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Absence of triangles in maximal supergravity amplitudes

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Abstract: From general arguments, we show that one-loop n-point amplitudes in colourless theories satisfy a new type of reduction formula. These lead to the existence of cancellations beyond supersymmetry. Using such reduction relations we prove the no-triangle hypothesis in maximal supergravity by showing that in four dimensions the n-point graviton amplitude contain only scalar box integral functions. We also discuss the reduction formulas in the context of gravity amplitudes with less and no supersymmetry.

Keywords: Extended Supersymmetry, Gauge Symmetry.

Contents

1. Introduction

The knowledge of perturbative gravity amplitudes and their UV-behaviour is to a large extent based on arguments from power counting rather than on explicit computations. Since the calculation of graviton scattering amplitudes is a lengthy and challenging subject — accurate power counting arguments incorporating all symmetries of the amplitude, e.g. gauge symmetry, supersymmetry etc, can be a preferred option. Much care should however be taken not to overlook symmetries that could drastically reduce an expected power counting behaviour.

The one-loop n-graviton amplitude in a Feynman diagram approach in $D = 4 - 2\epsilon$ dimensions takes the generic form

$$\begin{aligned} M_{n;1} &= \mu^{2\epsilon} \int \frac{d^D\ell}{(2\pi)^D} \frac{\prod_j^{2n}(q_{\mu_j}^{(2n,j)}\ell^{\mu_j}) + \prod_j^{2n-1}(q_{\mu_j}^{(2n-1,j)}\ell^{\mu_j}) + \cdots + K}{\ell_1^2 \cdots \ell_n^2} \\ &\equiv \mu^{2\epsilon} \int \frac{d^D\ell}{(2\pi)^D} \frac{\mathcal{P}_n(\ell)}{\ell_1^2 \cdots \ell_n^2} \,. \end{aligned} \tag{1.1}$$

Here $\ell_i^2 = (\ell - k_1 - \cdots - k_i)^2$ are the propagators along the loop and $q_{\mu_j}^{(i,j)}$ are functions of external momenta and polarisations. The integration is performed by separating [1] the four dimensional dependence and the $D-4 = -2\epsilon$ part of the loop momentum, $\ell = \bar\ell + \ell_{[-2\epsilon]}$ with $\ell_{[-2\epsilon]}^2 = \mu^2$ given by the infrared regulator so that

$$\int \frac{d^D\ell}{(2\pi)^D} = \int \frac{d^4\bar\ell}{(2\pi)^4} \int \frac{d^{-\epsilon}(\mu^2)}{(2\pi)^{-2\epsilon}} \,. \tag{1.2}$$

Momentum conservation implies that $k_1 + \cdots + k_n = 0$. K is a constant. Because of the two derivative nature of the cubic gravitational coupling, the numerator $\mathcal{P}_n(\ell)$ is a polynomial with at most $2n$ powers of loop momentum ℓ. Supersymmetry can be accounted for by a correcting factor of $\ell^{-\mathcal{N}}$, in a theory with $\mathcal{N}$ on-shell supercharges. The maximal order for the numerator polynomial for the n point amplitude in eq. (1.1) is thus $\ell^{2n-\mathcal{N}}$ in this power count.

According to the above naïve powercounting the n-graviton one-loop amplitude in $\mathcal{N} = 8$ supergravity has at most $2(n-4)$ powers of ℓ in the numerator polynomial. The loop momentum polynomial can be manipulated via a succession of (ordered) integral reductions cancelling one power of loop momentum at each step of reduction [2, 3, 1, 4]. This reduces all tensor integrals in the amplitude expression into a linear combination of scalar integrals. The naïve power counting in $\mathcal{N} = 8$ supergravity indicates that one-loop amplitudes should be expandable in a basis of *scalar box, triangle and bubble integral functions* as well as *rational pieces*.

Surprisingly in a number of explicit calculations using on-shell unitarity techniques [5–10] it has however been observed that this naïve power count does not reflect reality and that maximal supergravity amplitudes have a much better power counting. This has also been referred to as the "no-triangle" hypothesis of $\mathcal{N} = 8$ supergravity [9, 10]. The "no-triangle" hypothesis suggests that $\mathcal{N} = 8$ supergravity amplitudes in four dimensions are completely specified by a basis of box integral functions and in particular do neither contain *triangles* or *bubble functions* nor *rational pieces*. By analysing the dependence on the loop momenta in the cuts it was deduced in [10] that $n-4$ powers of loop momenta must have cancelled in the total amplitude in comparison to the counting in eq. (1.1), yielding the "no-triangle" hypothesis.

In this paper we demonstrate that for colourless gauge theories, like gravity, there are new reduction formulas in place for on-shell amplitudes where at each step of reduction *two powers of loop momentum are cancelled*. This contrasts integral reduction formulas for ordered theories where only *one power of loop momentum is cancelled*. The unordered integral reductions are instrumental in exhibiting the improved divergence structure of colourless gauge theories such as gravity. We will first discuss on very general grounds how a set of reduction formulas for unordered integral functions at one-loop can be induced by gauge invariance. Next we will discuss the consequences for maximal supergravity and explain why the naïve power counting is incorrect and how the generic n-graviton amplitude in four dimensions in maximal supergravity cannot contain any basis integral other than scalar box integral functions. This argument follows directly from an application of the new integral reduction formulas on the generic graviton amplitude where polarisation tensors has been expanded in terms of the momenta of the external legs. This constitute a direct proof of the "no-triangle" hypothesis for $\mathcal{N} = 8$ supergravity.

2. One-loop gravity amplitude in the string based formalism

In this section we will review the string based formalism to be employed later in the paper for proving the "no triangle" hypothesis of the $\mathcal{N} = 8$ n-graviton amplitudes at one-loop following [11, 12, 5, 13].

The string based formalism is very natural in theories with no colour factors since it in a simple way incorporates expressions for one-loop amplitudes which has complete crossing symmetry of all external legs. The crossing symmetry implies that the various colour ordered gravity amplitudes have the same tensorial structure and imply additional simplicity.

We will first consider an n-point one-loop φ^3 scalar field theory amplitude. We will introduce Feynman parameters a_i and exponentiate the propagators. Through this we can write the ordered scalar n-point one-loop φ^3 amplitude as

$$\begin{aligned} I_n &= \mu^{2\epsilon} \int \frac{d^D\ell}{\pi^{\frac{D}{2}}} \prod_{i=1}^{n} \frac{1}{(\ell - k_1 - \cdots - k_i)^2} \\ &= \mu^{2\epsilon} \int \frac{d^D\ell}{\pi^{\frac{D}{2}}} \prod_{i=1}^{n} \int_0^\infty d\alpha_i \, \exp\left(- \sum_{i=1}^{n} \alpha_i (\ell - k_1 - \cdots - k_i)^2 \right) \\ &= \mu^{2\epsilon} \int \frac{d^D\hat\ell}{\pi^{\frac{D}{2}}} e^{-T\hat\ell^2} \int_0^\infty \frac{dT}{T} T^{-n} \prod_{i=1}^{n} \int_0^1 da_i \, \delta\left(1 - \sum_{i=1}^{n} a_i\right) \exp(-T\, Q_n) \,. \end{aligned} \tag{2.1}$$

Here we have used μ to denote the infrared regulator. We have exponentiated the propagators by introducing the Schwinger parameter α_i. The Feynman parameters have next been rescaled by the proper-time of the loop defined as $T = \alpha_1 + \cdots + \alpha_n$ according to $a_i = \alpha_i/T \in [0,1]$ for $1 \leq i \leq n$. We have used the following definitions [1]

$$Q_n = \sum_{1\leq i<j\leq n} S_{ij}\, a_i a_j \,, \tag{2.2}$$

where

$$S_{ij} \equiv \frac{1}{2} (k_i + \cdots + k_{j-1})^2, \qquad i \neq j, \; S_{ii} \equiv 0 \,. \tag{2.3}$$

Performing the integration over the shifted loop momentum

$$\hat\ell = \ell - K_{[n]} \,, \qquad K_{[n]} = \sum_{i=1}^{n} k_i \sum_{j=1}^{i} a_j \,, \tag{2.4}$$

we obtain the following expression,

$$I_n = \Gamma\left(\frac{D-1}{2}\right) \int_0^\infty \frac{dT}{T} T^{-D/2+n} \int_0^1 da_n \int_0^{a_n} da_{n-1} \cdots \int_0^{a_2} da_1 \, \delta\Big(1 - \sum_{i=1}^{n} a_i\Big) \, e^{-T\, Q_n} \,. \tag{2.5}$$

