

The three-point function of planar quadrangulations

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The three-point function of planar quadrangulations

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Abstract. We compute the generating function of random planar quadrangulations with three marked vertices at prescribed pairwise distances. In the scaling limit of large quadrangulations, this discrete three-point function converges to a simple universal scaling function, which is the continuous three-point function of pure 2D quantum gravity. We give explicit expressions for this universal three-point function in both the grand-canonical and canonical ensembles. Various limiting regimes are studied when some of the distances become large or small. By considering the case where the marked vertices are aligned, we also obtain the probability law for the number of geodesic points, namely vertices that lie on a geodesic path between two given vertices, and at prescribed distances from these vertices.

Keywords: rigorous results in statistical mechanics, exact results, random graphs, networks

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1. Introduction**1.1. The problem**

Maps are fundamental objects of discrete mathematics, usually defined as cellular embeddings of graphs into surfaces. They raise many interesting combinatorial and probabilistic questions. The understanding of their statistical properties is also relevant to physics, where random maps provide discrete models for fluctuating surfaces, for instance in the context of two-dimensional quantum gravity [1]. Many results have been obtained over the years for the *enumeration* of various families of maps, including maps carrying spins or particles. Besides Tutte's original approach using some recursive decomposition of the maps [2], several new techniques of enumeration have been introduced over the years, using for instance random matrix integrals [3, 4] or, more recently, bijections with decorated trees like 'blossom trees' [5]–[10] or 'well-labeled trees' [11]–[14]. So far, most enumeration results dealt however with global properties of the maps, consisting in simply counting these maps or computing *global* correlations functions for some observables, obtained by averaging over the position of the points where these observables are measured (see [4]). To fully understand the structure of random maps, it is however necessary to have access to *local* correlations, in which one controls the distance between the points

where the measurement takes place [15, 16]. Very little is known at this time about local correlations but many results in this direction should be within reach by a proper use of the above-mentioned bijection with well-labeled trees. Indeed, in this approach, the coding of maps into trees makes explicit reference to the graph distance to some origin vertex, thus keeping track in the enumeration of a number of graph distances between vertices. First progress was achieved in [17], where the so-called canonical *two-point function* of quadrangulations was computed. This function gives the average number of pairs of vertices at prescribed distance from each other in the ensemble of quadrangulations (i.e. maps with only tetravalent faces) with a fixed number of faces. In particular, for large quadrangulations and at large distances, a sensible scaling limit can be obtained where the two-point function converges to some universal scaling function, which is the two-point function of pure 2D quantum gravity [15, 17]. Further recent progress was made in [18], where the question of geodesic paths between two vertices in random quadrangulations was addressed, and a number of results were obtained on the actual dependence of the statistics of geodesics on the distance between their endpoints.

In this paper, we address the question of the *three-point function* of planar quadrangulations, consisting in enumerating triply pointed quadrangulations, i.e. quadrangulations with three marked vertices *at prescribed pairwise distances*. Our main result is an explicit formula for the generating function of such triply pointed quadrangulations counted with a weight g per face. Our approach relies on an extension by Miermont of the bijection with well-labeled trees which allows us to treat the case of multiply pointed quadrangulations [19] via a coding by more general well-labeled maps, which can then be enumerated. Considering the scaling limit of large quadrangulations and large distances, we then give explicit expressions for the universal scaling form of this three-point function, in both grand-canonical and canonical ensembles. In the language of 2D quantum gravity, this constitutes the continuous three-point function for the universality class of the so-called pure gravity, with the planar topology. We discuss here its main properties.

The paper is organized as follows. In section 1.2, we start by briefly recalling some known results about the two-point function of quadrangulations, and its continuum limit. We then present in section 2 our results for the three-point function. We give in section 2.1 the explicit formula for the generating function $G(d_{12}, d_{23}, d_{31}; g)$ with a weight g per face of planar quadrangulations with three distinct vertices, distinguished as 1, 2 and 3, at prescribed pairwise distances d_{12} , d_{23} and d_{31} . We then derive in section 2.2 its universal scaling form both in the grand-canonical ensemble with a fixed ‘cosmological constant’ and in the canonical ensemble of maps with a fixed, but large number n of faces. The grand-canonical three-point function is obtained by letting g tend to its critical value $1/12$ and considering distances scaling as $(1/12 - g)^{-1/4}$. As for the canonical three-point function, it is obtained by letting the size n of the quadrangulations tend to infinity, with distances scaling as $n^{1/4}$. Various limits of the canonical three-point function are discussed in section 2.3, corresponding to the cases when one or several of the (rescaled) distances become large or small. Sections 3 and 4 are devoted to the actual derivation of our main formula for $G(d_{12}, d_{23}, d_{31}; g)$. We first recall in section 3.1 the Miermont bijection between, on the one hand, multiply pointed quadrangulations with sources and delays and, on the other hand, well-labeled maps. We show in section 3.2 how to use this bijection in the case of triply pointed maps and how to keep track of all pairwise distances by a proper choice of delays. This allows us to reduce our enumeration problem to that of counting

particular well-labeled maps with three faces, with a number of constraints on their labels. The generating function for such maps is finally obtained in section 4, where we make use of three main building blocks: the already known generating function for well-labeled trees, recalled in section 4.1, a new generating function for properly weighted Motzkin paths describing chains of such trees corresponding to well-labeled maps with two faces, made explicit in section 4.2 and another new generating function for so-called Y-diagrams appearing in a decomposition of well-labeled maps with three faces, made explicit in section 4.3, where we finally establish our formula for $G(d_{12}, d_{23}, d_{31}; g)$. An alternative derivation of the above generating function for the weighted Motzkin path is presented in the appendix. We discuss in section 4.4 a number of applications of our formula corresponding to the so-called (non-universal) ‘local limit’ of large quadrangulations, where we keep the distances finite but let the number n of faces tend to infinity. We discuss in detail the case where one of the three marked vertices is a *geodesic point*, i.e. lies on a geodesic path between the two others. We give in particular the probability law for the number of geodesic points at fixed distances from two marked vertices. We finally discuss possible extensions of our results and conclude in section 5.

1.2. The two-point function of quadrangulations

Before we present our results for the three-point function, it is useful to recall briefly some known facts about the two-point function of quadrangulations. This function is obtained by considering configurations of doubly pointed planar quadrangulations, i.e. planar quadrangulations with *two marked distinct and distinguished vertices*. As is customary, every configuration is counted with a weight g per face and some inverse symmetry factor, equal to the inverse of the order of its automorphism group. The generating function $G(i; g)$ of such weighted doubly pointed quadrangulations where the two marked points are at distance i from each other was found to be [13]

$$G(i; g) = \begin{cases} \log \left(\frac{R_i}{R_{i-1}} \right) & \text{for } i \geq 2 \\ \log R_1 & \text{for } i = 1, \end{cases} \quad (1.1)$$

with R_i given by [17]

$$R_i = R \frac{[i]_x [i+3]_x}{[i+1]_x [i+2]_x}, \quad i \geq 1. \quad (1.2)$$

Here and throughout the paper, we use the shorthand notation

$$[i]_x \equiv \frac{1-x^i}{1-x}. \quad (1.3)$$

The quantities R and x in (1.2) are power series in g determined by

$$g = \frac{x+1+1/x}{(x+4+1/x)^2}, \quad R = \frac{x+4+1/x}{x+1+1/x}, \quad (1.4)$$

so we have

$$R = \frac{1 - \sqrt{1 - 12g}}{6g}, \quad (1.5)$$

$$x = \frac{1 - 24g - \sqrt{1 - 12g} + \sqrt{6}\sqrt{72g^2 + 6g\sqrt{1 - 12g} - 1}}{2(6g + \sqrt{1 - 12g} - 1)}.$$

The quantity $G(i; g)$ constitutes the discrete ‘grand-canonical’ two-point function of quadrangulations and corresponds to an ensemble of quadrangulations with a varying number of faces, governed by g . We may instead decide to fix the number n of faces of the quadrangulation. The corresponding partition function can be obtained via a contour integral in g around 0 as

$$G(i; g)|_{g^n} = \frac{1}{2i\pi} \oint \frac{dg}{g^{n+1}} G(i; g). \tag{1.6}$$

This quantity constitutes the discrete ‘canonical’ two-point function of quadrangulations.

A sensible continuum limit is reached by letting g approach its critical value $1/12$ and jointly taking a large distance i , with the following scaling:

$$g = \frac{1}{12} (1 - \Lambda\epsilon), \quad i = D\epsilon^{-1/4}, \tag{1.7}$$

where $\epsilon \rightarrow 0$ and where Λ is the so-called ‘cosmological constant’. In this limit, we have

$$\log\left(\frac{R_i}{2}\right) \sim \epsilon^{1/2} \mathcal{F}\left(D; \sqrt{3/2}\Lambda^{1/4}\right) \quad \text{with } \mathcal{F}(D; \alpha) = -\frac{2\alpha^2}{3} \left(1 + \frac{3}{\sinh^2(\alpha D)}\right), \tag{1.8}$$

and so

$$G(i; g) \sim \epsilon^{3/4} \mathcal{G}\left(D; \sqrt{3/2}\Lambda^{1/4}\right) \quad \text{with } \mathcal{G}(D; \alpha) = \partial_D \mathcal{F}(D; \alpha) = 4\alpha^3 \frac{\cosh(\alpha D)}{\sinh^3(\alpha D)}. \tag{1.9}$$

This constitutes the continuous two-point function in the grand-canonical ensemble with cosmological constant Λ [15, 16]. Again, we may instead consider the canonical ensemble where the number of faces n of the quadrangulations is now fixed to a large value. A sensible limit is reached by letting n tend to infinity with the distance between the marked vertices scaling as $n^{1/4}$. The integral (1.6) translates at large n via a saddle point estimate into an integral over *real values* of some parameter ξ , upon setting

$$g = \frac{1}{12} \left(1 + \frac{\xi^2}{n}\right), \quad i = Dn^{1/4}. \tag{1.10}$$

Using equation (1.9) with $\epsilon = 1/n$ and $\Lambda = -\xi^2$, and performing a proper normalization of $G(i; g)|_{g^n}$ by the number of doubly pointed quadrangulations with size n , we then obtain a normalized probability density

$$\rho(D) = \frac{2}{i\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi \xi e^{-\xi^2} \mathcal{G}(D; \sqrt{-3i\xi/2}). \tag{1.11}$$

This constitutes the continuous canonical two-point function: the quantity $\rho(D) dD$ measures the infinitesimal probability, in the ensemble of large doubly pointed quadrangulations, that the two marked vertices be at a rescaled distance in the range $[D, D + dD]$. Its value (1.11) is expected to be universal for a large class of maps forming the universality class of so-called pure gravity. Finally, another quantity of interest is the (cumulative) distribution function

$$\Phi(D) = \frac{2}{i\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi \xi e^{-\xi^2} \mathcal{F}(D; \sqrt{-3i\xi/2}), \tag{1.12}$$

giving the probability that the distance is less than D . The functions $\rho(D)$ and $\Phi(D)$ are represented in figure 1. For small D , we have $\rho(D) \sim (3/7)D^3$.

The three-point function of planar quadrangulations

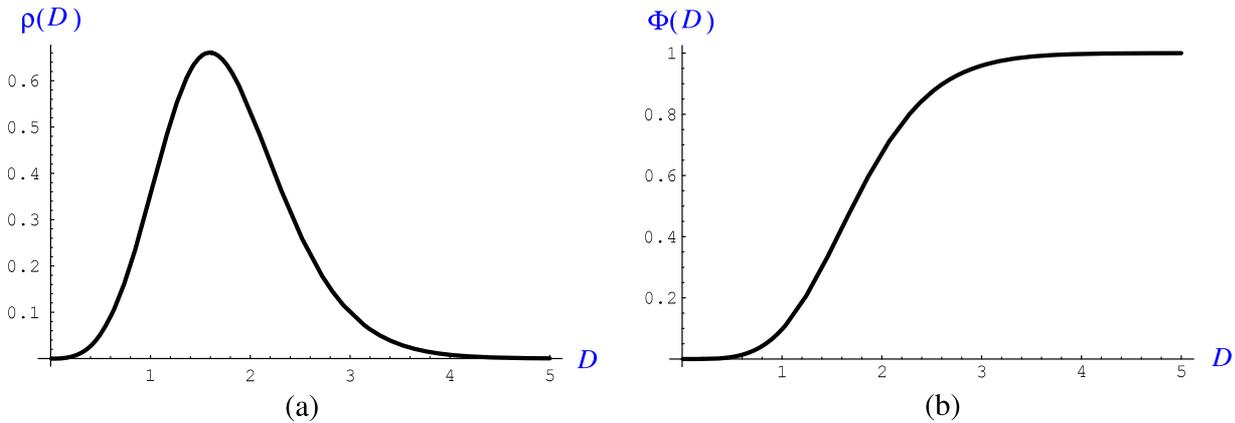


Figure 1. The universal canonical two-point function, given by (a) the probability density function $\rho(D)$, and (b) the corresponding distribution function $\Phi(D)$.

2. The three-point function

We now come to our main subject, namely the three-point function of quadrangulations. In this section, we present our main result, which is an explicit formula for the generating function of planar quadrangulations with three marked vertices at prescribed pairwise distances. We shall concentrate here on the properties of this three-point function, leaving its precise derivation to sections 3 and 4 below. An important application is the continuum scaling limit in which the three-point function takes a particularly simple universal form.

2.1. The three-point function of quadrangulations

We consider planar quadrangulations with *three distinct vertices* distinguished as 1, 2 and 3, and we denote by d_{12} , d_{23} and d_{31} their respective pairwise graph distances. Prescribing these distances, let $G(d_{12}, d_{23}, d_{31}; g)$ denote the generating function for the number of such triply pointed quadrangulations, counted with a weight g per face. Note that there are no symmetry factors for triply pointed quadrangulations as the latter have no symmetries. The function $G(d_{12}, d_{23}, d_{31}; g)$ is defined for strictly positive integer values of d_{12} , d_{23} and d_{31} that furthermore satisfy:

- the triangular inequalities: $d_{12} \leq d_{23} + d_{31}$ and its cyclic permutations;
- the condition that $d_{12} + d_{23} + d_{31}$ is even.

This latter condition stems from the fact that every planar quadrangulation is bipartite; hence the length of any cycle on the map is even. These conditions are best expressed through the parametrization

$$\begin{aligned} d_{12} &= s + t, \\ d_{23} &= t + u, \\ d_{31} &= u + s, \end{aligned} \tag{2.1}$$

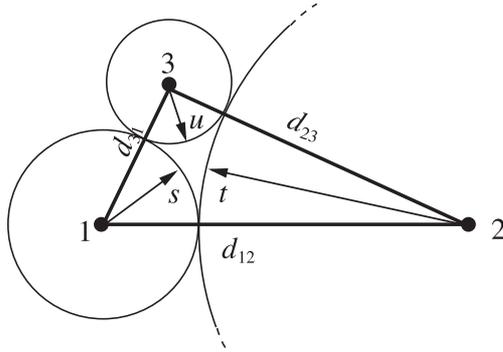


Figure 2. A schematic picture of the relations (2.1) and (2.2).

easily inverted into

$$\begin{aligned} s &= \frac{d_{12} - d_{23} + d_{31}}{2}, \\ t &= \frac{d_{12} + d_{23} - d_{31}}{2}, \\ u &= \frac{-d_{12} + d_{23} + d_{31}}{2}. \end{aligned} \tag{2.2}$$

With this parametrization, the above conditions are equivalent to requiring that s, t, u be non-negative integers and that at most one of them may vanish. An intuitive picture of this parametrization is as follows (see figure 2): drawing a triangle 123 in the plane with sides d_{12}, d_{23} and d_{31} , the quantities s, t and u can be viewed as the radii of three pairwise tangent circles centered at the points 1, 2, 3. Note that having one of the parameters vanishing, say u , corresponds in this picture to having 3 lying on the segment 12. On the quadrangulation, this simply means that 3 lies precisely on a geodesic path between 1 and 2, so that $d_{12} = d_{23} + d_{31}$. We shall say in this case that the three points are *aligned* and that 3 is a *geodesic point* between 1 and 2.

