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# Low energy expansion of the four-particle genus-one amplitude in type II superstring theory

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Abstract: A diagrammatic expansion of coefficients in the low-momentum expansion of the genus-one four-particle amplitude in type II superstring theory is developed. This is applied to determine coefficients up to order $s^6\,\mathbf{R}^4$ (where $s$ is a Mandelstam invariant and $\mathbf{R}$ the linearized super-curvature), and partial results are obtained beyond that order. This involves integrating powers of the scalar propagator on a toroidal world-sheet, as well as integrating over the modulus of the torus. At any given order in $s$ the coefficients of these terms are given by rational numbers multiplying multiple zeta values (or Euler-Zagier sums) that, up to the order studied here, reduce to products of Riemann zeta values. We are careful to disentangle the analytic pieces from logarithmic threshold terms, which involves a discussion of the conditions imposed by unitarity. We further consider the compactification of the amplitude on a circle of radius $r$, which results in a plethora of terms that are power-behaved in $r$. These coefficients provide boundary 'data' that must be matched by any non-perturbative expression for the low-energy expansion of the four-graviton amplitude. The paper includes an appendix by Don Zagier.

Keywords: Superstrings and Heterotic Strings, String Duality.

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## Contents

# 1. Introduction

The low-momentum expansion (or $\alpha'$ expansion) of string theory scattering amplitudes produces an infinite sequence of stringy corrections to Einstein supergravity. Although the coefficients in the momentum expansion of tree-level amplitudes are easily obtained to all orders, it is far more difficult to obtain the coefficients in the expansion of higher-genus contributions. Knowledge of such terms might be of use in constraining non-perturbative extensions of string amplitudes. However, determining these coefficients is technically challenging, as we will see.

We will here be interested in the low-momentum expansion of the genus-one scattering amplitude for four particles in the massless supergravity multiplet. Each external particle is labeled by its momentum $p_r$ $(r = 1, 2, 3, 4)$, where $p_r^2 = 0$, and its superhelicity $\zeta_r$, which takes 256 values (the dimensionality of the maximal supergravity multiplet). The genus-one amplitude has the form [1],

$$\mathbf{A}^{genus-1}_{\zeta_1,\zeta_2,\zeta_3,\zeta_4} = I\,\mathbf{R}^4_{\zeta_1,\zeta_2,\zeta_3,\zeta_4}\,, \tag{1.1}$$

where $I$ is the integral of a modular function,

$$I(s,t,u) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} F(s,t,u;\tau)\,, \tag{1.2}$$

where $F(s,t,u;\tau)$ is defined in eq. (2.1) and $s$, $t$, $u$ are Mandelstam invariants[1] and $\tau = \tau_1 + i\tau_2$ and $d^2\tau \equiv d\tau_1 d\tau_2 = d\tau d\bar\tau/2$. The integral is over a fundamental domain of $SL(2,\mathbb{Z})$, defined by

$$\mathcal{F} = \{|\tau_1| \le \tfrac{1}{2}, |\tau|^2 \ge 1\}\,. \tag{1.3}$$

The kinematical factor in (1.1) is given by (see (7.4.57) of [2])

$$\mathbf{R}^4_{\zeta_1,\zeta_2,\zeta_3,\zeta_4}(p_1,p_2,p_3,p_4) = \zeta_1^{AA'}\,\zeta_2^{BB'}\,\zeta_3^{CC'}\,\zeta_4^{DD'}\,K_{ABCD}\,\tilde K_{A'B'C'D'}\,, \tag{1.4}$$

where the indices $A, B$ on the polarization tensors $\zeta_r^{AB}$ run over both vector and spinor values (for example, the graviton polarization is $\zeta^{\mu\nu}$, where $\mu,\nu = 0, 1, \ldots, 9$) and the tensor $K\,\tilde K$ is defined in [2]. In the case of external gravitons $\mathbf{R}$ reduces to the linearized Weyl curvature, $R_{\mu\rho\nu\sigma} = -4p_{[\mu}\,\zeta_{\rho][\sigma}\,p_{\nu]}$, and the kinematic factor is $R^4$, which denotes the product of four Weyl curvatures contracted into each other by a well-known sixteen-index $t_8t_8$ tensor defined in appendix 9.A of [3]. The integrand in (1.1) is given by the expectation value of the product of the vertex operators for the four external states integrated over their insertion positions on the toroidal world-sheet of complex structure $\tau$.

The expression (1.1) is free from divergences. We can anticipate it has a low-momentum expansion that is the sum of an analytic part and a non-analytic part associated with threshold singularities,

$$I(s,t,u) = I_{an}(s,t,u) + I_{nonan}(s,t,u)\,. \tag{1.5}$$

[1]The Mandelstam invariants are defined by $s = -(p_1+p_2)^2$, $t = -(p_1+p_4)^2$ and $u = -(p_1+p_3)^2$ and satisfy the mass-shell constraint $s+t+u=0$.



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As shown in [4], the analytic part can be expressed in a power series in the Mandelstam invariants,

$$I_{an}(s,t,u) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \hat\sigma_2^p \hat\sigma_3^q \, J^{(p,q)}\,, \tag{1.6}$$

where

$$\hat\sigma_2 = \left(\frac{\alpha'}{4}\right)^2 (s^2+t^2+u^2) \qquad \hat\sigma_3 = \left(\frac{\alpha'}{4}\right)^3 (s^3+t^3+u^3) = 3\left(\frac{\alpha'}{4}\right)^3 stu\,, \tag{1.7}$$

and $J^{(p,q)}$ are constant coefficients that are to be determined. The series (1.6) is the most general power series in Mandelstam invariants that is symmetric in $s$, $t$ and $u$, subject to $s+t+u=0$, as follows by making use of the identity,

$$\hat\sigma_n = \left(\frac{\alpha'}{4}\right)^n \left(s^n+t^n+u^n\right) = n \sum_{2p+3q=n} \frac{(p+q-1)!}{p!q!} \left(\frac{\hat\sigma_2}{2}\right)^p \left(\frac{\hat\sigma_3}{3}\right)^q . \tag{1.8}$$

Each term in the series is a symmetric monomial in the Mandelstam invariants of order $r = 2p+3q$. For $r<6$ there is a single kinematic structure (a single term for a given value of $r$), but there is a two-fold degeneracy for $r=6$, associated with $J^{(3,0)}$ and $J^{(0,2)}$, and thereafter the degeneracy of the kinematic factors increases sporadically.

The nonanalytic terms have branch cuts with a structure that is determined by unitarity. These singularities arise as infrared effects of internal massless states (i.e., supergravity states) in the loop. The discontinuities across these branch cuts are given by products of string tree-level amplitudes integrated over the phase space of the (massless) states. The lowest order term is the one-loop contribution of maximal supergravity, while higher-order nonanalytic terms arise as stringy effects from higher-order terms in the expansion of the tree amplitudes. As will become apparent later, the resulting structure of the nonanalytic contribution to $I(s,t,u)$ is given by a series of terms,

$$I_{nonan}(s,t,u) = I_{\rm SUGRA} + I_{nonan\ (4)} + I_{nonan\,(6)} + I_{nonan\,(7)} + \cdots\,, \tag{1.9}$$

where $I_{\rm SUGRA}$ is the integral that arises in one-loop maximal supergravity. This has a complicated set of threshold singularities, that have discontinuities in a single Mandelstam variable, as well as terms that have multiple discontinuities (as will be reviewed in detail in section 4). The nature of the singularities of $I_{nonan}$ depends on the space-time dimension. In ten dimensions the higher-order threshold terms, $I_{nonan\,(r)}$, have logarithmic branch points, schematically of the form $s^r \ln(-\alpha' s/\mu_r)$, where $\mu_r$ is a constant scale. After compactification on a circle to nine non-compact dimensions they are half-integer powers, schematically of the form $(-\alpha' s)^{k+1/2}$ with integer $k$.

One motivation for evaluating the constants $J^{(p,q)}$ is to constrain the complete, non-perturbative, expression for the low-energy expansion of the four-particle amplitude. For example in the IIB case the exact $SL(2,\mathbb{Z})$-invariant amplitude must have analytic pieces of the form

$$\sum_{p,q=0}^{\infty} \mathcal{E}_{(p,q)}(\Omega,\bar\Omega)\, e^{(2p+3q-1)\phi/2}\, \hat\sigma_2^p \hat\sigma_3^q\, \mathbf{R}^4\,, \tag{1.10}$$

where $\Omega = \Omega_1 + i\Omega_2 = C^{(0)} + ie^{-\phi}$ is the complex coupling constant that is given in terms of the Ramond-Ramond pseudoscalar, $C^{(0)}$, and the dilaton $\phi$. The explicit power of the string coupling constant $g_s = e^{\phi}$ disappears in the Einstein frame and $\mathcal{E}_{(p,q)}(\Omega,\bar\Omega)$ is a modular function that contains both perturbative terms (that are power-behaved in $\Omega_2^{-1} = g_s$ for small $g_s$) and non-perturbative (instantonic) contributions (that behave like $e^{-2\pi n/g_s}$). The genus-one terms arise in (1.10) from the piece of $\mathcal{E}_{(p,q)}$ proportional to $e^{(1-2p-3q)\phi/2}\, J^{(p,q)}$. This amplitude may be interpreted in terms of a local effective action which can be schematically written in the form

$$\alpha'^{r-1} \int d^{10}x\, e^{(2p+3q-1)\phi/2} \sqrt{-G}\, \mathcal{E}_{(p,q)}(\Omega,\bar\Omega)\, D^{2r}\mathbf{R}^4 \,, \tag{1.11}$$

In this expression the derivatives are contracted into each other and act on the curvature tensors in $\mathbf{R}^4$ in a manner that is defined by the functions of the Mandelstam invariants given in (1.10). The functions $\mathcal{E}_{(p,q)}$ are known for $(p,q) = (0,0)$, $(1,0)$, $(0,1)$, and there are various conjectures at higher orders [5]. In order to test conjectured forms of $\mathcal{E}_{(p,q)}$ it is helpful to have as much information from string perturbation theory as possible.

The aim of the present paper is to develop the expansion of the genus-one amplitude more systematically in order to explicitly evaluate the coefficients of higher momentum terms. Since the type IIA and IIB theories have identical massless four-particle scattering amplitudes up to at least genus three, the calculations in this paper apply to both types of theory. In section 2 we will begin by reviewing the procedure of [4]. This involves integrating powers of propagators on the world-sheet torus with fixed modulus $\tau = \tau_1 + i\tau_2$, followed by an integral over $\tau$.

The ten-dimensional theory is considered in section 3, where the analytic terms in the expansion of the amplitude up to order $s^6\,\mathbf{R}^4$ are evaluated. In order to analyze these terms we have to separate the nonanalytic threshold singularities in a well-defined manner. Such terms arise from the degeneration limit of the torus, in which $\tau_2 \to \infty$, so we will divide the integral over $\mathcal{F}$ into two domains: (i) $\mathcal{F}_L$, where $\tau_2 \le L$; (ii) $\mathcal{R}_L$, where $\tau_2 \ge L$, with $L \gg 1$. Integration over the truncated fundamental domain $\mathcal{F}_L$ is carried out in section 3 by making use of a theorem of harmonic analysis [6], and gives rise to the analytic terms together with an $L$-dependent piece.

The integral over $\mathcal{R}_L$, considered in section 4, generates the nonanalytic threshold behaviour as well as canceling the $L$-dependence of the $\mathcal{F}_L$ integral. The lowest-order threshold behaviour is the same as in the supergravity field theory one-loop amplitude and has the schematic form $\alpha' s \log(-\alpha' s/\mu_1) + \alpha' t \log(-\alpha' t/\mu_1) + \alpha' u \log(-\alpha' u/\mu_1)$ where $s$, $t$ and $u$ are Mandelstam invariants and $\mu_1$ is a constant scale. The precise form of this threshold term is considerably more complicated, as will be seen in detail in section 4.2.1, but it still possesses the property that the scale of the logarithms cancels (using $s+t+u=0$), so there is no ambiguity associated with the normalization of the logarithms. However, an important new feature at order $\alpha'^4 s^4\,\mathbf{R}^4$ is the occurrence of the second nonanalytic massless threshold singularity, $I_{nonan\,(4)}$, that arises via unitarity from the presence of the tree-level $\mathbf{R}^4$ interaction, which will be discussed in section 4.1. This is again a symmetric function of the Mandelstam variables and has the schematic form $\alpha'^4 s^4 \log(-\alpha' s/\mu_4) + \dots$, where $\mu_4$ is another normalization constant. Changing the value of $\mu_4$ changes the definition of the coefficient of the analytic terms of the form $s^4 + \dots$, so the precise value of $\mu_4$ has to be determined. We will also determine the terms at order $s^5\,\mathbf{R}^4$ and $s^6\,\mathbf{R}^4$, where the next threshold singularity arises. We will also discuss partial results for terms at order $s^7\,\mathbf{R}^4$, and $s^8\,\mathbf{R}^4$. These ten-dimensional results are summarized in section 4.3.

In section 5 we will study the compactification of the loop amplitude on a circle of dimensionless radius $r$ to nine dimensions. The threshold singularities are now half-integral powers of $s, t$ and $u$, and can be uniquely disentangled from the contributions analytic in $s$, $t$ and $u$, that give rise to the local effective action. The expression for the compactified amplitude depends on $r$ as well as on the nine-dimensional Mandelstam invariants. Since T-duality equates the IIA theory at radius $r$ to the IIB theory at radius $1/r$ and the four-particle genus-one amplitude of the IIA and IIB theories are equal, the compactified genus-one amplitude is invariant under $r \to 1/r$. The coefficient of a generic term of order $s^k\,\mathbf{R}^4$ in the analytic part of the compactified genus-one amplitude contains powers ranging from $r^{2k-1}$ to $r^{1-2k}$ plus exponential terms $O(\exp(-r))$. The term linear in $r$ survives the ten-dimensional limit $r \to \infty$. The infinite series of terms proportional to $r^{2k-1}\,s^k$ for $k > 1$, diverge in the ten-dimensional $r \to \infty$ limit. However, these can be resummed and thereby convert the nine-dimensional normal thresholds, which contain half-integer powers of the Mandelstam invariants, into the ten-dimensional thresholds containing factors of $\log(-\alpha'\,s)$, $\log(-\alpha'\,t)$ and $\log(-\alpha'\,u)$ [7, 8]. In addition there are terms of the form $s^k\,\log(r^2)$, which are analytic in $s$ but not in $r$. The many coefficients of the low-energy expansion of the nine-dimensional and ten-dimensional genus-one amplitude that we determine have the form of rational numbers multiplying products of Riemann zeta values. These nine-dimensional results are summarized in section 5.2.

## 2. The general structure of the genus-one four-particle amplitude

Here we review the general structure of the genus-one amplitude in ten-dimensional Minkowski space and its $\alpha'$ expansion. The dynamical factor $F$ in (1.2) is given by an integral over the positions $\nu^{(i)} = \nu_1^{(i)} + i\nu_2^{(i)}$ of the four vertex operators on the torus,

$$\begin{aligned} F(s,t,u;\tau) &= \prod_{i=1}^{3} \int_{\mathcal{T}} \frac{d^2\nu^{(i)}}{\tau_2} \left(\chi_{12}\chi_{34}\right)^{\alpha' s} \left(\chi_{14}\chi_{23}\right)^{\alpha' t} \left(\chi_{13}\chi_{24}\right)^{\alpha' u} \\ &= \int_{\mathcal{T}} \prod_{i=1}^{3} \frac{d^2\nu^{(i)}}{\tau_2}\, e^{\mathcal{D}} = \int_{\mathcal{T}} \prod_{i=1}^{3} \frac{d^2\nu^{(i)}}{\tau_2}\, \exp(\alpha' s\Delta_s + \alpha' t\Delta_t + \alpha' u\Delta_u), \end{aligned} \tag{2.1}$$

where $d^2\nu^{(i)} \equiv d\nu_1^{(i)} d\nu_2^{(i)}$, $\nu^{(4)} = \tau$, and

$$\mathcal{D} = \alpha' s\Delta_s + \alpha' t\Delta_t + \alpha' u\Delta_u, \tag{2.2}$$

with

$$\Delta_s = \log(\chi_{12}\chi_{34}), \qquad \Delta_t = \log(\chi_{14}\chi_{23}), \qquad \Delta_u = \log(\chi_{13}\chi_{24}) \tag{2.3}$$



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and $\log \chi_{ij} \equiv \log \mathcal{P}(\nu^{(i)} - \nu^{(j)}|\tau)$ where

$$\mathcal{P}(\nu|\tau) = -\frac{1}{4}\left|\frac{\theta_1(\nu|\tau)}{\theta_1'(0|\tau)}\right|^2 + \frac{\pi\nu_2^2}{2\tau_2}\,, \tag{2.4}$$

is the scalar Green function on the torus. These Green functions are integrated over the domain $\mathcal{T}$ defined by

$$\mathcal{T} = \left\{ -\frac{1}{2} \le \nu_1 < \frac{1}{2},\ 0 \le \nu_2 < \tau_2 \right\} \tag{2.5}$$

Since $I_{nonan}$ has logarithmic branch points associated with thresholds for intermediate on-shell massless states, it has singularities in $s$, $t$ and $u$ that must be extracted from the complete expression before the analytic terms can be determined. These thresholds arise from the region of moduli space in which $\tau_2 \to \infty$, which is the degeneration limit of the torus. Our procedure for separating the threshold term will be to treat the region with $\tau_2 \ge L$ separately from the region $\tau_2 \le L$, where $L \gg 1$. In other words, we write the integral over the fundamental domain as the sum of two terms,

$$I(s,t,u) = I_{\mathcal{F}_L}(L;s,t,u) + I_{\mathcal{R}_L}(L;s,t,u)\,, \tag{2.6}$$

where

$$I_{\mathcal{F}_L}(L;s,t,u) = \int_{\mathcal{F}_L} \frac{d^2\tau}{\tau_2^2}\, F(s,t,u;\tau)\,, \qquad I_{\mathcal{R}_L}(L;s,t,u) = \int_{\mathcal{R}_L} \frac{d^2\tau}{\tau_2^2}\, F(s,t,u;\tau)\,. \tag{2.7}$$

The first term on the right-hand side is integrated over $\mathcal{F}_L$, which is the fundamental domain cut off at $\tau_2 \le L$, and the second over $R_L$, which is the semi-infinite rectangular domain $\tau_2 \ge L$, $-\frac{1}{2} \le \tau_1 \le \frac{1}{2}$. Clearly, the dependence on $L$ cancels from the full amplitude. The first term contains the analytic part of the amplitude, together with an $L$-dependent piece, which is also analytic in the Mandelstam invariants,

$$I_{\mathcal{F}_L}(L;s,t,u) = I_{an}(s,t,u) + R(L;s,t,u)\,. \tag{2.8}$$

The $L$-dependence is contained in the function $R(L;s,t,u)$, which has an expansion of the form

$$R(L;s,t,u) = \sum_r (d_1^r L^{r-1} + d_2^r L^{r-3} + \cdots + d_{r/2+1}^r \log(L/\mu_r))\, s^r + \cdots\,, \tag{2.9}$$

where ... denotes terms involving $t$ and $u$. The second term on the right-hand side of (2.6) contains the nonanalytic piece of $I$, together with an $L$-dependent piece that cancels the $L$-dependence of the first term,

$$I_{\mathcal{R}_L}(L;s,t,u) = I_{nonan}(s,t,u) - R(L;s,t,u)\,. \tag{2.10}$$

The integrand of $I_{R_L}$ can be evaluated by substituting the large-$\tau_2$ approximation and will give rise to the nonanalytic pieces, whereas the integral over $\mathcal{F}_L$ contains purely analytic pieces.

The low energy expansion of the analytic part, $I_{\mathcal{F}_L}$, in (2.7) is obtained by expanding the integrand $F(\tau,\bar\tau)$ in (2.1) in powers of the scalar Feynman propagator and then integrating over the toroidal world-sheet. However, in order to treat the thresholds consistently we will separate them by dividing the integral into the two pieces given in (2.6). The resulting expansion of the analytic piece is contained in

$$\begin{aligned} I_{\mathcal{F}_L}(L;s,t,u) &= I_{an}(s,t,u) + R(L;s,t,u) \\ &= \sum_{n=0}^{\infty} \int_{\mathcal{F}_L} \frac{d^2\tau}{\tau_2^2} \int_{\mathcal{T}} \prod_{i=1}^{3} \frac{d^2\nu^{(i)}}{\tau_2} \frac{1}{n!} \mathcal{D}^n \,. \end{aligned} \tag{2.11}$$

The quantity $\mathcal{D}$ is the linear combination,

$$\begin{aligned} \mathcal{D} &= \alpha' s(\mathcal{P}(\nu^{(12)}|\tau) + \mathcal{P}(\nu^{(34)}|\tau) - \mathcal{P}(\nu^{(13)}|\tau) - \mathcal{P}(\nu^{(24)}|\tau)) \\ &\quad + \alpha' t(\mathcal{P}(\nu^{(14)}|\tau) + \mathcal{P}(\nu^{(23)}|\tau) - \mathcal{P}(\nu^{(13)}|\tau) - \mathcal{P}(\nu^{(24)}|\tau)) \,, \end{aligned} \tag{2.12}$$

where $\mathcal{P}(\nu^{(ij)}|\tau)$ is defined in (2.4) and can be written as [4]

$$\mathcal{P}(\nu|\tau) = \frac{1}{4\pi} \sum_{(m,n)\neq(0,0)} \frac{\tau_2}{|m\tau+n|^2} \exp\left[\frac{2\pi i}{\tau_2}(m\nu_1\tau_2 - (m\tau_1+n)\nu_2)\right] + C(\tau,\bar\tau)\,, \tag{2.13}$$

where $C(\tau,\bar\tau)$ cancels out of the $SL(2,\mathbb{Z})$-invariant combination in (2.12). It can also be written as

$$\mathcal{P}(\nu|\tau) \equiv \hat{\mathcal{P}}^\infty(\nu|\tau) + \tilde{P}(\nu|\tau) \tag{2.14}$$

where $\hat{\mathcal{P}}^\infty = \lim_{\tau_2\to\infty} \mathcal{P}(\nu|\tau)$ is proportional to $\tau_2$ and is given by

$$\hat{\mathcal{P}}^\infty(\nu|\tau) = \frac{\tau_2}{4\pi} \sum_{n\neq 0} \frac{e^{2i\pi n\,\hat\nu_2}}{n^2} = \frac{\pi\tau_2}{2}\left(\hat\nu_2^2 - |\hat\nu_2| + \frac{1}{6}\right)\,, \tag{2.15}$$

where $\hat\nu_2 \equiv \nu_2/\tau_2$ and the second expression is defined in the range $-\frac{1}{2} \le \hat\nu_2 \le \frac{1}{2}$ and is periodically repeated outside this range. The quantity $\tilde{P}(\nu|\tau)$ in (2.14) is given by

$$\tilde{P}(\nu|\tau) = \frac{1}{4} \sum_{\substack{m\neq 0\\ k\in\mathbb{Z}}} \frac{1}{|m|} e^{2i\pi m(k\tau_1+\nu_1)}\, e^{-2\pi\tau_2|m||k-\hat\nu_2|}\,. \tag{2.16}$$

The decomposition (2.14) will be useful for expansions at large $\tau_2$.

One way of evaluating the coefficients $J^{(0,0)}, J^{(1,0)}, J^{(0,1)}, \cdots$ is to consider the derivatives of $I_{an}$ in the small $s$, $t$ and $u$ limit, as was done in [4]. Thus, the coefficients in the power series expansion

$$I_{an}(s,t,u) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} I_{an}^{(m,n)} \left(\frac{\alpha'}{4}\right)^{m+n} \frac{s^m t^n}{m!n!}\,, \tag{2.17}$$

are given by

$$I_{an}^{(m,n)} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} f_{an}^{(m,n)}(\tau,\bar\tau)\,, \tag{2.18}$$

where

$$f_{an}^{(m,n)}(\tau,\bar\tau) = \frac{1}{m!n!}\,\partial_s^m\partial_t^n\,F(s,t,u;\tau)\Big|_{s=t=u=0}$$
$$= \int_{\mathcal{T}}\prod_{i=1}^{3}\frac{d^2\nu^{(i)}}{\tau_2}\,(4\Delta_s-4\Delta_u)^m\,(4\Delta_t-4\Delta_u)^n \tag{2.19}$$

with $\Delta_s$, $\Delta_t$ and $\Delta_u$ are defined in (2.3).

Defining the quantity $j^{(p,q)}(\tau,\bar\tau)$ to be the integrand of $J_{\mathcal{F}_L}^{(p,q)}$, so that

$$J^{(p,q)} = \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\,j^{(p,q)}(\tau,\bar\tau)\,, \tag{2.20}$$

and comparing (2.17) with (1.6) leads to the implicit relation between the integrands $f_{an}^{(m,n)}(\tau,\bar\tau)$ and $j^{(p,q)}(\tau,\bar\tau)$,

$$\sum_{p,q=0}^{\infty}\hat\sigma_2^p\hat\sigma_3^q\,j^{(p,q)} = \sum_{m,n=0}^{\infty}\left(\frac{\alpha'}{4}\right)^{m+n}\frac{s^m t^n}{m!n!}\,f_{an}^{(m,n)}\,. \tag{2.21}$$

We note further that (1.6) implies

$$J^{(0,0)} = I_{an}^{(0,0)}\,,\qquad J^{(1,0)} = \frac{1}{4}\,I_{an}^{(2,0)}\,,\qquad J^{(0,1)} = -\frac{1}{6}\,I_{an}^{(1,2)}\,,\qquad J^{(2,0)} = \frac{1}{96}\,I_{an}^{(4,0)}\,,$$
$$J^{(1,1)} = -\frac{1}{144}\,I_{an}^{(4,1)}\,,\qquad J^{(3,0)} = \frac{1}{2880}\,I_{an}^{(5,1)}\,,\qquad J^{(0,2)} = \frac{1}{5760}\,I_{an}^{(6,0)} \tag{2.22}$$

## 3. The analytic terms in the ten-dimensional genus-one amplitude

In this section we will formulate the diagrammatic expansion for determining the coefficients, $J^{(p,q)}$, of the higher derivative terms starting from (2.17). This will then be used to determine the coefficients of terms up to $\alpha'^6\,s^6\,\mathbf{R}^4$. These coefficients are obtained by integrating the $j^{(p,q)}$'s, which are linear combinations of the $f_{an}^{(m,n)}$'s that have $m+n=2p+3q$, and are defined in (2.19). In other words, the coefficients are given as integrals of products of powers of propagators between different vertex operators attached to the torus.

