Hybrid formalism and topological amplitudes*

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Abstract

We study four-dimensional compactifications of type II superstrings on Calabi-Yau spaces in the hybrid formalism. Chiral and twisted-chiral interactions are re-derived, which involve the coupling of the compactification moduli to two powers of the Weyl-tensor and of the derivative of the universal tensor field-strength. We review the formalism and provide details of some of its technicalities.

1 Introduction

Type II string compactified on a Calabi–Yau 3-fold gives rise to $\mathcal{N} = 2$ supergravity in four dimension. Most calculations of string scattering amplitudes, and therefore of the construction of the low-energy-effective action, are based on the Ramond-Neveu-Schwarz (RNS) formulation of the superstring. A drawback of this formulation is that spacetime supersymmetry is not manifest and is achieved only after GSO projection.

An alternative formulation without these complications is the hybrid formulation. Hybrid string theory can be obtained by a field redefinition from the gauge-fixed RNS string or by covariantizing the Green-Schwarz (GS) string in light-cone gauge. In this sense, worldsheet reparametrizations are gauge-fixed in the hybrid formulation. Nevertheless, there is no need for ghost-like fields in the formalism since the theory can be formulated as a $\mathcal{N} = 4$ topological theory and amplitudes can be computed directly by the methods of topological string theory [1]. The theory consists of two completely decoupled twisted worldsheet SCFT, one describing the spacetime part,

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one the internal part. Despite being twisted, hybrid string theory describes the full theory, i.e., it computes also non-topological amplitudes. Hybrid type IIA and IIB string theories are distinguished by the relative twisting of the left- and right-moving sector of the internal SCFT. When working with either type one is therefore committed to a given fixed twisting.

The hybrid formulation was developed in a series of papers by N. Berkovits and various collaborators [1, 2, 3]. It was reviewed in [4]. One purpose of this note is our attempt to fill in details of some of the more technical aspects. This is done in Sections 2 and the Appendices B and C, the content of which is well known to the few experts in the field, but often not readily accessible.

The main application of hybrid strings in this note are presented in Sections 3 and 4. We extend the analysis of higher order derivative interactions to the twisted-chiral sector. The procedure is analogous to the computation in the chiral sector given [1]. Even though one is working with a fixed relative twisting, giving rise to either type IIA or type IIB, it is shown that the chiral and twisted-chiral couplings of each type II theory depend on both the A-model and B-model topological partition functions. In the effective action these amplitudes give rise to couplings of compactification moduli to two powers of the Weyl tensor or of the derivative of the universal tensor field-strength. In the RNS formulation these couplings were discussed in [5].

Another possible application is flux compactifications of the type II string with $\mathcal{N} = 1$ spacetime supersymmetry. The breaking $\mathcal{N} = 2 \to \mathcal{N} = 1$ results from auxiliary fields acquiring vacuum expectation values [6]. Due to its manifest spacetime supersymmetry, the hybrid formulation might be the most suitable. First steps in this direction were already taken in [7, 8].

2 Compactified string theory in RNS and hybrid variables

In this section we present a detailed account of the field mapping between the variables of the Ramond-Neveu-Schwarz (RNS) formulation of those of the hybrid formulation of the superstring. We consider here only Calabi-Yau compactifications to four spacetime dimensions and split all variables into a spacetime and an internal part. The internal part is practically the same for the RNS and the hybrid formulation, while the two descriptions of the spacetime are different.

2.1 Hybrid variables

Type II and heterotic string theory compactified on a Calabi-Yau three-fold can be formulated within a covariant version of the Green-Schwarz (GS) formulation [2, 3]. The spacetime part consists of four bosons x^m, two pairs of left-moving canonically conjugate Weyl fermions (p_α, θ^β) and $(\bar{p}^{\dot\alpha}, \bar{\theta}_{\dot\beta})$, both of conformal weight $(1, 0)$ and a chiral boson ρ with action

$$S = \frac{1}{\pi}\int d^2z \left\{ \frac{1}{2}\bar{\partial}x^m\partial x_m + p^\alpha\bar{\partial}\theta_\alpha + \bar{p}_{\dot\alpha}\bar{\partial}\bar{\theta}^{\dot\alpha} + \tilde{p}^\alpha\partial\tilde{\theta}_\alpha + \bar{\tilde{p}}_{\dot\alpha}\partial\bar{\tilde{\theta}}^{\dot\alpha} + \frac{1}{2}\bar{\partial}\rho\partial\rho \right\} . \quad (2.1)$$