We find the following expression for the three-point function:

$$G(d_{12}, d_{23}, d_{31}; g) = \Delta_s \Delta_t \Delta_u F(s, t, u; g), \tag{2.3}$$

where Δ_s denotes the finite difference operator acting on an arbitrary function $f(s)$ as

$$\Delta_s f(s) \equiv f(s) - f(s - 1), \tag{2.4}$$

and similarly for Δ_t and Δ_u , while $F(s, t, u; g)$ has the explicit form

$$\begin{aligned} &F(s, t, u; g) \\ &= \frac{[3]_x ([s + 1]_x [t + 1]_x [u + 1]_x [s + t + u + 3]_x)^2}{[1]_x^3 [s + t + 1]_x [s + t + 3]_x [t + u + 1]_x [t + u + 3]_x [u + s + 1]_x [u + s + 3]_x}. \end{aligned} \tag{2.5}$$

As before, we use the notation (1.3) with a parameter x related to g by equation (1.4), or more explicitly by equation (1.5). Note that $F(s, t, u; g)$ is manifestly symmetric in s, t, u , which is consistent with G being symmetric in the distances, as expected.

Relations (2.3) and (2.5) hold *a priori* only in the allowed range of distances, namely when s, t, u are non-negative integers and no more than one of them may vanish. In the case of aligned points, say when u vanishes, relation (2.3) reduces to

$$G(s+t, t, s; g) = \Delta_s \Delta_t F(s, t, 0; g) \tag{2.6}$$

as $F(s, t, -1; g)$ vanishes identically, while relation (2.5) reduces to

$$F(s, t, 0; g) = \frac{[3]_x [s+1]_x [t+1]_x [s+t+3]_x}{[1]_x [s+3]_x [t+3]_x [s+t+1]_x}. \tag{2.7}$$

It is convenient to slightly extend the domain of definition of G and F as follows: when exactly two of the parameters, say t and u , vanish, corresponding to 2 and 3 being identical, equations (2.3) and (2.5) give a result $G(s, 0, s; g) = 0$, which can still be considered as the correct result as we imposed that the three marked vertices have to be distinct. Finally, when $s = t = u = 0$, the above formulas yield the somewhat conventional value $G(0, 0, 0; g) = 1$. We shall therefore take the convention that $G(d, d, 0; g) = G(d, 0, d; g) = G(0, d, d; g) = \delta_{d,0}$. With this convention, equation (2.3) may be inverted into

$$F(s, t, u; g) = \sum_{s'=0}^s \sum_{t'=0}^t \sum_{u'=0}^u G(s'+t', t'+u', u'+s'; g), \tag{2.8}$$

so $F(s, t, u; g) - 1$ can be interpreted as the generating function of quadrangulations with three *distinct* marked vertices at distances $d_{12} = s' + t'$, $d_{23} = t' + u'$ and $d_{31} = u' + s'$ such that $s' \leq s$, $t' \leq t$ and $u' \leq u$. In particular, taking the limit $s, t, u \rightarrow \infty$ amounts to relaxing these constraints, i.e. considering three arbitrary distinct vertices. We have

$$\begin{aligned} \lim_{s,t,u \rightarrow \infty} F(s, t, u; g) &= \frac{1+x+x^2}{(1-x)^2} = \frac{1}{2} (1 + (1-12g)^{-1/2}) \\ &= 1 + \sum_{n \geq 1} \frac{3^n}{2} \binom{2n}{n} g^n, \end{aligned} \tag{2.9}$$

where we identify the coefficient of g^n as the number of triply pointed quadrangulations with n faces and three distinct marked vertices.

2.2. Universal continuum limit

The above formula for the three-point function is easily translated in the continuum limit, i.e. when g approaches its critical value $1/12$ and all distances are large, with the following scaling:

$$g = \frac{1}{12} (1 - \Lambda \epsilon), \quad d_{12} = D_{12} \epsilon^{-1/4}, \quad d_{23} = D_{23} \epsilon^{-1/4}, \quad d_{31} = D_{31} \epsilon^{-1/4}, \tag{2.10}$$

with $\epsilon \rightarrow 0$. Similarly, the parameters s, t, u defined in equation (2.1) have the scaling

$$s = S \epsilon^{-1/4}, \quad t = T \epsilon^{-1/4}, \quad u = U \epsilon^{-1/4}, \tag{2.11}$$

where S, T, U are related to D_{12}, D_{23} and D_{31} via relations similar to equations (2.1) and (2.2), namely

$$\begin{aligned} D_{12} &= S + T, & S &= \frac{D_{12} - D_{23} + D_{31}}{2}, \\ D_{23} &= T + U, & T &= \frac{D_{12} + D_{23} - D_{31}}{2}, \\ D_{31} &= U + S, & U &= \frac{-D_{12} + D_{23} + D_{31}}{2}. \end{aligned} \tag{2.12}$$

In this limit, we obtain

$$\begin{aligned} G(d_{12}, d_{23}, d_{31}; g) &\sim \epsilon^{1/4} 2\mathcal{G}(D_{12}, D_{23}, D_{31}; \sqrt{3/2}\Lambda^{1/4}), \\ F(s, t, u; g) &\sim \epsilon^{-1/2} \mathcal{F}(S, T, U; \sqrt{3/2}\Lambda^{1/4}), \end{aligned} \tag{2.13}$$

with finite scaling functions \mathcal{G} and \mathcal{F} depending on the rescaled distances and on the ‘cosmological constant’ Λ , encoded for later convenience in the parameter

$$\alpha = \sqrt{3/2}\Lambda^{1/4}. \tag{2.14}$$

Note the factor of 2 in the first line of equation (2.13) which we introduced to compensate for this relation holding only for discrete distances satisfying the parity condition that $d_{12} + d_{23} + d_{31}$ be even, while G should be taken as zero otherwise. The factor 2 ensures *a posteriori* that $\mathcal{G} = (2\mathcal{G} + 0)/2$ is a correct measure of G on average. Relation (2.3) becomes

$$\mathcal{G}(D_{12}, D_{23}, D_{31}; \alpha) = \frac{1}{2} \partial_S \partial_T \partial_U \mathcal{F}(S, T, U; \alpha), \tag{2.15}$$

while equation (2.5) translates immediately into

$$\mathcal{F}(S, T, U; \alpha) = \frac{3}{\alpha^2} \left(\frac{\sinh(\alpha(S + T + U)) \sinh(\alpha S) \sinh(\alpha T) \sinh(\alpha U)}{\sinh(\alpha(S + T)) \sinh(\alpha(T + U)) \sinh(\alpha(U + S))} \right)^2. \tag{2.16}$$

The above formulas (2.15) and (2.16) give the continuous three-point function in the grand-canonical ensemble with a varying size of the quadrangulations and a fixed cosmological constant Λ . This constitutes the grand-canonical three-point function of pure 2D quantum gravity.

If we wish instead to work in the canonical ensemble where the number of faces n of the quadrangulations is fixed to a large value, and the distances are scaled as $n^{1/4}$, we can use the above formulas with $\epsilon = 1/n$, and a varying cosmological constant $\Lambda = -\xi^2$. As seen for the two-point function, the coefficient of g^n in G or F is obtained by a contour integration in g which, asymptotically at large n , yields, via a saddle point estimate, an integral over real values of ξ , namely,

$$\begin{aligned} G(d_{12}, d_{23}, d_{31}; g)|_{g^n} &\sim \frac{12^n}{i\pi n} \int_{-\infty}^{\infty} d\xi \xi e^{-\xi^2} n^{-1/4} 2\mathcal{G}(D_{12}, D_{23}, D_{31}; \sqrt{-3i\xi/2}), \\ F(s, t, u; g)|_{g^n} &\sim \frac{12^n}{i\pi n} \int_{-\infty}^{\infty} d\xi \xi e^{-\xi^2} n^{1/2} \mathcal{F}(S, T, U; \sqrt{-3i\xi/2}), \end{aligned} \tag{2.17}$$

where we take the determination $\sqrt{-i\tau} = e^{-\text{sgn}(\tau)i\pi/4}\sqrt{|\tau|}$ for τ real. Normalizing these asymptotic values, we find the canonical three-point function

$$\rho(D_{12}, D_{23}, D_{31}) = \frac{2}{i\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi \xi e^{-\xi^2} \mathcal{G}(D_{12}, D_{23}, D_{31}; \sqrt{-3i\xi/2}), \quad (2.18)$$

where $\rho(D_{12}, D_{23}, D_{31}) dD_{12} dD_{23} dD_{31}$ is the infinitesimal probability that the pairwise rescaled distances between the three marked points in the ensemble of triply pointed random quadrangulations of fixed large size be respectively in the ranges $[D_{12}, D_{12} + dD_{12}]$, $[D_{23}, D_{23} + dD_{23}]$, $[D_{31}, D_{31} + dD_{31}]$. Equation (2.18) constitutes the canonical three-point function of pure 2D quantum gravity for the planar topology.

Equation (2.18) was obtained by dividing the first line of equation (2.17) by the number of triply pointed quadrangulations of size n , as given by (2.9):

$$\frac{3^n}{2} \binom{2n}{n} \sim \frac{12^n}{2\sqrt{\pi}n^{1/2}}, \quad (2.19)$$

and multiplying by $n^{3/4}/2$ which is the number of allowed triplets of discrete distances (d_{12}, d_{23}, d_{31}) falling into the above range of rescaled distances (note the factor $1/2$ coming from the parity of $d_{12} + d_{23} + d_{31}$). Similarly, the second line of equation (2.17) translates into the integrated three-point function

$$\Phi(S, T, U) = \frac{2}{i\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi \xi e^{-\xi^2} \mathcal{F}(S, T, U; \sqrt{-3i\xi/2}), \quad (2.20)$$

corresponding to the probability that the three marked points be at distances $S' + T'$, $T' + U'$ and $U' + S'$ with $S' \leq S$, $T' \leq T$ and $U' \leq U$. The functions Φ and ρ are related through

$$\rho(D_{12}, D_{23}, D_{31}) = \frac{1}{2} \partial_S \partial_T \partial_U \Phi(S, T, U), \quad (2.21)$$

or conversely

$$\Phi(S, T, U) = \int_0^S dS' \int_0^T dT' \int_0^U dU' 2\rho(S' + T', T' + U', U' + S'), \quad (2.22)$$

where the factor 2 may be alternatively understood as the Jacobian in the change of variables from (D_{12}, D_{23}, D_{31}) to (S, T, U) . Finally, equation (2.9) translates into $\lim_{S,T,U \rightarrow \infty} \Phi(S, T, U) = 1$, so we find the normalization

$$\int_{\mathcal{D}} dD_{12} dD_{23} dD_{31} \rho(D_{12}, D_{23}, D_{31}) = 1, \quad (2.23)$$

as it should. Here \mathcal{D} denotes the domain of positive real values of D_{12} , D_{23} and D_{31} satisfying the triangular inequalities.

Let us now discuss in more detail the properties of the above continuous grand-canonical or canonical three-point functions. As a preliminary exercise, let us see how to recover the two-point function from the value of the three-point function upon integrating over the position of one of the points, say 3, keeping the distance D_{12} fixed to a constant

value. We have

$$\begin{aligned}
 & \int_{\mathcal{D}_{12}} dD_{23} dD_{31} \mathcal{G}(D_{12}, D_{23}, D_{31}; \alpha) \\
 &= \int_0^\infty dS \int_0^\infty dT \int_0^\infty dU \partial_S \partial_T \partial_U \mathcal{F}(S, T, U; \alpha) \delta(S + T - D_{12}) \\
 &= \int_0^\infty dS \int_0^\infty dT \lim_{U \rightarrow \infty} (\partial_S \partial_T \mathcal{F}(S, T, U; \alpha)) \delta(S + T - D_{12}) \\
 &= \int_0^{D_{12}} dS \, 18 \left(\frac{\sinh(\alpha S) \sinh(\alpha(D_{12} - S))}{\sinh^2(\alpha D_{12})} \right)^2 \\
 &= \frac{9}{8 \sinh^4(\alpha D_{12})} \left(4D_{12} + 2D_{12} \cosh(2\alpha D_{12}) - 3 \frac{\sinh(2\alpha D_{12})}{\alpha} \right). \tag{2.24}
 \end{aligned}$$

Here \mathcal{D}_{12} is a shorthand notation for the domain of allowed distances D_{23}, D_{31} at a fixed value of D_{12} . Comparing this result with the value (1.9) for the continuous two-point function, we see that

$$\int_{\mathcal{D}_{12}} dD_{23} dD_{31} \mathcal{G}(D_{12}, D_{23}, D_{31}; \alpha) = -\frac{9}{16\alpha^3} \partial_\alpha \mathcal{G}(D_{12}; \alpha). \tag{2.25}$$

Taking α as in equation (2.14) for the grand-canonical ensemble, we have $-9/(16\alpha^3)\partial_\alpha = -\partial_\Lambda$ and we thus find the expected result that the integral of the grand-canonical three-point function over the position of one of the points reproduces precisely the grand-canonical two-point function for the remaining two points, up to the action of the trivial operator $-\partial_\Lambda$ corresponding precisely to the marking of the third point with no prescription on its position.

Similarly in the canonical ensemble, taking $\alpha = \sqrt{-3i\xi/2}$, we have $-9/(16\alpha^3)\partial_\alpha = 1/(2\xi)\partial_\xi$ so that

$$\begin{aligned}
 \int_{\mathcal{D}_{12}} dD_{23} dD_{31} \rho(D_{12}, D_{23}, D_{31}) &= \frac{2}{i\sqrt{\pi}} \int_{-\infty}^\infty d\xi \xi e^{-\xi^2} \frac{1}{2\xi} \partial_\xi \mathcal{G}(D_{12}; \sqrt{-3i\xi/2}) \\
 &= \rho(D_{12}) \tag{2.26}
 \end{aligned}$$

upon integrating by parts over ξ , with $\rho(D_{12})$ as in equation (1.11). As expected, we recover the canonical two-point function as the marginal probability density obtained by integrating the canonical three-point function over the position of one of the points.

In order to visualize the three-point function and in view of the above result, it is convenient to first fix one of the distances, say D_{12} , to a constant value and to plot the *conditional probability density*:

$$\rho(D_{23}, D_{31} | D_{12}) \equiv \frac{\rho(D_{12}, D_{23}, D_{31})}{\rho(D_{12})}. \tag{2.27}$$

The quantity $\rho(D_{23}, D_{31} | D_{12}) dD_{23} dD_{31}$ measures the infinitesimal conditional probability that point 3 be at distances from points 1 and 2 respectively in the ranges $[D_{23}, D_{23} + dD_{23}]$ and $[D_{31}, D_{31} + dD_{31}]$, given that the distance between 1 and 2 is D_{12} . This conditional probability density may then be plotted for various values of D_{12} , as illustrated in figure 3, where a number of properties of the three-point function can be observed.

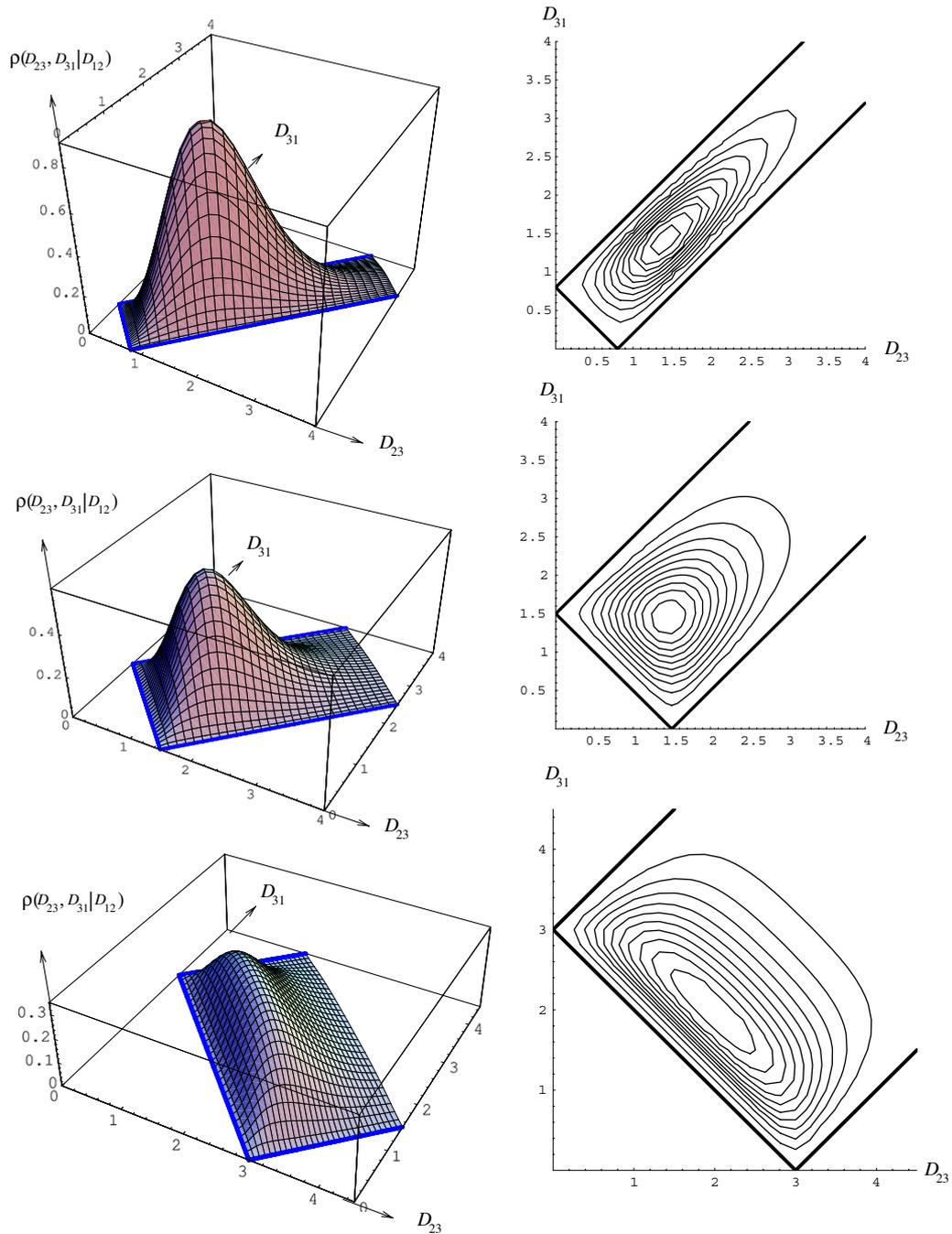


Figure 3. Left: plots of the conditional probability density $\rho(D_{23}, D_{31}|D_{12})$ for $D_{12} = 0.8, 1.5$ and 3.0 , from top to bottom. Right: the corresponding contour plots, where the thick lines indicate the boundary of the allowed domain \mathcal{D}_{12} for distances, as determined by the triangular inequalities. For the first (small enough) two values of D_{12} , the density is maximal for $D_{23} = D_{31} \sim 1.5$. For the larger value $D_{12} = 3.0$, the density becomes elongated and squeezed along the boundary line $D_{23} + D_{31} = D_{12}$.

