NON-RENORMALISATION THEOREMS IN SUPERSTRING AND SUPERGRAVITY THEORIES

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1. Introduction

The theoretical construction of unification models for particle physics has led to remarkable progress in the understanding of the fundamental interactions involved in particle physics phenomena at accelerator energy scale, or in cosmological phenomena responsible for the formation of the visible matter of our universe. However a good understanding of quantum gravity effects at either short distances or large (cosmological) scales is still lacking. It is expected that subtle quantum gravity effects could be at work behind some of the outstanding fundamental problems of modern cosmology and particle physics models and their ultra-violet completion (or may be the absence of constraints from the ultra-violet completion). For instance, because of our poor understanding of the rules for a correct quantisation of the gravitational forces one gets a landscape of vacua for unification models coupled to gravity, as was explained by Nima Arkani-Hamed, Frederik Denef and Michael Douglas at this school. Hopefully, the difficulties of charting the physically relevant vacua of string theory (or any other consistent theory of quantum gravity) would be resolved once the correct boundary conditions and quantization rules for quantum gravity has been better understood.

It is a remarkable feature of the string setup to provide a consistent theory for quantum gravity and its supersymmetric extensions [1–3]. String theory provides a consistent framework for analysing perturbative and non-perturbative aspects of quantum gravity. The low energy approximation of various compactifications of string theory leads to the supergravity theories and their quantum corrections. In string theory based models the various coupling constant depends on the moduli of the theory which are acted on by the perturbative and non-perturbative symmetries of string theory. These symmetries are the U-dualities [4] connecting all the different corners of the M-theory moduli space [5].

A typical amplitude computed within string theory is given by an integral over the moduli space of the punctured Riemann surface used for the definition of the amplitude [1,2,6]. This moduli space resums in a very compact expression the contributions from the many Feynman graphs one has to sum in the usual field theoretical analysis [7–10]. This compact formulation makes explicit some cancellations that are not a priori obvious using the traditional Feynman rules for constructing the amplitudes. Even if the maximal $N = 8$ supergravity is perturbatively ultra-violet finite, it will not be complete in the ultra-violet and the inclusion of extra non-perturbative states will be needed for getting a consistent theory. These extra states are charged under the U-dualities of M-theory and their decoupling from the supergravity massless states is singular [11].

In these lecture notes we will describe the various lessons than one can draw about the role of linearised on-shell supersymmetry and the string dualities in the analysis of amplitude computations in the gravitational sector of string theory and supergravity theories in various dimensions. We will review, in section 2 the role of the on-shell linearised extended supersymmetry on the ultra-violet behaviour of multi-graviton amplitudes. The pure spinor formalism of N. Berkovits [12–15], has provided a new understanding of the role of supersymmetry for the case of $N = 8$ supergravity in diverse dimensions, and allowed to construct a new set of higher-derivative gravitational F-terms [16]. We will then discuss, in section 3, some conditions for expecting that multi-graviton amplitude in $N = 8$ supergravity in flat space could have a much better ultra-violet behaviour and its relation with $N = 4$ super-Yang–Mills will be reviewed. Then in section 4 we will discuss the construction of higher-loop amplitudes in supergravity theory using the on-shell unitarity method [17]. In section 5 we will apply the relation between the supergravity in eleven dimensions and perturbative string theory provided by M-theory to get constraints on the low-energy behaviour of multi-graviton amplitudes. In section 4.2 and 4.3 we analyse the structure of the four-graviton one-loop and two-loop amplitudes in $N = 8$ supergravity, and in section 8 we describe various non-renormalisation conditions on some gravitational couplings to the low-energy effective action of string theory. In Appendix A we describe some of the structure of graviton amplitudes at the tree-level, genus-one, genus two and higher-order in string theory.

2. Ultra-violet divergences ...

A perturbative treatment of gravity [7, 8] in the background field method [9, 10], by linearization of the Einstein-Hilbert actions

$$\mathcal{L} = \frac{1}{\kappa_{(D)}^2} \int d^D x \sqrt{-g^{(D)}}\, \mathcal{R}^4[g] \tag{2.1}$$

around a specific background $g_{\mu\nu} = g^{(0)}_{\mu\nu} + \kappa_{(D)}\, h_{\mu\nu}$ give rise to an infinite set of effective vertices of two derivative nature. The Newton's constant in D-dimension has dimension $\kappa^2_{(D)} = (length)^{D-2}$ and an L-loop, n-graviton amplitude in D-dimension as mass dimension

$$[M_{n;L}] \sim \kappa_{(D)}^{2(L-1)+n}\, (mass)^{(D-2)L+2}\,. \tag{2.2}$$

One notices that the mass dimension of the amplitude is independent of the number of external legs n (except for the dependence on $\kappa_{(D)}$ from the normalisation of the external states) due to the two derivative coupling nature of the interactions.

2.1. ... in pure gravity

In pure gravity the one-loop four-graviton amplitude has dimension

$$[M_{4;1}^{(4)}] \sim \kappa_{(4)}^4 (mass)^4 \tag{2.3}$$

and has a logarithmically ultra-violet divergence which requires the introduction of a counter-term of dimension four

$$\delta_1 M_{4;1}^{(4)} = \alpha\, (\kappa_{(4)}^2\, R_{mnpq})^2 + \beta\, (\kappa_{(4)}^2\, R_{mn})^2 + \gamma\, (\kappa_{(4)}^2\, R)^2 \tag{2.4}$$

given by a precise linear combination of the square of the Riemann tensor, the Ricci tensor and the Ricci scalar. But for pure gravity this quantity vanishes on-shell and the divergence is accidentally zero [9]. This not true when the theory is coupled to matter and there a divergence at one-loop. At two-loop order the four-graviton amplitude has dimension

$$[M_{4;2}^{(4)}] \sim \kappa_{(4)}^6\, (mass)^6 \tag{2.5}$$

which requires the introduction of a counter-term of dimension six constructed from three powers of the Riemann tensor R_{mnpq}, the Ricci tensor R_{mn} and the Ricci scalar R, symbolically represented as

$$\delta_2 M_{4;2}^{(4)} = (\kappa_{(4)}^2\, R_{mnpq})^3 \tag{2.6}$$

and the theory of pure gravity is divergent at two-loop [18–20].

From the formula (2.2) one can read off the critical dimension for the appearance of ultra-violet divergences

$$D \geq D_c = 2 + \frac{2}{L}\,. \tag{2.7}$$

This formula indicates that pure gravity is finite only in two dimensions.

2.2. ... in extended supergravity

For at least $N = 1$ linearly realised on-shell supersymmetry in four dimensions R_{mnpq}^3 cannot be supersymmetrised [21, 22]. Therefore the two-loop four-graviton amplitude in four dimensions is finite for $N \geq 1$ supergravity.

In a four dimensional N extended supersymmetric theory the *on-shell* counter-terms have to correspond to an integral over full superspace (a D-term) of the form [23–26]

$$\delta\mathrm{L} = \kappa_{(4)}^{d+2N-4} \int d^4x d^{4N}\theta \, \det(E)\, \mathcal{L}(\mathbf{R}, \mathbf{T}) \qquad (2.8)$$

where $\det(E)$ is the determinant of the super-vielbein and $\mathcal{L}(\mathbf{R}, \mathbf{T})$ is a superspace density of length dimension $-d$ expressed in terms of the super-curvature $\mathbf{R}$ and the supertorsion $\mathbf{T}$. The superspace variables have the following dimensions $[x] = length$ and $[\theta] = (length)^{1/2}$.

The Bianchi identities for the superspace formalism with N on-shell linearly realised supersymmetry are expressed in term of a scalar superfield φ and in terms of the chiral superfield of spin $2 - N/2$ for $N \leq 4$ [24–28].

• For $1 \leq N \leq 3$ the superfield φ has the Weyl tensor[1] $C_{\alpha\beta\gamma\delta}$ appearing at the order θ^N

$$\varphi_{(\beta_{N+1}\cdots\beta_4)} = \phi_{(\beta_{N+1}\cdots\beta_4)} + \cdots + \frac{1}{N!}\theta^{\beta_1}_{a_1}\cdots\theta^{\beta_N}_{a_N}\epsilon^{a_1\cdots a_N}\, C_{\beta_1\cdots\beta_4} + \cdots \qquad (2.9)$$

where a_i are indices for the $SU(N)$ R-symmetry. This superfield has length dimension $-(2-N/2)$ and the *first* possible counter-term allowed by supersymmetry is

$$\delta\mathrm{L} = \kappa_{(4)}^4 \int d^4x d^{4N}\theta\, (\varphi_{\alpha_1\cdots\alpha_4}\bar\varphi^{\dot\alpha_1\cdots\dot\alpha_4})^2 \sim \frac{1}{\kappa_{(4)}^4}\int d^4x\, (\kappa_{(4)}^2\, C)^4 \qquad (2.10)$$

which is a *three-loop* contribution to the four-graviton amplitude. In this case because the dimension of the superfield depends on the number of linearly realised supersymmetries, its dimension balances the one from the fermionic measure and the order of the appearance of the counter-term is always three-loop.

• For $4 \leq N \leq 8$ the Bianchi identities are solved in terms of the scalar superfield of dimension 0 where the Weyl tensor appears at the order θ^4

$$\varphi_{[a_1\cdots a_4]} = \phi_{[a_1\cdots a_4]} + \cdots + \theta^{\beta_1}_{a_1}\cdots\theta^{\beta_4}_{a_4} C_{\beta_1\cdots\beta_4} + \cdots \qquad (2.11)$$

and the *first* possible counter-term allowed by linearized supersymmetry is

$$\delta\mathrm{L} = \kappa_{(4)}^{2N-4} \int d^4x d^{4N}\theta\; (\varphi_{a_1\cdots a_4}\bar\varphi^{a_1\cdots a_4})^{\frac{N}{2}} \sim \frac{1}{\kappa_{(4)}^4}\int d^2x\, (\kappa_{(4)}^2\, C)^N \qquad (2.12)$$

[1] In four dimensions we use the spinorial notation for the Riemann and Weyl tensors

spin/dimension	2	3/2	1	1/2	0
superfield	$W_{\alpha\beta\gamma\delta}$	$W^{i}_{\alpha\beta\gamma}$	$W^{ij}_{\alpha\beta}$	χ^{ijk}_{α}	φ^{ijkl}
$SU(8)_R$ rep.	1	8	28	56	70

Table 1

Basic superfields of linearized $N = 8$ supergravity. The lowest-component of these superfields are respectively given by the Weyl tensor $C_{\alpha\beta\gamma\delta}$, the 8 gravitino curvatures, 28 (Ramond-Ramond) field strengths, 56 Weyl spinors and 70 scalars.

In this case, because the superfield has dimension 0, the order at which the counter-term can appear is controlled by the dimension of the superspace integration. The linearized superspace integrals correspond to a three loop contribution to the four-graviton amplitude for $N = 4$, a four loop contribution to the four-graviton amplitude for $N = 5$, a five-loop contribution to the six-graviton amplitude for $N = 6$, a six-loop contribution to the seven-graviton amplitude for $N = 7$ and a seven-loop contribution to the eight-graviton amplitude for $N = 8$.

The absence of a three-loop divergence in four-graviton amplitude in four dimensions for $N = 8$ supergravity [29] indicates that more than sixteen on-shell linearized supersymmetries are controlling the perturbative computations of $N = 8$ supergravity. The precise number of supersymmetries is still unknown and we will describe below various constraints from string theory and dualities that can shed some light on this issue.

For the particular case of $N = 8$ supergravity the three-loop counter-term was constructed using the scalar superfield of the superspace formalism in four dimensions [27], but this superfield is not invariant under the U-duality symmetries of the theory which are expected to play an important role [11]. So we will be considering the construction of on-shell counter-terms that are invariant under the local $SU(8)$ R-symmetry and global E_7 of $N = 8$ supergravity [30].

• With increasing mass dimensions the first superfield invariant under the local $SU(8)$ R-symmetry and global E_7 of $N = 8$ supergravity is the dilatino superfield of length dimension $-1/2$ given in table 1

$$\chi^{ijk}_{\alpha} = D_{\alpha,l}\varphi^{ijkl} \tag{2.13}$$

from which one can construct an *eight loop* counter-term [23]

$$\delta \mathrm{L} = \kappa_{(4)}^{14} \int d^4x d^{32}\theta \, \det(E) \, (\chi^{\alpha}_{ijk}\bar{\chi}^{ijk}_{\alpha})^2 \sim \frac{1}{\kappa_{(4)}^{4}} \int d^4x \, \kappa_{(4)}^{18} \, D^{10} C^4 \, . \tag{2.14}$$

This superfield is invariant under the local $SU(8)$ R-symmetry of $N = 8$ supergravity [23] unlike the scalar superfield discussed. All the Bianchi identities of $N = 8$ supergravity can be expressed in terms of this spinor superfield [27]. But bearing in mind the properties of the maximal supergravity theories with 32 supercharges in every dimension where they can be defined, but the dimension 1/2 superfield does not appear in the N=1 $D = 11$ formulation. We are considering gravity amplitudes, so one wants to construct counter-term using only Lorentz invariant quantities expressed in terms of the curvature of the (super)graviton. In ten dimensions the dimension 1/2 superfield starts with the dilatino field $\chi_\alpha = \lambda_\alpha + \theta^\beta F_{\alpha\beta} + \cdots$ (from which the four dimensional χ_α^{ijk} is the dimensional reduction) and $SO(1,9)$ Lorentz invariance.

• The integrated graviton vertex operators in the pure spinor formalism for perturbative string theory were constructed by Berkovits [12–15]

$$V = \int d^2z \, \left(G_{MN}\partial x^M \bar{\partial} x^N + W_{\alpha\beta,+-} d_+^\alpha d_-^\beta + \cdots \right) \qquad (2.15)$$

from the superfield $W_{\alpha\beta,+-}$ with dimension $(length)^{-1}$ has the Weyl tensor at order θ^2

$$W_{\alpha\beta,a_1a_2} = F_{\alpha\beta,a_1a_2} + \cdots + \theta_{a_1}^\gamma \theta_{a_2}^\delta \, C_{\alpha\beta\gamma\delta} + \cdots \qquad (2.16)$$

The lowest component of this superfield are the Ramond-Ramond field-strengths. In ten-dimensional type IIb supergravity this superfield arises by taking two fermionic derivatives on the scalar superfield dilaton $\Phi = \tau + \cdots + \theta^2(R + \partial F_5) + \cdots$ [31], and in type IIa supergravity this superfield is readily obtained by a dimensional reduction from the mass dimension one, four-form superfield of eleven dimensional supergravity [32]

$$(\Gamma^{m_1\cdots m_3} D)_\alpha \, W_{m_1\cdots m_4} = 0 \qquad (2.17)$$

which has the following (schematic) expansion

$$W_{m_1\cdots m_4} = F_{m_1\cdots m_4} + \cdots + \theta^2 C + \cdots \qquad (2.18)$$

starting with the field-strength for the three-form potential $F_4 = dC_3$ and having the Weyl tensor at θ^2 order. By dimensional reduction of this superfield gives the ten dimensional and four dimensional superfields discussed above.

