t^2} \sum_{p\neq 0} \exp(-\pi p^2 v V/\tau_2) \right) \\ &= \frac{\pi^6\, b^0_{(p,q)}}{3} \int_{0_\chi}^{V\Lambda} \frac{dV}{V} + \frac{\pi^6\, b^0_{(p,q)}}{3v} \int_{0_\chi}^{V\Lambda} \frac{dV}{V^2}\,. \end{aligned} \tag{B.77}$$

The second term in (B.76) may be evaluated by first performing a Poisson resummation on one of the integers, which gives a sum over the winding number, $\hat{n}$, and the Kaluza–Klein charge, m. The integral over τ_1 projects onto the terms with $\hat{n}\, m = 0$, giving

$$\begin{aligned} h^{0\,(2)}_{(p,q)} &= \frac{2\pi^5\, b^0_4}{v^{\frac12}} \int_{0_\chi}^{V\Lambda} \frac{dV}{V^{\frac32}} \int_{\Lambda^2/V}^\infty \frac{dt}{t^{\frac32}} \\ &+ \frac{2\pi^5\, b^0_{(p,q)}}{v^{\frac12}} \int_{0_\chi}^{V\Lambda} \frac{dV}{V^{\frac32}} \int_{\Lambda^2/V}^\infty \frac{dt}{t^{\frac32}} \sum_{q\neq 0} \left(e^{-\pi v V q^2 t} + e^{-\pi q^2 t/(vV)} \right) \\ &= \frac{\pi^6\, b^0_{(p,q)}}{3} \frac{1}{v} \int_{0_\chi}^{V\Lambda} \frac{dV}{V^2} - \frac{\pi^4\, b^0_{(p,q)}}{v} \int_{0_\chi}^{V\Lambda} \frac{dV}{V^2} \sum_{\hat{m}\neq 0} \frac{1}{\hat{m}^2}\, e^{-\pi\, \hat{m}^2\, v\, (V/\Lambda)^2} \end{aligned} \tag{B.78}$$

where we have dropped the first term of the second line since it is smaller than $\exp(-\pi v \Lambda^2)$ and the last line follows by a further Poisson resummation of the second term of the second line.

$$\begin{aligned} h^0_{(p,q)} &= h^{0\,(1)}_{(p,q)} - h^{0\,(2)}_{(p,q)} = \frac{\pi^6\, b^0_{(p,q)}}{3} \int_{0_\chi}^{V\Lambda} \frac{dV}{V}\, e^{S/V} - \frac{\pi^4\, b^0_{(p,q)}}{v} \int_{0_\chi}^{V\Lambda} \frac{dV}{V^2}\, e^{S/V} \\ &= \frac{\pi^6\, b^0_{(p,q)}}{3} \log(\chi/\Lambda^2 C'_{(p,q)}) + O(\Lambda^{-1})\,, \end{aligned} \tag{B.79}$$

where $C'_{(p,q)}$ is an undetermined function of z, but is independent of v. After substituting the values of $b^0_{(3,0)}$ and $b^0_{(0,2)}$ we find

$$h^0_{(3,0)} = \frac{386316}{143}\, \zeta(6)\, \log(\chi/\Lambda^2\, C_{(3,0)})\,, \tag{B.80}$$

$$h^0_{(0,2)} = \frac{28518}{11}\, \zeta(6)\, \log(\chi/\Lambda^2\, C_{(0,2)})\,. \tag{B.81}$$

Note that in this case the scale of the logarithm is Λ, in contrast to the earlier cases, where it was $1/R_{11}^2$ – there is a new primitive divergence. This had to be the case since these terms are the $\hat{m} = 0 = \hat{n}$ part of the $L = 2$ eleven-dimensional supergravity amplitude, which is the only part that arises in the limit $R_{11} \to \infty$, where there is a $\log \Lambda$ divergence. The more conventional dimensional regularization argument that leads to the same coefficient for the $S^6 \log S\, \mathcal{R}^4$ term is given in appendix E.2. The pole residue in (E.11) matches perfectly with the above coefficient (once the differences in the conventions used for the normalization are taken into account).

In order to compare with the string result we will write

$$\begin{aligned} h^0_{(3,0)} &= \frac{386316}{143}\,\zeta(6)\left(\log(\chi R_{11}^2/C_{(3,0)}) - \log(\Lambda^2 R_{11}^2)\right) \\ &= \frac{386316}{143}\,\zeta(6)\left(\log(\chi g_A^2/C_{(3,0)}) - \log(\Lambda^2 v)\right)\,, \end{aligned} \tag{B.82}$$

where the first term on the right-hand side combines nicely with the contributions from $h^i_{(3,0)}$ with $i \neq 0$ to reproduce the correct threshold term. The left over part is to be subtracted by a new counterterm.

B.5.3 $I^{(d=10)}_{(3,0)}$, $I^{(d=10)}_{(0,2)}$ and counterterm contributions

The values of $h^{2k}_{(3,0)}$ and $h^{2k}_{(0,2)}$ for $k = 0, \ldots, 6$ determine the solutions,

$$I^{(d=10)}_{(3,0)} = 168\pi\,\zeta(6)\,v^{\frac{3}{2}}\,\Lambda^3 + \frac{100647}{715}\,\zeta(3)\,\zeta(6) - 3465\,\zeta(6)\,\log(\chi v/\tilde{C}_{(3,0)}) \tag{B.83}$$

and

$$I^{(d=10)}_{(0,2)} = 168\pi\,\zeta(6)\,v^{\frac{3}{2}}\,\Lambda^3 + \frac{15827}{110}\zeta(3)\zeta(6) - \frac{6615}{2}\,\zeta(6)\,\log(\chi v/\tilde{C}_{(0,2)})\,. \tag{B.84}$$