First, the three-point function vanishes at the boundary of the domain \mathcal{D}_{12} , namely when the three points are aligned. It also vanishes when the distances from point 3 to points 1 and 2 tend to infinity. The probability density is maximal when D_{23} and D_{31} are both equal to a particular value D_{\max} , depending on D_{12} . For small enough D_{12} (less than ~ 3 or so), this preferred value is roughly independent of D_{12} and comparable to the maximum of the two-point function $\rho(D)$, namely $D_{\max} \sim 1.5$. For larger D_{12} , the triangular inequalities impose that $D_{\max} \geq D_{12}/2$ so the value of D_{\max} must increase. In this case, the density profile becomes squeezed and elongated along the boundary line $D_{23} + D_{31} = D_{12}$ of the domain \mathcal{D}_{12} with, as we shall see below, a width of order $D_{12}^{-1/3}$ at large D_{12} .

2.3. Limiting behaviors

The conditional probability density $\rho(D_{23}, D_{31}|D_{12})$ has interesting limiting behaviors whenever D_{12} becomes small or large. For $D_{12} \rightarrow 0$, taking $S = \sigma D_{12}$ and $T = \tau D_{12}$, we find that

$$\begin{aligned} \mathcal{G}((\sigma + \tau)D_{12}, \tau D_{12} + U, \sigma D_{12} + U; \alpha) &= \frac{1}{2D_{12}^2} \partial_\sigma \partial_\tau \partial_U \mathcal{F}(\sigma D_{12}, \tau D_{12}, U; \alpha) \\ &= \frac{9D_{12}^2}{2} \frac{\sigma^2 \tau^2 (\sigma^2 + 4\sigma\tau + \tau^2)}{(\sigma + \tau)^4} \mathcal{G}(U; \alpha) + \mathcal{O}(D_{12}^3), \end{aligned} \tag{2.28}$$

where $\mathcal{G}(U; \alpha)$ is the two-point function, as given by equation (1.9). Taking $\sigma + \tau = 1$ and using the small D_{12} behavior $\rho(D_{12}) \sim (3/7)D_{12}^3$, we deduce that

$$\rho((1 - \sigma)D_{12} + U, \sigma D_{12} + U|D_{12}) \sim \frac{1}{2D_{12}} \times \rho(U) \times 21\sigma^2(1 - \sigma)^2(1 + 2\sigma(1 - \sigma)) \tag{2.29}$$

in terms of the canonical two-point function $\rho(U)$. Writing $U = (D_{23} + D_{31} - D_{12})/2$ and $\sigma = (1 + \omega)/2$ with $\omega = (D_{31} - D_{23})/D_{12}$, this equation may alternatively be written as

$$\begin{aligned} \rho(D_{23}, D_{31}|D_{12}) &\sim \frac{1}{D_{12}} \times \rho\left(\frac{D_{23} + D_{31} - D_{12}}{2}\right) \times \psi\left(\frac{D_{31} - D_{23}}{D_{12}}\right), \\ \text{with } \psi(\omega) &\equiv \frac{21}{64}(1 - \omega^2)^2(3 - \omega^2), \end{aligned} \tag{2.30}$$

involving a new function $\psi(\omega)$ properly normalized to 1 when ω varies from -1 to 1 . The conditional probability density $\rho(D_{23}, D_{31}|D_{12})$ therefore factorizes in the small D_{12} limit into the product of the canonical two-point function in the ‘longitudinal direction’ (corresponding to varying values of $D_{23} + D_{31}$) and the above probability density ψ in the ‘transverse direction’ (corresponding to varying $D_{31} - D_{23}$). This property is illustrated in figure 4 for a value $D_{12} = 0.05$.

Let us now consider the case $D_{12} \rightarrow \infty$, and more precisely the limit when *both* S and T tend simultaneously to infinity (with $D_{12} = S + T$). In this limit, we have

$$\begin{aligned} \mathcal{G}(D_{12}, (D_{12} - S) + U, S + U; \alpha) \\ \sim 24\alpha e^{-\alpha(2D_{12} + 5U)} \sinh^2(\alpha U) (3 \cosh(\alpha U) + 19 \sinh(\alpha U)), \end{aligned} \tag{2.31}$$

while the grand-canonical two-point function behaves as

$$\mathcal{G}(D_{12}; \alpha) \sim 16\alpha^3 e^{-2\alpha D_{12}}. \tag{2.32}$$

The three-point function of planar quadrangulations

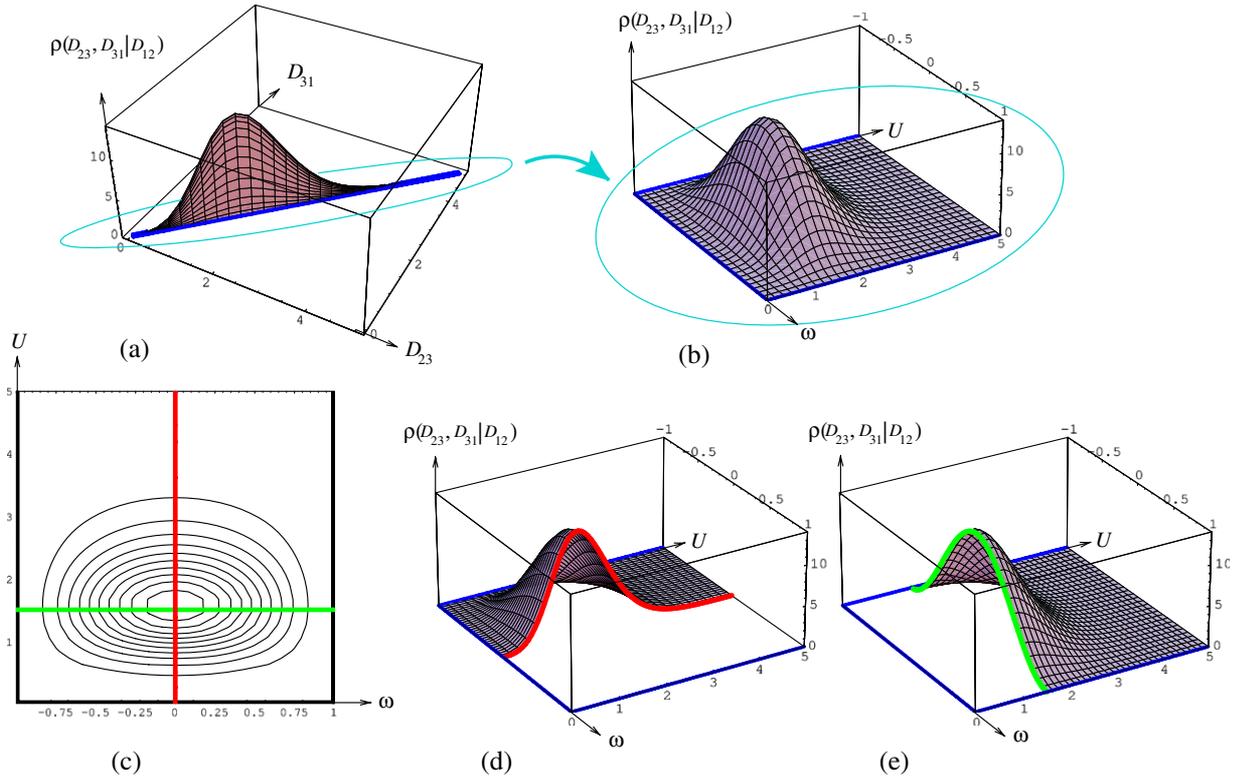


Figure 4. The conditional probability density $\rho(D_{23}, D_{31}|D_{12})$ for a small value of D_{12} , here $D_{12} = 0.05$ (a). The same density (b) and its contour plot (c), now expressed as a function of the longitudinal variable $U = (D_{23} + D_{31} - D_{12})/2$ and a *rescaled* transverse variable $\omega = (D_{31} - D_{23})/D_{12}$. As is apparent on taking longitudinal and transverse cut views along the thick lines in (c), the conditional probability factorizes into the product of the two-point function $\rho(U)$ (red curve in (d)) and the probability density $\psi(\omega)$ (green curve in (e)).

We deduce that

$$\rho(D_{12} - S + U, S + U | D_{12}) \sim \frac{\int_{-\infty}^{\infty} d\xi \xi e^{-\xi^2} 24\alpha e^{-\alpha(2D_{12}+5U)} \sinh^2(\alpha U) (3 \cosh(\alpha U) + 19 \sinh(\alpha U))}{\int_{-\infty}^{\infty} d\xi \xi e^{-\xi^2} 16\alpha^3 e^{-2\alpha D_{12}}}, \quad (2.33)$$

with $\alpha = \sqrt{-3i\xi}/2$. Note that this result is independent of S . Both integrals are dominated at large D_{12} by their saddle point at $\xi = (3D_{12}^2)^{1/3}i/2$, leading to

$$\rho(D_{23}, D_{31}|D_{12}) \sim \frac{1}{2D_{12}} \times (9D_{12})^{1/3} \varphi \left((9D_{12})^{1/3} \frac{D_{23} + D_{31} - D_{12}}{2} \right), \quad (2.34)$$

with $\varphi(\nu) \equiv \frac{4}{3} \sinh(\nu/2)^2 (11 e^{-2\nu} - 8 e^{-3\nu})$,

involving a new function $\varphi(\nu)$ properly normalized to 1 when ν varies from 0 to ∞ . In the limit of large D_{12} , the conditional probability density $\rho(D_{23}, D_{31}|D_{12})$ therefore becomes elongated and squeezed along the line $D_{23} + D_{31} = D_{12}$, with a profile given by the above

The three-point function of planar quadrangulations

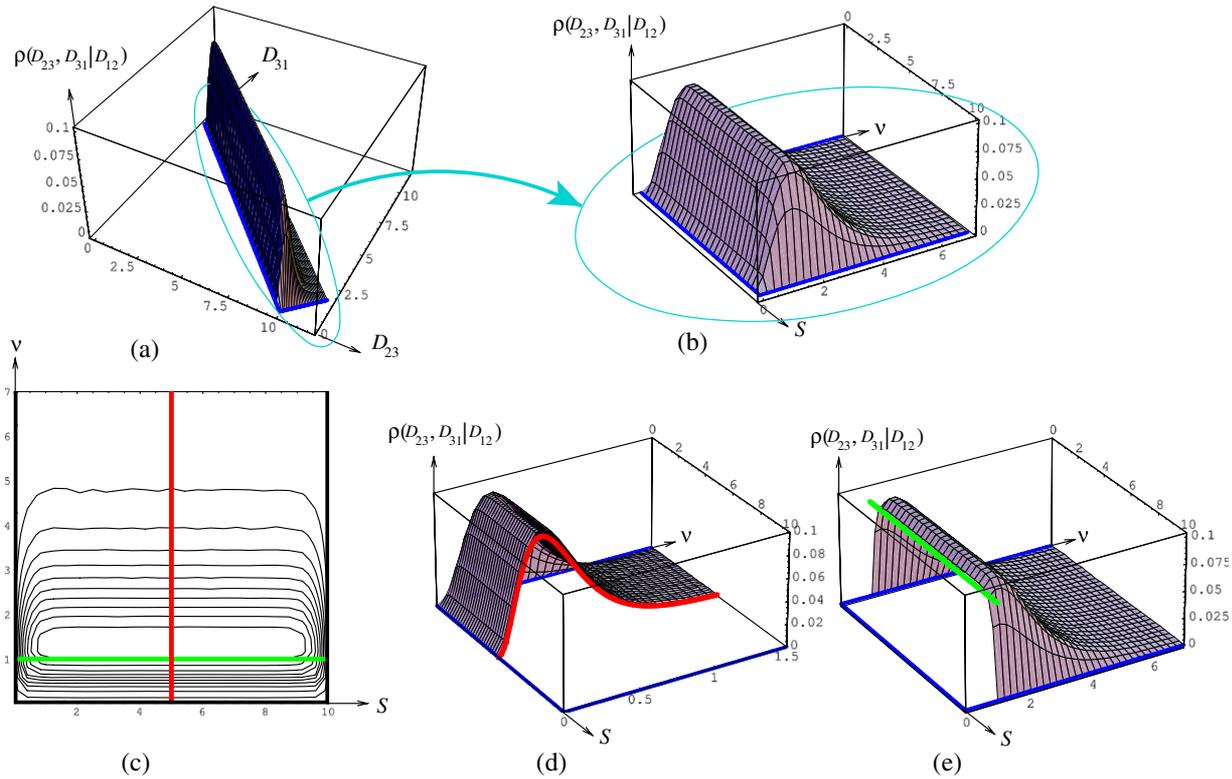


Figure 5. The conditional probability density $\rho(D_{23}, D_{31}|D_{12})$ for a large value of D_{12} , here $D_{12} = 10$ (a). The same density (b) and its contour plot (c), now expressed as a function of the transverse variable $S = (D_{12} + D_{31} - D_{23})/2$ and a *rescaled* longitudinal variable $\nu = (9D_{12})^{1/3}(D_{23} + D_{31} - D_{12})/2$. As is apparent on taking longitudinal and transverse cut views along the thick lines in (c), the conditional probability factorizes into the product of the probability density $\varphi(\nu)$ (red curve in (d)) and a uniform density in the S direction (green curve in (e)).

probability density φ in the longitudinal direction and a uniform probability density in the transverse direction. This property is illustrated in figure 5 for a value $D_{12} = 10$. Note that the extension in the longitudinal direction is of order $D_{12}^{-1/3}$, as announced.

To end this section, let us finally discuss the value of the three-point function whenever all distances become small or large. For S, T and U small (and of the same order), we find the limiting behavior

$$\Phi(S, T, U) \sim \frac{9}{28} \frac{(STU(S + T + U))^3 (S^2 + T^2 + U^2 + ST + TU + US)}{(S + T)^2 (T + U)^2 (U + S)^2}, \quad (2.35)$$

which is a homogeneous function of degree 8 of its arguments. Applying equation (2.21), we deduce that $\rho(D_{12}, D_{23}, D_{31})$ is, at small distances, a homogeneous function of degree 5 of its arguments. Note that for a manifold of fractal dimension d_F , we expect a degree $(d_F - 1) + (d_F - 2) = 2d_F - 3$ as the choice of point 2 at a fixed distance from point 1 will select a manifold of dimension $d_F - 1$ and the choice of point 3 at fixed distances from points 1 and 2 will then select a manifold of dimension $d_F - 2$. The value 5 above

is therefore compatible with the known value $d_F = 4$ for the fractal dimension of large quadrangulations.