• In four dimensions this superfield is obtained by dimensional reduction of the superfields of $N = 8$ supergravity. Using this "string favoured" building block in four dimensions one can construct the following *nine loop* counter-term [33]

$$\delta \mathrm{L} = \kappa_{(4)}^{16} \int d^4x d^{32}\theta \, \det(E) \, (W_{\alpha\beta})^4 \sim \frac{1}{\kappa_{(4)}^4} \int d^4x \, \kappa_{(4)}^{20} \, D^{12} C^4 \, . \qquad (2.19)$$

3. Critical dimension for (logarithmic) ultra-violet divergences

In order to determine the critical dimension for logarithmic ultra-violet divergences of gravity amplitudes one needs to know precisely how the loop integral diverges. On general grounds a L-loop n-point gravity amplitude in D-dimension will be decomposed as

$$M_{n;L} = \sum_i t_i \, I^{(D)}_{n;L\,(i)}[\ell^\nu]\,, \tag{3.1}$$

where t_i is some tensor constructed from the polarisations and the momenta of the external states, and $I^{(D)}_{n;L}[\ell^\nu]$ is an n-point L-loop integral defined in dimension D with ν powers of loop momenta in the numerators. Each individual integral can have a worse ultra-violet behaviour than the total amplitude $M_{n;L}$ where subtle cancellations are expected to occur.

If Λ is momentum cut-off, the low-energy expansion of the four-graviton amplitude at L loops and D dimensions will behaves as

$$M_{4;L} \sim \Lambda^{\delta_L}\, \mathcal{O}_{k_L} + \cdots \tag{3.2}$$

where $\mathcal{O}_{k_L}$ is an operator of dimension $(length)^{-k_L}$. The dimension of this operator can change with the loop order. The leading degree of ultra-violet divergence is

$$\delta_L = (D-2)L + 2 + k_L \tag{3.3}$$

so that the total dimension of the loop amplitude is $(D-2)L+2$. The ellipsis in the eq. (3.2) are for the sub-leading ultra-violet divergences. Ultra-violet divergences occur when $\delta_L \geq 0$

$$D \geq D_c = 2 + \frac{2+k_L}{L}\,. \tag{3.4}$$

For $N = 8$ supergravity all the four-graviton amplitudes have at least a factor of $\hat{R}^4$ (see eq. (4.9)) but more derivatives can be factorised and the low energy expansion starts contributing with an operator of dimension $8 + 2\beta_L$

$$\mathcal{O}_{k_L} = D^{2\beta_L}\, R^4\,. \tag{3.5}$$

In this case the critical dimension for the appearance of ultra-violet logarithmic divergences is

$$D_c = 2 + \frac{6+2\beta_L}{L}\,. \tag{3.6}$$

Explicit results for the four-graviton amplitude give that $\beta_1 = 0$ at one-loop [34], and $\beta_2 = 2$ at two loops [12, 35–39], and $\beta_L = 3$ at three loops[2] [29]. Assuming that $\beta_L = 3$ for $L \geq 3$ one concludes that the critical dimension for ultra-violet divergence is

$$D_c = 2 + \frac{12}{L} \quad \text{for} \quad L \geq 3\,. \tag{3.7}$$

Predicting a first divergence for $N = 8$ supergravity in four dimension at $L = 6$ loop. This formula predicts as well that the four-loop four-graviton amplitude diverges logarithmically in $D = 5$.

In ten dimensions using the non-minimal pure spinor formalism Berkovits showed in [16] up to $L = 6$ the four-graviton amplitudes are F-term satisfying the rule $\beta_L = L$. This result which only makes use of the fermionic zero mode saturation does not involve any massive string excitations and makes only use of the fact that the vertex operators are constructed using the mass dimension one superfield $W_{\alpha\beta}$ of eq. (2.16).

Since this superfield exists for all the formulations of the maximal ($N = 8$) supergravity in every dimensions, it was argued in [41] that this leads to the following critical dimension

$$D \geq D_c = 2 + \frac{18}{L} \quad \text{for} \quad L \geq 6\,. \tag{3.8}$$

And the first divergence of $N = 8$ supergravity in four dimensions is expected at nine loops.

All these formulæ give that only $D = 2$ supergravity is finite (for any number of supersymmetries). As long $2 + k_L$ in eq. (3.4) or $6 + 2\beta_L$ in eq. (3.6) is bounded when the loop order L increases the equation $D_c = 4$ will always have a solution at some loop order and an ultra-violet divergence will occurs in four dimensions. It was proposed in [33,41] that at each loop order two extra powers of the external momenta factors out the low energy limit of the L-loop amplitude, so that

$$\beta_L = L \quad \text{for} \quad L \geq 2 \tag{3.9}$$

giving the critical dimension

$$D \geq D_c = 4 + \frac{6}{L} \quad \text{for} \quad L \geq 2\,. \tag{3.10}$$

[2] Notice that β_L is the leading power of the amplitude in the low-energy expansion, and not all the various diagrams composing the $L = 3$ have an overall power of $D^6 R^4$ in the solution presented in [29]. The string based analysis presented in [16] assures that there is representation of the $L = 3$ amplitude with an explicit power of $D^6 R^4$. The field theory result of [29] can be rewritten in such a form as well [40].

Fig. 1. Effective interactions that could be giving an ultra-violet behaviour with the critical dimension $D_c = 4 + 6/L$

Since the critical dimension is always bigger than 4, then $N = 8$ supergravity would be finite in four dimensions.

When the condition $\beta_L = L$ holds the constant piece in D_c has changed its value from 2 to 4. This means that at each loop order the mass dimension of the loop integral increases by a factor of $(mass)^{D-2}$ typical of φ^4 scalar interactions or cubic derivative interactions of figure 1. These vertices are the elementary one of $N = 4$ super-Yang-Mills. The φ^4 vertex are needed from four-loop order [42,43]. These vertex appear in the construction of the multiloop amplitudes so that no triangle sub-graph are generated. Supersymmetry cannot be responsible for such reduction which are due to extra cancellations from pure gravity interactions [44–46] accounted for the general coordinate gauge invariance in one-loop amplitudes and the sum over all the permutation of the external legs in a theory without color ordering [47].

In four dimensions the loop amplitude has negative mass dimension -6

$$[M^{(4)}_{4;L}] \sim (mass)^{-6}\, D^{2L} R^4\,, \tag{3.11}$$

which means that the amplitude has no ultra-violet divergences but only IR divergences. At one-loop order, the amplitude is given by R^4 times the scalar box amplitude I_4 which has dimension $(mass)^{-4}$ in four dimension [34], at two-loop order the amplitude is given by D^4R^4 times the planar and non-planar double-box [35] of dimension $(mass)^{-6}$ [48, 49]. For more details on the structure of four-graviton loop amplitude we refer to section 4.

3.1. Is $N = 8$ equal to $(N = 4)^2$?

The formula (3.8) is the same critical formula as $N = 4$ super-Yang–Mills indicating a very close relation between $N = 8$ supergravity and $N = 4$ super-Yang–Mills. The relation is that the basis of integrals on which $N = 8$ supergravity amplitudes are expressed is the same as the one of $N=4$ super-Yang-Mills amplitudes once all the planar and non-planar contributions are included. We will see this in practice at the one-loop $L=1$ and two-loop order $L=2$ in section 4.

The n-point L-loop amplitude in D dimensions for $N=4$ super-Yang–Mills has the following mass dimension

$$[A_{n;L}] \sim g_{YM}^{2(L-1)+n} (mass)^{(D-4)L} \tag{3.12}$$

with a coupling constant g_{YM}^2 of dimensions $(length)^{D-4}$. In four dimensions the theory is logarithmic divergent. For $N=4$ super-Yang–Mills supersymmetric the finiteness in four dimensions is only a consequence of supersymmetry [50] and the supersymmetric cancellations give that the low-energy expansion of the amplitude is given by (for $L \geq 2$) [50]

$$A_{4;L} \sim \Lambda^{(D-4)L-6}\, D^2F^4 + \cdots \tag{3.13}$$

which is enough for the perturbative ultra-violet finiteness of the theory in four dimensions. This leads to the critical dimension for super-Yang–Mills amplitudes

$$D \geq D_c = 4 + \frac{6}{L} \,. \tag{3.14}$$

his formula indicates that the dimension six operator D^2F^4 factorises at higher loops which is enough for assuring that the theory is perturbatively finite and does not have logarithmic divergences. It is important to remark here that the dimension of the operator that is factorised does not depend on the loop order.

The KLT [51] relation express the supergravity tree-level amplitudes as sum of square of Yang–Mills amplitudes

$$M_n^{tree} = \sum_{\sigma\in S_n} p_{n-3}^{(i)}(s_{ij})\, A_n^{tree}(1,2,\cdots,n)\, A_n^{tree}(\sigma(1),\sigma(2),\cdots,\sigma(n)) \,. \tag{3.15}$$

In this expression σ is a permutation of the external legs and $p_{n-3}^{(i)}(s_{ij})$ are polynomials of order $n-3$ in the kinematic invariants $s_{ij}=(k_i+k_j)^2$ needed to cancel the spurious poles appearing when multiplying the two gauge theory amplitudes [52–54]. For instance the tree-level four gauge boson (colour stripped) amplitude is

$$A_4^{tree}(s,t) = \frac{1}{s\,t}\, t_8F^4 \tag{3.16}$$

which squared gives the tree-level four-graviton amplitude

$$M_4^{tree}(s,t) = s\, A_4^{tree}(s,t)\, A_4^{tree}(s,u) = \frac{1}{stu}\, t_8t_8R^4 \,. \tag{3.17}$$

These tree-level relations have some important implications at the loop level structure where the amplitude is developed on the same basis [17, 35, 47, 55] of integrals as for $N = 4$ super-Yang–Mills with for coefficients tensorial structure that are 'square' of the $N = 4$ super-Yang–Mills tree coefficients from the KLT relation [51]. These relations are specific to the on-shell amplitude computations and are not properties of the effective action of supergravity theories but are nevertheless a useful guide for constructing some higher derivative superinvariants for the ten- and eleven-dimensional supergravity effective action in [56] under linearized supersymmetry.

One important difference between the gravity and Yang–Mills is the absence of colours. At one-loop order the sum over all the external legs for a colorless theory played was needed for cancelling the triangle contributions [47]. At higher-loop order this requires that one sums over all the planar and non-planar contributions to a specific amplitude, and any good ultra-violet properties of $N = 8$ supergravity must rely on subtle cancellations between the planar and non planar sector. We will see examples of this when discussing the explicit example of higher-loop contributions in the next section.

In the context of $N = 4$ supergravity one expects $\beta_L = L/2$ giving a critical dimension

$$D_c = 3 + \frac{6}{L} \tag{3.18}$$

and a first divergence at $L = 6$ in four dimensions, which is higher than the three loop divergence prediction based on the linearized $N = 4$ supersymmetry.

In the following we will describe a construction of the higher-loop amplitude four-graviton amplitude in $N = 8$ supergravity and their regularisation using the string theory induced scheme [57, 58]. This construction will give a set of non renormalisation theorems for the higher derivative gravitational interactions. Since all the possible counter-term from the superspace formalism have a contribution to the four-graviton amplitude, we will be able to confront the predictions from supersymmetry and dualities for the ultra-violet behaviour of the multiloop amplitudes in supergravity in various dimensions.

4. Higher loop amplitudes in supergravity theories

For computing higher-loop amplitude in eleven dimensions one could envisage using background field method linearizing the eleven dimensions supergravity action of [59] around a specific background and making use of Feynman rules. This has been used to extract information about one- and two-loop amplitudes in gravity [10, 18–20] and some of the structure of the higher order corrections to M-theory [60, 61]. The method is very cumbersome an obscures many of the properties of the gravitational interactions (see [17] and a presentation of these points).

In the following we will describe how to construct the four-graviton L-loop amplitude of $N=1$ supergravity in eleven dimensions using on the on-shell unitarity method for supergravity developed by Zvi Bern and his collaborators [62].

4.1. Cut construction of the loop amplitudes

We construct the scattering S_{if}-matrix between initial state i and final state f from its discontinuities across the branch cuts in the complex energy plane (the Mandelstam plane). Because in an unitary local quantum field theory the S-matrix $S_{ij} = \delta_{if} + i\,T_{if}$ the S-matrix satisfies $SS^\dagger = \mathbb{I}$ the transition matrix T satisfies the relation

$$2i\,(T - T^\dagger)_{if} = \sum_k T_{ik} T^\dagger_{kf}. \tag{4.1}$$

where the sum on the right hand side is over all possible intermediate states. Perturbatively the states k are the one running in the loops. In the context of supergravity theories the sum will be over the massless supergravity multiplet.

This relation is valid for any unitary quantum field theory independently of any perturbative expansion.

A perturbative expansion relation (4.1) relates the value of the discontinuity of the scatting matrix across a branch cuts at a given loop order L to the integration over the intermediate states exchanged between lower loops order amplitude.

Unfortunately there are various ambiguities in reconstructing the real part of the S-matrix associated with the fact that the dispersion relations give rise to diverging expressions which need to be regularised and introduce some ambiguities in the rational part of the amplitude, and one has to use the dispersion relation for obtaining the real part of the amplitude [63]. The only ambiguity with a physical meaning are the one associated with the usual ultra-violet and infra-red divergences of the amplitude. A traditional way of dealing with this problem is to consider the unitarity method in the context of the dimensional regularisation [64, 65]. The application of this method in the context of the helicity formalism proved to be a very powerful tool for constructing higher-loop amplitude in gauge and supergravity theories [17, 66–68].

In some particular cases the amplitudes do not contain rational terms and can therefore be completely reconstructed by considering the behaviour of the S-matrix across the branch cuts.