A term at order $\alpha'^r$ is represented by a sum of 'Feynman' diagrams with a total of $r$ propagators joining pairs of vertices. Any propagator can join any of six distinct pairs of vertices separated by $\nu^{(ij)} = \nu^{(i)} - \nu^{(j)}$ $(i,j=1,2,3,4)$. A diagram is therefore labeled by a set of six numbers,

$$\{\ell\} = \{\ell_1,\ell_2,\ldots,\ell_6\}\,,\qquad \sum_{k=1}^{6} l_k = r\,, \tag{3.1}$$

which define the number of propagators that join each pair of vertices. The labeling of the diagram is indicated in the figures in appendix B. Each diagram is associated with a scalar function $D_{\{\ell\}}(\tau,\bar\tau)$ that is determined by integrating the positions, $\nu^{(i)}$, using the representation (2.13) of the propagator, which is periodic in both $\nu_1^{(i)}$ and $\nu_2^{(i)}$, leading to conservation of the torus momentum, $\mathbf{p} = m + n\tau \in \mathbb{Z}+\tau\mathbb{Z}$, at each vertex. The result is a function that depends on the topology of the diagram — i.e. on the $\ell_k$'s — and is given at order $\alpha'^r$ by

$$\mathrm{D}_{\{\ell\}}(\tau,\bar\tau) = \frac{\tau_2^r}{(4\pi)^r} \sum_{\mathbf{p}_1,\ldots,\mathbf{p}_r\in\mathbb{Z}+\tau\mathbb{Z}} \frac{\prod_{i=1}^4 \delta(\sum_{j\in I_i}\mathbf{p}_j)}{|\mathbf{p}_1|^2\cdots|\mathbf{p}_r|^2}\,, \tag{3.2}$$

where the topology of the diagram is subsumed in the values of the sets $I_i$ with $i = 1, 2, 3, 4$. The momentum conservation $\delta$-function is understood to mean

$$\delta(\mathbf{p}_1+\mathbf{p}_2) = \delta(m_1+m_2)\,\delta(n_1+n_2)\,. \tag{3.3}$$

This condition eliminates all one-particle reducible diagrams. In appendix B we will derive some detailed properties of the $\mathrm{D}_{\{\ell\}}$'s that contribute to the expansion up to order $s^8\,\mathbf{R}^4$. The net result is that $j^{(p,q)}(\tau,\bar\tau)$ is a linear combination,

$$j^{(p,q)}(\tau,\bar\tau) = \sum_{\{\ell\}} e^{(p,q)}_{\{\ell\}}\,\mathrm{D}_{\{\ell\}}(\tau,\bar\tau)\,, \tag{3.4}$$

where $e^{(p,q)}_{\{\ell\}}$ is a set of constant coefficients and the sum is over all diagrams with $\sum_{k=1}^6 l_k = r$.

In order to evaluate the integral of $j^{(p,q)}(\tau,\bar\tau)$ in (2.20) we would like to make use of a theorem reviewed in [6] and restated in appendix A.2. This theorem states that any function that is square integrable on $\mathcal{F}$ is the sum of three terms: (i) a function whose zero mode with respect to $\tau_1$ vanishes; (ii) a constant; (iii) a linear combination of incomplete theta series'. However, $j^{(p,q)}(\tau,\bar\tau)$ is not square integrable as it stands, since it is easy to see that it has a large-$\tau_2$ expansion of the form

$$j^{(p,q)} = a^0_{(p,q)}\,\tau_2^{2p+3q} + a^1_{(p,q)}\,\tau_2^{2p+3q-1} + \cdots + a^{4p+6q-1}_{(p,q)}\,\tau_2^{1-2p-3q} + O(\exp(-\tau_2))\,, \tag{3.5}$$

where $a^{2p+3q}_{(p,q)} \equiv J^{(p,q)}$. We can, however, proceed by subtracting the positive powers of $\tau_2$ in a manner consistent with modular invariance. This can be achieved by subtracting a suitably chosen quadratic form in nonholomorphic Eisenstein series,[2]

$$P^{(p,q)}(\{\hat E_r\}) = \sum_{s,s'} d^{(p,q)}_{ss'}\,\hat E_s\,\hat E_{s'}\,, \tag{3.6}$$

where $d^{(p,q)}_{ss'}$ are constant coefficients and

$$\hat E_s = \frac{1}{(4\pi)^s}\sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m+n\tau|^{2s}} = \frac{2\zeta(2s)\Gamma(s)}{\pi^{s/2}}\,\tau_2^s + \frac{2\zeta(2s-1)\Gamma(s-\frac12)}{\pi^{s-1/2}}\,\tau_2^{1-s} + O(e^{-2\pi\tau_2}) \tag{3.7}$$

(some properties of nonholomorphic Eisenstein series are reviewed in appendix A.1). In other words we choose the polynomial, $P^{(p,q)}$, so that it reproduces the terms with positive powers of $\tau_2$ in (3.5), so that

$$j^{(p,q)} = P^{(p,q)}(\{\hat E_r\}) + J^{(p,q)} + \delta j^{(p,q)}\,, \tag{3.8}$$

[2]This quadratic form may not be unique but any ambiguity is irrelevant in the following.

where $J^{(p,q)}$ is a constant and $\delta j^{(p,q)} \to 0$ as $\tau_2 \to \infty$. Since $\delta j^{(p,q)}$ is square integrable in the fundamental domain $\mathcal{F}$, the theorem applies to it, and since we also have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1\, \delta j^{(p,q)} = \frac{a_{(p,q)}^{2p+3q+1}}{\tau_2} + \cdots + a_{(p,q)}^{4p+6q-1}\, \tau_2^{1-2p-3q} + O(\exp(-\tau_2)) \neq 0\,, \tag{3.9}$$

the theorem implies that

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\, \delta j^{(p,q)} = 0\,. \tag{3.10}$$

Therefore, after cutting off the fundamental domain at large $\tau_2 = L$ we have

$$J^{(p,q)}_{\mathcal{F}_L} = \int_{\mathcal{F}_L} \frac{d^2\tau}{\tau_2^2}\, j^{(p,q)} = \frac{\pi}{3}\, J^{(p,q)} + \int_{\mathcal{F}_L} \frac{d^2\tau}{\tau_2^2}\, P^{(p,q)}(\{\hat{E}_2\}) + O(1/L) \tag{3.11}$$

(where we have used the fact that $\int_{\mathcal{F}} d^2\tau/\tau_2^2 = \pi/3$).

The function $P^{(p,q)}$ and constants $J^{(p,q)}$ that arise up to order order $s^6\, \mathbf{R}^4$ are given in (C.2)–(C.6) in appendix C. The integral of $P^{(p,q)}(\{E_r\})$ over the fundamental domain can be reduced to a boundary integral by using $\Delta \hat{E}_s = s(s-1)\, \hat{E}_s$ and integrating by parts, as is reviewed in appendix A.1. This leads to the following results.

- At order $\mathbf{R}^4$. In this case $j^{(0,0)} = \mathrm{D}_0 = 1$, so $J^{(0,0)} = \pi/3$, as it is given by the volume of a fundamental domain for $SL(2,\mathbb{Z})$.
- At order $\alpha'^2\, s^2\, \mathbf{R}^4$ it is easy to see that $j^{(1,0)} = \mathrm{D}_2 = \hat{E}_2$ (as was found in [4]). Substituting $\Delta \hat{E}_2 = 2\, \hat{E}_2$ for $P^{(1,0)}$ in the right-hand side of (3.11) gives a purely boundary contribution proportional to $L$

  $$J^{(1,0)}_{\mathcal{F}_L} = L\,. \tag{3.12}$$

  This $L$ dependence is canceled by the contribution from the region $\tau_2 \geq L$, so the result is

  $$J^{(1,0)} = 0\,. \tag{3.13}$$

- At order $s^3\, \mathbf{R}^4$, using (C.2) results in

  $$J^{(0,1)}_{\mathcal{F}_L} = \frac{\pi}{3} \frac{\zeta(3)}{3} + \frac{\pi^3}{567}\, L^2 + O(1/L)\,, \tag{3.14}$$

  which reproduces the result in [4]. The $L^2$ dependence is canceled by the contribution of the amplitude from the region $\tau_2 \geq L$, which was given in (3.24) of [4] and has the form

  $$\lim_{s,t\to 0} I_{nonan\,(3);\mathcal{R}_L}(s,t) = -\frac{\pi^3}{567}\, L^2\, \hat{\sigma}_3 + O(1/L)\,. \tag{3.15}$$

  At this order there are no threshold contributions so $I_{nonan\,(3)} = 0$, and the coefficient multiplying $s^3\, \mathbf{R}^4$ in the ten-dimensional effective action is

  $$J^{(0,1)} = \frac{\pi}{3} \frac{\zeta(3)}{3}\,. \tag{3.16}$$

- At order $\alpha'^4 s^4 \mathbf{R}^4$, using (C.3) and (A.10) we find

$$J^{(2,0)}_{\mathcal{F}_L} = \frac{4\pi^4}{42525} L^3 + \frac{2\pi}{45} \zeta(3) \log(L/\tilde\mu_4) + O(1/L) , \qquad (3.17)$$

where the scale $\tilde\mu_4$ is given by

$$\log \tilde\mu_4 = -\frac{1}{2} + \log(2) - \frac{\zeta'(3)}{\zeta(3)} + \frac{\zeta'(4)}{\zeta(4)} . \qquad (3.18)$$

The occurrence of the $\log L$ term originates from the presence of $\hat E_2^2$ in the expression for $j^{(2,0)}$ in (C.3), together with the use of the integral (A.8). In section 4.2.2 the $L^3$ and $\log(L)$ contributions will be shown to cancel with the contribution from $\tau_2 \geq L$. In particular, we will see that

$$\lim_{s,t\to 0} I_{nonan\,(4);\mathcal{R}_L}(s,t) = I_{nonan\,(4)}(s,t) - \left(\frac{4\pi^4}{42525} L^3 + \frac{2\pi}{45} \zeta(3) \log(L/\tilde\mu_4)\right) \hat\sigma_2^2 , \qquad (3.19)$$

where

$$I_{nonan\,(4)}(s,t) = -\frac{4\zeta(3)}{45\pi} \alpha'^4 s^4 \log(-\alpha' s/\mu_4) + perms . \qquad (3.20)$$

We will derive the coefficient of this term (which was also derived from the unitarity argument in the last subsection), as well as the value of $\mu_4$, in section 4.2.2. Since there is no constant $L$-independent term in (3.17) (apart from the $\log\tilde\mu_4$ associated with the $\log L$) we find

$$J^{(2,0)} = 0 . \qquad (3.21)$$

- At order $\alpha'^5 s^5 \mathbf{R}^4$, using (C.4), we find

$$J^{(1,1)}_{\mathcal{F}_L} = \frac{\pi}{3} \frac{97}{1080} \zeta(5) + \frac{L^4 \pi^5}{400950} + \frac{L \pi^2 \zeta(3)}{378} + O(1/L) . \qquad (3.22)$$

The power-behaved $L^4$ and $L$ contributions will again be seen to cancel the contribution from the $\tau_2 \geq L$ region,

$$\lim_{s,t\to 0} I_{nonan\,(5);\mathcal{R}_L}(s,t) = -\left(\frac{L^4 \pi^5}{400950} - \frac{L\pi^2 \zeta(3)}{378} + O(1/L)\right) \hat\sigma_2 \hat\sigma_3 , \qquad (3.23)$$

but there is no $\hat\sigma_2\hat\sigma_3 \log L$ term. This is in accord with the earlier unitarity argument, which shows that there is no threshold at the order $s^5$ so $I_{nonan\,(5)}(s,t) = 0$. The genus-one contribution to the ten-dimensional action at this order is given by the constant term in (3.22),

$$J^{(1,1)} = \frac{\pi}{3} \frac{97}{1080} \zeta(5) . \qquad (3.24)$$

- At order $\alpha'^6 s^6 \mathbf{R}^4$ there are two independent tensorial structures. For $\hat\sigma_2^3 \mathbf{R}^4$, using (C.5) and (A.11), we find

$$\begin{aligned} J^{(3,0)}_{\mathcal{F}_L} =& \frac{\pi}{3} \frac{\zeta(3)^2}{30} \\ &+ \frac{2\pi^6}{4729725} L^5 + \frac{2\pi^3 \zeta(3)}{4725} L^2 + \frac{11\pi}{630} \log(L/\tilde\mu_6) \zeta(5) + O(1/L) . \end{aligned} \qquad (3.25)$$

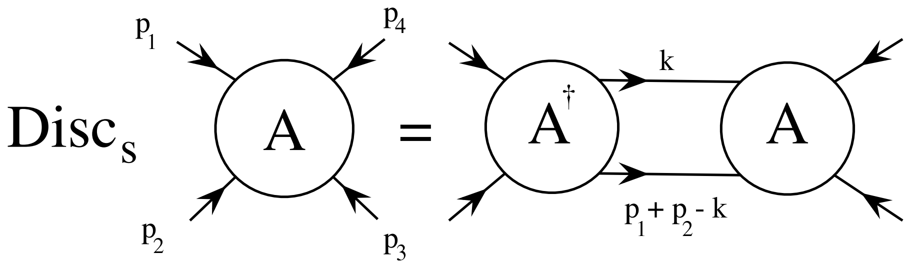


**Figure 1:** Unitarity equation relating the two-particle $s$-channel discontinuity of the amplitude to the square of the four-particle amplitude, integrated over the phase space of the intermediate particles and summed over the species of these particles.

For $\hat{\sigma}_3^2\,\mathbf{R}^4$, using (C.6) and (A.11), we find

$$\begin{aligned} J^{(0,2)}_{\mathcal{F}_L} =& \frac{\pi}{3}\,\frac{61\zeta(3)^2}{1080} \\ &+ \frac{1744\,\pi^6}{3192564375}\,L^5 + \frac{8\,\pi^3\,\zeta(3)}{14175}\,L^2 + \frac{\pi}{45}\,\log(L/\tilde{\mu}_6)\,\zeta(5) + O(1/L)\,. \end{aligned} \tag{3.26}$$

The scale $\tilde{\mu}_6$ in both of these equations is given by

$$\log\tilde{\mu}_6 = -\frac{7}{12} + \log(2) - \frac{\zeta'(5)}{\zeta(5)} + \frac{\zeta'(6)}{\zeta(6)}\,. \tag{3.27}$$

In this case the $\log L$ terms originate from $\hat{E}_3^2$ terms in $j^{(3,0)}$ and $j^{(0,2)}$, together with the integral in (A.8). The $L$-dependent terms in these equations will again be seen to cancel with contributions from the $\tau_2 \geq L$ part of the integral, resulting in a net nonanalytic term of the form $h^{(6)}(s,t,u)\,\log(-\alpha'\,s/\mu_6) + perms$, where $h^{(6)}$ is a monomial in $s$, $t$ and $u$ of order six (defined by (4.13). We see from (3.25) that the coefficient of the analytic $\hat{\sigma}_2^3\,\mathbf{R}^4$ term is

$$J^{(3,0)} = \frac{\pi}{3}\,\frac{\zeta(3)^2}{30}\,, \tag{3.28}$$

while from (3.26) we see that the coefficient of $\hat{\sigma}_3^2\,\mathbf{R}^4$ is

$$J^{(0,2)} = \frac{\pi}{3}\,\frac{61\zeta(3)^2}{1080}\,. \tag{3.29}$$

## 4. Ten-dimensional threshold terms

We now turn to discuss the nonanalytic terms in the low-energy expansion of the ten-dimensional amplitude. These are characterized by branch cuts with imaginary parts that are determined by unitarity in a standard manner that we will review in the following subsection. The scales of the logarithms are more difficult to evaluate and will require a direct calculation of the genus-one integral in the region $\tau_2 \geq L$ section 4.2.

### 4.1 Two-particle unitarity

Unitarity identifies the discontinuity of the amplitude across the $s$-channel two-particle cut (when $s \geq 0$, $t \leq 0$) with the product two amplitudes integrated over the phase space for the intermediate massless two-particle states and summed over the superhelicities of these states (as illustrated in figure 1).

Our normalisations for the string S-matrix are such that the tree-level and genus-one terms in the amplitude enter in the combination[3]

$$\mathbf{A} = \kappa_{(10)}^2\, g_s^4 \left( \frac{1}{g_s^2} \mathbf{A}^{\mathrm{tree}} + 2\pi \mathbf{A}^{genus-1} + O(g_s^2) \right) , \tag{4.1}$$

where $\kappa_{(10)}^2 = 2^6 \pi^7 \alpha'^4$ and $\mathbf{A}^{\mathrm{tree}}$ has the form shown in (4.7) and $\mathbf{A}^{genus-1} = \mathbf{A}^{genus-1}_{an} + \mathbf{A}^{genus-1}_{nonan}$. At lowest order in the string coupling constant the unitarity relation has the form

$$\begin{aligned}\mathrm{Disc_s}\, \mathbf{A}^{genus-1}_{\zeta_1,\zeta_2,\zeta_3,\zeta_4}(p_1,p_2,p_3,p_4) = &- i\, \frac{\kappa_{(10)}^2}{\alpha'}\, \frac{\pi}{2} \int \frac{d^{10}k}{(2\pi)^{10}}\, \delta^{(+)}(k^2)\, \delta^{(+)}((q-k)^2) \\ &\sum_{\{\zeta_r,\zeta_s\}} \mathbf{A}^{\mathrm{tree}}_{\zeta_1\zeta_2\zeta_r\zeta_s}(p_1,p_2,-k,k-q)\, \mathbf{A}^{\mathrm{tree}}_{\zeta_3\zeta_4\zeta_r\zeta_s}(p_3,p_4,k,q-1k)\,,\end{aligned} \tag{4.2}$$

where $\sum_{\{\zeta_r,\zeta_s\}}$ denotes the sum over all the two-particle massless $N = 8$ supergravity states, and $\delta^{(+)}(p^2) \equiv \delta^{(10)}(p^2)\theta(p^0)$ imposes the mass-shell condition on each intermediate massless state,

$$k^2 = 0\,, \qquad (q-k)^2 = 0\,. \tag{4.3}$$

Expanding both sides of (4.2) in powers of $\alpha'$ determines the discontinuity of the genus-one amplitude in terms of the square of the terms in the tree-level expansion.

The discontinuity of the genus-one amplitude is obtained by substituting the tree-level scattering amplitude into (4.2). Recall that this amplitude has the form [2]

$$\mathbf{A}^{\mathrm{tree}}_{\zeta_1,\zeta_2,\zeta_3,\zeta_4}(p_1,p_2,p_3,p_4) = C(s,t,u)\, \mathbf{R}^4_{\zeta_1,\zeta_2,\zeta_3,\zeta_4}(p_1,p_2,p_3,p_4)\,, \tag{4.4}$$

where $s,t,u$ are Mandelstam invariants, satisfying the mass-shell condition $s+t+u=0$, and

$$C(s,t,u) = -\frac{\Gamma\left(-\frac{\alpha' s}{4}\right)\Gamma\left(-\frac{\alpha' t}{4}\right)\Gamma\left(-\frac{\alpha' u}{4}\right)}{\Gamma\left(1+\frac{\alpha' s}{4}\right)\Gamma\left(1+\frac{\alpha' t}{4}\right)\Gamma\left(1+\frac{\alpha' u}{4}\right)}\,, \tag{4.5}$$

The unitarity relation takes a very special form in maximal supergravity (as it does in maximal Yang-Mills), because of the self-replicating relation derived in [9],

$$\sum_{\{\zeta_r,\zeta_s\}} \mathbf{R}^4_{\zeta_1\zeta_2\zeta_r\zeta_s}(p_1,p_2,k-q,-k)\, \mathbf{R}^4_{\zeta_3\zeta_4\zeta_r\zeta_s}(k,q-k,p_3,p_4) = s^4\, \mathbf{R}^4_{\zeta_1,\zeta_2,\zeta_3,\zeta_4}(p_1,p_2,p_3,p_4) \tag{4.6}$$

[3] In the $S$-matrix there is a power is $g_s$ for each external state. Here we are considering the four point amplitude, leading to an overall factor of $g_s^4$.

$(q = p_1 + p_2)$, which simplifies the left-hand side of (4.2) drastically.

The low-momentum expansion of (4.2) follows by substituting the expansion of $\mathbf{A}^{\rm tree}$ in the right-hand side (see, for example, [8]),

$$\begin{aligned}
\mathbf{A}^{\rm tree} &= \frac{3}{\hat\sigma_3}\,\exp\left(-\sum_{n=1}^{\infty}\frac{2\zeta(2n+1)}{2n+1}\,\hat\sigma_{2n+1}\right)\mathbf{R}^4 \\
&= \left(\frac{3}{\hat\sigma_3} + 2\zeta(3) + \zeta(5)\hat\sigma_2 + \frac{2}{3}\zeta(3)^2\hat\sigma_3 + \frac{1}{2}\zeta(7)\hat\sigma_2^2 + \frac{2}{3}\zeta(3)\zeta(5)\hat\sigma_2\hat\sigma_3 \right. \\
&\qquad \left. + \frac{1}{4}\zeta(9)\hat\sigma_2^3 + \frac{2}{27}\left(2\zeta(3)^3 + \zeta(9)\right)\hat\sigma_3^2 + \cdots\right)\mathbf{R}^4\,.
\end{aligned} \tag{4.7}$$

We see that the left-hand side of (4.6) is identified with the lowest order term in the integrand on the right-hand side of (4.2) since $\mathbf{A}^{\rm tree} \sim \mathbf{R}^4/stu$. This means that, to lowest order in $\alpha'$, ${\rm Disc_s}\,\mathbf{A}^{genus-1}$ is given by

$$ {\rm Disc_s}\mathbf{A}^{genus-1}(p_1,p_2,p_3,p_4) = -is^2\,\frac{\kappa_{(10)}^2}{\alpha'}\,\frac{\pi}{2}\mathbf{R}^4\int\frac{d^{10}k}{(2\pi)^{10}}\,\frac{\delta^{(+)}(k^2)\delta^{(+)}((q-k)^2)}{(p_1-k)^2(p_4+k)^2(p_2-k)^2(p_3+k)^2}\,, \tag{4.8}$$

where we have used the expressions for the Mandelstam invariants of the tree amplitudes on either side of the intermediate states,

$$ t' = -(p_1-k)^2\,, \qquad u' = -(p_2-k)^2\,, \qquad t'' = -(p_4+k)^2\,, \qquad u'' = -(p_3+k)^2\,. \tag{4.9}$$

This reproduces the $s$-channel discontinuity of the massless box diagram, which is of order $s$. In section 4.2.1 we will find the complete expression for the genus-one contribution, which is the supergravity contribution, $\mathbf{A}_{\rm SUGRA}$.

The next term in the $\alpha'$ expansion is obtained by substituting $2\zeta(3)\,\alpha'^3\,\mathbf{R}^4$ in one of the factors of $\mathbf{A}^{\rm tree}$ on the right-hand side of the unitarity relation and the lowest-order term in the other. This amounts to multiplying the right-hand side of (4.6) by $2\zeta(3)\,\alpha'^3\,st'u'$ and so the expression for the discontinuity at this order is obtained by multiplying the integrand of (4.8) by the same factor. The integral is proportional to $\zeta(3)\,\alpha'\,s\,S(0,0,1,1)$ that is evaluated in appendix E, giving

$$ {\rm Disc_s}\mathbf{A}^{genus-1}_{(4)}(p_1,p_2,p_3,p_4) = -2i\pi\,\frac{-4\pi\zeta(3)}{45}\,\left(\frac{\alpha'\,s}{4}\right)^4\,\mathbf{R}^4\,, \tag{4.10}$$

with similar expressions for the $t$-channel and $u$-channel discontinuities. This implies that $\mathbf{A}_{(4)}$ has the form

$$ \mathbf{A}^{genus-1}_{(4)}(p_1,p_2,p_3,p_4) = -\frac{4\pi\zeta(3)}{45}\left(\frac{\alpha'}{4}\right)^4\left(s^4\log\left(-\frac{\alpha' s}{\mu_4}\right) + t^4\log\left(-\frac{\alpha' t}{\mu_4}\right) + u^4\log\left(-\frac{\alpha' u}{\mu_4}\right)\right)\mathbf{R}^4, \tag{4.11}$$

where the scale, $\mu_4$, inside the logarithm is yet not determined — this must await the more detailed analysis of the amplitude in section 4.2.2.

There are no contributions to the discontinuity at order $(\alpha'\,s)^5$ since there are no terms in $\mathbf{A}^{\rm tree}$ of order $\alpha'\,s\,\mathbf{R}^4$. The next contribution is a discontinuity of order $\alpha'^6\,s^6\,\mathbf{R}^4$, obtained by expanding one of the $\mathbf{A}^{\rm tree}$ factors on the right-hand side of (4.2) to the next non-trivial order, which means substituting the $\zeta(5)\,\hat\sigma_2'\mathbf{R}^4$ term of (4.7) into (4.2). The resulting integral is proportional to $\zeta(5)\,(2\alpha'\,s\,S(0,0,3,1)+(\alpha'\,s)^3\,S(0,0,1,1))$ that is also evaluated in appendix E, giving

$$\mathrm{Disc_s}\mathbf{A}^{genus-1}_{(6)}(p_1,p_2,p_3,p_4)=-2i\pi\,\frac{-\pi\,\zeta(5)}{2520}\left(\frac{\alpha'}{4}\right)^6\,(87\,s^6+s^4\,(t-u)^2)\,\mathbf{R}^4\,, \qquad (4.12)$$

which is attributed to a function of the form

$$\mathbf{A}^{genus-1}_{(6)}(p_1,p_2,p_3,p_4)=-\frac{\pi\,\zeta(5)}{2520}\left(\frac{\alpha'}{4}\right)^6\left((87s^6+s^4(t-u)^2)\log\left(-\frac{\alpha' s}{\mu_6}\right)\right.$$
$$\left.+(87t^6+t^4(s-u)^2)\log\left(-\frac{\alpha' t}{\mu_6}\right)+(87u^6+u^4(s-t)^2)\log\left(-\frac{\alpha' u}{\mu_6}\right)\right)\mathbf{R}^4, \qquad (4.13)$$

where $t$-channel and $u$-channel contributions have again be added. The scale $\mu_6$ can again, in principle, be determined by explicit evaluation of the loop amplitude as in section 4.2.2 (although in this case we will not complete the evaluation).

At order $\alpha'^7\,s^7\,\log(-\alpha'\,s)$, and beyond, contributions of a new type arise. These come from the presence of higher-order terms in both factors of $\mathbf{A}^{\rm tree}$ in the unitarity equation. They correspond to terms in which there are stringy corrections to both propagators in the $t'$ and $t''$ channels. For example, the total $(\alpha'\,s)^7$ contribution is proportional to the sum, $\zeta(3)^2\,(\alpha'\,s)^2\,(2S(0,0,2,2)+S(1,1,1,1))$ in the notation of appendix E.

For future reference it is interesting to note that the overall coefficient of $\log\mu_4$ in (E.8) is

$$-\frac{\pi\zeta(3)}{45}\,\hat\sigma_2^2\,, \qquad (4.14)$$

while the overall coefficient of $\log\mu_6$ in (E.9) is

$$-\frac{\pi\zeta(5)}{630}\,(11\hat\sigma_2^3+14\,\hat\sigma_3^2)\,\log\mu_6\,. \qquad (4.15)$$

These are the same as the coefficients of the $r\,s^4\,\log(r^2)$, $s^6\,\log(r^2)$ and $s^6\,\log(r^2)$ terms that arise in the compactification on a circle of radius $r$ to be considered in section 5 and summarized in (5.27). The necessity for such a cancelation is discussed following (5.28).

## 4.2 Low-momentum expansion of threshold terms

The constant scales inside the logarithms are not determined by unitarity, but by a direct calculation of the amplitude in the large-$\tau_2$ region, which we turn to next.