The Λ^3 terms are canceled by the counter-term diagram (2.47) and replaced by finite contributions that are interpreted in the IIA string coordinates as genus-four perturbative contributions[16]. There are two distinct terms in (B.83) and (B.84) that have no power of v and are independent of Λ (they are finite terms). These correspond to genus-three IIA string contributions. The $\log(\chi v)$ term corresponds to the genus-three part of $E_{5/2}\, s^6 \log(\chi)$. The genus-one string part of this expression does not arise from two-loop supergravity diagrams considered in this paper, but it is easy to see from dimensional arguments that it should be obtained from the three-loop amplitude of eleven-dimensional supergravity. The Λ^3 terms are one-loop sub-divergences regularized by the counter-term diagram of equation (2.47)

$$I_{(3,0)} + I^{(d=10)}_{\triangleright\,(3,0)} + \delta_2 I_{(3,0)} = 210\,\zeta(8)\,v^{\frac{3}{2}} + \frac{100647}{715}\,\zeta(3)\,\zeta(6) - 3465\,\zeta(6)\,\log(\chi v/C_{(3,0)})\,, \tag{B.85}$$

and

$$I_{(0,2)} + I^{(d=10)}_{\triangleright\,(0,2)} + \delta_2 I_{(0,2)} = 210\,\zeta(8)\,v^{\frac{3}{2}} + \frac{15827}{110}\zeta(3)\zeta(6) - \frac{6615}{2}\,\zeta(6)\,\log(\chi v/C_{(0,2)})\,. \tag{B.86}$$

[16]Since at order $S^6\,\mathcal{R}^4$ the diagram regulating the one-loop sub-divergence gives a result proportional to $\sigma_6 = \sigma_2^3/4 + \sigma_3^2/3$, it is necessary that the Λ^3 coefficients for $I_{(3,0)}$ and $I_{(0,2)}$ are the same.

C. Quasi-zero mode modular functions

In this section we will evaluate the coefficient $\mathcal{E}^{(2)\,0}_{(2,0)}$ of the $\sigma_2^2\,\mathcal{R}^4$ term in (4.30) and the coefficients $\mathcal{E}^{(6)\,0}_{(3,0)}$ and $\mathcal{E}^{(6)\,0}_{(0,2)}$ of the $S^6\,\mathcal{R}^4$ terms in (4.32) and (4.33). These are the cases in which modular function in the integrand, $B_{(p,q)}(\tau) = b_{(p,q)}$, is a constant so the eigenvalue in the inhomogenious Laplace equations is zero and the source term vanishes. We will see by direct evaluation that in these cases the coefficients satisfy Laplace equations of the form

$$\Delta_\Omega \mathcal{E}^{(r)\,0}_{(p,q)} = \frac{D_{(p,q)}}{2}\,, \tag{C.1}$$

where $D_{(p,q)}$ are constants.

C.1 Evaluation of $\mathcal{E}^{(2)\,0}_{(2,0)}$

In the $(2,0)$ case we know that there is a two-loop supergravity threshold (which will be explicitly evaluated in section E). This is associated with the zero Kaluza–Klein modes in the loops, so here we will use the Kaluza–Klein basis for the sums, which means we need to evaluate

$$\mathcal{E}^{(2)\,0}_{(2,0)} = \frac{4\mathcal{V}_2^2}{\pi^2}\, I_{(2,0)\,0} = \frac{4\mathcal{V}_2^2}{\pi^2}\, b^0_{(2,0)}\, K \tag{C.2}$$

where $b^0_{(2,0)} = -13/21$ and

$$\begin{aligned} K &= \frac{\pi^3}{\mathcal{V}_2^2} \sum_{\{m_I,n_J\}\in\mathbb{Z}^4} \int_{0_\chi}^{V^\Lambda} \frac{dV}{V} \int_{\mathcal{F}_\Lambda} \frac{d^2\tau}{\tau_2^2}\, e^{-\pi\frac{G^{IJ}}{V\tau_2}[(m+\tau n)_I(m+\bar\tau n)_J]} \\ &= \frac{\pi^3}{\mathcal{V}_2^2} \sum_{\{m_I,n_J\}\in\mathbb{Z}^4} \int_{0_\chi}^{V^\Lambda} \frac{dV}{V} \int_{\mathcal{F}_\Lambda} \frac{d^2\tau}{\tau_2^2}\, e^{-\frac{\pi}{\mathcal{V}_2\Omega_2 V\tau_2}|m_1+n_1\tau+\Omega(m_2+n_2\tau)|^2 - 2\pi\frac{m_1n_2-m_2n_1}{\mathcal{V}_2 V}}\,. \end{aligned} \tag{C.3}$$

This will be analyzed by separating the integrand into sectors with different patterns of vanishing coefficients,

$$K \equiv \sum_{m_1,n_1,m_2,n_2} \hat K_{(m_1,n_1)(m_2,n_2)} = \sum_{m_1,n_1,m_2,n_2} \int_{0_\chi}^{V^\Lambda} \frac{dV}{V} \int_{\mathcal{F}_\Lambda} \frac{d^2\tau}{\tau_2^2}\, J_{(m_1,n_1)(m_2,n_2)}\,. \tag{C.4}$$

