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A New Class of Cellular Automata with a Discontinuous Glass Transition

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Abstract We introduce a new class of two-dimensional cellular automata with a bootstrap percolation-like dynamics. Each site can be either empty or occupied by a single particle and the dynamics follows a deterministic updating rule at discrete times which allows only emptying sites. We prove that the threshold density ρ_c for convergence to a completely empty configuration is non trivial, $0 < \rho_c < 1$, contrary to standard bootstrap percolation. Furthermore we prove that in the subcritical regime, $\rho < \rho_c$, emptying always occurs exponentially fast and that ρ_c coincides with the critical density for two-dimensional oriented site percolation on $\mathbb{Z}^2$. This is known to occur also for some cellular automata with oriented rules for which the transition is continuous in the value of the asymptotic density and the crossover length determining finite size effects diverges as a power law when the critical density is approached from below. Instead for our model we prove that the transition is *discontinuous* and at the same time the crossover length diverges *faster than any power law*. The proofs of the discontinuity and the lower bound on the crossover length use a conjecture on the critical behaviour for oriented percolation. The latter is supported by several numerical simulations and by analytical (though non rigorous) works through renormalization techniques. Finally, we will discuss why, due to the peculiar *mixed critical/first order character* of this transition, the model is particularly relevant to study glassy and jamming transitions. Indeed, we will show that it leads to a dynamical glass transition for a Kinetically Constrained Spin Model. Most of the results that we present are the rigorous proofs of physical arguments developed in a joint work with D.S. Fisher.

Keywords Bootstrap percolation · Glass transition · Cellular automata · Finite size scaling

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1 Introduction

We introduce a new class of two-dimensional cellular automata, i.e. systems of particles on $\mathbb{Z}^2$ with the constraint that on each site there is at most one particle at a given time. A configuration at time t is therefore defined by giving for each $x \in \mathbb{Z}^2$ the occupation variable $\eta_t(x) \in \{0, 1\}$ representing an empty or occupied site, respectively. At time $t = 0$ sites are independently occupied with probability ρ and empty with probability $1 - \rho$. Dynamics is given by a deterministic updating rule at discrete times with the following properties: it allows only emptying sites; it is local in time and space, namely η_{t+1} is completely determined by η_t and $\eta_{t+1}(x)$ depends only on the value of η_t on a finite set of sites around x.

We will be primarily interested in the configuration which is reached in the infinite time limit. We will first identify the value of the critical density ρ_c, namely the supremum over the initial densities which lead almost surely to an empty configuration. In particular we will prove that $\rho_c = p_c^{OP}$, where p_c^{OP} is the critical probability for oriented site percolation on $\mathbb{Z}^2$. Furthermore, we will analyze the speed at which the system is emptied in the subcritical regime and prove that emptying always occurs exponentially fast for $\rho < p_c^{OP}$. Then, we will determine upper and lower bounds for the crossover length below which finite size effects are relevant when $\rho \nearrow \rho_c$. These bounds establish that the crossover length diverges as the critical density is approached from below and divergence is faster than power law. Finally, we will analyze the behaviour around criticality of the final density of occupied sites, ρ_∞. We will prove that the transition is discontinuous: $\rho_\infty(\rho)$ is zero if $\rho < \rho_c$ and $\rho_\infty(\rho_c) > 0$. We underline that both discontinuity and the lower bound on the crossover length are proved modulo a conjecture (Conjecture 3.1) for the critical behaviour of oriented site percolation (actually, for the proof of discontinuity we will only need a milder version of Conjecture 3.1 which is stated as Conjecture 3.2). This conjecture states a property which is due to the anisotropic character of oriented percolation and it is widely accepted in physical literature, where it has been verified both by analytical works through renormalization techniques and numerical simulations. However, we are not aware of a rigorous mathematical proof.

One of the main interests of this new model relies on the peculiar feature of its transition: there is a diverging lengthscale as for standard continuous critical transitions, but at the same time the density of the final cluster $\rho_\infty(\rho)$, that plays the role of the order parameter, is discontinuous. This discontinuous/critical character, to our knowledge, has never been found so far in any cellular automata or in other type of phase transitions for short range finite dimensional lattices.[1]

Among the most studied cellular automata we recall bootstrap percolation [1] and oriented cellular automata [14]. In bootstrap percolation the updating rule is defined as follows[2]: a site can be emptied only if the number of its occupied nearest neighbours is smaller than a threshold, l. In this case, it has been proved [1, 14] that on $\mathbb{Z}^d$, the critical density is either 1 or 0 depending on l: $\rho_c = 0$ for $l < d$, $\rho_c = 1$ for $l \geq d$. On the other hand, oriented cellular automata on $\mathbb{Z}^d$ are defined as follows: site x can be emptied only if $(x + e_1, \ldots, x + e_d)$ are all empty, where e_i are the coordinate unit vectors. In this case it has been proven [14] that the critical density coincides with the critical probability for oriented site percolation and the transition is continuous, namely $\rho_\infty(\rho_c) = 0$.

[1] On the other hand such type of transition is found in some problems on non-finite dimensional lattices, e.g. the k-core problem [13] or bootstrap percolation on random graphs [7]. It has also been established for long-range systems in one dimension [2, 3].

[2] Note that the model is usually defined in this way in physical literature, while in mathematical literature the role of empty and filled sites is exchanged: dynamics allows only filling sites and a site can be filled only if the number of its neighbours is greater than l. The same is true for oriented models defined below.

Our model shows a behaviour that is different from both bootstrap and oriented cellular automata, since the transition occurs at a finite density and it is discontinuous. Models with such a critical/first order transition have long been quested in physical literature since they are considered to be relevant for the study of the liquid/glass and more general jamming transitions. In the last section we will discuss the behavior of a Kinetically Constrained Spin Model, the so called Spiral Model (SM) [16], which has a stochastic evolution with dynamical rules related to those of our cellular automata. We will show that the present results for the cellular automata imply that SM has a dynamical transition with the basic properties expected for glass and jamming transitions. For a more detailed discussion of the physical problem we refer to our joint work with D.S. Fisher [15, 16]. Most of the results that we present are the rigorous proofs of physical arguments developed in [15, 16] for several jamming percolation models. The cellular automaton we consider in this paper (and the related Spiral Model [16]) is one of the simplest in this class. Originally, in [15], we focused on the so called Knight models for which some of our physical arguments cannot be turned into rigorous ones as pointed out in [17]. In particular our original conjecture [15] that the transition for Knights occurs at p_c^{OP} should not be correct [16, 17]. However, as discussed in [16], numerical simulations suggest that the physical behavior of the Knight models around its transition (which is located before p_c^{OP}) is the same of SM.

2 Setting and Notation

2.1 The Model

The model is defined on the 2-dimensional square lattice, $\mathbb{Z}^2$. We denote by e_1 and e_2 the coordinate unit vectors, by x, y, z the sites of $\mathbb{Z}^2$ and by $|x-y|$ the Euclidean distance between x and y. The configuration space is $\Omega = \{0,1\}^{\mathbb{Z}^2}$, i.e. any configuration $\eta \in \Omega$ is a collection $\{\eta(x)\}_{x\in\mathbb{Z}^2}$, with $\eta(x) \in (0,1)$, where 0 and 1 represent an empty or occupied site, respectively. At time $t=0$ the system is started from a configuration $\eta_0 \in \Omega$ chosen at random according to Bernoulli product measure μ^ρ, namely $\eta_0(x)$ are i.i.d. variables and $\mu^\rho(\eta_0(x)=1)=\rho$. Therefore ρ will be called the initial density. The evolution is then given by a deterministic process at discrete time steps $t = 0, 1, 2, \dots$ and the configuration at time t, η_t, is completely determined by the configuration at time $t-1$ according to the updating rule

$$\eta_t = T\eta_{t-1}, \tag{2.1}$$

with the evolution operator $T : \Omega \to \Omega$ defined as

$$T\eta(x) := \begin{cases} 0 & \text{if } \eta(x) = 0, \\ 0 & \text{if } \eta(x) = 1 \text{ and } \eta \in \mathcal{A}_x, \\ 1 & \text{if } \eta(x) = 1 \text{ and } \eta \notin \mathcal{A}_x, \end{cases} \tag{2.2}$$

with

$$\mathcal{A}_x := (\mathcal{E}_x^{NE-SW} \cap \mathcal{E}_x^{NW-SE}), \tag{2.3}$$

where

$$\mathcal{E}_x^{NE-SW} = \mathcal{V}_{NE_x} \cup \mathcal{V}_{SW_x},$$
$$\mathcal{E}_x^{NW-SE} = \mathcal{V}_{NW_x} \cup \mathcal{V}_{SE_x},$$

and, for any $A \subset \mathbb{Z}^2$, we denote by $\mathcal{V}_A$ the event that all sites in A are empty, $\mathcal{V}_A : (\eta : \eta(x) = 0 \ \forall x \in A)$, and the sets NE_x, SW_x, NW_x and SE_x are defined as

$$NE_x := (x + e_2, x + e_1 + e_2),$$

$$SW_x := (x - e_2, x - e_1 - e_2),$$

$$NW_x := (x - e_1, x - e_1 + e_2),$$

$$SE_x := (x + e_1, x + e_1 - e_2).$$

In words the updating rule defined by (2.1) and (2.2) can be described as follows. Let the *North-East* (NE_x), *South-West* (SW_x), *North-West* (NW_x) and *South-East* (SE_x) neighbours of x be the couples depicted in Fig. 1. If x is empty at time $t - 1$, it will be also empty at time t (and at any subsequent time). Otherwise, if x is occupied at time $t - 1$, it will be empty at time t if and only if at time $t - 1$ the following local constraint is satisfied: both its North-East *or* both its South-West neighbours are empty *and* both its North-West *or* both its South-East neighbours are empty too. See Fig. 1a (Fig. 1b) for an example in which the constraint is (is not) satisfied. As it will become clear in the proofs of Theorems 3.4 and 3.5, the fact that in order to empty x we necessarily have to satisfy a requirement in the NE–SW *and* an (independent) requirement in the NW–SE direction is the key ingredient which makes the behaviour of this model quantitatively different from the oriented cellular automata in [14].

One can also give the following alternative equivalent definition of the dynamics. Let $\mathcal{I}_x$ be the collection of the four subsets of $\mathbb{Z}^2$ each containing two adjacent couples of the above defined neighbours of x, namely

$$\mathcal{I}_x := \{NE_x \cup SE_x;\ SE_x \cup SW_x;\ SW_x \cup NW_x;\ NW_x \cup NE_x\}.$$

With this notation it is immediate to verify that definition (2.3) is equivalent to requiring that at least one of the sets $A \in \mathcal{I}_x$ is completely empty, namely

$$\mathcal{A}_x := \bigcup_{A \in \mathcal{I}_x} \mathcal{V}_A.$$

The following properties can be readily verified. The dynamics is attractive with respect to the partial order $\eta^1 \prec \eta^2$ if $\eta^1(x) \le \eta^2(x) \ \forall\, x \in \mathbb{Z}^2$. Attractiveness here means that if we start the process from two different configurations η_0^1 and η_0^2 with $\eta_0^1 \le \eta_0^2$ at each subsequent time the partial order will be preserved. The updating rule is short-range, indeed $\mathcal{A}_x$ depends only on the value of η on the first and second neighbours of x. Furthermore it is both invariant under translations and under rotations of 90 degrees (and multiples). Indeed, if for all $y \in \mathbb{Z}^2$ we define the translation operator $\tau_y : \Omega \to \Omega$ as $(\tau_y \eta)_z = \eta_{y+z}$, it is immediate to verify that $\tau_y \eta \in (\notin) \mathcal{A}_{x+y}$ if and only if $\eta \in (\notin) \mathcal{A}_x$. On the other hand, if we let $f_{-90} : \mathbb{Z}^2 \to \mathbb{Z}^2$ be the operator which acts as $f_{-90}(x_1 e_1 + x_2 e_2) := -x_2 e_1 + x_1 e_2$ and we define the rotation operator $\mathcal{R}_{90} : \Omega \to \Omega$ which acts as $(\mathcal{R}_{90}\eta)(x) = \eta(f_{-90}(x))$, it is immediate to verify that $\mathcal{R}_{90}\eta \in (\notin) \mathcal{A}_{f_{-90}(x)}$ if and only if $\eta \in (\notin) \mathcal{A}_x$.

For future purposes it is also useful to define the model on a finite volume $\Lambda \subset \mathbb{Z}^2$, i.e. to define an evolution operator $T_\Lambda : \Omega_\Lambda \to \Omega_\Lambda$, where Ω_Λ is the configuration space

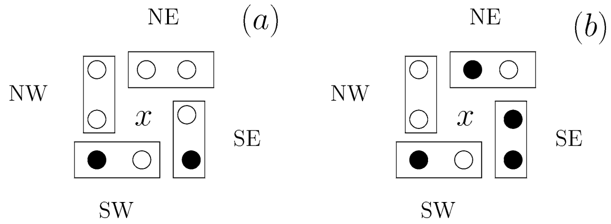

Fig. 1 Site x and the four couples of its North-East (NE), South-East (SE), North-West (NW) and South-West (SW) neighbours. **a** If x is occupied it will be empty at next time step. Indeed its NE and NW neighbours are all empty, thus $\eta \in \mathcal{V}_{NE} \subset \mathcal{E}^{NE-SW}$ and $\eta \in \mathcal{V}_{NW} \subset \mathcal{E}^{NW-SE}$. Therefore $\eta \in \mathcal{A}_x$. **b** If x is occupied it will remains occupied at next time step. Indeed neither the NE nor the SW neighbours are completely empty, thus $\eta \notin \mathcal{E}^{NE-SW}$ and therefore $\eta \notin \mathcal{A}_x$

$\Omega_\Lambda := \{0,1\}^\Lambda$. A natural way to do this is to fix a configuration $\omega \in \Omega_{\mathbb{Z}^2\setminus\Lambda}$ and to consider the evolution operator $T_{\Lambda,\omega}$ with fixed boundary condition ω, i.e. for each $x \in \Lambda$ we let

$$T_{\Lambda,\omega}\eta(x) := T(\eta\cdot\omega)(x), \tag{2.4}$$

where $\eta\cdot\omega \in \Omega$ is the configuration which equals η inside Λ and ω outside. Note that $T_{\Lambda,\omega}$ depends only on the value of ω on the sites $y \in \mathbb{Z}\setminus\Lambda$ such that $y \in (NE_x \cup SE_x \cup SE_x \cup NW_x)$ for at least one $x \in \Lambda$. Note also that the configuration reached under $T_{\omega,\Lambda}$ after $|\Lambda|$ steps is stationary, namely

$$T_{\omega,\Lambda}^{|\Lambda|}\eta = T_{\omega,\Lambda}^{|\Lambda|+n}\eta, \tag{2.5}$$

for any $n \geq 0$ (this trivially follows from the fact that we are evolving deterministically on a finite region and that only emptying of sites is allowed). A choice which we will often consider is the case of filled boundary conditions, namely $\omega(x) = 1$ for all $x \in \mathbb{Z}\setminus\Lambda$, and we will denote by T_Λ^f the corresponding evolution operator.