The singularities due to massless two-particle thresholds arise at genus one from the degeneration of the torus in the limit $\tau_2\to\infty$. As indicated in figure 2(a), the zeroth order contribution is simply the type II supergravity box diagram with massless states in all four internal lines, where the threshold behaviour of the S-matrix has the form $a_1\,\alpha'\,s\,\log(-\alpha'\,s/\mu_1)\,\mathbf{R}^4$. The constant $\mu_1$ cancels out using $s+t+u=0$. Figure 2(b) shows the discontinuity receives higher derivative corrections on one side of the cut, due to

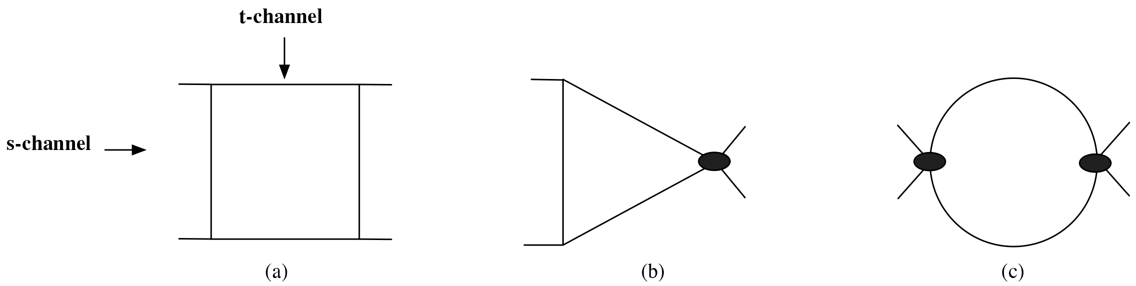


**Figure 2:** Degeneration limits of the genus-one string amplitude. (a) The limit in which all internal legs are massless propagators. This is the limit that gives the one-loop ten-dimensional supergravity contribution. (b) The triangle limit, in which there are three internal massless propagators. (c) The bubble limit in which two massless propagators are picked out. The blobs in (a) and (b) represent higher derivative contact interactions induced by stringy corrections.



$\mathbf{R}^4$, $s^3\,\mathbf{R}^4$, and higher-order contact interactions indicated by the blob. These correspond to threshold terms of the form $\alpha'^4\, s^4\, \log(-\alpha'\, s/\mu_4)$, $\alpha'^6\, s^6\, \log(-\alpha'\, s/\mu_6)$, $\alpha'^7\, s^7\, \log(-\alpha'\, s/\mu_7)$, $\dots$. There are also corrections with higher-order contributions on both sides of the cut, as seen in figure 2(c). These lead to higher-order threshold terms.

An important and well-known technical problem in the analysis of these threshold contributions is that the integral representation (1.2) is not well defined for any values of the Mandelstam invariants since the branch points in the $s$, $t$ and $u$ channels coincide. This is remedied by splitting the integration range of the vertex operator insertion points into three regions $\mathcal{T}_{st}$, $\mathcal{T}_{us}$ and $\mathcal{T}_{tu}$, that generate cuts in the $(s,t)$, the $(s,u)$ and $(t,u)$ channels, respectively. The integrand in the $\mathcal{T}_{st}$ region is real for $s,t<0$ (so that $u>0$) and may be defined in the physical $s$-channel region ($s>0$, $t<0$) by analytic continuation. Similarly, in the $\mathcal{T}_{us}$ region the integrand is defined for $s,u<0$, while in the $\mathcal{T}_{tu}$ region it is defined for $t,u<0$. Further discussion of the integral representation of the genus-one expression may be found in [10].

### 4.2.1 The massless supergravity amplitude

In the limit $\tau_2\to\infty$ the leading contribution in the $\mathcal{T}_{st}$ region is given by the integral [8]

$$I_{\mathcal{T}_{st}} = \int_L^\infty \frac{d\tau_2}{\tau_2^2}\int_{\mathcal{T}_{st}} \prod_{i=1}^3 d\omega_i\, e^{\alpha'\pi\tau_2\, Q(s,t)}\,, \tag{4.16}$$

where $\mathcal{T}_{st}=\{0\le\omega_1\le\omega_2\le\omega_3\le 1\}$, with $\omega_i=\nu_2^{(i)}/\tau_2$. The expression for $Q(s,t)$

$$Q(s,t) = s\omega_1(\omega_3-\omega_2) + t(\omega_2-\omega_1)(1-\omega_3)\,, \tag{4.17}$$

arises by taking the asymptotic limit of the propagators, $\lim_{\tau_2\to\infty}\Delta_s \equiv \Delta_s^\infty = \hat{\mathcal{P}}^\infty(\nu^{(12)}) + \hat{\mathcal{P}}^\infty(\nu^{(34)})$, in the expression (2.1) for the one-loop amplitude (with $s$ and $t$ negative). We need to add the contribution to the amplitude in the regions $\mathcal{T}_{us}=\{0\le\omega_2\le\omega_1\le\omega_3\le 1\}$ with $Q(s,u)=s(\omega_3-\omega_1)\omega_2+u(\omega_1-\omega_2)(1-\omega_3)$ and $\mathcal{T}_{tu}=\{0\le\omega_1\le\omega_3\le\omega_2\le 1\}$ with $Q(t,u)=u(\omega_2-\omega_1)\omega_1+t(\omega_3-\omega_1)(1-\omega_2)$. This is equivalent to evaluating the sum of the supergravity box diagrams figure 2(a) in the $(s,t)$, $(s,u)$ and $(t,u)$ channels.

It is very complicated to evaluate the detailed form of this term using the cutoff at $\tau_2 \geq L$, so we will review the method used in [11] for evaluating the integral using dimensional regularization. This will generally give a different definition of the scale, $\mu_1$, in terms of the form $s \log(-\alpha' s/\mu_1)$. However, in this case the scale cancels out (using $s+t+u=0$) and so the result is identical. In $D-2\epsilon$ dimensions, the one-loop amplitude becomes[4]

$$I^{(d=D-2\epsilon)}_{\mathcal{T}_{st}}(s,t) = c(D,\epsilon) \int_0^1 d\xi \, \frac{(-\alpha' s\xi)^{\frac{D}{2}-\epsilon-3} - (-\alpha' t(1-\xi))^{\frac{D}{2}-\epsilon-3}}{\alpha' t + \alpha' u\xi}\,, \tag{4.18}$$

where

$$c(D,\epsilon) = \pi^{\frac{D}{2}-4-\epsilon} \frac{\Gamma(4-\frac{D}{2}+\epsilon)\Gamma(\frac{D}{2}-2-\epsilon)^2}{(D-6-2\epsilon)\Gamma(D-4-2\epsilon)}\,. \tag{4.19}$$

In order to perform the $\epsilon$ expansion we separate the integral into the two contributions

$$I^{(d=D-2\epsilon)}_{\mathcal{T}_{st}}(s,t) = K_s(s,t) - K_t(s,t)\,, \tag{4.20}$$

where

$$K_s(s,t) = c(D,\epsilon) \int_0^1 d\xi \, \frac{(-\alpha' s\xi)^{\frac{D}{2}-\epsilon-3} - (-\alpha' s\xi_*)^{\frac{D}{2}-\epsilon-3}}{\alpha' t - \alpha'(s+t)\xi} \tag{4.21}$$

with $\xi_* = t/(s+t)$. In this manner it is clear that the integral over $\xi$ does not develop a singularity in the limit $\epsilon \to 0$ as long as $D > 4$. In such dimensions $\epsilon$ singularities can only arise from the factor $c(D,\epsilon)$. These correspond to ultraviolet divergences that first arise as a $\epsilon$ pole in ten dimensions, $c(10,\epsilon) \to -\pi/(5!\,\epsilon)$ as $\epsilon \to 0$. For $D \leq 4$ the integral may diverge at the $\xi = 0$ endpoint, leading to infrared divergences that are seen as singularities in the $\epsilon \to 0$ limit. We now consider the ten and nine dimensional cases separately.

- In ten dimensions we have

$$\begin{aligned} I^{(d=10-2\epsilon)}_{\mathcal{T}_{st}}(s,t) &= -\frac{1}{5!\,\epsilon}\,\pi^{1-\epsilon}\,e^{-\gamma_E \epsilon}\left(1 + \frac{46}{15}\,\epsilon + O(\epsilon^2)\right)\left(\frac{\alpha' u}{2} - \epsilon I^{(1)} + O(\epsilon^2)\right) \\ &= -\frac{\alpha' u \pi}{240\,\epsilon} + \frac{1}{240}\alpha' u\,\pi\left(\gamma_E + \log\pi - \frac{46}{15}\right) + \frac{\pi}{5!} I^{(d=10)}_{\mathcal{T}_{st}}(s,t) + O(\epsilon^2)\,, \end{aligned} \tag{4.22}$$

where $\gamma_E$ is Euler's constant. The full result is given by adding the $(s,t)$, $(t,u)$ and $(s,u)$ contributions and using the on-shell condition. The pole in $\epsilon$ cancels in the sum, leaving the ultraviolet finite result for the full amplitude in the $\epsilon \to 0$ limit,

$$I^{(d=10)}_{\mathrm{SUGRA}}(s,t,u) = I^{(d=10)}_{\mathcal{T}_{st}}(s,t) + I^{(d=10)}_{\mathcal{T}_{tu}}(t,u) + I^{(d=10)}_{\mathcal{T}_{us}}(u,s)\,, \tag{4.23}$$

where the function $I^{(d=10)}_{\mathcal{T}_{st}}$ is given by

$$I^{(d=10)}_{\mathcal{T}_{st}}(s,t) = \int_0^1 \frac{d\xi}{\alpha' t + \alpha' u\xi}\Big[(-\alpha' t(1-\xi))^2 \log(-\alpha' t(1-\xi)) - (-\alpha' s\xi)^2 \log(-\alpha' s\xi)\Big]. \tag{4.24}$$

[4]We have made use of the change of variables $w_1 = \eta\xi_1$, $w_2 = (1-\eta)(1-\xi_2) + \eta\xi_1$ and $w_3 = 1-\eta+\eta\xi_1$.

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This expression is real in the region $s < 0$, $t < 0$ and has the appropriate imaginary part in other regions of the Mandelstam invariants. For example, in the physical region, $s > 0$ and $t, u < 0$, the integrand in (4.24) is nonsingular in the whole range of $\xi$, so the only singularities are branch points due to the explicit $\log(-\alpha' s)$ factor. The pole of the denominator at $\xi = -t/u$ is canceled by a zero in the numerator and the integrand is finite in the range of integration, as is evident from the form of $K_s(s,t)$ in (4.21). Other singularities with double discontinuities can be found by analytic continuation.

Evaluating the integrals in (4.24) explicitly leads to

$$\begin{aligned} I^{(d=10)}_{\mathcal{T}_{st}}(s,t) = & \frac{\alpha' u}{4} + \alpha' \frac{st}{2u} + \alpha' \frac{s^2(s+3t)\log(-\alpha' s) + t^2(t+3s)\log(-\alpha' t)}{2u^2} \\ & -\alpha' \frac{s^2 t^2}{u^3} \left( \mathcal{L}_2\left(-\frac{u}{t}\right) + \mathcal{L}_2\left(-\frac{u}{s}\right) \right) , \end{aligned} \tag{4.25}$$

where

$$\mathcal{L}_2(x) = \mathrm{Li}_2(x) + \log(x)\log(1-x) = \frac{\pi^2}{6} - \mathrm{Li}_2(1-x) \tag{4.26}$$

is a real function for all $x \geq 0$ [12]. Although $I^{(d=10)}_{\mathcal{T}_{st}}(s,t)$ is a complicated expression it is easy to see that it has the important scaling property

$$I^{(d=10)}_{\mathcal{T}_{st}}(Ls, Lt) = L\, I^{(1)}(s,t) - \frac{\alpha' u}{2}\, L\log(L)\,, \tag{4.27}$$

which ensures that the scale of the logarithm does not contribute after summation over the $(s,t)$, $(t,u)$ and $(s,u)$ terms.

• For completeness, we also include the case of nine dimensions, where there are no divergences when $\epsilon \to 0$ and we can set $\epsilon = 0$ directly in the expression for the amplitude, giving

$$\begin{aligned} I^{(d=9)}_{\mathcal{T}_{st}}(s,t) = & \frac{2}{3}\frac{(-\alpha' s)^{3/2} + (-\alpha' t)^{3/2}}{\alpha' u} - 2\frac{(-\alpha' s)^{3/2} t + (-\alpha' t)^{3/2} s}{\alpha' u^2} \\ & + \frac{3(-\alpha' s)^{3/2}(-\alpha' t)^{3/2}}{(\alpha' u)^{5/2}} \log\left( \frac{(\sqrt{-\alpha' t} + \sqrt{\alpha' u})(\sqrt{-\alpha' s} - \sqrt{\alpha' u})}{(\sqrt{-\alpha' t} - \sqrt{\alpha' u})(\sqrt{-\alpha' s} + \sqrt{\alpha' u})} \right) . \end{aligned} \tag{4.28}$$

Once again the full expression for $I^{(d=19)}_{nonan}(s,t,u)$ is obtained by adding the contributions of $I^{(d=10)}_{\mathcal{T}_{tu}}$ and $I^{(d=10)}_{\mathcal{T}_{us}}$ to $I^{(d=10)}_{\mathcal{T}_{st}}$.

### 4.2.2 String corrections and higher-order massless thresholds

At higher order in the derivative expansion there are further massless threshold effects due to the higher-order contact terms in the blob in figures 2(b) and 2(c). These lead to factors of the form $s^r \log(-s/\mu_r)$, where the scale of the logarithm does not cancel. Luckily the threshold structure of these terms is much simpler than that of the box diagram in figure 2(a) since they only possess singularities in $s$ rather than overlapping singularities. So we can go back to the cutoff procedure outlined earlier in order to evaluate the expressions, taking care to verify that the dependence on the cutoff $L$ indeed cancels.

The supergravity contribution considered in the last subsection was obtained by replacing the propagators in the exponent of (2.1) by their leading form in the large-$\tau_2$ limit. In order to analyze threshold terms at higher-order in the momentum expansion we need to keep non-leading terms at large $\tau_2$. To do this we write

$$\Delta_s \sim \Delta_s^\infty + \delta_s \,, \tag{4.29}$$

where the correction, $\delta_s$, to the asymptotic value is given by the $k = 0$ part of $\tilde{P}(\nu|\tau)$ defined in (2.16),

$$\delta_s = \sum_{m\neq 0} \frac{1}{4|m|} \left( e^{2i\pi(m\nu_1^{(12)}+i|m\nu_2^{(12)}|)} + e^{2i\pi(m\nu_1^{(34)}+i|m\nu_2^{(34)}|)} \right) \,, \tag{4.30}$$

with similar expressions for $\delta_{t,u}$. In the $\tau_2 \geq L$ region the terms with $k \neq 0$ are suppressed by powers of $e^{-L}$. The asymptotic formula for the amplitude is therefore given by [4],

$$\begin{aligned} I_{\mathcal{T}_{st};\mathcal{R}_L}(s,t) = & \int_L^\infty \frac{d\tau_2}{\tau_2^2} \int_{\mathcal{T}_{st}} \prod_{i=1}^3 \frac{d^2\nu}{\tau_2} \exp\left(\alpha' s(\Delta_s^\infty - \Delta_u^\infty) + \alpha' t(\tilde{\Delta}_t^\infty - \tilde{\Delta}_u^\infty)\right) \\ & \exp\left(\alpha' s(\delta_s - \delta_u) + \alpha' t(\delta_t - \delta_u)\right) \,. \end{aligned} \tag{4.31}$$

The higher-order threshold contributions containing factors of $\log(-sL)$ are obtained by expanding the integrand in powers of $\delta$, $e^{\alpha'\delta} = 1 + \alpha'\delta + \alpha'^2\delta^2/2 + \ldots$. The factors of $e^{\alpha'\,\Delta^\infty}$ in the integrand contribute to terms with positive powers of $L$.

In particular, the terms of order $\alpha'^4\, s^4 \log(-\alpha' s/\mu_4)$, which will be denoted $I^{(4)}$, is obtained from the $\delta_s^2$ contribution of the integrand in (4.31). Neither $\delta_t$ or $\delta_u$ contributes to this threshold behaviour term (although $\delta_t$ does contribute to the analogous $t$-channel term, $\alpha'^4\, t^4 \log(-\alpha' t/\mu_4)$). This gives

$$\begin{aligned} I^{(4)}_{\mathcal{T}_{st};\mathcal{R}_L}(s,t) = & \frac{1}{2} \sum_{m\neq 0} \frac{(-\alpha' s)^2}{(4m)^2} \int_L^\infty \frac{d\tau_2}{\tau_2^2} \int_{\mathcal{T}_{st}} \prod_{i=1}^3 d\omega_i e^{\alpha'\pi\tau_2 Q(s,t) - 4\pi|m|\tau_2(\omega_2-\omega_1)} \\ & + \frac{1}{2} \sum_{m\neq 0} \frac{(-\alpha' s)^2}{(4m)^2} \int_L^\infty \frac{d\tau_2}{\tau_2^2} \int_{\mathcal{T}_{st}} \prod_{i=1}^3 d\omega_i e^{\alpha'\pi\tau_2 Q(s,t) - 4\pi|m|\tau_2(1-\omega_3)} \\ = & \sum_{m\neq 0} \frac{(-\alpha' s)^2}{(4m)^2} g^{(4)}(L;s,t) \,, \end{aligned} \tag{4.32}$$

where we note that the two terms on the right-hand side of the first equality are equal and

$$g^{(4)}(L;s,t) = \int_L^\infty \frac{d\tau_2}{\tau_2^2} \int_0^1 d\eta\, d\xi_1 d\xi_2\, \eta(1-\eta)\, e^{\alpha'\pi\tau_2 Q(s,t) - 4\pi|m|\tau_2(1-\xi_2)(1-\eta)} \,. \tag{4.33}$$

Noting further that $Q$ is given in terms of the variables $\eta$, $\xi_1$ and $\xi_2$ by

$$Q(s,t) = s\,\eta(1-\eta)\,\xi_1\,\xi_2 + t\,\eta(1-\eta)\,(1-\xi_1)\,(1-\xi_2) \,, \tag{4.34}$$

and performing the $\xi_2$ integral gives

$$g^{(4)}(L;s,t) = \int_L^\infty \frac{d\tau_2}{\tau_2^3} \int_0^1 d\eta\, d\xi_1\, \eta\, \frac{e^{\alpha'\pi s\tau_2\eta(1-\eta)\xi_1} - e^{-4\pi m\tau_2(1-\eta)} e^{\alpha'\pi t\tau_2\eta(1-\eta)(1-\xi_1)}}{\alpha'\eta\pi(s\xi_1 - t(1-\xi_1)) + 4\pi m} \,. \tag{4.35}$$

The term of order $s^4$ in the amplitude comes from the $O(\alpha'^3)$ correction to the tree amplitude represented by the blob in 2(b). This is obtained by expanding $g^{(4)}$ to $O(s^2)$. At this order it is easy to see that the term proportional to $e^{-4\pi m\tau_2(1-\eta)}$ in the numerator of (4.35) is proportional to $e^{-L}$ and is negligible. The term of the form $s^2\log(-\alpha' s)$ arises by expanding $e^{\alpha'\pi s\tau_2\eta(1-\eta)\xi_1}$ to quadratic order and replacing the denominator by $4\pi m$. As a result we find the $O(s^2)$ terms,

$$g^{(4)}(L;s,t) = -\frac{1}{4\pi m}\,\frac{(\alpha' s\,\pi)^2}{180}\,\big(\log(-\alpha' s\,\pi\, L/c_e) - 2/5\big)\,L^3 + O(s^3, L^{-1})\,, \tag{4.36}$$

where $c_e = \exp(-\gamma_E)$. In obtaining this we have used the fact that

$$\lim_{x\to 0}\int_L^\infty \frac{dt}{t^3}\,e^{-xt} = \frac{1}{2L^2} - \frac{x}{L} - \frac{x^2}{2}\,(\log(xL) - 3/2 + \gamma) + o(x^2). \tag{4.37}$$

Substituting (4.36) into (4.32) gives the contribution

$$I^{(4)}_{\mathcal{T}_{st};\mathcal{R}_L}(s,t) = -\frac{2\pi\zeta(3)}{45}\left(\frac{\alpha'}{4}\right)^4 s^4\log(-\alpha' s\,L/\hat\mu_4) + \cdots\,, \tag{4.38}$$

where the scale of the logarithm at this order is given by

$$\log\hat\mu_4 = \frac{2}{5} - \log(\pi/c_e)\ . \tag{4.39}$$

The ellipses indicate that a similar contribution that arises by using the $\delta_t^2$ term in the expansion of $e^{\alpha' t\delta_t}$, which gives a threshold singularity in the $t$ channel in which figure 2(b) is transformed by interchanging $s$ and $t$. There are similarly contributions from $\mathcal{T}_{us}$ and $\mathcal{T}_{tu}$ regions containing analogous terms with $s$, $t$ and $u$-channel thresholds.

Therefore the total contribution from the upper part of the fundamental domain $\tau_2 \geq L$ to the terms of order $s^4$ is given by the sum

$$\begin{aligned} I^{(4)}_{\mathcal{R}_L}(s,t,u) & I^{(4)}_{\mathcal{T}_{st};\mathcal{R}_L}(s,t) + I^{(4)}_{\mathcal{T}_{tu};\mathcal{R}_L}(t,u) + I^{(4)}_{\mathcal{T}_{us};\mathcal{R}_L}(u,s) \\ &= -\frac{4\zeta(3)}{45\pi}\left(\frac{\alpha'}{4}\right)^4 s^4\,\log(-\alpha' sL\pi/c_e) - 2/5) + (s\to t) + (s\to u)\,. \end{aligned} \tag{4.40}$$

The dependence on both $L^3$ and $\log(L)$ cancel in the sum of this contribution and the contribution from $\tau_2 \leq L$ given in (3.17), giving

$$\begin{aligned} I^{(4)}_{nonan}(s,t,u) = -\frac{4\pi\zeta(3)}{45}\left(\frac{\alpha'}{4}\right)^4 & \big[s^4\,\log(-\alpha' s/\mu_4) \\ & +t^4\,\log(-\alpha' t/\mu_4) + u^4\,\log(-\alpha' u/\mu_4) - 6/5\big]\ , \end{aligned} \tag{4.41}$$

where

$$\log\mu_4 = \log\hat\mu_4 - \log\tilde\mu_4 = \frac{9}{10} - \log\left(\frac{2}{\pi c_e}\right) + \frac{\zeta'(3)}{\zeta(3)} - \frac{\zeta'(4)}{\zeta(4)}\,, \tag{4.42}$$

which is the sum of the scale from the lower part of the fundamental domain, given in (3.18), and the scale in the logarithm obtained from the above computation from the large-$\tau_2$ part of the fundamental domain (4.39).

Contributions to terms of order $s^5$ arise both from expanding the function $g^{(4)}(L; s, t)$ to one further order in $s$ and by bringing down one further power of $\delta$ from the exponential in (4.31). In this case the there are no terms proportional to $\log L$, which reflects the fact that there is no threshold term at this order, as follows from the unitarity argument in section 4.1. In order to evaluate the thresholds at order $\alpha'^6\, s^6 \log(-\alpha' s/\mu_6)$, it is necessary to expand $g^{(4)}(L; s, t)$ to two further orders in $s$, which is fairly complicated. In addition, there are contributions from yet higher powers of $\delta$. Although we have not evaluated these terms, in principle, they will determine a scale $\log \hat{\mu}_6$ (analogous to $\log \hat{\mu}_4$ in (4.39)) that combines with $\log \tilde{\mu}_6$ (3.27) to give the scale $\log \mu_6$.

At higher orders in $\alpha'$ string corrections arise on both sides of the threshold discontinuity as we saw in section 4.1. The qualitative structure of these threshold contributions matches that of the discontinuities that we determined earlier from unitarity. Firstly, there are contributions from the triangle diagram in figure 2(b) beyond the ones considered above. The first of these is of order $\alpha'\, s^7 \log(-\alpha'\, s)$. In addition, a new class of contributions arises, in which both the propagators on the left and right of the $s$-channel cut are canceled and the result is the diagram in figure 2(c), which has two blobs representing the higher derivative vertices.

### 4.3 Summary of the expansion of the ten-dimensional genus-one amplitude

To put these results in perspective we will here summarize the low-energy expansion of the genus-one contribution to the analytic part of the amplitude up to order $s^6\, \mathbf{R}^4$. At tree-level and genus one, type IIA and type IIB massless four-particle amplitudes are completely equivalent, so the results apply equally to both cases.

The analytic terms to this order are summarized by

$$\begin{aligned} \mathbf{A}^{genus-1}_{an}(\hat{\sigma}_2, \hat{\sigma}_3) &= \frac{\pi}{3}\left(\sum_{p,q} J^{(p,q)}\, \hat{\sigma}_2^p \hat{\sigma}_3^q\right) \mathbf{R}^4 \\ &= \frac{\pi}{3}\Bigg(1 + 0\,\hat{\sigma}_2 + \frac{\zeta(3)}{3}\hat{\sigma}_3 + 0\,\hat{\sigma}_2^2 + \frac{97}{1080}\,\zeta(5)\,\hat{\sigma}_2\hat{\sigma}_3 \\ &\qquad + \frac{1}{30}\,\zeta(3)^2\hat{\sigma}_2^3 + \frac{61}{1080}\zeta(3)^2\hat{\sigma}_3^2 + \cdots\Bigg)\mathbf{R}^4\,, \end{aligned} \tag{4.43}$$

(where we have indicated explicitly the vanishing $\hat{\sigma}_2$ and $\hat{\sigma}_2^2$ coefficients), while the nonanalytic terms are contained in

$$\begin{aligned} \mathbf{A}^{genus-1}_{nonan}(\hat{\sigma}_2, \hat{\sigma}_3) &= \mathbf{A}^{genus-1}_{\rm SUGRA} \\ &+\left(\frac{\alpha'}{4}\right)^4 \frac{4\zeta(3)\pi}{45}\left(s^4 \log\left(-\frac{\alpha' s}{\mu_4}\right) + t^4 \log\left(-\frac{\alpha' t}{\mu_4}\right) + u^4 \log\left(-\frac{\alpha' u}{\mu_4}\right)\right)\mathbf{R}^4 \\ &+\left(\frac{\alpha'}{4}\right)^6 \frac{\pi\,\zeta(5)}{2520}\left((87\,s^6 + s^4\,(t-u)^2)\,\log\left(-\frac{\alpha' s}{\mu_6}\right)\right. \\ &\left. + (87t^6 + t^4(s-u)^2)\log\left(-\frac{\alpha' t}{\mu_6}\right) + (87u^6 + u^4(s-t)^2)\log\left(-\frac{\alpha' u}{\mu_6}\right)\right)\mathbf{R}^4 + \cdots\,, \end{aligned} \tag{4.44}$$

where $\mathbf{A}^{genus-1}_{\text{SUGRA}}$ is the result obtained in ten-dimensional maximal supergravity that we described earlier where we also discussed the scales $\mu_4$ and $\mu_6$. Notice that the terms in equations (4.43) and (4.44) satisfy a 'transcendentality condition' in which $\zeta(k)$ is associated with a weight $k$, $\pi$ has weight 1 and $\log(x)$ also has weight 1. The total weight of a term of order $(\alpha')^q$ is $q+1$.

Clearly, the separation of the amplitude into $\mathbf{A}_{an}$ and $\mathbf{A}_{nonan}$ depends on the definition of the scales inside the logarithmic terms in (4.44). The sum of the terms that depend on $\mu_4$ and $\mu_6$ is

$$-\frac{4\pi\zeta(3)}{45}\,\hat\sigma_2^2\,\log\mu_4\,\mathbf{R}^4-\frac{\zeta(5)\pi}{630}\,\left(11\hat\sigma_2^3+14\hat\sigma_3^2\right)\,\log\mu_6\,\mathbf{R}^4\,. \tag{4.45}$$

Notice that although changing the scale inside the logarithms alters the analytic part of the amplitude, the pattern of Riemann zeta values is different in the coefficients of the terms in $\mathbf{A}^{genus-1}_{an}$ from those in $\mathbf{A}^{genus-1}_{nonan}$. In this sense, there is an objective meaning to the separation between the analytic and the nonanalytic parts as given in (4.43) and (4.44). For example, at order $(\alpha'\,s)^6$ one has a finite piece proportional to $\zeta(3)^2$ while the logarithmic term is multiplied by $\zeta(5)$ in agreement with the 'transcendentality' property noted earlier. So our choice of scale in the argument of the logarithmic terms is not only the natural scale that arises from the calculation of section 3, but is natural if $\zeta(3)^2$ and $\zeta(5)$ are considered to be distinct coefficients.