It is convenient to decompose the sums as follows,

$$\begin{aligned} &\sum_{m_1,n_1,m_2,n_2} J_{(m_1,n_1)(m_2,n_2)} \\ &\quad = J_{(0,0)(0,0)} + \sum_{(m_1,n_1)\neq(0,0)} J_{(m_1,n_1)(0,0)} + \sum_{(m_2,n_2)\neq(0,0)} \sum_{m_1,n_1} J_{(m_1,n_1)(m_2,n_2)} \\ &\quad = J_{(0,0)(0,0)} + \sum_{(p,q)} \sum_{k_1\neq 0} J_{(k_1p,k_1q)(0,0)} + \sum_{(p,q)} \sum_{k_2\neq 0} \sum_{m_1,n_1} J_{(m_2,n_2)(k_2p,k_2q)}\,, \end{aligned} \tag{C.5}$$

where p, q are relatively prime. We may now perform the 'unfolding trick', which replaces the integral of the sum over p and q over $\mathcal{F}_\Lambda$ by an integral of only the $(p,q)=(1,0)$ term over the rectangle $\mathcal{R}_\Lambda$: $\{0 \le \tau_2 \le \Lambda^2/V,\ 1/2 \le \tau_1 \le 1/2\}$. In principle, in the presence of the upper cutoff $\tau_2 \leq \Lambda^2/V$ this unfolding leads to a very complicated τ_1 and V-dependent lower cutoff on τ_2. However, as we will see, the results we need are not sensitive to the lower end of the τ_2 integral and we will set this to zero. This gives

$$\begin{aligned}
&\int_{\mathcal{F}_\Lambda} \frac{d^2\tau}{\tau_2^2} \sum_{m_1,n_1,m_2,n_2} J_{(m_1,n_1)(m_2,n_2)} \\
&\quad = \int_{\mathcal{F}_\Lambda} \frac{d^2\tau}{\tau_2^2} J_{(0,0)(0,0)} + \int_{\mathcal{R}} \frac{d^2\tau}{\tau_2^2} \left(\sum_{k_1\neq 0} J_{(k_1,0)(0,0)} + \sum_{k_2\neq 0} \sum_{m_1,n_1} J_{(m_1,n_1)(k_2,0)} \right) \\
&\quad = \int_{\mathcal{F}_\Lambda} \frac{d^2\tau}{\tau_2^2} J_{(0,0)(0,0)} + \int_{\mathcal{R}_\Lambda} \frac{d^2\tau}{\tau_2^2} \left(\sum_{(k_1,k_2)\neq(0,0)} J_{(k_1,0)(k_2,0)} + \sum_{n_1,k_2\neq 0} \sum_{m_1} J_{(m_1,n_1)(k_2,0)} \right) ,
\end{aligned} \tag{C.6}$$

This is a decomposition into the sum of singular, degenerate and non-degenerate orbits of $SL(2,\mathbb{Z})$ in the language of [7].

Consider first the $m_I = n_I = 0$ term, which contains the $\log(\chi)$ factor. In this case $J = \pi/3 + O(V/\Lambda^2)$ and the result is

$$K_{(0,0)(0,0)} = \frac{\pi^3}{\mathcal{V}_2^2} \frac{\pi}{3} \log(\chi/C\Lambda^2) + O(\mathcal{V}_2^{-1}) \,, \tag{C.7}$$

where we have only kept the leading term in the limit $\mathcal{V}_2 \to 0$, which is the part that behaves as $\mathcal{V}_2^{-2}$.

The second term in (C.6) leads to

$$\begin{aligned}
\sum_{k_1,k_2\neq 0} K_{(k_1,0)(k_2,0)} &= \frac{\pi^3}{\mathcal{V}_2^2} \int_0^{V\Lambda} \frac{dV}{V} \int_0^{\Lambda^2/V} \frac{d\tau_2}{\tau_2^2} \sum_{k_1,k_2\neq 0} \exp\left(-\frac{\pi}{\mathcal{V}_2\Omega_2 V\tau_2} |k_1 + k_2\Omega_2|^2 \right) \\
&= \frac{\pi^3}{\mathcal{V}_2^2} \int_0^{V\Lambda} dV \int_{\Lambda^{-2}}^{\infty} d\hat{y} \sum_{k_1,k_2\neq 0} \exp\left(-\frac{\pi\hat{y}}{\mathcal{V}_2\Omega_2} |k_1 + k_2\Omega_2|^2 \right) ,
\end{aligned} \tag{C.8}$$

where we have defined $\hat{y} = (V\tau_2)^{-1}$. It is easy to see that this depends linearly on Λ^2 and has an overall power of $\mathcal{V}_2^{-1}$, so it does not contribute to the term proportional to $\mathcal{V}_2^{-2}$ and can be ignored here.

The last term in (C.6) leads to

$$\sum_{k_1,k_2\neq 0} K_{(m_1,n_1)(k_2,0)} = \frac{2\pi^3}{\mathcal{V}_2^2} \sum_{n_1>0,k_2\neq 0} \sum_{m_1+0}^{n_1-1} \int_0^{V\Lambda} \frac{dV}{V} \int_{\mathcal{R}_\Lambda} \frac{d^2\tau}{\tau_2^2} \, e^{-\frac{\pi}{\mathcal{V}_2\Omega_2 V\tau_2}|m_1+n_1\tau+\Omega k_2|^2 + \frac{2\pi}{V\mathcal{V}_2} n_1 k_2} . \tag{C.9}$$

Integrating over τ_1 gives the expression

$$\begin{aligned}
\sum_{k_1,k_2\neq 0} K_{(m_1,n_1)(k_2,0)} &= \frac{\pi^3}{\mathcal{V}_2^2} (\mathcal{V}_2\Omega_2)^{\frac{1}{2}} \sum_{n_1\neq 0,k_2\neq 0} \int_0^{V\Lambda} \frac{dV}{V^{\frac{1}{2}}} \int_0^{\Lambda^2/V} d\tau_2 \tau_2^{-\frac{3}{2}} \, e^{-\frac{\pi}{\mathcal{V}_2\Omega_2 V\tau_2}(n_1^2\tau_2^2 + k_2^2\Omega_2^2)} \\
&= \frac{\pi^3}{\mathcal{V}_2^2} (\mathcal{V}_2\Omega_2)^{\frac{1}{2}} \sum_{n_1\neq 0,k_2\neq 0} \int_0^{V\Lambda} dV \int_0^{\Lambda^2} dy y^{-\frac{3}{2}} \, e^{-\frac{\pi}{\mathcal{V}_2\Omega_2}(\frac{1}{V^2} n_1^2 y + \frac{1}{y} k_2^2\Omega_2^2)} ,
\end{aligned} \tag{C.10}$$