Finally, when D_{12} , D_{23} and D_{31} are large and of the same order, we have

$$\mathcal{G}(D_{12}, D_{23}, D_{31}; \alpha) \sim 66\alpha e^{-\alpha(D_{12}+D_{23}+D_{31})}, \quad (2.36)$$

and we find the asymptotic behavior

$$\rho(D_{12}, D_{23}, D_{31}) \sim \frac{99}{\sqrt{6}}(D_{12} + D_{23} + D_{31}) e^{-(3/4)^{5/3}}(D_{12} + D_{23} + D_{31})^{4/3}. \quad (2.37)$$

3. Triply pointed quadrangulations and well-labeled maps

Let us now come to the derivation of our main result, as given by equations (2.3) and (2.5). For this derivation, we shall rely on a new bijection discovered recently by Miermont between multiply pointed quadrangulations, i.e. quadrangulations with a number, say p , of marked vertices, and *well-labeled maps* with p faces, as will be defined below. This bijection extends a well-known bijection obtained by Schaeffer [11, 12] between pointed quadrangulations and well-labeled trees (or g -trees) corresponding to the case $p = 1$. In this section, we first recall the Miermont bijection and show how to use it to treat the case of planar triply pointed quadrangulations with prescribed pairwise distances between the three marked vertices. We then exploit these results in section 4 to obtain explicit formulas for various generating functions, leading eventually to expressions (2.3) and (2.5).

3.1. The Miermont bijection

The Miermont bijection works as follows. We start with a *bipartite* quadrangulation of genus h with n faces, together with a marked p -tuple of *distinct* vertices, hereafter denoted $1, 2, \dots, p$ and referred to as the *sources* of the quadrangulation. We shall denote by d_{ij} , $i, j = 1, \dots, p$, the distance in the quadrangulation between the sources i and j . For the construction below to work, we must impose that $d_{ij} \geq 2$ for all $i \neq j$, i.e. no two marked vertices are immediate neighbors on the map.

With each source i , we furthermore associate an integer τ_i , hereafter referred to as the *delay* of the source i , and which will act as a penalty when measuring distances from this source. Delays are subject to the following constraints:

$$\begin{aligned} |\tau_i - \tau_j| &< d_{ij}, & 1 \leq i \neq j \leq p, \\ \tau_i - \tau_j + d_{ij} &\text{ is even,} & 1 \leq i, j \leq p. \end{aligned} \quad (3.1)$$

Note that delays satisfying these constraints can always be found, for instance by taking $\tau_i = 0$ (respectively 1) for all sources belonging to the white (respectively black) sublattice of the naturally colored bipartite quadrangulation.

Given a quadrangulation with sources and delays as above, we may associate with each vertex v of the quadrangulation a label $\ell(v)$ defined as

$$\ell(v) = \min_{j=1, \dots, p} (\tau_j + d_j(v)), \quad (3.2)$$

where $d_j(v)$ denotes the distance in the quadrangulation from the vertex v to the source j . The label of any vertex is therefore the minimal value for all sources of a ‘delayed

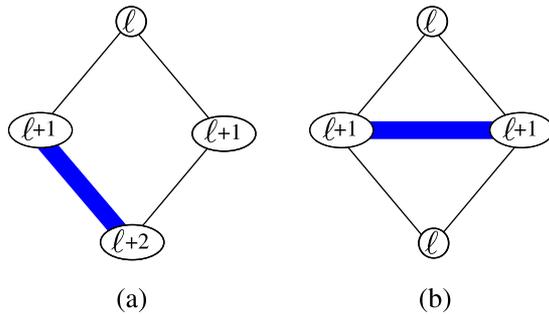


Figure 6. The two possible types of faces according to the labels of their incident vertices: (a) simple faces and (b) confluent faces. With each face, we associate an edge (thick line) as shown.

distance' equal, for each source, to the actual graph distance to this source incremented by the delay of the source. The first condition in (3.1) ensures that the label of the source i is τ_i (with the minimum in (3.2) reached only for $j = i$). Clearly, the labels of two adjacent vertices cannot differ by more than 1. Moreover, the parity of $\tau_j + d_j(v)$ necessarily changes between neighbors on a bipartite map and, from the second condition in (3.1), this parity is independent of j . This implies that the parity of $\ell(v)$ must also change between neighbors, leading to the crucial property

$$|\ell(v) - \ell(v')| = 1, \tag{3.3}$$

for any pair of adjacent vertices v and v' . The faces of the quadrangulation are therefore of two types: simple faces with a cyclic sequence of labels around the face of the form $\ell, \ell + 1, \ell + 2, \ell + 1$ and confluent faces with a sequence of the form $\ell, \ell + 1, \ell, \ell + 1$. As in Schaeffer's original construction, we associate with each of the n faces of the (now labeled) quadrangulation an edge as follows: for each simple face, we select the edge of type $\ell + 2 \rightarrow \ell + 1$ encountered clockwise around the face (see figure 6(a)); for each confluent face, we draw a new edge between the two vertices labeled $\ell + 1$ (see figure 6(b)).

It was shown by Miermont that the graph spanned by these n edges is a map that connects all the original vertices but the sources, which become isolated in the process (see figure 7 for an example). This map has moreover the same genus h as the original quadrangulation, and has p faces, each of them enclosing exactly one of the original sources and being referred to as face $1, 2, \dots, p$ accordingly. Keeping the labels on the vertices of the map, we see that, by construction, these labels vary by at most 1 along any edge of the map. Maps with integer labels satisfying this latter constraint will be called *well-labeled* maps. Finally, it can be shown that, for all the vertices incident to face i in the well-labeled map, the minimum in (3.2) is reached for $j = i$ (and possibly for some other value of j). In particular, all these vertices have a label larger than or equal to $\tau_i + 1$, and this value is reached by at least one of them, namely,

$$\min_{v \text{ incident to face } i} \ell(v) = \tau_i + 1. \tag{3.4}$$

It was shown that the construction above provides a bijection between, on the one hand, quadrangulations of genus h with n faces, p distinguished sources and *prescribed*

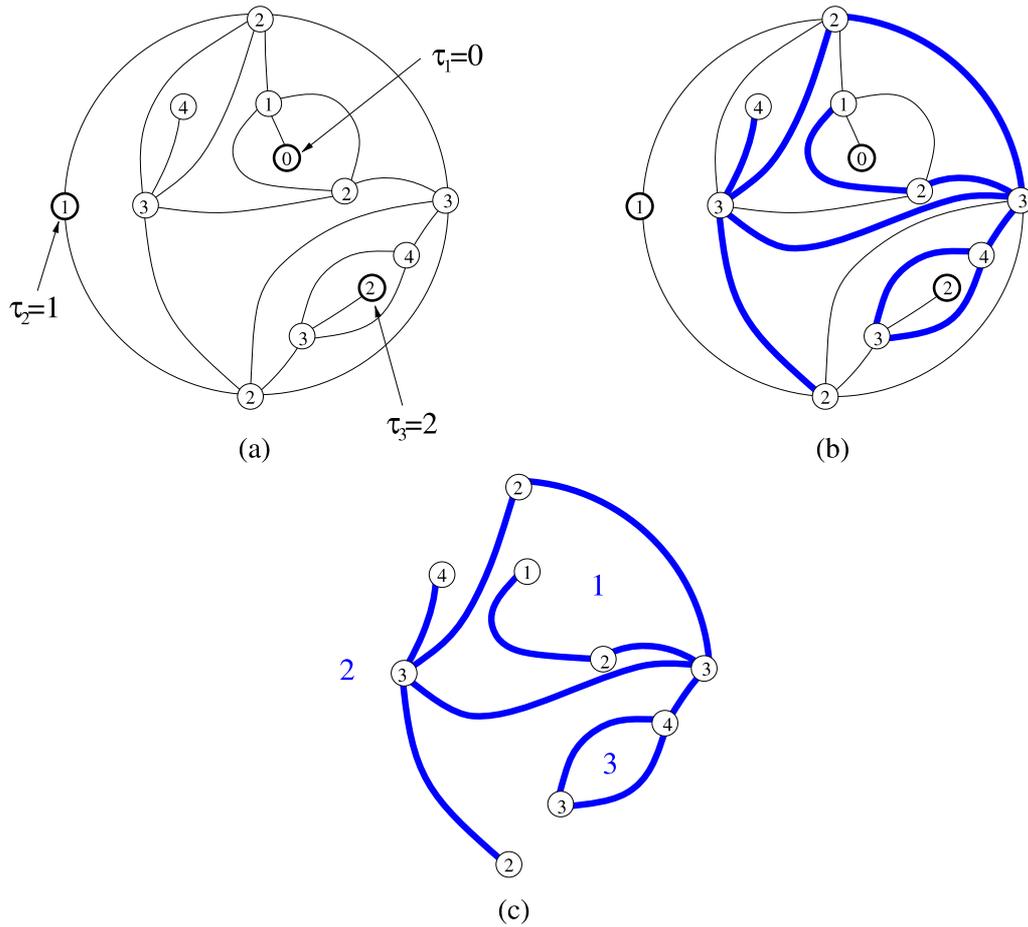


Figure 7. An example (a) of triply pointed quadrangulations where the marked vertices are indicated by thick circles. Taking delays $\tau_1 = 0$, $\tau_2 = 1$ and $\tau_3 = 2$ as indicated, each vertex v receives its label $\ell(v)$ as defined by (3.2). Applying (b) the rules of figure 6 on all the labeled faces results in a well-labeled map (c) with three distinguished faces $i = 1, 2, 3$. As apparent in (c), the minimal label among vertices incident to face i is $\tau_i + 1$.

delays $\{\tau_i\}$ satisfying (3.1), and on the other hand, well-labeled maps of genus h with n edges and p distinguished faces whose labels satisfy (3.4).

To recover the original quadrangulation from the well-labeled map, one can proceed as follows: for each face i of the map, we visit all its incident corners successively by going counterclockwise around the face. Calling $\tau_i + 1$ the minimal label of vertices incident to face i , we then connect by an arch each corner with label $\ell > \tau_i + 1$ to the first corner labeled $\ell - 1$ encountered counterclockwise around the face (see figure 8 for an example). Corners labeled $\tau_i + 1$ are finally connected by arches to an extra vertex labeled τ_i at the center of the face i . The set of all arches forms the edges of the original quadrangulation and the added vertices correspond to its sources.

An important property of the above coding of multiply pointed quadrangulations by well-labeled maps is that it keeps track of a number of distances in the original map. Indeed, as already mentioned, all the vertices incident to face i in the well-labeled map

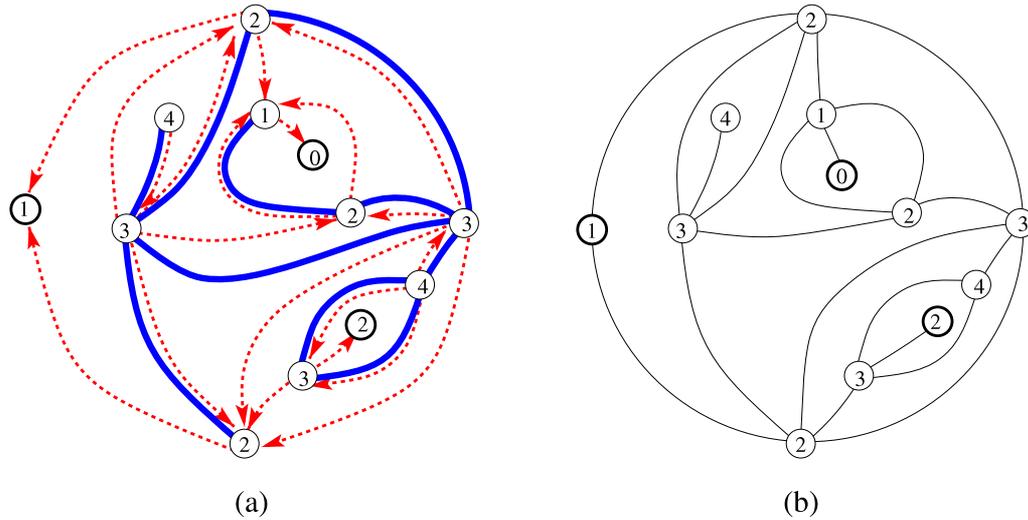


Figure 8. The inverse construction leading back from a well-labeled map, here with $p = 3$ faces, to a multiply (here triply) pointed quadrangulation. As explained in the text, calling $\tau_i + 1$ the minimal label of vertices incident to face i , an extra vertex (thick circle) with label τ_i is added at the center of each face i . Each corner with label ℓ is connected (a) by an arch (dashed line) to its successor with label $\ell - 1$. The set of these arches reconstructs the edges of the quadrangulation (b), while the added vertices form the sources of the quadrangulation.

correspond to vertices of the original quadrangulation for which the minimum in (3.2) is reached for $j = i$. This ensures the crucial property

$$d_i(v) = \ell(v) - \tau_i \quad \text{for } v \text{ incident to face } i. \tag{3.5}$$

Finally, the above bijection holds for prescribed values of the delays but one may as well consider all *pairs made of a quadrangulation with p marked vertices and of a set of delays* satisfying (3.1). In that case, in order to keep a finite number of configurations of delays, we must impose an additional condition, say for instance $\min_{i=1,\dots,p} \tau_i = 0$. The resulting configurations are clearly in bijection with well-labeled maps with p faces and arbitrary integer labels such that the minimal label is 1 and the delay τ_i associated with each source i is recovered via $\tau_i = \min_{v \text{ incident to face } i} \ell(v) - 1$.

3.2. Application to triply pointed quadrangulations

Let us now apply the above results to the case of planar quadrangulations with three distinct marked vertices, distinguished as 1, 2 and 3. Note first that a planar quadrangulation is automatically bipartite, as required by the above construction. We shall distinguish two cases of marked vertices:

- (i) the generic case where the three marked vertices are not aligned;
- (ii) the particular case where these three vertices are aligned.

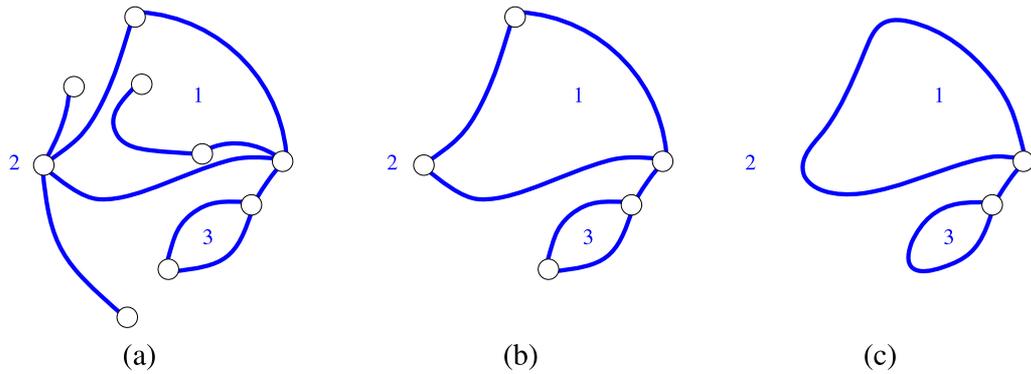


Figure 9. A planar map with three faces (a), its skeleton map (b) and its backbone map (c).

Let us first concentrate on case (i), leaving the discussion of case (ii) to the end of this section. Note first that, since the three points are not aligned, their pairwise distances d_{ij} , $1 \leq i \neq j \leq 3$, are all larger than or equal to 2. Indeed, if at least one of these distances, say d_{12} , was equal to 1, i.e. if 1 and 2 were immediate neighbors, then, from the triangular inequalities, we would have $|d_{31} - d_{23}| \leq 1$ and, from the bipartite nature of the map, this would imply $d_{31} = d_{23} + 1$ or $d_{23} = d_{31} + 1$. In the first case, 2 would be on a geodesic path between 1 and 3 while in the second case, 1 would be on a geodesic path between 2 and 3. In both cases, the three vertices would be aligned.

Having $d_{ij} \geq 2$ for all $1 \leq i \neq j \leq 3$, we may therefore apply Miermont's construction with $p = 3$. The resulting well-labeled map is now a planar map with three faces that are distinguished. Such maps can be classified as follows: it is convenient to view the map as made of a *skeleton* map decorated by a number of attached tree components (see figure 9). The skeleton map is obtained by iteratively erasing all the edges of the map which are incident to a univalent vertex until no univalent vertex is left. The suppressed edges form tree components which, once attached to the skeleton, reproduce the original map. The skeletons themselves can be classified according to their *backbone* map, which is the map obtained by erasing all the bivalent vertices of the skeleton and merging their incident edges into a single edge (see figure 9). Clearly, by construction, the backbones are planar maps with three (distinguished) faces and no univalent or bivalent vertex. There are only a finite number of such maps, all displayed in figure 10. This allows us to classify all well-labeled maps obtained in the Miermont bijection according to their underlying backbone.