The chiral boson is periodic with period $\rho \sim \rho + 2\pi i$ and

$$\rho(z)\rho(w) = -\ln(z - w) , \quad (2.2)$$

From these fields one constructs the generators

$$
\begin{aligned}
T &= -\frac{1}{2}\partial x^m \partial x_m - p^\alpha \partial\theta_\alpha - \bar{p}_{\dot\alpha}\partial\bar\theta^{\dot\alpha} - \frac{1}{2}\partial\rho\partial\rho - \frac{1}{2}Q_\rho \partial^2\rho\,, \\
G^- &= \frac{1}{\sqrt{32}}e^\rho\, d^2\,, \qquad G^+ = -\frac{1}{\sqrt{32}}e^{-\rho}\,\bar d^2\,, \\
J &= \partial\rho\,,
\end{aligned}
\tag{2.3}
$$

We have defined the fermionic currents (cf. Appendix A)

$$
\begin{aligned}
d_\alpha &= p_\alpha + i\bar\theta^{\dot\alpha}\partial x_{\alpha\dot\alpha} - \bar\theta^2\partial\theta_\alpha + \frac{1}{2}\theta_\alpha\partial\bar\theta^2\,, \\
\bar d^{\dot\alpha} &= \bar p^{\dot\alpha} + i\theta_\alpha \partial x^{\alpha\dot\alpha} - \theta^2\partial\bar\theta^{\dot\alpha} + \frac{1}{2}\bar\theta^{\dot\alpha}\partial\theta^2\,.
\end{aligned}
\tag{2.4}
$$

In the definition of the energy-momentum tensor a background charge Q_ρ for the chiral boson ρ is included. It is obtained from the coupling of the field ρ to the world-sheet curvature. This coupling is not visible in conformal gauge. The background charge implies the conformal weights $\mathrm{wt}(\exp(q\rho)) = -\frac{1}{2}q(q+Q_\rho)$, and therefore $\mathrm{wt}(\exp(\pm\rho)) = -\frac{1}{2}(1 \pm Q_\rho)$ and $\mathrm{wt}(G^\pm) = \frac{1}{2}(3 \pm Q_\rho)$. Also the central charge of the Virasoro algebra depends on the value of Q_ρ. It is $c_x + c_{p,\theta} + c_{\bar p,\bar\theta} + c_\rho = 4-4-4+(1+3Q_\rho^2) = 3\,(Q_\rho^2-1)$. For $Q_\rho = 0$, (T, G^+, G^-, J) generate an untwisted $c=-3$, $\mathcal{N}=2$ superconformal algebra while for non-vanishing Q_ρ the algebra is twisted. It is topological for $Q_\rho = \pm 1$. When checking the algebra for this case, in particular, the correct overall sign on the right-hand side of

$$
G^+(z)G^-(w) \sim \frac{\frac{c}{3}}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{T(w)}{z-w}\,,
\tag{2.5}
$$

for $c=-3$, the relative minus sign in the definitions of $G^\pm$ is crucial. It is also consistent with the requirement $(G^+)^\dagger = G^-$ if we define $(e^\rho)^\dagger = -e^{-\rho}$. The hermiticity properties of the hybrid variables are further discussed in Appendices A and B. As explained in section 2.3, the field mapping from the RNS variables determines the background charge as $Q_\rho = -1$. The four-dimensional part is therefore a twisted $c=-3$, $\mathcal{N}=2$ superconformal algebra.