where $y = V\tau_2$. Since each term in the sum is dominated by the V cutoff, we will perform a Poisson resummation of the integer n_1 after adding and subtracting the $n_1 = 0$ term, which is proportional to $\int dV \sim \Lambda^2$. Since we are here not keeping terms that are powers of the cutoff (since they will not have the appropriate power of $\mathcal{V}_2^{-2}$) we will drop this term. After the Poisson resummation the result is (again dropping terms that are positive powers of Λ and are therefore not of order $\mathcal{V}_2^{-2}$)

$$\begin{aligned}\sum_{k_1,k_2\neq 0} K_{(m_1,n_1)(k_2,0)} &\sim \frac{\pi^3\Omega_2}{\mathcal{V}_2}\sum_{n_1\neq 0}\sum_{\hat{k}_2\neq 0}\int_0^{V_\Lambda} dV\, V\int_0^{\Lambda^2} dy y^{-2}\, e^{-\frac{\pi\Omega_2\mathcal{V}_2 V^2}{y}(\hat{n}^1)^2-\pi\frac{\Omega_2}{\mathcal{V}_2 y}k_2^2}\\ &= \frac{2\pi^2}{\mathcal{V}_2^2}\,\zeta(2)\sum_{k_2\neq 0}\int_0^{\Lambda^2}\frac{dy}{y}\, e^{-\pi\frac{\Omega_2}{\mathcal{V}_2 y}k_2^2} = -2\frac{\pi^2}{\mathcal{V}_2^2}\,\zeta(2)\,\log(\Lambda^2\mathcal{V}_2/\Omega_2 C)\,,\end{aligned} \tag{C.11}$$

Therefore, the total contribution to $I^{(2)}_{(2,0)}$ proportional to $\mathcal{V}_2^{-2}$ (which therefore does not have a power of Λ) gives a contribution

$$K = -2\frac{\pi^2}{\mathcal{V}_2^2}\,\zeta(2)\,\log(\chi\mathcal{V}_2/\Omega_2 C)\,, \tag{C.12}$$

so that from (C.2) we have

$$\mathcal{E}^{(2)\,0}_{(2,0)} = \frac{104}{21}\,\zeta(2)\,\log(\chi\mathcal{V}_2/\Omega_2 C)\,, \tag{C.13}$$

so that

$$\Delta_\Omega\mathcal{E}^{(2)\,0}_{(2,0)} = \frac{104}{21}\,\zeta(2)\,. \tag{C.14}$$

C.2 Evaluation of $\mathcal{E}^{(6)\,0}_{(3,0)}$ and $\mathcal{E}^{(6)\,0}_{(0,2)}$

The terms of order $S^6\,\mathcal{R}^4$ will contribute to a logarithmic eleven-dimensional threshold term, which means that the zero winding number sector $\hat{m}^I = \hat{n}^J = 0$ possesses the singularity. In the winding number basis the expressions we need to evaluate are

$$\begin{aligned}\mathcal{E}^{(6)\,0}_{(3,0)} &= \frac{16}{3\pi^2}\,I_{(3,0)\,0} = \frac{16}{\pi^2}\,b^0_{(3,0)}\,\hat{K}\,,\\ \mathcal{E}^{(6)\,0}_{(0,2)} &= \frac{16}{3\pi^2}\,I_{(0,2)\,0} = \frac{16}{\pi^2}\,b^0_{(2,0)}\,\hat{K}\,.\end{aligned} \tag{C.15}$$

where

$$\hat{K} = \pi^5\sum_{\{\hat{m}_I,\hat{n}_J\}}\int_{0_\chi}^{V^\Lambda}\frac{dV}{V}\int_{\mathcal{F}_\Lambda}\frac{d^2\tau}{\tau_2^2}\, e^{-\pi\frac{VG_{IJ}}{\tau_2}\left[(\hat{m}+\tau\hat{n})^I(\hat{m}+\bar{\tau}\hat{n})^J\right]}\,. \tag{C.16}$$

We will now decompose the sums in the same manner as in (C.5) ,(C.6), writing

$$\hat{K} \equiv \sum_{\hat{m}^1,\hat{n}^1,\hat{m}^2,\hat{n}^2}\hat{K}_{(\hat{m}^1,\hat{n}^1)(\hat{m}^2,\hat{n}^2)} = \sum_{\hat{m}^1,\hat{n}^1,\hat{m}^2,\hat{n}^2}\int_{0_\chi}^{V^\Lambda}\frac{dV}{V}\int_{\mathcal{F}_\Lambda}\frac{d^2\tau}{\tau_2^2}\,\hat{J}_{(\hat{m}^1,\hat{n}^1)(\hat{m}^2,\hat{n}^2)}\,. \tag{C.17}$$

and

$$\int_{\mathcal{F}_\Lambda} \frac{d^2\tau}{\tau_2^2} \sum_{\hat m^1,\hat n^1,\hat m^2,\hat n^2} \hat J_{(\hat m^1,\hat n^1)(\hat m^2,\hat n^2)}$$

$$= \int_{\mathcal{F}_\Lambda} \frac{d^2\tau}{\tau_2^2} \hat J_{(0,0)(0,0)} + \int_{\mathcal{R}_\Lambda} \frac{d^2\tau}{\tau_2^2} \left(\sum_{(\hat k_1,\hat k_2)\neq(0,0)} \hat J_{(\hat k_1,0)(\hat k_2,0)} + \sum_{\hat n^1,\hat k_2\neq 0} \sum_{\hat m^1} \hat J_{(\hat m^1,\hat n^1)(\hat k_2,0)} \right),$$

The zero winding number term is given by

$$\hat K_{(0,0)(0,0)} = \pi^5 \frac{\pi}{3} \log(\chi/C\Lambda^2) + O(\mathcal{V}_2 - 1), \tag{C.18}$$