The above classification holds for arbitrary planar maps with three (distinguished) faces and makes no reference to labels or delays. For arbitrary fixed values of the delays, the maps in the Miermont bijection must be equipped with labels satisfying (3.4) and one recovers all quadrangulations with three marked vertices at pairwise distances compatible with these delays by considering all possible well-labeled maps with such labels. This requires us in particular to consider all possible backbones in the above classification.

The key point of our derivation that will allow us to keep track of all pairwise distances is to impose an extra condition relating the delays at the sources to the pairwise distances between these sources. In other words, we *choose particular values for the delays* related

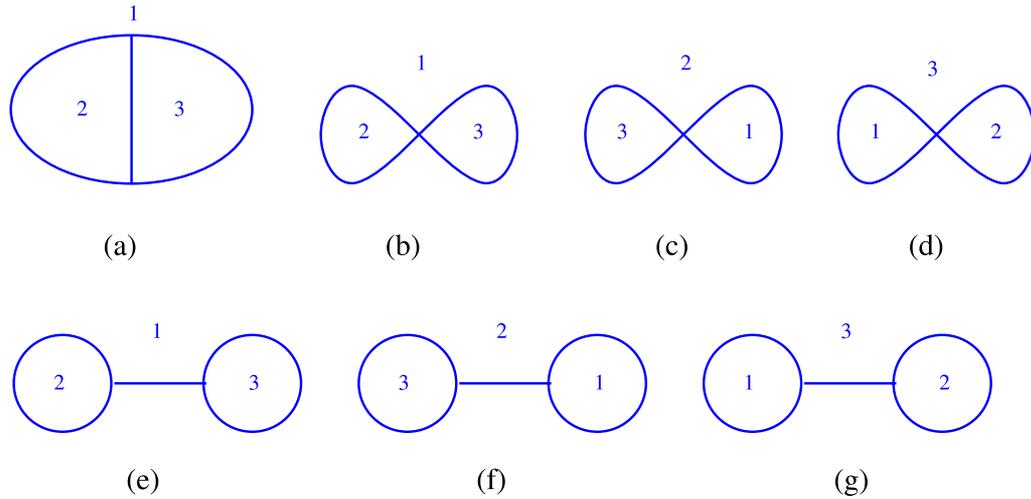


Figure 10. The seven possible backbones of planar maps with three faces distinguished as 1, 2 and 3. It will prove useful to view backbones of type (b), (c) and (d) as degenerate limits of the backbone (a) when one of its three edges is shrunk to a point.

to the pairwise distances of three marked vertices on the quadrangulation. This particular choice of delays will impose extra conditions on the associated well-labeled maps that will restrict the possible choice of backbone and imply additional constraints for the labels. More precisely, let us choose for the delays τ_1 , τ_2 and τ_3 the values $-s$, $-t$, $-u$ with s, t, u as in equation (2.2), namely,

$$\begin{aligned} \tau_1 = -s &= \frac{-d_{12} + d_{23} - d_{31}}{2}, \\ \tau_2 = -t &= \frac{-d_{12} - d_{23} + d_{31}}{2}, \\ \tau_3 = -u &= \frac{d_{12} - d_{23} - d_{31}}{2}. \end{aligned} \tag{3.6}$$

Note that s , t and u are strictly positive integers since the three points are not aligned, so the delays are strictly negative. We have $|\tau_1 - \tau_2| = |d_{23} - d_{31}| \leq d_{12}$ from the triangular inequality and again, since we assumed that the points are not aligned, we cannot have equality. This property and the same properties under cyclic permutations of the indices ensure that the first condition in (3.1) is satisfied. Moreover, $\tau_1 - \tau_2 + d_{12} = 2t$ is even and so are the two similar quantities obtained by cyclic permutations of the indices, so the second condition in (3.1) is also satisfied.

Let us now analyze in more detail the constraints on the labels of the well-labeled maps that result from this particular choice of delays. First, we must impose the general condition (3.4), namely that all labels of vertices incident to face 1 (respectively face 2 and 3) be larger than or equal to $1 - s$ (respectively $1 - t$ and $1 - u$), this value being reached by at least one of them. In particular, for any vertex v incident to face 1 (respectively 2 and 3), the quantity $\ell(v) - \tau_1 = \ell(v) + s$ (respectively $\ell(v) + t$ and $\ell(v) + u$) measures the distance from this vertex to the source 1 (respectively 2 and 3). Let us now consider the *boundary* between, say, faces 1 and 2, defined as the set of vertices and edges *incident to*

both faces 1 and 2. Note that such a boundary is part of the skeleton, which is the union of all boundaries for all pairs of faces. For any vertex on the boundary between faces 1 and 2, the quantity $(\ell(v) + s) + (\ell(v) + t) = 2\ell(v) + d_{12}$ measures the length of a particular path joining the source 1 to the source 2 and passing through v . This length must be larger than or equal to the distance d_{12} between 1 and 2, which implies that $\ell(v) \geq 0$. The same is true for vertices on the boundary between faces 1 and 3 or between 2 and 3. We therefore have the crucial property that *the labels of vertices on the skeleton are non-negative*.

Now consider a geodesic path on the quadrangulation between sources 1 and 2. This path must necessarily intersect the skeleton of the well-labeled map at a vertex v_1 incident to face 1 and at a vertex v_2 incident to face 2. We have then $d_{12} = (\ell(v_1) + s) + d(v_1, v_2) + (\ell(v_2) + t)$, where $d(v_1, v_2)$ is the distance between v_1 and v_2 in the quadrangulation. As $s + t = d_{12}$, this implies that $\ell(v_1) + \ell(v_2) + d(v_1, v_2) = 0$, and, since v_1 and v_2 lie on the skeleton, all these terms are non-negative. We deduce that $d(v_1, v_2) = 0$, i.e. v_1 and v_2 are identical, and moreover $\ell(v_1) = \ell(v_2) = 0$. The fact that $v_1 = v_2$ requires that the boundary between faces 1 and 2 be non-empty, which rules out well-labeled maps whose backbone is of type (g) in figure 10. Similarly, backbones of type (e) and (f) are ruled out by considering a geodesic path between 2 and 3 or between 1 and 3 respectively. We are therefore left with backbones of the type (a), (b), (c) or (d) only.

In the case of a backbone of type (a), we moreover deduce that there must be a label 0 on the boundary between 1 and 2. The same property holds of course for the other boundaries too. We end up with the following constraint: *each of the three boundaries between faces 1 and 2, 2 and 3, and 1 and 3 must contain a vertex v with label $\ell(v) = 0$* . The same constraint holds in the case of a backbone of type (b), (c) or (d). In this case, one of the boundaries is made of a single vertex (which also belongs to the two other boundaries and corresponds to the four-valent vertex of the backbone or of the skeleton). We deduce that this vertex must have a label 0.

To summarize the above analysis, the well-labeled maps obtained for our particular choice (3.6) of delays satisfy (see figure 11 for an illustration)

$$\begin{aligned}
 \min_{\substack{v \text{ incident} \\ \text{to face 1}}} \ell(v) = 1 - s, & \quad \min_{\substack{v \text{ incident} \\ \text{to face 2}}} \ell(v) = 1 - t, & \quad \min_{\substack{v \text{ incident} \\ \text{to face 3}}} \ell(v) = 1 - u, \\
 \min_{\substack{v \text{ incident} \\ \text{to faces 1 and 2}}} \ell(v) = & \quad \min_{\substack{v \text{ incident} \\ \text{to faces 2 and 3}}} \ell(v) = & \quad \min_{\substack{v \text{ incident} \\ \text{to faces 3 and 1}}} \ell(v) = 0.
 \end{aligned} \tag{3.7}$$

By the convention that the minimum of the empty set is $+\infty$, the last three conditions imply that any two faces have a non-empty boundary; hence the backbone is necessarily of type (a), (b), (c) or (d). Note also that the first three conditions are simply the general requirement (3.4) in the Miermont bijection, while the last three are specific to our choice of delays. For all these conditions to be compatible, it is crucial that s , t and u be strictly positive.

Let us start conversely from a well-labeled map with three (distinguished) faces whose labels satisfy (3.7) for some strictly positive integer values of s , t and u . The labels satisfy in particular the general condition (3.4) for delays $\tau_1 = -s$, $\tau_2 = -t$ and $\tau_3 = -u$, and applying the inverse construction discussed in section 3.1 will produce a triply pointed quadrangulation with three distinct sources (necessarily at a distance larger than or equal to 2) and with associated delays $-s$, $-t$ and $-u$. Again a geodesic

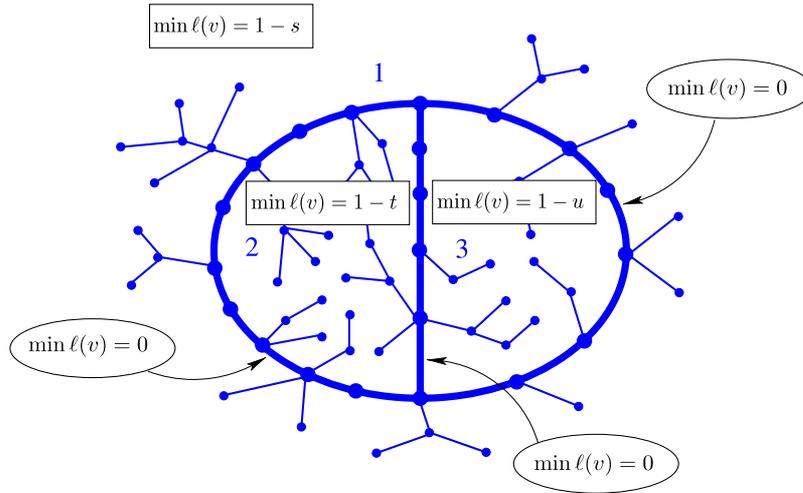


Figure 11. A schematic picture of a well-labeled map whose backbone is of type (a) in the classification of figure 10, and the set of constraints on its labels for our special choice of delays (3.6). This constraints are ‘face constraints’ (in squared boxes) for the vertices incident to each face, and ‘boundary constraints’ (in ellipses) for the vertices on the boundary between two faces (thick lines). The case of maps whose backbone is of type (b), (c) or (d) is obtained from this picture by simply shrinking one of the three boundary lines into a single vertex.

path from the source 1 to the source 2 will necessarily intersect the skeleton of the well-labeled map at a vertex v_1 incident to face 1 and at a vertex v_2 incident to face 2, so that $d_{12} \geq \ell(v_1) + s + \ell(v_2) + t \geq s + t$. Now, from the fourth condition in (3.7), there exists a vertex v at the boundary between faces 1 and 2 with label $\ell(v) = 0$ and the union of a geodesic path from the source 1 to v and a geodesic path from v to the source 2 forms a path from 1 to 2 of length $\ell(v) + s + \ell(v) + t = s + t$. We deduce that the distance between the sources 1 and 2 in the quadrangulation is given by $d_{12} = s + t$, and similarly $d_{23} = t + u$ and $d_{31} = u + s$. In particular, as s , t , and u are strictly positive, the sources cannot be aligned.

To conclude, we now have a bijection between, on the one hand, triply pointed quadrangulations whose three marked vertices are not aligned and have prescribed pairwise distances d_{12} , d_{23} and d_{31} , and on the other hand, well-labeled maps with three faces whose labels satisfy (3.7) for s , t and u related to d_{12} , d_{23} and d_{31} via relation (2.2).

It remains to discuss the case (ii) of aligned marked vertices. Let us consider the case where, say, 3 lies on a geodesic path between 1 and 2, i.e. $d_{12} = d_{23} + d_{31}$, or equivalently $u = 0$. In this case, as the three points are distinct, we have $d_{12} \geq 2$ and we may apply the Miermont bijection to our map, now considered as a doubly pointed quadrangulation with sources 1 and 2 only, complemented by an extra marked vertex 3 lying on a geodesic between 1 and 2. For the two sources, we again choose specific delays, namely $\tau_1 = -s = -d_{31}$ and $\tau_2 = -t = -d_{23}$. We have $|d_{31} - d_{23}| < d_{23} + d_{31}$ and $d_{31} - d_{23} + d_{23} + d_{31} = 2d_{31}$ is even, so the delays satisfy (3.1). The associated well-labeled map now has two (distinguished) faces and its skeleton is a simple loop, made of all vertices and edges on the boundary of faces 1 and 2 (note that the backbone in this

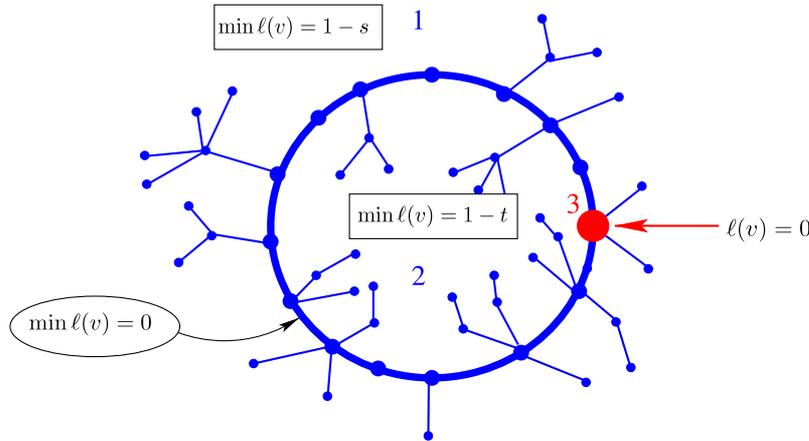


Figure 12. A schematic picture of a well-labeled map with two faces (a) and a marked vertex 3 (big circle) as obtained from the Miermont bijection in the case of aligned marked vertices. For our special choice of delays, the constraints on labels are two ‘face constraints’ (in squared boxes) for the vertices incident to each of the two faces, one ‘boundary constraint’ (ellipse) for the vertices on the boundary between the two faces (thick line), and a ‘vertex constraint’ that $\ell = 0$ for the vertex 3. Note that this picture can be considered as a degenerate limit of figure 11 when the two boundaries with face 3 reduce to a single vertex.

case is not strictly speaking a map as it has no vertex). The same argument as before shows that, for our particular choice of delays, all labels of vertices on the skeleton are non-negative and at least one of them vanishes. Now the vertex 3 is necessarily incident to one of the two faces, say face 2. Consider then a geodesic path from 1 to 3. This path must meet the boundary between faces 1 and 2 at a vertex v and we have then $d_{31} = \ell(v) + s + d_3(v)$ where $d_3(v)$ is the distance from v to the vertex 3. As $s = d_{31}$, we deduce that $\ell(v) + d_3(v) = 0$ and, as both quantities are non-negative, $\ell(v) = 0$ and $d_3(v) = 0$. The vertex 3 necessarily lies on the skeleton and has the minimum label 0. We therefore end up with a well-labeled map with two (distinguished) faces, with labels satisfying (see figure 12 for an illustration)

$$\min_{\substack{v \text{ incident} \\ \text{to face 1}}} \ell(v) = 1 - s, \quad \min_{\substack{v \text{ incident} \\ \text{to face 2}}} \ell(v) = 1 - t, \quad \min_{\substack{v \text{ incident} \\ \text{to faces 1 and 2}}} \ell(v) = 0, \quad (3.8)$$

and with a marked vertex 3 among those vertices of the skeleton having label 0.

Starting conversely from a well-labeled map with two faces, and labels satisfying (3.8) for some strictly positive integer values of s and t , and with a marked vertex 3 on the skeleton with label 0, we can use the inverse construction to recover a quadrangulation with two sources and a marked vertex. All paths between the sources 1 and 2 necessarily cross a vertex v on the boundary between faces 1 and 2 so their length is larger than $(\ell(v) + s) + (\ell(v) + t)$, and hence larger than $s + t$. As the marked vertex 3 belongs to the boundary and has label 0, it is at distance $d_{31} = s$ and $d_{23} = t$ from the sources 1 and 2 and the distance between the two sources is exactly $d_{12} = s + t$. In particular, the three points are aligned with 3 lying between 1 and 2.