The Calabi-Yau compactification is described by an internal $\mathcal{N}=2$ SCFT. The generators (T_C, G_C^+, G_C^-, J_C) form an untwisted $c=9$, $\mathcal{N}=2$ superconformal algebra and commute with (2.3). The generators $(\mathcal{T}, \mathcal{G}^\pm, \mathcal{J})$ of the combined system are obtained by adding[1] the twisted generators $(T_C + \frac{1}{2}\partial J_C, G_C^+, G_C^-, J_C)$ to those of (2.3),

$$
\mathcal{T} = T + T_C + \frac{1}{2}\partial J_C\,, \quad \mathcal{G}^\pm = G^\pm + G_C^\pm\,, \quad \mathcal{J} = J + J_C\,.
\tag{2.6}
$$

They form a twisted $c=6$, $\mathcal{N}=2$ superconformal algebra. The current J_C can be represented in terms of a free boson H as $J_C = i\sqrt{3}H$. The generators $G_C^\pm$ can then be written in the form $G_C^+ = e^{+\frac{i}{\sqrt{3}}H}G'$ and $G_C^- = e^{-\frac{i}{\sqrt{3}}H}\bar G'$ where G' and $\bar G'$ are uncharged under J_C. The conformal weight of $e^{\frac{iq}{\sqrt{3}}H}$ is $\frac{q}{6}(q-3)$.

[1]When working with the explicit realizations (2.3) of $G^\pm$ cocycle factors must be included in order for the space-time and the internal part of $\mathcal{G}^\pm$ to anticommute. The explicit expressions are given in (2.12).

For a twisted algebra, the conformal anomaly vanishes (though the other currents are anomalous). There are, therefore, two options: either, one untwists the resulting algebra, couples the system to a set of $c = -6$, $\mathcal{N} = 2$ superconformal ghosts (thereby canceling the central charge) and calculates scattering amplitudes utilizing the $\mathcal{N} = 2$ prescription [9]. Alternatively, one embeds the twisted $c = 6$, $\mathcal{N} = 2$ SCFT into a (small version of the) twisted $\mathcal{N} = 4$ algebra and uses the topological prescription [1, 10] to compute the spectrum and correlation functions. This is the method we follow.

The embedding into a twisted small $\mathcal{N} = 4$ superconformal algebra[2] proceeds as follows: The U(1)-current $\mathcal{J} = J + J_C$ is augmented to a triplet of currents $(\mathcal{J}^{++}, \mathcal{J}, \mathcal{J}^{--})$. The $\mathcal{J}$-charge of $\mathcal{J}^{\pm\pm}$ is ± 2 and the conformal weights are $\mathrm{wt}(\mathcal{J}^{++}) = 0$ and $\mathrm{wt}(\mathcal{J}^{--}) = 2$. They satisfy the SU(2) relation

$$\mathcal{J}^{++}(z)\mathcal{J}^{--}(w) \sim \frac{1}{(z-w)^2} + \frac{\mathcal{J}(w)}{(z-w)} \,. \tag{2.7}$$

There are two SU(2) doublets of fermionic generators: $(\mathcal{G}^+, \widetilde{\mathcal{G}}^-)$ and $(\mathcal{G}^-, \widetilde{\mathcal{G}}^+)$ that transform in the $\mathbf{2}$ and $\mathbf{2}^*$ of SU(2), respectively. The $\widetilde{\mathcal{G}}^{\pm}$ are defined via the operator products

$$\mathcal{J}^{\pm\pm}(z)\mathcal{G}^{\mp}(w) \sim \mp\frac{\widetilde{\mathcal{G}}^{\pm}(w)}{z-w} \,, \quad \mathcal{J}^{\pm\pm}(z)\widetilde{\mathcal{G}}^{\mp}(w) \sim \pm\frac{\mathcal{G}^{\pm}(w)}{z-w} \,. \tag{2.8}$$

and have $\mathrm{wt}(\tilde{\mathcal{G}}^+) = 1$ and $\mathrm{wt}(\tilde{\mathcal{G}}^-) = 2$. The other OPEs of $\mathcal{J}^{\pm\pm}$ with the fermionic generators are finite. The notation $\widetilde{\mathcal{O}}$ refers to a more general operator conjugation $\mathcal{O} \to \widetilde{\mathcal{O}}$, for which (2.8) is a special case. It is explained in Appendix B.