2.2 Main Issues

Before presenting our results, let us informally introduce the main issues that we will address. We underline once more that these are akin to those examined in previous works for bootstrap percolation and oriented models [6, 14]. However, the answers will be qualitatively different.

- We will determine the critical density ρ_c such that, a.s. with respect to the initial distribution μ^ρ, if $\rho < \rho_c$ all the lattice gets eventually emptied under the updating rule, while for $\rho > \rho_c$ this does not occur. The precise definition of ρ_c follows. Consider on $\{0,1\}$ the discrete topology and on Ω the Borel σ-algebra Σ. Let $\mathcal{M}$ be the set of measures on (Ω,Σ) and μ_t^ρ be the evoluted of the initial distribution $\mu_0^\rho = \mu^\rho$ according to the above deterministic rules. Due to attractiveness it is immediate to conclude that μ_t^ρ converges weakly to a probability distribution $\mu_\infty^\rho \in \mathcal{M}$. Following [14] we can indeed define a partial order among $\mu,\nu \in \mathcal{M}$ as $\mu \leq \nu$ if $\int f(\eta)\,d\mu(\eta) \leq \int f(\eta)d\nu(\eta)$, $\forall f:\Omega\to\mathbb{R}$ and f increasing. The fact that 0's are stable implies that $\mu_0^\rho \geq \mu_1^\rho \geq \cdots$. This, together with the compactness of Ω and $\mathcal{M}$, assures the weak convergence of μ_t^ρ to a probability distribution $\mu_\infty^\rho \in \mathcal{M}$. We can therefore define the critical density ρ_c as

$$\rho_c := \sup(\rho : \rho_\infty(\rho) = 0), \tag{2.6}$$

where $\rho_\infty(\rho)$, henceforth referred to as the *final density*, is defined as

$$\rho_\infty(\rho) := \mu_\infty^\rho(\eta(0)). \tag{2.7}$$

- We will analyze the speed at which the system is emptied in the subcritical regime, $\rho < \rho_c$. Let t_E be the first time at which the origin gets emptied

$$t_E := \inf(t \geq 0 : \eta_t(0) = 0). \tag{2.8}$$

Following notation in [14] we let $P_\rho(\cdot)$ be the probability measure on $(\{0,1\}^{\mathbb{Z}^2,\{0,1,2,\ldots\}}, \Sigma^1)$, where Σ^1 is the Borel σ-algebra on $\{0,1\}^{\mathbb{Z}^2,\{0,1,2,\ldots\}}$. With this notation we define the speed γ as

$$\gamma(\rho) := \sup(\beta \geq 0 : \exists C < \infty \quad \text{s.t.} \quad P_\rho(t_E > t) \leq Ce^{-\beta t}) \tag{2.9}$$

and the corresponding critical density as

$$\tilde{\rho}_c = \sup(\rho : \gamma(\rho) > 0). \tag{2.10}$$

It is immediate from above definitions to check that $\tilde{\rho}_c \leq \rho_c$. We will prove that the equality is verified, namely emptying always occurs exponentially fast in the subcritical regime $\rho < \rho_c$.

- We will analyze the final density at criticality and establish that the transition is discontinuous ($\rho_\infty(\rho_c) > 0$).
- We will analyze the finite size scaling. Let $\Lambda_{2L} \subset \mathbb{Z}^2$ and $\Lambda_{L/2} \subset \Lambda_L$ be two square regions centered around the origin and of linear size $2L$ and $L/2$, respectively. We denote by $\eta^s(\eta)$ the stationary configuration which is reached after $(2L)^2$ steps when we evolve from η with filled boundary conditions on Λ_{2L}, $\eta^s(\eta) := (T^f_{\Lambda_{2L}})^{4L^2}\eta$. Finally, we let $E(L,\rho)$ be the probability that $\Lambda_{L/2}$ is empty in η^s

$$E(L,\rho) = \mu^\rho(\eta^s(x) = 0, \quad \forall x \in \Lambda_{L/2}). \tag{2.11}$$

As we shall show $\lim_{L\to\infty} E(L,\rho) = 1$ for $\rho < \rho_c$ and $\lim_{\rho \nearrow \rho_c} E(L,\rho) \neq 1$ when L is kept fixed to any finite value. We will therefore study the scaling as $\rho \nearrow \rho_c$ of the crossover length $\Xi(\rho)$ defined as

$$\Xi(\rho) := \inf(L : E(L,\rho) \geq 1/2). \tag{2.12}$$

Another possible definition of the crossover length would have corresponded to defining $E(L,\rho)$ as the probability that the origin is empty in η^s. This requirement, less stringent than the previous one, leads to the same results for $\Xi(\rho)$ at leading order, see Sect. 6.

3 Results

Let us first recall some definitions and results for oriented site percolation on $\mathbb{Z}^2$ which we will be used in the following. We say that $(x_1, x_2, \ldots, x_n)$ is an *oriented path* in $\mathbb{Z}^2$ if $x_{i+1} - x_i \in (e_1, e_2)$ and, given a configuration $\eta \in \{0,1\}^{\mathbb{Z}^2}$, we let $x \to y$ if there exists an oriented path connecting x and y (i.e. with $x_1 = x$ and $x_n = y$) such that all its sites are occupied ($\eta(x_i) = 1$ for all $i = 1, \dots, n$). For a given $x \in \mathbb{Z}^2$, we define its *occupied oriented cluster* to be the random set

$$\mathcal{C}_x^{OP}(\eta) := (y \in \mathbb{Z}^2 : x \to y). \tag{3.13}$$

Finally, for each n, we define the random set Γ_x^n as

$$\Gamma_x^n := (y \in \mathbb{Z}^2 : x \to y \text{ and } \exists m \text{ s.t. } y - x = me_1 + ne_2). \tag{3.14}$$

With the above notation the percolation probability $\alpha(p)^{OP}$ is defined as $\alpha(p)^{OP} := \mu^p(\eta : |\mathcal{C}_0^{OP}| = \infty)$. As it has been proven, see [8], $\alpha(p)^{OP}$ is zero at small p and strictly positive at high enough p: the system undergoes a phase transition. The critical density, defined as $p_c^{OP} := \inf(p : \alpha(p)^{OP} > 0)$, has been proven to be non trivial, $0 < p_c^{OP} < 1$, see [8] for some upper and lower bounds. We also recall that extensive numerical simulations lead to $p_c^{OP} \sim 0.705489(4)$ [11]. Furthermore the transition is continuous in the percolation probability, namely $\alpha(p_c^{OP}) = 0$ [9] and in the subcritical regime an exponential bound has been proven [8]: at any $\rho < p_c^{OP}$ there exists $\xi_{OP}(\rho) < \infty$ such that for $n \to \infty$ the following holds

$$\mu^\rho(\Gamma_x^n \neq \emptyset) \leq e^{-n/\xi_{OP}}. \tag{3.15}$$

Finally, we recall the conjecture for the critical behavior. Let $\Lambda_{a,b}$ be a rectangular region with two sides (∂R_1 and ∂R_2) of length a parallel to $e_1 + e_2$, and two sides (∂R_3 and ∂R_4) of length b parallel to $-e_1 + e_2$. Let also $\mu_{N,z}^\rho$ be the Bernoulli measure on Λ_{N,N^z} conditioned by having both sides of length N, $\partial\mathcal{R}_1$ and $\partial\mathcal{R}_2$, completely empty. The following properties are expected to hold for the probability of finding an occupied oriented path crossing the rectangle in the direction parallel to $e_1 + e_2$, i.e. connecting the two non empty borders ∂R_3 and ∂R_4:

Conjecture 3.1 *There exists z, c_{OP}^u, c_{OP}^l and $\xi(\rho)$ with $0 < z < 1$, $0 < c_{OP}^u < 1$, $0 < c_{OP}^l < 1$, $\xi(\rho) < \infty$ for $\rho < p_c^{OP}$ and $\lim_{\rho \nearrow p_c^{OP}} \xi = \infty$ s.t.*

$$\lim_{L\to\infty} \mu_{L,z}^{p_c^{OP}}(\exists x \in \partial\mathcal{R}_3 \textit{ and } y \in \partial\mathcal{R}_4 \textit{ s.t. } x \to y) = c_{OP}^u, \tag{3.16}$$

$$\lim_{\rho \nearrow p_c^{OP}} \mu_{\xi,z}^{\rho}(\exists x \in \partial\mathcal{R}_3 \textit{ and } y \in \partial\mathcal{R}_4 \textit{ s.t. } x \to y) = c_{OP}^l. \tag{3.17}$$

The above conjecture is given for granted in physical literature, where it has been verified both by numerical simulations and by analytical works based on renormalization techniques (see [11] for a review). The physical arguments supporting this conjecture are based on finite size scaling and the anisotropy of oriented percolation that gives rise to two different correlation lengths in the parallel and perpendicular direction w.r.t. the orientation of the lattice, namely in the $e_1 + e_2$ and $-e_1 + e_2$ directions. This explains why in finite size effects an anisotropy critical exponent z emerges such that at $\rho = p_c^{OP}$ the probability of finding a spanning cluster on a system of finite size $L \times L^z$ converges to a constant which is bounded away from zero and one when $L \to \infty$. To our knowledge, a rigorous proof of Conjecture 3.1 has not yet been provided. However, in [8] it has been proven that the opening edge of the percolating cluster is zero at criticality, which implies anisotropy of the percolating clusters. Our results, unless where explicitly stated (Theorems 3.4 and 3.5(ii)), *do not* rely on the above conjecture.

Finally, we recall that the minimal ξ_{OP} and ξ for which (3.15) and (3.16) hold, namely the parallel correlation length, is expected to diverges when $p \nearrow p_c^{OP}$ as $(\rho - p_c^{OP})^\alpha$ with $\alpha \simeq 1.73$. The lowest value of z for which the result of Conjecture 3.1 is expected to hold, i.e. the anisotropy critical exponent, is $z \simeq 0.63$.

Before stating our results, let us give a milder version of Conjecture 3.1 which will be sufficient to prove discontinuity of the transition (the stronger version 3.1 will be used only to prove the upper bound for the correlation length).

Fix $\ell_0 > 0$ and consider the sequence of increasing rectangles $\mathcal{R}_i := \Lambda_{\ell_i, 1/12\ell_i}$ with $\ell_i = 2\ell_{i-1}$ and denote the two short sides parallel to the $-e_1 + e_2$ direction by ∂R_3^i and ∂R_4^i. Let S_i be the event that $\mathcal{R}_i$ is crossed in the parallel direction, namely

$$S_i := (\eta : \exists\, x \in \partial\mathcal{R}_3^i \text{ and } y \in \partial\mathcal{R}_4^i \text{ s.t. } x \to y). \tag{3.18}$$

Conjecture 3.2 $\sum_{i=1}^{\infty} |\log(\mu^\rho(S_i))| < \infty$ *at* $\rho = p_c^{OP}$.

The fact that Conjecture 3.1 implies Conjecture 3.2 follows immediately by cutting $\mathcal{R}_i$ into $O(\ell_i^{1-z})$ slices of size $\ell_i \times \ell_i^z$ and using (3.16) to bound the probability that each slice is not spanned by a cluster.

We are now ready to state our results. We have proved that the critical densities defined in (2.6) and (2.10) are equal and furthermore they coincide with the critical probability for oriented site percolation (and therefore also of oriented cellular automata [14]) on $\mathbb{Z}^2$, namely

Theorem 3.3 $\rho_c = \tilde{\rho}_c = p_c^{OP}$.

However the critical properties are different from oriented percolation: the transition is here discontinuous in the final density and the crossover length diverges faster than any power law at criticality. More precisely

Theorem 3.4 *If Conjecture* 3.2 *holds*, $\rho_\infty(\rho_c) > 0$.

Theorem 3.5 (i) $\lim_{\rho \nearrow \rho_c} \xi_{OP}(\rho)^{-2-\epsilon} \log \Xi(\rho) = 0$ *for any* $\epsilon > 0$.

(ii) *If Conjecture* 3.1 *holds*, $\Xi(\rho) \geq c_1 \xi \exp[c_2\, \xi(\rho)^{1-z}]$ *with* $c_1 = 1/(2\sqrt{2})$ *and* $c_2 = |\log(1 - c_{OP}^l)|/2$,

where $\xi_{OP}(\rho)$ is the smallest constant for which (3.15) holds and z, ξ are the smallest constant which satisfy (3.17). Note that, if the conjectured power law behavior for ξ and ξ_{OP} holds then our bounds imply a faster than power law divergence for Ξ. This property, as well as the discontinuity of the final density, makes the character of this transition completely different from the one of oriented percolation and oriented cellular automata.

4 Critical Density: Proof of Theorem 3.3

Proof of Theorem 3.3 The proof follows from the inequality $\tilde{\rho}_c \leq \rho_c$ (which can be readily verified from definitions (2.6) and (2.10)) and the following Lemmas 4.1 and 4.2. □

Lemma 4.1 $\rho_c \le p_c^{OP}$.

Lemma 4.2 $\gamma(\rho) > 0$ *for* $\rho < p_c^{OP}$. *Therefore* $\tilde{\rho}_c \ge p_c^{OP}$.

4.1 Upper Bound for ρ_c: Proof of Lemma 4.1

In order to establish an upper bound for ρ_c we first identify a set of configurations in which the origin is occupied and it can be never emptied at any finite time because it belongs to a proper infinite cluster of occupied sites. In this case we will say that the origin is *frozen*. Then we prove that a cluster which makes the origin frozen exists with finite probability under the initial distribution $\mu_0^\rho = \mu^\rho$ for $\rho > p_c^{OP}$. This follows from the fact that the origin can be frozen via two infinite independent clusters which, under a proper geometrical transformation, can be put into a one to one correspondence with infinite occupied clusters of oriented percolation. We stress that these clusters, which are sufficient to prove the desired upper bound for the critical density, *are not* the only possible clusters which can freeze the origin, as will become clear in the proof of Theorem 3.4.