The appearance of rational terms in the amplitude is connected to the number of loop momenta in the Feynman integrals (and therefore the ultra-violet

behaviour) of the amplitude by various steps of the Passarino-Veltman reduction [69].
Consider an n-point one-loop amplitude in D dimensions with φ^3 vertices

$$I^{(D)}_{n;\nu}(k_1, \cdots, k_n) = \int d^D\ell \, \frac{\mathcal{P}_\nu(\ell)}{\ell_1^2 \cdots \ell_n^2} \tag{4.2}$$

where $\ell_i^2 = (\ell - k_1 - \cdots - k_i)^2$ are the various propagators along the loop and $\mathcal{P}_\nu(\ell)$ in the numerator is a polynomial of degree ν in the loop momentum ℓ.

The dimension of the integral in eq. (4.2) is

$$[I^{(D)}_{n;\nu}] \sim (mass)^{D+\nu-2n} \, \mathcal{O}_\nu(h, k) \tag{4.3}$$

the superficial degree of divergence $\delta_{1,n} = D + \nu - 2n$ is a function of the dimension D, the number of external states and the order ν of the polynomial. The amplitude is ultra-violet finite when $2n \geq D + \nu$, therefore by analysing the ultra-violet behaviour of the integral in arbitrary dimensions and with an increasing number of external legs one can determine the power of ν of loop momenta in the numerators of the amplitudes.

For on-shell massless external states using the identity

$$\frac{2\ell \cdot k_1}{\ell^2 \, (\ell - k_1)^2} = \frac{1}{(\ell - k_1)^2} - \frac{1}{\ell^2} \tag{4.4}$$

one reduces [69] the integral I_n in eq. (4.2) into the difference of two $n-1$-point loop amplitude with a numerator of degree $\nu - 1$

$$I^{(D)}_{n;\nu}(k_1, \cdots, k_n) = I^{(D)}_{n-1,\nu-1}(k_1 + k_2, \cdots, k_n) - I^{(D)}_{n-1;\nu-1}(k_2, \cdots, k_n + k_1) \tag{4.5}$$

On the right hand side one sees the two massive external legs with momentum $k_1 + k_n$ and $k_1 + k_2$ due to the cancellation of the propagators with the identity (4.4).

For gravity or gauge theories the higher power of loop momenta in $\mathcal{P}_\nu(\ell)$ is given by the cubic vertex, which means that $\mathcal{P}_\nu(\ell) \leq n$ for gauge theories and $\mathcal{P}_\nu(\ell) \leq 2n$ for gravity. For $N = 4$ super-Yang–Mills a vertex brings a power of the loop momenta, and the saturation of eight fermionic zero modes (needed by the $N = 4$ supersymmetry) cancels four loops momenta leading to $\nu \leq n - 4$. In gravity each vertices bring two powers of loop momenta, and for supergravity theories with N supersymmetries it is conjectured that [45]

$$\nu \leq 2n - N - (n - 4) \,, \tag{4.6}$$

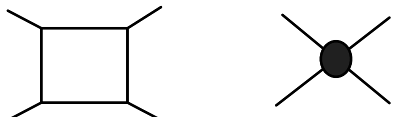

Fig. 2. The one-loop four-graviton amplitudes in $N = 8$ supergravity is given by the box diagram in figure (a). In $D \leq 8$ the amplitude has a ultra-violet divergence which is subtracted by the counter-term represented in figure (b).

where supersymmetry only guarantees that $\nu \leq 2n - N$. Therefore $n - 4$ extra cancellations of loop momentum are needed.

One remarks that there is a special case where the extra $n-4$ cancellations are not needed. This is the case of the amplitudes with more than four external states the one-loop amplitude which contains at most $n - 4$ powers of propagators ℓ_i in the numerator and reduce directly to a massive boxes

$$\int d^D\ell \frac{\ell_5^2 \cdots \ell_n^4}{\ell_1^2 \cdots \ell_n^2} = \int d^D\ell \frac{1}{\ell_1^2 \cdots \ell_4^2} \,. \tag{4.7}$$

In this case one concludes that

• Clearly theories with $\nu < n$ are one-loop cut constructable since no rational terms can be obtained by a succession of Passarino-Veltman reductions.
• $N = 4$ super-Yang–Mills does not have rational term and is cut-constructable.
• For $N = 8$ supergravity it is conjectured that $\nu \leq n-4 < n$, which means that one-loop n-graviton amplitudes under a Passarino-Veltman reduction but do not contain integral functions more singular than (massive) boxes and in particular do not contain triangles or bubble functions [45, 70]. At one-loop order this fact has been linked to gauge invariance and the summation over all permutations of the external legs for a colorless theory [47]. Because of the absence of colors in gravity on has to sum over all planar and non-planar contributions at higher-loop order. This plays an essential role in the cancellation of various unwanted contributions that would bring a worst ultra-violet behaviour.

When using the on-shell unitarity method for constructing the higher-loop amplitude in $N = 8$ supergravity, one has to sum over all the 256 massless states of the graviton supermultiplet. These states are the same in every dimensions and the construction of the amplitudes are valid in every dimensions where $N = 8$ supergravity can be defined.

4.2. The one-loop amplitude

The amplitude between four massless state of the supergravity multiplet of $N = 8$ supergravity in D dimensions can be constructed completely from its 2-particle cut with the result [62]

$$M_{4;1}(k_1, \cdots, k_4) = \frac{\kappa_{(D)}^4}{(2\pi)^D} \hat{R}^4 \, [I_4^{(D)}(S,T) + I_4^{(D)}(S,U) + I_4^{(D)}(T,U)] \quad (4.8)$$

where $\hat{R}^4$ is kinematic factor for four massless external states defined in [1]

$$\hat{R}^4 = h_1^{AA'} h_2^{BB'} h_3^{CC'} h_4^{DD'} \, K_{ABCD} \tilde{K}_{A'B'C'D'} \, , \quad (4.9)$$

the indices A, B on the superhelicitcty run over both vector and spinor values and span the 256 states of the massless $N = 8$ gravity supermultiplet. $I_4^{(D)}$ is the D-dimension massless box integral ($I_{4;0}^{(D)}$ in the notations of eq. (4.2)) given by

$$I_4^{(D)}(S,T) = \int_{\Lambda^{-2}}^{\infty} \frac{dt}{t} \, t^{4-\frac{D}{2}} \int_0^1 \prod_{i=1}^{4} d\nu_i \, e^{\pi t Q_4(k_i)} \quad (4.10)$$

where

$$Q_n(k_1, \cdots, k_n) = \sum_{1 \le i < j \le n} k_i \cdot k_j \, (\nu_{ij}^2 - |\nu_{ij}|) \, . \quad (4.11)$$

The absolute value in Q_n forces the breaking of the integral into three different physical regions where the integral converges [71, 72]. The (s,t)-region for $\mathcal{T}_{ST} = \{0 \le \nu_1 \le \nu_2 \le \nu_3 \le \nu_4 \le 1\}$, the (s,u)-region for $\mathcal{T}_{SU} = \{0 \le \nu_2 \le \nu_1 \le \nu_3 \le \nu_4 \le 1\}$ and the (t,u)-region for $\mathcal{T}_{TU} = \{0 \le \nu_1 \le \nu_3 \le \nu_2 \le \nu_4 \le 1\}$. The integral is to be evaluated with $s, t < 0$ where it converges and then analytically continued to the physical region.

The expression (4.8) for the amplitude for four massless $N = 8$ states of the supergravity multiplet has been constructed using the on-shell unitarity method and is valid in any dimensions $D \le 11$. In dimension $D \le 10$ this expression has been obtained by Green et al. [34] by compactifying the genus-one string loop amplitude and decoupling the massive string modes, Kaluza–Klein and winding modes.[3]

[3]The string one-loop amplitude is of course ultra-violet finite, and gives a resulting total amplitude which also ultra-violet finite because the residue of the $1/\epsilon$ in $5 \le D \le 10$ and $\frac{1}{\epsilon^2}$ pole in $D = 4$ is proportional to $s + t + u = 0$ and vanishes on-shell.

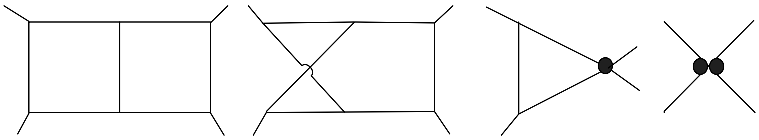

Fig. 3. The two-loop four-graviton amplitudes in $N = 8$ supergravity is given by the double box diagrams in figure (a) and (b). Figure (c) represents the contribution induces by the one-loop counter-term of figure 2(b). Figure (d) represents the new primitives ultra-violet divergences arising for $D \geq 7$.

The lower bound on the integral is an ultra-violet cut-off [4] and the amplitude $M_{1;4}$ has the leading ultra-violet behaviour

$$[M_{4;1}] \sim \frac{\kappa_{(D)}^4}{(2\pi)^D} \hat{R}^4 \, (mass)^{D-8} \tag{4.12}$$

and the superficial degree of divergence of the 4 graviton amplitude is given by the behaviour of the scalar box in D dimensions

$$\delta_{4;1} = D - 8 \, . \tag{4.13}$$

In $D = 11$ this amplitude has a cubic divergence which will be regulated by the addition of a local counter-term

$$\delta_1 M_{4;1} = c_4 \, \frac{\kappa_{(11)}^4}{(2\pi)^{11}} \, \frac{\pi^3}{2} \, \ell_P^3 \, \hat{R}^4 \, . \tag{4.14}$$

The precise renormalisation scheme will be described in section 6.

4.3. Two-loop amplitude

The on-shell unitarity method [62] gives that the two-loop four-graviton amplitude for $N = 8$ supergravity in D-dimension is expressed as a the sum of scalar double-box amplitude represented in figure 3 and given by

$$M_{4;2} = i \frac{\kappa_{(D)}^6}{(2\pi)^{2D}} \hat{R}^4 \left[S^2 I^{(S)} + T^2 I^{(T)} + U^2 I^{(U)} \right] \tag{4.15}$$

which is given by a sum of contributions from the S, T and U-channel with

$$I^{(S)} = \frac{1}{2} \left(I^P(S,T) + I^P(S,U) + I^{NP}(S,T) + I^{NP}(S,U) \right) \, , \tag{4.16}$$

[4]In $D \leq 4$ the amplitude develops infra-red (IR) divergences which will not be discussed in here.

with analogous expressions for $I^{(T)}$ and $I^{(U)}$. The loop integral $I^P(S,T)$ is the planar two-loop φ^3 contribution

$$I^P(S,T) = \int \frac{d^D p d^D q}{p^2(p-k_1)^2(p-k_{12})^2(q-k_{12})^2(q-k_4)^2 q^2(p+q)^2} \quad (4.17)$$

and $I^{NP}(S,T)$ is the non-planar two-loop φ^3 contribution

$$I^{NP}(S,T) = \int \frac{d^D p d^D q}{p^2(p-k_1)^2(p-k_{12})^2(q-k_4)^2 q^2(p+q)^2(p+q-k_3)^2} . \quad (4.18)$$

At this order we see the appearance of a non-planar contribution to the amplitude. In the case of gravity there is no colour and the planar and non-planar contributions have to be summed in the total amplitude. The two-loop string amplitude is given by a single expression in eq. (A. 17) expressed as the integral over the moduli space genus two Riemann surfaces [36, 37]. At this order the field theory limit has a remaining ‘modular’ symmetry putting together planar and the non-planar contribution in a single contribution [58, 73].

Finally because there are no diagrams with three external legs on the same loop proper-time, the two loop amplitude does not contain any triangle. This is a consequence of the vanishing of the factor $|\mathcal{Y}_s|^2$ in eq. (A. 19) when three external legs are on the same loop proper-time in the integrand of the field theory limit of the genus two [73].

In eleven dimensions the one-loop counter-term in eq. (4.14) induces the following contribution at two-loop order

$$\delta_{2a} M_{4;2} = i c_4 \, \frac{\pi^3}{2} \ell_P^3 \, \frac{\kappa_{(D)}^6}{(2\pi)^{2D}} \, \hat{R}^4 \, I_{3;1}^{(D)} , \quad (4.19)$$

regulating the one-loop ultraviolet sub-divergence of the amplitude.

In eleven dimension a new set of primitive divergences arise at two-loop order

$$M_{4;2} \sim \Lambda^8 \mathcal{D}^4 \hat{R}^4 + \Lambda^6 \mathcal{D}^6 \hat{R}^4 + \Lambda^4 \mathcal{D}^8 \hat{R}^4 + \Lambda^2 \mathcal{D}^{10} \hat{R}^4 + \log(\Lambda) \mathcal{D}^{12} \hat{R}^4 . \quad (4.20)$$

these divergences are subtracted by local counter-term

$$\delta_{2b} M_{4;2} = \frac{\kappa_{(11)}^6}{(2\pi)^{22}} \sum_{k=2}^{6} c_k \, \mathcal{D}^{2k} \, \hat{R}^4 \ , \quad (4.21)$$

the value of this counter-term will be determined in section 7 in the context of the duality compatible renormalisation scheme we use for regulating the ultraviolet divergences of the supergravity amplitudes.

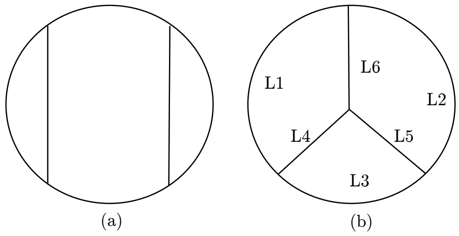

Fig. 4. The vacuum diagrams for the three-loop four-graviton amplitudes in $N = 8$ supergravity has two different topologies depicted in figure (a) and (b). The vacuum diagram in figure (a) leads to the class of scalar φ^3 triple ladder and cross-ladder boxes. They have a prefactor of $D^8 R^4$. The vacuum diagram in figure (b) leads to various diagrams that are no purely scalar φ^3.