To conclude, we now have a bijection between, on the one hand, triply pointed quadrangulations whose three marked vertices are aligned with, say 3 lying between 1 and 2, and have prescribed pairwise distances d_{12} , d_{23} and $d_{31} = d_{12} - d_{23}$, and on the other hand, well-labeled maps with two faces whose labels satisfy (3.8) for $s = d_{31}$ and $t = d_{23}$, and with a marked vertex with label 0 on the skeleton.

Note finally that the case of aligned vertices is not fundamentally different from the generic case of non-aligned vertices. Indeed, the well-labeled map with two faces and a marked vertex of figure 12 may be viewed as a degenerate form of the generic case of figure 11 for which two of the boundaries, say that between faces 1 and 3 and that between faces 2 and 3, reduce to a single vertex, so that face 3 in practice disappears and its boundary reduces to the single vertex 3.

4. Enumeration of triply pointed quadrangulations

Thanks to the above bijections, enumerating triply pointed planar quadrangulations with marked points at prescribed pairwise distances amounts to enumerating well-labeled maps having three or two faces, and satisfying (3.7) or (3.8) respectively. In practice, the generic case corresponds to a well-labeled map whose backbone is of type (a) in figure 10. One then recovers all the other possible cases (backbone of type (b), (c), (d) or aligned vertices) by simply allowing one or two of the boundaries to reduce to a single vertex.

In all this section, we shall consider generating functions with a weight g per edge of the well-labeled map, corresponding to a weight g per face of the quadrangulation. Our approach is as follows. In section 4.1 we recall known results about the enumeration of well-labeled trees, that is well-labeled planar maps with one face. In section 4.2 we consider well-labeled maps with two faces, which correspond to triply pointed quadrangulations whose marked points are aligned. Considering this simpler situation first allows us to introduce some definitions and derive a formula that will be instrumental in section 4.3, where we consider the general case and derive the expressions (2.3) and (2.5) for the three-point function. Finally section 4.4 is devoted to other applications of our formulas, here in the ‘local limit’ for infinitely large quadrangulations (in contrast with the continuum limit studied previously).

4.1. Generating functions for well-labeled trees

The first building block in our enumeration task is the generating function $R_i(g)$ of well-labeled trees planted at a corner with label i and such that all labels are strictly positive. With this definition we clearly have $R_i = 0$ for $i \leq 0$ while, for $i > 0$, R_i satisfies the recursion relation

$$R_i = \frac{1}{1 - g(R_{i-1} + R_i + R_{i+1})}, \quad (4.1)$$

where by convention, we set $R_i|_{g^0} = 1$ for $i > 0$. This recursion relation was solved in [17], with the result

$$R_i = R \frac{[i]_x [i+3]_x}{[i+1]_x [i+2]_x} \quad \text{for } i > 0 \quad \text{with } [i]_x \equiv \frac{1-x^i}{1-x}, \quad (4.2)$$

where the quantities R and x are related to g by equation (1.4), and more precisely equation (1.5). As already mentioned in the introduction, the quantities $\log(R_i/R_{i-1})$ for $i > 1$ and $\log(R_1)$ for $i = 1$ can be interpreted as the generating functions for doubly pointed quadrangulations with two marked vertices at distance i from each other (and counted with their inverse symmetry factor). This property is a direct consequence of Schaeffer's bijection between pointed planar quadrangulations and well-labeled trees.

Here, from conditions (3.7) or (3.8), we will be led to consider well-labeled trees planted at a corner with some label $\ell \geq 0$ and such that all labels be larger than, say, $1 - s$ for some strictly positive integer value of s . The corresponding generating function is nothing but $R_{\ell+s}$, as obtained by shifting all labels on the tree by s . Note that, in trees counted by $R_{\ell+s}$, the label $1 - s$ may or may not be reached.

4.2. Generating functions for well-labeled maps with two faces

The second building block in our enumeration task corresponds to quadrangulations with three marked aligned vertices. By the bijection of section 3, we must actually enumerate well-labeled maps with two faces 1 and 2, with labels satisfying (3.8), and with a marked vertex on the skeleton having label 0.

Consider such a well-labeled map. Its skeleton consists of a single loop of arbitrary length m , and the sequence of labels along it, read from the marked vertex, can be viewed as the sequence of heights in a *Motzkin path* \mathcal{M} of length m , i.e. a path going from 0 to 0 in m steps that never dips below height 0 and is made only of $+1$, 0 or -1 steps (see figure 13 for an illustration). Let us denote this sequence as $(0 = \ell_0, \ell_1, \dots, \ell_{m-1}, \ell_m = 0)$. The whole well-labeled map is uniquely decomposed into the skeleton and its attached tree components: we see that there are two well-labeled trees attached at each vertex of the skeleton, one surrounded by face 1 with all labels larger than $1 - s$, the other surrounded by face 2 with all labels larger than $1 - t$. The generating functions for such trees are $R_{\ell+s}$ and $R_{\ell+t}$ respectively, where ℓ is the label of the attachment vertex on the skeleton. This leads us to consider the generating function for Motzkin paths with such attached trees:

$$X_{s,t} \equiv \sum_{m \geq 0} \sum_{\substack{\text{Motzkin paths of length } m \\ \mathcal{M}=(0=\ell_0, \ell_1, \dots, \ell_m=0)}} \prod_{k=0}^{m-1} g R_{\ell_k+s} R_{\ell_k+t}, \tag{4.3}$$

where a weight g is attached to each step of the path (corresponding to an edge of the skeleton), in addition to the weight g per edge in the well-labeled trees already counted in R_i . Note that there is no weight attached to the last endpoint ($k = m$) of the path as it should be identified with the first one ($k = 0$), so that we do not overcount the trees attached there. By convention the path of length $m = 0$ is counted with a weight 1.

Conversely, a Motzkin path with such attached trees can be transformed back into a well-labeled map with two faces, but this map does not necessarily satisfy (3.8) in general, because we have not imposed that the minimum label in face 1 (respectively face 2) be exactly $1 - s$ (respectively $1 - t$). Nevertheless, we readily see that these global constraints are realized by considering the generating function

$$\Delta_s \Delta_t X_{s,t} = X_{s,t} - X_{s-1,t} - X_{s,t-1} + X_{s-1,t-1}, \tag{4.4}$$

which corresponds to subtracting the contribution of maps where the labels in face 1 are larger than $2 - s = 1 - (s - 1)$ or the labels in face 2 are larger than $2 - t = 1 - (t - 1)$. In

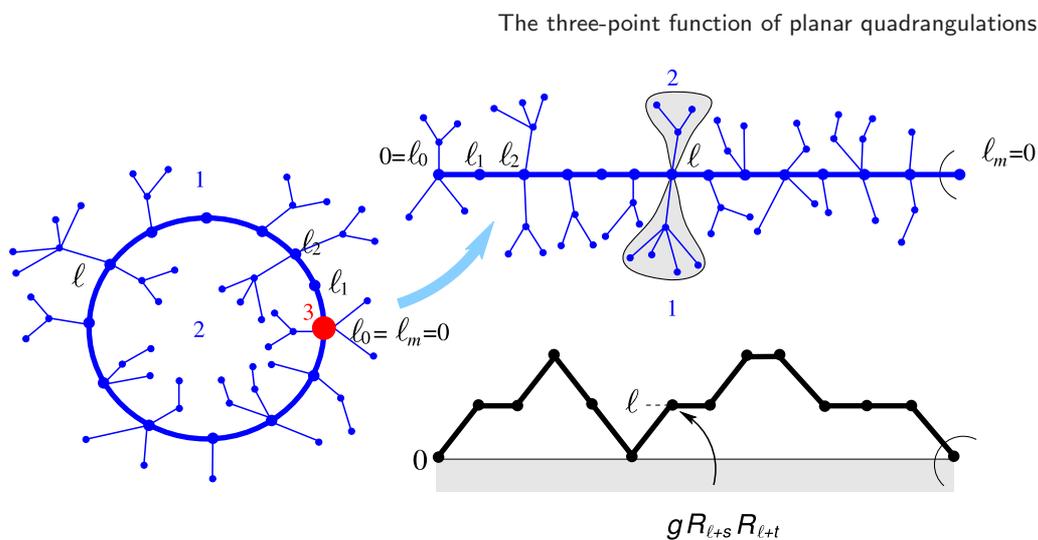


Figure 13. Illustration of the expression for the generating function of well-labeled maps with two faces, corresponding to triply pointed quadrangulations whose marked vertices are aligned. Opening the skeleton (thick edges) at the marked vertex 3, we obtain a chain with non-negative labels $(\ell_0, \ell_1, \dots, \ell_m)$ and with attached well-labeled trees. Since $\ell_0 = \ell_m = 0$, the sequence of labels along the chain forms a Motzkin path. In the enumeration, each vertex with height ℓ in the Motzkin path receives a weight $R_{\ell+s}$, which counts the possible configurations of the incident well-labeled tree in face 1, a weight $R_{\ell+t}$ for the incident tree in face 2 and a weight g for the subsequent edge on the chain.

the end, using the bijection of section 3, we find that $\Delta_s \Delta_t X_{s,t}$ is the generating function for triply pointed quadrangulations whose marked vertices are aligned, with pairwise distances s , t and $s+t$.

It turns out that, with the particular form (4.2) for R_i , $X_{s,t}$ has a very simple expression. We indeed have the remarkable combinatorial identity

$$X_{s,t} = \frac{[3]_x [s+1]_x [t+1]_x [s+t+3]_x}{[1]_x [s+3]_x [t+3]_x [s+t+1]_x}, \tag{4.5}$$

which holds for all non-negative integer values of s and t . This establishes the restricted results (2.6) and (2.7).

Let us discuss how (4.5) can be proved. First, from its definition (4.3) as a sum of weighted Motzkin paths, $X_{s,t}$ satisfies the recursion relation

$$X_{s,t} = 1 + gR_s R_t X_{s,t} (1 + gR_{s+1} R_{t+1} X_{s+1,t+1}), \tag{4.6}$$

obtained by decomposing the Motzkin path according to its first step. The Motzkin path can be of length 0 and receives by convention the weight 1, or it can start with a 0 or +1 step yielding a first factor $gR_s R_t$. If it starts with a 0 step, the rest of the path is again a Motzkin path with generating function $X_{s,t}$. If it starts with a step +1, the rest of the path is now a path from height 1 to height 0 which, upon considering the *first* return at height 0, can be decomposed into a path from 1 to 1 with heights larger than or equal to 1, a step down from 1 to 0 and a Motzkin path from 0 to 0. These three components have respective weights $X_{s+1,t+1}$, $gR_{s+1} R_{t+1}$ and $X_{s,t}$. Gathering all the possibilities above

leads to equation (4.6), which uniquely determines $X_{s,t}$ for all $s, t \geq 0$ as power series in g , with $X_{s,t} = 1 + \mathcal{O}(g)$. Now it is a straightforward exercise to check that the form (4.5) of $X_{s,t}$ does indeed satisfy (4.6) for R_i given by (4.2) with the relation (1.4), and has the expansion $X_{s,t} = 1 + \mathcal{O}(g)$. This completes the proof.

The above argument does not allow us to derive the explicit form (4.5) *ab initio* from its definition (4.3) but only checks a posteriori that this form matches the definition of $X_{s,t}$. A more constructive and historical proof of the formula (4.5) is presented in the appendix where we give two methods for enumerating so-called ‘quadrangulations with a geodesic boundary’ of length $2(s+t)$. Equating the two results precisely yields the expression (4.5) for $X_{s,t}$.

Yet another proof is provided by the discussion in section 4.4 below, where it will be shown that $\Delta_s \Delta_t \log X_{s,t}$ is the generating function for well-labeled maps with two faces satisfying (3.8), but without a marked vertex on the skeleton (this induces non-trivial symmetry factors). By the Miermont bijection, with the particular choice of delays $\tau_1 = -s$ and $\tau_2 = -t$, these correspond to doubly pointed quadrangulations where the marked vertices are at distance $s+t$. But the generating function for these doubly pointed quadrangulations is given by equation (1.1) with $i = s+t$, and we arrive at the equality

$$\log \left(\frac{R_{s+t}}{R_{s+t-1}} \right) = \Delta_s \Delta_t \log X_{s,t} = \log \left(\frac{X_{s,t} X_{s-1,t-1}}{X_{s,t-1} X_{s-1,t}} \right). \quad (4.7)$$

It is now an easy exercise to derive (4.5) from this equality using (4.2) and the initial values $X_{s,0} = X_{0,t} = 1$.

Finally, let us remark that

$$\lim_{s,t \rightarrow \infty} X_{s,t} = 1 + x + x^2 = \frac{x}{gR^2}. \quad (4.8)$$

Thus the basic quantity x can be interpreted as a generating function for Motzkin paths with attached well-labeled trees with no lower bound on the labels.

4.3. Generating functions for well-labeled maps with three faces

We are now ready to enumerate triply pointed quadrangulations in the generic case, corresponding to well-labeled maps with three faces 1, 2, 3 satisfying (3.7). Let us consider such a well-labeled map. As discussed in section 3, its backbone is necessarily of one of the types (a), (b), (c) or (d) illustrated in figure 10.

We first consider the generic type (a), and denote by v and v' the two trivalent vertices of the backbone, say with v such that faces 1, 2, 3 appear in counterclockwise order around it. Then the skeleton consists of three chains connecting v to v' , corresponding to the three boundaries 1–2, 2–3 and 3–1. By (3.7), the minimal label on each boundary must be 0. Let us then denote by v_{12} (resp. v'_{12}) the first vertex with label 0 encountered when following the boundary 1–2 starting from v (resp. v'). Similarly we define v_{23} , v'_{23} , v_{31} and v'_{31} . We can now perform a decomposition of the well-labeled map by cutting it at the vertices v_{12} , v'_{12} , v_{23} , v'_{23} , v_{31} and v'_{31} , as illustrated on figure 14. More precisely for each cut vertex, the cutting line is drawn between the two corners following immediately the two incident skeleton edges, going counterclockwise around the vertex. With this convention, each half of the cut vertex has one incident skeleton edge, and one attached

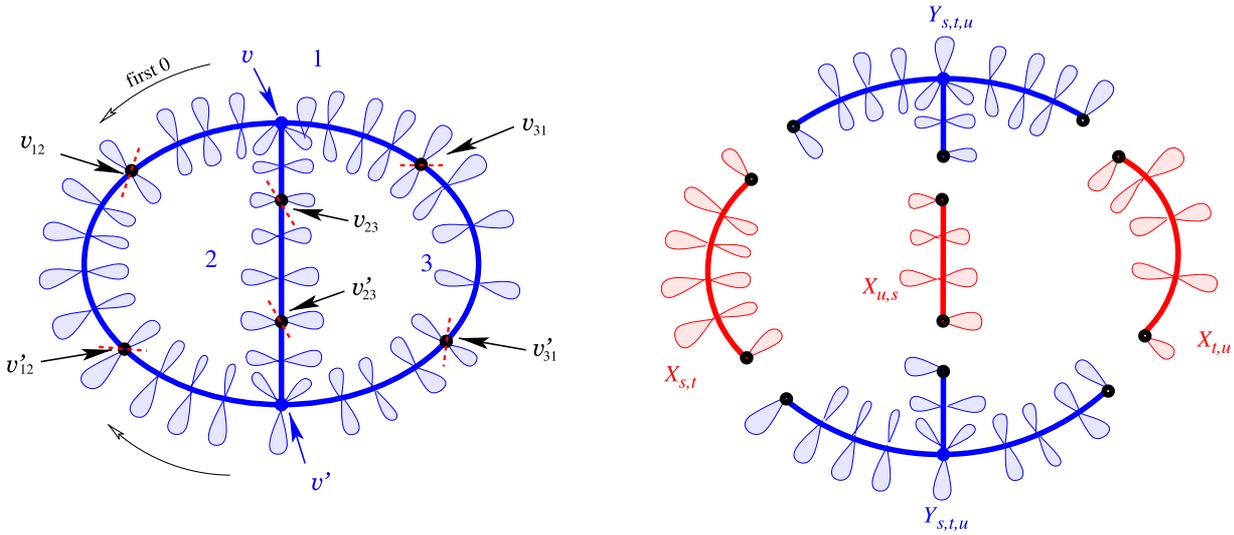


Figure 14. The decomposition of a generic well-labeled map with three faces into chains and Y-diagrams. We represented the skeleton of the map by thick lines and schematized the attached trees by blobs. The skeleton is the union of the three boundaries 1–2, 2–3 and 3–1 which meet at vertices v and v' . On each boundary, we select the vertices with label 0 closest to v and v' (denoted as $v_{12}, v'_{12}, v_{23}, v'_{23}, v_{31}$ and v'_{31} with obvious conventions). We cut the map at these selected vertices as shown by the dashed lines. We end up with five pieces in general, namely three chains (in red) and two Y-diagrams (in blue). Taking into account the constraints on the labels of the attached trees (see the text), this decomposition translates into the relation $F(s, t, u; g) = X_{s,t}X_{t,u}X_{u,s}(Y_{s,t,u})^2$.

well-labeled tree. In general the skeleton is split into five components:

- three (linear) chains connecting v_{12} to v'_{12} , v_{23} to v'_{23} , and v_{31} to v'_{31} respectively;
- two *Y-diagrams*, made of three chains connecting a central vertex (v or v') to distinct endpoints (v_{12}, v_{23}, v_{31} or $v'_{12}, v'_{23}, v'_{31}$).