Let us start by introducing some additional notation. We say that $(x_1, x_2, \dots, x_n)$ is a *North-East* (*NE*) path in $\mathbb{Z}^2$ if $x_{i+1} \in NE_{x_i}$ for all $i \in (1, \dots, n-1)$, and we let $x \xrightarrow{NE} y$ if there is a NE path of sites connecting x and y (i.e. with $x_1 = x$ and $x_n = y$) such that each site in the path is occupied ($\eta(x_i) = 1$ for all $i = 1, \dots, n$). Also, we define the *North-East occupied cluster* of site x to be the random set

$$\mathcal{C}_x^{NE} := (y \in \mathbb{Z}^2 : x \xrightarrow{NE} y)$$

and, for each n, we also define the random set $\Gamma_x^{NE,n}$ as

$$\Gamma_x^{NE,n} := (y \in \mathbb{Z}^2 : x \xrightarrow{NE} y \text{ and } \exists m \text{ s.t. } y - x = me_1 + ne_2). \tag{4.19}$$

We make analogous definitions for the *South-West*, *North-West* and *South-East paths*, the correspondent occupied clusters $\mathcal{C}_x^{SW}$, $\mathcal{C}_x^{NW}$ and $\mathcal{C}_x^{SE}$ and for the random sets $\Gamma_x^{NW,n}$, $\Gamma_x^{SW,n}$ and $\Gamma_x^{SE,n}$. Note that if x is empty its occupied cluster in all the directions is empty, instead if x is occupied each occupied cluster contains at least x. In Fig. 2 we depict as an example the North-East (inside the dashed line) and South-West (inside the continuous line) occupied clusters of a given occupied site x. With this notation we define

$$\mathcal{F}_x^{NE-SW} := (\eta : |\mathcal{C}_x^{NE}| = \infty \text{ and } |\mathcal{C}_x^{SW}| = \infty) \tag{4.20}$$

and it is immediate to verify that the origin is frozen on any configuration $\eta \in \mathcal{F}_x^{NE-SW}$, namely

Lemma 4.3 $\rho_\infty(\rho) \ge \mu^\rho(\mathcal{F}_0^{NE-SW})$.

Proof The result follows immediately once we prove that, for any given $\tau > 0$, the inequality $t_E > \tau$ holds, where t_E is the first time at which the origin gets emptied, see Definition 2.8. Since for hypothesis $|\mathcal{C}_0^{NE}| = \infty$ and $|\mathcal{C}_0^{SW}| = \infty$, there exist y and w such that $y \in \mathcal{C}_0^{NE}$, $w \in \mathcal{C}_0^{SW}$, the length L_y of the minimal NE occupied path $x_1 = 0, \dots, x_{L_y} = y$ connecting 0 to y verifies $L_y > \tau + 1$ and the length L_w of the minimal SW occupied path $\tilde{x}_1 = 0, \dots, \tilde{x}_{L_w} = w$ connecting 0 to w verifies $L_w > \tau + 1$. Let $\bar{x}_2$ be the

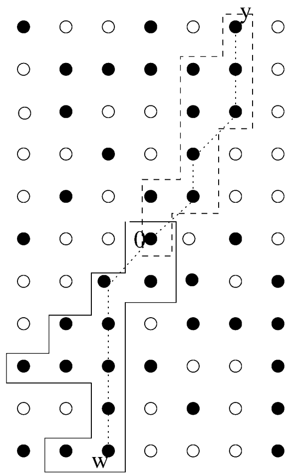

Fig. 2 Sites inside the continuous (*dashed*) *line* form the South-West (North-East) occupied cluster for the origin, 0. It is immediate to check that in order to empty 0 we have (at least) to destroy either the NE occupied path connecting 0 to y or the SW occupied path connecting 0 to w (sites indicated by the *dotted line*). This requires a number of steps which is at least the minimum of the lengths of these two paths (each path have to be emptied sequentially from its external border unless the other one has been already emptied)

site which is emptied first among x_2 and $\tilde{x}_2$. It is immediate to verify (see Fig. 2) that $t_E > \inf(t \geq 0 : \eta_t(\bar{x}_2) = 0) \geq \min(L_w, L_y) - 1 > \tau$, which concludes the proof. □

By definition of NE and SW neighbours, it is easy to verify that (except for x itself) there do not exist sites that can be connected to x both by a NE and a SW path, thus it follows immediately that

$$\mu^\rho(\mathcal{F}_0^{NE-SW}) \geq \mu^\rho(|\mathcal{C}_0^{NE}| = \infty)\mu^\rho(|\mathcal{C}_0^{SW}| = \infty). \tag{4.21}$$

We will now prove that the probabilities of such infinite NE or SW occupied clusters can be rewritten in terms of the probability of infinite clusters for oriented percolation, namely

Lemma 4.4 $\mu^\rho(|\mathcal{C}_0^{NE}| = \infty) = \mu^\rho(|\mathcal{C}_0^{SW}| = \infty) = \mu(|\mathcal{C}_0^{OP}| = \infty)$.

Proof Let $v_1 = e_1 + e_2$ and $v_2 = e_2$, each $x \in \mathbb{Z}^2$ can be written in a unique way as $x = mv_1 + nv_2$. We can therefore define the operator $R^{NE} : \Omega \to \Omega$ which acts as $(R^{NE}\eta)(me_1 + ne_2) = \eta(mv_1 + nv_2)$. It is immediate to verify that $\mu^\rho(\eta) = \mu^\rho(R^{NE}\eta)$ and that $|\mathcal{C}_0^{NE}(\eta)| = |\mathcal{C}_0^{OP}(R^{NE}\eta)|$ for any η. Thus $\mu^\rho(|\mathcal{C}_0^{NE}| = \infty) = \mu(|\mathcal{C}_0^{OP}(\eta)| = \infty)$. The result $\mu^\rho(|\mathcal{C}_0^{SW}| = \infty) = \mu(|\mathcal{C}_0^{OP}| = \infty)$ can be proved analogously. □

We are now ready to conclude the proof of the upper bound for ρ_c.

Proof of Lemma 4.1 The result follows from Lemma 4.3, (4.21), Lemma 4.4 and the definition of p_c^{OP} which implies $\mu(|\mathcal{C}_0^{OP}(\eta)| = \infty) > 0$ for $\rho > p_c^{OP}$. □

4.2 Lower Bound for $\tilde{\rho}_c$: Proof of Lemma 4.2

The central result of this section is Lemma 4.7. This contains a lower bound for the probability that a certain finite region can be emptied (except for some special sets at its corners) when evolution occurs with fixed filled boundary conditions and $\rho < p_c^{OP}$. Since this lower bound can be made arbitrarily near to one provided the size of the region is taken sufficiently large, the result $\rho_c \geq p_c^{OP}$ will easily follow (Corollary 4.8). Some additional work involving

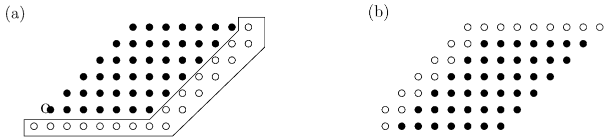

Fig. 3 **a** The region $\mathcal{R}_{a,b}$ (here $a=7$, $b=6$) and (inside the *continuous line*) the boundary region $\partial\mathcal{R}_{a,b}$ which guarantees that we can empty $\mathcal{R}_{a,b}$ (Proposition 4.5). **b** The alternative choice of boundary conditions which is described in Remark 4.6

a renormalization technique in the same spirit of the one used for bootstrap percolation in [14] will be used to prove the stronger result $\tilde{\rho}_c \geq p_c^{OP}$ (Lemma 4.2).

Let S_a be a segment of length a with left vertex in the origin,

$$S_a := \bigcup_{x=0}^{a-1} i e_1$$

and $\mathcal{R}_{a,b}$ be the quadrangular region with two sides parallel to the e_1 direction and two sides parallel to the e_1+e_2 direction which is obtained by shifting b times S_a of e_1+e_2 depicted in Fig. 3, namely

$$\mathcal{R}_{a,b} := \bigcup_{i=0}^{b-1} [S_a + i(e_1+e_2)],$$

where for each $x \in \mathbb{Z}^2$ and $A \subset \mathbb{Z}^2$ we let $x+A \subset \mathbb{Z}^2$ be $x+A := (y : y = x+z \text{ with } z \in A)$. As it is immediate to verify, if we impose empty boundary conditions on the first external segment parallel to the bottom border and on the first two segments parallel to the right border (empty sites inside the continuous line in Fig. 3a), $\mathcal{R}_{a,b}$ is completely emptied in (at most) $|\mathcal{R}_{a,b}| = ab$ steps. More precisely, if we define the bottom right border as

$$\partial\mathcal{R}_{a,b} := (S_a - e_2 - e_1) \cup [ae_1 + (b-1)(e_1+e_2)] \bigcup_{i=1}^{b} [S_2 + ae_1 + (i-2)(e_1+e_2)]$$

and we recall that, for each $A \subset \mathbb{Z}^2$, $\mathcal{V}_A$ is the set of configurations which are empty on all sites in A, the following holds

Proposition 4.5 *If $\eta \in \mathcal{V}_{\partial\mathcal{R}_{a,b}}$, then $T^{ab}\eta \in \mathcal{V}_{\mathcal{R}_{a,b}}$.*

Proof Starting from the bottom right corner of $\mathcal{R}_{a,b}$ we can erase all particles in S_a from right to left, thanks to the fact that their *SE* and *SW* neighbours are empty. Then we can erase all particles in $S_a + e_2 + e_1$ starting again from the rightmost site and so on until emptying the whole region. □

Remark 4.6 Analogously, it is easy to verify that an alternative choice of boundary conditions which guarantees that we can empty $\mathcal{R}_{a,b}$ is to impose empty sites on the external segment parallel to the top border and on to the two external segments parallel to the left border (see Fig. 3b), namely on

$$(S_a + L(e_1+e_2)) \cup (-e_1) \bigcup_{i=1}^{b} [S_2 - 2e_1 + i(e_1+e_2)].$$

With a slight abuse of notation we denote the region with the shape of $\mathcal{R}_{L,L}$ which is centered around the origin and its corresponding border by $\mathcal{R}_L$ and ∂R_L:

$$\mathcal{R}_L := \mathcal{R}_{L,L} - L/2e_1 - L/2e_2,$$

$$\partial R_L := \partial\mathcal{R}_{L,L} - L/2e_1 - L/2e_2,$$

namely $\mathcal{R}_L$ is the region delimited by vertexes A, B, C, D in Fig. 7 (here and in the following, without lack of generality, we choose L such that $L/4$ is integer). Then we denote by $\widetilde{\mathcal{R}}_L$ the region inside the bold dashed line in Fig. 7, which is obtained from $\mathcal{R}_L$ by subtracting at the bottom left and top right corners two regions, $\mathcal{R}_{bl}$ and $\mathcal{R}_{tr}$, which have the shape of $\mathcal{R}_{L/4}$, namely

$$\widetilde{\mathcal{R}}_L := \mathcal{R}_L \setminus (\mathcal{R}_{bl} \cup \mathcal{R}_{tr}),$$

with

$$\mathcal{R}_{tr} := \mathcal{R}_{L/4,L/4} + L/4(e_1 + e_2),$$

$$\mathcal{R}_{bl} := \mathcal{R}_{L/4,L/4} - Le_1 - L/2e_2.$$

Let η be a configuration on $\mathcal{R}_L$ and denote by η^s the stationary configuration reached upon evolving η with fixed filled boundary conditions on $\mathcal{R}_L$, $\eta^s := (T^f_{\mathcal{R}_L})^{L^2}\eta$. We say that η is *good* if $\widetilde{\mathcal{R}}_L$ is completely empty on η^s, and we denote by G^L the set of good configurations, i.e.

$$G^L := (\eta \in \Omega_{\mathcal{R}_L} : \eta^s \in \mathcal{V}_{\widetilde{\mathcal{R}}_L}).$$

Given a configuration η on $\mathbb{Z}^2$ we denote by $\eta_{\mathcal{R}_L}$ its restriction to $\mathcal{R}_L$. If $\eta_{\mathcal{R}_L}$ is good, then the evolution on the infinite lattice also empties in at most L^2 steps the region $\widetilde{\mathcal{R}}_L$, namely $T^{L^2}\eta \in \mathcal{V}_{\widetilde{\mathcal{R}}_L}$, as can be easily proved by using attractiveness of the dynamics.

The following holds on the probability that a region is good

Lemma 4.7 *For any $\rho < p_c^{OP}$ and for any $\epsilon > 0$ there exists $L(\rho,\epsilon) < \infty$ such that for $L = L(\rho,\epsilon)$*

$$\mu^\rho(G^L) > 1 - \epsilon.$$

We postpone the proof of this main Lemma 4.7 and derive its consequences for ρ_c and $\tilde{\rho}_c$.

Corollary 4.8 *$\rho_c \geq p_c^{OP}$, thus $\rho_c = p_c^{OP}$.*

Proof Since the origin belongs to $\widetilde{\mathcal{R}}_L$, if $\mathcal{R}_L$ is good then the origin is certainly empty at time L^2, which implies

$$\mu^\rho(T^{L^2}\eta(0) = 0) \geq \mu^\rho(G_L). \tag{4.22}$$

This, together with the definition (2.7) and the result of Lemma 4.7, guarantees that if $\rho < p_c^{OP}$ for any given $\epsilon > 0$ we can choose $L > L(\rho,\epsilon)$ such that

$$1 - \mu^\rho_\infty(\eta(0)) \geq \mu^\rho(T^{L^2}\eta(0) = 0) > 1 - \epsilon. \tag{4.23}$$

Thus for any ϵ we have $0 \leq \mu^{\rho}_{\infty}(\eta(0)) \leq \epsilon$ and therefore $\mu^{\rho}_{\infty}(\eta(0)) = 0$, which implies $\rho_c \geq p_c^{OP}$ (recall definition (2.6) for ρ_c). The identification of ρ_c with p_c^{OP} immediately follows from the latter result and Lemma 4.1. □

In order to prove the stronger result of Lemma 4.2 we now have to introduce a renormalization procedure in the same spirit of [14]. Fix an integer scale L and let $\mathbb{Z}^2(L) \equiv L\mathbb{Z}^2$. We consider a partition of $\mathbb{Z}^2$ into disjoint regions $\mathcal{R}_L^z := \mathcal{R}_L + z$, $z \in \mathbb{Z}^2(L)$. In the following we will refer to $\mathbb{Z}^2(L)$ as the *renormalized lattice* and, given configuration $\eta \in \Omega_{\mathbb{Z}^2}$, we will say that a site $z \in \mathbb{Z}^2(L)$ is good if the configuration $\eta_{\mathcal{R}_L^z}$ restricted to the tile $\mathcal{R}_L^z \subset \mathbb{Z}^2$ corresponding to z is good. Note that the events that two different sites z and z' are good are independent.