## 5. The genus-one four-particle amplitude in nine dimensions

We now turn to consider the compactification of the amplitude on a circle of radius $r$ so that nine-space-time dimensions are non-compact. We will specialize to the case in which the momenta of the scattering particles, $p_r^\mu$, have zero components in the compact direction, so the external states are Kaluza-Klein ground states. In nine dimensions the genus-one normal thresholds have square root singularities instead of logarithms. This separates the threshold terms in the amplitude from the analytic terms in a very clear manner. The massive Kaluza-Klein modes also generate massive square root thresholds that are expanded, when $\alpha' s \gg 1/r^2$, in an infinite series of terms of the form $(\alpha' s r^2)^n$ and therefore enter into higher order terms in the low energy expansion. The logarithmic singularities characteristic of ten dimensions are recovered in the ten-dimensional limit by a condensation of these Kaluza-Klein thresholds [7].

### 5.1 General method

The expression for the integral $I$ in the compactified genus-one amplitude (1.1) now has the form

$$I^{(d=9)}(s,t,u;r)=\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\,\sqrt{\tau_2}\,\Gamma_{(1,1)}(r)\,F(s,t,u;\tau)\,, \tag{5.1}$$

where the lattice sum factor is given by [11]

$$\sqrt{\tau_2}\,\Gamma_{(1,1)}(r)=\sqrt{\tau_2}\sum_{(m,\hat n)\in\mathbb{Z}^2}e^{-\pi\tau_2\left((\frac{m}{r})^2+(\hat n r)^2\right)+2i\pi\tau_1\hat m n}\,. \tag{5.2}$$

The loop amplitude can now be expressed in terms of an expansion in half-integer powers of $\alpha' s$, $\alpha' t$ and $\alpha' u$. We can separate this into two terms,

$$I^{(d=9)}(s,t,u;r) = I^{(d=9)}_{an}(s,t,u;r) + I^{(d=9)}_{nonan}(s,t,u;r)\,. \tag{5.3}$$

The analytic part has an expansion with integer powers of the Mandelstam invariants of the form

$$I^{(d=9)}_{an}(s,t,u;r) = \sum_{p,q=0}^{\infty} J^{(p,q)}(r)\,\hat\sigma_2^p \hat\sigma_3^q \tag{5.4}$$

and is analytic in $s$, $t$ and $u$. We will see that the coefficients can be expanded at large $r$ as sums of powers of $r$, together with $r\log r^2$ terms and exponentials,

$$\begin{aligned} J^{(p,q)}(r) \,=\, & J_1^{(p,q)}\, r^{4p+6q-1} + J_2^{(p,q)}\, r^{4p+6q-3} + \cdots + J_{4p+6q}^{(p,q)}\, r^{-4p-6q+1} \\ & + K^{(p,q)}\, r\log(r^2\lambda_{2p+3q}) + O(e^{-r})\,, \end{aligned} \tag{5.5}$$

where the $\lambda_k$'s are constants. In the ten-dimensional theory only the term linear in $r$ survives with coefficient $J_{2p+3q}^{(p,q)}$. The nonanalytic part has the form

$$I^{(d=9)}_{nonan}(s,t,u;r) = b_1 s^{\frac{1}{2}} + b_4 s^{\frac{7}{2}} + \cdots\,, \tag{5.6}$$

where terms involving $t$ and $u$ have not been included. The coefficients $b_i$ are independent of $r$. In the following we will be interested in determining the coefficients $J_r^{(p,q)}$ in the expansion of $I_{an}^{(d=9)}$, but will not consider the coefficients $b_n$ of the non-analytic terms in any detail (although they are relatively easy to extract).

We will first reexpress the lattice sum by a Poisson resummation that replaces the sum over $m$ by a sum over $\hat p$ and two relatively prime integers $(\hat m, \hat n)$,

$$\sqrt{\tau_2} \sum_{(m,\hat n)\in\mathbb{Z}^2} e^{-\pi\tau_2\left((\frac{m}{r})^2+(\hat n r)^2\right)+2i\pi\tau_1 m\hat n} = r \sum_{\substack{\hat p\in\mathbb{Z}\\ (\hat m,\hat n)=1}} e^{-\pi\,\hat p^2\, r^2\, \frac{|\hat m+\hat n\tau|^2}{\tau_2}}\,. \tag{5.7}$$

Thanks to the modular invariance of the kinematical factor $F(s,t,u;\tau)$, after separating the zero-winding term ($\hat p = 0$) one can unfold the integral onto the semi-infinite strip [6], which gives

$$\begin{aligned} I^{(d=9)}(s,t,u;r) \,=\, & r\int_{\mathcal F} \frac{d^2\tau}{\tau_2^2}\, F(s,t,u;\tau) \\ & + r\sum_{\hat p\neq 0}\int_0^\infty \frac{d\tau_2}{\tau_2^2}\, e^{-\pi\hat p^2\frac{r^2}{\tau_2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1\, F(s,t,u;\tau)\,. \end{aligned} \tag{5.8}$$

The first term is the zero-winding term, which is the term that survives the $r\to\infty$ limit,

$$r\int_{\mathcal F}\frac{d^2\tau}{\tau_2^2}\; F(s,t,u;\tau) = r\, I^{(d=10)}(s,t,u)\,, \tag{5.9}$$

where the low-energy expansion of $I^{(d=10)}(s,t,u)$ was considered in sections 3 and 4.



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Proceeding as for the ten-dimensional amplitude in section 3, the integral $I^{(d=9)}_{\hat p\neq 0}$ can be split into an analytic and non-analytic parts by dividing the $\tau_2$ integral into two domains, so that $I^{(d=9)} = I^{(d=9)}_{(1)} + I^{(d=9)}_{(2)}$, where

$$I^{(d=9)}_{(1)} = r \sum_{\hat p\neq 0} \int_0^L \frac{d\tau_2}{\tau_2^2}\, e^{-\pi \hat p^2 \frac{r^2}{\tau_2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1\, F(s,t,u;\tau) \tag{5.10}$$

contains the analytic part of the amplitude, and

$$I^{(d=9)}_{(2)} = r \sum_{\hat p\neq 0} \int_L^\infty \frac{d\tau_2}{\tau_2^2}\, e^{-\pi \hat p^2 \frac{r^2}{\tau_2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1\, F(s,t,u;\tau)\,, \tag{5.11}$$

contains the non-analytic threshold terms. As in the ten-dimensional case, the $L$ dependence will cancel from the final expressions.

The analytic contributions in equation (5.10) can be analyzed by expanding $\int d\tau_1 F(s,t,u;\tau)$ in powers of $\hat\sigma_2$ and $\hat\sigma_3$,

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} d\tau_1 F(s,t,u;\tau) = \sum_{p,q=0}^{\infty} \hat\sigma_2^p\, \hat\sigma_3^q\, j^{(p,q)}\,. \tag{5.12}$$

As in section 3 the coefficients at any order in $s$, $t$, $u$, are determined by properties of the $\mathrm{D}^{(0)}_{\{l\}}$ functions determined in appendices B and D. In those appendices we expanded these functions in powers of $1/\tau_2$. In the simplest cases we displayed all power-behaved terms while in others we only displayed the positive powers and constant terms.

For the terms that are constant or negative powers of $\tau_2$ — i.e., of the form $a_k\, \tau_2^k$ with $k \le 0$ — both the $\tau_2$ integral and the $\hat p$ sum in (5.10) are easy to evaluate. For these terms we may simply take $L \to \infty$ and $I^{(d=9)}_{(2)}$ in (5.11) is zero. The result is a succession of terms of the form

$$\hat\sigma_2^p\, \hat\sigma_3^q \sum_{k\le 0} J_k^{(p,q)}\, r^k\,, \tag{5.13}$$

with $k \le 0$. The contribution $J_0^{(p,q)}$ is the ten dimensional contribution $J^{(p,q)}$ discussed in section 3.

The terms in the expansion of $\int d\tau_1 F(s,t,u;\tau)$ that are positive powers of $\tau_2$ of the form $a_k\, \tau_2^k$ with $k \ge 2$ have to be treated separately since, in the limit $\tau_2 \to \infty$, the $\hat p$ sum in (5.10) and (5.11) contains the factor $e^{-\pi \hat p^2 r^2/\tau_2} \sim 1$, so it needs a Poisson resummation. This involves first adding and subtracting a term with $\hat p = 0$. The subtracted term,

$$I^{(d=9)}_{(2)\hat p=0} = r \int_L^\infty \frac{d\tau_2}{\tau_2^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1\, F(s,t,u;\tau)\,, \tag{5.14}$$

is precisely the term studied in the last section that has thresholds of the form $s^k \log(-\alpha' s L/\hat\mu_k)$ with a coefficient that ensures that it cancels the $s^k \log(-\alpha'\, s/\mu_k)$ thresholds contained in the term $r\, I^{(d=10)}$ in (5.8). This ensures that there are no $\log(-\alpha'\, s)$ terms in the nine-dimensional expression, in accord with unitarity. After the Poisson resummation the integer $\hat{p}$ is replaced by the integer $p$ and we have

$$I_{(2)}^{(d=9)} = \sum_{p\in\mathbb{Z}} \int_L^\infty \frac{d\tau_2}{\tau_2^{\frac{3}{2}}}\, e^{-\pi\, p^2 \frac{\tau_2}{r^2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1\, F(s,t,u;\tau)\,. \tag{5.15}$$

The $p=0$ contribution,

$$I_{(2);p=0}^{(d=9)} = \int_L^\infty \frac{d\tau_2}{\tau_2^{\frac{3}{2}}} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1\, F(s,t,u;\tau)\,, \tag{5.16}$$

contains the nine-dimensional threshold terms of the form $(-s)^{k+1/2}$, which we will not be considering in any detail in the following. For the $p\neq 0$ terms we can again take $L\to\infty$, in which case $I_{(2)}^{(d=9)}$ vanishes and $I_{(1)}^{(d=9)}$ gives a series of terms of the form (5.13) with $k\geq 1$.

The terms that we have calculated of the form (5.13) will be summarized in (5.27) in subsection 5.2.

However, there is a subtlety in considering the $\tau_2$ integral of terms that are linear in $\tau_2$, i.e., terms of the form $\int d\tau_1 F(s,t,u;\tau)\sim a_1\tau_2$ in (5.8). To see this note that in this case the $\hat{p}=0$ term we need to add and subtract before doing the Poisson resummation in $I_{(1)}^{(d=9)}$ is

$$I_{\hat{p}=0;a_1}^{(d=9)} = a_1 r \int_0^L \frac{d\tau_2}{\tau_2}\;, \tag{5.17}$$

which diverges. This means that for terms linear in $\tau_2$ we cannot perform a Poisson resummation in $I_{(1)}^{(d=9)}$ but we still have to perform a Poisson resummation in $I_{(2)}^{(d=9)}$ since each term in the $\hat{p}$ sum has a factor $e^{-\pi\hat{p}^2\frac{r^2}{\tau_2}}\sim 1$ at large $\tau_2$. Explicitly, we have

$$I_{(1);a_1}^{(d=9)} \equiv a_1\, r \sum_{\hat{p}\neq 0} \int_0^L \frac{d\tau_2}{\tau_2}\, e^{-\pi\hat{p}^2\frac{r^2}{\tau_2}} \equiv a_1\, r\, f_1(L/r^2)\,, \tag{5.18}$$

where we have rescaled $\tau_2$ in order to write the expression in terms of a function $f_1(L/r^2)$ which will be of later use. After a Poisson resummation, the $p\neq 0$ terms in $I_{(2);a_1}^{(d=9)}$ are given by

$$I_{(2);p\neq 0;a_1}^{(d=9)} = a_1 \sum_{p\neq 0}^{\infty} \int_L^\infty \frac{d\tau_2}{\tau_2^{1/2}}\, e^{-\pi\, p^2\frac{\tau_2}{r^2}} \equiv a_1 r f_2(L/r^2)\,, \tag{5.19}$$

where we have again rescaled $\tau_2$ in order to write the expression in terms of a function $f_2(L/r^2)$, which will be of later use.

The dependence on $L/r^2$ in $f_1$ in equation (5.18) will cancel with that of the term $f_2$ in equation (5.19). To see this, consider the derivatives of these terms with respect to $L$. We have

$$r\,\frac{\partial f_1(L/r^2)}{\partial L} = \frac{r}{L}\sum_{\hat{p}\neq 0} e^{-\pi\hat{p}^2\frac{r^2}{L}}\,. \tag{5.20}$$

On the other hand Poisson resumming the integer $\hat{p}$ in (5.19)

$$r\,\frac{\partial f_2(L/r^2)}{\partial L} = -\frac{1}{L^{\frac12}}\sum_{p\neq 0} e^{-\pi p^2\frac{L}{r^2}}$$
$$= \frac{1}{L^{\frac12}} - \frac{r}{L} - \frac{r}{L}\sum_{\hat{p}\neq 0} e^{-\pi\hat{p}^2\frac{r^2}{L}}\,. \tag{5.21}$$

In order to see what this means we can use the fact that

$$\frac{\partial}{\partial L}\Big(I^{(d=9)}_{(1)p\neq 0;a_1} + I^{(d=9)}_{(2)\hat{p}\neq 0;a_1}\Big) = a_1 r\,\frac{\partial f_1(L/r^2)}{\partial L} + a_1 r\,\frac{\partial f_2(L/r^2)}{\partial L} = \frac{a_1}{L^{1/2}} - a_1\frac{r}{L}\,. \tag{5.22}$$

Integrating over $L$, we find

$$I^{(d=9)}_{(1)\hat{p}\neq 0;a_1} + I^{(d=9)}_{(2)\hat{p}\neq 0;a_1} = a_1 r\left(2\frac{L^{\frac12}}{r} + \log\tilde{\lambda} + \log(r^2/L)\right). \tag{5.23}$$

where the constant $\tilde{\lambda}$ can be determined by integrating between $r^2$ and $L$,

$$\log\tilde{\lambda} = -2 + \int_0^1 \frac{dt}{t}\sum_{p\neq 0} e^{-\pi p^2/t} + \int_1^\infty \frac{dt}{\sqrt{t}}\sum_{p\neq 0} e^{-\pi p^2 t} \tag{5.24}$$
$$= \gamma_E - \log(4\pi)\,.$$

So the $L$-dependent terms cancel, apart from $-a_1 r\log L$ and $2a_1 L^{1/2}$. These two terms cancel with the $L$-dependent parts of the $\hat{p}=0$ threshold terms that subtract the $\log(-\alpha' s)$ ten dimensional thresholds and the $p=0$ threshold terms in (5.16) that add in the $(-\alpha' s)^{-\frac12}$ thresholds.

## 5.2 Summary of the expansion of the nine-dimensional genus-one amplitude

By using the method outlined above and developed in detail in the appendices, we find the following terms in the momentum expansion of the analytic terms in the integral, $I$, that defines the four-particle amplitude in (1.1) up to order $s^8\,\mathbf{R}^4$ (with partial results beyond)

$$\begin{aligned}
I^{(d=9)}_{an}(r;s,t) =&\, r\,I^{(d=10)}_{an} + 2\sum_{p=1}^{\infty} r\int_0^\infty \frac{d\tau_2}{\tau_2^2}e^{-\frac{\pi p^2 r^2}{\tau_2}}\,j^{(p,q)}\,\hat{\sigma}_2^p\,\hat{\sigma}_3^q\\
=&\, r\,I^{(d=10)}_{an} + 2\sum_{p=1}^{\infty} r\int_0^\infty \frac{d\tau_2}{\tau_2^2}e^{-\frac{\pi p^2 r^2}{\tau_2}}\Big[1 + \hat{\sigma}_2\Big(\frac{\pi^2}{45}\tau_2^2 + \frac{\zeta(3)}{\pi\tau_2}\Big)\\
&+ \hat{\sigma}_3\Big(\frac{2\pi^3\tau_2^3}{567} + \frac{\zeta(3)}{3} + \frac{5\zeta(5)}{4\pi^2\tau_2^2} + O(e^{-\tau_2})\Big)\\
&+ \hat{\sigma}_2^2\Big(\frac{8}{315}\zeta(4)\tau_2^4 + \frac{2\pi\zeta(3)}{45}\,\tau_2 + \frac{5\zeta(5)}{12\pi\tau_2} + \frac{\zeta(3)^2}{4\pi^2\tau_2^2} + \frac{\zeta(7)}{4\pi^3\tau_2^3} + O(e^{-\tau_2})\Big)\\
&+ \hat{\sigma}_2\hat{\sigma}_3\Big(\frac{4\pi^5}{66825}\tau_2^5 + \frac{\pi^2\zeta(3)}{63}\,\tau_2^2 + \frac{29\zeta(5)}{135} + O(\tau_2^{-1})\Big)\\
&+ \hat{\sigma}_2^3\Big(\frac{2}{1001}\zeta(6)\tau_2^6 + \frac{4\pi^3\zeta(3)}{4725}\,\tau_2^3 + \frac{11\pi\zeta(5)}{630}\,\tau_2 + \frac{\zeta(3)^3}{30} + O(\tau_2^{-1})\Big)\\
&+ \hat{\sigma}_3^2\Big(\frac{1744}{675675}\zeta(6)\tau_2^6 + \frac{16\pi^3\zeta(3)}{14175}\,\tau_2^3 + \frac{4\pi\zeta(5)}{180}\,\tau_2 + \frac{61\zeta(3)^2}{6144} + O(\tau_2^{-1})\Big)\Big]\,,
\end{aligned} \tag{5.25}$$

where $I^{(p,q)}_{(d=10)}$ is the ten-dimensional integral considered in section 3 (the coefficient of $\mathbf{R}^4$ in (4.43)). These expansions in powers of $1/\tau_2$ are valid for large $\tau_2$. We have displayed all powers of $\tau_2$ up to order $\hat{\sigma}_2^2$, but at higher orders in $s,t,u$ there are further inverse powers of $\tau_2$ beyond the order displayed, which we have not calculated. There are also exponentially suppressed contributions of order $e^{-\tau_2}$ at order $\hat{\sigma}_3$ and above.

It is easy to integrate over $\tau_2$ using the formula

$$\sum_{\hat{p}=1}^{\infty} r\int_0^\infty \frac{d\tau_2}{\tau_2^2} e^{-\frac{\pi\hat{p}^2r^2}{\tau_2}}\tau_2^n = r^{2n-1}\zeta^*(2-2n)\,. \tag{5.26}$$

where $\zeta^*(x)=\pi^{-x/2}\,\zeta(x)\,\Gamma(x/2)$. The contributions with $n>1$ can be obtained by using the analytic continuation $\zeta^*(2-2n)=\zeta^*(2n-1)$, which is equivalent to performing the Poisson resummation described in section 5.1. For terms linear in $\tau_2$ ($n=1$) the integral requires greater care, as was also emphasized in section 5.1, where we were led to (5.22). The result is

$$\begin{aligned}
I^{(d=9)}_{an}(r;s,t) \;=\; \frac{\pi}{3}\Bigg[& r+r^{-1}+\hat{\sigma}_2\left(\frac{\zeta(3)}{15}r^3+\frac{\zeta(3)}{15}r^{-3}\right)\\
&+\hat{\sigma}_3\left(\frac{\zeta(5)}{63}r^5+\frac{\zeta(3)}{3}r+\frac{\zeta(3)}{3}r^{-1}+\frac{\zeta(5)}{63}r^{-5}\right)\\
&+\hat{\sigma}_2^2\left(\frac{\zeta(7)}{315}r^7+\frac{2\zeta(3)}{15}r\log(r^2\lambda_4)+\frac{\zeta(5)}{36}r^{-3}+\frac{\zeta(3)^2}{315}r^{-5}+\frac{\zeta(7)}{1050}r^{-7}\right)\\
&+\hat{\sigma}_2\hat{\sigma}_3\left(\frac{7\zeta(9)}{2970}r^9+\frac{\zeta(3)^2}{21}r^3+\frac{97\zeta(5)}{1080}r+\frac{29\zeta(5)}{135}r^{-1}+O(r^{-3})\right)\\
&+\hat{\sigma}_2^3\left(\frac{3\zeta(11)}{8008}r^{11}+\frac{2\zeta(3)\zeta(5)}{525}r^5+\frac{11\zeta(5)}{210}r\log(r^2\lambda_6)+\frac{\zeta(3)^2}{30}r+\frac{\zeta(3)^2}{30}r^{-1}+O(r^{-3})\right)\\
&+\hat{\sigma}_3^2\Bigg(\frac{109\zeta(11)}{225225}r^{11}+\frac{8\zeta(3)\zeta(5)}{1575}r^5+\frac{\zeta(5)}{15}r\log(r^2\lambda_6)+\frac{61\zeta(3)^2}{1080}r+\frac{61\zeta(3)^2}{6144}r^{-1}\\
&\qquad +O(r^{-3})\Bigg)+O(e^{-r})\Bigg]
\end{aligned} \tag{5.27}$$

Note that the $r\to 1/r$ symmetry is manifest only in the first two lines of this expression, in which there are no $e^{-r}$ terms and each power of $r$ is accompanied by a corresponding inverse power of $r$ with identical coefficient. At order $\hat{\sigma}_2^2$ and beyond, there is no such pairing of terms and here terms that are exponentially suppressed at large $r$ play an essential rôle in guaranteeing the $r\to 1/r$ symmetry.

The $(\alpha'\,s)^k\,r\,\log(r^2\,\lambda_k)$ contributions are the ones we found in (5.22) that arise when there is a linear dependence on $\tau_2$ and are connected with the ten-dimensional threshold contributions discussed in section 4.2.2. The scale $\lambda_k$ in such logarithms is determined by combining $\log(\tilde{\lambda})$ from (5.24) and the scale $\log(\hat{\mu}_k)$ of the non-analytic contribution discussed in section 4.2.2,

$$\log\lambda_k \equiv \log(\tilde{\lambda}/\tilde{\mu}_k)\,. \tag{5.28}$$

It is striking that the coefficients of the logarithm terms for $\hat{\sigma}_2^2$, $\hat{\sigma}_2^3$ and $\hat{\sigma}_3^2$ agree with values based on duality with eleven-dimensional supergravity compactified on a two-torus [5].

The presence of the terms with coefficients $r\,\log r^2$ is essential in ensuring a smooth ten-dimensional limit as $r \to \infty$. To see this, recall that in this limit an infinite series of terms with positive powers of $r^2\,s$ must resum in a manner that cancels the nine-dimensional square root thresholds [7], generating the ten-dimensional logarithmic thresholds, of the form $r\,s^k\,\log(-r^2\alpha'\,s)$. The resummation therefore produces $r\,\log r^2$ terms with coefficients that are the same as the ten-dimensional massless threshold terms (4.44) computed in section 4. More precisely, the $r\log r^2$ terms produced in the resummation must appear with the same coefficients as $\log\mu_4$ and $\log\mu_6$ in (4.45). From (4.45) we see that these $r\log r^2$ contributions are precisely canceled by the terms $\hat\sigma_2^2\,r\,\log(r^2)$, $\hat\sigma_3^2\,r\,\log(r^2)$ and $\hat\sigma_2^3\,r\,\log(r^2)$ in (5.27), as required.

The terms in equations (5.27) satisfy an extended transcendentality condition in which a power of $r^{\pm(1+2m)}$ (with $m \geq 0$) contributes weight $-m$ and $\log(r^2)$ has weight 1. As before, $\zeta(k)$ has weight $k$ and $\pi$ has weight 1. The total weight of any term of order $(\alpha')^q$ is once again equal to $q+1$.

Although we have displayed results up to order $(\alpha'\,s)^6\,\mathbf{R}^4$, we have also obtained partial results at all orders. Thus, we have evaluated the coefficients of all terms of the form $\hat\sigma_2^n$ (such as $\hat\sigma_2^4$, which corresponds to one of the kinematic structures appearing at order $(\alpha'\,s)^8\,\mathbf{R}^4$), in terms of harmonic sums or multiple zeta values, using general expressions derived in the appendices. It is notable that, at least in all cases studied here, these reduce to the product of Riemann zeta values. This leads to the interesting possibility that the coefficients of the momentum expansion of the genus-one four-particle amplitude are all rational numbers multiplying products of Riemann zeta values. Even more interesting is the possibility (motivated by results from eleven-dimensional supergravity on $S^1$ and on $T^2$ [5]) that this property holds for all genera in string perturbation theory.

We have also determined the coefficients $u_k$ of terms of the form $u_k\,(\alpha'\,s)^k\zeta(2k-1)r^{2k-1}$, which contains the leading power of $r$ for a given value of $k$. These coefficients follow from the methods described in appendix D, and the results agree with the type IIA expressions derived by taking the one-loop amplitude of eleven dimensional supergravity on $T^2$ [13, 8]. The subleading contributions of the form $v_k\,(\alpha'\,s)^k\,r^{2k-7}$ match those obtained from the two-loop amplitude of eleven dimensional supergravity on $T^2$, at least up to (and including) order $(\alpha'\,s)^6\,\mathbf{R}^4$ [5].