We will remark that if one performs the integration over the proper-time T then the familiar expression for an ordered n-point one-loop scalar amplitude as used in [1] is obtained

$$I_n = \Gamma\left(\frac{D-1}{2}\right) \Gamma\left(n - \frac{D}{2}\right) \int_0^1 da_n \int_0^{a_n} da_{n-1} \cdots \int_0^{a_2} da_1 \, \delta\Big(1 - \sum_{i=1}^{n} a_i\Big)\, Q_n^{\frac{D}{2}-n} \,. \tag{2.6}$$

If we perform a change of variables according to

$$\nu_i = \sum_{j=1}^{i} a_j \,, \tag{2.7}$$

we can rewrite the expression for Q_n in eq. (2.2) in the following way

$$Q_n = \sum_{1\leq i<j\leq n} (k_i \cdot k_j) \Big[(\nu_i - \nu_j)^2 - |\nu_i - \nu_j|\Big] . \tag{2.8}$$

This will be the expression that we will use in the main part of the paper. In the ordered amplitude eq. (2.5) all the ν_i are ordered according to $0 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n$. Using the ν_i variables the total momentum along the loop is given by the expression

$$K_{[n]} = \sum_{i=1}^{n} k_i \, \nu_i \, . \tag{2.9}$$

In this representation the ν_i are the dual variables to the loop momenta and any power of ν_i in the integrand of the integral represents powers of the loop momentum in the amplitude.

The unordered amplitudes can be obtained by summing the expression eq. (2.5) over all orderings of the external legs

$$\begin{aligned}\mathcal{I}_n &= \mu^{2\epsilon} \sum_{\sigma\in\mathfrak{S}_n} \int \frac{d^D\ell}{\pi^{\frac{D}{2}}} \prod_{i=1}^{n} \frac{1}{(\ell - k_{\sigma(1)} - \cdots - k_{\sigma(i)})^2} \\ &= \Gamma\left(\frac{D-1}{2}\right) \int_0^\infty \frac{dT}{T}\, T^{-D/2+n} \int_0^1 d\nu_1 \cdots d\nu_n \, \frac{1}{n} \left[\sum_{i=1}^{n} \delta(\nu_i = 1)\right] e^{-T\,Q_n} . \end{aligned} \tag{2.10}$$

Here $\mathfrak{S}_n$ is the set of all permutations of the n orderings of the external legs. Because of the summation over all orderings of the external legs, the ν_i variables with $1 \leq i \leq n$ are freely integrated between $[0, 1]$. By translational invariance along the loop and momentum conservation the integration depends only on $n-1$ variables. We can fix this translational invariance around the loop by freezing one of the ν_i to 1. In order to preserve the symmetry among the ν_i variables we insert a symmetrised delta-function.

For an ordering of the external legs specified by a permutation $\sigma \in \mathfrak{S}_n$ of n objects, the mapping between the ν_i and a_i variables takes the form

$$\nu_i = \sum_{j=1}^{i} a_{\sigma(j)} \, . \tag{2.11}$$

The absolute value in Q_n in eq. (2.8) allows us to formally break up the domain of integration for the integral into various regions of analyticity in the complex energy momentum plane. These regions can be seen to correspond exactly to the possible physical orderings of the external legs. For instance, for the case of the four-point amplitude the integral $\mathcal{I}_4$ becomes

$$\begin{aligned}\mathcal{I}_4 &= \Gamma\left(\frac{D-1}{2}\right) \int_0^\infty \frac{dT}{T}\, T^{-D/2+4} \int_0^1 d\nu_1 \cdots d\nu_3 \, \delta(\nu_4 = 1)\, e^{-T\,Q_4} \\ &= I_4(s,t) + I_4(s,u) + I_4(t,u) \, , \end{aligned} \tag{2.12}$$

where $I_4(s,t)$ is the scalar box function evaluated in the physical region $s, t < 0$

$$I_4(s,t) = \frac{1}{st} \left[\frac{2}{\epsilon^2} \left((-s)^{-\epsilon} + (-t)^{-\epsilon}\right) - \ln^2(-s/-t) - \pi^2 + \mathcal{O}(\epsilon)\right] , \tag{2.13}$$

with identical definitions for $I_4(s,u)$ and $I_4(t,u)$.

For interacting colourless gauge theories like gravity the general structure of the one-loop n-point amplitude in D dimensions is given by [11, 12, 5]

$$\mathcal{M}_{n;1} = \int_0^\infty \frac{dT}{T} T^{-D/2+n} \int_0^1 d\nu_1 \cdots d\nu_n \, \frac{1}{n} \left[\sum_{i=1}^n \delta(\nu_i = 1)\right] \mathcal{P}(h_{r_i s_i}, k_i; \nu_i) \, e^{-T\, Q_n} \,. \quad (2.14)$$

Here $h_{r_i s_i}$ is the polarisation tensor of the ith graviton with momentum k_i. Comparing with eq. (2.10) the new ingredient is the quantity $\mathcal{P}(h_{r_i s_i}, k_i; \nu_i)$ arising from the three, four and higher point interaction vertex that encodes the tensorial structure of the amplitude.

One important property of the representation (2.14) is that the orderings of the external legs all have the *same* tensorial structure. This is a consequence of the form of the expression for $\mathcal{M}_{n;1}$ where the polarisation dependence enters the function $\mathcal{P}(h_{r_i s_i})$ which is integrated over the unconstrained variables $[0,1]$.

As in the case of the scalar amplitude that we described earlier the various regions of analyticities in the complex energy plane arise from expanding the absolute value constraints in Q_n and the sign function G_F.

For n-graviton one-loop amplitudes in supergravity theories with $\mathcal{N}$ on-shell supersymmetries (counting the number of supersymmetries in units of four dimensional Majorana supercharges) with $0 \leq \mathcal{N} \leq 8$, one way to derive the coefficient $\mathcal{P}(h_{r_i s_i})$ is to extract it from the following generating function

$$\mathcal{P}(h_{r_i s_i}, k_i; \nu_i) = \prod_{I=1}^{\mathcal{N}/4} \prod_{m=1}^{4} d\psi_I^m d\bar{\psi}_I^m \times \quad (2.15)$$
$$\times \prod_{\alpha,\dot\alpha=1}^{2} \prod_{i=1}^{n} \int d\theta_i^\alpha d\bar{\theta}_i^{\dot\alpha} e^{\sum_{i=1}^n (\theta_i^1\, h_i\cdot\psi + \theta_i^2\, (ik_i\cdot\psi)) + c.c.} e^{\mathcal{F}+\bar{\mathcal{F}}}\Big|_{\text{multilinear}} .$$

In this equation θ_i^α and $\dot{\theta}_i^{\dot\alpha}$ are $\alpha, \dot\alpha = 1, 2$ anticommuting variables and ψ_I^m and $\bar{\psi}_I^m$ are $1 \leq m \leq 4$ and $1 \leq I, J \leq \mathcal{N}/4$ space-time fermionic zero modes (counting the number of supersymmetries in units of four dimensional Majorana supercharges).