4.4. Higher-loop amplitudes

The four-graviton three-loop amplitude in $N = 8$ supergravity has been constructed in ref. [29] and has the two classes of vacuum diagram represented in figure 4. It was shown in ref. [29] that the low-energy limit of this amplitude is given by

$$M_{4;3} \sim \Lambda^{3(D-4)-6} \, D^6 \hat{R}^4 + \cdots \qquad (4.22)$$

One class of diagrams is given by all the scalar φ^3 planar and non-planar triple-box diagrams one obtains by putting the legs external legs on the vacuum diagram of figure 4(a) without generating diagrams containing triangles. These diagrams have an explicit factor of $D^8 R^4$ and are much too ultraviolet convergent to be the leading contribution to the low-energy limit of the $L = 3$ amplitude. At this order appears a new class of diagrams which are not purely scalar φ^3 diagrams but with some powers of the loop momenta in the numerator of the integrals. They arise from the vacuum diagram of figure 4(b). Although the solution presented in [29] did not have an explicit factor of $D^6 R^4$ (some of the diagrams only have an explicit $D^4 \hat{R}^4$ factor in front of the loop integrals) the structure of the genus 3 amplitude derived in [16] guarantees that there is an expression of the three loop result ith an explicit $D^6 R^4$ in front the loop integrals, with loop integrals obtained by putting the external legs on the diagram in figure 4(b). The resulting four-point loop integrals must have a maximun of two powers of loop propagators in the numerators to have the dimension $[I_{4;3}^{(D)}] = (mass)^{15}$.

The construction of the gravitational F-terms by Berkovits in [16] guarantees that up to six loops the four-graviton amplitude will start contributing from $D^{2L} R^4$ in the low energy limit.

The construction of the higher loop amplitude in supergravity is a difficult task essentially because one has to sum over all the permutations of the external states and include both planar and non-planar contributions. It was found in [58] that the $L = 2$ amplitude has an hidden modular symmetry inherited from the symmetries of the string theory genus two moduli space. This symmetry puts together the planar and non-planar contributions in a single an compact expression [58, 73]. One wonder about the existence of some remnant of the higher-genus string amplitude modular symmetries organising the field theory contributions by putting together planar and non-planar contributions and facilitating the construction of the higher loop contributions.

5. Duality constraints

5.1. The M-theory conjecture

The eleven dimensional supergravity [59] has a Lagrangian of the form

$$\mathcal{S}^{(11)}_{CJS} = \frac{1}{2\,\kappa^2_{(11)}} \int d^{11}x\,\sqrt{-G}\,\left[\mathcal{R}_{(11)} + \frac{1}{2}|G_4|^2 + \frac{1}{6} C_3 \wedge G_4 \wedge G_4 + \text{fermions}\right] \tag{5.1}$$

where $\mathcal{R}_{(11)}$ is the Ricci scalar in eleven dimensions and $G_4 = dC_3$ the four form-field strength. The coupling constant $\kappa^2_{(11)} = (2\pi)^8\,\ell_P^9$.

This Lagrangian can be seen as a consequence of the κ-symmetry invariance of the M2-brane [74] and has an on-shell superspace description in eleven dimensions [32].

The M-theory conjecture states that this Lagrangian is the kinetic part of the effective action a fundamental theory. The microscopic degrees of freedom of M-theory are not known so far.

Even if the cancellations described in the previous section occur to all orders the eleven dimensional supergravity will have ultraviolet divergences, and needs to be regulated. We will regulate the theory by adding *local counter-terms* to the eleven dimension supergravity effective action

$$\delta\mathcal{S}_{CJS} = \frac{1}{2\kappa^2_{(11)}} \int d^{11}x\,\sqrt{-G}\,\sum_k c_k\,\ell_P^{2(k-1)}\,\mathcal{R}^k + \cdots \tag{5.2}$$

where $\mathcal{R}^i$ represents an higher dimensional operators composed by powers of Riemann tensors, or derivatives on the Riemann tensor $\mathcal{D}^m\mathcal{R}^n$ with $n+2m = k$. (There are of course counter-terms depending on the four-form field-strength but they will not be discussed here.) The knowledge of the microscopic degrees of freedom of M-theory and its symmetries would dictates the infinite series of counter-terms to add to the theory for making it ultraviolet finite and determine the values of the constants c_k in eq. (5.2). This cutoff should be determined by the microscopic degree of freedom of M-theory and related to the tension of the M2-brane $T_{M2} \sim 1/\ell_P^3$ or the M5-brane $T_{M5} \sim 1/\ell_P^6$. The $N = 8$ supersymmetric cancellations of loop momenta in one-loop amplitudes assure that the higher power of one-loop sub-divergences is given by Λ^3. Therefore one only expects in eleven dimensions primitive divergences of the type Λ^{3n}. These divergences are subtracted by the following infinite set of counter-term to the M-theory action

$$S_{M-theory} = \frac{1}{\ell_P^9} \int d^{11}x \left[\mathcal{R}_{(11)} + \sum_{k\geq 0} c_k \, \ell_P^{6k+6} \, R^{3k+4} \right] . \tag{5.3}$$

The infinite set of coefficients is constraints by the duality symmetries of M-theory, and have been determined up to order $k = 2$ in [57, 58, 73, 75]

$$\begin{aligned} S_{M-theory} = \frac{1}{\ell_P^9} \int d^{11}x \Big[&\mathcal{R}_{(11)} + 4\zeta(2)\, R^4 + 2\zeta(4)\, D^6 R^4 \\ &+ \frac{196}{142}\, \zeta(6)\, D^{12} R^4 + \cdots \Big] \end{aligned} \tag{5.4}$$

For instance the first non-zero correction, the $\ell_P^6 \, \mathcal{R}^4$ term in (5.2) can be seen as originating from membrane effects [56, 76–80] induced for instance by world-sheet higher-loop effects on the world-volume of the M2-brane. It would be interesting to test if the pattern of the higher-derivative corrections appearing in $\delta\mathcal{S}$ determined in the references [57, 58, 73] and reviewed in the section 6 and 7 can be reproduced from the infinite dimensional group of symmetries of M-theory considered in [81–83].

In the following we will use a renormalisation scheme for regulating the loop amplitudes by matching the result to perturbative string data[5] which will allow us to derive a set of counter-terms to the effective action.

Taking this theory on a circle of radius[6] $R_{11} \, \ell_P$ the theory is identified with type IIa string theory if the masses are measured with the following metric [5, 84]

$$\frac{G_{MN}}{\ell_P^2} dx^M dx^N = \frac{1}{R_{11}} \, \frac{g^A_{\mu\nu}}{\ell_s^2} \, dx^\mu dx^\nu + R_{11}^2 (dx^{11} - C_\mu dx^\mu)^2 \tag{5.5}$$

where $M = 0, \ldots, 10$, $\mu = 0, \ldots, 9$, $g^A_{\mu\nu}$ is the type IIa sigma model metric, and C_μ is the 1-form RR-potential carrying the D0-brane charge. The radius R_{11} is

[5]Considering this theory on a circle of finite value of R_{11} the S-matrix $\mathcal{S}(R_{11}, \ell_P)$ can be expanded in powers of R_{11} (there are as well $\exp(-1/R_{11})$ effects which are D-branes which will be commented on below) which will be matched with corresponding quantities from string perturbation.

[6]We always take the radii as dimensionless quantities.

related to the string coupling constant by the relation

$$R_{11}^3 = (g_s^A)^2 \tag{5.6}$$

which together with the following relation between the eleven dimensional Planck length and the string scale

$$\ell_P = (g_s^A)^{\frac{1}{3}}\, \ell_s, \tag{5.7}$$

gives the dictionary between M-theory variables and string variables. These relations relate the strong coupling limit $g_s \to \infty$ of type IIa string to the eleven dimensional theory $R_{11} \to \infty$.

Plugging these relations into eq. (5.1) the Einstein-Hilbert term transform into [5]

$$S^{(IIa)} = \frac{1}{2\,\kappa_{(10)}^2} \int d^{10}x\, \sqrt{-g^A}\, \frac{1}{(g_s^A)^2}\, [\mathcal{R}_{(10)} + \cdots] \tag{5.8}$$

the Einstein-Hilbert term in ten dimensions in the string frame, and the ellipsis are for the various non-gravitational contributions arising from the reduction of the action (5.1) leading to the type IIa supergravity action in ten dimensions. We have $2\kappa_{(10)}^2 = (2\pi)^7\, \alpha'^4$.

Because of the specific dependence on R_{11} in the metric (5.5) the massless supergraviton multiplet in eleven dimensions gives the ten dimensional supergraviton multiplets as well as Ramond 1-form carrying $D0$-brane charges.

The type IIb theory in ten dimensions can be obtained as well by considering the M-theory on a torus of vanishing volume $\mathcal{V} \to 0$ and fixed complex structure Ω [85, 86]. The complex structure $\Omega = \Omega_1 + i\Omega_2$ with $\Omega_2 = R_{10}/R_{11}$ becomes the complexified coupling constant of the type IIb superstring $\tau = C^{(0)} + i/g_s^B$ where $C^{(0)}$ is the RR 0-form potential which couples to D-instantons. For external states without momenta in the internal directions the resulting amplitude will be invariant under the $Sl(2,\mathbb{Z})$ symmetry inherited by large diffeomorphism of the compactification torus. This (geometric) symmetry becomes in the type IIb limit the non-perturbative S-duality symmetry of the theory.

Under this reduction the higher derivative corrections in eq. (5.2) to the effective action in eleven dimensions transforms as

$$\frac{\ell_P^{2k-2}}{2\,\kappa_{(11)}^2} \int d^{11}x \sqrt{-G}\, c_k\, \mathcal{R}^k \to \frac{\ell_s^{2k-2}}{2\,\kappa_{(10)}^2} \int d^{10}x\, \sqrt{-g^A}\, c_k\, (g_s^A)^{\frac{2(k-4)}{3}}\, \mathcal{R}^k \tag{5.9}$$

giving contributions to be interpreted as string perturbative contributions to the type IIa string effective action only if [87]

$$k = 4 + 3m \qquad \text{with} \qquad m \geq 0\,. \tag{5.10}$$

When the condition (5.10) is not satisfied the coefficient c_k is *set to zero* and give is the same condition as in (5.3). And when this condition is satisfied an higher derivative correction to the classical action of M-theory is expected. The precise value for this coefficient will be determined by matching the value of the perturbative string genus $m + 1$ contribution for this operator.

The contributions in eq. (5.2) will contribute to higher-loop amplitude computations as counter-term to the ultraviolet divergences of the graviton amplitudes.

For instance the $\mathcal{R}^4$ term

$$\delta^{(4)}\mathcal{S} = \frac{1}{2\kappa_{(11)}^2} \int d^{11}x\sqrt{-G}\, c_4\, \frac{\kappa_{(11)}^2}{\ell_P^3}\, \mathcal{R}^4 \tag{5.11}$$

will be a one-loop $L = 1$ counter-term for the $\Lambda^3 \sim 1/\ell_P^3$ divergence in the four-graviton amplitude. The contributions that do not match the condition in eq. (5.10) have a zero coefficient $c_k = 0$ which means that a primitive ultraviolet divergence of higher-loops amplitudes in eleven dimensions is subtracted with a zero remainder. One interesting case that occurs in the dimension 20 operators $\mathcal{D}^{12}\mathcal{R}^4$.[7] Its contribution to the effective action of M-theory can be understood as a counter-term to the logarithmic divergence of the two-loop $L = 2$ four-graviton amplitude in eleven dimensions

$$\delta^{(10)}\mathcal{S} = \frac{1}{2\kappa_{(11)}^2} \int d^{11}x\sqrt{-G}\, c_{10}\, \kappa_{(11)}^4\, \mathcal{D}^{12}\mathcal{R}^4\,, \tag{5.12}$$

which is the counter-term to a superficial $\Lambda^9 \sim 1/\ell_P^9$ divergence of the three-loop $L = 3$ four-graviton amplitude in eleven dimensions

$$\delta^{(10)}\mathcal{S} = \frac{1}{2\kappa_{(11)}^2} \int d^{11}x\sqrt{-G}\, c_{10}\, \frac{\kappa_{(11)}^6}{\ell_P^9}\, \mathcal{D}^{12}\mathcal{R}^4\,. \tag{5.13}$$

Both of these points of view will be discussed in the section 7.

5.2. *An all order argument for* $\beta_L = L$

The general structure of an L-loop amplitude in eleven dimensions is

$$M_{4;L} = \sum_i \mathcal{D}^{2n_i}\, \hat{R}^4\, I_{4;L(i)}^{(11)}\,, \tag{5.14}$$

[7] On-shell there are two types of couplings [72, 73] given by $(S^2 + T^2 + U^2)^3\mathcal{R}^4$ and $(S^3 + T^3 + U^3)^2\,\mathcal{R}^4$. Each coupling receives distinct contribution in string theory at tree-level order (A. 6) and at loop order (A. 12).

where $I^{(11)}_{4;L(i)}$ is a Feynman loop integral (not necessarily of scalar type for $L \geq 3$) so that the low-energy expansion is given by

$$M_{4;L} \sim \mathcal{D}^{2\beta_L} \hat{R}^4 \, \Lambda^{9L-6-2\beta_L} + \cdots \,, \tag{5.15}$$

where the ellipsis are for sub-leading ultraviolet divergences. After compactification on a circle of radius $R_{11}\,\ell_P$, there are Kaluza–Klein states of mass $n^2/(R_{11}\,\ell_P)^2$ running in the loops, and the low-energy expansion of the loop amplitude becomes [33]

$$M_{4;L} \sim \sum_{n\geq 0} \sum_{\nu=0}^{9L-6-2\beta_L} \Lambda^{9L-6-2\beta_L-\nu} (R_{11}\ell_P)^{-\nu} \, (R_{11}\,\ell_P \mathcal{D})^{2n} \mathcal{D}^{2\beta_L} \hat{R}^4 + \cdots \tag{5.16}$$

converting this expression to the string frame using the metric of eq. (5.5) and the relation $R_{11}^3 = (g_s^A)^2$ we get that the amplitude contributes in ten dimensions to

$$M^{(11\to 10)}_{4;L} \sim \sum_{n\geq 0} \sum_{\nu=0}^{9L-6-2\beta_L} (\Lambda\ell_P)^{9L-6-2\beta_L-\nu} \, (g_s^A)^{2(h-1)} \, \mathcal{D}^{2(\beta_L+n)} R^4 + \cdots \tag{5.17}$$

with

$$h = n + \frac{\beta_L - \nu}{3} \,. \tag{5.18}$$

The quantity h is the highest genus order at which the higher derivative contribution $D^{2(\beta_L+n)}R^4$ can occur in string perturbation in ten dimensions.

Since $\nu \geq 0$ we have that $h \leq n + \beta_L/3 \leq n + \beta_L$. We have seen in section 4 that the four-graviton one-loop amplitude has $\beta_1 = 0$ and that $\beta_L \geq 2$ for $L \geq 2$, therefore only the Kaluza–Klein contributions from the $L = 1$ loop contribution gives the maximal genus contribution $h = k$ for the operator $\mathcal{D}^{2k}R^4$ which implies that the low-energy limit of the supergravity and string theory amplitudes statisfy the rule[8]

$$\beta_L = L \,. \tag{5.19}$$

This argument indicates that the operator $\mathcal{D}^{2k}R^4$ to string effective action in ten dimensions receives a perturbative contributions until genus k.