The nontrivial OPEs of the supercurrents are

$$\mathcal{G}^+(z)\widetilde{\mathcal{G}}^+(w) \sim \frac{2\mathcal{J}^{++}(w)}{(z-w)^2} + \frac{\partial\mathcal{J}^{++}(w)}{z-w} \,, \quad \widetilde{\mathcal{G}}^-(z)\mathcal{G}^-(w) \sim \frac{2\mathcal{J}^{--}(w)}{(z-w)^2} + \frac{\partial\mathcal{J}^{--}(w)}{z-w} \,. \tag{2.9}$$

and

$$\mathcal{G}^+(z)\mathcal{G}^-(w) \sim \frac{2}{(z-w)^3} + \frac{\mathcal{J}(w)}{(z-w)^2} + \frac{\mathcal{T}(w)}{z-w} \,, \tag{2.10}$$

and the very same OPE for $\widetilde{\mathcal{G}}^+(z)$ and $\widetilde{\mathcal{G}}^-(w)$. The explicit form of the currents and super-currents is

$$\mathcal{J}^{\pm\pm}(z) = c_{\pm} e^{\pm\int^z \mathcal{J}} = c_{\pm} e^{\pm(\rho + i\sqrt{3}H)} \,, \tag{2.11}$$

and

$$\begin{aligned} \mathcal{G}^+ &= -\left(\tfrac{1}{\sqrt{32}} e^{-\rho}\bar{d}^2 + c_+ G_C^+\right) \\ \mathcal{G}^- &= \quad \tfrac{1}{\sqrt{32}} e^{\rho} d^2 + c_- G_C^- \\ \tilde{\mathcal{G}}^+ &= -\left(\tfrac{1}{\sqrt{32}} c_+ e^{2\rho + i\sqrt{3}H} d^2 + e^{\rho} G_C^{++}\right) \\ \tilde{\mathcal{G}}^- &= -\left(\tfrac{1}{\sqrt{32}} c_- e^{-2\rho - i\sqrt{3}H} \bar{d}^2 + e^{-\rho} G_C^{--}\right) \end{aligned} \tag{2.12}$$

[2]Small $\mathcal{N} = 4$ superconformal algebras were constructed in [11]. Our conventions are based on the algebra presented in [12].

Here $G_C^{\pm\pm}$ are defined[3] as $G_C^{\pm\pm} = e^{\pm i\sqrt{3}H}(G_C^{\mp})$ and $c_\pm = e^{\pm i\pi \oint \mathcal{J}} = e^{\pm i\pi(p_\rho + \sqrt{3}p_H)}$.[4] The various signs and cocycle factors $c_\pm$ are necessary in order to guarantee the hermiticity relations $(\mathcal{J}^{++})^\dagger = \mathcal{J}^{--}$, $(\mathcal{G}^+)^\dagger = \mathcal{G}^-$ and $(\tilde{\mathcal{G}}^+)^\dagger = \tilde{\mathcal{G}}^-$, the appropriate Grassmann parity of the generators, and for correctly reproducing the algebra.

In type II theories the spacetime fields are supplemented by two pairs of right-moving canonically conjugate Weyl fermions and a periodic right-moving chiral boson. We will use the subscripts "L" and "R" in order to distinguish left-moving from right-moving fields and adopt the notation $|A|^2 = A_L A_R$. For notational simplicity we discuss mostly type IIB string theory, for which the left- and right-movers are twisted in the same way. For type IIA theories the right-moving part of the algebra is obtained by the opposite twisting as compared to IIB. Operationally, the expressions for IIA can be obtained from those of IIB by replacing $(J_C)_R \to -(J_C)_R$ (thereby reversing the background charge) in above definitions of the currents and by reversing, e.g., $(G_C^\pm)_R \to (G_C^\mp)_R$. The spacetime part remains unaffected.

2.2 RNS variables

In the RNS representation the spacetime fields are (x^m, ψ^m) with $m = 1, \cdots, 4$. They contribute with $c^{x,\psi} = 6$ to the central charge of the Virasoro algebra. We will concentrate on the left-moving sector in what follows.