In this expression one has to extract the multilinear part in the polarisations h_i and $\bar{h}_i$ with $1 \leq i \leq n$ where we have decomposed the polarisations of the graviton $h_{r_i s_i}$ as a symmetric product of two spin one polarisations h_{r_i} and $\bar{h}_{r_i}$, i.e., $h_{r_i s_i} = (h_{r_i}\bar{h}_{r_i} + h_{r_i}\bar{h}_{s_i})/2$. The expression for $\exp(\mathcal{F})$ is given by [11, 12, 5, 13]

$$\begin{aligned}\mathcal{F} = {} & \frac{T}{2}\sum_{i\neq j}(h_i\cdot h_j)\,\theta_i\theta_j\,\partial_i\partial_j G_B(\nu_i-\nu_j) + \frac{i}{2}\sum_{i\neq j}(k_i\cdot h_j\,\theta_i - k_j\cdot h_i\,\theta_j)\partial_i G_B(\nu_i-\nu_j)\\ & +\frac{1}{2}\sum_{i\neq j}(h_i\cdot h_j)\,G_F(\nu_i-\nu_j) - \frac{i}{2}\sum_{i\neq j}(k_i\cdot h_j\,\theta_j - k_j\cdot h_i\,\theta_i)\,G_F(\nu_i-\nu_j)\\ & +\frac{1}{2}\sum_{i\neq j}\theta_i\theta_j\,(k_i\cdot k_j)\,G_F(\nu_i-\nu_j)\,. \end{aligned} \quad (2.16)$$

One can use an equivalent definition for $\bar{\mathcal{F}}$ where all the polarisations are taken to be $\bar{h}_i$, and the fermionic variables $\bar{\theta}_i$. We have introduced a scalar world-line Green's function

$G_B(x)$ and the $G_F(x)$ function defined by

$$G_B(x) = x^2 - |x|, \qquad G_F(x) = \mathrm{sign}(x)\,, \tag{2.17}$$

so that

$$\sum_{1\leq i<j\leq n} (k_i\cdot k_j)\, G_B(\nu_i-\nu_j) = Q_n\,. \tag{2.18}$$

The origin of each term in this expression can be traced back to the various contractions between the world-sheet variables. [The following definitions can be obtained by considering the $\alpha'\to 0$ limit of the corresponding superstring quantities as detailed in the appendix of [13]. They can be derived as well without any reference to string theory and are obtainable by considering the Green's function in the worldline formalism [11, 12]].

The bosonic contractions for $i\neq j$

$$\begin{aligned}\langle h_i\cdot\partial X(\nu_i)\, h_j\cdot\partial X(\nu_j)\rangle &= 2(h_i\cdot h_j)\,(\delta(\nu_i-\nu_j)-1)\,,\\ \langle h_i\cdot\partial X(\nu_i)\, \bar h_j\cdot\bar\partial X(\nu_j)\rangle &= 2(h_i\cdot \bar h_j)\,(\delta(\nu_i-\nu_j)-1)\,,\\ \langle h_i\cdot\partial X(\nu_i)\, k_j\cdot X(\nu_j)\rangle &= (h_i\cdot k_j)\,\partial_{\nu_i}G_B(\nu_i-\nu_j)\,.\end{aligned} \tag{2.19}$$

The fermionic contractions for $i\neq j$

$$\langle v_i\cdot\psi(\nu_i)\, v_j\cdot\psi(\nu_j)\rangle = (v_i\cdot v_j)\, G_F(\nu_i-\nu_j)\,. \tag{2.20}$$

Here v_i stands for an external polarisation h_i or momentum k_i. We will return to the structure of one-loop amplitudes in supergravity theories in section 5.

When extracting the multilinear part in the polarisations h_i and $\bar h_i$ which define $\mathcal{P}(h_i,\bar h_j,k_i,\nu_i)$ in eq. (2.15) one finds that the result is expressible solely in terms of the first derivative of the Green function $\partial_x G_B(x)$ and the second derivative of the Green function

$$\partial_x^2 G_B(x) = 2\,(\delta(x)-1)\,. \tag{2.21}$$

We want to stress that to the contrary of [11, 12, 5] we are *not* systematically integrating out the double derivatives of the Green function G_B.

We will make use of an expansion of the polarisations of the external states in a basis of independent momenta

$$h_i = \sum_{j=1}^{n-1} c_i^{\,j}\, k_j + q^\perp\,. \tag{2.22}$$

Here $q^\perp$ is a vector orthogonal to the $(n-1)$ linearly independent external momenta. The momentum k_j is only needed for dimensions $D>4$. An identical definition for the $\bar h_i$ polarisations can be employed. Using the relation eq. (2.18) one easily derives that

$$\begin{aligned}\sum_{j=1}^{n}(h_i\cdot k_j)\,\partial_i G_B(\nu_i-\nu_j) &= \sum_{j=1}^{n}(h_i\cdot k_j)\,[-2\,\nu_j - G_F(\nu_i-\nu_j)]\\ &= \sum_{r=1}^{n-1} c_i^{\,r}\left[\partial_r Q_n + \sum_{j=1}^{n}(k_r\cdot k_j)\,(G_F(\nu_r-\nu_j)-G_F(\nu_i-\nu_j))\right],\end{aligned} \tag{2.23}$$

which implies that $\mathcal{P}$ is only a function of the first derivatives of Q_n and G_F. Hence the amplitude takes the symbolic form

$$\mathcal{M}^{\mathcal{N}}_{n;1} = \sum_{\substack{r+s+u=2n-\mathcal{N}\\ 0\leq u\leq n}} \sum_{l=0}^{u} t^{l}_{r,s}\, \mathcal{I}^{[D+2(u-l)]}_{n-l}[\underline{\mathrm{I}}_r, \underline{\mathrm{J}}_s]\,. \tag{2.24}$$

It is clear that the integration over the fermionic variables in eq. (2.15) will bring powers of the polarisations multiplied by the G_F function in eq. (2.17) or the second derivative of the propagator $\partial_x^2 G_B$ in (2.21). Thereby the number of powers of single derivatives on the propagators are reduced. This reduces the number of single derivatives of Q_n via the relation in eq. (2.18). The integration over the $4\mathcal{N}$ fermionic variables in eq. (2.15) implies the following restriction on the powers ∂Q_n and G_F in eq. (2.24)

$$r + s \leq 2n - \mathcal{N}\,. \tag{2.25}$$

The cancellation of $\mathcal{N}$ powers of loop momenta in the n point one-loop amplitude depends on the number of fermionic zero modes which are independent of the nature of the external states. As a result of this an amplitude with other external states than gravitons from the massless supergravity multiplet will take the same form as in eq. (2.24), since the integrals $\mathcal{I}_n[\underline{\mathrm{I}}_r, \underline{\mathrm{J}}_s]$ that arise from the correlators of the world-line fields in eq. (2.19) and eq. (2.20) have the same form independently of the external massless states.

Because of the relation between the loop momentum and the total momentum $K_{[n]}$ defined in eq. (2.9) as well as the relation

$$\partial_{\nu_i} Q_n = -2\, k_i \cdot K_{[n]} - \sum_{m=1}^{n} (k_i\cdot k_m)\, G_F(\nu_i - \nu_m)\,, \tag{2.26}$$

the constraint in eq. (2.25) is equivalent to the statement that one-loop n-graviton amplitudes with $\mathcal{N}$ supersymmetries have the maximum power of loop momentum given by

$$\mathcal{P}_n(h_i, \bar h_i, k_i; \nu_i) \;\sim\; \ell^{2n-\mathcal{N}}, \quad \text{for} \quad \ell \gg 1\,. \tag{2.27}$$

This cancellation arises from the saturation of the $\mathcal{N}$ zero modes for the fermions running in the loop and is independent of the number of external states and the dimension as long as the number of supersymmetries are preserved. The constraint in eq. (2.25) will turn to be important for the analysis leading to the proof of the "no-triangle" hypothesis of the $\mathcal{N} = 8$ supergravity amplitude in section 5.

We have not integrated over the one-loop proper-time in eq. (2.14) because the contributions from first term of the first line in eq. (2.16) will give rise to expressions with higher powers of T to be integrated over. This will give rise two different types of contributions. The delta-function part from eq. (2.21) will give rise to an amplitude with less points and the constant piece in (2.21) will give rise to dimension shifted integrals. These will be discussed in detail in section 5.1.

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3. Reduction formulas

In this section we will derive new integral reduction identities relevant for the colourless n-point one-loop amplitude. We will define the *unordered* n-point integral in D dimensions

$$\mathcal{I}_n[\underline{\mathrm{I}}_r,\underline{\mathrm{J}}_s] \equiv \int_0^1 d^{n-1}\nu\, Q_n^{D/2-n} \prod_{i\in\underline{\mathrm{I}}_r} \partial_{\nu_i} Q_n \prod_{x\in\underline{\mathrm{J}}_s} G_F(x)\,. \tag{3.1}$$

These integrals will appear generically in amplitudes in the context of a string based formalism [11 − 13]. In this paper the amplitude are evaluated in $D = 4 - 2\epsilon$ dimensions.