The second term in (C.18) leads to

$$\sum_{k_1,k_2\neq 0} \hat K_{(k_1,0)(k_2,0)} = \pi^5 \int_0^{V^\Lambda} \frac{dV}{V} \int_{0_\chi}^{\Lambda^2/V} \frac{d\tau_2}{\tau_2^2} \sum_{\hat k_1,\hat k_2\neq 0} \exp\left(-\pi \frac{\mathcal{V}_2 V}{\Omega_2\tau_2} |\hat k_1 + \hat k_2\Omega_2|^2\right)$$

$$= \pi^5 \int_{0_\chi}^{V^\Lambda} dV \int_{\Lambda^{-2}}^{\infty} d\hat y \sum_{k_1,k_2\neq 0} \exp\left(-\pi \frac{\mathcal{V}_2}{\Omega_2} V^2 \hat y |\hat k_1 + \hat k_2\Omega_2|^2\right), \tag{C.19}$$

where $\hat y = (V\tau_2)^{-1}$.

The last term in (C.18) leads to

$$\sum_{k_1,k_2\neq 0} \hat K_{(\hat m^1,\hat n^1)(k_2,0)} = \pi^5 \sum_{\hat n^1,\hat k_2\neq 0} \sum_{\hat m^1} \int_{0_\chi}^{V^\Lambda} \frac{dV}{V} \int_{\mathcal{R}_\Lambda} \frac{d^2\tau}{\tau_2^2} e^{-\frac{\pi\mathcal{V}_2 V}{\Omega_2\tau_2} |\hat m^1 + \hat n^1\tau + \Omega\hat k_2|^2 + 2\pi V\mathcal{V}_2 \hat n^1 \hat k_2}. \tag{C.20}$$

Integrating over τ_1 gives

$$\sum_{k_1,k_2\neq 0} \hat K_{(\hat m^1,\hat n^1)(\hat k_2,0)} = \pi^5 \left(\frac{\Omega_2}{\mathcal{V}_2}\right)^{\frac12} \sum_{\hat n^1\neq 0,\hat k_2\neq 0} \int_{0_\chi}^{V^\Lambda} \frac{dV}{V^{\frac32}} \int_0^{\Lambda^2/V} d\tau_2 \tau_2^{-\frac32} e^{-\frac{\pi\mathcal{V}_2 V}{\Omega_2\tau_2}((\hat n^1)^2\tau_2^2 + \hat k_2^2\Omega_2^2)}$$

$$= \pi^5 \left(\frac{\Omega_2}{\mathcal{V}_2}\right)^{\frac12} \sum_{\hat n_1\neq 0,\hat k_2\neq 0} \int \frac{dxdy}{xy^{\frac32}} e^{-\pi\mathcal{V}_2\Omega_2\hat k_2^2 x - \pi\frac{\mathcal{V}_2}{\Omega_2}(\hat n^1)^2 y}, \tag{C.21}$$

where $x = V/\tau_2$ and $y = V\tau_2$. The y integral may be performed without worrying about the cutoff and gives

$$\int_0^\infty \frac{dy}{y^{\frac32}} \sum_{\hat n^1\neq 0} e^{-\pi\frac{\mathcal{V}_2}{\Omega_2}(\hat n^1)^2 y} = \frac{\pi}{3} \left(\frac{\mathcal{V}_2}{\Omega_2}\right)^{\frac12}, \tag{C.22}$$

where we have used the analytic continuation of the Riemann zeta function to write $\hat\sum |n_1| = -1/6$. The x integral in (C.21) gives

$$\int_{0_\chi}^{\Lambda^2} \frac{dx}{x} \sum_{\hat k_2\neq 0} e^{-\pi\mathcal{V}_2\Omega_2\hat k_2^2 x} = \Sigma_1(\mathcal{V}_2\Omega_2) - \log(\chi/C\,\Lambda^2) = -\log\left(\mathcal{V}_2\,\Omega_2\,\Lambda^2/\hat C\right) - \log(\chi/\hat C\Lambda^2), \tag{C.23}$$

where we have subtracted the $\hat{k}_2 = 0$ term (proportional to $\log(\chi)$) from the sum that defines Σ_1 in (B.7) and discarded the term proportional to $S^{-\frac{1}{2}}$, which is accompanied by a factor of $\mathcal{V}_2^{-\frac{1}{2}}$.

Substituting (C.22) and (C.23) into (C.21) and combining this with (C.18), which is the other contribution that does not have a power of Λ, gives the total contribution

$$\hat{K} = \pi^5 \frac{\pi}{3} \log\left(\frac{\mathcal{V}_2 \Omega_2 \Lambda^2}{\hat{C}}\right) , \tag{C.24}$$

so that, from (C.15),

$$\begin{aligned} \mathcal{E}^{(6)\,0}_{(3,0)} &= \frac{12264}{715}\, 64\, \zeta(2)^2 \log\left(\frac{\mathcal{V}_2 \Omega_2 \Lambda^2}{\hat{C}}\right) , \\ \mathcal{E}^{(6)\,0}_{(0,2)} &= \frac{2716}{165}\, 64\, \zeta(2)^2 \log\left(\frac{\mathcal{V}_2 \Omega_2 \Lambda^2}{\hat{C}}\right) . \end{aligned} \tag{C.25}$$

Note that, as had to be the case, the $\log(\chi)$ in the zero-winding sector cancels with the effects of non-zero winding. The Laplace equations satisfied by these coefficients are

$$\Delta_\Omega \mathcal{E}^{(6)\,0}_{(3,0)} = \frac{12264}{715}\, 64\, \zeta(2)^2 , \qquad \Delta_\Omega \mathcal{E}^{(6)\,0}_{(0,2)} = \frac{2716}{165}\, 64\, \zeta(2)^2 . \tag{C.26}$$