It is convenient to bosonize the (Euclideanized) worldsheet fermions,

$$\psi^1 \pm i\psi^2 = e^{\pm i\varphi^1}\,, \qquad \psi^3 \pm i\psi^4 = e^{\pm i\varphi^2}\,. \tag{2.13}$$

As usual we suppress cocycle factors. The bosonized expression for the SO(4)-spin fields of positive and negative chirality are

$$S^\alpha = e^{\pm\frac{i}{2}(\varphi^1+\varphi^2)}\,, \qquad \bar{S}^{\dot\alpha} = e^{\pm\frac{i}{2}(\varphi^1-\varphi^2)}\,. \tag{2.14}$$

The internal sector (the Calabi-Yau threefold) is accounted for by a $c = 9$ CFT with $\mathcal{N} = 2$ worldsheet superconformal symmetry generated by $\breve{T}_C$, $\breve{G}_C{}^\pm$, and $\breve{J}_C$. Their relation to the generators introduced in the previous section is explained in sec. 2.3. The U(1) R-current $\breve{J}_C$ can be expressed in terms of a free chiral boson $\breve{H}$ as

$$\breve{J}_C = i\sqrt{3}\,\partial\breve{H}\,, \quad \breve{H}(z)\breve{H}(w) = -\ln(z-w)\,. \tag{2.15}$$

Any field $\mathcal{O}^{(q)}$ with R-charge q can be decomposed as $\mathcal{O}^{(q)} = \exp(\frac{iq}{\sqrt{3}}\breve{H})\mathcal{O}'$ where $\mathcal{O}'$ is uncharged with respect to $\breve{J}_C$. For the generators $\breve{G}_C^\pm$ this part is independent of $\breve{H}$.

Covariant quantization requires fixing the local reparametrization invariance of $\mathcal{N} = 1$ worldsheet supergravity. This introduces the (b, c) and (β, γ) ghost systems. With $c^{\text{gh}} = -15$ the total central charge vanishes. Following [13], we 'bosonize' the ghosts

$$\begin{aligned} b &= e^{-\sigma}\,, & c &= e^{\sigma}\,, \\ \beta &= e^{-\phi}\partial\xi = e^{-\phi+\chi}\partial\chi\,, & \gamma &= e^{\phi}\eta = e^{\phi-\chi}\,, \\ \xi &= e^{\chi}\,, & \eta &= e^{-\chi}\,. \end{aligned} \tag{2.16}$$

[3] The expression $A(B(w))$ denotes the residue in the OPE of $A(z)$ with $B(w)$ and equals the (anti)commutator $[\oint A, B(w)\}$. The notation is $\oint A \equiv= \frac{1}{2\pi i}\oint dz A(z)$.

[4] The momentum modes $p_\rho = \oint \partial\rho$ and $p_H = i\oint \partial H$ satisfy the commutation relations $[p_\rho, \rho] = -1$ and $[p_H, H] = -i$. Their hermiticity properties are discussed in Appendix B and imply $(c_+)^\dagger = c_-$.

The total energy-momentum tensor is

$$\begin{aligned} T_{\mathrm{RNS}} = & -\frac{1}{2}\partial x^m \partial x_m - \frac{1}{2}\psi^m \partial \psi_m + \breve{T}_C \\ & + \frac{1}{2}\left[(\partial\sigma)^2 + (\partial\chi)^2 - (\partial\phi)^2\right] - \frac{1}{2}\,\partial^2(2\phi - 3\sigma - \chi)\,. \end{aligned} \tag{2.17}$$

The generators (T_{RNS}, b, j_{BRST}, J^{gh}) form a twisted $\mathcal{N} = 2$ algebra. The U(1)-current is $J^{\mathrm{gh}} = -(bc + \xi\eta)$. The BRST-current j_{BRST} is given in Appendix B.

2.3 Field redefinition from RNS to hybrid variables

The RNS variables are mapped to the hybrid variables in a two-step procedure. From the RNS variables one first forms a set of variables, called the "chiral GS-variables" in [2, 1]. In this section, we refer to these variables. The hybrid variables of the previous section are obtained in a second step by performing a field redefinition on the chiral GS-variables. We will suppress this field redefinition in the following and refer to Appendix B for a detailed account.