In the above integral formula the proper-time has been integrated out and all orderings of the external legs are integrated over. We have defined the set of indices $\underline{\mathrm{I}}_r = \{i_1,\dots,i_r\}$ with $i_1, \dots, i_r$ indices taking values in $\{1,\dots,n\}$. We have defined the set of the differences of positions $\underline{\mathrm{J}}_s = \{\nu_{j_1}-\nu_{k_1},\dots,\nu_{j_s}-\nu_{k_s}\}$ with $j_1, \dots, j_s$ and $k_1, \dots, k_s$ indices taking values in $\{1,\dots,n\}$ which are the arguments of the sign function $G_F(x)$ in the integral. The integration is performed over the position ν_i of the external legs along the loop with the measure

$$d^{n-1}\nu = d\nu_1\cdots d\nu_n\,\frac{1}{n}\sum_{i=1}^{n}\delta(\nu_i-1)\,. \tag{3.2}$$

The translational invariance around the loop is fixed by freezing one of the ν_i to 1. In order to preserve the symmetry along the ν_i variables we insert a symmetrised delta-function. [For a given ordering, say the canonical ordering $(123\cdots n)$, the ν_i are related to the (rescaled) Feynman parameters a_i used in ref. [1] by the linear relations $\nu_i = \sum_{1\le m\le i} a_m$ given in (2.7).]

The quantity Q_n is defined by

$$Q_n \equiv \sum_{1\le i<j\le n} (k_i\cdot k_j)\, G_B(\nu_i-\nu_j)\,. \tag{3.3}$$

The functions $G_B(x)$ and $G_F(x)$ are defined by

$$G_B(x) = x^2 - |x|\,, \qquad G_F(x) = \mathrm{sign}(x)\ . \tag{3.4}$$

[These quantities are the infinite tension limit ($\alpha'\to 0$) bosonic and fermionic string propagators of the genus one amplitude (supersymmetry does however not play a rôle here) [11 − 13].]

- We will begin by considering n-point integrals with $r+1$ insertions of ∂Q_n factors and no insertions of G_F functions, so that $\mathcal{I}_n[\underline{\mathrm{I}}_{r+1}] \equiv \mathcal{I}_n[\underline{\mathrm{I}}_{r+1},\emptyset]$

$$\mathcal{I}_n[\underline{\mathrm{I}}_{r+1}] = \int_0^1 d^{n-1}\nu\, Q_n^{D/2-n} \prod_{i\in\underline{\mathrm{I}}_{r+1}} \partial_{\nu_i} Q_n\,, \tag{3.5}$$

  where $\underline{\mathrm{I}}_{r+1} \equiv \{i_1,\dots,i_{r+1}\}$. Now assume that i_{r+1} has multiplicity m in $\underline{\mathrm{I}}_{r+1}$, i.e. $\underline{\mathrm{I}}_{r+1} = \{i_{r+1}\}^m \cup \underline{\mathrm{I}}_{r-m+1}$ with $i_{r+1} \notin \underline{\mathrm{I}}_{r-m+1}$. We will set as well $\underline{\mathrm{I}}_r = \underline{\mathrm{I}}_{r-m+1} \cup \{i_{r+1}\}^{m-1}$, $\underline{\mathrm{I}}_{r-1}^{(r+1)} = \underline{\mathrm{I}}_{r-m+1}\cup\{i_{r+1}\}^{m-2}$, and finally $\underline{\mathrm{I}}_{r-1}^{(j)} \equiv \{i_1,\dots,\hat{i}_j,\dots,i_{r-m+1}\}\cup$

$\{i_{r+1}\}^{m-2}$, i.e. the list of $r-1$ indices with i_j omitted. Using that $Q_n^{D/2-n}\partial_\nu Q_n = (D/2-n+1)^{-1}\partial_\nu Q_n^{D/2-n+1}$ and integrating by parts in eq. (3.5) we get that

$$\mathcal{I}_n[\underline{\mathrm{I}}_{r+1}] = \frac{2}{D/2-n+1}\Bigg[\sum_{j\in\underline{\mathrm{I}}_{r-m+1}} (k_{i_{r+1}}\cdot k_j)\left(-\mathcal{I}_{n-1}^{(i_{r+1}j)}[\underline{\mathrm{I}}_{r-1}^{(j)}] + \mathcal{I}_n^{[D+2]}[\underline{\mathrm{I}}_{r-1}^{(j)}]\right) \\ +(m-1)\sum_{s=1}^{n}(k_{i_{r+1}}\cdot k_s)\,\mathcal{I}_{n-1}^{(i_{r+1}s)}[\underline{\mathrm{I}}_{r-1}^{(r+1)}]\Bigg]. \tag{3.6}$$

Because of the relation eq. (2.26) a power of $\partial_\nu Q_n$ corresponds to a power of loop momenta ℓ since $\partial_{\nu_i} Q_n \sim k_i\cdot\ell$. The above relation expresses the n-point amplitude $\mathcal{I}_n[\underline{\mathrm{I}}_{r+1}]$ with $r+1$ insertions of loop momentum factors as a linear combination of the $n-1$-point amplitudes with $r-1$ insertions of loop momenta $\mathcal{I}_{n-1}^{(i_{r+1}j)}[\underline{\mathrm{I}}_{r-1}^{(j)}]$ and $\mathcal{I}_{n-1}^{(i_{r+1}s)}[\underline{\mathrm{I}}_{r-1}^{(r+1)}]$. We have

$$\mathcal{I}_{n-1}^{(ij)}[\underline{\mathrm{I}}_{r-1}^{(j)}] \equiv \int_0^1 d^{n-1}\nu\, Q_n^{D/2-n}\,\delta(\nu_i-\nu_j)\prod_{s\in\underline{\mathrm{I}}_{r-1}^{(j)}}\partial_{\nu_s}Q_n\,, \tag{3.7}$$

and the n-point dimension shifted integral $\mathcal{I}_n^{[D+2]}[\underline{\mathrm{I}}_{r-1}^{(j)}]$ evaluated in dimensions $D+2$ is

$$\mathcal{I}_n^{[D+2]}[\underline{\mathrm{I}}_r] = \int_0^1 d^{n-1}\nu\, Q^{(D+2)/2-n}\prod_{i\in\underline{\mathrm{I}}_r}\partial_{\nu_i}Q_n\,. \tag{3.8}$$

Integrals with more than one mass are defined in the same way with several delta function insertions. The integration by part produces a boundary term

$$\partial\mathcal{I}_n[\underline{\mathrm{I}}_{r+1}] = \frac{1}{D/2-n+1}\int_0^1 d^{n-1}\nu\, Q_n^{D/2-n+1} \\ \times\left[\delta(\nu_{i_{r+1}}=1)-\delta(\nu_{i_{r+1}}=0)\right]\prod_{i\in\underline{\mathrm{I}}_r}\partial_{\nu_i}Q_n\,, \tag{3.9}$$

which is vanishing because of the 1-periodicity of Q_n in each of the ν_i variables, $Q_n(\nu_1,\dots,\nu_i+1,\dots) = Q_n(\nu_1,\dots,\nu_i,\dots)$ since $G_B(1-x)=G_B(x)$ for $0\leq x\leq 1$ and $G_B(0)=G_B(1)=0$.

These reduction formulas hence express any n-point integrals with r powers of loop momenta *summed over all orderings of the external legs* as linear combinations of $(n-1)$-point one-mass integrals with $r-2$ powers of loop momenta and possibly dimension shifted integrals. This can be summarised by the schematic rule

$$\mathcal{I}_n[(\partial Q_n)^r] \rightsquigarrow \mathcal{I}_{n-1}^{\text{mass}}[(\partial Q_n)^{r-2}] + \mathcal{I}_n^{[D+2]}[(\partial Q_n)^{r-2}]\,. \tag{3.10}$$

- When some factors of $G_F(x)$ are present in the integrand we have to distinguish between the following cases

  ▷ If all the $i\in\underline{\mathrm{I}}_r$ are such that ν_i is not an argument of $G_F(x)$ for any $x\in\underline{\mathrm{J}}_s$, then the same manipulations leading to eq. (3.6) and eq. (3.10) apply with no changes.