[8] Infra-red singularities could reduce the derivative order to which the amplitude contributes in the low-energy limit. We are assuming that no infra-red singularities are encountered when taking the low-energy limit of the loop amplitude. The issue of infra-red singularities in the low-energy of the amplitude does not arise in ten dimensions but could be a problem for the compactified cases [11].

The same argument implies that the operators $\mathcal{D}^{2k} R^4 \, r_A^{-2n}$ to the type IIa string effective action in nine dimensions receives perturbative contributions until genus $k+n$.

We will see in the next section how the duality symmetries of M-theory relate the higher-order Kaluza–Klein contributions of the $L=1$ loop amplitude to the contributions from higher-loop orders.

6. The contributions from the one-loop amplitude

We consider the reduction on a d dimensional torus $\mathbb{T}^d$ of the four-graviton $L=1$ loop amplitude $M_{1;4}$ given in eq. (4.8). The result is given by the sum of the scalar integrals for each channels in eq. (4.10)

$$I_4^{(11-d)}(S,T) = \frac{\pi^{\frac{11-d}{2}}}{\ell_P^d \, \mathcal{V}} \int_{\Lambda^{-2}}^{\infty} \frac{dt}{t} \, t^{\frac{d-3}{2}} \int_{\mathcal{T}_{ST}} \prod_{r=1}^{4} d\nu_r \sum_{\{m\}\in\mathbb{Z}^d} e^{-\pi \, t \, G^{IJ} m_I m_J + \pi \, t \, Q_4} \, . \tag{6.1}$$

The masses of the Kaluza–Klein state running in the loop is denoted $G^{IJ} m_I m_J$ and the volume of the torus is $\ell_P^d \, \mathcal{V}_d$. This expression contains a non-analytic contribution from the massless supergravity states in dimensions $11-d$, and analytic terms. The non-analytic part is the usual field theory contribution from the massless states given by

$$I_{4,\text{nonana}}^{(11-d)}(S,T) \sim \int_0^1 \prod_{r=1}^{3} d\nu_r \, (Q_4)^{\frac{d-3}{2}} \, . \tag{6.2}$$

For $d=0$ this is the eleven dimensional supergravity contribution $M_{4;1} \sim (-S)^{3/2}$, for $d=1$ this is the ten dimension supergravity contribution $M_{4;1}^{(10)} \sim S \log(-S)$ of eq. (A. 13), and for $d=2$ this is the nine dimensional contribution $M_{4;1}^{(9)} \sim (-S)^{-1/2}$ (see [33, 57, 58] for a detailed discussion on the relation between these various contributions by considering the decompactification limits $d=2 \to d=1$ and $d=1 \to d=0$).

The expression for $I_4^{(11-d)}$ has the same leading ultraviolet divergence Λ^3 as the parent integral $M_{4;1}$, with Λ an ultraviolet cut-off measured in eleven dimensional Planck units. The ultraviolet divergences arise from the momentum independent part for small values of the proper time $t = \Lambda^{-2} \sim 0$, and correspond to a local ultraviolet divergence in eleven dimensions.

In order to isolate the divergences one must perform a Poisson resummation over the Kaluza–Klein modes m_I to get [57, 58]

$$
\begin{aligned}
I_o^{(11-d)}(S,T) &= \pi^{\frac{11-d}{2}} \int_0^{\Lambda^2} d\hat{t}\,\hat{t}^{\frac{1}{2}} \sum_{\{\hat{m}\}\in\mathbb{Z}^d} e^{-\pi\hat{t}\,G_{IJ}\hat{m}^I\hat{m}^J} \\
&= \Lambda^3 + \frac{1}{\mathcal{V}^3} \sum_{\substack{\{\hat{m}\}\in\mathbb{Z}^d\\ \{\hat{m}\}\neq(0,\cdots,0)}} \frac{1}{(\hat{m}^I\,\hat{G}_{IJ}\,\hat{m}^J)^{\frac{3}{2}}}\,,
\end{aligned} \tag{6.3}
$$

where $G_{IJ} = \mathcal{V}\,\hat{G}_{IJ}$ is the metric of the d-torus and $\det\hat{G}_{IJ} = 1$. The ultraviolet divergence is now localised in the zero winding sector $\hat{m}_I = 0$. The finite part is the contribution from the non zero winding modes given by

$$
\mathcal{E}_s(G_{IJ}) \equiv \sum_{\substack{\{\hat{m}\}\in\mathbb{Z}^d\\ \{\hat{m}\}\neq(0,\cdots,0)}} \frac{1}{(\hat{m}^I\,\hat{G}_{IJ}\,\hat{m}^J)^s}\,. \tag{6.4}
$$

This expression is invariant under the large diffeomorphism $Sl(d,\mathbb{Z})$ of the d dimensional torus. The higher order terms in the external momenta expansion gives

$$
\tilde{I}_4^{(11-d)}(S,T) = 2\pi^{7-d-n}\,(\ell_P^d\mathcal{V})^{n+\frac{d-5}{2}}\,\frac{\mathcal{G}_{ST}^n}{n!}\,\Gamma\left(\frac{d-3}{2}+n\right)\,\mathcal{E}_{\frac{d-3}{2}+n}(\hat{G}^{IJ}) \tag{6.5}
$$

where

$$
\mathcal{G}_{ST}^n \equiv \int_{\mathcal{T}_{ST}} \prod_{i=1}^4 d\nu_i\,(Q_4)^n\,. \tag{6.6}
$$

The superficial divergence of the one-loop amplitude is subtracted by adding to the eleven dimensional action the *local* counter-term of equation (5.11) which contributes to the one-loop amplitude by the following contribution

$$
\delta_1 M_{4;1}^{(11)} = c_4\,\frac{\kappa_{(11)}^4}{(2\pi)^{11}}\,\frac{\pi^3}{2}\ell_P^3\,\hat{R}^4\,. \tag{6.7}
$$

We now determine the value of c_4 by matching the total one-loop amplitude with corresponding string theory expressions reviewed in section Appendix A.

6.1. The circle compactification

For the case of a circle compactification $\mathbb{T}^1 = S^1$ of radius R_{11} one gets [57, 58]

$$
M_{4;1} + \delta_1 M_{4;1} = \frac{\kappa_{(11)}^4}{\ell_P^3}\,\hat{R}^4\,[(\Lambda\,\ell_P)^3 + c_4 + \frac{2\zeta_3}{R_{11}^3}] + \cdots \tag{6.8}
$$

where the ellipsis are for higher derivative contributions discussed in eq. (6.13) and eq. (6.15) that are independent of the cut-off. Using the dictionary in eq. (5.5) the amplitude translates into the ten dimensional expression

$$M_{4;1} + \delta_1 M_{4;1} \sim \kappa_{(10)}^2 \, \hat{R}^4 \, [(\ell_P \Lambda)^3 + c_4 + \frac{2\zeta_3}{(g_s^A)^2}] + \cdots \tag{6.9}$$

where we recognise tree-level and one-loop string contributions. Comparing with the value of the $\hat{R}^4$ contribution at genus-one in string theory in eq. (A. 12) we deduce that

$$(\ell_P \Lambda)^3 + c_4 = \frac{2\pi^2}{3}. \tag{6.10}$$

Another equivalent way of formulating the same regularisation scheme is to use the T-duality properties of the perturbative string amplitudes, which we discuss now.

6.2. The torus compactification

The $\mathbb{T}^2$ torus compactification of the one-loop four-point amplitude $M_{1;4}$ gives the following perturbative contributions [57, 58]

$$M_{4;1} + \delta_1 M_{4;1} \sim \kappa_{(10)}^2 \, \hat{R}^4 \, \left[\frac{1}{r_A} \, \left((\ell_P \, \Lambda)^3 + c_4 \right) + r_A \, \left(\frac{2\zeta_3}{(g_s^A)^2} + 4\zeta_2 \right) + \cdot \right] \, , \tag{6.11}$$

where we used that $r_A = R_{10}\sqrt{R_{11}}$. The T-duality invariance of the four-graviton amplitude at tree-level and genus-one in string theory requires that the above expression is invariant under the transformation $r_B \to r_A = 1/r_B$. Although the tree-level contribution is invariant thanks to the transformation rules of the ten dimensional dilaton or as consequence of the M-theory dictionary given in eq. (5.5)

$$\frac{r_A}{(g_s^A)^2} = \frac{R_{10}\sqrt{R_{11}}}{R_{11}^3} = \left(\frac{R_{10}}{R_{11}} \right)^2 \, \frac{1}{R_{10}\sqrt{R_{11}}} = \frac{r_B}{(g_s^B)^2} \, , \tag{6.12}$$

the one-loop contribution is invariant only if the condition (6.10) is satisfied.

The rest of the higher order derivative contributions to string genus-one amplitude in (6.9) give ultraviolet finite contributions which read in the type IIa frame

$$M_{4;1} + \delta_1 M_{4;1} \sim \cdots + \kappa_{(10)}^2 \, 8\pi^{\frac{3}{2}} \sum_{n=2}^{\infty} \frac{\Gamma(n-\frac{1}{2})\zeta_{2n-1}}{n!} \, r_A^{2n-1} \, (\ell_s \mathcal{D}^{2n} \hat{R}^4) + \cdots \, , \tag{6.13}$$

and in the type IIb frame

$$M_{4;1}+\delta_1 M_{4;1} \sim \cdots+\kappa_{(10)}^2\, 8\pi^{\frac{3}{2}} \sum_{n=2}^{\infty} \frac{\Gamma(n-\frac{1}{2})\zeta_{2n-1}}{n!}\, r_B^{-2n+1}\, (\ell_s \mathcal{D}^{2n} \hat{R}^4)+\cdots\,, \tag{6.14}$$

where $\mathcal{D}^{2n} = \mathcal{G}_{ST}^n + \mathcal{G}_{TU}^n + \mathcal{G}_{SU}^n$. The ellipsis represent the contributions given in eq. (6.9) and in eq. (6.15). These expressions match the corresponding contributions to the derivative expansion of the genus-one amplitude compactified to nine dimensions on a circle of radius r_A or r_B derived in [72], but they are not invariant under the T-duality symmetry $r_A \to r_B = 1/r_A$. We will see below that the missing contributions are provided by the higher loop $L \geq 2$ contributions in eleven dimensions.

Finally the last piece from $L = 1$ amplitude are the higher-derivative contributions that give higher string genus contributions in the type IIa frame

$$M_{4;1}+\delta_1 M_{4;1} \sim \cdots+\kappa_{(10)}^2\, 8\pi^2\, r_A \sum_{n=2}^{\infty} \frac{\Gamma(n-1)\zeta_{2n-2}}{n!}\, (g_s^A)^{2n-2}\, (\ell_s \mathcal{D}^{2n} \hat{R}^4)\,, \tag{6.15}$$

and in the type IIb frame

$$M_{4;1} + \delta_1 M_{4;1} \sim \cdots + \kappa_{(10)}^2\, 8\pi^2 \sum_{n=2}^{\infty} \frac{\Gamma(n-1)\zeta_{2n-2}}{n!}\, \frac{(g_s^B)^{2n-2}}{r_B^{2n+1}}\, (\ell_s \mathcal{D}^{2n} \hat{R}^4)\,. \tag{6.16}$$

These contributions are genus n contributions to the operator $\mathcal{D}^{2n}\mathcal{R}^4$, satisfying the relation $\beta_n = n$ derived in section 5.2. The value of the coefficient for the $\mathcal{D}^4 \hat{R}^4$ term matches the genus two contributions derived from string theory in eq. (A. 17) [58, 88].

We will return to these contributions in section 8 when we will discuss non renormalisation theorems in string theory and supergravity.

6.3. The three-torus compactification

Taking the amplitude on a three-torus $\mathbb{T}^3$ one gets for the $\hat{R}^4$ term in the type IIa variables

$$M_{4;1} + \delta_1 M_{4;1} \sim \kappa_{(10)}^2\, \Big([(\ell_P \Lambda)^3 + c_1]\, U_2 + \mathcal{E}_{\frac{3}{2}}(\hat{G}_{IJ}) \Big)\, \hat{R}^4 \tag{6.17}$$

where $\mathcal{E}_{3/2}(\hat{G})$ is the $Sl(3, \mathbb{Z})$ modular form defined in eq. (6.4), which has the perturbative expansion [89, 90]

$$\mathcal{E}_{\frac{3}{2}}(\hat{G}) = 2\zeta_3\, \frac{T_2}{(g_s^A)^2} - \pi \log(T_2|\eta(T)|^2) - \frac{4\pi}{3}\, \log(T_2^2/g_s^A) + n.p. \tag{6.18}$$

where the first term corresponds to the tree-level contribution for the type IIa string on two torus of volume T_2, the second term gives the contributions from the wrapped F-string on the two torus and contains the genus-one perturbative contributions, the third term is a logarithmic term expressed in terms of the the eight dimensional dilaton $\exp(-2\phi^{(8)}) = T_2 \exp(-2\phi^{(10)})$. This contribution arises from the massless threshold $\hat{R}^4 \log(s)$ in eight dimensions after performing a Weyl rescaling to get to the Einstein frame. The non perturbative effects are the D-instanton effects and the (p, q)-string wrapped around the two-torus.