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# A. Some mathematical background

## A.1 Eisenstein series

The non-holomorphic Eisenstein Series $E_s$ are defined by

$$E_s = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m+n\tau|^{2s}} \ . \tag{A.1}$$

We will also use the notation $\hat{E}_s = E_s/(4\pi)^s$ and $E_s^* = \Gamma(s)\pi^{-s}\ E_s$. The function $E_s^*$ can be expanded at large $\tau_2$ as follows

$$\begin{aligned} E_s^*(\tau,\bar\tau) \ &= 2\zeta^*(2s)\,\tau_2^s + 2\zeta^*(2s-1)\,\tau_2^{1-s} \\ &+4\tau_2^{\frac{1}{2}} \sum_{N\neq 0} |N|^{s-\frac{1}{2}}\,\hat\sigma_{1-2s}(|N|)\,K_{s-\frac{1}{2}}(2\pi|N|\tau_2)\,e^{2i\pi N\tau_1} \end{aligned} \tag{A.2}$$

where $\hat\sigma_k(n) = \sum_{d|n} d^k$ is the $k$th divisor function of $n$, $K_s(z)$ are the Bessel functions of the second kind, and

$$\zeta^*(s) = \frac{\zeta(s)\Gamma(s/2)}{\pi^{s/2}} = \sum_{p\geq 1}\int_0^\infty \frac{dt}{t}\, t^{\frac{s}{2}}\, e^{-\pi p^2 t} \ . \tag{A.3}$$

This function satisfies the functional equation $\zeta^*(s) = \zeta^*(1-s)$ as is easily shown using the Poisson resummation formula [14]. The function $E_s^*$ obeys an analogous relation $E_s^* = E_{1-s}^*$ and satisfy the Laplace equation

$$\Delta_\tau E_s^*(\tau,\bar\tau) = s(s-1)E_s^*(\tau,\bar\tau) \ . \tag{A.4}$$

In the main text we need to integrate the product of a pair of Eisenstein series that appears in the integrals of the functions $P^{(p,q)}(\{E_r\})$ over $\mathcal{F}_L$. We may use (A.4) to replace one factor of $\hat{E}_s$ in the integral by $\Delta\hat{E}_s$ and then integrate by parts to give the result

$$\frac{1}{4\zeta^*(2s)\zeta^*(2s')}\int_{\mathcal{F}_L} \frac{d^2\tau}{\tau_2^2} E_s^* E_{s'}^* = \frac{L^{s+s'-1}}{s+s'-1} - \frac{L^{1-s-s'}}{s+s'-1}\phi(s)\phi(s') + \frac{L^{s-s'}}{s-s'}\phi(s') - \frac{L^{s'-s}}{s-s'}\phi(s) + o(1), \tag{A.5}$$

where

$$\phi(s) = \zeta^*(2s-1)/\zeta^*(2s)\,. \tag{A.6}$$

In section 3 we need to use the special cases

$$\phi(2) = \frac{\pi\zeta(3)}{2\zeta(4)}\,, \qquad \phi(3) = \frac{3\pi\zeta(5)}{8\zeta(6)}\,. \tag{A.7}$$

For $s > s'$ and in the $L\to\infty$ limit, the right-hand side of (A.5) contains two terms proportional to $L^{s+s'-1}$ and $L^{s-s'}$, respectively. When $s = s'$ the integral may be evaluated by taking the $s\to s'$ limit of (A.5).

$$\frac{1}{4\zeta^*(2s)^2}\int_{\mathcal{F}_L} \frac{d^2\tau}{\tau_2^2}\,(E_s^*)^2 = \frac{L^{2s-1}}{2s-1} - \frac{L^{1-2s}}{2s-1}\phi^2(s) + 2\phi(s)\,\log(L/\tilde\mu_{2s}) + o(1)\,, \tag{A.8}$$

where

$$\log(\tilde{\mu}_{2s}) = \frac{1}{2}\frac{\phi'(s)}{\phi(s)} = \frac{\zeta'(2s)}{\zeta(2s)} + \frac{\Gamma'(s-1/2)}{2\Gamma(s-1/2)} - \frac{\Gamma'(s)}{2\Gamma(s)}\,. \tag{A.9}$$

For the cases of interest in the main text we need

$$\log\tilde{\mu}_4 = \frac{1}{2} - \log(2) + \frac{\zeta'(3)}{\zeta(3)} - \frac{\zeta'(4)}{\zeta(4)} \tag{A.10}$$

$$\log\tilde{\mu}_6 = \frac{7}{12} - \log(2) + \frac{\zeta'(5)}{\zeta(5)} - \frac{\zeta'(6)}{\zeta(6)} \tag{A.11}$$

### A.2 Space of square integrable functions

Consider a modular function $f(\tau,\bar{\tau})$ with the following zero mode expansion

$$f^0(\tau_2) \equiv \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 f^0(\tau,\bar{\tau}) = \sum_{k=1}^{n}\frac{a_k}{\tau_2^k} + \sum_{N\neq 0}\sum_{k=0}^{m}\frac{b_k(|N|)}{(2\pi\tau_2)^k}\,e^{-2\pi|N|\tau_2}\,. \tag{A.12}$$

The sums over $k$ are of finite range, $n$ and $m$ are in general different. The coefficients $a_k$ and $b_k(|N|)$ are constrained by the modular invariance of the function $f(\tau,\bar{\tau})$. Prototypes of such functions are the modular functions $\delta j^{(p,q)}$ arising from the derivative expansion of the string loop amplitude, where all the positive powers in $\tau_2$ in the zero mode expansion have been subtracted by polynomials in the Eisenstein series as in appendix C.

In this case the integral of $f(\tau,\bar{\tau})$ over a fundamental domain of $SL(2,\mathbb{Z})$ converges since this function is square integrable in the fundamental domain. We apply the following lemma given, for example, on page 256 of [6]:

**Lemma.** *The space of square integrable functions on $L^2(\mathcal{F})$ on a fundamental domain $\mathcal{F} = SL(2,\mathbb{Z})\backslash H$ is given by the orthogonal decomposition*

$$L^2(\mathcal{F}) = L_0^2(\mathcal{F}) \oplus \mathbb{C} \oplus \theta_0 \tag{A.13}$$

*where*

$$L_0^2(\mathcal{F}) = \{f \in L^2(\mathcal{F}) | \int_{-1/2}^{1/2} d\tau_1 f(\tau) = 0 for\ almost\ all\ \tau_2 > 0\} \tag{A.14}$$

*and $\theta_0$ denotes the closed subspace of $L^2(\mathcal{F})$ generated by the incomplete theta series,*

$$T\psi(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi(\Im m(\gamma\tau)) \text{ for } \tau\in H \text{ and } \Gamma_\infty = \{\pm\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} | n\in\mathbb{Z}\} \tag{A.15}$$

*such that*

$$\int_0^\infty \frac{d\tau_2}{\tau_2^2}\psi(\tau_2) = 0 = \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\,T\psi(\tau)\,,$$

*where $\psi$ is smooth with compact support on $\mathbb{R}^+$.*

Since $f^0$ in (A.12) does not have any component on $L_0^2(\mathcal{F})$ and does not contain any constant term it belongs to the space $\theta_0$. For any incomplete theta series in $\theta_0$ the integral over the fundamental domain vanishes. Therefore for any $f\in\theta_0$ we have

$$\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\,f(\tau) = 0\,.$$

### A.3 Harmonic sums

In appendix B we will find different types of harmonic sums that arise upon integrating over the vertex positions. In this subsection we start by discussing the simplest cases.

#### A.3.1 $S(m,n)$ and $S(\alpha_1,\cdots,\alpha_n;\beta)$

Consider the sum

$$S(m,n) = \sum_{k_1,\ldots,k_m\neq 0} \frac{\delta(\sum_{1\leq i\leq m} k_i)}{|k_1\cdots k_m|(|k_1|+\cdots+|k_m|)^n}\,, \qquad m\geq 2\,. \tag{A.16}$$

As shown in section A.3.2, this sum reduces to a sum over multi-zeta values (MZV's), with the result (for $m \geq 2$)

$$S(m,n) = m! \sum_{\substack{a_1,\ldots,a_r\in\{1,2\}\\ a_1+\cdots+a_r=m-2}} 2^{2(r+1)-m-n}\,\zeta(n+2,a_1,\ldots,a_r)\,, \tag{A.17}$$

where $\zeta(n_1,\ldots,n_r)$ is a multiple zeta value of weight $w=\sum_{i=1}^r s_i$ and depth $r$ defined by

$$\zeta(s_1,\ldots,s_r) = \sum_{n_1>n_2>\cdots>n_r\geq 1} \frac{1}{n_1^{s_1}\cdots n_r^{s_r}}\,. \tag{A.18}$$

In general, $\zeta(m,n)$ does not reduce to a polynomial in zeta values.

In the following we give details of $S(3,n)$, $S(4,n)$ and $S(5,1)$, specializing to the cases needed in the main text.

- $S(2,n)$ is given by
$$S(2,n) = 2^{1-n}\,\zeta(n+2)\,. \tag{A.19}$$

- $S(3,n)$ is given by
$$\begin{aligned} S(3,n) &= \frac{3}{2^{n-2}}\,\zeta(n+2,1) \\ &= \frac{3}{2^{n-1}}\left((n+2)\zeta(n+3) - \sum_{k=1}^{n}\zeta(n+2-k)\zeta(k+1)\right)\,, \end{aligned} \tag{A.20}$$

 which has been reduced to zeta values using the identity [15]

$$\zeta(n,1) = \frac{n}{2}\zeta(n+1) - \frac{1}{2}\sum_{k=1}^{n-2}\zeta(n-k)\zeta(k+1) \tag{A.21}$$

 In particular, using various expressions for MZV given in the references [15 − 20] we have

$$S(3,1) = \frac{3}{2}\,\zeta(4) \tag{A.22}$$
$$S(3,2) = 6\zeta(5) - 3\zeta(2)\zeta(3) \tag{A.23}$$
$$S(3,3) = \frac{9}{8}\zeta(6) - \frac{3}{4}\zeta(3)^2 \tag{A.24}$$

 where we have used $\zeta(2)\zeta(4) = 7\zeta(6)/4$ and $\zeta(2)^3 = 35\zeta(6)/8$.

- $S(4,n)$ is given by

$$S(4,n) \;=\; \frac{4!}{2^n}\;\left(\zeta(n+2,2)+4\zeta(n+2,1,1)\right)\;.$$

In particular, we find

$$S(4,1) \;=\; 30\zeta(5)-12\zeta(2)\zeta(3) \tag{A.25}$$

$$S(4,2) \;=\; \frac{53}{2}\zeta(6)-18\zeta(3)^2 \tag{A.26}$$

$$S(4,3) \;=\; -\frac{4}{5}\,\zeta(3)\,\zeta(4)-21\,\zeta(2)\,\zeta(5)+27\,\zeta(7) \tag{A.27}$$

$$S(4,4) \;=\; -\frac{3}{5}\zeta(6,2)+3\,\zeta(2)\,\zeta(3)^2-15\,\zeta(3)\,\zeta(5)+\frac{463}{40}\,\zeta(8) \tag{A.28}$$

In the last expression we see the appearance, at weight 8, of the MZV $\zeta(6,2)$ that does not reduce to a polynomial in zeta values.

Another sum that appears is

$$S(\alpha_1,\ldots,\alpha_n;\beta)=\sum_{m_i\geq 1}\frac{1}{m_1^{\alpha_1}\cdots m_n^{\alpha_n}\,(m_1+\cdots+m_n)^{\beta}}\;. \tag{A.29}$$

Ordering the $\alpha_i$ as $\alpha_n\geq\alpha_{n-1}\geq\cdots\geq\alpha_1$ and introducing the variables $\mu_1=m_1$, and $\mu_r=\mu_{r-1}+m_r$ for $2\leq r\leq n$, this sum can be written as

$$S(\alpha_1,\ldots,\alpha_n;\beta)=\sum_{\mu_n>\mu_{n-1}>\cdots>\mu_1\geq 1}\frac{1}{\mu_n^{\beta}(\mu_n-\mu_{n-1})^{\alpha_n}\cdots(\mu_2-\mu_1)^{\alpha_2}\mu_1^{\alpha_1}}\;. \tag{A.30}$$

A repeated use of the identity [21]

$$\frac{1}{m^i n^j}=\sum_{\substack{r+s=i+j\\ r,s>0}}\frac{\binom{r-1}{i-1}}{(m+n)^r n^s}+\frac{\binom{r-1}{j-1}}{(m+n)^r m^s}\,,\qquad i,j>0\,, \tag{A.31}$$

gives, for $\alpha_i>0$ and $\beta>0$,

$$\begin{aligned}S(\alpha_n,\ldots,\alpha_1;\beta) \;=\; &\sum_{\substack{r_1+s_1=\alpha_2+r_0\\ r_1,s_1>0}}\;\sum_{\substack{r_2+s_2=\alpha_3+r_1\\ r_2,s_2>0}}\cdots\sum_{\substack{r_{n-1}+s_{n-1}=\alpha_n+r_{n-2}\\ r_{n-1},s_{n-1}>0}}\\ &\times\prod_{i=1}^{n-1}\left[\binom{r_i-1}{\alpha_{n+1}-1}+\binom{r_i-1}{r_{n-1}-1}\right]\,\delta_{r_0,\alpha_1}\zeta(\beta+r_{n-1},s_{n-1},\ldots,s_1)\end{aligned}$$

In the special case $\beta=0$ we find

$$S(\alpha_n,\ldots,\alpha_1;0)=\prod_{i=1}^{n}\zeta(\alpha_i)\,. \tag{A.32}$$

The $n = 2$ case gives the Witten zeta-function $S(\alpha_2, \alpha_1, \beta) = W(\alpha_2, \alpha_1, \beta)$, with

$$W(\alpha_1, \alpha_2, \beta) = \sum_{m,n=1}^{\infty} \frac{1}{m^{\alpha_2} n^{\alpha_1} (m+n)^{\beta}} = \sum_{\substack{r+s=\alpha_2+\alpha_1 \\ r,s>0}} \left[ \binom{r-1}{\alpha_2 - 1} + \binom{r-1}{\alpha_1 - 1} \right] \zeta(\beta + r, s)\,, \qquad \text{(A.33)}$$

so

$$W(0, \alpha, \beta) = W(\alpha, 0, \beta) = \zeta(\beta, \alpha)\,, \qquad W(\alpha, \beta, 0) = \zeta(\alpha)\zeta(\beta)\,. \qquad \text{(A.34)}$$

If $\alpha_i = 0$ for $p$ values of $i$ we have

$$S(\alpha_n, \ldots, \alpha_{p+1}, 0, \ldots, 0; \beta) = \zeta(\beta, \alpha_n, \ldots, \alpha_{p+1}, 0, \ldots, 0) \qquad \text{(A.35)}$$

### A.3.2 Evaluation of $S(m, n)$ by Don Zagier

In this appendix we give a proof due to Don Zagier of the general formula (A.17) for the values of the sums

$$S(m, n) = \sum_{k_1, \ldots, k_m \neq 0} \frac{\delta(\sum_{1 \le i \le m} k_i)}{|k_1 \cdots k_m| (|k_1| + \cdots + |k_m|)^n}\,, \qquad (m, n \ge 0)\,. \qquad \text{(A.36)}$$

(Note that $S(m, n) = 0$ if $m < 2$, since then the sum is empty.) Denoting by $r$ and $m - r$ the number of $i$ with $k_i > 0$ and $k_i < 0$, respectively, and by $l$ the sum of the positive $k_i$, we can rewrite $S(m, n)$ as

$$S(m, n) = \sum_{r=0}^{m} \binom{m}{r} \sum_{l=1}^{\infty} \frac{S_r(l) S_{m-r}(l)}{(2l)^n} \qquad \text{(A.37)}$$

where

$$S_r(l) = \sum_{\substack{k_1, \cdots k_r \ge 1 \\ k_1 + \cdots + k_r = l}} \frac{1}{k_1 \cdots k_r} \qquad (= 0 \quad \text{if} \quad r = 0)\,. \qquad \text{(A.38)}$$

Observing that $S_r(l)$ is the coefficient of $x^l$ in the series expression of $\mathrm{Li}_1(x)^r$, where $\mathrm{Li}_1(x) = \sum_{k \ge 1} x^k / k = -\log(1 - x)$, we obtain

$$2^n\, S(m, n) = \sum_{l=1}^{\infty} \frac{1}{l^n} \sum_{r=0}^{m} \binom{m}{r} \mathrm{coeff}_{x^l y^l} \left[ \mathrm{Li}_1(x)^r\, \mathrm{Li}_1(y)^{m-r} \right] = \sum_{l=1}^{\infty} \frac{1}{l^n} \mathrm{coeff}_{x^l y^l} \left[ \left( \mathrm{Li}_1(x) + \mathrm{Li}_1(y) \right)^m \right] \qquad \text{(A.39)}$$

Hence the generating function $\sum_{m \ge 0} S(m, n)\, X^m / m!$ is given by

$$2^n \sum_{m \ge 0} S(m, n) \frac{X^m}{m!} = \sum_{l=1}^{\infty} \frac{1}{l^n} \mathrm{coeff}_{x^l y^l} \left[ \exp\left( X(\mathrm{Li}_1(x) + \mathrm{Li}_1(y)) \right) \right] = \sum_{l=1}^{\infty} \frac{1}{l^n} \mathrm{coeff}_{x^l y^l} \left[ (1 - x)^{-X} (1 - y)^{-X} \right]$$

$$
\begin{aligned}
&= \sum_{l=1}^{\infty} \frac{1}{l^n} \binom{X+l-1}{l}^2 \\
&= \sum_{l=1}^{\infty} \frac{X^2}{l^{n+2}} \prod_{h=1}^{l-1} \left(1+\frac{X}{h}\right)^2 \\
&= \sum_{l=1}^{\infty} \frac{X^2}{l^{n+2}} \prod_{h=1}^{l-1} \left(1+4\sum_{a\in\{1,2\}} \frac{(X/2)^a}{h^a}\right) \\
&= \sum_{r\geq 0} \sum_{\substack{l>h_1>\cdots>h_r>0 \\ a_1,\ldots,a_r\in\{1,2\}}} \frac{4^r\, X^2 (X/2)^{a_1+\cdots+a_r}}{l^{n+2}\, h_1^{a_1}\cdots h_r^{a_r}} \,. \qquad \text{(A.40)}
\end{aligned}
$$

Comparing the coefficients of $X^m$ on both sides gives the desired formula

$$
S(m,n) = \frac{m!}{2^{m+n-2}} \sum_{\substack{a_1,\ldots,a_r\in\{1,2\} \\ a_1+\cdots+a_r=m-2}} 2^{2r}\, \zeta(n+2,a_1,\ldots,a_r)\,. \qquad \text{(A.41)}
$$

## B. Properties of $\mathrm{D}_{\ell_1,\ldots,\ell_6}$

The coefficients of the terms in the analytic part of the momentum expansion are determined in terms of the functions $\mathrm{D}_{\ell_1,\ldots,\ell_6}$ ($\sum_{k=1}^{6} l_k = r$) associated with the diagrams shown in figures 3–6. In the main part of the paper these enter in two separate manners.

1) In section 5 we considered compactification on a circle of radius $r$ and considered the coefficients of terms at each order up to $s^6$ that are power-behaved in $r$ for large $r$. In this case the coefficients are determined by knowledge of the terms in $\mathrm{D}_{\ell_1,\ldots,\ell_6}$ that are power-behaved in $\tau_2$ for large $\tau_2$. Such terms are obtained by expanding the $\tau_1$,

$$
\mathrm{D}^{(0)}_{\{\ell\}}(\tau_2) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{D}_{\{\ell\}}(\tau,\bar\tau) \qquad \text{(B.1)}
$$

in powers of $\tau_2^{-1}$. This will also be carried out in this appendix for all the $D_{\{\ell\}}$'s that enter up to order $s^6$ (although we will not evaluate the coefficients of the negative powers of $r$ at order $s^5$ and beyond).

2) In section 3 we saw that the values of the coefficients of terms in the low-momentum expansion in ten dimensions are determined in terms of a constant that survives the $\tau_2 \to \infty$ limit after subtracting the positive powers of $\tau_2$ with a polynomial in Eisenstein series, $P^{(p,q)}(\{\hat E_r\})$. In this appendix we will here derive $P^{(p,q)}$ and the value of the constant for each $D_{\{\ell\}}$ function that enters up to order $s^6$. In fact, the part of $D_{\{\ell\}}$ that has positive powers of $\tau_2$, or is constant in the large-$\tau_2$ limit, is independent of $\tau_1$ so it is also determined by $D^{(0)}_{\{\ell\}}$.

For these reasons, our priority here is to compute the $\tau_1$ zero modes of the $D_{\{\ell\}}$'s. For this purpose we will need to make extensive use of the representation of the propagator as the sum $\hat{\mathcal{P}}^{\infty}(\nu) + \tilde P(\nu)$ given in (2.14). A few special cases were evaluated in [4].

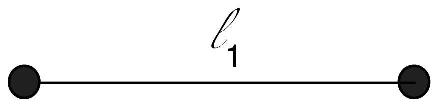


**Figure 3:** The two-vertex diagram with $\ell_1$ lines connecting the two vertices defines $\mathrm{D}_{\ell_1}$. Each line is associated with a propagator on the torus with momentum $\mathbf{p}_i = m_i + n_i\tau$ $(i = 1, \dots, l_1)$.

### B.1 The two-vertex case

The two-vertex diagram with $\ell_1$ lines is associated with a modular function of weight $\ell_1$, which we will denote by

$$\mathrm{D}_{\ell_1} \equiv \mathrm{D}_{\ell_1,0,0,0,0,0} \,, \tag{B.2}$$

in which case we have

$$\mathrm{D}_{\ell_1} = \sum_{(m_i,n_i)\neq(0,0)} \delta(\sum_{r=1}^{l_1} \mathbf{p}_r) \prod_{i=1}^{\ell_1} \frac{1}{4\pi} \frac{\tau_2}{|m_i + n_i\tau|^2} \,. \tag{B.3}$$

The zero mode is given by[5]

$$\begin{aligned}
\mathrm{D}^{(0)}_{\ell_1} &= \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \int \frac{d\nu^{(1)} d\nu^{(2)}}{\tau_2^2} \mathcal{P}_{12}^{\ell_1} \\
&= \sum_{r+s=\ell_1} \binom{\ell_1}{r} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \int \frac{d\nu^{(1)} d\nu^{(2)}}{\tau_2^2} (\hat{\mathcal{P}}^{\infty}_{12})^r (\tilde{P}_{12})^s \\
&= \sum_{r+s=\ell_1} \binom{\ell_1}{r} \int_0^1 d\hat\nu_2^{(1)} d\hat\nu_2^{(2)} (\hat{\mathcal{P}}^{\infty}_{12})^r \\
&\quad \times \sum_{\substack{m_1,\dots,m_s\neq 0\\ k_1\cdots k_s\in\mathbb{Z}}} \frac{\delta(\sum_i m_i)\delta(\sum_i m_i k_i)}{4^s |m_1\cdots m_s|} e^{-2\pi\tau_2 \sum_i |m_i||k_i - \hat\nu_2^{(12)}|} \,.
\end{aligned} \tag{B.4}$$

A direct evaluation of the integrals using the identity

$$\int_0^1 dx_1 \int_0^{x_1} dx_2 f(x_1 - x_2) = \int_0^1 dx\,(1-x)\,f(x) \,, \tag{B.5}$$

with $f(x) = (\hat{\mathcal{P}}^{\infty}(x))^n \exp(-2\pi\tau_2 \sum_i |m_i||k_i - x|)$, and using the periodicity of the asymptotic propagator $\hat{\mathcal{P}}^{\infty}(x)$, leads to

$$\begin{aligned}
\mathrm{D}^{(0)}_{\ell_1} &= \left(\frac{\pi\tau_2}{12}\right)^{\ell_1} {}_2\mathrm{F}_1(1, -\ell_1, 3/2; 3/2) + \\
&\quad + \frac{2}{4^{\ell_1}} \sum_{k=0}^{\ell_1 - 2} \sum_{k_1+k_2+k_3=k} \frac{(-1)^{k_2}}{6^{k_3}} \frac{\ell_1!(2k_1+k_2)!}{(\ell_1-k)!k_1!k_2!k_3!} (2\pi\tau_2)^{k_3-k_1-1} \times \\
&\quad \times S(\ell_1 - k, 2k_1 + k_2 + 1) + O(\exp(-\tau_2)) \,,
\end{aligned} \tag{B.6}$$

[5] In this section we use the condensed notation $\mathcal{P}_{ij} = \mathcal{P}(\nu^{(ij)})$, $\hat{\mathcal{P}}^{\infty}_{ij} = \hat{\mathcal{P}}^{\infty}(\nu^{(ij)})$ and $\tilde{P}_{ij} = \tilde{P}(\nu^{(ij)})$.

where the sums $S(m,n)$ are defined in (A.16) and evaluated in section A.3.2.

Now we will write down the power-behaved terms in the large $\tau_2$ expansion for some particular cases. The first non-trivial case is $\mathrm{D}_2 = \hat{E}_2$. The next cases are $\mathrm{D}_3$, $\mathrm{D}_4$ (called $B_2$ and $C_4$ in [4]). We will also need $\mathrm{D}_5$ and $\mathrm{D}_6$. Their zero-mode expansions are given by

$$\mathrm{D}_2^{(0)} = \frac{1}{(4\pi)^2}\left[2\zeta(4)\tau_2^2 + \frac{\pi\zeta(3)}{\tau_2}\right] \tag{B.7}$$

$$\mathrm{D}_3^{(0)} = \frac{1}{(4\pi)^3}\left[2\zeta(6)\tau_2^3 + \pi^3\zeta(3) + \frac{3\pi}{4}\frac{\zeta(5)}{\tau_2^2}\right] + O(e^{-\tau_2}) \tag{B.8}$$

$$\mathrm{D}_4^{(0)} = \frac{1}{(4\pi)^4}\left[10\zeta(8)\tau_2^4 + \frac{2\pi^5}{3}\zeta(3)\,\tau_2 + \frac{10\pi^3\zeta(5)}{\tau_2} - \frac{3\pi^2\zeta(3)^2}{\tau_2^2} + \frac{9\pi}{4}\frac{\zeta(7)}{\tau_2^3}\right] + O(e^{-\tau_2}) \tag{B.9}$$

$$\begin{aligned}\mathrm{D}_5^{(0)} = \frac{1}{(4\pi)^5}\Bigg[&20\zeta(10)\tau_2^5 + \frac{10\,\pi^7\,\zeta(3)}{27}\,\tau_2^2 + \frac{95\,\pi^5\,\zeta(5)}{6} \\ &+\frac{900\,\zeta(4)\,\zeta(3)^2}{\tau_2} + \frac{105\,\pi^3\,\zeta(7)}{4\,\tau_2^2} - \frac{135\,\zeta(2)\,\zeta(3)\,\zeta(5)}{\tau_2^3} + \frac{225\,\pi\,\zeta(9)}{16\,\tau_2^4}\Bigg] + O(e^{-\tau_2})\end{aligned} \tag{B.10}$$

$$\begin{aligned}\mathrm{D}_6^{(0)} = \frac{1}{(4\pi)^6}\Bigg[&\frac{46375\,\zeta(12)\,\tau_2^6}{691} + \frac{5\,\pi^9\,\tau_2^3\,\zeta(3)}{27} + 2365\,\zeta(6)\,\zeta(3)^2 + \frac{140\,\pi^7\,\zeta(5)}{9}\,\tau_2 \\ &-\frac{3\pi^5(34020\zeta(4)\,\zeta(3) + 42120\,\zeta(2)\,\zeta(5) - 117115\,\zeta(7))}{32\,\tau_2} - \frac{12150\,\zeta(4)\,\zeta(3)\,\zeta(5)}{\tau_2^2} \\ &+\frac{45\,\pi^3\,(2\,\zeta(3)^3 - 14\,\zeta(3)\,\zeta(6) + 9\,\zeta(9))}{2\,\tau_2^3} \\ &-\frac{4050\,\zeta(2)\,\left(\zeta(5)^2 + 2\,\zeta(3)\,\zeta(7)\right)}{8\,\tau_2^4} + \frac{4725\,\pi\,\zeta(11)}{32\,\tau_2^5}\Bigg] + O(e^{-\tau_2})\end{aligned} \tag{B.11}$$

It is notable that all the coefficients appearing in this expansion are products of zeta values multiplied by rational coefficients. Thanks to this, the positive powers of $\tau_2$ in each $\mathrm{D}_{\{\ell\}}$ function can be matched with a polynomial in Eisenstein series. We find

$$\mathrm{D}_2 = \hat{E}_2 \tag{B.12}$$

$$\mathrm{D}_3 = \hat{E}_3 + \frac{\zeta(3)}{64} + \mathrm{D}_3^{fin} \tag{B.13}$$

$$\mathrm{D}_4 = -30\hat{E}_4 + 15\,\hat{E}_2^2 + \mathrm{D}_4^{fin} \tag{B.14}$$

$$\mathrm{D}_5 = -375\,\hat{E}_5 + 175\,\hat{E}_2\hat{E}_3 + \frac{155}{12288}\,\zeta(5) + \mathrm{D}_5^{fin} \tag{B.15}$$

$$\mathrm{D}_6 = -\frac{8297625}{691}\,\hat{E}_6 + 4900\,\hat{E}_3^2 + 875\hat{E}_2\hat{E}_4 + \frac{25}{4096}\,\zeta(3)^2 + \mathrm{D}_6^{fin} \tag{B.16}$$

where $\mathrm{D}_{\ell_1}^{fin}$ is such that $\lim_{\tau_2\to\infty}\mathrm{D}_{\ell_1}^{fin} = 0$.