▷ If i_{r+1} has multiplicity one in $\underline{\mathrm{I}}_{r+1} = \underline{\mathrm{I}}_r \cup \{i_{r+1}\}$ with $i_{r+1} \notin \underline{\mathrm{I}}_r$ and $\underline{\mathrm{J}}_1 = \{\nu_{i_{r+1}} - \nu_j\}$ then

$$\mathcal{I}_n[\underline{\mathrm{I}}_{r+1}, \underline{\mathrm{J}}_1] = \frac{1}{D/2-n+1}\int_0^1 d^{n-1}\nu \partial_{\nu_{i_{r+1}}} Q_n^{D/2-n+1} \times \\ \times G_F(\nu_{i_{r+1}} - \nu_j) \prod_{i\in\underline{\mathrm{I}}_r} \partial_{\nu_i} Q_n. \tag{3.11}$$

Integrating this by parts leads to

$$\mathcal{I}_n[\underline{\mathrm{I}}_{r+1}, \underline{\mathrm{J}}_1] = \frac{2}{D/2-n+1} \times \\ \times \Bigg[\sum_{j\in\underline{\mathrm{I}}_r} (k_{i_{r+1}} \cdot k_j)(-\mathcal{I}_{n-1}^{(i_{r+1}j)}[\underline{\mathrm{I}}_{r-1}^{(j)}, \underline{\mathrm{J}}_1] + \mathcal{I}_n^{[D+2]}[\underline{\mathrm{I}}_{r-1}^{(j)}, \underline{\mathrm{J}}_1]) \\ + \Big((n-1)\mathcal{I}_{n-1}^{(i_{r+1}j)}[\underline{\mathrm{I}}_r] - \sum_{l=1}^{n} \mathcal{I}_n^{(i_{k+1}l)}[\underline{\mathrm{I}}_r^{(r+1)}]\Big)\Bigg]. \tag{3.12}$$

This is easily generalised to other cases, with higher multiplicity of i_{k+1} and with additional G_F contributions. We have the special cases,

$$\begin{aligned} \mathcal{I}_n[\{i\}, \{\nu_r - \nu_s\}] &= 0\,, \text{ for } i \notin \{r, s\}\,, \\ \mathcal{I}_n[\{i\}, \{\nu_i - \nu_j\}] &= -\frac{2}{D/2-n+1}\mathcal{I}_{n-1}^{(ij)}[\emptyset]\,, \\ \mathcal{I}_n[\emptyset, \{\nu_r - \nu_s\}] &= 0\,. \end{aligned} \tag{3.13}$$

An obvious generalisation of these identities imply that $\mathcal{I}_n[\partial Q_n, (G_F)^s] = 0$ for $s \geq 2$ and $\mathcal{I}_n[(G_F)^s] = 0$ for all $s > 0$.

We can conclude that when some G_F factors are present at best only *one power of loop momentum* is cancelled at each step of reduction as in the usual integral reductions case [2, 3, 1, 4]. This can be summarised schematically by the rule

$$\mathcal{I}_n[(\partial Q_n)^r, G_F] \rightsquigarrow \mathcal{I}_{n-1}^{\text{mass}}[(\partial Q_n)^{r-1}] + \mathcal{I}_{n-1}^{\text{mass}}[(\partial Q_n)^{r-2}, G_F] + \mathcal{I}_n^{[D+2]}[(\partial Q_n)^{r-2}, G_F]\,. \tag{3.14}$$

We have shown that in the unordered integral, because of the sum over all the permutations of the external legs, new integral reduction formulas, given schematically by eq. (3.10) and eq. (3.14), are valid.

Before closing this section we will make a few remarks concerning the reduction formulas.

The main reason for introducing the integrals in eq. (3.1) is because they are the building blocks of the 'string based' method. The ordered integrals considered in [1] are expressed in terms of the Feynman parameters a_i which are linearly related to the ν_i parameters with the relation (2.7) and to the $\partial_{\nu_i} Q_n$ using the relation (2.26). The main difference with the analysis in that paper lie in the fact that we considering unordered integral expressions where the absence of boundaries imply the vanishing of total derivative contributions. We would like to stress that these identities are crucial for the observed extra simplicity of unordered gravity amplitudes. It is important that the crossing symmetry of gravity amplitudes assure that each ordered amplitude have the same tensorial structure leading to the generic structure for the amplitude as given in eq. (2.24) allowing us to use the reduction formulas derived in that section.

These rules are the general cases of the identities used in ref. [13] for the cancellation of triangle contributions to the five graviton amplitude at one-loop in $\mathcal{N} = 8$ supergravity.

4. Structure of the supergravity amplitude

The n-graviton amplitudes in supergravity with $\mathcal{N}$ supersymmetries at one-loop have the following representation that we described in section 2

$$\mathcal{M}_{n;1}^{\mathcal{N}} = \Gamma(n - D/2) \times \int_0^1 d^{n-1}\nu P(\varepsilon_{ij}, k_i, \nu_i)\, Q_n^{D/2-n} \,. \tag{4.1}$$

Here $\varepsilon_{ij} = (h_i\bar{h}_j + h_j\bar{h}_i)/2$ are the polarisations of the gravitons and the integral contains a polynomial of order $2n-\mathcal{N}$ in ν_i given by $P(\varepsilon_{ij}, k_i, \nu_i) = P(H_i \cdot K_{[n]}, Y_{ij}\, G_F(\nu_i - \nu_j), (h_i \cdot h_j)\, \delta(\nu_i - \nu_j))$, where

$$K_{[n]} = \sum_{m=1}^{n} k_m\, \nu_m \,, \tag{4.2}$$

is the total loop momentum, H_i is h_i or $\bar{h}_i$ and $Y_{ij} \in \{(H_i \cdot H_j),\ (H_i \cdot k_j),\ (k_i \cdot k_j)\}$ for $i \neq j$. The structure of $P(\varepsilon_{ij}, k_i, \nu_i)$ can easily be determined using the string based rules [11 − 13] and is given by bosonic contractions, plane wave factor contractions and fermionic contractions as described in section 2. We will discuss these in turn [we refer to the appendix A of [13] for conventions and derivation of these field theory limits].

The bosonic contractions for $i \neq j$

$$\begin{aligned} \langle h_i \cdot \partial X(\nu_i)\, h_j \cdot \partial X(\nu_j)\rangle &= 2(h_i \cdot h_j)\, (\delta(\nu_i - \nu_j) - 1)\,, \\ \langle h_i \cdot \partial X(\nu_i)\, \bar{h}_j \cdot \bar{\partial} X(\nu_j)\rangle &= 2(h_i \cdot \bar{h}_j)\, (\delta(\nu_i - \nu_j) - 1)\,. \end{aligned} \tag{4.3}$$

The δ-function leads to the reducible contributions which are present in the amplitude from five-point order [13]. These give rise to the higher-point vertices to the field theory loop amplitude contribution. The zero mode of the bosonic world-sheet coordinate contributes to the constant piece. This piece once plugged into the first contribution in eq. (2.16) leads to a dimension shifted integral after having integrated over the proper-time T in eq. (2.14). An amplitude with $\mathcal{N}$ supersymmetries contains the contributions $\mathcal{I}_n^{[D+2k]}[\underline{\mathrm{I}}_r, \underline{\mathrm{J}}_s]$ for $0 \leq k \leq n - \mathcal{N}/2$. [This corresponds to the contributions $\mathcal{A}_n^{(2)\infty}$ in eq. (2.7) and eq. (5.4) of reference [13]]. These contributions are proportional to $h_i \cdot \bar{h}_j$ for $i \neq j$, which is not invariant under the (linearised) gauge transformations $h_i \to h_i + k_i\lambda$. Gauge invariance is recovered when momentum dependent contributions from integrating by parts in the reduction formulas eq. (5.4) are taken into account.

The contraction with plane-wave factors

$$\sum_{\substack{1\leq m\leq n\\ m\neq i}} (h_i\cdot k_m)\,\partial_{\nu_i} G_B(\nu_i-\nu_m) = -2h_i\cdot K_{[n]} - \sum_{m=1}^{n} (h_i\cdot k_m)\,G_F(\nu_i-\nu_m)\,. \tag{4.4}$$

Here we have used that a given external state has to be contracted with all the plane-wave factors of the other external states.

The fermionic contractions for $i\neq j$

$$\langle v_i\cdot\psi(\nu_i)\,v_j\cdot\psi(\nu_j)\rangle = (v_i\cdot v_j)\,G_F(\nu_i-\nu_j)\,. \tag{4.5}$$

Here v_i stands for an external polarisation h_i or momentum k_i.