Following [2, 1] we define the following superspace variables:[5]

$$\begin{aligned} \theta^\alpha &= c\xi\, e^{-\frac{3}{2}\phi}\, \bar{\Sigma}\, S^\alpha\,, \qquad & \bar{\theta}^{\dot{\alpha}} &= e^{\frac{\phi}{2}}\, \Sigma\, \bar{S}^{\dot{\alpha}}\,, \\ p_\alpha &= b\eta\, e^{\frac{3}{2}\phi}\, \Sigma\, S_\alpha\,, \qquad & \bar{p}_{\dot{\alpha}} &= e^{-\frac{\phi}{2}}\, \bar{\Sigma}\, \bar{S}_{\dot{\alpha}}\,, \end{aligned} \tag{2.18}$$

where

$$\Sigma = e^{\frac{i}{2}\sqrt{3}\breve{H}}\,, \quad \bar{\Sigma} = e^{-\frac{i}{2}\sqrt{3}\breve{H}}\,. \tag{2.19}$$

In this definition, θ^α (p_α) carries charge $q_C = -\frac{3}{2}$ ($+\frac{3}{2}$). Here and in the following, q_C denotes the charge under $\oint \breve{J}_C$, the U(1) R-symmetry of the internal $c = 9$ SCFT.

In order to implement a complete split between the spacetime and the internal part one must require that the hybrid variables (2.18) do not transform under the $c = 9$ SCFT generators. For instance, the variables (2.18) should not carry a charge with respect to the U(1) R-symmetry. This can be realized by shifting the U(1) charge by the picture-counting operator,

$$\mathcal{P} = -\beta\gamma + \xi\eta = -\partial\phi + \partial\chi\,. \tag{2.20}$$

The variable θ^α, for instance, has picture $-\frac{1}{2}$. This motivates the following definition of the shifted U(1) current

$$J_C = \breve{J}_C - 3\mathcal{P} = \breve{J}_C + 3\,\partial(\phi - \chi)\,. \tag{2.21}$$

More generally, the fields of the internal part are transformed by the field transformation [2, 1]

$$F_C = e^{\mathcal{W}}\breve{F}_C e^{-\mathcal{W}}\,, \quad \mathcal{W} = \oint (\phi - \chi)\breve{J}_C\,. \tag{2.22}$$

For the other generators of the internal $\mathcal{N} = 2$ algebra this implies

$$\begin{aligned} G_C^+ &= e^{(\phi-\chi)}\breve{G}_C^+\,, \\ G_C^- &= e^{-(\phi-\chi)}\breve{G}_C^-\,, \\ T_C &= \breve{T}_C + \partial(\phi - \chi)\,\breve{J}_C + \frac{3}{2}\,(\partial\phi - \partial\chi)^2\,. \end{aligned} \tag{2.23}$$

[5]The RNS variables are actually subject to the rescaling given in Appendix B.1. We neglect this issue here.

These are the generators that couple to the chiral GS-variables defined in (2.18). The generators coupling to the hybrid variables are related to these by the field redefinition discussed in Appendix B, which does not affect the algebraic structure discussed in the following. The currents (T_C, G_C^+, G_C^-, J_C) generate an untwisted $\mathcal{N}=2$ superconformal algebra. The shift by the picture changing current in the relation (2.21) amounts to a background charge $Q_{J_C}=-3$ for the current J_C. The RNS ghost-current

$$J^{\mathrm{gh}} = -(bc+\xi\eta) = \partial\sigma - \partial\chi\,, \tag{2.24}$$

which is obtained from the ghost current of the "small Hilbert space" $-(bc+\beta\gamma) = \partial\sigma - \partial\phi$ by adding the picture-counting operator (2.20), is mapped to a combination of the current $J=\partial\rho$ and the shifted internal U(1) R-current [2, 1],

$$J = \partial\rho = J^{\mathrm{gh}} - J_C = \partial\sigma + 2\partial\chi - 3\partial\phi - \breve{J}_C\,. \tag{2.25}$$

This equation defines the chiral boson ρ in terms of the RNS variables. The mapping is such that the ρ-system[6] acquires a background charge $Q_\rho = -1$ and that it has regular OPEs with the internal generators (T_C, G_C^+, G_C^-, J_C). The superspace variables θ, $\bar\theta$, p and $\bar p$, and the redefined internal operators (2.22) all have zero ρ-charge. This, in particular, means that (2.22) leads to a complete decoupling of the internal sector from the chiral GS-variables.