D. Weak coupling expansion of the generalized modular functions

In the main text we found modular functions which are defined by Poisson equations in the fundamental domain, of the general form

$$[\Delta_\Omega - s(s-1)]\, \mathcal{E}(\Omega) = S(\Omega) , \qquad s \geq 0 . \tag{D.1}$$

Here we determine the perturbative part of $\mathcal{E}$ for a general source term S with a zero mode expansion given by

$$S(\Omega) = \sum_{n=0}^{N} \alpha_n \Omega_2^{n_0 - n} + S_{cusp}(\Omega_2) . \tag{D.2}$$

We assume that the polynomial part does not contain Ω_2^s or Ω_2^{1-s} (either because $s > N - 1 - n_0,\ s > n_0$ or because $\alpha_s = \alpha_{1-s} = 0$). For the relevant cases n_0 will be an integer or half-integer number. S_{cusp} is an exponentially suppressed contribution, which nevertheless will contribute to the perturbative (power-behaved) part of $\mathcal{E}$.

The general structure of the zero mode expansion of the solution $\mathcal{E}(\Omega)$ of (D.1) is the sum of the particular solution with the source term and a solution of the homogeneous equation

$$\mathcal{E}(\Omega) = \sum_{n=0}^{N} \frac{\alpha_n\, \Omega_2^{n_0 - n}}{(n_0 - n)(n_0 - n - 1) - s(s-1)} + \alpha \Omega_2^s + \beta \Omega_2^{1-s} + \mathcal{O}(\exp(-\Omega_2)) \tag{D.3}$$

The parameters α and β are integration constants which are fixed by boundary conditions. For the cases appeared in the main text, one must impose that $\alpha = 0$ because s is such that Ω_2^s is more singular than the tree-level contribution in the weak coupling limit.

D.1 General method for determining β terms

The value of β is determined as in section 5.4 of [8] by integrating over the cutoff fundamental domain for $SL(2,\mathbb{Z})$ the product of $\mathcal{E}$ with the Eisenstein series E_s –which is a solution of the homogeneous equation associated with (D.1). Then we perform the partial integrations as

$$0 = \int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2} \left([\Delta - s(s-1)]\, E_s\right)\mathcal{E} = \int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2}\, E_s\, S + \int_{\partial\mathcal{F}_L} \left(\bar{\partial}E_s\mathcal{E} - E_s\partial\mathcal{E}\right) \tag{D.4}$$

Computing the boundary term, we find

$$\begin{aligned}\int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2}\, E_s\, S(\Omega) = 2\zeta(2s)\,(1-2s)\,\beta \\ + 2\zeta(2s)\sum_{n=0}^{N} \frac{\alpha_n(n_0-n-s)}{(n_0-n)(n_0-n-1)-s(s-1)}\, L^{s+n_0-n-1} \\ + \mathcal{O}(L^{-1})\end{aligned} \tag{D.5}$$

where we have only displayed the terms that do not vanish when the cutoff $L \to \infty$. Since

$$E_s(\Omega) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Im\mathrm{m}(\gamma\cdot\Omega)^s\,, \tag{D.6}$$

the integral on the left hand side can be evaluated by unfolding the Eisenstein series E_s, giving that in the limit of large L

$$\begin{aligned}\int_0^L \frac{d\Omega_2}{\Omega_2^2}\, 2\zeta(2s)\,\Omega_2^s\, S(\Omega) = 2\zeta(2s)\,(1-2s)\,\beta \\ + 2\zeta(2s)\sum_{n=0}^{N} \frac{\alpha_n(n_0-n-s)}{(n_0-n)(n_0-n-1)-s(s-1)}\, L^{s+n_0-n-1} \\ + \mathcal{O}(L^{-1})\end{aligned} \tag{D.7}$$

The power-behaved terms in L in (D.5) cancel against the contributions from the power-behaved terms in S, so that the value of β is determined by the projection of S_{cusp} on E_s:

$$(1-2s)\,\beta = \int_0^\infty \frac{d\Omega_2}{\Omega_2^2}\, \Omega_2^s\, S_{cusp}(\Omega_2)\ . \tag{D.8}$$

D.2 β coefficients arising from a source $E_{s_1}E_{s_2}$

For the particular case of a source term given by the product of two Eisenstein series, $E_{s_1}E_{s_2}$, the β-coefficient can be given in a closed form. Substituting the well known large Ω_2 expansion of the Eisenstein series (see e.g. [8]) into the right-hand side of (D.8) we find

$$\begin{aligned}(2s-1)\,\beta^{(s)}_{(s_1,s_2)} &= \int_0^\infty \frac{dt}{t^{1-s}}\, \frac{32\pi^{s_1+s_2}}{\Gamma(s_1)\Gamma(s_2)} \sum_{n>0} \frac{\sigma_{1-2s_1}(n)\sigma_{1-2s_2}(n)}{n^{1-s_1-s_2}}\, K_{s_1-\frac{1}{2}}(2\pi n\Omega_2) K_{s_2-\frac{1}{2}}(2\pi n\Omega_2) \\ &= 4\pi^{s_1+s_2-s} \sum_{n>0} \frac{\sigma_{1-2s_1}(n)\sigma_{1-2s_2}(n)}{n^{s+1-s_1-s_2}} \\ &\quad\times \frac{\Gamma\left(\frac{s-s_1-s_2+1}{2}\right)\Gamma\left(\frac{s+s_1-s_2}{2}\right)\Gamma\left(\frac{s-s_1+s_2}{2}\right)\Gamma\left(\frac{s+s_1+s_2-1}{2}\right)}{\Gamma(s)\Gamma(s_1)\Gamma(s_2)}\,,\end{aligned} \tag{D.9}$$

where we have used the result for the integral of the product of two Bessel functions,

$$\int_0^\infty dt t^{m-1} \quad K_{n-\frac{1}{2}}(t) K_{p-\frac{1}{2}}(t) = \frac{1}{2^{3-m}\Gamma(m)} \Gamma\left(\frac{m-n-p+1}{2}\right)\Gamma\left(\frac{m+n-p}{2}\right)\Gamma\left(\frac{m-n+p}{2}\right)\Gamma\left(\frac{m+n+p-1}{2}\right) \tag{D.10}$$