Let z be site of the renormalized lattice. If its South, South-East and East neighbours, i.e. $z - e_2$, $z + e_1 - e_2$ and $z + e_1$, are good then after (at most) $2|\mathcal{R}_L| + |\mathcal{R}_{L/4}| = L^2 33/16$ steps the region corresponding to z on the original lattice, $\mathcal{R}_L^z$, is completely empty. More precisely

Proposition 4.9 *If* $\eta_{\mathcal{R}_L^{z+e_1}} \in G_L$, $\eta_{\mathcal{R}_L^{z+e_1-e_2}} \in G_L$ *and* $\eta_{\mathcal{R}_L^{z-e_2}} \in G_L$ *then* $T^{CL^2}\eta \in \mathcal{V}_{\mathcal{R}_L^z}$ *for* $C = 33/16$.

Proof From the definition of good configurations it follows that at time L^2 all sites belonging to $\widetilde{\mathcal{R}}_L^{z+e_1} \cup \widetilde{\mathcal{R}}_L^{z-e_2} \cup \widetilde{\mathcal{R}}_L^{z+e_1-e_2}$ are empty, i.e.

$$T^{L^2}\eta \in \mathcal{V}_A \quad \text{with } A := \widetilde{\mathcal{R}}_L^{z+e_1} \cup \widetilde{\mathcal{R}}_L^{z-e_2} \cup \widetilde{\mathcal{R}}_L^{z+e_1-e_2}. \tag{4.24}$$

If we denote by $\mathcal{R}_{tr}^{z-e_2}$ ($\mathcal{R}_{bl}^{z+e_1}$) the top right (bottom left) region of linear size $L/4$ which belongs to $\mathcal{R}_L^{z-e_2}$ ($\mathcal{R}_L^{z+e_1}$) and by $\partial\mathcal{R}_{tr}^{z-e_2}$ ($\partial\mathcal{R}_{bl}^{z+e_1}$) the corresponding bottom right borders, it is immediate to verify that (4.24) implies that at time L^2 both $\partial\mathcal{R}_{tr}^{z-e_2}$ and $\partial\mathcal{R}_{bl}^{z+e_1}$ are completely empty. This, together with Proposition 4.5, implies that at time $L^2 + (L/4)^2$ both $\mathcal{R}_{tr}^{z-e_2}$ and $\mathcal{R}_{bl}^{z+e_1}$ will be empty. The latter result, together with (4.24), guarantees that the bottom right border of $\mathcal{R}_L^z$, $\partial\mathcal{R}_L^z$, is empty at time $L^2 + (L/4)^2$. By using again Proposition 4.5 it follows that at time $L^2 + L^2/16 + L^2$ the entire region $\mathcal{R}_L^z$ will also be empty. □

We say that $(x_1, x_2, \dots, x_n)$ with $x_i \in \mathbb{Z}^2(L)$ is a *South–South-East–East* (*S–SE–E*) *path* if $x_{i+1} \in (e_1, e_1 - e_2, -e_2)$ for all $i \in (1, \dots, n-1)$. For a given configuration η let $x \overset{S-SE-E}{\longrightarrow} y$ if there is a S–SE–E path of sites connecting x and y (i.e. with $x_1 = x$ and $x_n = y$) such that each site in the path is not good. Also, we define the *S–SE–E bad cluster* of the origin to be the random set

$$\mathcal{C}^b := (y \in \mathbb{Z}^2(L) : 0 \overset{S-SE-E}{\longrightarrow} y).$$

Finally, we define the *range of the bad cluster* of the origin, R^b, as

$$R^b := \sup(k : \mathcal{C}^b \cap (D_k \cup D_{k+1}) \neq \emptyset),$$

where

$$D_k = (y : y = a(e_1 + e_2) - ke_2 \text{ with } a \in (0, \dots, k))$$

(and we let $\sup(\emptyset) = -\infty$). See Fig. 4 for an example of $\mathcal{C}^b$ and R^b.

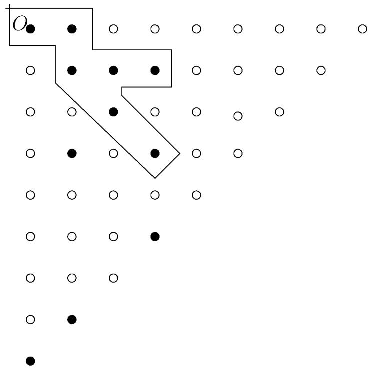

Fig. 4 The renormalized lattice: *black* and *white circles* stand for bad and good sites, respectively. Inside the *continuous line* we depict the S–SE–E bad cluster of the origin, $\mathcal{C}^b$. For this configuration the range of the bad cluster verifies $R^b = 7$

Proof of Lemma 4.2 Let τ be the renormalized time defined by the following relation

$$t(\tau) := 33/16L^2\tau. \tag{4.25}$$

For a given configuration η, we denote by $\mathcal{C}^b_\tau$ and R^b_τ the cluster and range of the origin at the renormalized time τ, i.e. for the configuration $\eta_{t(\tau)}$. Let $R^b_0 = k$. At time $\tau = 1$ Proposition 4.9 guarantees that all the tiles of the original lattice corresponding to renormalized sites in D_k and which belong to the cluster of the origin are empty. Indeed each of these sites has its South, South-East and East neighbours which are good at time zero since they are in D_k (if they were bad they would also belong to the cluster of the origin and this would imply $R^b_0 \geq k + 1$ in contrast with our assumption $R^b_0 = k$). This implies therefore $R^b_1 \leq k - 1 = R^b_0 - 1$. The same procedure can be applied at each subsequent (renormalized) time step, yielding

$$R^b_{\tau+1} \leq R^b_\tau - 1 \tag{4.26}$$

and finally

$$R^b_k \leq 0. \tag{4.27}$$

The latter equation implies that the South, the South-East and the East neighbours of the origin are good when the renormalized time coincides with the range of the bad cluster of the origin at time zero. Therefore, using again Proposition 4.9, we get that the origin is certainly empty at time $t = 33/16L^2(R^b_0 + 1)$ and therefore the first time at which the origin gets emptied, t_E, verifies

$$P_\rho(t_E > t) \leq \mu^\rho(R^b > t16/33L^{-2} - 1). \tag{4.28}$$

We can now use a Peierls type estimate to evaluate the probability that the range of the origin is larger than a certain value

$$\mu^\rho(R^b > s - 1) \leq \sum_{l=s}^{\infty}(1 - \mu^\rho(G_L))^{l+1}3^l. \tag{4.29}$$

If $\rho < p_c^{OP}$ and we choose $\epsilon < 1/3$, Lemma 4.7 guarantees the existence of an $L(\rho, \epsilon)$ such that if we let $L = L(\rho, \epsilon)$ we have $\alpha := (1 - \mu^\rho(G_L))3 < 3\epsilon < 1$. Therefore we obtain and exponential decrease for the above probability,

$$\mu^\rho(R^b > s - 1) \leq (1 - \alpha)^{-1}\exp(-s|\log(\alpha)|). \tag{4.30}$$

This, together with (4.28), allows to conclude

$$P_\rho(t_E > t) \le C\exp(-t\beta(\rho)) \tag{4.31}$$

with $\beta(\rho) = |\log(\alpha)|16/33L(\rho, 1/3)^{-2}$. Thus the the speed $\gamma(\rho)$ at which the lattice is emptied (see definition (2.9)) satisfies $\gamma(\rho) \ge \beta(\rho) > 0$ at any $\rho < p_c^{OP}$ and we conclude that $\tilde{\rho}_c \ge p_c^{OP}$. Note that, as a byproduct, we have also derived a lower bound on the velocity in terms of the crossover length of oriented percolation (via expression (4.41) for $L(\rho, \epsilon)$). □

We are now left with the proof of the main Lemma 4.7 which will be achieved in several steps. Let us start by proving some results on the sufficient conditions which allow to enlarge proper empty regions.

Let $\mathcal{Q}_L^{NW-NE}$ and $\mathcal{Q}_L^{SW-SE}$ be the two quadrangular regions inside the continuous lines of Fig. 5,

$$\mathcal{Q}_L^{NW-NE} = \bigcup_{i=1}^{L} (S_{2L-2(i-1)} - (L-i+1)e_1 + (i-1)e_2),$$

$$\mathcal{Q}_L^{SW-SE} = \bigcup_{i=1}^{L} (S_{2L-2(i-1)} - (L-i+1)e_1 - ie_2),$$

and $\mathcal{O}_L$ be the octagon centered around the origin formed by their union

$$\mathcal{O}_L := \mathcal{Q}_L^{NW-NE} \cup \mathcal{Q}_L^{SW-SE}.$$

If $\mathcal{O}_L$ is empty a sufficient condition in order to expand the empty region of one step, i.e. to empty the region $\mathcal{O}_{L+1}$, is that the four key sites

$$K_L^{NW} := -(L+1)e_1, \qquad K_L^{NE} := e_1 + Le_2,$$
$$K_L^{SW} := -(L+1)e_2, \qquad K_L^{SE} := (L+1)e_1 - e_2,$$

are all empty, namely

Proposition 4.10 *If* $\eta(x) \in \mathcal{V}_{\mathcal{O}_L}$ *and* $\eta(K_L^{NE}) = \eta(K_L^{NW}) = \eta(K_L^{SW}) = \eta(K_L^{SE}) = 0$, *then* $T^L\eta \in \mathcal{V}_{\mathcal{O}_{L+1}}$.

Proof From Proposition 4.5, it is immediate to verify that we can empty all sites of the form $K_L^{NW} + (i-1)(e_1+e_2)$ with $1 \le i \le L+1$. The same occurs for all sites $K_L^{SE} - (i-1)(e_1+e_2)$ with again $1 \le i \le L+1$, as can be immediately be verified by using Remark 4.6. Then it is easy to verify that we can subsequently empty all sites of the form $K_L^{NE} + (i-1)(e_1 - e_2)$ ($K_L^{SW} + (i-1)(-e_1+e_2)$) with $1 \le i \le L+1$ since they all have the *NW* and *SW* (*NE* and *SE*) neighbours which are empty. □

We will now prove that, if $\mathcal{O}_L$ is empty, a sufficient condition in order to guarantee that K_L^{NW} is empty after L^2 steps is that its *NE* occupied cluster does not survive after L steps, namely $\Gamma_{K_L^{NW}}^{NE,L}$ is an empty set (and analogous results in the other directions). More precisely if we define the events

$$\mathcal{K}_L^{NW} := (\eta : \Gamma_{K_L^{NW}}^{NE,L} = \emptyset), \qquad \mathcal{K}_L^{NE} := (\eta : \Gamma_{K_L^{NE}}^{SE,L} = \emptyset),$$
$$\mathcal{K}_L^{SE} := (\eta : \Gamma_{K_L^{SE}}^{SW,L} = \emptyset), \qquad \mathcal{K}_L^{SW} := (\eta : \Gamma_{K_L^{SW}}^{NW,L} = \emptyset), \tag{4.32}$$

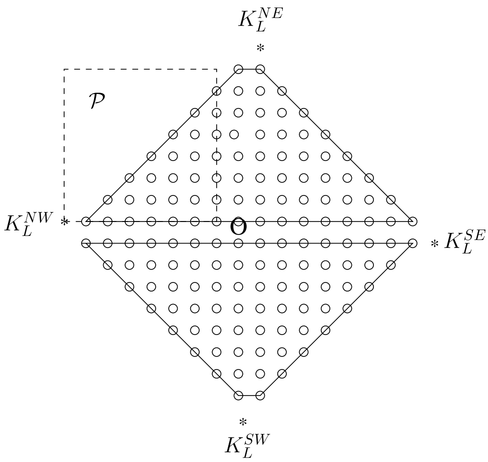

Fig. 5 The octagon $\mathcal{O}_L$ composed by $\mathcal{Q}_L^{NW-NE}$ and $\mathcal{Q}_L^{SW-SE}$ (*top* and *bottom regions* inside the *continuous lines*, respectively) and the key external sites $K_L^{NW}, K_L^{NE}, K_L^{SE}, K_L^{SW}$

the following holds:

Lemma 4.11 (i) *If* $\eta \in \mathcal{V}_{\mathcal{O}_L}$ *and* $\eta \in \mathcal{K}_L^{NW}$ *then* $T^{L^2}\eta(K_L^{NW}) = 0$.
(ii) *If* $\eta \in \mathcal{V}_{\mathcal{O}_L}$ *and* $\eta \in \mathcal{K}_L^{NE}$ *then* $T^{L^2}\eta(K_L^{NE}) = 0$.
(iii) *If* $\eta \in \mathcal{V}_{\mathcal{O}_L}$ *and* $\eta \in \mathcal{K}_L^{SW}$ *then* $T^{L^2}\eta(K_L^{SW}) = 0$.
(iv) *If* $\eta \in \mathcal{V}_{\mathcal{O}_L}$ *and* $\eta \in \mathcal{K}_L^{SE}$ *then* $T^{L^2}\eta(K_L^{SE}) = 0$.

Proof We will prove only result (i), the proofs of the other results follow along the same lines. Let $\mathcal{P}$ be the square region of size $L \times L$ with vertexes $(-e_1, K_L^{NW}, K_L^{NW} + Le_2, -e_1 + Le_2)$ (region inside the dashed line in Fig. 5). Recall the definition (2.4) and consider the evolution operator $T_{\omega,\mathcal{P}}$ restricted to $\mathcal{P}$ and with fixed boundary conditions $\omega(x) = 0$ for $x \in \mathcal{O}_L$ and $\omega(x) = 1$ for $x \in \mathbb{Z}^2 \setminus \mathcal{O}_L$. For simplicity of notation we will call $\widetilde{T}$ such operator. It is immediate to verify that

$$\text{If } \eta \in \mathcal{V}_{\mathcal{O}_L} \text{ and } \widetilde{T}^{L^2}\eta(K_L^{NW}) = 0, \quad \text{then } T^{L^2}\eta(K_L^{NW}) = 0. \tag{4.33}$$

We are therefore left with proving that the hypothesis in (i) imply those in (4.33), which we will do by contradiction. First we need to introduce some additional notation.

Let η^s be the stationary configuration reached under $\widetilde{T}$ after L^2 steps, $\eta^s := \widetilde{T}^{L^2}$ (stationarity follows from (2.5) and $|\mathcal{P}| = L^2$) and $\mathcal{P}^*$ be the rectangular region $\mathcal{P}^*$: $\mathcal{P} \bigcup_{n=0}^{L} ne_2 \bigcup_{n=0}^{L+1} (-ne_1 - e_2)$, we define the following random set

$$\begin{aligned} \mathcal{B} &:= (x \in \mathcal{P} : \eta^s(x) = 1; \\ &\eta^s(y) = 0 \text{ if } y \in \mathcal{P}^* \text{ and } y = x + (n+m)e_1 + me_2 \quad \text{with } n \geq 1, m \geq 0; \\ &\eta^s(y) = 0 \text{ if } y \in \mathcal{P}^* \text{ and } y = x + (n+m)e_1 - (m+1)e_2 \quad \text{with } n \geq 1, m \geq 0). \end{aligned} \tag{4.34}$$

The following properties, whose proof is postponed, hold

Proposition 4.12 *If* $x \in \mathcal{B}$, *then*

(i) $\eta^s(x + e_2) = 1$ *or* $\eta^s(x + e_1 + e_2) = 1$ (*or both*).
Let $b(x)$ *be rightmost among these two sites which is occupied* (*in* η^s)
(ii) *If* $b(x) \in \mathcal{P}$, *then* $b(x) \in \mathcal{B}$.