In D dimensions we can expand each of the polarisation tensors h_i and $\bar h_i$ in a basis of independent momenta

$$h_i = \sum_{j=1}^{n-1} c_i{}^j\,k_j + q^\perp\,. \tag{4.6}$$

Here $q^\perp$ is a vector orthogonal to the $(n-1)$ linearly independent external momenta k_j. We use an identical definition for the $\bar h_i$ polarisations. Using that $\partial_{\nu_i} Q_n = -2k_i\cdot K_{[n]} - \sum_{m=1}^n (k_i\cdot k_m)\,G_F(\nu_i-\nu_m)$ one gets that

$$h_i\cdot K_{[n]} = \sum_{j=1}^{n-1}\frac{c_i{}^j}{2}\Big[-\partial_{\nu_j}Q_n + \sum_{m=1}^{n}(k_j\cdot k_m)G_F(\nu_j-\nu_m)\Big]\,. \tag{4.7}$$

Hence the amplitude eq. (4.1) reduces to a sum of integral contributions of eq. (3.1) involving $\partial_\nu Q_n$ and G_F factors

$$\mathcal{M}^{\mathcal{N}}_{n;1} = \sum_{\substack{r+s+u=2n-\mathcal{N}\\ 0\leq u\leq n}}\ \sum_{l=0}^{u} t^l_{r,s}\,\mathcal{I}^{[D+2(u-l)]}_{n-l}[\underline{\mathrm{I}}_r,\underline{\mathrm{J}}_s]\,. \tag{4.8}$$

In this equation $t_{r,s}$ is the tensorial structure to be discussed in more details elsewhere [14], and $\mathcal{I}_n[\underline{\mathrm{I}}_r,\underline{\mathrm{J}}_s]$ are the loop integrals on which we can apply the reduction formulas in eq. (3.10) and eq. (3.14).

5. Maximal supergravity

For the case of $\mathcal{N}=8$ supergravity the highest contribution of loop momentum is for $r=2(n-4)$ and $s=0$. There are no powers of G_F in the integral. A direct application of n steps of the reduction rules eq. (3.10) gives

$$\begin{aligned}\mathcal{I}_n[(\partial Q_n)^{2(n-4)}] &\rightsquigarrow \mathcal{I}^{\rm mass}_{n-1}[(\partial Q_n)^{2(n-5)}] + \mathcal{I}^{[D+2]}_{n}[(\partial Q_n)^{2(n-5)}]\\ &\rightsquigarrow\cdots\rightsquigarrow \mathcal{I}^{\rm mass}_4[\emptyset] + \sum_{m=1}^{n-4}\mathcal{I}^{[D+2m]}_{4+m}[\emptyset]\,.\end{aligned} \tag{5.1}$$

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Hence we end up with scalar box integral functions plus dimension shifted scalar integrals. We demonstrate that these cancel when one includes the dimension shifted contribution from the contractions of polarisations in eq. (4.3).

For $s \neq 0$, and $r > s$ one first applies s steps of the reduction formula eq. 3.14 to get

$$\mathcal{I}_n[(\partial Q_n)^r, (G_F)^s] \rightsquigarrow \mathcal{I}^{\text{mass}}_{n-s}[(\partial Q_n)^{r-s}] + \mathcal{I}^{[D+2]}_n[(\partial Q_n)^{r-s}]\,, \tag{5.2}$$

then one applies $(r-s)/2$ steps of the reduction formula eq. 3.10 to get to

$$\begin{aligned}\mathcal{I}_n[(\partial Q_n)^r, (G_F)^s] &\rightsquigarrow \mathcal{I}^{\text{mass}}_{n-s}[(\partial Q_n)^{r-s}] + \mathcal{I}^{[D+2]}_n[(\partial Q_n)^{r-s}] \rightsquigarrow \cdots \\ &\rightsquigarrow \mathcal{I}^{\text{mass}}_{n-(r+s)/2}[\emptyset] + \sum_{m=1}^{n-4} \mathcal{I}^{[D+2m]}_{4+m}[\emptyset]\,.\end{aligned} \tag{5.3}$$

Since $r+s = 2(n-4)$ in $\mathcal{N} = 8$ supergravity one ends with scalar box integral functions plus dimension shifted scalar integrals which again will cancel in the total amplitude. For $s > r$ there are only vanishing contributions from the reduction formulas via eq. (3.13) and its generalisation.

5.1 Cancellation of the dimension shifted integrals

The dimension shifted scalar integrals generated by the reduction formula have the momentum space representation

$$\mathcal{I}^{[4+2m]}_{4+m}[\emptyset] = -i\frac{(-1)^k(4\pi)^{2+m-\epsilon}}{\Gamma(2+\epsilon)} \times \int_0^\infty \frac{d^{4-\epsilon}\ell d^{2m}\ell_\perp}{(2\pi)^{4+2m-2\epsilon}} \prod_{i=1}^{4+m} \frac{1}{(\ell - k_1 - \cdots k_i)^2 + \ell_\perp^2}\,. \tag{5.4}$$

The $4+m$-point scalar loop integral with four dimensional kinematics in eq. (5.4) is evaluated in $4+2m$ dimensions, does not carry any UV or IR divergences and is finite when $\epsilon \to 0$.

The reduction formulas derived in the previous section do not contain any information about the gauge invariance and the number of supersymmetries of the theory. In the total physical amplitude the higher dimensional contributions generated by these reduction formulas will combine with the ones from the contractions in eq. (4.3) so that they appear with gauge invariant coefficients, as we show below. For $\mathcal{N} = 8$ supergravity amplitudes these contributions sum up to total derivatives and do not contribute to the physical amplitude.

For two given external states labelled i and j and with H_i for the polarisations h_i or $\bar{h}_i$, the zero-mode part of each contraction in eq. (4.3) contributes to the amplitude as follows

$$\begin{aligned}\langle H_i \cdot \partial_{\nu_i} X\, H_j \cdot \bar\partial_{\nu_j} X\rangle &\to -2(H_i \cdot H_j)\, Q_n \\ &= -2\, Q_n \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} c_i{}^r c_j{}^s\, (k_r \cdot k_s) \\ &= \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} c_i{}^r c_j{}^s\, Q_n\, [\partial_{\nu_r}\partial_{\nu_s} Q_n - 2\delta(\nu_r - \nu_s)]\,.\end{aligned} \tag{5.5}$$

In the second line we have used the expansion of the polarisations in the basis of momenta defined in eq. (4.6). The dimension shifted integrals arises from the two derivatives acting on Q_n. We will now show that this contribution will cancel against a corresponding contribution from the contractions of the polarisations with the plane-wave factors in eq. (4.4)

$$\sum_{\substack{1\leq p\leq n\\ p\neq i}}\sum_{\substack{1\leq q\leq n\\ q\neq j}}\langle h_i\cdot\partial_{\nu_i}X\,k_p\cdot X\rangle\,\langle\bar h_j\cdot\bar\partial X_{\nu_j}\,k_q\cdot X\rangle\to\sum_{r=1}^{n-1}\sum_{s=1}^{n-1}c_i{}^r c_j{}^s\partial_{\nu_r}Q_n\partial_{\nu_s}Q_n\,. \tag{5.6}$$

Since the contributions eq. (5.5) corresponds to the contraction between two polarisation of the external states and eq. (5.6) corresponds to the contraction between the polarisations and the plane-wave factors these contributions arise in the amplitude with the following coefficients

$$\begin{aligned}\text{eq. (5.5)}+\text{eq. (5.6)}\to&\sum_{r=1}^{n-1}\sum_{s=1}^{n-1}c_i{}^r c_j{}^s\left[\partial_{\nu_r}Q_n\partial_{\nu_s}Q_n+Q_n\,\partial_{\nu_r}\partial_{\nu_s}Q_n-2Q_n\,\delta(\nu_r-\nu_s)\right] \qquad (5.7)\\ =&\sum_{s=1}^{n-1}c_i{}^r c_j{}^s\left[\partial_{\nu_r}(Q_n\partial_{\nu_s}Q_n)-2Q_n\,\delta(\nu_r-\nu_s)\right].\end{aligned}$$

The second given by the delta-function collapses two external legs and generates a massive scalar integral with one leg less. The first term generated by integration by parts produces a dimension shifted integral of the same structure as the one generated by the reduction formula in eq. (3.6).

We will now show how this works in few specific examples.