### B.2 The three-vertex case

The three-vertex diagrams are associated with the functions $\mathrm{D}_{\ell_1,\ell_2,\ell_3} \equiv \mathrm{D}_{\ell_1,\ell_2,\ell_3;0,0,0}$, which

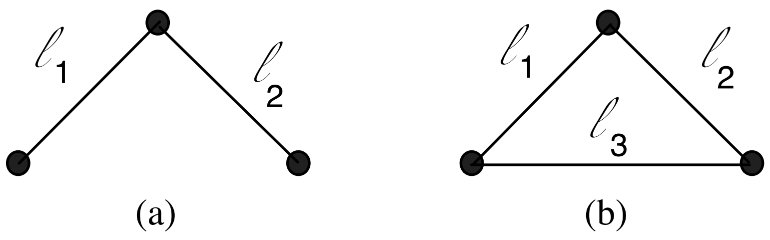


**Figure 4:** The two possible three-vertex diagrams, with $\ell_1$, $\ell_2$ and $\ell_3$ lines joining the vertices defines $\mathrm{D}_{\ell_1,\ell_2,\ell_3}$. Setting $l_3 = 0$ in (b) reduces the figure to the product of two two-vertex diagrams $\mathrm{D}_{\ell_1,\ell_2,\emptyset} = \mathrm{D}_{\ell_1} \times \mathrm{D}_{\ell_2}$ shown in (a).



has the form

$$\mathrm{D}_{\ell_1,\ell_2,\ell_3} = \sum_{(m_i^{(j)},n_i^{(j)})\neq(0,0)} \prod_{1\leq r<s\leq 3} \delta\left(\sum_{k=1}^{l_r} \mathbf{p}_k^{(r)} - \sum_{k=1}^{l_s} \mathbf{p}_k^{(s)}\right) \prod_{j=1}^{3}\prod_{i=1}^{\ell_j} \frac{1}{4\pi} \frac{\tau_2}{|m_i^{(j)} + n_i^{(j)}\tau|^2}\,, \tag{B.17}$$

where $\mathbf{p}_k^{(r)}$ is the momentum of the $k$th line in the $r$th leg. Momentum conservation at each vertex implies that when one set of integers is empty the diagram reduces to the product of two two-vertex diagrams, $\mathrm{D}_{\ell_1,\ell_2,\emptyset} = \mathrm{D}_{\ell_1} \times \mathrm{D}_{\ell_2}$. Some particular cases are

$$\begin{aligned} \mathrm{D}_{1,1,1} &= \hat{E}_3\,, \\ \mathrm{D}_{0,2,2} &= \hat{E}_2^2\,, \\ \mathrm{D}_{0,2,3} &= \mathrm{D}_3 \times \hat{E}_2\,. \end{aligned} \tag{B.18}$$

For the general case where all the $\ell_i$ are non zero we extract the zero mode contribution as in the previous subsection by splitting the string propagator into its asymptotic part and the finite part at large $\tau_2$. Starting with

$$\mathrm{D}^{(0)}_{\ell_1,\ell_2,\ell_3} = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \int \frac{d\nu^{(1)} d\nu^{(2)} d\nu^{(3)}}{\tau_2^3}\, \mathcal{P}_{12}^{\ell_1}\mathcal{P}_{23}^{\ell_2}\mathcal{P}_{13}^{\ell_3}\,, \tag{B.19}$$

where none of the integers $n_i$ is zero and splitting the propagator, we find the general expression

$$\begin{aligned} \mathrm{D}^{(0)}_{\ell_1,\ell_2,\ell_3} = &\sum_{\ell_i=r_i+s_i} \prod_{i=1}^{3} \frac{\ell_i!}{r_i! s_i!} \int_0^1 d\hat{\nu}_2^{(i)}\, (\hat{\mathcal{P}}_{12}^{\infty})^{r_1} (\hat{\mathcal{P}}_{23}^{\infty})^{r_2} (\hat{\mathcal{P}}_{13}^{\infty})^{r_3} \times \\ &\times \sum_{\substack{m_i^{(ij)}\\ k_l^{(ij)}}} \prod_{1\leq i<j\leq 3} \frac{\delta(\sum_i m_i^{(ij)})}{|m_1^{(ij)}|\cdots|m_{s_{ij}}^{(ij)}|} \frac{\delta(\sum_{i,j,l} m_l^{(ij)} k_l^{(ij)})}{4^{s_1+s_2+s_3}} \times \\ &\times e^{-2\pi\tau_2 \sum |m_l^{(ij)}||k_l^{(ij)} - \hat{\nu}_2^{(ij)}|}\,. \end{aligned} \tag{B.20}$$

It is convenient to decompose this into different contributions:

$$\mathrm{D}^{(0)}_{\ell_1,\ell_2,\ell_3} = \mathrm{D}^{\infty}_{\ell_1,\ell_2,\ell_3} + \mathrm{D}^{(a)}_{\ell_1,\ell_2,\ell_3} + \mathrm{D}^{(b)}_{\ell_1,\ell_2,\ell_3} + \mathrm{D}^{(c)}_{\ell_1,\ell_2,\ell_3} + \mathrm{D}^{(c)}_{\ell_2,\ell_1,\ell_3} + \mathrm{D}^{(c)}_{\ell_3,\ell_2,\ell_1} \tag{B.21}$$

where $\mathrm{D}^{\infty}$ represents the contribution where $s_i = 0$ for all $i = 1, 2, 3$, $\mathrm{D}^{(a)}$ and $\mathrm{D}^{(b)}$ represent the cases where $s_i \geq 1$ for all $i = 1, 2, 3$ and $s_1 = 0$ with $s_2, s_3 \geq 2$, and $\mathrm{D}^{(c)}$ represents the cases where $s_1 = s_2 = 0$, $s_3 \geq 2$. The resulting expressions involve sums of the form

$$S(s_1, s_2; s_3; \alpha, \beta) = \sum_{(\underline{m^1}, \underline{m^2}, \underline{m^3})} \frac{\delta(\sum_j m^1_i - \sum_j m^2_j)\delta(\sum_j m^1_i - \sum_j m^3_j)}{\prod_{j=1}^{s_1} |m^1_j| \prod_{j=1}^{s_2} |m^2_j| \prod_{j=1}^{s_3} |m^3_j|} \times \frac{1}{(|\underline{m^1}| + |\underline{m^3}|)^{\alpha}(|\underline{m^2}| + |\underline{m^3}|)^{\beta}}, \tag{B.22}$$

where $|\underline{m^i}| = \sum_j |m^i_j|$. This is not one of the expressions that we analyzed in the earlier appendices.

For the term $\mathrm{D}^{\infty}_{\ell_1,\ell_2,\ell_3}$ we find

$$\mathrm{D}^{\infty}_{\ell_1,\ell_2,\ell_3} = 2 \left(\frac{\pi\tau_2}{2}\right)^{\ell_{123}} \sum_{a_i+b_i+c_i=\ell_i} \prod_{i=1}^{3} \frac{\ell_i!}{a_i!b_i!c_i!} \frac{(-1)^{b_{123}}}{6^{c_{123}}} \times \frac{(2a_2+b_2)!(2a_3+b_3)!}{(2(a_2+a_3)+b_2+b_3+1)!} \frac{1}{(2a_{123}+b_{123}+2)}, \tag{B.23}$$

where we are using the compact notation,

$$x_{123} = x_1 + x_2 + x_3, \qquad \text{with } \ \mathrm{x} = \mathrm{a}, \mathrm{b}, \mathrm{c} \text{ or } \mathrm{n}. \tag{B.24}$$

When all $s_1$, $s_2$ and $s_3$ are greater than 1, or $s_1 = 0$ and $s_2$ and $s_3$ greater than 2 we have

$$\mathrm{D}^{(a)}_{\ell_1,\ell_2,\ell_3} = \frac{2}{4^{\ell_{123}}} \widehat{\sum} \prod_{i=1}^{3} \frac{r_i!}{a_i!b_i!c_i!} \frac{(-1)^{b_{123}}}{6^{c_{123}}} (2\pi\tau_2)^{c_{123}-a_{123}-2} \times \times \frac{(2a_3+b_3)!l_1!l_2!}{q_1!q_2!} S(s_1, s_2; s_3; l_1+1, l_2+1), \tag{B.25}$$

where the summation is over

$$\ell_i = r_i + s_i, \tag{B.26}$$
$$r_i = a_i + b_i + c_i, \tag{B.27}$$
$$q_1 + q_2 = 2a_3 + b_3, \tag{B.28}$$
$$l_i = 2a_i + b_i + q_i, \tag{B.29}$$

and

$$\mathrm{D}^{(b)}_{\ell_1,\ell_2,\ell_3} = \frac{2}{4^{\ell_{123}}} \widehat{\sum} \prod_{i=1}^{3} \frac{r_i!}{a_i!b_i!c_i!} \frac{(-1)^{b_{123}}}{6^{c_{123}}} (2\pi\tau_2)^{c_{123}-a_{123}-2} \frac{(2a_3+b_3)!}{q_1!q_2!} \times \left[\sum_{r=0}^{l_1} (-1)^{q_2} \frac{l_1!(l_2+r)!}{r!} S(s_3, s_2; s_1; l_1-r+1, l_2+r+1) + (1 \leftrightarrow 2)\right] \tag{B.30}$$

and when $s_1 \geq 2$ and $s_2 = s_3 = 0$ in (B.19)

$$\mathrm{D}^{(c)}_{\ell_1,\ell_2,\ell_3} = \frac{2}{4^{\ell_{123}}} \widehat{\sum_{\substack{s_2=0\\ s_3=0}}} \prod_{i=1}^{3} \frac{r_i!}{a_i! b_i! c_i!} \frac{(-1)^{b_{123}}}{6^{c_{123}}} (2\pi\tau_2)^{c_{123}-a_{123}-2} \times \\ \times (-1)^{q_2} \frac{(2a_3+b_3)!}{q_1! q_2! (l_2+1)} \Big[(-1)^{b_3}\, l_1! S(s_1, l_1+1)\, (2\pi\tau_2)^{l_2+1} \\ + (1-(-1)^{b_3})\, (2a_{123}+b_{123}+1)! S(s_1, 2a_{123}+b_{123}+2)\Big]\,. \tag{B.31}$$

Substituting these expressions for $D^\infty$, $D^{(a)}$, $D^{(b)}$ and $D^{(c)}$ into (B.21) gives the expression for $D_{\ell_1,\ell_2\ell 3}$. This completes the general expression for the three-vertex diagrams. We list a few explicit examples obtained by using the above general formulas and simplifying the harmonic sums (B.22). Thankfully, the functions $S(s_1, s_2, s_3; \alpha, \beta)$ that we have not evaluated drop out, apart from the special cases $\alpha = \beta = 0$ and $\alpha = \beta = 1$, which are simple to evaluate directly. The result for the zero mode expansion of the three vertex functions is

$$\mathrm{D}^{(0)}_{1,1,1} = \frac{1}{(4\pi)^3}\left(2\zeta(6)\tau_2^3 + \frac{3\pi}{4}\frac{\zeta(5)}{\tau_2^2}\right) \tag{B.32}$$

$$\mathrm{D}^{(0)}_{1,1,2} = \frac{1}{(4\pi)^4}\left(\frac{4}{3}\zeta(8)\,\tau_2^4 + \frac{2\zeta(4)\zeta(3)}{\pi}\tau_2 + \frac{5\pi}{2}\frac{\zeta(2)\zeta(3)}{\tau_2} \right. \\ \left. + \left(\frac{9}{2}\zeta(2)\zeta(3)^2 - 10\zeta(8)\right)\frac{1}{\tau_2^2} + \frac{9\pi}{16}\frac{\zeta(7)}{\tau_2^3}\right) + O(e^{-\tau_2}) \tag{B.33}$$

$$\mathrm{D}^{(0)}_{1,1,3} = \frac{1}{(4\pi)^5}\left(\frac{42}{5}\zeta(10)\tau_2^5 + 21\zeta(3)\zeta(6)\,\tau_2^2 + \frac{33\pi}{2}\zeta(4)\zeta(5)\right) + O\left(\frac{1}{\tau_2}\right) \tag{B.34}$$

$$\mathrm{D}^{(0)}_{1,2,2} = \frac{1}{(4\pi)^5}\left(\frac{8}{5}\zeta(10)\tau_2^5 + 4\pi\zeta(3)\zeta(6)\,\tau_2^2 + 24\pi\zeta(4)\zeta(5)\right) + O\left(\frac{1}{\tau_2}\right) \tag{B.35}$$

$$\mathrm{D}^{(0)}_{1,1,4} = \frac{1}{(4\pi)^6}\left(\frac{9940}{691}\zeta(12)\tau_2^6 + 140\pi\zeta(3)\zeta(8)\tau_2^3 + \frac{525\pi}{2}\zeta(5)\zeta(6)\tau_2 \right. \\ \left. + \frac{1449}{2}\zeta(3)^2\zeta(6) - \frac{450450}{691}\zeta(12)\right) + O\left(\frac{1}{\tau_2}\right) \tag{B.36}$$

$$\mathrm{D}^{(0)}_{2,2,2} = \frac{1}{(4\pi)^6}\left(\frac{10615}{691}\zeta(12)\tau_2^6 + 30\pi\zeta(3)\zeta(8)\tau_2^3 + 177\pi\zeta(5)\zeta(6)\tau_2 \right. \\ \left. + 945\zeta(3)^2\zeta(6) - \frac{1576575}{1382}\zeta(12)\right) + O\left(\frac{1}{\tau_2}\right) \tag{B.37}$$

$$\mathrm{D}^{(0)}_{1,2,3} = \frac{1}{(4\pi)^6}\left(\frac{4470}{691}\zeta(12)\tau_2^6 + 30\pi\zeta(3)\zeta(8)\tau_2^3 + \frac{519\pi}{4}\zeta(5)\zeta(6)\tau_2 \right. \\ \left. + \frac{63}{2764}(33859\zeta(3)^2\zeta(6) - 35750\zeta(12))\right) + O\left(\frac{1}{\tau_2}\right). \tag{B.38}$$

The notation $O(1/\tau_2)$ indicates the presence of terms that are suppressed by inverse powers of $\tau_2$ that we have not calculated (and there are also exponentially suppressed terms which are not evaluated).

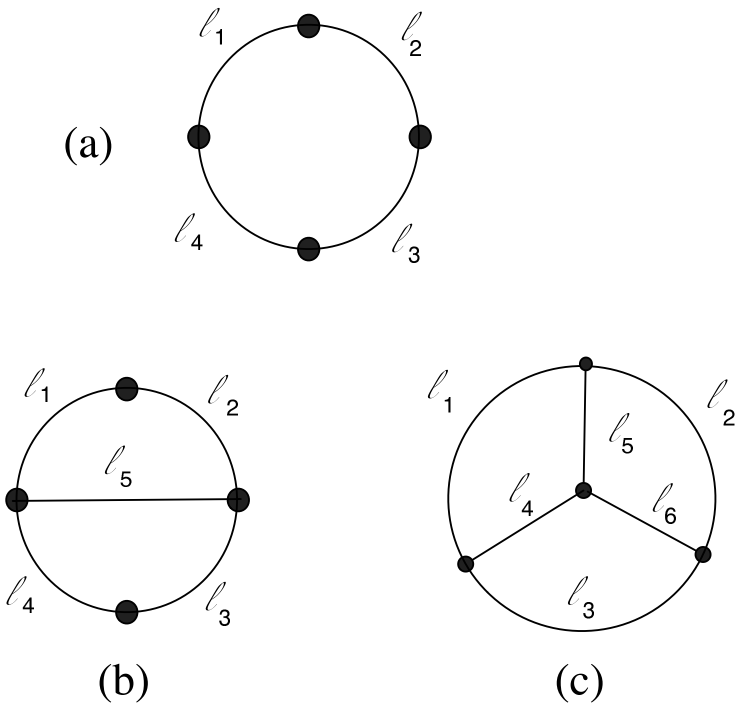


**Figure 5:** The three possible non-degenerate four-vertex diagrams. (a) defines $\mathrm{D}_{\ell_1,\ell_2,\ell_3,\ell_4}$, (b) defines $\mathrm{D}_{\ell_1,\ell_2,\ell_3,\ell_4;\ell_5}$, (c) defines $\mathrm{D}_{\ell_1,\ell_2,\ell_3;\ell_4,\ell_5,\ell_6}$.

As before, we can associate a quadratic function of Eisenstein series that reproduces the positive powers of $\tau_2$ of these zero modes,

$$\mathrm{D}_{1,1,1} = \hat{E}_3 \tag{B.39}$$

$$\mathrm{D}_{1,1,2} = -\frac{1}{2}\hat{E}_4 + \frac{1}{2}\hat{E}_2^2 + \mathrm{D}^{fin}_{1,1,2} \tag{B.40}$$

$$32\mathrm{D}_{1,1,3} - 24\mathrm{D}_{1,2,2} = -\frac{2592}{5}\,\hat{E}_5 + 288\,\hat{E}_2\hat{E}_3 - \frac{11}{1920}\zeta(5) + 32\mathrm{D}^{fin}_{1,1,3} - 24\mathrm{D}^{fin}_{1,2,2} \tag{B.41}$$

$$\begin{aligned}\mathrm{D}_{1,1,4}+\mathrm{D}_{2,2,2}-2\mathrm{D}_{1,2,3} &= -\frac{162575}{691}\,\hat{E}_6 + 60\,\hat{E}_3^2 + 55\,\hat{E}_2\hat{E}_4 - \frac{7}{245760}\zeta(3)^2 \\ &\quad +\mathrm{D}^{fin}_{1,1,4} + \mathrm{D}^{fin}_{2,2,2} - 2\mathrm{D}^{fin}_{1,2,3}\end{aligned} \tag{B.42}$$

### B.3 The four-vertex case

The general four-vertex diagrams shown in figure 5 have between four and six non-zero $\ell_k$'s. We will begin by computing the general expression for the four-vertex function of figure 5(a), in which $\ell_5 = \ell_6 = 0$ and $\ell_1, \ell_2, \ell_3, \ell_4$ are non-zero and arbitrary.

#### B.3.1 $\ell_5 = \ell_6 = 0$

In this case we define $\mathrm{D}_{\ell_1,\ell_2,\ell_3,\ell_4} \equiv \mathrm{D}_{\ell_1,\ell_2,\ell_3,\ell_4;0,0}$, and each four-vertex diagram is the modular function

$$\mathrm{D}_{\ell_1,\ell_2,\ell_3,\ell_4} \equiv \sum_{(m_i^{(j)},n_i^{(j)})\neq(0,0)} \prod_{1\le r<s\le 4} \delta\left(\sum_{k=1}^{l_r}\mathbf{p}^{(r)} - \sum_{k=1}^{l_s}\mathbf{p}_k^{(s)}\right) \prod_{j=1}^{4}\prod_{i=1}^{\ell_j} \frac{1}{4\pi}\frac{\tau_2}{|m_i^{(j)}+n_i^{(j)}\tau|^2} \tag{B.43}$$

In the particular case where one of the $\ell_i$ is zero, say $\ell_4=0$, this reduces to a product of three two-vertex functions, $\mathrm{D}_{\ell_1,\ell_2,\ell_3,0}=\mathrm{D}_{\ell_1}\times\mathrm{D}_{\ell_2}\times\mathrm{D}_{\ell_3}$, as in figure 6(c), or $\mathrm{D}_{\ell_1}\times\mathrm{D}_{\ell_2}$ as in 6(a).

For the general case where all the $\ell_i$ ($i=1,2,3,4$) are non zero we extract the zero mode contribution as in the previous subsections. We write

$$\mathrm{D}^{(0)}_{\ell_1,\ell_2,\ell_3,\ell_4} = \int_{-\frac12}^{\frac12} d\tau_1 \int \frac{d\nu^{(1)}d\nu^{(2)}d\nu^{(3)}}{\tau_2^3}\, \mathcal{P}_{12}^{\ell_1}\mathcal{P}_{23}^{\ell_2}\mathcal{P}_{34}^{\ell_{34}}\mathcal{P}_{14}^{\ell_4} \tag{B.44}$$

with $\nu_4=0$. Then we have to compute the integrals

$$\begin{aligned}\mathrm{D}^{(0)}_{\ell_1,\ell_2,\ell_3,\ell_4} = & \sum_{\ell_i=r_i+s_i}\prod_{i=1}^{4}\frac{\ell_i!}{r_i!s_i!}\int_0^1 d\hat\nu_2^{(i)}\,(\hat{\mathcal{P}}_{12}^{\infty})^{r_1}(\hat{\mathcal{P}}_{23}^{\infty})^{r_2}(\hat{\mathcal{P}}_{34}^{\infty})^{r_3}(\hat{\mathcal{P}}_{14}^{\infty})^{r_4}\times\\ &\times\sum_{\substack{m_i^{(ij)}\\ k_l^{(ij)}}}\prod_{1\le i<j\le 4}\frac{\delta(\sum_i m_i^{(ij)})}{|m_1^{(ij)}|\cdots|m_{s_{ij}}^{(ij)}|}\frac{\delta(\sum_{i,j,l}m_l^{(ij)}k_l^{(ij)})}{4^{s_1+s_2+s_3+s_4}}\times\\ &\times e^{-2\pi\tau_2\sum|m_l^{(ij)}||k_l^{(ij)}-\hat\nu_2^{(ij)}|}\end{aligned} \tag{B.45}$$

The final result can be separated into five contributions:

$$\mathrm{D}^{(0)}_{\ell_1,\ell_2,\ell_3,\ell_4} = \mathrm{D}^{\infty}_{\ell_1,\ell_2,\ell_3,\ell_4} + \mathrm{D}^{(1)}_{\ell_1,\ell_2,\ell_3,\ell_4} + \mathrm{D}^{(a)}_{\ell_1,\ell_2,\ell_3,\ell_4} + \mathrm{D}^{(b)}_{\ell_1,\ell_2,\ell_3,\ell_4} + \mathrm{D}^{(c)}_{\ell_1,\ell_2,\ell_3,\ell_4} \tag{B.46}$$

where $\mathrm{D}^{\infty}$ accounts for the case where $s_i=0$ for all $i=1,2,3,4$; $\mathrm{D}^{(a)}$ contains the case where three $s_i$ vanish; $\mathrm{D}^{(b)}$ contains the case where two $s_i$, namely $s_i$ and $s_{i+1}$, vanish; $\mathrm{D}^{(c)}$ contains the case where $s_i$ and $s_{i+2}$ vanish; and $\mathrm{D}^{(1)}$ contains all other contributions, namely those where $s_i\ge 1$ for all $i=1,2,3,4$ and those where one of the $s_i$ is zero.

We are interested in the contributions which are not exponentially suppressed, which means that the integration region includes points where $\sum|m_l^{(ij)}||k_l^{(ij)}-\hat\nu_2^{(ij)}|=0$. Note that this is possible only if all $k_l^{(ij)}$, with a given $(ij)$ are equal to each other, i.e. $k_l^{(ij)}\equiv k^{(ij)}$, for $(ij)=12,23,34$ and 14. As a result, $\delta(\sum_{i,j,l}m_l^{(ij)}k_l^{(ij)})$ becomes proportional to $\delta(\sum_i m_i^{(ij)})$ so it gives no further restriction to the above sums in (B.46). Therefore, we have to sum over four integers $(k^{(12)},k^{(23)},k^{(34)},k^{(14)})$. Since $|\hat\nu_2^{(ij)}|\le 1$, the only contributions come from the terms with $k^{(ij)}=0,1,-1$. In order to identify such contributions, it is convenient to decompose the integrals into four contributions:

$$\begin{aligned}\int_0^1\int_0^1\int_0^1 d\hat\nu_2^{(1)}d\hat\nu_2^{(2)}d\hat\nu_2^{(3)} &= \int_0^1 d\hat\nu_2^{(1)}\int_0^{\hat\nu_2^{(1)}}d\hat\nu_2^{(2)}\int_0^{\hat\nu_2^{(2)}}d\hat\nu_2^{(3)} + \int_0^1 d\hat\nu_2^{(1)}\int_0^{\hat\nu_2^{(1)}}d\hat\nu_2^{(2)}\int_{\hat\nu_2^{(2)}}^1 d\hat\nu_2^{(3)}\\ &+\int_0^1 d\hat\nu_2^{(2)}\int_0^{\hat\nu_2^{(2)}}d\hat\nu_2^{(1)}\int_0^{\hat\nu_2^{(2)}}d\hat\nu_2^{(3)} + \int_0^1 d\hat\nu_2^{(2)}\int_0^{\hat\nu_2^{(2)}}d\hat\nu_2^{(1)}\int_{\hat\nu_2^{(2)}}^1 d\hat\nu_2^{(3)}\\ &\equiv J_1+J_2+J_3+J_4\end{aligned} \tag{B.47}$$

Now it is easy to recognize the possible contributions of $(k^{(12)}, k^{(23)}, k^{(34)}, k^{(14)})$. For example, for $J_1$, the values of $k^{(ij)}$ which will give rise to contributions to the zero mode that are not exponentially suppressed are $(0,0,0,0), (1,1,1,1), (0,0,1,1), (1,0,0,1), (0,1,0,1)$. The next step is to consider each separate contribution and perform the integral. Computing the integrals in all cases, we find an explicit expression for the general four-vertex function. We will here omit the explicit expression since it is very cumbersome. The expression involves, in addition to the harmonic sums of the previous subsection $S(m,n)$ and $S(s_1,s_2,s_3;\alpha,\beta)$, new harmonic sums that are given by

$$\begin{aligned} H_1(s_1,s_2,s_3,s_4;\alpha,\beta,\gamma) &= \sum_{(\underline{m}^1,\underline{m}^2,\underline{m}^3,\underline{m}^4)} \frac{\delta(\sum_j m^1_i - \sum_j m^2_j)\,\delta(\sum_j m^3_i - \sum_j m^4_j)}{\prod_{a=1}^4 \left(\prod_{j=1}^{s_a} |m^a_j|\right)} \\ &\quad \times \frac{1}{(|\underline{m}^1| + |\underline{m}^4|)^\alpha (|\underline{m}^2| + |\underline{m}^4|)^\beta (|\underline{m}^3| + |\underline{m}^4|)^\gamma} \qquad \text{(B.48)} \\ H_2(s_1,s_2,s_3,s_4;\alpha,\beta,\gamma) &= \sum_{(\underline{m}^1,\underline{m}^2,\underline{m}^3,\underline{m}^4)} \frac{\delta(\sum_j m^1_i - \sum_j m^2_j)\,\delta(\sum_j m^3_i - \sum_j m^4_j)}{\prod_{a=1}^4 \left(\prod_{j=1}^{s_a} |m^a_j|\right)} \\ &\quad \times \frac{1}{(|\underline{m}^2| + |\underline{m}^3|)^\alpha (|\underline{m}^1| + |\underline{m}^4|)^\beta (|\underline{m}^3| + |\underline{m}^4|)^\gamma} \qquad \text{(B.49)} \\ H_3(s_1,s_2,s_3,s_4;\alpha,\beta,\gamma) &= \sum_{(\underline{m}^1,\underline{m}^2,\underline{m}^3,\underline{m}^4)} \frac{\delta(\sum_j m^1_i - \sum_j m^2_j)\,\delta(\sum_j m^3_i - \sum_j m^4_j)}{\prod_{a=1}^4 \left(\prod_{j=1}^{s_a} |m^a_j|\right)} \\ &\quad \times \frac{1}{(|\underline{m}^2| + |\underline{m}^4|)^\alpha (|\underline{m}^1| + |\underline{m}^2|)^\beta (|\underline{m}^3| + |\underline{m}^4|)^\gamma} \qquad \text{(B.50)} \end{aligned}$$

where $|\underline{m}^1| = \sum_j |m^{12}_j|$, $|\underline{m}^2| = \sum_j |m^{23}_j|$, $|\underline{m}^3| = \sum_j |m^{34}_j|$, $|\underline{m}^4| = \sum_j |m^{14}_j|$. In addition, the result involves particular cases of these harmonic sums:

$$\begin{aligned} H_0(s_1,s_2;\alpha,\beta,\gamma) &\equiv H_3(0,0,s_1,s_2;\alpha,\beta,\gamma) = H_3(0,s_1,0,s_2;\gamma,\alpha,\beta) \\ S_0(s_1,s_2;\alpha,\beta) &\equiv S(0,s_2,s_1;\alpha,\beta) \qquad \text{(B.51)} \end{aligned}$$