There are a number of particular cases. If there is a unique vertex with label 0 on one of the boundaries, say 1–2, then $v_{12} = v'_{12}$, and the corresponding linear chain can be seen as an isolated vertex without attached trees, i.e. it is *trivial*. If, say, v has label 0, then $v = v_{12} = v_{23} = v_{31}$, and the corresponding Y-diagram can also be considered as trivial, made of an isolated vertex without attached trees (using our convention to cut within the corners following immediately the incident skeleton edges counterclockwise, the three trees attached to v go in different chains). For a backbone of type (a), we have $v \neq v'$; hence it is not possible that both Y-diagrams and one of the chains be simultaneously trivial. Such a situation however occurs when we consider degenerate backbones types (b), (c), (d), in which case both Y-diagrams and exactly one of the chains are trivial, or when we consider well-labeled maps with two faces, for which both Y-diagrams and exactly two of the chains are trivial. It is then convenient to also include the case of the fully trivial map reduced to a single vertex, when both Y-diagrams and three chains are trivial. To summarize, gathering all possible cases, an arbitrary triply pointed quadrangulation is in

correspondence with a well-labeled map that itself can be decomposed into a combination of three chains and two Y-diagrams (possibly reduced to isolated vertices).

Let us now investigate the constraints on the labels of attached well-labeled trees inherited from the first line of (3.7). It is convenient to slightly relax these conditions and consider well-labeled maps satisfying

$$\min_{\substack{v \text{ incident} \\ \text{to face 1}}} \ell(v) \geq 1 - s, \quad \min_{\substack{v \text{ incident} \\ \text{to face 2}}} \ell(v) \geq 1 - t, \quad \min_{\substack{v \text{ incident} \\ \text{to face 3}}} \ell(v) \geq 1 - u. \quad (4.9)$$

We shall call $F(s, t, u; g)$ the generating function of such maps. Then the generating function for maps satisfying (3.7) is simply given by $\Delta_s \Delta_t \Delta_u F$. With the relaxed conditions (4.9), the constraints on labels simply factorize into individual constraints for each attached well-labeled tree.

The case of chains is easy: the sequence of labels along the chain between v_{12} and v'_{12} , say, can be seen as a Motzkin path, and all attached trees previously surrounded by face 1 (resp. 2) have labels larger than $1 - s$ (resp. $1 - t$). Upon closing the chain, this is the same object as that considered in the previous section since, for non-trivial chains, there is only one attached tree at the endpoint v_{12} (resp. v'_{12}), with labels larger than $1 - s$ (resp. $1 - t$). Therefore the generating function for chains with attached well-labeled trees satisfying these constraints is nothing but $X_{s,t}$, including the trivial chain with weight 1. By the same reasoning, upon a common cyclic permutation of 1, 2, 3 and s, t, u , we find that the generating function for all possible chains between v_{23} and v'_{23} (resp. v_{31} and v'_{31}) is $X_{t,u}$ (resp. $X_{u,s}$).

We next need to consider the new situation of a Y-diagram, say the one containing v . As we said above, it is made of three chains (branches) connecting the central vertex v to the endpoints v_{12}, v_{23}, v_{31} . If v has label 0 then the Y-diagram is trivial; otherwise every branch has a non-zero length, since $\ell(v_{12}) = \ell(v_{23}) = \ell(v_{31}) = 0$. On each branch, the sequence of labels can be viewed as a path going from height $\ell(v)$ to 0, made only of +1, 0 or -1 steps, and not reaching height 0 before the last endpoint. On the branch to v_{12} , every intermediate vertex has two attached trees, and by (4.9) the one (within face 1) has labels larger than $1 - s$ and the other (within face 2) has labels larger than $1 - t$. The final vertex v_{12} has only one attached tree with labels larger than $1 - t$. Similar properties are found for the other branches, upon a simultaneous cyclic permutation of 1, 2, 3 and s, t, u . Finally v , being incident to all three faces, has three attached trees with labels respectively larger than $1 - s, 1 - t$ and $1 - u$. Let us denote by $Y_{s,t,u}$ the generating function for Y-diagrams satisfying such constraints. This function is determined by the recursive equation

$$Y_{s,t,u} = 1 + g^3 R_s R_{s+1} R_t R_{t+1} R_u R_{u+1} X_{s+1,t+1} X_{t+1,u+1} X_{u+1,s+1} Y_{s+1,t+1,u+1}, \quad (4.10)$$

which is derived in the same spirit as equation (4.6) for $X_{s,t}$. Indeed, let us consider the label of v . If $\ell(v) = 0$ then the Y-diagram is reduced to an isolated vertex with weight 1. Otherwise $\ell(v) \geq 1$, and we cut the Y-diagram at each first vertex with label 1 encountered when following each branch starting from v (using the same procedure as above for dispatching attached trees). We then obtain four pieces: another Y-diagram (which is trivial iff $\ell(v) = 1$) and three chains. The Y-diagram has labels 1 on its endpoints; otherwise it satisfies the same constraints regarding the labels on its attached trees. By decreasing all labels by 1, we find that such Y-diagrams are enumerated by $Y_{s+1,t+1,u+1}$.

The chains can be seen as paths from 1 to 0, that do not reach height 0 before the endpoint. By cutting out the last step and decreasing all labels by 1, we obtain Motzkin paths enumerated by $X_{s+1,t+1}$, $X_{t+1,u+1}$, $X_{u+1,s+1}$, for the respective endpoints v_{12} , v_{23} and v_{31} . Finally the last steps contribute with respective weights $gR_{s+1}R_t$, $gR_{t+1}R_u$ and $gR_{u+1}R_s$. Collecting all contributions, equation (4.10) follows.

We found a remarkably simple form for $Y_{s,t,u}$:

$$Y_{s,t,u} = \frac{[s+3]_x [t+3]_x [u+3]_x [s+t+u+3]_x}{[3]_x [s+t+3]_x [t+u+3]_x [u+s+3]_x}, \tag{4.11}$$

which can be readily checked by substituting into (4.10), and noting that it is the unique solution satisfying $Y_{s,t,u} = 1 + \mathcal{O}(g)$. However we do not have a more ‘combinatorial’ derivation for this formula, similar to those mentioned in the previous section for $X_{s,t}$. Note that the form (4.2) for R_i itself still lacks such a combinatorial explanation. Alternatively, we may also write $Y_{s,t,u}$ as a sum:

$$Y_{s,t,u} = \sum_{\ell=0}^{\infty} \tilde{X}_{\ell,s,t} \tilde{X}_{\ell,t,u} \tilde{X}_{\ell,u,s}, \tag{4.12}$$

$$\tilde{X}_{\ell,s,t} = \frac{x^\ell [s+1]_x [s+2]_x [t]_x [t+3]_x [2\ell+s+t+3]_x}{[s+t+3]_x [\ell+s+1]_x [\ell+s+2]_x [\ell+t]_x [\ell+t+3]_x},$$

where $\tilde{X}_{\ell,s,t}$ satisfies $\tilde{X}_{\ell,s,t} = gR_{s+1}R_t X_{s+1,t+1} \tilde{X}_{\ell-1,s+1,t+1}$, and corresponds to the generating function for branches from v to v_{12} where v has a prescribed label ℓ .

So far we have derived the generating function for Y-diagrams containing v ; however we easily see that Y-diagrams containing v' have the same generating function $Y_{s,t,u}$ (which is symmetric in s, t, u). Combining the expressions for X and Y , we arrive at the following expression for $F(s, t, u; g)$:

$$F(s, t, u; g) = X_{s,t} X_{t,u} X_{u,s} (Y_{s,t,u})^2$$

$$= \frac{[3]_x ([s+1]_x [t+1]_x [u+1]_x [s+t+u+3]_x)^2}{[1]_x^3 [s+t+1]_x [s+t+3]_x [t+u+1]_x [t+u+3]_x [u+s+1]_x [u+s+3]_x}, \tag{4.13}$$

and we readily recognize (2.5). As mentioned above, the desired generating function for maps satisfying (3.7) is $\Delta_s \Delta_t \Delta_u F(s, t, u; g)$. By the discussion of section 3, this is precisely the three-point function $G(d_{12}, d_{23}, d_{31}; g)$. This completes the proof of (2.3).

4.4. Local limit for large quadrangulations and statistics of geodesic points

As a final application of our formulas for triply pointed quadrangulations, we can consider the ‘local limit’ of large quadrangulations, obtained by considering the canonical ensemble of quadrangulations with n faces and letting n tend to ∞ , *keeping the distances between the marked vertices finite*. As opposed to the scaling limit discussed in section 2.2, results in this local limit are expected to be non-universal and specific to quadrangulations, and would be different for other classes of maps. Again, we can extract the term g^n of the various generating functions by contour integrations in g of the type of equation (1.6). In the local limit of large quadrangulations, these integrals can be evaluated exactly using

saddle point estimates (see for instance [18] for a general discussion of this technique). By proper normalizations, our enumeration results then translate directly into *average properties in canonical ensembles* of simply or doubly pointed large (meaning strictly speaking with fixed size $n \rightarrow \infty$) quadrangulations, i.e. quadrangulations with one or two marked vertices, here referred to as sources (see again [18] for a discussion on these ensembles).

For instance, from the explicit form (2.5), we can extract the *average number* $\langle c(d_{12}, d_{23}, d_{31}) \rangle$ of couples of vertices (2, 3) at finite distance d_{23} from each other and at respective distances d_{12} and d_{31} from the source (denoted as 1) in the ensemble of large simply pointed quadrangulations. We find the following formula:

$$\begin{aligned} \langle c(d_{12}, d_{23}, d_{31}) \rangle &= \Delta_s \Delta_t \Delta_u f(s, t, u), \quad \text{where} \\ f(s, t, u) &= \frac{9}{140} \frac{((1+s)(1+t)(1+u)(3+s+t+u))^2}{(1+s+t)(3+s+t)(1+t+u)(3+t+u)(1+u+s)(3+u+s)} \\ &\quad \times (29 + 20(s+t+u) + 5(s^2 + t^2 + u^2 + st + tu + us)) \\ &\quad \times ((st + tu + us + stu)(4 + s + t + u) - stu), \end{aligned} \tag{4.14}$$

and s, t, u are related to d_{12}, d_{23}, d_{31} via (2.2).

Taking $u = 0$, this quantity reduces to $\Delta_s \Delta_t f(s, t, 0)$ with

$$f(s, t, 0) = \frac{9}{140} \frac{(1+s)(1+t)(3+s+t)}{(3+s)(3+t)(1+s+t)} st (29 + 20(s+t) + 5(s^2 + t^2 + st)) (4 + s + t), \tag{4.15}$$

and measures, in the ensemble of large simply pointed quadrangulations, the average number of pairs made of a first vertex at distance $s + t$ from the source and a second vertex lying on a geodesic path between the source and the first vertex, at distance s from the former. Upon dividing by the known average number N_{s+t} of vertices at distance $s + t$ from the source in large simply pointed quadrangulations (see for instance [13] for the value of N_{s+t}), this gives the average number

$$\begin{aligned} \langle c(s) \rangle_{s+t} &= \frac{1}{N_{s+t}} \Delta_s \Delta_t f(s, t, 0), \\ \text{with } N_{s+t} &= \frac{3}{35} (s+t+1)(5(s+t)^2 + 10(s+t) + 2), \end{aligned} \tag{4.16}$$

of geodesic points, i.e. vertices lying on a geodesic path between the two sources, and at distance s from the first one, in the ensemble of large doubly pointed quadrangulations where the two sources are at distance $s + t$ from each other. Fixing this distance to a constant value $s + t = d$ and letting s vary, the ‘profile of geodesic points’ $\langle c(s) \rangle_d$ versus s/d is represented in figure 15(a) for $d = 10, 100$ and 1000 . When t becomes large, keeping s finite, we have

$$f(s, t, 0) \sim \frac{9}{28} \frac{s(1+s)}{3+s} t^4, \tag{4.17}$$

and thus

$$\Delta_s \Delta_t f(s, t, 0) \sim \frac{9}{7} \frac{s(5+s)}{(3+s)(2+s)} t^3, \tag{4.18}$$

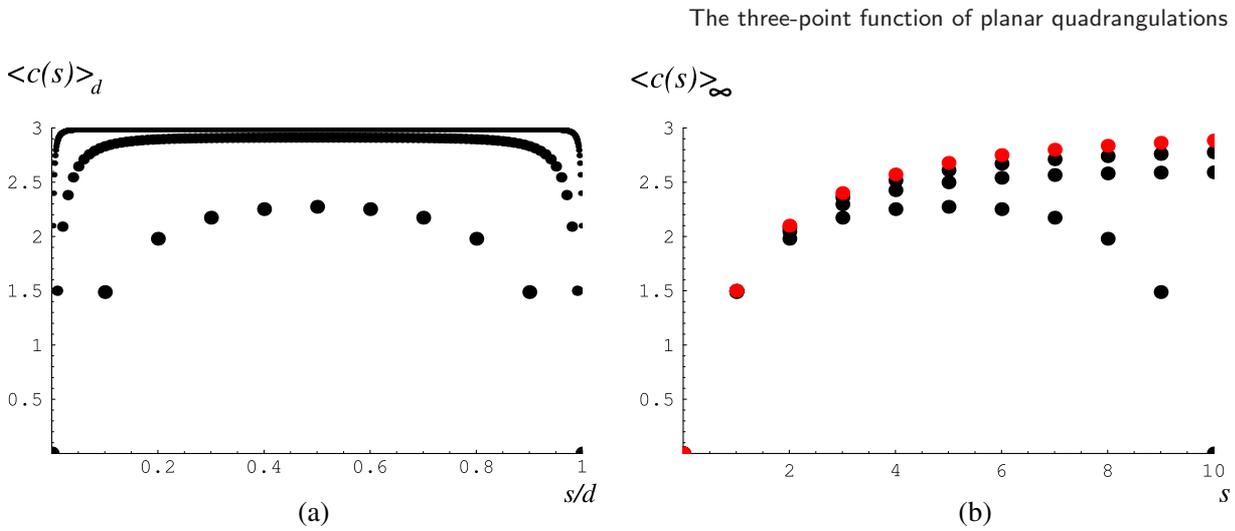


Figure 15. The ‘profile of geodesic points’ $\langle c(s) \rangle_d$, measuring the average number c of vertices lying on a geodesic path between the two sources, and at distance s from the first source, in the ensemble of large doubly pointed quadrangulations with the two sources at distance d from each other. This profile is represented in (a) versus s/d for $d = 10, 100$ and 1000 (from bottom to top), for all allowed values of s . The edge of the profile corresponding to values $s = 1, \dots, 10$ is represented versus s in (b) for $d = 10, 20$ and 50 (lower values, ordered from bottom to top) and in the limit $d \rightarrow \infty$ (top values in red).

which compared to the average number $(3/7)t^3$ of vertices at a large distance $(s + t) \sim t$, gives an average number

$$\langle c(s) \rangle_\infty = \frac{3s(5 + s)}{(3 + s)(2 + s)} \tag{4.19}$$

of geodesic points at finite distance s from the first source when the second source is far away. This limiting form is represented in figure 15(b), where it is compared with the profile at finite values $d = 10, 20$ and 50 of the distance between the two sources. In particular, far from the first source, i.e. when s becomes large, equation (4.19) gives an average number of 3 geodesic points far away from both sources, as is apparent in figures 15(a) and (b).