By using the above properties, we will conclude the proof of Lemma 4.11 by contradiction. Let us suppose that the left hand side of (4.33) does not hold, namely $\eta^s(K_L^{NW}) = 1$. Since $\eta \in \mathcal{V}_{\mathcal{O}_L}$ it follows immediately that $K_L^{NW} \in \mathcal{B}$. Thus we can define a sequence $\{x_i\}$ with $i \in (1, \dots, L)$ with $x_1 = K_L^{NW}$ and $x_i = b(x_{i-1})$ for $i \leq L$ and it is immediate to verify that $x_i \in NE_{x_{i-1}}$ and $\eta^s(x_i) = 1$. Therefore, under the hypothesis $\eta^s(K_L^{NW}) = 1$, we have identified a NE occupied path $x_1, \dots, x_L$ of L sites for K_L^{NW} such that $x_L - K_L^{NW} = me_1 + Le_2$, which implies $\Gamma_{K^{NW}}^{NE,L} \neq \emptyset$. Since this result contradicts the hypothesis of the lemma we conclude that $\eta^s(K_L^{NW}) \neq 1$, thus the condition in the left hand side of (4.33) is verified and the proof is concluded. □

We are therefore left with proving the properties of the random set of sites $\mathcal{B}$.

Proof of Proposition 4.12 (i) Follows immediately from the fact that the South-East neighbours of x, $SE_x = (x + e_1, x + e_1 - e_2)$, are both empty in η^s ($x \in \mathcal{B}$ implies $x \in \mathcal{P}$ and therefore $SE_x \subset \mathcal{P}^*$ and both these sites are empty thanks to the definition of $\mathcal{B}$). Thus at least one of the two sites $NE_x = (x + e_2, x + e_1 + e_2)$ should be occupied, otherwise x would be empty at time $s+1$ (which is forbidden by the stationarity of η^s). We let $b(x)$ be the rightmost among these occupied sites.

(ii) The first property defining $\mathcal{B}$, $\eta^s(b(x)) = 1$, is satisfied by definition of $b(x)$. Let us prove that the second and third property are verified too. If $b(x) = x + e_1 + e_2$, then

$$b(x) + (n+m)e_1 + me_2 = x + (n+m+1)e_1 + (m+1)e_2,$$

$$b(x) + (n+m)e_1 - (m+1)e_2 = x + (n+1+m)e_1 - me_2$$

and the properties defining $\mathcal{B}$ are immediately verified. On the other hand, if $b(x) = x + e_2$,

$$b(x) + (n+m)e_1 + me_2 = x + ((n-1) + (m+1))e_1 + (m+1)e_2,$$

$$b(x) + (n+m)e_1 - (m+1)e_2 = x + ((n+1) + (m-1))e_1 - ((m-1)+1)e_2.$$

Thus if $n \geq 2$ (for all $m \geq 0$) the second property defining $\mathcal{B}$ for site $b(x)$ follows from the fact that $x \in \mathcal{B}$ and the same holds for the third property if $m \geq 1$ (for all $n \geq 1$). The third property is also easily established if $m = 0$, since in this case $b(x) + (n+m)e_1 - (m+1)e_2 = b(x) + ne_1 - e_2 = x + ne_1$, which is again empty since $x \in \mathcal{B}$. Some additional care is required to verify the only remaining case, i.e. the validity of second property in the case $n = 1$. Notice that since $b(x) = x + e_2$, this implies $\eta^s(x + e_2 + e_1) = 0$ (by definition $b(x)$ is the rightmost occupied site in NE_x). Thus the second property is verified in the case $n = 1$ and $m = 0$. Let us consider the case $b(x) + (n+m)e_1 + me_2$ with $n = 1$ and $m = 1$. It is easily verified that this site is also empty in the stationary configuration η^s since its SE neighbours are of the kind $x + (1+m)e_1 + me_2$ or $x + (2+m)e_1 + me_2$ (and therefore empty since $x \in \mathcal{B}$), one of its SW neighbours is also of the form $x + (1+m)e_1 + me_2$ and the other one is $b(x) + (1 + (m-1))e_1 + (m-1)e_2$. The latter, for the choice $m = 1$, is $b(x) + e_1$ which we verified to be empty. The same property can be verified by induction for all the other values $m \geq 0$ and $n = 1$. □

From Proposition 4.10 and Lemma 4.11 we can now derive the following lower bound (which will be used to prove Lemma 4.7) on the probability that after $L_n := 4(L+n)^2$ steps we have enlarged of (at least) n steps the empty octagonal region

Lemma 4.13 *If $\rho < p_c^{OP}$, then*

$$\mu^\rho(T^{L_n}\eta \in \mathcal{V}_{\mathcal{O}_{L+n}} | \eta \in \mathcal{V}_{\mathcal{O}_L}) \geq \prod_{i=0}^{n-1}[1-\exp(-(L+i)/\xi_{OP})]^4.$$

Proof *Case* $n=1$. From Proposition 4.10 and Lemma 4.11 it follows that

$$\mu^\rho(T^{L_1}\eta \in \mathcal{V}_{\mathcal{O}_{L+1}} | \eta \in \mathcal{V}_{\mathcal{O}_L}) \geq [\mu^\rho(\mathcal{K}_L^{NW})]^4,$$

where we used the fact that the events $\mathcal{V}_{\mathcal{O}_L}$, $\mathcal{K}_L^{NW}$, $\mathcal{K}_L^{NE}$, $\mathcal{K}_L^{SE}$ and $\mathcal{K}_L^{SW}$ are independent and we used the equalities

$$\mu^\rho(\mathcal{K}_L^{NW}) = \mu^\rho(\mathcal{K}_L^{NE}) = \mu^\rho(\mathcal{K}_L^{SE}) = \mu^\rho(\mathcal{K}_L^{SW}), \tag{4.35}$$

which easily follow from symmetry properties. Along the same lines of the Proof of Lemma 4.4, it is now easy to prove that the probability of this event coincides with the analogous quantity for oriented percolation, namely

$$\mu^\rho(\mathcal{K}_L^{NW}) = \mu^\rho(\Gamma^{NE,L}_{K_L^{NW}} = \emptyset) = \mu^\rho(\Gamma^{L}_{K_L^{NW}} = \emptyset). \tag{4.36}$$

Then the proof is completed by using the exponential bound (3.15).

Case $n=2$. From Proposition 4.10 and Lemma 4.11 we get

$$\begin{aligned}&\mu^\rho(T^{L_2}\eta \in \mathcal{V}_{\mathcal{O}_{L+2}} | \eta \in \mathcal{V}_{\mathcal{O}_L}) \geq \\ &\mu^\rho(\mathcal{K}_L^{NE} \cap \mathcal{K}_L^{SE} \cap \mathcal{K}_L^{SW} \cap \mathcal{K}_L^{NW} \cap \mathcal{K}_{L+1}^{NE} \cap \mathcal{K}_{L+1}^{SE} \cap \mathcal{K}_{L+1}^{SW} \cap \mathcal{K}_{L+1}^{NW}),\end{aligned} \tag{4.37}$$

where the couples of events at size L and $L+1$ are now not independent, $\mu^\rho(\mathcal{K}_L^{NE} \cap \mathcal{K}_{L+1}^{NE}) \neq \mu^\rho(\mathcal{K}_L^{NE})\mu^\rho(\mathcal{K}_{L+1}^{NE})$. However, since all the events that we consider are of the form (4.32) and therefore non increasing with respect to the partial order $\eta \prec \eta'$ if $\eta(x) \leq \eta'(x)$ $\forall x$, we can apply FKG inequality [10] and get

$$\begin{aligned}&\mu^\rho(T^{L_2}\eta \in \mathcal{V}_{\mathcal{O}_{L+n}} | \eta \in \mathcal{V}_{\mathcal{O}_L}) \geq \\ &\mu^\rho(\mathcal{K}_L^{NE})\mu^\rho(\mathcal{K}_L^{SE})\mu^\rho(\mathcal{K}_L^{NW})\mu^\rho(\mathcal{K}_L^{SW})\mu^\rho(\mathcal{K}_{L+1}^{NE})\mu^\rho(\mathcal{K}_{L+1}^{NW})\mu^\rho(\mathcal{K}_{L+1}^{SE})\mu^\rho(\mathcal{K}_{L+1}^{SW}) \geq \\ &[\mu^\rho(\mathcal{K}_L^{NW})]^4[\mu^\rho(\mathcal{K}_{L+1}^{NW})]^4 \geq [1-\exp(-L/\xi_{OP})]^4[1-\exp(-(L+1)/\xi_{OP})]^4,\end{aligned} \tag{4.38}$$

where again we used the symmetry properties (4.35), the mapping to oriented percolation (4.36) and the fact that $\rho < p_c^{OP}$.

Case $n>2$. The proof follows the same lines of the case $n=2$. □

In the proof of the main Lemma 4.7 we will also need a condition which is sufficient to guarantee the expansion of an empty region of the type $\mathcal{Q}_L^{NE-NW}$ to the larger region $\mathcal{T}_L := (S_{2L} - Le_1) \cup (e_2 + \mathcal{Q}_L^{NE-NW})$ (region inside the continuous line of Fig. 6). In particular, we will need a sufficient condition which does not involve the configuration inside the rectangular region $\Lambda_{2m,n}$ with vertexes $(L+2)e_2 - (m-1)e_1, (L+2)e_2 + me_1, -(m-1)e_1 + (L+2-n)e_2, me_1 + (L+2-n)e_2$ (region inside the dashed–dotted line in Fig. 6).

Lemma 4.14 *Let $m, n < L/2$. If $\eta \in \mathcal{V}_{\mathcal{Q}_L^{NE-NW}}$, $\Gamma^{NE,L/2}_{K_L^{NW}} = \emptyset$ and $\Gamma_x^{SE,L/2} = \emptyset$ for all x such that $(xe_2 = L+1-n, 0 \leq xe_1 \leq m)$ and for all x such that $(xe_1 = m, L+1-n \leq xe_2 \leq L+1)$, then $T^{L^2}\eta \in \mathcal{V}_{\mathcal{T}_L}$.*

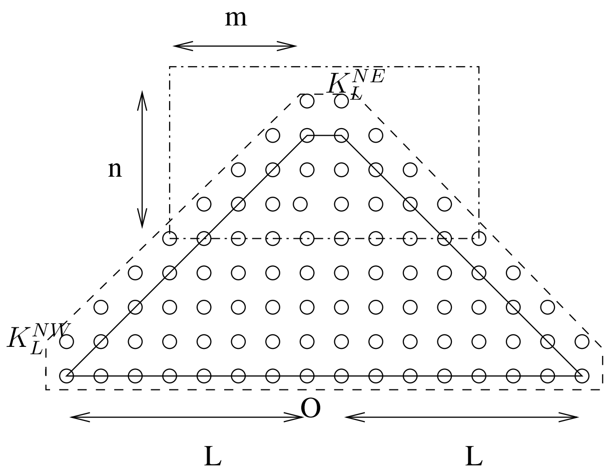

Fig. 6 Inside the *dashed line* we depict the region $\mathcal{T}_L := (S_{2L} - Le_1) \cup (e_2 + \mathcal{Q}_L^{NE-NW})$ which is emptied provided $\mathcal{Q}_L^{NE-NW}$ (region inside the *continuous line*) is empty and the other hypothesis of Lemma 4.14 hold. The region inside the *dashed–dotted line* is $\Lambda_{2m,n}$, the rectangular region whose internal configuration is *not* involved in the hypothesis of Lemma 4.14

Proof Following the same lines of the proof of Proposition 4.10, it is immediate to verify that if K_L^{NW} and K_L^{NE} are empty, then $S_{2L-Le_1} \cup (e_2 + \mathcal{Q}_L^{NE-NW})$ is emptied in at most L steps. Since the hypothesis exclude the existence of a NE (SE) path for K_L^{NW} (K_L^{NE}) of length larger or equal than L, the same arguments used in Lemma 4.11 allow to conclude that both this sites are emptied in (at most) L^2 steps and the proof is concluded. Note that all the hypothesis do not involve the value of the occupation variables inside $\Lambda_{2m,n}$. □

Remark 4.15 Analogous sufficient conditions in order to expand the bottom (right or left) half of $\mathcal{O}_L$, towards the bottom (right or left direction, respectively) can be established by applying the invariance of constraints under rotations of 90 degrees.

We are now ready to prove Lemma 4.7.

Proof of Lemma 4.7 Choose two integers ℓ and L such that $L = c\ell$ with $c > 1$ also integer. Then divide $\mathcal{R}_L$ into a grid of c columns and c rows of width ℓ. Let $Q_1, \dots, Q_{c^2}$ be the squares identified by this grid and inside each square consider a centered octagon $\mathcal{O}_r$ with $r < \ell/2$ and let $\mathcal{O}_r^i$ be the octagon associated to Q_i (octagon inside the continuous line in Fig. 8a).

We will now state a series of requirements (i–viii) which involve only the configuration inside $\mathcal{R}_L$ and are sufficient in order to guarantee that we can empty $\widetilde{\mathcal{R}}_L$ in (at most) L^2 steps. The sufficiency of these conditions can be directly proven by using step by step the previously proved properties (as detailed below):

(i) There exists at lest one Q_i with $c < i < c^2 - c$ such that $\mathcal{O}_r^i$ is completely empty. Denote by I be the smallest integer such this property holds and let K_r^{NW}, K_r^{NE}, K_r^{SW} and K_r^{SE} be the NW, NE, SW and SE key sites corresponding to the octagon $\mathcal{O}_r^I$. Note that the conditions on i exclude that Q_I belongs to the top or bottom row of squares.
(ii) The configuration in Q_I belongs to the events $\mathcal{K}_r^{NW}$, $\mathcal{K}_r^{NE}$,$\mathcal{K}_r^{SW}$ and $\mathcal{K}_r^{SE}$. This, as has been proved in Lemma 4.11, is sufficient in order to expand $\mathcal{O}_r^I$ of one step, i.e. to reach an empty octagon $\mathcal{O}_{r+1}$.
(iii) The configuration in Q_I verifies also the constraints (Lemma 4.11) required to expand further the empty octagon until reaching the border of Q_I. Let $\mathcal{O}_{\ell/2}^I$ be the empty region reached via this procedure (region inside the dashed line in Fig. 8b).