Running a `Mathematica` program with the complete expression, we find the following large-$\tau_2$ expansions for the zero modes,

$$\mathrm{D}^{(0)}_{1,1,1,1} = \frac{1}{(4\pi)^4}\left(2\zeta(8)\tau_2^4 + \frac{5\pi\zeta(7)}{8\tau_2^3}\right) \qquad \text{(B.52)}$$

$$\begin{aligned} \mathrm{D}^{(0)}_{1,1,1,2} = \frac{1}{(4\pi)^5}\bigg(&\frac{6\zeta(10)}{5}\tau_2^5 + 2\pi\zeta(3)\zeta(6)\tau_2^2 - \frac{1}{2}\pi\zeta(4)\zeta(5) \\ &+\frac{21\pi\zeta(2)\zeta(7)}{8\tau_2^2} - \frac{3\zeta(2)\zeta(3)\zeta(5)}{\tau_2^3} + \frac{43\pi\zeta(9)}{64\tau_2^4}\bigg) + O(e^{-\tau_2}) \qquad \text{(B.53)} \end{aligned}$$

$$\mathrm{D}^{(0)}_{1,1,2,2} = \frac{1}{(4\pi)^6}\left(\frac{612}{691}\zeta(12)\tau_2^6 + \frac{8\pi}{3}\zeta(3)\zeta(8)\tau_2^3 - \pi\zeta(5)\zeta(6)\tau_2 + \frac{7}{12}\zeta(3)^2\zeta(6)\right) + O\left(\frac{1}{\tau_2}\right) \text{(B.54)}$$

$$\mathrm{D}^{(0)}_{1,2,1,2} = \frac{1}{(4\pi)^6}\left(\frac{612}{691}\zeta(12)\tau_2^6 + \frac{8\pi}{3}\zeta(3)\zeta(8)\tau_2^3 - \pi\zeta(5)\zeta(6)\tau_2 + 21\zeta(3)^2\zeta(6)\right) + O\left(\frac{1}{\tau_2}\right) \text{(B.55)}$$

$$\mathrm{D}^{(0)}_{1,1,1,3} = \frac{1}{(4\pi)^6}\left(\frac{5625}{691}\zeta(12)\tau_2^6 + 20\pi\zeta(3)\zeta(8)\tau_2^3 - \frac{15\pi}{4}\zeta(5)\zeta(6)\tau_2 + \frac{63}{4}\zeta(3)^2\zeta(6)\right) + O\left(\frac{1}{\tau_2}\right), \qquad \text{(B.56)}$$

From these expressions we can determine the expression quadratic in $\hat{E}_r$'s that reproduces the terms with positive powers of $\tau_2$, so that,

$$\mathrm{D}_{1,1,1,1} = \hat{E}_4 \tag{B.57}$$

$$\mathrm{D}_{1,1,1,2} = -\frac{8}{5}\hat{E}_5 + \hat{E}_2\hat{E}_3 + \mathrm{D}^{fin}_{1,1,1,2} \tag{B.58}$$

$$\mathrm{D}_{1,1,2,2} = -\frac{3658}{2073}\hat{E}_6 - \frac{1}{3}\hat{E}_3^2 + \frac{4}{3}\hat{E}_2\hat{E}_4 + \frac{\zeta(3)^2}{184320} + \mathrm{D}^{fin}_{1,1,2,2} \tag{B.59}$$

$$\mathrm{D}_{1,2,1,2} = -\frac{3658}{2073}\hat{E}_6 - \frac{1}{3}\hat{E}_3^2 + \frac{4}{3}\hat{E}_2\hat{E}_4 + \frac{\zeta(3)^2}{184320} + \mathrm{D}^{fin}_{1,2,1,2} \tag{B.60}$$

$$\mathrm{D}_{1,1,1,3} = -\frac{10415}{691}\hat{E}_6 - \frac{5}{4}\hat{E}_3^2 + 10\,\hat{E}_2\hat{E}_4 + \frac{\zeta(3)^2}{245760} + \mathrm{D}^{fin}_{1,1,1,3} \tag{B.61}$$

We saw in section 3 and in appendix A.2 that the constant part in the large $\tau_2$ expansion is needed in order to determine the coefficient of $s^k\,\mathbf{R}^4$ in the ten-dimensional theory.

In addition to the preceding results, which are needed in order to determine the coefficients of the terms up to order $s^6\,\mathbf{R}^4$ that are summarized in the main text, we have also evaluated certain terms at higher order. For the vertex functions appearing in the calculation of the genus-one coefficient of $s^7\,\mathbf{R}^4$ we find

$$\begin{aligned} \mathrm{D}^{(0)}_{1,1,1,4}\Big|_{cste} &= -\frac{167}{123863040}\,\zeta(7); \quad \mathrm{D}^{(0)}_{1,1,2,3}\Big|_{cste} = \frac{727}{82575360}\,\zeta(7); \\ \mathrm{D}^{(0)}_{1,2,1,3}\Big|_{cste} &= \frac{727}{82575360}\,\zeta(7); \quad \mathrm{D}^{(0)}_{1,2,2,2}\Big|_{cste} = \frac{733}{61931520}\,\zeta(7) \end{aligned} \tag{B.62}$$

For $s^8\,\mathbf{R}^4$, we find

$$\mathrm{D}^{(0)}_{1,1,1,5}\Big|_{cste} = \frac{223}{49545216}\zeta(3)\zeta(5)\ , \qquad \mathrm{D}^{(0)}_{2,2,2,2}\Big|_{cste} = -\frac{173}{15482880}\zeta(3)\zeta(5) \tag{B.63}$$

$$\mathrm{D}^{(0)}_{1,2,1,4}\Big|_{cste} = \mathrm{D}^{(0)}_{1,1,2,4}\Big|_{cste} = -\frac{199}{61931520}\zeta(3)\zeta(5)\ , \quad \mathrm{D}^{(0)}_{1,1,3,3}\Big|_{cste} = -\frac{69}{9175040}\zeta(3)\zeta(5), \tag{B.64}$$

$$\mathrm{D}^{(0)}_{1,3,1,3}\Big|_{cste} = -\frac{449}{55050240}\zeta(3)\zeta(5)\ , \quad \mathrm{D}^{(0)}_{1,2,2,3}\Big|_{cste} = \mathrm{D}^{(0)}_{2,1,2,3}\Big|_{cste} = -\frac{89}{10321920}\zeta(3)\zeta(5). \tag{B.65}$$

These coefficients are strikingly simple — a rational number times $\zeta(3)\zeta(5)$ — even though they arise after summing a huge number of terms.

### B.3.2 $\ell_5$ and/or $\ell_6 \neq 0$

Let us now consider particular four-vertex diagrams where $\ell_5$ and/or $\ell_6$ are different from zero. In some cases, they reduce to products of lower-point vertex diagrams, as in figure 6(d), given by $\mathrm{D}_{\ell_1\ell_2\ell_3} \times \mathrm{D}_{\ell_4}$.

More generally, the diagrams are those of figure 5(b) and figure 5(c). Although we have not evaluated the D functions in these cases for arbitrary nonzero values of $\ell_r$, we have computed the positive powers of $\tau_2$ and the constant part in two special cases that are needed for evaluating terms of order $s^k\,\mathbf{R}^4$ up to $k=6$. One of these is the diagram in figure 5(c) with all $l_r=1$, for which the zero mode is

$$\mathrm{D}^{(0)}_{1,1,1;1,1,1} = \frac{1}{(4\pi)^6}\left(\frac{138}{691}\,\zeta(12)\,\tau_2^6 + 6\pi\,\zeta(5)\zeta(6)\tau_2 + O(1/\tau_2)\right)\,, \tag{B.66}$$

which leads to the expression

$$\mathrm{D}_{1,1,1;1,1,1} = -\frac{2791}{691}\,\hat{E}_6 + 2\,\hat{E}_3^2 + \mathrm{D}^{fin}_{1,1,1;1,1,1}\,. \tag{B.67}$$

The other special case that we have evaluated is that of figure 5(b) with $\ell_r=1$, for which the zero mode is

$$\mathrm{D}^{(0)}_{1,1,1,1;1} = \frac{1}{(4\pi)^5}\,\frac{4}{5}\zeta(10)\tau_2^5 + \frac{\zeta(5)}{30720} + O(1/\tau_2)\,, \tag{B.68}$$

which leads to

$$\mathrm{D}_{1,1,1,1;1} = \frac{2}{5}\hat{E}_5 + \frac{\zeta(5)}{30720} + \mathrm{D}^{fin}_{1,1,1,1;1}\,. \tag{B.69}$$

However, we will see in the next section that in order to determine $j^{(0,2)}$ we also need to evaluate $\mathrm{D}_{1,1,1,1;2}$ and $\mathrm{D}_{1,1,1,2;1}$. These will be obtained by a slightly different procedure in appendix D.

## C. Diagrammatic expansion of the coefficients

We will here present the expansions (3.4) of the $j^{(p,q)}$'s as linear combinations of $\mathrm{D}_{\{\ell\}}$'s for values of $p$ and $q$ up to order $2p+3q=8$. We will also substitute the expressions derived in the last section for $\mathrm{D}_{\{\ell\}}$ in terms of Eisenstein series that are used in evaluating the coefficients in the ten-dimensional case and the large-$\tau_2$ power series expansions that are needed for obtaining the $r$-dependent coefficients in the nine-dimensional case.

We will present the expressions for each $j^{(p,q)}$ function in the form

$$\begin{aligned} j^{(p,q)} &= \sum_{\{\ell\}} e^{(p,q)}_{\{\ell\}}\,\mathrm{D}_{\{\ell\}} \\ &= P^{(p,q)}(\{\hat{E}_r\}) + J^{(p,q)} + \delta j^{(p,q)}\,, \end{aligned} \tag{C.1}$$

where $\delta j^{(p,q)}$ decreases by at least a power of $\tau_2$ at large $\tau_2$. The first equality shows the diagrammatic decomposition and the second equality shows how the positive powers of $\tau_2$ are incorporated in a quadratic form in Eisenstein series and will be appropriate for the ten-dimensional calculations. Furthermore, substituting the expansions of $\mathrm{D}^{(0)}_{\{\ell\}}$ derived in the previous appendix into the first line of (C.1) gives the expressions contained in the nine-dimensional case in (5.25).

The list of functions of interest to us is the following:

$$\begin{aligned} j^{(0,1)} &= \frac{4^3}{3!}\,(8\mathrm{D}_{1,1,1} + 2\mathrm{D}_3) \\ &= \frac{4^3}{3!}\left(10\hat{E}_3 + \frac{\zeta(3)}{32}\right) + \delta j^{(0,1)}\,, \end{aligned} \tag{C.2}$$



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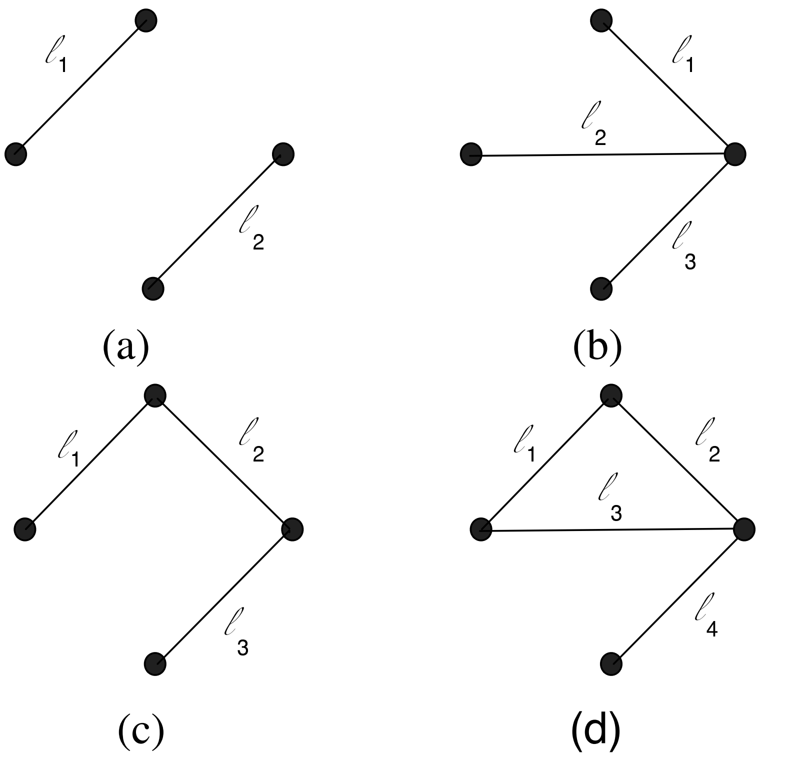

**Figure 6:** This figure shows all the degenerate four-vertex diagrams that reduce to products of two-vertex and three-vertex diagrams.

where the $\mathrm{D}_3$ contribution[6] is represented by the two-vertex diagram of figure 3 of appendix B.1 with $\ell_1 = 2$, and $\delta j^{(0,1)}$.

$$\begin{aligned} j^{(2,0)} &= \frac{4^4}{4!}\left(9\mathrm{D}_2^2 + 6\mathrm{D}_{1,1,1,1} + \mathrm{D}_4\right) \\ &= \frac{4^4}{4!}\left(24\hat{E}_2^2 - 24\hat{E}_4\right) + \delta j^{(2,0)}\,, \end{aligned} \tag{C.3}$$

where $\mathrm{D}_4$ is the $\ell_1 = 4$ contribution to figure 3.

$$\begin{aligned} j^{(1,1)} &= \frac{5}{6}\frac{4^5}{5!}\left(2\mathrm{D}_5 + 96\mathrm{D}_2\mathrm{D}_{1,1,1} + 28\mathrm{D}_3\mathrm{D}_2 + 32\mathrm{D}_{1,1,3} - 24\mathrm{D}_{1,2,2} + 24\mathrm{D}_{1,1,1,2} - 48\mathrm{D}_{1,1,1,1;1}\right) \\ &= \frac{5}{6}\frac{4^5}{5!}\left(-\frac{9864}{5}\hat{E}_5 + 1080\hat{E}_3\hat{E}_2 + \frac{97}{7680}\zeta(5)\right) + \delta j^{(1,1)}\,, \end{aligned} \tag{C.4}$$

where $\mathrm{D}_{\ell_1,\ell_2\ell_3}$, $\mathrm{D}_{\ell_1,\ell_2\ell_3,\ell_4}$ and $\mathrm{D}_{\ell_1,\ell_2\ell_3,\ell_4;\ell_5}$ are defined by figure 4(b), figure 5(a) and figure 5(b), respectively.

$$\begin{aligned} j^{(3,0)} &= \frac{4^6}{6!}\left(-5\mathrm{D}_3^2 + \frac{45}{2}\mathrm{D}_2\,\mathrm{D}_4 + 45\mathrm{D}_{1,1,1}^3 + \frac{1}{2}\mathrm{D}_6 + 60\mathrm{D}_{1,1,1,3} - 90\mathrm{D}_{1,1,2,2} + 45\mathrm{D}_{1,2,1,2}\right) \\ &= \frac{4^6}{6!}\left(-\frac{9501120}{691}\hat{E}_6 + 3960\hat{E}_3^2 + 2880\,\hat{E}_2\hat{E}_4 + \frac{3}{512}\,\zeta(3)^2\right) + \delta j^{(3,0)}\,. \end{aligned} \tag{C.5}$$

[6] In [4] $\mathrm{D}_3$ was denoted $B_2$.

$$
\begin{aligned}
j^{(0,2)} =& \frac{4^6}{6!}\left(\frac{2}{3}\mathrm{D}_6+\frac{100}{3}\mathrm{D}_3^2-10\mathrm{D}_2\mathrm{D}_4-20\mathrm{D}_2^3+160\mathrm{D}_{1,1,1}\mathrm{D}_3\right.\\
&+40(\mathrm{D}_{2,2,2}+\mathrm{D}_{1,1,4}-2\mathrm{D}_{1,2,3})-20(3\mathrm{D}_{1,2,1,2}-12\mathrm{D}_{1,1,2,2}+4\mathrm{D}_{1,1,1,3})\\
&\left.+80\mathrm{D}_{1,1,1;1,1,1}+240(\mathrm{D}_{1,1,1,1;2}-2\mathrm{D}_{1,1,1,2;1})\right)\\
=& \frac{4^6}{6!}\left(-\frac{12345120}{691}\,\hat{E}_6+5040\,\hat{E}_3^2+3840\,\hat{E}_2\hat{E}_4+\frac{61}{6144}\,\zeta(3)^2\right)+\delta j^{(0,2)}\,,
\end{aligned}
\tag{C.6}
$$

where $\mathrm{D}_{\ell_1,\ell_2\ell_3,\ell_4;\ell_5,\ell_6}$ is defined by figure 5(c)

All of the D functions that arise in these expressions were evaluated in appendix B apart from $\mathrm{D}_{1,1,1,1;2}$ and $\mathrm{D}_{1,1,1,2;1}$, which will be determined by a different procedure in appendix D.

We also note the sum of diagrams that arise at the next two orders, even though we will not evaluate the D functions that arise in these cases in this paper. The coefficient multiplying $\hat{\sigma}_2^2\,\hat{\sigma}_3\,\mathbf{R}^4$ is given by

$$
\begin{aligned}
j^{(2,1)} \;=\; \frac{4^7}{7!}\,\frac{2\cdot 4^5}{3^3\cdot 5}&\left(24\mathrm{D}_{1,1,5}-60\mathrm{D}_{1,2,4}+40\mathrm{D}_{1,3,3}+90\mathrm{D}_{1,1,1,4}-120\mathrm{D}_{1,1,2,3}+180\mathrm{D}_{1,2,1,3}\right.\\
&-90\mathrm{D}_{1,2,2,2}+30\mathrm{D}_2(16\mathrm{D}_{1,1,3}-12\mathrm{D}_{1,2,2}+5\mathrm{D}_2\mathrm{D}_3)+120\mathrm{D}_{1,1,1}\mathrm{D}_4\\
&+25\mathrm{D}_3\mathrm{D}_4+39\mathrm{D}_2\mathrm{D}_5+\mathrm{D}_7-480\mathrm{D}_{1,1,1,3;1}\\
&\left.+360(\mathrm{D}_{1,1,2,2;1}-\mathrm{D}_{1,2,1,2;1}+\mathrm{D}_{1,2,2,1;1})\right).
\end{aligned}
\tag{C.7}
$$

The coefficient multiplying $\hat{\sigma}_2^4\mathbf{R}^4$ is given by

$$
\begin{aligned}
j^{(4,0)} =& \frac{4^8}{8!}\,\frac{128}{135}\left(\mathrm{D}_8+84\mathrm{D}_2\mathrm{D}_6-56\mathrm{D}_3\mathrm{D}_5+105\mathrm{D}_4^2+1260\mathrm{D}_2^2\mathrm{D}_4-560\mathrm{D}_2\mathrm{D}_3^2\right.\\
&+336\mathrm{D}_{1,1,1,5}-1680\mathrm{D}_{1,1,2,4}+1120\mathrm{D}_{1,1,3,3}+840\mathrm{D}_{1,2,1,4}\\
&\left.-3360\mathrm{D}_{1,2,2,3}+560\mathrm{D}_{1,3,1,3}+1680\mathrm{D}_{2,1,2,3}+630\mathrm{D}_{2,2,2,2}\right).
\end{aligned}
\tag{C.8}
$$

The coefficient multiplying $\hat{\sigma}_2\hat{\sigma}_3^2\,\mathbf{R}^4$ is

$$
\begin{aligned}
j^{(1,2)} =& \frac{4^8}{8!}\,\frac{128}{135}\left(\mathrm{D}_8+700\,\mathrm{D}_2\,\mathrm{D}_3^2-420\,\mathrm{D}_2^2\,\mathrm{D}_4-35\,\mathrm{D}_4^2+420\,\mathrm{D}_{1,1,1}\,\mathrm{D}_5+154\,\mathrm{D}_3\,\mathrm{D}_5\right.\\
&+14\,\mathrm{D}_2\,\mathrm{D}_6+1680\,\mathrm{D}_3\,\mathrm{D}_{1,1,3}+1260\,\mathrm{D}_2\,\mathrm{D}_{1,1,4}+70\,\mathrm{D}_{1,1,6}\\
&-1260\,\mathrm{D}_3\,\mathrm{D}_{1,2,2}-2520\,\mathrm{D}_2\,\mathrm{D}_{1,2,3}\\
&-210\,\mathrm{D}_{1,2,5}+70\,\mathrm{D}_{1,3,4}+1260\,\mathrm{D}_2\,\mathrm{D}_{2,2,2}+210\,\mathrm{D}_{2,2,4}\\
&-140\,\mathrm{D}_{2,3,3}-84\,\mathrm{D}_{1,1,1,5}+1470\,\mathrm{D}_{1,1,2,4}-1120\,\mathrm{D}_{1,1,3,3}\\
&-210\,\mathrm{D}_{1,2,1,4}+2940\,\mathrm{D}_{1,2,2,3}-140\,\mathrm{D}_{1,3,1,3}-1680\,\mathrm{D}_{2,1,2,3}\\
&-630\,\mathrm{D}_{2,2,2,2}+420\,\mathrm{D}_{1,1,1,1;4}-1680\,\mathrm{D}_{1,1,1,2;3}-1260\,\mathrm{D}_{1,2,2,1;2}\\
&+3360\,\mathrm{D}_{1,1,1,3;2}-2100\,\mathrm{D}_{1,1,1,4;1}-1260\,\mathrm{D}_{1,1,2,2;2}+840\,\mathrm{D}_{1,1,2,3;1}\\
&+2520\,\mathrm{D}_{1,2,1,2;2}-4200\,\mathrm{D}_{1,2,1,3;1}+1260\,\mathrm{D}_{1,2,2,2;1}+840\,\mathrm{D}_{1,2,3,1;1}\\
&\left.+1680\,\mathrm{D}_{1,1,3;1,1,1}+1260\,\mathrm{D}_{2,1,2;1,1,1}-2520\,\mathrm{D}_{2,2,1;1,1,1}\right)
\end{aligned}
\tag{C.9}
$$

All the D's in the last two equations have been evaluated apart from $\mathrm{D}_{\ell_1,\ell_2,\ell_3;\ell_4,\ell_5,\ell_6}$ and $\mathrm{D}_{\ell_1,\ell_2,\ell_3;\ell_4,\ell_5,\ell_6}$ with all $\ell_k$'s $\neq 0$. These coefficients that have not been evaluated arise from the 'Mercedes' diagram, figure 5(c).

## D. The zero mode of $\mathrm{D}_{\ell_1,\ldots,\ell_6}$ by another method

The calculations of appendix B become very difficult at relatively low orders, as is evident from the fact that we did not evaluate certain D functions that were needed in order to determine $j^{(0,2)}$ in (B). Here we will here present an alternative method for evaluating these functions, even though this also has technical difficulties.

The different diagrams that arise in the low energy expansion can be computed by using the representation (2.13) of the propagator. We proceed by considering the following generalization of the D functions,

$$C_{s_1,s_2,\ldots,s_{N+1}} = \sum_{(m_i,n_i)\neq(0,0)} \prod_{i=1}^{N+1} \frac{\tau_2^{s_i}}{|m_i+n_i\tau|^{2s_i}} \delta\Big(\sum_k m_k\Big)\delta\Big(\sum_k n_k\Big)\,, \tag{D.1}$$

where $s_i$ are integers and the sum over $k$ involves a subset of the $\{m_i, n_i\}$, according to the topology of the diagram, which needs to be specified in order to define this expression completely. Certain special cases of these functions correspond to certain $D$ functions. For example,

$$\mathrm{D}_3 = \frac{1}{(4\pi)^3} C_{1,1,1}\,, \qquad \mathrm{D}_{1,1,1,1;1} = \frac{1}{(4\pi)^5} C_{2,2,1} \tag{D.2}$$

(although information on the topology of the diagram needs to be specified in completely defining the function $C_{2,2,1}$).

### D.1 Leading and subleading powers of $\tau_2$

We will start by determining the first two terms in the expansion of (D.1) at large $\tau_2$. To illustrate the method, we start with a particular example,

$$C_{s_1,s_2,s_3} = \sum_{\substack{(m_1,n_1)\neq(0,0)\\(m_2,n_2)\neq(0,0)}} \frac{\tau_2^{s_1+s_2+s_3}}{|m_1+n_1\tau|^{2s_1}|m_2+n_2\tau|^{2s_2}|m_1+m_2+(n_1+n_2)\tau|^{2s_3}}\,. \tag{D.3}$$

The zero mode expansion has the general form

$$\begin{aligned} C^{(0)}_{s_1,s_2,s_3} &= a_{s_1s_2s_3}\ \tau_2^{s_1+s_2+s_3} + b^1_{s_1s_2s_3}\ \tau_2^{1+s_1-s_2-s_3} + b^2_{s_1s_2s_3}\ \tau_2^{1+s_2-s_1-s_3} \\ &\quad + b^3_{s_1s_2s_3}\ \tau_2^{1+s_3-s_1-s_2} + \cdots + c_{s_1s_2s_3}\ \tau_2^{1-s_1-s_2-s_3} + O\big(\exp(-c\tau_2)\big)\,. \end{aligned} \tag{D.4}$$

So we begin by computing $a_{s_1s_2s_3}$ and $b^i_{s_1s_2s_3}$.

The leading term comes by setting $n_1 = n_2 = 0$ in the sums. This gives

$$\begin{aligned} a_{s_1s_2s_3} &= {\sum_{m_1,m_2}}' \frac{1}{|m_1|^{2s_1}|m_2|^{2s_2}|m_1+m_2|^{2s_3}} \\ &= 2W(2s_1,2s_2,2s_3) + 2W(2s_2,2s_3,2s_1) + 2W(2s_3,2s_1,2s_2)\,, \end{aligned} \tag{D.5}$$

where $\sum'_{m_1,m_2}$ excludes $m_1 m_2(m_1+m_2)=0$ and $W(a,b,c)$ is the Witten zeta-function of (A.33) whose values are tabulated using the methods of the appendix A.3 . For example, one gets $a_{1,1,1}=2\zeta(6)$.