Beside its average $\langle c(s) \rangle_{s+t}$, we have access to the full probability law for the number c of geodesic points. Indeed, consider triply pointed quadrangulations where the three sources are aligned, with the third one lying on a geodesic path between the first two. As discussed in section 3.2, these are in bijection with well-labeled maps of the type of figure 12, with two faces and labels satisfying (3.8). As already discussed, these maps are counted by $\Delta_s \Delta_t X_{s,t}$, where $X_{s,t}$ accounts for the closed chain of vertices forming the boundary of the two faces. Now, for a fixed well-labeled map, the third marked vertex (vertex 3 in figure 12) can sit on any of the vertices of the boundary with label 0 and the number of geodesic points is given by the number of such vertices. Maps whose boundary has exactly c such vertices are simply enumerated by $\Delta_s \Delta_t X_{s,t}^{(c)}$, where

$$X_{s,t}^{(c)} = \frac{1}{c} \left(\frac{X_{s,t} - 1}{X_{s,t}} \right)^c. \tag{4.20}$$

Indeed, the quantity $(X_{s,t} - 1)/X_{s,t} = gR_s R_t (1 + gR_{s+1} R_{t+1} X_{s+1,t+1})$ is the generating function for weighted Motzkin paths having all intermediate heights strictly above 0 and $X_{s,t}^{(c)}$ simply counts cyclic sequences of exactly c such paths. In particular, we have the consistency relation

$$\sum_{c \geq 1} c X_{s,t}^{(c)} = X_{s,t}, \tag{4.21}$$

since in $X_{s,t}$, one of the c vertices with label 0 is marked on the chain. We also have the normalization

$$\sum_{c \geq 1} X_{s,t}^{(c)} = \log X_{s,t}, \tag{4.22}$$

which allows us to recover the generating function for doubly pointed maps with two marked vertices at distance $s + t$ via

$$\Delta_s \Delta_t \log X_{s,t} = \log \left(\frac{X_{s,t} X_{s-1,t-1}}{X_{s,t-1} X_{s-1,t}} \right) = \log \left(\frac{[s+t+3]_x [s+t]_x^2}{[s+t-1]_x [s+t+2]_x^2} \right) = \log \left(\frac{R_{s+t}}{R_{s+t-1}} \right). \tag{4.23}$$

As mentioned in section 4.2, this identity can be used to actually derive formula (4.5).

Combining (4.20) and (4.23), we deduce that, in the canonical ensemble of doubly pointed quadrangulations with n faces whose two sources are at distance $s + t$ from each other, the probability $p_{s+t;n}(c, s)$ that there be exactly c vertices lying on geodesics between the two sources at distance s from the first one reads

$$p_{s+t;n}(c, s) = \frac{\Delta_s \Delta_t X_{s,t}^{(c)} |_{g^n}}{\log(R_{s+t}/R_{s+t-1}) |_{g^n}}. \tag{4.24}$$

Again we can consider the local limit of large quadrangulations by sending $n \rightarrow \infty$ and keeping s and t finite, leading, after extracting the g^n terms by a saddle point technique, to a probability

$$p_{s+t}(c, s) = \frac{1}{N_{s+t}} \Delta_s \Delta_t \frac{f(s, t, 0)}{A_{s,t}^2} \left(\frac{A_{s,t} - 1}{A_{s,t}} \right)^{c-1}$$

with $A_{s,t} = 3 \frac{(s+1)(t+1)(s+t+3)}{(s+3)(t+3)(s+t+1)},$ (4.25)

and with $f(s, t, 0)$ and N_{s+t} as in (4.15) and (4.16). In particular, when t becomes large, we find a probability

$$p_\infty(c, s) = \frac{s+3}{2} \left(\frac{2s}{3(s+1)} \right)^c - \frac{s+2}{2} \left(\frac{2(s-1)}{3s} \right)^c \tag{4.26}$$

that there be c geodesic points at distance s from the first source if the two sources are far apart. Finally, far away from the first source, i.e. when s becomes large, this probability reduces to

$$p_\infty(c) = \frac{1}{2} \left(\frac{2}{3} \right)^c, \tag{4.27}$$

which is the probability law for the number c of geodesic points at any fixed distance, far away from the two sources.

5. Discussion

In this paper, we computed the generating function for triply pointed planar quadrangulations with three marked vertices at prescribed pairwise distances. We then derived its universal scaling form and analyzed its behavior in various limiting regimes. The main ingredient in our derivation is the Miermont bijection between triply pointed planar quadrangulations with sources and delays and well-labeled planar maps with three faces. To keep track of all distances between the marked points, we had to supplement this bijection with a particular choice of delays, resulting in a more restricted set of well-labeled maps amenable to a direct enumeration. The combinatorial building blocks in this enumeration were the already known generating function R_i for well-labeled trees and new generating functions $X_{s,t}$ and $Y_{s,t,u}$ describing configurations of chains or Y-diagrams of such trees. These new functions have very simple forms (4.5) and (4.11), the first one being also used to address the statistics of geodesic points in doubly pointed quadrangulations. We believe that these formulas will be useful for other enumeration problems related to well-labeled maps, and consequently to quadrangulations, including maps with higher genus. In this respect, is it tempting to view $X_{s,t}$ as a propagator and $Y_{s,t,u}$ as a vertex, whose combination into Feynman diagrams builds general well-labeled maps with a number of constraints in their labels.

A natural question is of course that of the p -point function for $p > 3$, which would require us to compute the generating function for multiply pointed quadrangulations with p sources at prescribed pairwise distances. Even if the Miermont bijection works in this case with arbitrary delays, it does not seem possible in general to keep track of all pairwise distances via a proper choice of delays as we did for $p = 3$. Indeed, there are only p free values for the delays but $p(p-1)/2$ pairwise distances, so, when $p > 3$, we cannot encode all pairwise distances in the delays. More precisely, we expect that our construction will work only if the p marked vertices have pairwise distances which can be realized as distances between the centers of p pairwise tangent hyperspheres in $p-1$ dimensions. In other words, this requires that we can write $d_{ij} = s_i + s_j$ for some set of non-negative integers s_i , $i = 1, \dots, p$. This is clearly not a generic situation but it includes for instance the case of regular simplices, for which all pairwise distances are the same. Multiply pointed quadrangulations whose p sources have prescribed pairwise distances of the above restricted form can be enumerated explicitly along the same lines as in the case $p = 3$. For instance, taking $(s_1, s_2, s_3, s_4) = (s, t, u, v)$ in the case $p = 4$, one finds a generating function $\Delta_s \Delta_t \Delta_u \Delta_v F(s, t, u, v; g)$ with $F(s, t, u, v; g) = X_{s,t} X_{s,u} X_{s,v} X_{t,u} X_{t,v} X_{u,v} Y_{s,t,u} Y_{s,t,v} Y_{s,u,v} Y_{t,u,v}$.

Finally, another extension concerns classes of maps more general than quadrangulations. The discrete two-point function is known for instance for bipartite maps with prescribed face degrees [17] and has a structure similar to that of equations (1.1) and (1.2), but now with $[i]$ taking a more general explicit form. We expect the Miermont bijection to naturally extend to these maps, using the mobile rules of [13]. We further expect many of our arguments to hold in this case. In particular, the derivation of $X_{s,t}$ in appendix should extend to this situation and lead to the same formula (4.5), now with the modified

[*i*]. As for $Y_{s,t,u}$, we had no combinatorial derivation but we may still speculate that the form (4.11) remains unchanged.

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Appendix. Derivation of formula (4.5) for $X_{s,t}$

We give here a derivation of the explicit formula (4.5) for $X_{s,t}$ based on the enumeration of *quadrangulations with a geodesic boundary*. Such objects were introduced and enumerated in [18] in the context of a general study of geodesic paths in quadrangulations. They can be defined as follows: a quadrangulation with a geodesic boundary of length $2i$ ($i \geq 2$) is a planar map with a marked face of degree $2i$ and with all other faces of degree 4 (squares), and with two marked (and distinguished) vertices incident to the marked face *at distance i from each other in the map*. This last requirement is equivalent to demanding that the boundary of the marked face be made of two paths of length i joining the marked vertices, which are moreover geodesic paths in the map. These paths may possibly meet at common vertices (necessarily at the same distance from the marked vertices) or even along common edges. It is convenient, when drawing this map in the plane, to choose the marked face as the external face. Ignoring this face results in a quadrangulation having a boundary made of the two geodesic paths above joining the marked vertices (see figure A.1(a) for an illustration).

To enumerate these maps, we may transform them into true planar quadrangulations by adding edges in the external face linking the two boundaries of the quadrangulation. We shall refer to these boundaries as the left and right boundaries by viewing the quadrangulation with the first marked vertex at the top. We then connect by an edge winding around the quadrangulation each vertex of the right boundary at distance $k = 2, \dots, i-1$ from the first marked vertex to the vertex of the left boundary at distance $k-1$ from the first marked vertex (see figures A.1(b) and (c)). As a result, the external face is divided into $i-1$ squares. Note that the added edges do not modify the distances to the two marked vertices of any vertex in the map. In particular, the two boundaries remain geodesic paths.

We then can use the Miermont bijection to transform this quadrangulation into a well-labeled map. Let us do this in two ways:

- (i) By considering the map as a pointed map with only one source (the first marked vertex) and delay $\tau_1 = 0$ (this corresponds to the original Schaeffer construction). This construction will create a particular type of well-labeled map with one face, i.e. a well-labeled tree referred to as a 'spine tree' in [18].
- (ii) By considering the map as a doubly pointed map with two sources (the two marked vertices) and delays $\tau_1 = -s$ and $\tau_2 = -t$ for arbitrary strictly positive values of s and t satisfying $s + t = i$. This will produce a particular type of well-labeled map with two faces.

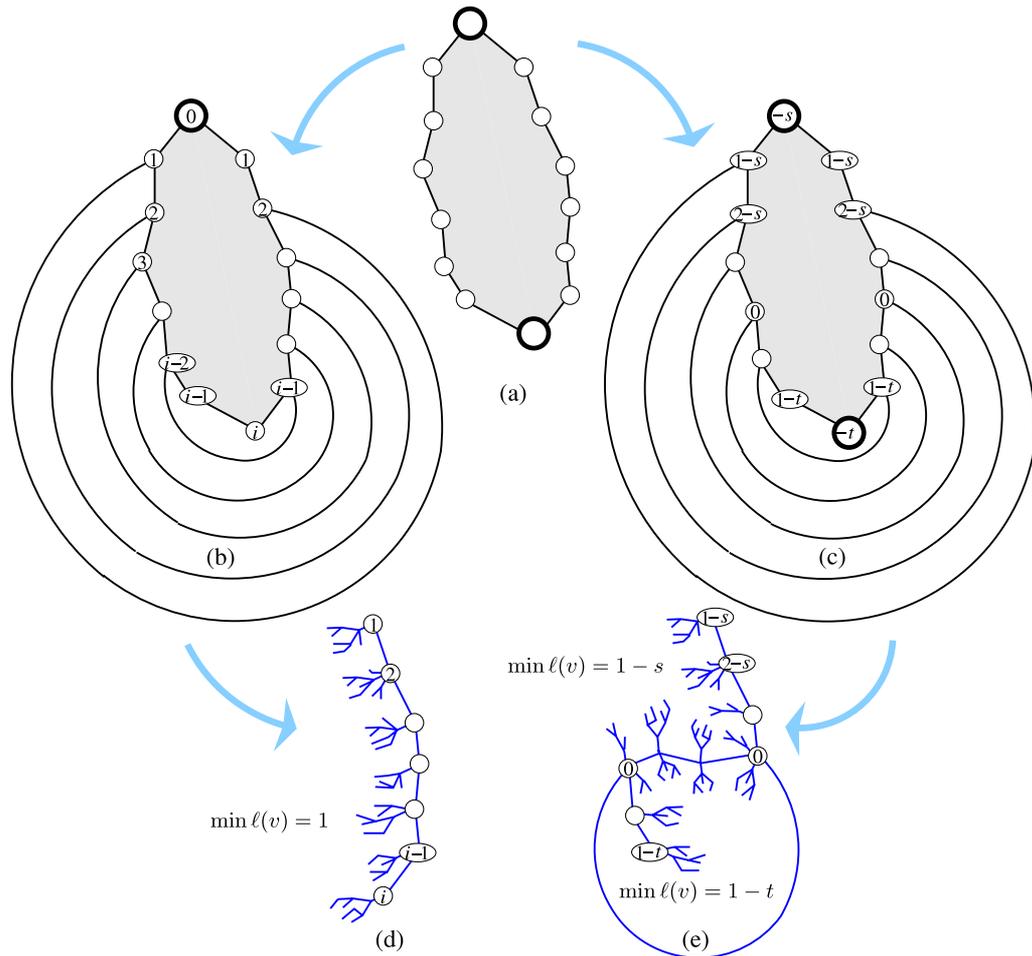


Figure A.1. A schematic picture (a) of a quadrangulation with a geodesic boundary of length $2i$. The underlying quadrangulation is symbolized by the gray background. Its boundary is made of two paths of length i joining two marked vertices (thick circles) which are geodesic paths in the whole quadrangulation. The configuration can be transformed into a true quadrangulation without boundary by adding winding edges as in (b) or (c). Such quadrangulation can be considered as a simply pointed quadrangulation (b) whose source is the top marked vertex to which we assign a delay 0. Performing the Miermont bijection creates a well-labeled tree whose structure is displayed in (d). The quadrangulation can be considered alternatively as a doubly pointed quadrangulation (c) whose sources are the two marked vertices to which we assign delays $-s$ and $-t$ (with $s+t = i$). Performing the Miermont bijection now creates a well-labeled map with two faces whose structure is displayed in (e).

In the first case, the labels of the vertices of the two boundaries are their distances to the first marked vertex, ranging from 0 to a maximal value i for the second marked vertex. Applying the construction of figure 6, each of the added squares selects an edge of the right boundary (see figure A.1(d)). All of the edges of this boundary except the first one are selected, creating a chain of length $i - 1$ whose vertices have labels $1, 2, \dots, i$. The rest

of the quadrangulation gives rise to tree components that are attached only to the left side of the chain. All of these tree components are well-labeled trees with labels larger than $\tau_1 + 1 = 1$, i.e. with strictly positive labels. Note that the minimal label 1 is automatically attained at the first vertex of the chain. The generating function for the well-labeled map obtained with the above structure, counted with a weight g per edge, is simply

$$g^{i-1} \prod_{k=1}^i R_k = g^{i-1} R^i \frac{[1]_x [i+3]_x}{[3]_x [i+1]_x}. \tag{A.1}$$

Thanks to the Schaeffer bijection, this is also the generating function for our quadrangulations with a geodesic boundary (up to a factor g^{i-1} for the spurious added squares).

In the second approach, the labels of the vertices on the boundaries are, when going away from the first marked vertex towards the second marked vertex, $-s, 1-s, 2-s, \dots, 0$ for the first $s+1$ vertices and then $-1, -2, \dots, -t$ for the last t . Applying again the construction of figure 6, the first $s-1$ added squares select edges of the right boundary while the last $t-1$ ones select edges of the left boundary (see figure A.1(e)). The first s edges of right boundary except the first one are selected, creating a chain of length $s-1$ whose vertices have labels $1-s, 2-s, \dots, 0$, while the last t edges of left boundary except the last one are selected, creating a chain of length $t-1$ whose vertices have labels $0, -1, \dots, 1-t$. Finally, the s th added square is a confluent face that gives rise to a winding edge connecting the two vertices labeled 0 on the two boundaries. The rest of the quadrangulation creates a path joining these two vertices with label 0, and a number of tree components attached to both sides of this path, to the left side of the chain of length $s-1$ and to the right side of the chain of length $t-1$ (see figure A.1(e)). All these tree components are well-labeled trees with labels larger than $\tau_1 + 1 = 1-s$ or $\tau_2 + 1 = 1-t$ according to which face they lie in (see figure A.1). Note again that the minimal label $1-s$ or $1-t$ is automatically attained on the chains. The generating function for the labeled maps obtained with the above structure is now

$$g^{s+t-1} \prod_{\ell=1-s}^0 R_{\ell+s} \prod_{\ell=1-t}^0 R_{\ell+t} \times X_{s,t} = g^{s+t-1} R^{s+t} \frac{[1]_x [s+3]_x [1]_x [t+3]_x}{[3]_x [s+1]_x [3]_x [t+1]_x} X_{s,t}, \tag{A.2}$$

where the factor $X_{s,t}$ comes from the path joining the two vertices labeled 0 on the boundaries. Thanks to the Miermont bijection, this is also the generating function for our quadrangulations with a geodesic boundary of length $2(s+t)$ (up to a factor g^{s+t-1} for the spurious added squares). Taking $i = s+t$, the formulas (A.1) and (A.2) thus enumerate the same objects; hence they must be equal. Equating (A.1) and (A.2) with $i = s+t$ yields

$$X_{s,t} = \frac{[3]_x [s+1]_x [t+1]_x [s+t+3]_x}{[1]_x [s+3]_x [t+3]_x [s+t+1]_x}, \tag{A.3}$$

which is nothing but (4.5).

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