- At five point order the $\mathcal{N}=8$ amplitude in eq. (4.8) involves only $\mathcal{I}_5[(\partial Q_5)^r,(G_F)^s]$ with $r+s=2$. The contribution $\mathcal{I}_5[(\partial Q_5)^2]$ arises from the contractions of eight left-moving and right-moving fermions and corresponds to the contributions $\mathcal{A}_5^{(2)\infty}$ and the contribution $|\mathcal{A}_5^{(1)\infty}-t_{10}\cdot F^5|^2$ in eq. (2.7) of ref. [13]

$$\delta\mathcal{M}_{5;1}^{\mathcal{N}=8}=\sum_{i,j=1}^{5}t_8F_{\hat\imath}^4t_8F_{\hat\jmath}^4\,\langle(h_i\cdot\partial X)\,(\bar h_j\cdot\bar\partial X)\prod_{r=1}^{5}e^{ik_r\cdot X}\rangle\,, \tag{5.8}$$

  where $t_8F_{\hat\imath}^4$ is defined in eq. (2.1) of ref. [13]. Focusing on the contractions eq. (5.5) and eq. (5.6), this expression leads to the following contributions

$$\begin{aligned}\delta\mathcal{M}_{5;1}^{\mathcal{N}=8}\Big|_{(5.5)+(5.6)}&=\sum_{i,j=1}^{5}t_8F_{\hat\imath}^4t_8F_{\hat\jmath}^4\,c_i{}^r c_j{}^s\;\times \qquad (5.9)\\ &\quad\times\Gamma(5-D/2)\int_0^1 d^4\nu Q_5^{D/2-5}\left[\partial_{\nu_r}Q_5\partial_{\nu_s}Q_5+\frac{1}{D/2-4}Q_5\partial_{\nu_r}\partial_{\nu_s}Q_5\right]\\ &=\sum_{i,j=1}^{5}t_8F_{\hat\imath}^4t_8F_{\hat\jmath}^4\,c_i{}^r c_j{}^s\,\Gamma(3-D/2)\int_0^1 d^4\nu\,\partial_{\nu_r}\partial_{\nu_s}Q_5^{D/2-3}\,,\end{aligned}$$

which again leads to a vanishing total derivative by the 1-periodicity of Q_5 with respect to each of the ν_i variables. This implies that in the five-point $\mathcal{N}=8$ supergravity amplitude the highest contribution in loop momentum $\mathcal{I}_5[(\partial Q_5)^2]$ has been completely cancelled against the dimension shifted contribution $\mathcal{I}_5[Q_5\partial^2 Q_5]$. The other contribution $\mathcal{I}_5[\partial Q_5, G_F]$ does not receive contributions from the dimension shifted integral (thanks to eq. (3.13)) and is hence completely reducible to scalar box integrals defined in $D=4-2\epsilon$ dimensions. $\mathcal{I}_5[(G_F)^2]$ is vanishing by the generalisation of eq. (3.13).

- The six point $\mathcal{N}=8$ amplitude in eq. (4.8) involves the contributions $\mathcal{I}_6[(\partial Q_6)^r, (G_F)^s]$ with $r+s=4$. The contribution $\mathcal{I}_6[(\partial Q_6)^4]$ arises from the contractions of eight left-moving and right-moving fermions [from the square of the last term in $\mathcal{A}_6^{(1)}$ in eq. (5.1) and the last contribution to $\mathcal{A}_6^{(2)}$ in eq. (5.4) in ref. [13]] and is given by

$$\delta\mathcal{M}_{6;1}^{\mathcal{N}=8} = \sum_{ij,pq} t_{ij,pq}\,\langle (h_i\cdot\partial X)(h_j\cdot\partial X)\,(\bar h_p\cdot\bar\partial X)\,(\bar h_q\cdot\bar\partial X)\prod_{r=1}^{6} e^{ik_r\cdot X}\rangle\,, \quad (5.10)$$

where $t_{ij,pq}=t_8F^4_{\hat\imath,\hat\jmath}t_8F^4_{\hat p,\hat q}$, and with $t_8F^4_{\hat\imath,\hat\jmath}$ defined as in ref. [13] [It is a tensorial structure composed from four of the external polarisations and four of the external momenta]. The $H_i\cdot H_j$ contractions and the contractions with the plane-wave factor $\exp(ik\cdot X)$ lead to

$$\begin{aligned}\delta_1\mathcal{M}_{6;1}^{\mathcal{N}=8} = \sum_{r,s,t,u} t^{rstu}\,\Gamma(6-D/2)\int_0^1 d^5\nu\, Q_6^{D/2-6} \times \Big[&\partial_{\nu_r}Q_6\partial_{\nu_s}Q_6\,\partial_{\nu_t}Q_6\partial_{\nu_u}Q_6 \quad (5.11)\\ &+\frac{Q_6^2}{(D/2-4)(D/2-5)}\times\\ &\times\big(\partial_{\nu_r}\partial_{\nu_s}Q_6\,\partial_{\nu_t}\partial_{\nu_u}Q_6+\partial_{\nu_r}\partial_{\nu_u}Q_6\,\partial_{\nu_t}\partial_{\nu_s}Q_6+\partial_{\nu_r}\partial_{\nu_t}Q_6\,\partial_{\nu_s}\partial_{\nu_u}Q_6\big)\\ &+\frac{Q_6}{D/2-5}\big(\partial_{\nu_r}Q_6\partial_{\nu_s}Q_6\,\partial_{\nu_t}\partial_{\nu_u}Q_6+\partial_{\nu_t}Q_6\partial_{\nu_u}Q_6\,\partial_{\nu_r}\partial_{\nu_s}Q_6+\\ &+\partial_{\nu_s}Q_6\partial_{\nu_u}Q_6\,\partial_{\nu_r}\partial_{\nu_t}Q_6\partial_{\nu_t}Q_6\partial_{\nu_s}Q_6\,\partial_{\nu_r}\partial_{\nu_u}Q_6+\\ &+\partial_{\nu_r}Q_6\partial_{\nu_u}Q_6\,\partial_{\nu_t}\partial_{\nu_s}Q_6+\partial_{\nu_r}Q_6\partial_{\nu_t}Q_6\,\partial_{\nu_s}\partial_{\nu_u}Q_6\big)\Big]\,.\end{aligned}$$

Here we have defined $t^{rstu}\equiv t_{ijpq}\,c_i{}^r c_j{}^t c_p{}^s c_q{}^u$. Using that the third (and higher) derivative term $\partial_{\nu_i}\partial_{\nu_j}\partial_{\nu_k}Q_n=0$ vanishes, we can rewrite the previous expression as a total derivative

$$\begin{aligned}\delta_1\mathcal{M}_{6;1}^{\mathcal{N}=8} &= \sum_{r,s,t,u} t^{rstu}\,\Gamma(4-D/2)\int_0^1 d^5\nu\;\partial_{\nu_r}\partial_{\nu_s}\big(Q_6^{D/2-4}\partial_{\nu_t}Q_6\partial_{\nu_u}Q_6\big) \quad (5.12)\\ &+\sum_{r,s,t,u} t^{rstu}\,\Gamma(4-D/2)\int_0^1 d^5\nu\,\partial_{\nu_r}(Q_6^{D/2-4}\;\partial_{\nu_s}Q_6\partial_{\nu_t}\partial_{\nu_u}Q_6)\,.\end{aligned}$$

As before the boundary terms vanish by the 1-periodicity of the integrand. Therefore the highest power of loop momentum $\mathcal{I}_6[(\partial Q_6)^4]$ in the $\mathcal{N}=8$ supergravity six-point amplitude has completely cancelled against the dimension shifted contributions.