The term with coefficient $b^i_{s_1 s_2 s_3}$ comes from $n_i=0$ (where we define $n_3=n_1+n_2$). Consider in particular the contribution with $n_1=0$. This is given by

$$V^1_{s_1 s_2 s_3} \equiv \sum_{m_1,n_2\neq 0}\sum_{m_2} \frac{\tau_2^{s_1+s_2+s_3}}{|m_1|^{2s_1}|m_2+n_2\tau|^{2s_2}|m_1+m_2+n_2\tau|^{2s_3}}\,. \tag{D.6}$$

Applying the Poisson resummation formula for the summation over $m_2$ gives

$$V^1_{s_1 s_2 s_3} = \sum_{m_1,n_2\neq 0}\sum_{w} \frac{\tau_2^{s_1+s_2+s_3}}{|m_1|^{2s_1}}\int_{-\infty}^{+\infty} d\mu \frac{e^{2\pi i w(\mu-n_2\tau_1)}}{|\mu+in_2\tau_2|^{2s_2}|m_1+\mu+in_2\tau_2|^{2s_3}}\,. \tag{D.7}$$

Now the only dependence on $\tau_1$ is in the factor $e^{-2\pi i w n_2\tau_1}$. The zero mode is independent of $\tau_1$, so it arises from the $w=0$ term in the sum. Introducing a new integration variable $\nu=\mu/(n_2\tau_2)$ leads to

$$V^1_{s_1 s_2 s_3}\Big|_{\rm pert} = \sum_{m_1,n_2\neq 0} \frac{\tau_2^{1+s_1-s_2-s_3}}{|m_1|^{2s_1}|n_2|^{2s_2+2s_3-1}}\int_{-\infty}^{+\infty} d\nu \frac{1}{(\nu^2+1)^{s_2}((\nu+\frac{m_1}{n_2\tau_2})^2+1)^{s_3}}\,. \tag{D.8}$$

We now consider the limit of large $\tau_2$. When the sum over $m_1$ of $1/|m_1|^{2s_1}$ is convergent (i.e. $s_1>\frac{1}{2}$) the leading term of the expansion of the integrand in powers of $1/\tau_2$ is finite. For the leading term the integral reduces to

$$\int_{-\infty}^{+\infty} d\nu \frac{1}{(\nu^2+1)^{s_2+s_3}} = \frac{\sqrt{\pi}\,\Gamma(s_2+s_3-\frac{1}{2})}{\Gamma(s_2+s_3)}\,,$$

so

$$V^1_{s_1 s_2 s_3}\Big|_{\rm pert} = \tau_2^{1+s_1-s_2-s_3}\, 2\zeta(2s_1) 2\zeta(2s_2+2s_3-1)\frac{\sqrt{\pi}\,\Gamma(s_2+s_3-\frac{1}{2})}{\Gamma(s_2+s_3)}\,. \tag{D.9}$$

Therefore

$$b^1_{s_1 s_2 s_3} = \frac{4\sqrt{\pi}\,\Gamma(s_2+s_3-\frac{1}{2})}{\Gamma(s_2+s_3)}\zeta(2s_1)\zeta(2s_2+2s_3-1)\ . \tag{D.10}$$

As a check, we now compare with the expansions that were calculated earlier. For $\mathrm{D}^{(0)}_3$, we find

$$(4\pi)^3\,\mathrm{D}^{(0)}_3 = C^{(0)}_{1,1,1} = 2\zeta(6)\tau_2^3 + \pi^3\zeta(3) + O(\tau_2^{-2})\,, \tag{D.11}$$

in agreement with the expansion given in section B.1. We have multiplied $b^1_{1,1,1}$ by 3 since there are three identical subleading contributions $b^i_{1,1,1}$, $i=1,2,3$. Similarly, we reproduce the leading and subleading terms of the diagrams $\mathrm{D}_{1,1,1,2}=C_{1,1,3}/(4\pi)^5$, $\mathrm{D}_{1,1,1,1;1}=C_{2,2,1}/(4\pi)^5$.

We now turn to consider the more general modular function given by (D.1). The expansion now has the form

$$C_{s_1,s_2,\ldots,s_{N+1}} = a_{s_1,\ldots,s_{N+1}}\ \tau_2^{s_1+\cdots+s_{N+1}} + b^{ij}_{s_1,\ldots,s_{N+1}}\ \tau_2^{1+s_1+\cdots-s_i-s_j+\cdots+s_{N+1}} + \cdots\,, \tag{D.12}$$

where

$$a_{s_1,\ldots,s_{N+1}} = \sum_{m_1,\ldots,m_N}{}' \frac{1}{|m_1|^{2s_1}\cdots|m_N|^{2s_N}|m_1+\cdots+m_N|^{2s_{N+1}}} \,. \tag{D.13}$$

The second term arises by setting to zero all $n_k$ except $n_i$ and $n_j$ (with the understanding that $n_{N+1} \equiv \sum_k n_k$). The corresponding contribution $V^{i,j}$ is the following one:

$$V^{N,N+1}_{s_1,\ldots,s_{N+1}} \equiv \sum_{\substack{m_i\neq 0, i<N \\ n_N\neq 0}} \sum_{m_N} \frac{\tau_2^{s_1+\cdots+s_{N+1}}}{|m_1|^{2s_1}\cdots|m_{N-1}|^{2s_{N-1}}|m_N+n_N\tau|^{2s_N}|m_1+\cdots+m_N+n_N\tau|^{2s_{N+1}}} \tag{D.14}$$

Performing a Poisson resummation in $m_N$ we arrive at

$$V^{N,N+1}_{s_1,\ldots,s_{N+1}}\Big|_{\text{pert}} = \sum_{m_1,,\ldots,m_{N-1}\neq 0} \sum_{n_N\neq 0} \frac{\tau_2^{1+s_1+\cdots+s_{N-1}-s_N-s_{N+1}}\, J}{|m_1|^{2s_1}\cdots|m_{N-1}|^{2s_{N-1}}|n_N|^{2s_N+2s_{N+1}-1}} \,, \tag{D.15}$$

$$J = \int_{-\infty}^{+\infty} d\nu \frac{1}{(\nu^2+1)^{s_N}\left((\nu+\frac{1}{n_2\tau_2}(m_1+\cdots+m_{N-1}))^2+1\right)^{s_{N+1}}} \,.$$

Now we would like to extract the leading term in the expansion in powers of $1/\tau_2$ of $J$. When $s_1,\ldots,s_{N-1} > 1/2$, the sums over $m_i$ of $1/|m_i|^{2s_i}$ are convergent and the leading term is simply obtained by setting $\tau_2=\infty$ inside the integral. This gives

$$J = \frac{\sqrt{\pi}\,\Gamma(s_N+s_{N+1}-\frac{1}{2})}{\Gamma(s_N+s_{N+1})} \,,$$

so that

$$b^{N,N+1}_{s_1,\ldots,s_{N+1}} = \frac{2^N\sqrt{\pi}\,\Gamma(s_N+s_{N+1}-\frac{1}{2})}{\Gamma(s_N+s_{N+1})} \zeta(2s_1)\cdots\zeta(2s_{N-1})\zeta(2s_N+2s_{N+1}-1) \,. \tag{D.16}$$

Applying (D.13) and (D.16), we reproduce the first two terms in the expansion of $\mathrm{D}_4^{(0)}$ given in section B.1.

$$(4\pi)^4 \mathrm{D}_4^{(0)} = C^{(0)}_{1,1,1,1} = 10\zeta(8)\tau_2^4 + \frac{2\pi^5}{3}\zeta(3)\tau_2 + \cdots \tag{D.17}$$

We have included a combinatorial factor 4!/2!2! multiplying the subleading term, which counts the number of identical subleading terms obtained by setting different $n_i$ to zero. Similarly, we reproduce the first two terms in the expansion of $\mathrm{D}_5 = C_{1,1,1,1,1}/(4\pi)^5$ and $\mathrm{D}_{1,1,3} = C_{1,1,1,2}/(4\pi)^5$ (where the precise definition of these $C$ functions requires a specification of the topology of the diagram in addition to the values of the integers $s_i$).

The method outlined in this subsection is particularly useful for diagrams that correspond to D functions of the general form $\mathrm{D}^{(0)}_{\ell_1,\ell_2,\ell_3;\ell_4,\ell_5,\ell_6}$ where five or six of the $\ell_i$ are different from zero, for which we did not derive the general formula in section B. As an application we compute the first two terms in the expansion of $\mathrm{D}^{(0)}_{1,1,1;1,1,1}$, defined as the zero $\tau_1$ mode of

$$\mathrm{D}_{1,1,1;1,1,1} = \frac{1}{(4\pi)^6}\sum_{m_i,n_i}{}' \frac{\tau_2^6}{|m_1+n_1\tau|^2|m_2+n_2\tau|^2|m_3+n_3\tau|^2|m_1+m_2+(n_1+n_2)\tau|^2}$$
$$\times \frac{1}{|m_1+m_3+(n_1+n_3)\tau|^2|m_1+m_2+m_3+(n_1+n_2+n_3)\tau|^2} \,. \tag{D.18}$$

The expansion of the zero $\tau_1$ mode has the form

$$\mathrm{D}^{(0)}_{1,1,1;1,1,1} = \frac{1}{(4\pi)^6}\left(a\tau_2^6 + b\tau_2 + O(\tau_2^{-1})\right). \tag{D.19}$$

The leading term arises by setting all $n_i = 0$, giving the sum

$$a = \sum_{m_1,m_2,m_3}{}' \frac{1}{m_1^2 m_2^2 m_3^2 (m_1-m_2)^2(m_1-m_3)^2(m_1-m_2-m_3)^2}. \tag{D.20}$$

The symbol $\sum'$ indicates that the sum does not contain those $m_i$ where the denominator vanishes. Computing this sum gives

$$a = \frac{138}{691}\zeta(12). \tag{D.21}$$

The subleading term arises from four identical sums, in which: a) $n_1 = n_2 = 0$; b) $n_1 = n_3 = 0$; c) $n_1 = n_3,\ n_2 = 0$; d) $n_1 = n_2,\ n_3 = 0$. Following the above procedure for case a) we perform a Poisson resummation on $m_3$, leading to the result

$$b = 4\sum_{m_1,m_2,n_3}{}'\frac{1}{m_1^2 m_2^2(m_1-m_2)^2}\frac{1}{(n_3)^5}\int_{-\infty}^{+\infty} d\nu \frac{1}{(\nu^2+1)^3} = 6\pi\ \zeta(6)\zeta(5)\ . \tag{D.22}$$

### D.2 Systematic large-$\tau_2$ expansion

We return to the general modular function (D.1). The power-behaved terms of the zero mode expansion are $\tau_2^{s_1+s_2+\cdots+s_{N+1}}, \ldots, \tau_2^{1-s_1-s_2-\cdots-s_{N+1}}$. In order to obtain the coefficients of these terms one proceeds as follows. The different contributions can be organized in terms of the different subsets $\{n_k\}$ where all $n_k$ are zero (the most suppressed contribution is the one where none of the $n_i$ vanishes). These subsets can easily be visualized by considering a graphical representation of the modular function (see figure 5) and remove propagator lines in all possible ways leaving diagrams containing closed loops only. For example, in figure 5(a) one could cut all propagators in $\ell_4$ setting $n_1 = \cdots = n_{\ell_4} = 0$, as long as $\ell_{1,3} \geq 2$ and $\ell_2$ is either zero or $\ell_2 \geq 2$, so that only closed loops remain in the diagram. For a given subset $\{n_k\}$ of vanishing $n_k$, one performs Poisson resummation in all remaining $m_i$ variables with $i \neq k$. The resulting integral is then computed explicitly. The next step is to perform the remaining summations explicitly. This step becomes more complicated in diagrams with a large number of propagator lines. In this step one can drop contributions which are exponentially suppressed at large $\tau_2$.

As an example, we come back to (D.8) with $s_1 = s_2 = s_3 = 1$. Using

$$\int_{-\infty}^{+\infty} d\nu \frac{1}{(\nu^2+1)((\nu+a)^2+1)} = \frac{2\pi}{4+a^2}, \tag{D.23}$$

we find

$$V^1_{111} = 2\pi \sum_{m_1,n_2\neq 0} \frac{1}{m_1^2|n_2|}\frac{\tau_2^2}{4n_2^2\tau_2^2 + m_1^2}. \tag{D.24}$$

Now use

$$\sum_{m_1\neq 0} \frac{\tau_2^2}{m_1^2(4n_2^2\tau_2^2+m_1^2)} = \frac{\pi^2}{12n_2^2} + \frac{1}{16n_2^4\tau_2^2} - \frac{\pi}{8n_2^3\tau_2}\coth(2n_2\pi\tau_2)\ . \tag{D.25}$$

Noting that $n_2^{-3}\coth(2n_2\pi\tau_2) \to |n_2|^{-3}$, modulo exponentially suppressed terms, as $\tau_2 \to \infty$, it is straightforward to perform the remaining sum over $n_2$, giving zeta values. Adding the multiplicity factor 3 and the leading $\tau_2^3$ term, we obtain

$$(4\pi)^3\,\mathrm{D}_3^{(0)} = C^{(0)}_{1,1,1} = 2\zeta(6)\tau_2^3 + \pi^3\zeta(3) - \frac{\pi^6}{60\tau_2} + \frac{3\pi\zeta(5)}{4\tau_2^2} + V'_{111}\,, \tag{D.26}$$

where $V'_{111}$ represents the contribution where none of the $n_i$ vanishes. This is computed as follows. By performing Poisson resummation in $m_1$ and $m_2$ we find

$$V'_{111} = {\sum_{n_1,n_2}}' \int_{-\infty}^{+\infty} d\mu_1 d\mu_2 \frac{\tau_2^3}{|\mu_1+in_1\tau_2|^2|\mu_2+in_2\tau_2|^2|\mu_1+\mu_2+i(n_1+n_2)\tau_2|^2} + O(\exp(-2\pi\tau_2))\,. \tag{D.27}$$

Introducing new integration variables $\mu_1 = \nu_1|n_1|\tau_2$ and $\mu_2 = \nu_2|n_2|\tau_2$ and computing the integrals, we find

$$V'_{111} = \frac{\pi^2}{\tau_2}\ S(3,1) = \frac{3\pi^2}{2\tau_2}\zeta(4) = \frac{\pi^6}{60\tau_2}\,, \tag{D.28}$$

where $S(m,n)$ is defined in (A.16) and we have used (A.22). Thus $V'_{111}$ cancels the similar contribution in (D.26). The final result reproduces (B.8) obtained by using the asymptotic form of the propagator.

We now calculate the first terms in the expansion of $\mathrm{D}_{1,1,1,1;2}$ and $\mathrm{D}_{1,1,1,2;1}$, which are given by

$$\mathrm{D}_{1,1,1,1;2} = \sum_{\substack{m_1,m_2,m_3\\ n_1,n_2,n_3}} \frac{\tau_2^6}{|m_1+n_1\tau|^4|m_2+n_2\tau|^4|m_3+n_3\tau|^2|m_1+m_2+m_3+(n_1+n_2+n_3)\tau|^2}\,, \tag{D.29}$$

and

$$\mathrm{D}_{1,1,1,2;1} = \sum_{\substack{m_1,m_2,m_3\\ n_1,n_2,n_3}} \frac{\tau_2^6}{|m_1+n_1\tau|^4|m_2+n_2\tau|^2|m_3+n_3\tau|^2} \times \frac{1}{|m_1+m_2+(n_1+n_2)\tau|^2|m_2+m_3+(n_2+n_3)\tau|^2}\,. \tag{D.30}$$

The leading term is of order $\tau_2^6$ and arises from the contribution with $n_1 = n_2 = n_3 = 0$. There are various contributions according to which $n_i$ are zero. Our aim is to compute the expansion up to the term constant in $\tau_2$. Keeping only these contributions, we find

$$\begin{aligned}(4\pi)^6\ \mathrm{D}_{1,1,1,1;2} &= \frac{5047\zeta(12)}{691}\ \tau_2^6 + V_{12} + 4V_{13} + 2V_1 + O(\tau_2^{-1})\\ (4\pi)^6\ \mathrm{D}_{1,1,1,2;1} &= \frac{802\zeta(12)}{691}\ \tau_2^6 + \tilde V_{12} + 2\tilde V_{13} + \tilde V_{23} + \tilde V_1 + O(\tau_2^{-1})\end{aligned} \tag{D.31}$$

Here $V_{ij}$ indicates the contribution with $n_i = n_j = 0$ and $V_i$ the contribution with $n_i = 0$. To compute the leading term of order $\tau_2^6$ we have used (for $m_1 \neq 0$)

$$\begin{aligned}
\sum_{m_2}{}' \frac{1}{(m_1+m_2)^2 m_2^2} &= \frac{4\zeta(2)}{m_1^2} - \frac{6}{m_1^4} \\
\sum_{m_2}{}' \frac{1}{(m_1+m_2)^4 m_2^2} &= \frac{2\zeta(4)}{m_1^2} + \frac{8\zeta(2)}{m_1^4} - \frac{15}{m_1^6} \\
\sum_{m_2}{}' \frac{1}{(m_1+m_2)^4 m_2^4} &= \frac{4\zeta(4)}{m_1^4} + \frac{40\zeta(2)}{m_1^6} - \frac{70}{m_1^8}
\end{aligned} \tag{D.32}$$

Next, for each $V_{ij}$, $\tilde{V}_{ij}$ we perform Poisson resummation in the $m_k$ with $k \neq i, j$ and integrate over the resulting continuous variable $\nu_k$. We find

$$\begin{aligned}
V_{12} &= 2\pi \sum_{m_1,m_2,n_3} \frac{\tau_2^5}{m_1^4 m_2^4 |n_3|} \frac{1}{4n_3^2\tau_2^2 + (m_1+m_2)^2} \\
V_{13} &= \frac{\pi}{2} \sum_{m_1,m_3,n_2} \frac{\tau_2^3}{m_1^4 m_3^2 |n_2|^3} \frac{12 n_2^2 \tau_2^2 + (m_1-m_3)^2}{(4n_2^2\tau_2^2 + (m_1-m_3)^2)^2} \\
\tilde{V}_{12} &= 2\pi \sum_{m_1,m_2,n_3} \frac{\tau_2^5}{m_1^4 m_2^2 (m_1-m_2)^2 |n_3|} \frac{1}{4n_3^2\tau_2^2 + m_2^2} \\
\tilde{V}_{13} &= 2\pi \sum_{m_1,m_3,n_2} \frac{\tau_2^5}{m_1^4 m_3^2 |n_2|} \frac{m_1^2 + m_3^2 - m_1 m_3 + 12 n_2^2 \tau_2^2}{(4n_2^2\tau_2^2 + m_1^2)(4n_2^2\tau_2^2 + m_3^2)(4n_2^2\tau_2^2 + (m_1-m_3)^2)} \\
\tilde{V}_{23} &= \frac{\pi}{2} \sum_{m_2,m_3,n_1} \frac{\tau_2^3}{(m_2-m_3)^2 m_2^2 m_3^2 |n_1|^3} \frac{12 n_1^2 \tau_2^2 + m_2^2}{(4n_1^2\tau_2^2 + m_2^2)^2}
\end{aligned} \tag{D.33}$$

Using the identities (D.32) we find

$$\begin{aligned}
V_{12} &= \frac{14\pi}{3}\zeta(8)\zeta(3)\tau_2^3 - \frac{7\pi}{2}\zeta(5)\tau_2 + \frac{\pi^6}{180}\zeta(6) + O(\tau_2^{-1}) \\
V_{13} &= \frac{21\pi}{4}\zeta(5)\tau_2 - \frac{\pi^6}{90}\zeta(6) + O(\tau_2^{-1}) \\
\tilde{V}_{12} &= \frac{4\pi}{3}\zeta(8)\zeta(3)\tau_2^3 - \frac{5\pi}{4}\zeta(5)\tau_2 + \frac{\pi^6}{360}\zeta(6) + O(\tau_2^{-1}) \\
\tilde{V}_{13} &= \frac{21\pi}{4}\zeta(5)\tau_2 - \frac{\pi^6}{90}\zeta(6) + O(\tau_2^{-1}) \\
\tilde{V}_{23} &= \frac{3\pi}{2}\zeta(5)\tau_2 + O(\tau_2^{-1})
\end{aligned} \tag{D.34}$$

To compute the contribution $V_1$ we Poisson resum the integers $m_2$ and $m_3$. This gives

$$V_1 = \sum_{m_1,n_2,n_3} \frac{\tau_2^6}{m_1^4} \int_{-\infty}^{+\infty} d\mu_2 d\mu_3 \frac{1}{|\mu_2 + i n_2\tau_2|^4 |\mu_3 + i n_3 \tau_2|^2 |m_1 + \mu_2 - \mu_3 + i(n_2-n_3)\tau_2|^2} \tag{D.35}$$

where we have dropped exponentially suppressed terms. After introducing $\nu_{2,3}$ integration variables by $\mu_{2,3} = \nu_{2,3}|n_{2,3}|\tau_2$ we get

$$V_1 = \sum_{m_1,n_2,n_3} \frac{1}{m_1^4|n_2|^3|n_3|}\int_{-\infty}^{+\infty} d\nu_2 d\nu_3 \frac{1}{(\nu_2^2+1)^2(\nu_3^2+1)((\frac{m_1}{\tau_2}+\nu_2|n_2|-\nu_3|n_3|)^2+(n_2-n_3)^2)} \tag{D.36}$$

Since we are interested in the leading term $O(\tau_2^0)$ we can set $m_1 = 0$ in the integrand. The sum over $m_1$ then gives $2\zeta(4)$. Computing the integrals, one finds sums which can be reduced to Witten zeta functions. The final result is

$$V_1 = \frac{7\pi^6}{360} - \frac{\pi^6}{180}\zeta(3)^2 \tag{D.37}$$

Similarly, we find $\tilde{V}_1 = V_1$.

To summarize, we have found

$$\begin{aligned} \mathrm{D}_{1,1,1,1;2} &= \frac{5047\zeta(12)}{(4\pi)^6\, 691}\,\tau_2^6 + \frac{14\pi\zeta(8)\zeta(3)}{(4\pi)^6\, 3}\tau_2^3 + \frac{35\pi\zeta(6)\zeta(5)}{(4\pi)^6\, 2}\tau_2 - \frac{2\zeta(6)\zeta(3)^2}{(4\pi)^6\, 21} + O(\tau_2^{-1}) \\ &= -\frac{27965}{2073}\,\hat{E}_6 + \frac{35}{6}\,\hat{E}_3^2 + \frac{7}{3}\,\hat{E}_2\hat{E}_4 - \frac{2\zeta(6)\zeta(3)^2}{(4\pi)^6\, 21} + \mathrm{D}^{fin}_{1,1,1,1;2} \\ \mathrm{D}_{1,1,1,2;1} &= \frac{802\zeta(12)}{(4\pi)^6\, 691}\,\tau_2^6 + \frac{4\pi\zeta(8)\zeta(3)}{(4\pi)^6\, 3}\tau_2^3 + \frac{43\pi\zeta(6)\zeta(5)}{(4\pi)^6\, 4}\tau_2 - \frac{\zeta(6)\zeta(3)^2}{(4\pi)^6\, 21} + O(\tau_2^{-1}) \\ &= -\frac{3435}{691}\,\hat{E}_6 + \frac{43}{12}\,\hat{E}_3^2 + \frac{2}{3}\,\hat{E}_2\hat{E}_4 - \frac{\zeta(6)\zeta(3)^2}{(4\pi)^6\, 21} + \mathrm{D}^{fin}_{1,1,1,2;1} \end{aligned} \tag{D.38}$$

The results (D.38), together with (D.19), (D.21), (D.22) are used in (C.6) — combined with the other vertex functions found earlier — to find the corresponding $\hat{\sigma}_3^2$ terms in (5.25).

## E. Phase-space integrals for two-particle unitarity

We will here evaluate the phase-space integrals that arise in the unitarity analysis of section 4.1. This will involve a number of basic integrals that result from expanding the tree amplitudes in (4.2) in powers of $\hat{\sigma}'_2 = (\alpha'/4)^2\,(s^2 + t'^2 + u'^2)$, $\hat{\sigma}'_3 = (\alpha'/4)^3\,3st'u'$, $\hat{\sigma}''_2 = (\alpha'/4)^2\,(s^2 + t''^2 + u''^2)$ and $\hat{\sigma}''_3 = (\alpha'/4)^3\,3st''u''$, where $t'$ $u'$, $t''$, $u''$ are defined in terms of the internal and external momenta in (4.9). This leads to integrals of the general form

$$S(a,b,c,d) = \int d^{10}k\,\delta^{(+)}(k^2)\,\delta^{(+)}((p_1+p_2-k)^2)\,(t')^{a-1}\,(u')^{b-1}\,(t'')^{c-1}(u'')^{d-1}\,, \tag{E.1}$$

where $a, b, c, d$ are integers.

The $\delta^{(+)}$ functions impose the mass-shell conditions. If we choose the centre of mass frame in which $q^\mu = p_1^\mu + p_2^\mu = (s^{\frac{1}{2}}, \vec{0}_9)$ (where $\vec{0}_k$ is the $k$-dimensional zero vector) we can write

$$\delta^{(+)}((q-k)^2) = \frac{1}{2s}\delta\!\left(k^0 - \frac{1}{2}\,s^{\frac{1}{2}}\right), \qquad \delta^{(+)}(k^2) = \frac{1}{s}\,\delta(k^0 - |\vec{k}|)\,. \tag{E.2}$$

so that $k^\mu$ has the form

$$k^\mu = (k^0, \vec{k}) = \frac{s^{\frac{1}{2}}}{2}\,(1, \vec{n}_9)\,, \tag{E.3}$$

where $\vec{n}_9$ is the unit nine-vector. The integral (E.1) may be evaluated by choosing the momenta, which satisfy the on-shell masslessness condition, to take the following form,

$$\begin{aligned}
p_1^\mu &= \frac{s^{\frac{1}{2}}}{2}\,(1,\,1\,,\vec{0}_8) & p_2^\mu &= \frac{s^{\frac{1}{2}}}{2}\,(1,\,-1\,,\vec{0}_8)\\
p_3^\mu &= \frac{s^{\frac{1}{2}}}{2}\,(-1,\,\cos\rho,\,\sin\rho,\,\vec{0}_7)\,, & p_4^\mu &= \frac{s^{\frac{1}{2}}}{2}\,(-1,\,-\cos\rho,\,-\sin\rho,\,\vec{0}_7)\\
k^\mu &= \frac{s^{\frac{1}{2}}}{2}\,(1,\,\cos\theta,\,\sin\theta\cos\phi,\,\sin\theta\sin\phi\,\vec{n}_7)\,,
\end{aligned} \tag{E.4}$$

where $\vec{n}_7$ is the unit seven-vector and the scattering angle, $\rho$, is given by

$$\cos\rho = \frac{t-u}{s}\,. \tag{E.5}$$

Changing variables from $k^i$ $(i = 1, \ldots, 9)$ to $\theta$, $\phi$, $\vec{n}_7$, the measure of integration becomes

$$d^{10}k\,\delta^{(+)}((q-k)^2)\,\delta^{(+)}(k^2) = \frac{s^3}{2^6}\,(\sin\theta)^7\,(\sin\phi)^6\,d\theta\,d\phi\,d^7\vec{n}_7\,\delta(\vec{n}_7^2 - 1)\,. \tag{E.6}$$

In this parametrization we have

$$\begin{aligned}
t' &= 2p_1\cdot k = -\frac{s}{2}\,(1-\cos\theta)\,, \quad & t'' &= -2p_4\cdot k = -\frac{s}{2}\,(1+\cos\theta\cos\rho+\sin\theta\cos\phi\sin\rho)\,,\\
u' &= 2p_2\cdot k = -\frac{s}{2}\,(1+\cos\theta)\,, \quad & u'' &= -2p_3\cdot k = -\frac{s}{2}\,(1-\cos\theta\cos\rho-\sin\theta\cos\phi\sin\rho)\,.
\end{aligned} \tag{E.7}$$

The integral over the seven dimensional unit vector $\vec{n}_7$ gives an overall factor of the volume of the six-dimensional sphere $\mathrm{vol}(S^6) = 16\pi^3/15$.

Substituting the term of order $\zeta(3)\,\mathbf{R}^4$ introduces a factor of $st''u''$ into one of the tree amplitudes on the right-hand side of (4.2), with the lowest order term in the other, leading to an integral of the form (E.1) with $a = b = 0$ and $c = d = 1$,

$$S(0,0,1,1) = 2\zeta(4)\,s\,, \tag{E.8}$$

which determines the coefficient of the threshold terms of order $\alpha'^4\,s^4\,\log(-s/\mu_4)$.

Similarly, substituting the expansion of one tree-level amplitude at order $\zeta(5)\,\hat{\sigma}_2\,\mathbf{R}^4$ and the lowest order term in the other in (4.2) leads to an integral of the form (E.1) with $a = b = 0$ and $c = d = 2$,

$$S(0,0,3,1) = \frac{\zeta(4)}{56}\,s\,(31\,s^2 + (t-u)^2)\,, \tag{E.9}$$

which determines the coefficient of the threshold terms of order $\alpha'^6\,s^6\,\log(-s/\mu_6)$.

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