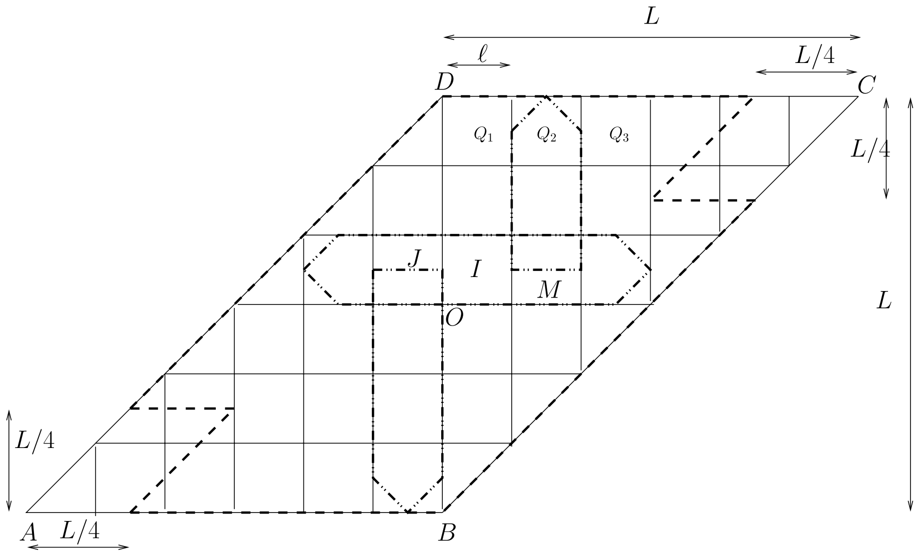

Fig. 7 The regions $\mathcal{R}_L$ and its partition in the grid of c^2 squares, with $c = L/\ell$ (here $c = 6$). The region $\widetilde{\mathcal{R}}_L$ is depicted inside the *dashed line*. The horizontal region plus the two ending triangles which is delimited by the *dashed–dotted line* is R_I (see condition (v) in the text). Instead, the two vertical regions delimited by the *dashed–dotted line* form C_I (see condition (iv) and (vi) in the text)

(iv) Consider the left and right half of $\mathcal{O}^I_{\ell/2}$. The sufficient conditions of Lemma 4.14 (rotated of 90 and 270 degrees, see Remark 4.15) with $m = n = r$ hold in order to expand these regions (towards the left and the right, respectively) via subsequent one site steps until emptying all sites in the region R_I composed by a row of squares plus two triangular region at the right and left which are at distance $\ell/2$ from the border of $\mathcal{R}_L$ parallel to $e_1 + e_2$ (R_I is the horizontal region delimited by the dashed–dotted line in Fig. 7). Note that at each step this sufficient conditions does not involve any of the occupation variables inside $\mathcal{O}^j$ for $j \neq I$.

(v) Let $Q_J \subset R_I$ ($Q_M \subset R_I$) be the leftmost (rightmost) square such that a vertical line drawn from J (M) crosses the bottom (top) border of $\mathcal{R}_L$, as shown in Fig. 7 (the existence of such Q_J and Q_M is guaranteed, since I does not belong neither to the top nor to the bottom row). The sufficient conditions of Proposition 4.14 and Remark 4.15 with $m = n = r$ holds in order to expand with subsequent one site steps the bottom half (top half) of $\mathcal{O}^J_{\ell/2}$ ($\mathcal{O}^M_{\ell/2}$) until touching the bottom (top) border of $\mathcal{R}_L$. Note that again at each step these conditions do not involve any of the occupation variables inside $\mathcal{O}^j$ for $j \neq I$. Let C_I be the region emptied via this procedure (which is composed by the two vertical regions delimited by the dashed–dotted line in Fig. 7).

(vi) For each square Q_j such that $Q_j \cap C_I \neq \emptyset$ we repeat the same procedure applied in (iv) to empty an horizontal region with the same shape of R_I by imposing necessary conditions which do not involve any of the occupation variables inside $\mathcal{O}^j$ for $j \neq I$.

In Fig. 8b we depict (inside the bold continuous line) the region which is emptied thanks to all the previous requirements. This includes the region $\mathcal{R}_L \setminus (R_l \cup R_r)$ (region inside the dashed line of Fig. 8), where R_l and R_r are two lateral strips of width $\ell 3/2$ (regions delimited by vertexes A, E, H, D and F, B, C, G in Fig. 8).

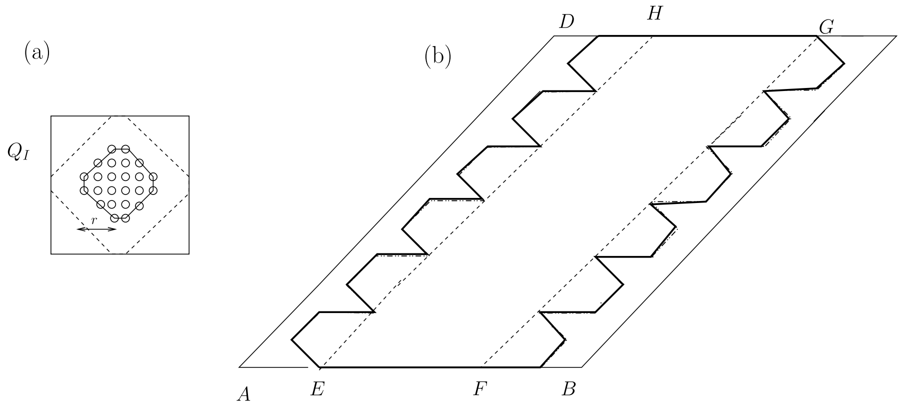

Fig. 8 **a** Zoom on square Q_I: the empty internal octagon $\mathcal{O}_r^I$ (guaranteed by condition (i)) and (inside the *dashed line*) the empty region $\mathcal{O}_{\ell/2}^I$ reached thanks to conditions (ii) and (iii). **b** Region inside the *bold continuous line* is emptied thanks to the conditions (i–vii)

Let

$$
\begin{aligned}
A_i &:= (x : x = -(L - 3/2\ell + i)e_1 - L/2e_2 + a(e_1 + e_2), \\
&\quad \text{with } (i-1)\lfloor c/6 \rfloor \le a \le i\lfloor c/6 \rfloor), \\
B_i &:= (x : x = -(L - 3/2\ell + i)e_1 - L/2e_2 + a(e_1 + e_2), \\
&\quad \text{with } i\lfloor c/6 \rfloor + 1 \le a \le L), \\
D_i &:= (x : x = (L - 3/2\ell + i)e_1 + L/2e_2 - a(e_1 + e_2), \\
&\quad \text{with } (i-1)\lfloor c/6 \rfloor \le a \le i\lfloor c/6 \rfloor), \\
E_i &:= (x : x = (L - 3/2\ell + i)e_1 + L/2e_2 - a(e_1 + e_2), \\
&\quad \text{with } i\lfloor c/6 \rfloor + 1 \le a \le L).
\end{aligned}
$$

Note that $\bigcup_{i=1}^{3/2\ell}(A_i \cup B_i) = R_l$ and $\bigcup_{i=1}^{3/2\ell}(D_i \cup E_i) = R_r$.

(vii) There exists at least one empty site inside each A_i for all $i \in [1, 3/2\ell]$. This implies that we can for sure empty all sites in $\bigcup_{i=1}^{3/2\ell} B_i$. The proof can be immediately obtained by subsequent applications of Proposition 4.5.

(viii) There exists at least one empty site inside each D_i for all $i \in [1, 3/2\ell]$. This implies that we can for sure empty all sites in $\bigcup_{i=1}^{3/2\ell} E_i$. The proof can be immediately obtained by subsequent applications of Remark 4.6.

The proof of the sufficiency of conditions (i–viii) in order to empty $\widetilde{\mathcal{R}}_L$ is then completed by noticing that the union between the region $\mathcal{R}_L \setminus (R_l \cup R_r)$ (emptied via conditions (i–vi)) and the regions $\bigcup_{i=1}^{3/2\ell} B_i$ and $\bigcup_{i=1}^{3/2\ell} E_i$ (emptied via conditions (vii) and (viii), respectively) covers $\widetilde{\mathcal{R}}_L$, namely $\widetilde{\mathcal{R}}_L \subset (\mathcal{R}_L \setminus (R_l \cup R_r)) \bigcup_{i=1}^{3/2\ell}(B_i \cup E_i)$.

In order to complete the proof, we are now left with evaluating the probability of such conditions. If we now denote by $P(j)$ the probability (w.r.t. μ^ρ) that the property stated in point (j) is satisfied we get

$$P(\mathrm{i}) = 1 - (1 - (1-\rho)^{r^2})^{c^2-2c},$$

$$P(\mathrm{ii}\cap\mathrm{iii}) \geq \prod_{j=1}^{\ell-r}\left[1-\exp\left(-\frac{j+r}{\xi}\right)\right]^4,$$

$$P(\mathrm{iv})P(v)P(\mathrm{vi}) \geq \left[1-\exp\left(-\frac{\ell-r}{\xi}\right)\right]^{6rL+2rL^2},$$

$$P(\mathrm{vii}) = P(\mathrm{viii}) = (1-\rho^{\lfloor c/6\rfloor})^{3/2l},$$

where the second and third bound follow using Lemma 4.13 and Lemma 4.14 and for simplicity of notation here and in the following we drop the index *OP* from the oriented percolation correlation length.

Note that we have chosen conditions (i–viii) in order that the event defined by (i) is independent from all the others, since it is the only condition which involves the configuration on the small octagons centered inside the squares of the grid, i.e. the $\mathcal{O}_r$'s. On the other hand the events required by conditions (ii–viii) are positively correlated (they are all non increasing event under the partial order $\eta \prec \eta'$ if $\eta(x) \leq \eta'(x)$ $\forall x$). Thus by using again FKG inequality and the above inequalities we get

$$\mu^\rho((T^f_{\mathcal{R}_L})^{L^2}\eta \in \mathcal{V}_{\tilde{\mathcal{R}}_L}) = \mu^\rho(G_L) \geq P(\mathrm{i})P(\mathrm{ii}\cap\mathrm{iii})P(\mathrm{iv})P(\mathrm{v})P(\mathrm{vi})P(\mathrm{vii})^2. \tag{4.39}$$

Let $r := 2\xi\log\xi \gg \xi$, $c := (1-\rho)^{-3\xi^2\log\xi^2}$, $\ell := \xi^4\log\xi$ and define the function $\tilde{L}(\rho)$ as follows

$$\tilde{L}(\rho) := (1-\rho)^{-3\xi^2\log\xi^2}\xi^4\log\xi. \tag{4.40}$$

From the above inequality we get in leading order as $\xi\to\infty$

$$P(\mathrm{i}) > 1-\exp(-(1-\rho)^{-2(\xi\log\xi)^2}),$$
$$P(\mathrm{ii}) \geq \exp(-4/\xi),$$
$$P(\mathrm{iv})P(\mathrm{v})P(\mathrm{vi}) \geq 1-\exp(-\xi^3\log\xi),$$
$$P(\mathrm{vii}) = 1-\xi^4\log\xi\exp(-|\log\rho|/4(1-\rho)^{-3\xi^2\log\xi^2}).$$

It is immediate to verify that in the limit $\rho \nearrow p_c^{OP}$, since $\xi\to\infty$ all these quantities go to one and for any $\epsilon>0$ there exists $\rho(\epsilon)$ (with $\rho(\epsilon)\nearrow p_c^{OP}$ when $\epsilon\searrow 0$) such that $\bar{\xi} := \xi(\rho(\epsilon))$ is sufficiently large to guarantee that the product on the left hand side of 4.39 is bounded from below by $1-\epsilon$ for $\rho\geq\rho(\epsilon)$. This implies the result of Lemma 4.7 with

$$L(\rho,\epsilon) = \tilde{L}(\rho), \tag{4.41}$$

for $\rho\geq\rho(\epsilon)$, where $\tilde{L}(\rho)$ has been defined in (4.41). The result for all the densities, $\rho<\rho(\epsilon)<p_c^{OP}$, and with the choice $L(\rho,\epsilon)=\tilde{L}(\rho(\epsilon))$ trivially follows from attractiveness which implies monotonicity of the probability of the good event (on a fixed size) under ρ. □

5 Discontinuity of Transition: Proof of Theorem 3.4

The proof is composed of two steps. First we construct a set of configurations such that the origin is *frozen*, i.e. it cannot be emptied at any finite time. Then we prove that this set has finite probability at $\rho = p_c^{OP}$. This is the same strategy which we already used to prove the upper bound of ρ_c. However the clusters which block the origin will now be of different type, the key feature being the existence of North-East and North-West occupied clusters which are mutually blocked. It is thanks to these structures that the properties of the transition are completely different from those of oriented percolation. On the other hand, we will see that an important ingredient which guarantees a finite weight to these configurations is the anisotropy of typical blocked clusters in each one of the two directions, i.e. anisotropy of conventional oriented percolation (it is here that Conjecture 3.2 is used).