The $Sl(3, \mathbb{Z})$ invariance of the coupling, inherited from the large diffeomorphism of the torus, is part of the eight dimensional U-duality group $Sl(3, \mathbb{Z}) \times Sl(2, \mathbb{Z})_U$ where U is the complex structure of the two torus on which the type IIa string is compactified. As for the case of the compactification to nine dimensions the string perturbative answer has to be invariant under the T-duality sub-group $Sl(2, \mathbb{Z})_T \times Sl(2, \mathbb{Z})_U$, and symmetric under the exchange between T and U. The perturbative part of eq. (6.17) is invariant only if the condition (6.10) is satisfied and the dependence on the complex structure U is given by $\log(U_2|\eta(U)|^2)$. These requirements allow the determination of the U-duality invariant coupling uniquely and reproduce the result given by [89]

$$\int d^9x \sqrt{-g^{(8)}} \left(\mathcal{E}_{\frac{3}{2}}(M) - \pi \log(U_2|\eta(U)|^2)\right) \hat{R}^4 , \tag{6.19}$$

with the $Sl(3, \mathbb{Z})$ modular forms defined by

$$\mathcal{E}_s(M) = \sum_{(m_1,m_2,m_3)\neq(0,0,0)} \frac{\nu_8^{-\frac{s}{3}}}{\left(\frac{|m_1+m_2\Omega+m_3B|^2}{\Omega_2} + \frac{m_3^2}{\nu_8}\right)^s} \tag{6.20}$$

where $1/\sqrt{\nu_8} = V^2/\ell_P^2 = g_S^{-1/2}\, T_2$ is the dimensionless compactification volume measured in Planck length unit, and $B = B_R + \Omega B_{NS}$ is the combinaison of the RR and NS B-field. The dependence on the B-field is needed in order to consider wrapped M2-branes along the internal directions which are not included in the construction from the multiloop amplitude we have described. The total eight dimensional coupling

$$\mathcal{E}^{(8d)}_{(0,0)} = \mathcal{E}_{\frac{3}{2}}(M) - 2\pi \log(U_2|\eta(U)|^2) \tag{6.21}$$

is a zero mode of the $SO(3)\backslash Sl(3, \mathbb{Z}) \times SO(2)\backslash Sl(2, \mathbb{Z})$ Laplacian associated with the U-duality group in eight dimensions [89]

$$\Delta_{Sl(3)\times Sl(2)}\, \mathcal{E}^{(8d)}_{(0,0)} = 8\pi\, . \tag{6.22}$$

The $D^4\hat{R}^4$ contribution from eq. (6.5) gives the coupling [90]

$$(T_2\,(g_s^A)^2)^{\frac{1}{3}}\,\mathcal{E}_2(\hat{G}^{-1})\,D^4\hat{R}^4 \sim \left[\zeta_4\,(g_s^A)^2 + 90\pi\,T_2\,E_2(U) + n.p.\right]\,\hat{R}^4 \quad (6.23)$$

where $E_s(U)$ are the usual $Sl(2,\mathbb{Z})$ Eisenstein series depending on the complex structure of the torus on which type IIa string is compactified. We recognise the genus two contributions given in eq. (6.15) and the genus-one contribution depending on the moduli T and U of the torus on which type IIa string is compactified. This expression fails to be invariant under T-duality but this symmetry will be recovered once the $L = 2$ contribution has been added.

7. The contributions from the two-loop amplitude

In this section we describe the contributions from the $L = 2$ amplitude compactified on a circle and on a torus. The analysis of the two loop amplitude brings new technical difficulties which we will not comment on and refer to the papers [58, 73, 75] for details. We only describe its consequences on the M-theory and string theory low energy effective action.

• The $\mathcal{D}^4\hat{R}^4$ term is the leading contribution to the low-energy limit of the $L = 2$ amplitude of eq. (4.15).
Taking the two-loop amplitude on a circle of radius $R_{11}\,\ell_P$ one gets at this order

$$I_1 \sim \left((\ell_P\Lambda)^8 + c_{2;0} + \pi^5\zeta_5\,[(\ell_P\Lambda)^3 + c_4]\,\frac{1}{R_{11}^5}\right)\,\mathcal{D}^4\hat{R}^4\,. \quad (7.1)$$

We have included the contribution from the new counter-term $c_{2;0}$ of eq. (4.21) needed to regulate the new primitive divergence at this order, and the contributions from the triangle diagram with the one-loop counter-term of figure 3(c). Converted into string variables using the relation between the M-theory parameters and the string variables given in eq. (5.5) one gets

$$I_1 \sim \left([(\Lambda\,\ell_P)^8 + c_{2;0}]\,(g_s^A)^{\frac{10}{3}} + \pi^5\zeta_5\,[(\Lambda\,\ell_P)^3 + c_4]\,\frac{1}{(g_s^A)^2}\right)\,\mathcal{D}^4\hat{R}^4\,. \quad (7.2)$$

The first term does not have any meaning in string perturbation theory, therefore the leading divergence has to be subtracted with no finite remainder

$$(\Lambda\,\ell_P)^8 + c_{2;0} = 0\,. \quad (7.3)$$

The second term is a string three loop contribution, which contribution which is determined by the regulated one-loop divergence in eq. (6.10) giving the correct value of the tree-level contribution to the $\mathcal{D}^4\hat{R}^4$ term [58].

Taking the amplitude on a torus one gets

$$I_1 \sim \left([(\Lambda\,\ell_P)^8 + c_{2;0}] + \frac{\pi^5}{2}[(\Lambda,\ell_P)^3 + c_4]\,\frac{E_{\frac{5}{2}}(\Omega)}{\mathcal{V}^{\frac{5}{2}}} + \frac{2\pi^4\zeta_3\zeta_4}{\mathcal{V}^4} \right) \mathcal{D}^4\hat{R}^4\,. \tag{7.4}$$

The second term in this expression decompactifies to a finite $E_{5/2}(\Omega)\,\mathcal{D}^4\hat{R}^4$ contribution to the ten dimensional type IIb limit $\mathcal{V} \to 0$ with Ω kept constant. The last term in this expression gives a genus 1 contribution to the effective action of type II string in nine dimensions. This contribution together with the $\mathcal{D}^4\hat{R}^4$ from the $L = 1$ amplitude in eq. (6.13) gives the complete contribution to the string genus-one amplitude at this order [71, 72]

$$\delta I_1 \sim \frac{\zeta_3}{15}(r_A^3 + \frac{1}{r_A^3})\,\mathcal{D}^4\hat{R}^4\,. \tag{7.5}$$

One notices that the $L = 2$ contribution provided by the genus-one $1/r_A^3\,\mathcal{D}^4\hat{R}^4$ missing in eq. (6.13) for the T-duality symmetry $r_A \to r_B = 1/r_A$ of the nine dimensional contribution to the low energy expansion of the genus-one string amplitude on a circle.

This is the manifestation of a generic phenomena where the compactification of the higher-loop four-graviton amplitudes will gives genus one contributions to the higher-derivative operators $\mathcal{D}^{2k}\hat{R}^4$ completing the expression in eq. (6.13) in a T-duality invariant expression.

The complete $\mathcal{D}^4\,\hat{R}^4$ in type IIb variables is given in the Einstein frame by [73]

$$\int d^9x\,\sqrt{-g^{(9)}}\,\mathcal{E}^{(9d)}_{(1,0)}\,\mathcal{D}^4\,\hat{R}^4\,. \tag{7.6}$$

with

$$\mathcal{E}^{(9d)}_{(1,0)} = \nu_9^{-\frac{3}{7}}E_{\frac{3}{2}}(\Omega) + \frac{2\pi^2}{3}\,\nu_9^{\frac{4}{7}}\,, \tag{7.7}$$

where $\nu_9^{1/2} = r_B/g_s^{1/4}$ is the radius of the circle measures Planck length unit. This coupling satisfies the differential equation [73]

$$[\Delta_\Omega + \frac{7}{4}\nu_9(\partial_{\nu_9}\nu_9\partial_{\nu_9}) + \frac{1}{2}\nu_9\partial_{\nu_9}]\,\mathcal{E}^{(9d)}_{(1,0)} = \frac{30}{7}\,\mathcal{E}^{(9d)}_{(1,0)}\,. \tag{7.8}$$

Considering the amplitude on $\mathbb{T}^3$ we obtain

$$\mathcal{E}_{\frac{5}{2}}(G) = 2\left(\frac{T_2}{(g_s^A)^2}\right)^{\frac{5}{3}}\zeta_5 + \frac{4}{3}\left(\frac{(g_s^A)^2}{T_2}\right)^{\frac{1}{3}}\,E_2(T)$$

which added to the $L = 1$ contribution in eq. (6.18) gives an expression invariant under the exchange of T and U moduli. The complete U-duality $Sl(3,\mathbb{Z}) \times Sl(2,\mathbb{Z})$ invariant expression has been derived in [90] with the result in the string frame being

$$\int d^8x \sqrt{-g^{(8)}}\, g_s^{-\frac{4}{3}}\, T_2^{-\frac{2}{3}} \, \Big(\mathcal{E}_{\frac{5}{2}}(M) - 8\, E_{-\frac{1}{2}}(M)\, E_2(U)\Big)\, \mathcal{D}^4\, \mathcal{R}^4\,. \tag{7.9}$$

where $\mathcal{E}_s(M)$ are the $Sl(3,\mathbb{Z})$ modular forms defined in eq. (6.20). The total eight dimensional coupling

$$\mathcal{E}^{(8d)}_{(1,0)} = \mathcal{E}_{\frac{5}{2}}(M) - 8\, E_{-\frac{1}{2}}(M)\, E_2(U)\,, \tag{7.10}$$

is an eigenfunction of the $Sl(3,\mathbb{Z}) \times Sl(2,\mathbb{Z})$ Laplacian [73]

$$\Delta_{Sl(3)\times Sl(2)}\, \mathcal{E}^{(8d)}_{(1,0)} = \frac{10}{3}\, \mathcal{E}^{(8d)}_{(1,0)}\,. \tag{7.11}$$

• Expanding the $L = 2$ loop to the next order, we obtain for the $\mathcal{D}^6 \hat{R}^4$ term on a circle [73, 75]

$$I_2 \sim \left([(\Lambda\, \ell_P)^6 + c_{2;1}] + [(\Lambda\, \ell_P)^3 + c_{2;1}] \frac{\pi\zeta_3}{3\, R_{11}^3} + \frac{\zeta_3^2}{2R_{11}^6} \right) \mathcal{D}^6 \hat{R}^4\,. \tag{7.12}$$

Converted to string variables and using the relation (6.10) we have

$$I_2 \sim \left([(\Lambda\, \ell_P)^6 + c_{2;1}]\, (g_s^A)^2 + \zeta_2\zeta_3 + \frac{\zeta_3^2}{2(g_s^A)^2} \right) \mathcal{D}^6 \hat{R}^4\,. \tag{7.13}$$

where we recognise a genus two, genus-one and tree-level string contribution. In order to determine the precise value for the new counter-term $c_{2;1}$ one would need to know the value of the genus 2 contribution to the $\mathcal{D}^6 \hat{R}^4$ in ten dimensions. Since this value is not known it will be determined later using the duality relations. The finite remainder after subtracting the divergence will provide a new corrections to the M-theory effective action of eq. (5.2) in agreement with the considerations of section 5.2 and the reference [87].

Taking the amplitude on a torus one gets [73, 75]

$$I_2 \sim \left([(\Lambda\, \ell_P)^6 + c_{2;1}] + [(\Lambda\, \ell_P)^3 + c_{2;1}]\, \frac{E_{\frac{3}{2}}(\Omega)}{\mathcal{V}^{\frac{3}{2}}} + \frac{\mathcal{E}_{(\frac{3}{2},\frac{3}{2})}(\Omega)}{\mathcal{V}^3} \right) \mathcal{D}^6 \hat{R}^4\,. \tag{7.14}$$

The contribution to the $\mathcal{D}^6\hat{R}^4$ to the ten dimensional effective action for the type IIb string $\mathcal{E}_{(3/2,3/2)}$ has been determined in [75] and has the following weak-coupling expansion

$$\mathcal{E}_{(\frac{3}{2},\frac{3}{2})}(\Omega) = 4\zeta_3^2\Omega_2^3 + 8\zeta_2\zeta_3\,\Omega_2 + \frac{48}{5}\zeta_2^2\Omega_2^{-1} + \frac{8}{9}\zeta_6\Omega_2^{-3} + O(e^{-2\pi\Omega_2})\,. \tag{7.15}$$

Converting eq. (7.14) into type IIa string variables and using the relation (6.10) we have

$$\begin{aligned} I_2 \sim \big(4\zeta_3^2\,\frac{r_A}{(g_s^A)^2} &+ 8\zeta_2\zeta_3\,(r_A + \frac{1}{r_A}) \\ &+ 16\zeta_2^2\,\frac{(g_s^A)^2}{r_A} + [(\Lambda\,\ell_P)^6 + c_{2;1}]\,r_A\,(g_s^A)^2 + \frac{48}{5}\,\zeta_2^2\,\frac{(g_s^A)^2}{r_A^3} \\ &+ \frac{8}{9}\,\zeta_6\,\frac{(g_s^A)^4}{r_A^5}\big)\,\mathcal{D}^6\hat{R}^4\,. \end{aligned} \tag{7.16}$$

The first line gives the string tree-level and the genus-one contributions in nine dimensions to $\mathcal{D}^6\hat{R}^4$ couplings. The value of these couplings match the results extracted from string perturbation theory (see the appendix Appendix A and [71, 72]). Again we stress the fact that the genus-one contribution is invariant under the T-duality transformation $r_A \to r_B = 1/r_A$ thanks to the relation in eq. (6.10) determining the one-loop counter-term.

The second line of this expression gives string genus two contributions. Counting the number of fermionic zero modes involved in the gravity amplitude, the genus two type IIa and type IIb string contributions are the same at the $\mathcal{D}^6\hat{R}^4$ order[9] and the second line in eq. (7.16) should be invariant under the T-duality symmetry $r_A \to r_B = 1/r_A$. The first term in this expression is invariant thanks to the transformation rules of the eight-dimensional dilaton in eq. (6.12). The second and the third terms are exchanged only if

$$(\Lambda\,\ell_P)^6 + c_{2;1} = \frac{48}{5}\,\zeta_2^2\,, \tag{7.17}$$

which determines the value of the $\mathcal{D}^6\hat{R}^4$ in the M-theory effective action (5.2).

The third line in eq. (7.16) is a genus three contribution. This expression is invariant under T-duality once summed with the three loop contribution from the

[9] In the RNS formalism, the chirality dependence of the gravity amplitudes enters from the odd/odd spin structure contributions. Because string perturbations has $2(g-1)$ odd moduli at genus g, the four-graviton amplitude in type II superstring can get contributions from the odd/odd spin structure sector from genus three. Using the non-minimal pure spinor formalism Berkovits showed that the chirality dependences arises from genus five [16].

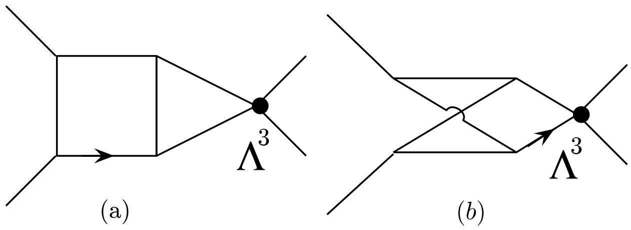

Fig. 5. Sub-divergences of the $L = 3$ amplitude constructed from figure 4(b). Diagram (a) and (b) contribute to a $\Lambda^3/\mathcal{V}^6\,\mathcal{D}^6\,\hat{R}^4$. The arrow indicates the quadratic dependence on the loop momenta in the numerator of the expression for the two-loop integrals.