Using the fact that $\sigma_a(pq)=\sigma_a(p)\sigma_a(q)$ for p and q prime and the fact that all integers can be decomposed over a product of primes, one easily establishes that

$$\sum_{n>0} \frac{\sigma_a(n)\sigma_b(n)}{n^r} = \frac{\zeta(r)\zeta(r-a)\zeta(r-b)\zeta(r-a-b)}{\zeta(2r-a-b)}\,, \tag{D.11}$$

whereby

$$\beta^{(s)}_{(s_1,s_2)} = \frac{4\pi^{s_1+s_2}}{\Gamma(s_1)\Gamma(s_2)}\,\frac{\zeta^*(s-s_1-s_2+1)\zeta^*(s+s_1-s_2)\zeta^*(s-s_1+s_2)\zeta^*(s+s_1+s_2-1)}{(2s-1)\,\zeta^*(2s)}\,, \tag{D.12}$$

with $\zeta^*(s)=\zeta(s)\Gamma(s/2)/\pi^{s/2}$.

E. Two-loop four-graviton supergravity amplitude in various dimensions

In this appendix we will consider the two-loop four-graviton amplitude of maximal supergravity in Minkowski space in nine, ten and eleven dimensions using dimensional regularization. These results make contact at various points with our discussion of eleven-dimensional supergravity compactified on a circle and on a two-torus. We follow the analysis described in [27, 28] and [29] based on a dimensional regularisation adapted to the ten and nine dimensional case. Although the ten-dimensional and eleven-dimensional results are in the literature [16] we include them here for completeness.

E.1 Ten dimensions

The amplitude in $D=10-2\epsilon$ dimensions takes the form

$$\begin{aligned} A_4^{(10-2\epsilon)} = \mathcal{R}^4 \Big((-S)^2 &\left[I^{P\,(10-2\epsilon)}(S,T) + I^{P\,(10-2\epsilon)}(S,U) + I^{NP\,(10-2\epsilon)}(S,T) + I^{NP\,(10-2\epsilon)}(S,U)\right] \\ + (-T)^2 &\left[I^{P\,(10-2\epsilon)}(T,S) + I^{P\,(10-2\epsilon)}(T,U) + I^{NP\,(10-2\epsilon)}(T,S) + I^{NP\,(10-2\epsilon)}(T,U)\right] \\ + (-U)^2 &\left[I^{P\,(10-2\epsilon)}(U,T) + I^{P\,(10-2\epsilon)}(U,S) + I^{NP\,(10-2\epsilon)}(U,T) + I^{NP\,(10-2\epsilon)}(U,S)\right] \Big)\,. \end{aligned} \tag{E.1}$$

The contributions I^P and I^{NP} are the scalar field theory double-box diagrams as described in section 2.

These integrals can be analyzed efficiently using the `Mathematica` package described in [30]. The integrals can be reduced by a repeated use of the first and second Barnes' lemma given in the appendix D of [27, 31] to reproduce the result of the appendix C of [16] The planar amplitude $I^P(S,T)$ is given by

$$I^{P\,(10-2\epsilon)}(S,T) = \frac{(-S\mu^2)^{3-2\epsilon}}{(4\pi)^{10}} \Big(-\frac{2}{7!\cdot 5!}\,\frac{4S+T}{S\,\epsilon^2} - \frac{63\,T^3 - 252\,S\,T^2 - 55\,S^2\,T + 704\,S^3}{700\cdot 9!\,S^3\,\epsilon} + \mathcal{O}(\epsilon^0)\Big)\,, \tag{E.2}$$

and the non-planar amplitude takes to form

$$I^{NP\,(10-2\epsilon)}(S,T) = \frac{(-S\mu^2)^{3-2\epsilon}}{(4\pi)^{10}} \left(\frac{7}{7!\cdot 5!}\frac{1}{\epsilon^2} + \frac{1}{30\cdot 9!}\,\frac{917S^2 + 2\,T\,U}{S^2\,\epsilon} + \mathcal{O}(\epsilon^0) \right)\,, \tag{E.3}$$

where μ is an arbitrary scale and we have made use of the on-shell condition $S+T+U=0$. The terms of $O(\epsilon^0)$ are functions of the dimensionless ratio T/S.

The $1/\epsilon^2$ pole cancels (on-shell) in the S-channel part of the amplitude, $A^{(S)}$, giving

$$\begin{aligned} & I^{P\,(10-2\epsilon)}(S,T) + I^{P\,(10-2\epsilon)}(S,U) + I^{NP\,(10-2\epsilon)}(S,T) + I^{NP\,(10-2\epsilon)}(S,U) \\ &= -\frac{1}{(4\pi)^{10}}\,\frac{50\,S^3 + 5STU}{6!^2\epsilon} + \mathcal{O}(\epsilon^0)\,. \end{aligned} \tag{E.4}$$

The ϵ pole contributes to terms proportional to $\log(-S\mu^2)$, which are not symmetric in the Mandelstam invariants. However, summing all the contributions and using the mass-shell constraint the total amplitude takes the symmetric form

$$A_4^{(10-2\epsilon)} = \frac{13}{(4\pi)^{10}\,466560\,\epsilon}\,\sigma_2\,\sigma_3\mathcal{R}^4 + \mathcal{O}(\epsilon^0)\,. \tag{E.5}$$

The striking cancelation of the $1/\epsilon^2$ pole separately in the S, T and U-channels, corresponds to the cancelation of the terms proportional to $\log^2(-S)$, $\log^2(-T)$ and $\log^2(-U)$ in the circle compactification of the term of order $S^5\,\mathcal{R}^4$ analyzed in appendix B.4 and of the $(\log\Omega_2)^2$ dependence in the $\mathcal{E}_{(1,1)}(\Omega)$ coefficient of equation (4.31).