Before entering the proof of Theorem 3.4, let us establish a result which will be used here as well as in the proof of Theorem 3.5(ii). Recall that we denote by $\Lambda_{a,b}$ a rectangular region with sides of length a parallel to $e_1 + e_2$ and sides of length b parallel to $-e_1 + e_2$. Let R_1 be a rectangular region $\Lambda_{a,b}$ centered at the origin and let $X := x_1, \dots, x_n$ with $x_i \in R_1$ be a North-East path which spans R_1, i.e. with x_1 and x_n belonging to the two sides of R_1 which are parallel to $-e_1 + e_2$ (see Fig. 9). Choose c and d such that $2c < a$ and $d > b$ and let R_2 and R_3 be two rectangular regions of the form $\Lambda_{c,d}$ which are centered on the line $e_1 + e_2$ with centers at $(a - c)/(2\sqrt{2})(e_1 + e_2)$ and $-(a - c)/(2\sqrt{2})(e_1 + e_2)$ (we consider without loss of generality that a, b, c, d are integer multiples of $2\sqrt{2}$). Note that both R_2 and R_3 intersect R_1 and both sides of R_1 which are parallel to $-e_1 + e_2$ lie on the two more far apart sides of R_2 and R_3 along this direction. Finally, let $Y := (y_1, \dots, y_m)$ and $Z := (z_1, \dots, z_{m'})$ with $y_i \in R_2$ and $z_i \in R_3$ be two North-West paths which span R_2 and R_3 respectively. Lemma 5.1 below states that if the three above defined paths X, Y and Z are occupied, the subset of X which belongs to $R_1 \setminus (R_2 \cup R_3)$ cannot be erased before erasing at least one site which belongs either to Y or to Z (actually along the same lines of the proof below one can prove the stronger result that one needs to erase at least $M := (d - b)/(2\sqrt{2})$ sites in either Y or Z). Let $\tilde{X} := X \setminus (R_2 \cup R_3)$ (sites inside the dashed contour in Fig. 9) and τ_A be the first time at which at least one site in A is empty. The following holds

Lemma 5.1 *If $\eta_0(w) = 1$ for all $w \in (X \cup Y \cup Z)$, $\tau_{\tilde{X}} \geq \tau_{Y \cup Z} + 1$.*

Proof In this proof for simplicity of notation $\tau_{\tilde{X}}$ will be sometimes simply denoted by τ. Let l and u to be the indexes such that $x_l \in R_1 \setminus R_3$, $x_{l-1} \in R_3$, $x_u \in R_1 \setminus R_2$ and $x_{u+1} \in R_2$ (it is immediate to verify that $(u - l) \geq (a - 2c)/\sqrt{2}$). The definition of $\tau_{\tilde{X}}$ implies that $\eta_{\tau-1}(x_{i+1}) = \eta_{\tau-1}(x_{i+1}) = 1$ for $l < i < u$, thus $\eta_\tau(x_i) = 1$. This in turn implies that $\eta_\tau(x_u) = 0$ or $\eta_\tau(x_l) = 0$ (or both). We will now prove that if the first possibility occurs this implies that at least one site of Y should be emptied before τ. The other case, $\eta_\tau(x_l) = 0$, can be treated analogously leading to the result that at least one site of Z should be emptied before τ. In both cases the result of the lemma follows. The first observation is that, since we consider the case $\eta_\tau(x_u) = 0$, this implies $\eta_{\tau-1}(x_{u+1}) = 0$. This follows from the fact that $\eta_{\tau-1}(x_{u-1}) = 1$, therefore the *SW* neighbours of x_l are not both empty at $\tau - 1$. Thus both its *NE* (which include x_{u+1} by definition) should be empty at $\tau - 1$, otherwise the emptying of x_u at time τ could not occur. This procedure can be iterated to prove that if s_{u+i} is the first time at which x_{u+i} is emptied it verifies

$$s_{u+i} \leq s_{u+i-1} - 1 \leq \cdots \leq s_u - 1 = \tau - 1, \tag{5.42}$$

for $0 < i \leq n - u$. Since both X and Y span $\tilde{R} := R_1 \cap R_2$ and X connects the two sides of $\tilde{R}$ which are parallel to $-e_1 + e_2$ and Y those that are parallel to $e_1 + e_2$, if we denote by ℓ_X (ℓ_Y) the continuous line obtained by joining the sites in X (in Y), it is immediate to verify that ℓ_X and ℓ_Y do intersect. Denote by P an intersection point for ℓ_X and ℓ_Y. Since ℓ_X is by

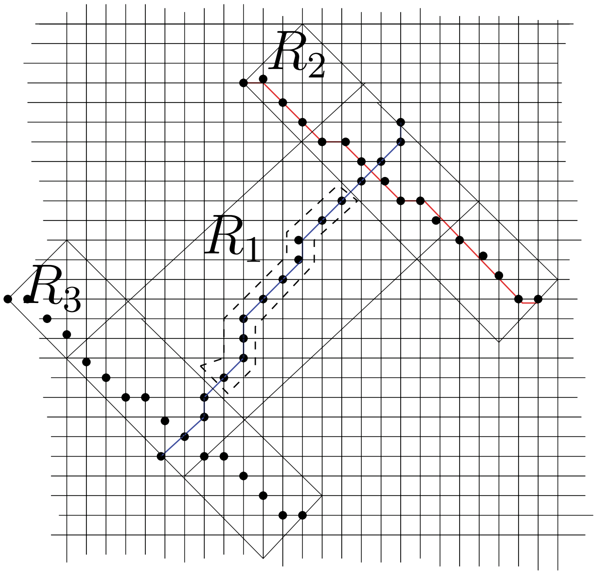

Fig. 9 The three rectangles R_1, R_2 and R_3 with the corresponding occupied clusters, X, Y and Z. The *red* (*blue*) *line* corresponds to ℓ_X and ℓ_Y and their intersection point is P. Here $P \notin X$, $P \notin Y$, therefore we are in case (b) for the proof. *Black dots* inside the *dashed contour* belong to $\tilde{X}$, i.e. the set of sites which are guaranteed to be occupied before at least one site in $Y \cup Z$ has been emptied

construction composed only by segments of the form e_2 and $e_1 + e_2$, while ℓ_Y is composed by segments of the form $-e_1$ and $-e_1 + e_2$ it can be easily verified that (a) either P belongs to the lattice (and therefore to both X and Y), (b) or it does not belong to the lattice but $P \pm (e_1 + e_2)/2$ belong to X and $P \pm (e_1 - e_2)$ belong to Y (this is for example the case for the paths depicted in Fig. 9). Let us treat case (a) and (b) separately.

(a) Since $P \in X$ and $P \in R_2$, we can identify an index j such that $u < j \leq n$ and $P = x_j$. Therefore by using 5.42, we get that the first time at which P is emptied, $s(P)$, verifies $s(P) = s_j \leq \tau_{\tilde{X}} - 1$. Since P also belongs to Y we have $\tau_{Y \cup Z} \leq s_j$ and therefore $\tau_{\tilde{X}} \geq \tau_{Y \cup Z} + 1$.
(b) Since $P - (e_1 + e_2)/2 \in X$ and $P \in (R_2 \cup x_u)$, we can identify an index j such that $u \leq j \leq n$ and $P - (e_1 + e_2)/2 = x_j$. By using (5.42), we get that $s_j \leq \tau_{\tilde{X}} - 1$ and $s_{j-1} > s_j$. Therefore both the NE neighbours of x_j should be empty at time $s_j - 1$ (since its SW neighbours are not both empty). These NE neighbours include $P + (e_1 - e_2)/2$, which therefore would be empty at time $s_j - 1$. Since $P + (e_1 - e_2)/2 \in Y$, we have proven that $\tau_{Y \cup Z} \leq s_j - 1$. Putting above results together we conclude that $\tau_{\tilde{X}} \geq s_j + 1 \geq \tau_{Y \cup Z} + 2$.

□

We can now proceed to prove the discontinuity of the transition.

Proof of Theorem 3.4 Fix $\ell_0 = \ell_1 > 0$ and let Λ_0 be the rectangle with the shape of $\Lambda_{\ell_0, 1/12\ell_0}$ centered at the origin and Λ_1^1 and Λ_1^2 be two rectangles with the shape of $\Lambda_{\ell_1/12, \ell_1}$ centered on the line $e_1 + e_2$ at distance $\ell_0 - 1/24\ell_0$ from the origin, as shown in Fig. 10. Let also $\ell_i = 2\ell_{i-2}$ and consider two infinite sequences of the rectangular regions Λ_i^1 and Λ_i^2 with the shape of $\Lambda_{\ell_i, 1/12\ell_i}$ for i even and of $\Lambda_{1/12\ell_i, \ell_i}$ for i odd. As shown in Fig. 10, the centers c_i^j of Λ_i^j for i odd (even) lie all on the $e_1 + e_2$ ($-e_1 + e_2$) line and their distance form the origin satisfies $|c_i^j| = \ell_{i-1}/2 - \ell_i/24$.

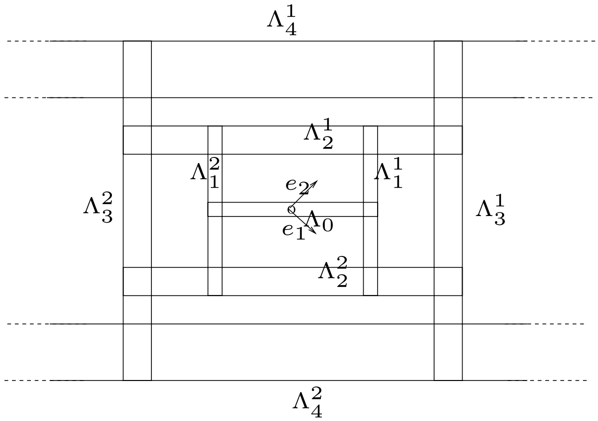

Fig. 10 We draw the first elements of the two sequences of increasing rectangles $\mathcal{R}_i^{1,2}$. The figure is rotated of 45 degrees for sake of space, the coordinate directions e_1 and e_2 are indicated

Let S_i^j for i even (for i odd) be the event that there exists an occupied *NE* (*NW*) path which connects the two short sides of Λ_i^j. Let also S_0 be the event that there exists an occupied NE path which contains the origin and connects the two short sides of Λ_0.

The origin is frozen if $\eta \in S_0 \bigcap_{i=1}^{\infty} S_i^1 \bigcap_{i=1}^{\infty} S_i^2$, namely

$$\rho_\infty(\rho) \geq \mu^\rho\left(S_0 \bigcap_{i=1}^{\infty} S_i^1 \bigcap_{i=1}^{\infty} S_i^2\right). \tag{5.43}$$

Indeed if we let A_i^j be the subset of the path spanning Λ_i^j which belongs to $\Lambda_i^j \setminus (\Lambda_{i+1}^1 \cup \Lambda_{i+1}^2)$ and τ_i be the first time at which at least one of the sites in A_i^1 or A_i^2 is emptied, by using Lemma 5.1 (and the analogous result for the structure rotated of 90 degrees) it can be easily established that: (i) $\tau_i \geq \tau_{i+1} + 1$; (ii) the origin cannot be emptied before time τ_1. For any $\tau > 0$ if we choose $i > \tau$, from (i) (and the fact that $\tau_i \geq 1$) we conclude immediately that $\tau_1 \geq i > \tau$. Therefore from (ii) the result $T^\tau \eta(0) = 1$ immediately follows. This, together with the arbitrariness of τ immediately leads to (5.43). Therefore, by using FKG inequality and Definition 2.6 we get

$$\rho_\infty(\rho) \geq \mu^\rho(\mathcal{S}_0) \prod_{i=1,\dots,\infty} \mu^\rho(\mathcal{S}_i^1)\mu^\rho(\mathcal{S}_i^2). \tag{5.44}$$

By using the same mapping of NE oriented paths to paths of oriented percolation used for Lemma 4.1 (and an analogous version in the NW direction), it is then immediate to verify that $\mu^\rho(S_i^j) = \mu^\rho(S_i)$, where S_i are the oriented percolation events defined in (3.18). Therefore the result $\rho_\infty(\rho_c) > 0$ follows from Conjecture 3.2. □

Note that the structure which we have used to block the origin does not contain any infinite cluster neither in the North-East nor in the South-West directions, therefore it could be emptied if we had chosen to block just in one direction (e.g. by taking $\mathcal{A}_x := (\eta : \eta \in (\mathcal{E}_x^{NE} \cup \mathcal{E}_x^{SW}))$ or $\mathcal{A}_x := (\eta : \eta \in (\mathcal{E}_x^{NW} \cup \mathcal{E}_x^{SE}))$). Oriented percolation corresponds to this choice of blocking just in one direction, this is why our proof of discontinuity does not apply to this case where it is indeed well known that the transition is continuous.

6 Finite Size Effects: Proof of Theorem 3.5

In this section we will prove Theorem 3.5 which provides upper and lower bounds for the scaling of the crossover length $\Xi(\rho)$ (see Definition 2.12) in terms of the correlation length of oriented percolation.

Proof of Theorem 3.5(*i*) Let Λ_{2L} and $\Lambda_{L/2}$ be two squares centered around the origin and of linear size $2L$ and $L/2$, respectively. If we recall the definitions given in Sect. 4.2 for regions R_L and $\widetilde{R}_L$, it is immediate to verify that $\Lambda_{L/2} \subset \tilde{R}_L$ and $R_L \subset \Lambda_{2L}$. This implies that if we take a configuration η in Λ_{2L} and we evolve it with occupied boundary conditions, $\Lambda_{L/2}$ is completely empty if the restriction of η to R_L is a good configuration. Thus $E(L,\rho) \geq \mu^{\rho}(G^L)$ and the upper bound for the crossover length follows immediately from Lemma 4.7 and (4.41). □

Proof of Theorem 3.5(*ii*) Let s and n be two positive integers and consider a square centered around the origin of linear size $4ns$ with two sides parallel to $e_1 + e_2$ and the others parallel to $-e_1 + e_2$, namely a region $\Lambda_{4ns,4ns}$. Inside this square we draw a renormalized lattice with sides parallel to the square sides and minimal step $4s$ as shown in Fig. 11 and, without lack of generality, we let the origin belong to this renormalized lattice. Then we draw around the origin the structure shown in Fig. 11a which is composed by the intersection of four rectangles, two of the form $\Lambda_{6s,s}$ (dashed–dotted line in Fig. 11a) and two of the form $\Lambda_{s,6s}$ (dashed line in Fig. 11a). Analogously, around the four sites n.n to the origin, we draw the structure obtained by a reflection from the one around the origin (so that for each of these sites one of the four rectangles coincides with the one of the origin). Then we continue this procedure until depicting one such structure centered on each site of the renormalized lattice in such a way that the structures on two neighbouring sites coincide always up to a reflection. This leads to the final structure depicted in Fig. 11b which contains $2N = O(2n^2)$ rectangles: half are of the form $\Lambda_{6s,s}$ and will be denoted by $\mathcal{R}_1, \ldots, \mathcal{R}_N$ with $\mathcal{R}_1$ being one of rectangles around the origin; and half are of the form $\Lambda_{s,6s}$ and will be denoted by $\mathcal{R}_{N+1}, \ldots, \mathcal{R}_{2N}$. For $1 \leq i \leq N$ ($i > N$), we let $\mathcal{S}_i$ be the event that there exists an occupied *NE* (*NW*) path which connects the two short sides of $\mathcal{R}_i$. If we evolve the dynamics with occupied boundary conditions on the square $\Lambda_{2\sqrt{2}ns}$ inscribed inside $\Lambda_{4ns,4ns}$ (square inside the dotted line in Fig. 11b), the inner square $\Lambda_{\sqrt{2}/2ns} \subset \Lambda_{2\sqrt{2}ns}$ can never be completely emptied (provided n is sufficiently large, $n > 12$, in order that $\Lambda_{\sqrt{2}/2ns}$ contains at least one rectangle R_i). In other words, recalling definition (2.11), the following holds

$$E(\sqrt{2}ns, \rho) \leq 1 - \mu^{\rho}\left(\bigcap_{i=1}^{2N} \mathcal{S}_i\right). \tag{6.45}$$

The key observation to prove 6.45 is the following. Choose a rectangle $\mathcal{R}_i$ and focus on the four perpendicular rectangles by which it is intersected. Let $\mathcal{R}_j$ and $\mathcal{R}'_j$ be the two that are more far apart (see Fig. 11b). If $i \leq N$ ($i > N$), let A_i be the subset of the NE (NW) spanning cluster for $\mathcal{R}_i$ which is contained in $\mathcal{R}_i \setminus (\mathcal{R}_j \cup \mathcal{R}'_j)$. The union over all rectangles of the sets A_i has the property that none of these sites can ever be emptied. This is an easy consequence of the result in Lemma 5.1 (and of having imposed occupied boundary conditions on $\Lambda_{2\sqrt{2}ns}$). Therefore since $\Lambda_{\sqrt{2}/2ns} \cap (\bigcup_{i=1}^{2N} A_i) \neq \emptyset$ (provided $n > 12$), (6.45) immediately follows.