The contribution $\mathcal{I}_6[(\partial Q_6)^3, G_F]$ combines with the dimension shifted integrals according to

$$\begin{aligned}
\delta_2 \mathcal{M}^{\mathcal{N}=8}_{6;1} &= \sum_{r,s,u} t^{rsu}\Gamma(6-D/2)\int_0^1 d^5\nu Q_6^{D/2-6} G_F(\nu_p-\nu_q)[\partial_{\nu_r}Q_6\partial_{\nu_s}Q_6\partial_{\nu_u}Q_6 \\
&\quad + \frac{Q_6}{(D/2-5)}(\partial_{\nu_r}\partial_{\nu_s}Q_6\partial_{\nu_u}Q_6 + \partial_{\nu_r}\partial_{\nu_u}Q_6\partial_{\nu_s}Q_6 + \partial_{\nu_r}Q_6\partial_{\nu_s}\partial_{\nu_u}Q_6)] \\
&= -\sum_{r,s,u} t^{rsu}\,\Gamma(3-D/2)\int_0^1 d^5\nu\, G_F(\nu_p-\nu_q)\,\partial_{\nu_r}\partial_{\nu_s}\partial_{\nu_u}Q_6^{D/2-3} \\
&= 2\sum_{s,u}(t^{psu}-t^{qsu})\Gamma(3-D/2)\int_0^1 d^5\nu\,\partial_{\nu_s}\partial_{\nu_u}(\delta(\nu_p-\nu_q)\delta(\nu_r-\nu_p)\,Q_6^{D/2-3})\,,
\end{aligned} \tag{5.13}$$

where ν_p and ν_q denotes positions of external states. This expression vanishes as a total derivative. The dimension shifted contributions in the quadratic hexagon $\mathcal{I}_6[(\partial Q_6)^2, (G_F)^2]$ are treated in a similar fashion as follows

$$\begin{aligned}
\delta_3 \mathcal{M}^{\mathcal{N}=8}_{6;1} &= \sum_{r,s} t^{rs}\,\Gamma(6-D/2)\int_0^1 d^5\nu Q_6^{D/2-6}\,\times \\
&\qquad \times \prod_{x\in \underline{\mathrm{J}}_2} G_F(x)\left[\partial_{\nu_r}Q_6\partial_{\nu_s}Q_6 + \frac{Q_6}{(D/2-5)}\,\partial_{\nu_r}\partial_{\nu_s}Q_6\right] \\
&= \sum_{r,s} t^{rs}\,\Gamma(4-D/2)\int_0^1 d^5\nu \prod_{x\in \underline{\mathrm{J}}_2} G_F(x)\,\partial_{\nu_r}\partial_{\nu_s}Q_6^{D/2-4} \\
&= -2\sum_{s}\Gamma(3-D/2)\int_0^1 d^5\nu\,\partial_{\nu_s}Q_6^{D/2-4} \\
&\quad \times[(t^{ps}-t^{qs})\delta(\nu_p-\nu_q)G_F(\nu_u-\nu_v)+(t^{us}-t^{vs})\delta(\nu_u-\nu_v)G_F(\nu_p-\nu_q)]\,,
\end{aligned} \tag{5.14}$$

and is given by a sum of linear one mass pentagons, which are completely reducible to scalar boxes. The last two contributions are the linear hexagon $\mathcal{I}_6[\partial Q_6, (G_F)^3]$ and the scalar hexagon $\mathcal{I}_6[(G_F)^4]$ which are vanishing.

We have thus shown that the dimension shifted contributions from the contractions in eq. (4.3) cancel against the ones from the reduction formulas. We would like to stress that this mechanism does not require any supersymmetry.

This implies that the only basis functions for the one-loop n-graviton amplitude in $\mathcal{N}=8$ supergravity are four-dimensional scalar box integral functions. This proves the "no-triangle" hypothesis.

6. Gravity with less or no supersymmetries

For less supersymmetry or in the case of pure gravity the application of the reduction formula eq. (3.10) and eq. (3.14) leads to

$$\mathcal{I}_n[(\partial Q_n)^r, (G_F)^s] \to \mathcal{I}^{\text{mass}}_{\mathcal{N}/2}[\emptyset]\,. \tag{6.1}$$

The endpoint of the chain of reductions is given by a scalar $\mathcal{N}/2$-point integral function. The dimension shifted integrals cancel in the physical amplitude following the previous arguments.

We can conclude the following from eq. (6.1):

- Theories with $\mathcal{N} \geq 3$ contain integral functions down to scalar bubbles and are hence one-loop cut constructible. This confirms the analysis of ref. [10] from considerations of on-shell unitarity cuts of the amplitude.

- For $\mathcal{N} = 0$ gravity: one-loop amplitudes are reducible down to rational parts as are QCD amplitudes. Since in four dimensions only the two-point (bubble) integral has UV logarithmic divergences, we conclude that one-loop gravity amplitudes are at most logarithmically diverging. For pure gravity this divergence cancels on-shell [15] but is present when coupled to matter [16].

7. Discussion and conclusions

In this paper we have explored the integral expansion of the one-loop n-point graviton amplitude in pure gravity and in supersymmetric extensions. It has been shown that these unordered amplitudes are constrained by new integral reduction formulas for colourless gauge theories in four dimensions. Decomposing the polarisation tensors in the amplitudes in a basis of independent momenta enables the use of these reduction formulas in a form that does not require the need to invert any Gram determinant in the kinematic variables. The Gram determinant generally vanishes for special kinematic configurations associated with particle productions at thresholds or planar dependence between external momenta. This is particularly suitable for a numerical analysis of the ϵ expansion of the amplitude [14]. For maximal $\mathcal{N} = 8$ supergravity this leads to "no-triangle" properties of the n-point supergravity amplitudes. This shows that the n-graviton amplitude at one-loop is completely specified by scalar box integral functions. The proof of the 'no triangle hypothesis' of one-loop amplitudes in $\mathcal{N} = 8$ supergravity was mainly discussed in the case of external graviton states but is generalisable to all matter states in the supergravity multiplet. This is because the constraint eq. (2.25) arising from the cancellation of $\mathcal{N}$ powers of loop momenta in the loop amplitude do not depend on the nature of the external states due to supersymmetry. Because the integrals $\mathcal{I}_n[\underline{\mathrm{I}}_r, \underline{\mathrm{J}}_s]$ which arise from the correlations of the world-line fields in eq. (2.19) and eq. (2.20) have the same form whatever the external massless states, any one-loop n-point amplitude between states of the massless supergravity multiplet will thus lead to amplitudes of the form eq. (2.24). For these amplitudes the unordered reduction formula can be directly applied. In $\mathcal{N} = 4$ supergravity it means that the n-graviton amplitude contains only integral functions up to scalar bubbles and thus is constructible from its cuts in $D = 4 - 2\epsilon$. For pure gravity our result yields an amplitude consisting of scalar box, triangle and bubble integrals as well as rational pieces.

The lack of colour in massless QED, means that one can as well apply the reduction formulas eq. (3.10) and eq. (3.14) to the light-light n-photon scattering at one-loop. For instance, a six-photon one-loop amplitude has at most six powers of loop momenta, $\mathcal{I}_6[\underline{\mathrm{I}}_r, \underline{\mathrm{J}}_s]$

with $r + s = 6$. Here the reduction formulas imply that the four dimensional one-loop amplitude can be expanded in terms of scalar box and triangle integral functions in $D = 4 - 2\epsilon$ dimensions. This is in agreement with the results of [18, 19].

Because of the colour factors in Yang-Mills theory, n-gluon amplitudes in $\mathcal{N} = 4$ super-Yang-Mills have to be reduced using the usual colour-ordered reduction formulas [1, 20]. Here only one power of loop momentum is cancelled at each step of reduction and the n-point amplitude is completely determined by scalar box integral functions. It is interesting to note the following: in colourless theories the sum over all orderings exactly produces the extra cancellations of loop momenta required to arrive at the same structure of the n-point one-loop $\mathcal{N} = 8$ supergravity amplitude as of the n-point one-loop $\mathcal{N} = 4$ super-Yang-Mills amplitude. This directly explains the similarity of the UV and IR structure in one-loop n-point amplitudes in $\mathcal{N} = 8$ supergravity and $\mathcal{N} = 4$ super-Yang-Mills.

The absence of triangles in one-loop $\mathcal{N} = 8$ supergravity amplitudes restricts the form of the multi-loop amplitudes [21], and is a necessary (but not sufficient) requirement for the absence of the three-loop divergence in four dimensions [22] and the possible perturbative finiteness of $\mathcal{N} = 8$ supergravity in four dimensions [9, 23, 21, 24, 10]. The results of this paper adds to the empirical knowledge of perturbative $\mathcal{N} = 8$ supergravity and $\mathcal{N} = 4$ super-Yang-Mills. It would be interesting to investigate if similar unordered integral reductions are possible to employ in the analysis of higher loop integrals. If so this could add another important clue in the of understanding of why higher-loop amplitudes in $\mathcal{N} = 8$ supergravity and $\mathcal{N} = 4$ super-Yang-Mills seemingly have the same UV-behaviour [23, 21] in four dimensions.

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