Under a rescaling $(S,T,U)\to\Omega_2\,(S,T,U)$ the amplitude behaves as

$$\Omega_2^{-5}\,A_4^{(10-2\epsilon)} \to A_4^{(10-2\epsilon)} + \frac{13}{(4\pi)^{10}\,233280}\,\sigma_2\sigma_3\,\log\Omega_2\,. \tag{E.6}$$

The $\log\Omega_2$ term (properly normalized) should be related to with the $\log\Omega_2$ term in (4.31).

E.1.1 The triangle counterterm diagram

The ten-dimensional two-loop amplitude receives an extra contribution from the triangle diagram with the one-loop counterterm $A_\triangleright$ of equation (2.47). In the dimensional regularisation scheme the triangle loop amplitude in $D = 10-2\epsilon$ reads

$$\begin{aligned} I_\triangleright^{(10-2\epsilon)}(S) &= -\frac{1}{(2\pi)^{10}}\int \frac{d^{10-2\epsilon}\ell}{\ell^2(\ell-p_1)^2(\ell-p_1-p_2)^2} \\ &= -\frac{\Gamma(-2+\epsilon)}{2\,(3-\epsilon)^2(4\pi)^5}\,(-S)^{2-2\epsilon} \end{aligned} \tag{E.7}$$

This pole in ϵ leads to a contribution of order $S^4\,\log(-S)$. This is interpreted in the $\mathcal{S}^1$ compactification to type IIA string theory as a genus-two threshold contribution. Unitarity requires the presence of this term since the discontinuity across the threshold is the product of the genus-one $\mathcal{R}^4$ term and the leading contribution from the tree-level amplitude. To pick out the coefficient we can perform a rescaling of the Mandelstam variables $(S,T,U)\to\Omega_2\,(S,T,U)$ this amplitude behaves as

$$\Omega_2^{-4}\,A_\triangleright^{(10-2\epsilon)} \to A_\triangleright^{(10-2\epsilon)} + \frac{1}{(4\pi)^5\,72}\,\sigma_2^2\,\log\Omega_2\,, \tag{E.8}$$

which corresponds to the two-loop $S^4\,\log\Omega_2$ term that would be contained in a modular function describing the terms at order $S^4\,\mathcal{R}^4$ in ten dimensions.

E.2 Eleven dimensions

In similar fashion the planar and non-planar scalar field theory integrals that contribute to the two-loop amplitude in $D = 11 - 2\epsilon$ dimensions are found to be given by

$$I^{P\,(11-2\epsilon)}(S,T) \,=\, (-S)^{-2\epsilon}\,\frac{\pi}{(4\pi)^{11}}\,\frac{1}{\epsilon}\,\frac{2100\,S^4 - 880\,S^3\,T + 215\,S^2\,T^2 + 30\,S\,T^3 + 12\,T^4}{9451728000} + \mathcal{O}(\epsilon^0) \tag{E.9}$$

$$I^{NP\,(11-2\epsilon)}(S,T) = (-S)^{-2\epsilon}\,\frac{\pi}{(4\pi)^{11}}\,\frac{1}{\epsilon}\,\frac{40383\,S^4 - 1138\ S^2\,T\,U + 144\ U^2\,T^2}{79394515200} + \mathcal{O}(\epsilon^0) \tag{E.10}$$

The resulting amplitude is

$$A_4^{(11-2\epsilon)} = \frac{\pi}{(4\pi)^{11}}\,\frac{1}{\epsilon}\,\frac{1971\,\sigma_2^3 + 2522\,\sigma_3^2}{5003856000} + \mathcal{O}(\epsilon^0)\ . \tag{E.11}$$

in agreement with the results of [16].

The $1/\epsilon$ pole in (E.11) gives a $S^6\,\mathcal{R}^4$ term in the amplitude in eleven-dimensional Minkowski space that should correspond to the zero-winding sector of the two-loop amplitude at order $S^6\,\mathcal{R}^4$ in the compactified theory. In section B.5.1 we determined the zero-winding coefficients $h^0_{(3,0)}$ and $h^0_{(0,2)}$ (see (B.80) and (B.81)). Referring back to the normalizations in equation (2.55), we see that these zero-winding terms agree precisely with (E.11) with $1/\epsilon$ replaced by $\log \Lambda^2/C$ (where C is an undetermined constant).

E.3 Nine dimensions

The planar and non-planar diagrams in nine dimensions also have logarithmic branch points. These arise from the coalescence of the square root branch points of the individual one-loop integrals. For completeness, we note the result of an analysis of the the planar and non-planar diagrams in $D = 9 - 2\epsilon$ dimensions analogous to the one of the preceding sub-sections, giving

$$\begin{aligned} I^{P,(9-2\epsilon)}(S,T) &= S^{2-2\epsilon}\,\frac{\pi}{(4\pi)^9}\,\frac{-45\,S^2 + 18\,S\,T + 2\,T^2}{399168\,\epsilon\,S^2} + \mathcal{O}(\epsilon^0) \\ I^{NP,(9-2\epsilon)}(S,T,U) &= s^{2-2\epsilon}\,\frac{-\pi}{(4\pi)^9\,332640}\,\frac{75\,S^2 + 2\,T\,U}{S^2\,\epsilon} + \mathcal{O}(\epsilon^0)\ . \end{aligned} \tag{E.12}$$

Collecting all the contributions one finds in agreement with [16]

$$A_4^{(9-2\epsilon)} = -\frac{1}{8\epsilon}\,\frac{1}{(4\pi)^9}\,\frac{13\pi}{9072}\,\sigma_2^2\,\mathcal{R}^4 + \mathcal{O}(\epsilon^0)\ . \tag{E.13}$$

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