$L = 1$ contribution in eq. (6.15) to give

$$\frac{8}{9}\,\zeta_6\,\left(r_A + \frac{1}{r_A^5}\right)\,(g_s^A)^4\,\mathcal{D}^6\hat{R}^4\,. \tag{7.18}$$

The $L = 3$ three-loop supergravity amplitude in eleven dimensions will contribute to the $\mathcal{D}^6\hat{R}^4$ coupling only from the class of diagrams constructed from the vacuum diagram given in figure 4(b) because the diagrams obtained from figure 4(a) all have a prefactor of $\mathcal{D}^8\,\hat{R}^4$ [29]. This amplitude satisfies the $\beta_3 = 3$ rule and has mass dimension

$$[M_{4,3}] \sim (mass)^{15}\,\mathcal{D}^6\hat{R}^4\,. \tag{7.19}$$

The sub-divergence in figure 5(a,b) can respectively contribute to a term of the type

$$\frac{\Lambda^3}{\mathcal{V}^6}\,\mathcal{D}^6\,\hat{R}^4\,, \tag{7.20}$$

contributing to a genus-one term completing the genus-one term $r_A^5\zeta_5\,\mathcal{D}^6\hat{R}^4$ from the $L = 1$ amplitude given in eq. (6.13) into a T-duality invariant expression

$$\zeta(5)\,\left(r_A^5 + \frac{1}{r_A^5}\right)\,\mathcal{D}^6\hat{R}^4\,. \tag{7.21}$$

This amplitude can only contribute to the ten dimensional type IIb effective action from a contribution of the type [73]

$$I_3 \sim \frac{\Lambda^9}{\mathcal{V}^3}\,f(\Omega)\,, \tag{7.22}$$

but a simple dimensional analysis on the various diagrams entering the $L = 3$ amplitude shows that no such contributions can be found in the solution given in [29] and the coupling to the ten dimensional type IIb effective action given by the eq. (7.15) is not renormalised.

The complete $\mathcal{D}^6\,\hat{R}^4$ coupling in nine dimensions in the Einstein frame is given by [73]

$$\int d^9x\,\sqrt{-g^{(9)}}\,\mathcal{E}^{(9d)}_{(0,1)}\,\mathcal{D}^6\,\hat{R}^4\,, \tag{7.23}$$

with

$$\begin{aligned}\mathcal{E}^{(9d)}_{(0,1)} &= \nu_9^{-\frac{6}{7}}\,\mathcal{E}_{(0,1)} + 4\zeta(2)\,\nu_9^{\frac{1}{7}}\,E_{\frac{3}{2}} + \frac{12\zeta(2)}{63}\,\nu_9^{\frac{15}{7}}\,E_{\frac{5}{2}}\\ &+ \frac{24\zeta(2)\zeta(5)}{63}\,\nu_9^{-\frac{20}{7}} + \frac{48\zeta(2)^2}{5}\,\nu_9^{\frac{8}{7}}\,.\end{aligned} \tag{7.24}$$

satisfying the Laplace equation

$$[\Delta_\Omega + \frac{7}{4}\nu_9(\partial_{\nu_9}\nu_9\partial_{\nu_9}) + \frac{1}{2}\nu_9\partial_{\nu_9}]\,\mathcal{E}^{(9d)}_{(0,1)} = 12\,\mathcal{E}^{(9d)}_{(0,1)} - 6\left(\mathcal{E}^{(9d)}_{(0,0)}\right)^2\,, \tag{7.25}$$

where $\mathcal{E}^{(9d)}_{(0,0)}$ is the nine dimensional $\hat{R}^4$ coupling

$$\mathcal{E}^{(9d)}_{(0,0)} = \nu_9^{-\frac{3}{7}}\,E_{\frac{3}{2}}(\Omega) + 4\zeta_2\,\nu_9^{\frac{4}{7}}\,. \tag{7.26}$$

The $D^6\,\hat{R}^4$ coupling in eight dimensions is given in the string frame [91]

$$\int d^8x\,\sqrt{-g^{(8)}}\,\frac{g_s^2}{T_2}\,\mathcal{E}^{(8d)}_{(0,1)}\,\mathcal{D}^6\,\hat{R}^4 \tag{7.27}$$

with

$$\mathcal{E}^{(8d)}_{(0,1)} = \mathcal{E}_{(\frac{3}{2},\frac{3}{2})}(M) + \frac{20}{3}\,E_{-\frac{3}{2}}(M)\,E_3(U) + \frac{1}{2}E_{\frac{3}{2}}(M)\,E_1(U) + f(U,\bar{U}) \tag{7.28}$$

where the $Sl(3,\mathbb{Z})$ modular forms $\mathcal{E}_{(3/2,3/2)}(M)$ is defined by

$$\Delta_{Sl(3)}\,\mathcal{E}_{(\frac{3}{2},\frac{3}{2})}(M) = 12\,\mathcal{E}_{(\frac{3}{2},\frac{3}{2})}(M) - \frac{3}{2}\,E^2_{\frac{3}{2}}(M)\,. \tag{7.29}$$

and the function $f(U,\bar{U})$ is defined by the differential equation with a source term [91]

$$\Delta_U\,f(U,\bar{U}) = 12\,f(U,\bar{U}) - 6\,E_1^2(U)\,. \tag{7.30}$$

The total eight dimensional coupling satisfies the Laplace equation with for source term the $\hat{R}^4$ the eight dimensional coupling given in eq. (6.21)

$$\Delta_{Sl(3)\times Sl(2)}\,\mathcal{E}^{(8d)}_{(0,1)} = 12\,\mathcal{E}^{(8d)}_{(0,1)} + \frac{3}{2}\,\left(\mathcal{E}^{(8d)}_{(0,0)}\right)^2\,. \tag{7.31}$$

8. Non-renormalisation theorems

We have described the structure of gravity amplitudes in supergravity and superstring theory, and in particular formulated various constraints that these amplitudes should satisfy in maximal supergravity.

For the case of maximal supergravity theories with $N = 8$ supersymmetry one expects that the low-energy limit of an L-loop four-graviton amplitude will start contributing from $\mathcal{D}^{2L} R^4$, summarised by the rule $\beta_L = L$. In section 4.4 we gave an argument based on supersymmetry for the validity of this rule up-to $L = 6$ and in section 5.2 we gave a argument based on dimensional analysis and the duality relations of M-theory for an all order confirmation of this rule.

When one applies the rule $\beta_L = L$ one deduces important non renormalisation theorems that the couplings to the low-energy effective action of superstring theories have to satisfy. At the level of amplitude computations in supergravity such conditions imply that the ratio of the four-graviton L-loop amplitude to the tree-amplitude would behave as

$$\left[\frac{M_{4;L}}{M_{4;tree}}\right] = \mathcal{D}^{2L}\, stu\, (mass)^{(D-4)L-6}\,, \tag{8.1}$$

indicating that in four dimensions, the L-loop amplitude would have to be the sum of an L-loop dimensionless Feynman integral times a power of the external momenta $\mathcal{D}^{2L}$ increasing with the loop order. Up to genus three it has been shown that the polarisation dependence of the four-graviton amplitude is the same as the tree amplitude [1, 6, 36, 37, 92] and that the structure of the amplitude has the above mentioned structure.

The conjectured S-duality invariance of the type IIb effective action implies that the higher-derivative terms of the type $\mathcal{D}^{2k}\,\hat{R}^4$ must have coefficients that are modular functions $f_k(\Omega)$ under the action of $Sl(2,\mathbb{Z})$ on the complexified coupling constant $\Omega = C^{(0)} + i/g_s$ where $C^{(0)}$ is the Ramond-Ramond 0-form potential. This conjecture can hold only if the polarisation dependence of the four-graviton amplitude at all orders in the genus expansion is the same.

The non renormalisation conditions for the R^4 couplings to the ten dimensional type IIa and IIb effective action [57, 93] beyond one-loop is guaranteed by the fact that higher-loop amplitudes in $N = 8$ supergravity have $\beta_L \geq 2$ for $L \geq 2$.

The exactness of the type IIb $E_{5/2}(\Omega)\,\mathcal{D}^4 R^4$ coupling extracted from the two-loop amplitude in eleven dimensions in [58] (see section 7 of this text as well) is assured by the fact that the rule $\beta_L \geq 3$ for $L \geq 3$ is satisfied by the four-graviton amplitudes in $N = 8$ supergravity [29] and the higher genus (F-terms) computation by Berkovits in [16].[10]

The dimensional analysis argument of section 5.2 implies that the higher-derivative operators $\mathcal{D}^{2k} R^4$ do not receive perturbative corrections beyond genus k in string perturbation theory. This fact has been confirmed up to $k \leq 5$ for the analysis of superstring amplitudes in [16] and supergravity computation in [33, 73].

One important consequence of the structure of the $L = 1$ and the $L = 2$ loop amplitudes and the relation between the eleven dimensional M-theory and string theory is that the higher-order couplings satisfy some differential equations.

The R^4 and $\mathcal{D}^4 R^4$ couplings to the ten dimensional type IIb effective action in ten dimensions satisfy the differential equation

$$\Delta\, f_{\frac{3}{2}+p}(\Omega) = \frac{(3+2p)(1+2p)}{4}\; f_{\frac{3}{2}+p}(\Omega) \tag{8.2}$$

where $\Delta = 4\Omega_2^2 \partial_\Omega \bar\partial_{\bar\Omega}$ is the $Sl(2,\mathbb{Z})$ laplacian and with respectively $p = 0$ and $p = 1$ as a trivial consequence of the structure[11] of the $L = 1$ [57] and the $L = 2$ loop amplitudes [58, 73]. This equation together with the boundary condition

$$\lim_{\Omega_2 \to \infty} \Omega_2^{\frac{1}{2}-p}\, f_{\frac{3}{2}+p}(\Omega) = \text{Cste} \tag{8.3}$$

with the constant given by the expansion of the tree-level amplitude given in eq. (A. 6), determine uniquely the couplings.

At higher-order in the derivative expansion the supersymmetry constraints on the couplings will change their structure with the appearance of source terms [75]. The $\mathcal{D}^6 R^4$ coupling to the type IIb effective action in ten dimensions satisfies the equation

$$\Delta\, f_3(\Omega) = 12\, f_3(\Omega) - 6 E_{\frac{3}{2}}(\Omega)^2 \tag{8.4}$$

with the boundary condition

$$\lim_{\Omega_2 \to \infty} \Omega_2^{-1}\, f_3(\Omega) = 4\zeta_3^2\,, \tag{8.5}$$

has a unique solution the function $f_3(\Omega) = \mathcal{E}_{(3/2,3/2)}(\Omega)$ found in [75] which has the weak coupling expansion given in eq. (7.15). For compactification of

[10] The issue of infra-red singularities in the low-energy of the amplitude does not arise in ten dimensions but could be a problem for the compactified cases [11].

[11] These equations have been shown to be a consequence of the on-shell supersymmetry of the type IIb supergravity in ten dimensions in [94, 95].

string theory to lower dimensions these couplings satisfy equivalent differential equations for the Laplacian associated with the U-duality group. In eight dimensions the U-duality group is $Sl(3,\mathbb{Z}) \times Sl(2,\mathbb{Z})$ and the action of the $SO(3)\backslash Sl(3) \times SO(2)\backslash Sl(2)$ Laplacian on the previous coupling is given by [73, 89–91]

$$\Delta_{Sl(3)\times Sl(2)}\, \mathcal{E}^{(8d)}_{(p,0)} = \frac{(3+2p)2p}{3} \mathcal{E}^{(8d)}_{(p,0)} \tag{8.6}$$

for the $\mathcal{R}^4$ ($p=0$) and the D^4R^4 ($p=1$) and

$$\Delta_{Sl(3)\times Sl(2)}\, \mathcal{E}^{(8d)}_{(0,1)} = 12\, \mathcal{E}^{(8d)}_{(0,1)} - \frac{3}{2}\left(\mathcal{E}^{(8d)}_{(0,0)}(M)\right)^2 \,, \tag{8.7}$$

for the $D^6\hat{R}^4$ term.

The structure of the differential equations for the $\hat{R}^4$ coupling in eq. (6.22), for the $D^4\,\hat{R}^4$ in eqs. (7.8) and (7.11) and for the $D^6\,\hat{R}^4$ in eqs. (7.25) and (7.31) have the same structure as the one for the ten dimensional type IIb theory. These equations that are expected to be a consequence of $N=8$ supersymmetry which is preserved by the torus compactification we have considered.

The presence of the source term in (8.4) can be motivated from the structure of the on-shell supersymmetry for the type IIb effective actions as follows. The on-shell supersymmetry variations of the α' corrections to the effective action of the type IIb superstring

$$S = S^{(0)} + \alpha'^3\, S^{(3)} + \alpha'^5\, S^{(5)} + \alpha'^6\, S^{(6)} + \cdots \,, \tag{8.8}$$

requires the modification of the on-shell supersymmetry transformations [56] at each order

$$\delta_\epsilon = \delta_\epsilon^{(0)} + \alpha'^3\, \delta_\epsilon^{(3)} + \alpha'^5\, \delta_\epsilon^{(5)} + \alpha'^6\, \delta_\epsilon^{(6)} + \cdots \,, \tag{8.9}$$

according to the following pattern

$$\begin{aligned} \delta_\epsilon^{(0)} S^{(0)} &= 0 \\ \delta_\epsilon^{(0)} S^{(3)} &= \delta_\epsilon^{(3)}\, S^{(0)} \\ \delta_\epsilon^{(0)} S^{(5)} &= \delta_\epsilon^{(5)}\, S^{(0)} \\ \delta_\epsilon^{(0)} S^{(6)} + \delta_\epsilon^{(3)} S^{(3)} &= \delta_\epsilon^{(6)}\, S^{(0)} \end{aligned} \tag{8.10}$$

At the order α'^6 for the first time, the lowest order modifications to the on-shell supersymmetry transformations enter in the equations giving rise to the source term in eq. (8.4). The structure of the differential equations at higher order is expected to follow a similar pattern where coupling are given by a finite sum of modular functions $\mathcal{E}^{(i)}_s(\Omega) = \sum_{i=1}^{n_s} e^{(i)}_s(\Omega)$ with each function $e^{(i)}_s(\Omega)$ satisfying a differential equation with a source as in eq. (8.4) [73].