Again we can use FKG inequality and conclude that

$$E(\sqrt{2}ns, \rho) \leq 1 - \mu^{\rho}(\mathcal{S}_1)^{2N}. \tag{6.46}$$

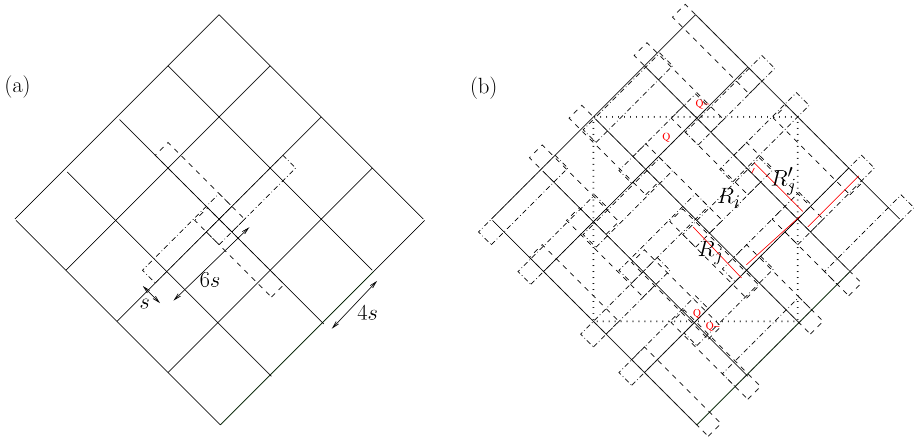

Fig. 11 **a** $\Lambda_{4ns,4ns}$ (here $n=4$) and the structure around the origin. **b** The *dashed* and *dotted–dashed rectangles* form the structure described in the text obtained by drawing around each renormalized site either the same structure as for the origin or the one obtained by a reflection symmetry. The *dotted square* corresponds to $\Lambda_{2\sqrt{2}ns}$

Note that now all rectangles have the same size, namely $\mu^\rho(\mathcal{S}_i)$ does not depend on i. The probability that there does not exist a NE cluster spanning $\mathcal{R}_1$ is clearly bounded from above by the product of the probabilities that on each of the s^{1-z} slices of the form Λ_{6s,s^z} which compose $\mathcal{R}_1$ there does not exist such a cluster when empty boundary conditions are imposed on the long sides of the slice. Therefore if we recall Conjecture 3.1 and we let $s=\xi(\rho)$, we get

$$\lim_{\rho\nearrow\rho_c} 1-\mu^\rho(\mathcal{S}_1)\le \lim_{\rho\nearrow\rho_c}(1-c^l_{OP})^{\xi^{1-z}}. \tag{6.47}$$

By using this inequality inside (6.46) we immediately get

$$\lim_{\rho\nearrow\rho_c} E(\sqrt{2}n\xi(\rho),\rho)\le \lim_{\rho\nearrow\rho_c} 1-\exp[-8(n+1)^2\exp(-a\xi^{1-z})], \tag{6.48}$$

where $a=|\log(1-c^l_{OP})|$ (and we used the fact that $(1-c^l_{OP})^{\xi^{1-z}}\to 0$ and $\log(1-x)>-2x$ for $0<x<1/2$ and $2N<2(n+1)^2$). Therefore, if we let $n(\rho)=1/4\exp(a\xi(\rho)^{1-z}/2)$ we get

$$\lim_{\rho\nearrow\rho_c} E(c_1\xi\exp(c_2\xi^{1-z}),\rho)<1/2, \tag{6.49}$$

with $c_1=1/(2\sqrt{2})$ and $c_2=|\log(1-c^l_{OP})|/2$. By recalling the definition of the crossover length (2.12) the proof is concluded. □

As discussed in the Sect. 2 another possible definition of the crossover length would have corresponded to defining $E(L,\rho)$ in (2.12) as the probability that the origin is empty in the stationary configuration which is reached after L^2 steps when we evolve the configuration with filled boundary conditions on Λ_L. In this case the previous proof would have to be modified. First one has to translate the structure in order that the origin is at the center of rectangle R_1. Then the events $\mathcal{S}_i$ for $i\ge 2$ remain unchanged but $\mathcal{S}_1$ should now be the event that there exist a structure which contains the origin similar to the one used in the previous section to prove discontinuity (see Fig. 10) and this up to the size of R_1 (i.e. with the longest rectangles Λ_i spanning R_1 in the parallel direction). Then by combining the arguments used to prove Theorem 3.4 and the above Theorem 3.5(ii) it is not difficult to see that the origin is blocked by this structure, thus leading again to inequality (6.45). Finally, to establish the upper bound (6.49) for the new $E(L,\rho)$, one needs to use a conjecture on the properties of oriented percolation which is slightly more general than Conjecture 3.2 and is again expected to be correct on the basis of finite size scaling and numerical simulations.

7 A Related Kinetically Constrained Spin Model

In this section we define a Kinetically Constrained Spin Model that we introduced in [15], which is related to the cellular automata we have considered in previous sections. Configurations η are again sets of occupation variables $\eta_x \in \{0,1\}$ for $x \in \mathbb{Z}^2$ distributed at time 0 with μ^ρ, but evolution is not deterministic. Dynamics is given by a continuous time Markov process with generator $\mathcal{L}$ acting on local functions $f:\Omega \to \mathbb{R}$ as

$$\mathcal{L}f(\eta) = \sum_{x\in\Lambda} c_x(\eta)[f(\eta^x) - f(\eta)], \tag{7.50}$$

with

$$\eta_z^x := \begin{cases} 1-\eta_x & \text{if } z = x, \\ \eta_z & \text{if } z \neq x. \end{cases} \tag{7.51}$$

The rates $c_x(\eta)$ are such that the flip in x can occur only if the configuration satisfies the same constraint that we required for the cellular automata in order to empty the same site, namely

$$c_x(\eta) := \begin{cases} 0 & \text{if } \eta_x \notin \mathcal{A}_x, \\ \rho & \text{if } \eta \in \mathcal{A}_x \text{ and } \ \eta_x = 0, \\ 1-\rho & \text{if } \eta \in \mathcal{A}_x \text{ and } \ \eta_x = 1. \end{cases} \tag{7.52}$$

It is immediate to check that the process satisfies detailed balance with respect to μ^ρ, which is therefore a stationary measure for the process. This property is the same as for the process without constraints, namely the case $c_x(\eta) = \rho(1-\eta(x)) + (1-\rho)\eta(x)$, but important differences occur due to the presence of constraints. In particular for our model μ^ρ is not the unique invariant measure. For example, since there exist configurations which are invariant under dynamics, any measure concentrated on such configurations is invariant too. A direct relation can be immediately established with the cellular automata studied in previous sections: configurations which are left invariant by the stochastic evolution are all the possible final configurations under the deterministic cellular automata evolution (since all sites in such configurations are either empty or such that if $\eta(x) = 1$ than $\eta \notin \mathcal{A}_x$). A natural issue is whether on the infinite lattice, despite the existence of several invariant measures and of blocked configurations, the long time limit of all correlation functions under the Markov process approaches those of the Bernoulli product measure for almost all initial conditions. By the spectral theorem this occurs if and only if zero is a *simple* eigenvalue of the generator of the dynamics, i.e. if $\mathcal{L}f_0(\eta) = 0$ with $f_0 \in L_2(\mu_\rho)$ implies that f_0 is constant on almost all configurations, i.e. on all except possibly a set of measure zero w.r.t. μ^ρ (see [12] and, for a specific discussion on Markov processes with kinetic constraints, see Theorem 2.3 and Proposition 2.5 of [4]).

By the result in Lemma 4.1 it is immediate to conclude that the process is for sure not ergodic for $\rho > p_c^{OP}$, since for example the characteristic function of set $\mathcal{F}_0^{NE-SW}$ is an invariant function which is not constant. On the other hand, Lemma 4.2 establishes irreducibility for $\rho < p_c^{OP}$, namely that a.s. in μ^ρ there exists $\forall x$ a path $\eta_1 = \eta, \ldots, \eta_n = \eta^x$ such that $\eta_{i+1} = \eta_i^y$ and $c_y(\eta_i) = 1$.[3] Ergodicity for $\rho < p_c^{OP}$ can then be immediately established thanks to irreducibility and the product form of Bernoulli measure (see [4] Proposition 2.5). Therefore at p_c^{OP} an ergodicity breaking transition occurs. Furthermore, by using the result of our Lemma 4.7, in [5] it has been proved that the spectral gap of this process is strictly positive at any $\rho < p_c^{OP}$, i.e. correlations decay exponentially in time to their equilibrium value.

Finally, for the connection with the physics of liquid/glass transition, let us analyze the Edwards–Anderson order parameter q which corresponds to the long time limit of the connected spin–spin correlation function,

$$q = \sum_\eta \mu^\rho(\eta) q(\eta) = \sum_\eta \mu^\rho(\eta) \lim_{t\to\infty} \lim_{|\Lambda|\to\infty} \mathbb{E}_\eta \left[\sum_{x\in\Lambda} |\Lambda|^{-1} \eta(x)\eta_t(x) - \rho^2 \right], \tag{7.53}$$

where $\mathbb{E}_\eta$ denotes the mean over the Markov process started at $\eta_0 = \eta$. Ergodicity guarantees that $q = 0$ for $\rho < p_c^{OP}$. For $\rho \geq p_c^{OP}$ one can obtain for a fixed initial configuration η the following inequality

$$q(\eta) \geq (1-\tilde{\rho})^2 \tilde{\rho}_{ob}/(1-\tilde{\rho}_{ob}),$$

where $\tilde{\rho}$ is the fraction of occupied sites in η and $\tilde{\rho}_{ob}$ is the fraction of occupied sites remained after performing on η the emptying process defined by the cellular automaton. Under the hypothesis that both these quantities have vanishing fluctuations in the large L limit with respect to the Bernoulli measure for the initial configuration, one finds that for $\rho \geq p_c^{OP}$ it holds

$$q \geq (1-\rho)^2 \frac{\rho_\infty(\rho)}{1-\rho_\infty(\rho)} \tag{7.54}$$

and therefore $q(\rho_c) > 0$.

Finally, as explained in [15], the fact that the crossover length for the cellular automata diverges faster than exponentially toward the critical density should correspond to (at least) an analogous divergence in the relaxation times for this stochastic model.

As discussed in [15], the first order/critical character of this dynamical transition is similar to the character experimentally detected for liquids/glass and more general jamming transitions. To our knowledge, this is the first example of a finite dimensional system with no quenched disorder with such a dynamical transition.

Acknowledgements We thank D.S. Fisher for a very important collaboration on this subject and R. Schonmann for very useful discussions.

[3] This follows immediately from the proof of Lemma 4.2: we have shown that for the cellular automata if we consider a sufficiently large finite lattice Λ_L around x there exists a path from η which subsequently empties all sites in Λ_L. For all the moves $\eta_i \to \eta_{i+1} = \eta_i^y$ in this path the rate c_i^y is non zero since the constraint for the cellular automata and the stochastic process are the same. We can construct analogously a path which goes form η^x to a configuration which is empty in Λ_L. Then, by connecting the two paths we get an allowed (i.e. with strictly positive rates) path from η to η^x.

References

1. Aizenman, M., Lebowitz, J.L.: Metastability effects in bootstrap percolation. J. Phys. A **21**, 3801–3813 (1988)
2. Aizenman, M., Newman, C.M.: Discontinuity of the percolation density in one-dimensional $1 = (xy)2$ percolation models. Commun. Math. Phys. **107**, 611–647 (1986)
3. Aizenman, M., Chayes, J.T., Chayes, L., Newman, C.M.: Discontinuity of the magnetization in one-dimensional $1/|x-y|2$ Ising and Potts models. J. Stat. Phys. **50**(1–2), 1–40 (1988)
4. Cancrini, N., Martinelli, F., Roberto, C., Toninelli, C.: Kinetically constrained spin models. Probab. Theory Relat. Fields (2007, in press). Preprint, math/0610106
5. Cancrini, N., Martinelli, F., Roberto, C., Toninelli, C.: Facilitated spin models: recent and new results. (2007, preprint)
6. Cerf, R., Manzo, F.: The threshold regime of finite volume bootstrap percolation. Stoch. Process. Appl. **101**, 69–82 (2002)
7. Chalupa, J., Leath, P.L., Reich, G.R.: Bootstrap percolation on a Bethe lattice. J. Phys. C **12**, L31 (1979)
8. Durrett, R.: Oriented Percolation in two dimensions. Ann. Probab. **12**, 999 (1982)
9. Bezuidenhout, C., Grimmett, G.: The critical contact process dies out. Ann. Probab. **18**, 1462–1482 (1990)
10. Fortuin, C.M., Kasteleyn, P.W., Ginibre, J.: Correlation inequalities on some partially ordered sets. Commun. Math. Phys. **22**, 89–103 (1971)
11. Hinrichsen, H.: Nonequilibrium critical phenomena and phase transitions in absorbing states. Adv. Phys. **49**, 815–965 (2000)
12. Liggett, T.M.: Interacting Particle Systems. Springer, New York (1985)
13. Pittel, B., Spencer, J., Wormald, N.: Sudden emergence of a giant k-core on a random graph. J. Comb. Theory Ser. B **67**, 111–151 (1996)
14. Schonmann, R.H.: On the behaviours of some cellular automata related to bootstrap percolation. Ann. Probab. **20**, 174–193 (1992)
15. Toninelli, C., Biroli, G., Fisher, D.S.: Jamming percolation and glass transition in lattice models. Phys. Rev. Lett. **96**, 035702 (2006)
16. Toninelli, C., Biroli, G., Fisher, D.S.: On the universality of jamming percolation: reply to the comment of Jeng and Schwarz. Phys. Rev. Lett. **98**, 129602 (2007)
17. Jeng, M., Schwarz, J.M.: Comment on jamming percolation and glass transitions in lattice models. Phys. Rev. Lett. **98**, 129601 (2007)