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Appendix A. The string S-matrix

In this appendix we collect various data from perturbative string calculations that are needed for comparison with the predictions from the $L = 1$ and $L = 2$ amplitudes of eleven dimensional supergravity on a circle and a torus.

The unitary string theory S-matrix for four-graviton scattering has the following perturbative expansion [1, 71, 72, 88]

$$S(\hat\sigma_2, \hat\sigma_3) = \kappa_{(10)}^2\, g_s^4 \left(\frac{1}{g_s^2} A^{\text{tree}}(\hat\sigma_2, \hat\sigma_3) + 2\pi A^{g=1} + + \pi\, g_s^2\, A^{g=2} + \cdots\right), \tag{A. 1}$$

where A^{tree}, $A^{g=1}$ and $A^{g=2}$ are respectively the tree-level, genus-one and genus two amplitudes for four massless states ($s+t+u=0$) described in the following section.

Appendix A.1. The tree-amplitude

The tree-level amplitude is given by

$$A^{\text{tree}} = \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)\Gamma(-\alpha' u)}{\Gamma(1+\alpha' s)\Gamma(1+\alpha' t)\Gamma(1+\alpha' u)}\, \hat R^4 \tag{A. 2}$$

where $\hat R^4$ defined in eq. (4.9).

Separating the supergravity contribution from the effect of massive string modes

$$A^{\rm tree} = \left(\frac{1}{\hat\sigma_3} + T\right)\, \hat R^4 \tag{A. 3}$$

We introduce the symmetric polynomials of the Mandelstam variables

$$\begin{aligned}\hat\sigma_n &\equiv \left(\frac{\alpha'}{4}\right)^n (s^n + t^n + u^n) \\ &= n \sum_{2p+3q=n} \frac{(p+q-1)!}{p!q!} \left(\frac{\hat\sigma_2}{2}\right)^p \left(\frac{\hat\sigma_3}{3}\right)^q ,\end{aligned} \tag{A. 4}$$

which for $n \geq 4$ are all expressible as polynomials in $\hat\sigma_2$ and $\hat\sigma_3$ because of the on-shell condition $s+t+u=0$ [71]. The α' expansion of the dynamical factor T takes the form

$$T = \sum_{p,q=0}^{\infty} T_{(p,q)} \hat\sigma_2^p \hat\sigma_3^q \,, \tag{A. 5}$$

so that [1, 71, 72]

$$\begin{aligned}T &= 2\zeta_3 + \zeta_5\, \hat\sigma_2 + \frac{2}{3}\zeta_3^2\, \hat\sigma_3 + \frac{1}{2}\zeta_7\, \hat\sigma_2^2 + \frac{2}{3}\zeta_3\zeta_5\, \hat\sigma_2\hat\sigma_3 \\ &\quad + \frac{1}{4}\zeta_9\, \hat\sigma_2^3 + \frac{2}{27}\left(2\zeta_3^3 + \zeta_9\right)\, \hat\sigma_3^2 + \cdots .\end{aligned} \tag{A. 6}$$

One remark here is that this expansion satisfies a transcendentality principle by giving a weight n to the Riemann zeta value ζ_n, the tree-level coefficient of the operator of order $\alpha'^{n+3}\, \mathcal{D}^{2n} \hat R^4$ is given by a polynomials in the odd zeta value of total weight $n+3$.

Appendix A.2. The genus-one amplitude

The genus-one amplitude is given by

$$A^{g=1} = \hat R^4 \int_{\mathcal{F}_{(1)}} \frac{d^2\tau}{\tau_2^2} \int_{\mathcal{T}_{(1)}} \prod_{i=1}^{3} \frac{d^2\nu^{(i)}}{\tau_2}\, e^{\rm D_{(1)}} \tag{A. 7}$$

where the integrations are performed over the domains

$$\begin{aligned}\mathcal{F}_{(1)} &= \{|\tau_1| \leq 1/2, |\tau|^2 \geq 1\} \\ \mathcal{T}_{(1)} &= \{-\frac{1}{2} \leq \nu_1 < \frac{1}{2},\ 0 \leq \nu_2 < \tau_2\}\end{aligned} \tag{A. 8}$$

and

$$\mathrm{D}_{(1)} = \frac{\alpha'}{2} \sum_{1\leq i<j\leq 4} k_i\cdot k_j\, \mathcal{P}_{(1)}(\nu^{(ij)}|\tau)\,, \tag{A. 9}$$

where $\mathcal{P}_{(1)}(\nu^{(ij)}|\tau)$ is the two dimensional propagator which can be written as [71]

$$\mathcal{P}_{(1)}(\nu|\tau) = \frac{1}{4\pi} \sum_{(m,n)\neq(0,0)} \frac{\tau_2}{|m\tau+n|^2}\, \exp\left[\frac{2\pi i}{\tau_2}\Im\mathrm{m}((m\tau+n)\bar\nu)\right] + C(\tau,\bar\tau)\,. \tag{A. 10}$$

The piece $C(\tau,\bar\tau)$ cancels out of the $Sl(2,\mathbb{Z})$-invariant combination in (A. 9) due to the on-shell condition $s+t+u=0$.

The low-energy expansion of $A^{g=1}$ is complicated by the presence of massless thresholds giving rise to non-analytic contributions. Separating the analytic and the non-analytic pieces as

$$A^{g=1} = (A^{g=1}_{\text{ana}} + A^{g=1}_{\text{nonana}})\,\hat R^4\,, \tag{A. 11}$$

one finds for the analytic contributions [71, 72]

$$\begin{aligned} A^{g=1}_{\text{ana}}(\hat\sigma_2,\hat\sigma_3) = \frac{2\zeta_2}{\pi}\big(1 &+ \frac{\zeta_3}{3}\hat\sigma_3 + 0\,\hat\sigma_2^2 + \frac{97}{7776}\,\zeta_5\,\hat\sigma_2\hat\sigma_3 \\ &+ \frac{1}{30}\,\zeta_3^2\hat\sigma_2^3 + \frac{61}{10801}\zeta_3^2\hat\sigma_3^2 + \cdots\big) \end{aligned} \tag{A. 12}$$

The non-analytic contributions take the following form (see [71, 72] for details)

$$\begin{aligned} A^{g=1}_{\text{nonana}}(\hat\sigma_2,\hat\sigma_3) &= \frac{\pi}{240}\,\frac{(\alpha'\,s)(s+3t)}{u}\log(-\alpha'\,s) + \frac{4\pi\zeta_3}{45}\,(\alpha'\,s/4)^4\,\log(-\frac{\alpha'\,s}{\mu_4}) \\ &+ \frac{\pi\,\zeta_5}{2520\cdot 4^6}\,(87\,(\alpha'\,s)^6 + (\alpha'\,s)^4\,(\alpha' t-\alpha' u)^2)\,\log(-\frac{\alpha'\,s}{\mu_6}) + \cdots\,. \end{aligned} \tag{A. 13}$$

The unitarity relation of the string S-matrix implies that the 2-particle s-channel discontinuity takes the form

$$\begin{aligned} &\mathrm{Disc_s}\, A^{g=1}(p_1,p_2,p_3,p_4) = -i\,\frac{\kappa^2_{(10)}}{\alpha'}\,\frac{\pi}{2}\int \frac{d^{10}k}{(2\pi)^{10}}\,\delta^{(+)}(k^2)\,\delta^{(+)}((q-k)^2) \\ &\sum_{\{h_r,h_s\}} A^{\text{tree}}(p_1,p_2,(-k)^{h_r},(k-q)^{h_s})\,A^{\text{tree}}(p_3,p_4,(k)^{-h_r},(q-k)^{-h_s})\,, \end{aligned} \tag{A. 14}$$

where the sum inside the cut is over all states within the supergraviton mulitplet and $\delta^{(+)}(p^2) = \delta^{(10)}(p^2)\theta(p^0)$ imposes the mass-shell condition on each intermediate state. The sum over all the helicities is performed easily thanks to the recycling identity for the $\hat{R}^4$ factor of eq. (4.9) derived in [62]

$$\begin{aligned}\sum_{\{h_r,h_s\}} &\hat{R}^4((p_1)^{h_1},(p_2)^{h_2},(k-q)^{h_r},(-k)^{h_s})\\ \times\ &\hat{R}^4((k)^{-h_r},(q-k)^{-h_s},(p_3)^{h_3},(p_4)^{h_4})\\ =\ &s^4\,\hat{R}^4((p_1)^{h_1},(p_2)^{h_2},(p_3)^{h_3},(p_4)^{h_4})\,.\end{aligned} \tag{A. 15}$$

The first contribution in eq. (A. 13) corresponds to the supergravity massless threshold obtained by injecting in the unitarity relation the $\hat{R}^4/\hat{\sigma}_3$ of eq. (A. 3) for each tree-level factor, the higher-order α' corrections arise by the α' corrections from the factor T in eq. (A. 13).

Giving a transcendentality weight 1 to π and $\log(x)$, the analytic contribution to the operators ${\alpha'}^{n+3}\,\mathcal{D}^{2n}\hat{R}^4$ in eq. (A. 12) have a total weight $n+1$. The coefficients are of the form of the volume of the fundamental domain [96] $\mathrm{vol}(\mathcal{F}_{(1)}) = 2\zeta_2/\pi$ defined in eq. (A. 8), times a polynomials in the odd zeta values of total weight n. The non-analytic contributions ${\alpha'}^{n+3}\,s^n\,\log(-\alpha'\,s)\,\hat{R}^4$ in eq. (A. 13) have a total weight coefficient n and are of the form $\mathrm{vol}(\mathcal{F}_{(1)})$ times a polynomial in the odd zeta values of total weight $n-1$.

At each order in the α' expansion of the genus one amplitude one brings down an extra factor of the two dimensional propagator $\mathcal{P}_{(1)}$ of eq. (A. 10) which is a modular form of weight one. The fact that the coefficients are only polynomials in odd zeta values is a consequence of the unitarity relation (A. 14) and the structure of the tree-level amplitude in eq. (A. 6).

The analytic contribution to the genus-one four-graviton amplitude in type II superstring compactified on a circle of radius $\ell_s\,r$ is given by [72]

$$\begin{aligned}I_{an}^{(d=9)}(r;s,t) &= \frac{\pi}{3}\Big[r + r^{-1} + \hat{\sigma}_2\left(\frac{\zeta(3)}{15}\,r^3 + \frac{\zeta(3)}{15}\,r^{-3}\right)\\ &+\ \hat{\sigma}_3\left(\frac{\zeta(5)}{63}\,r^5 + \frac{\zeta(3)}{3}\,r + \frac{\zeta(3)}{3}\,r^{-1} + \frac{\zeta(5)}{63}\,r^{-5}\right)\\ &+\ \hat{\sigma}_2^2\left(\frac{\zeta(7)}{315}\,r^7 + \frac{2\zeta(3)}{15}\,r\,\log(r^2\,\lambda_4) + \frac{\zeta(5)}{36}\,r^{-3} + \frac{\zeta(3)^2}{315}\,r^{-5} + \frac{\zeta(7)}{1050}\,r^{-7}\right.\\ &\left.\quad + O(r^{-3})\right) + O(e^{-r})\Big]\end{aligned} \tag{A. 16}$$

Appendix A.3. The genus two amplitude

The genus two amplitude is given by [36–39, 88]

$$A^{g=2} \;=\; \hat{R}^4 \int_{\mathcal{F}_{(2)}} \frac{d^3\Omega\wedge d^3\bar\Omega}{(\det\,\Im\mathrm{m}\Omega)^3} \int_{\mathcal{T}_{(2)}} \frac{|\mathcal{Y}_S|^2}{(\det\,\Im\mathrm{m}\Omega)^2}\, e^{\mathrm{D}_{(2)}} \qquad \text{(A. 17)}$$

with

$$\mathrm{D}_{(2)} = \frac{\alpha'}{2} \sum_{1\leq i<j\leq 4} k_i\cdot k_j\, \mathcal{P}_{(2)}(\nu^{(ij)}|\Omega) \qquad \text{(A. 18)}$$

where $\mathcal{P}_{(2)}(\nu^{(ij)}|\Omega)$ is the genus two propagator given in [36,37,88] and $\mathcal{Y}_S$ is a $(2,0)$-form given by

$$\mathcal{Y}_s = (t-u)\,\Delta(1,2)\Delta(3,4) + (s-t)\,\Delta(1,3)\Delta(4,2) + (u-s)\,\Delta(1,4)\Delta(2,3) \qquad \text{(A. 19)}$$

expressed in term of anti-symmetric combination of the Abelian differentials $\Delta(i,j) = \omega_1(z_i)\omega_2(z_j) - \omega_1(z_j)\omega_2(z_i)$. Finally Ω is the genus two period matrix, where the domains of integration $\mathcal{F}_{(2)}$ and $\mathcal{T}_{(2)}$ are defined in [36,37,88]. The leading term in the low energy expansion of this genus two amplitude is given by

$$A^{g=2} \;=\; \frac{4}{3\pi}\,\zeta_4\,\hat\sigma_2^2\hat{R}^4 \;+O(\alpha') \qquad \text{(A. 20)}$$

where we used the fact that the value of the volume of the genus two moduli space domain is given by $\mathrm{vol}(\mathcal{F}_{(2)}) = \zeta_4/(3\pi)$ [96].

Appendix A.4. Higher genus contributions

At higher genus order using the non-minimal pure spinor formalism [13, 14, 16] Berkovits showed that the leading behaviour of the low-energy expansion of the four graviton amplitude is given by F-terms of the schematic form

$$A^g \;=\; \int d^{16}\theta_L d^{16}\theta_R\, \theta_L^{12-2g}\, \theta_R^{12-2g}\, W^4_{\alpha\beta} \int_{\Sigma_g} (\cdots) + O(\alpha') \qquad \text{(A. 21)}$$

$$\propto\;\; \mathcal{D}^{2g}\,\hat{R}^4 + O(\alpha')\,. \qquad \text{(A. 22)}$$

where $W_{\alpha\beta}$ is the dimension 1 superfield introduced in eq. (2.16) appearing in the graviton vertex operators, and the ellipsis $(\cdots)$ is for the contributions from the other fields but do not contain any dependence on the superspace fermionic coordinates θ_L and θ_R integrated over the genus g Riemann surface.

This expression is valid up to genus 6, and at genus 6 the zero mode factor gives the D-term contribution discussed in section 2.2.

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