The field redefinitions (2.18) are such that the RNS generators $(T_{\mathrm{RNS}}\,, b\,, j_{\mathrm{BRST}}\,, J^{\mathrm{gh}})$ map to the hybrid generators of the $\mathcal{N}=2$ algebra

$$T_{\mathrm{RNS}} = \mathcal{T}\,, \quad b = \mathcal{G}^-\,, \quad j_{\mathrm{BRST}} = \mathcal{G}^+\,, \quad J^{\mathrm{gh}} = \mathcal{J}\,. \tag{2.26}$$

We hasten to add that in order to arrive at this correspondence one must correctly take into account the field mapping from the chiral GS-variables to the hybrid variables (cf. Appendix B).

It is straightforward to express the raising and lowering operators (2.11) of the $\mathcal{N}=4$ algebra in terms of RNS variables, since one can verify that these are not affected by the additional field redefinition, mapping hybrid to chiral GS-variables as discussed in Appendix B. From (2.18) one therefore concludes

$$\mathcal{J}^{++} = c\,\eta\,, \qquad \mathcal{J}^{--} = b\,\xi\,. \tag{2.27}$$

Using this it is easy to verify that the generators $\widetilde{\mathcal{G}}^\pm$, defined through (2.8), are expressed in RNS variables by

$$\widetilde{\mathcal{G}}^- = [Q_{\mathrm{BRST}}, b\,\xi] = b\,Z + \xi\,T_{\mathrm{RNS}}\,, \quad \widetilde{\mathcal{G}}^+ = \eta\,, \tag{2.28}$$

where Z is the picture changing operator of the RNS formalism, given in (B.20). We summarize the dictionary between the RNS and the hybrid currents in the following table:

$$\begin{aligned} \mathcal{T} &= T_{\mathrm{RNS}}\,, \\ \mathcal{J}^{++} = c\,\eta\,, \quad \mathcal{J}^{--} &= b\,\xi\,, \quad \mathcal{J} = J^{\mathrm{gh}} = -(bc+\xi\eta)\,, \\ \mathcal{G}^+ = j_{\mathrm{BRST}}\,, \quad \widetilde{\mathcal{G}}^+ &= \eta\,, \quad \mathcal{G}^- = b\,, \quad \widetilde{\mathcal{G}}^- = b\,Z + \xi T_{\mathrm{RNS}}\,. \end{aligned} \tag{2.29}$$

[6]The current which satisfies $T(z)j(w) \sim \frac{Q_\rho}{(z-w)^3} + \ldots$ and leads to $\bar\partial j = \frac{1}{8}Q_\rho\sqrt{g}R$ is $j = -\partial\rho = -J$.

So far we have concentrated on the left-moving (holomorphic) sector of the theory. For the heterotic string the right-moving sector is treated in the same way as in the RNS formulation: it is simply the bosonic string. For the type II string, however, the distinction between type IIA and IIB needs to be discussed. Since the construction presented above involves twisting the internal $c = 9$ SCFT, the distinction between IIA and IIB is analogous to the one in topological string theory where one deals with the so-called A and B twists (which are related by mirror symmetry). In type IIB, the left- and right-moving sectors are treated identically and the distinction is merely in the notation, i.e., to replace all fields $\phi_L(z)$ by $\phi_R(\bar{z})$. In type IIA, however, the twists in the two sectors are opposite. The two possible twists differ in the shift of the conformal weight, which is either $h \to h - \frac{1}{2}q$ or $h \to h + \frac{1}{2}q$. Above we have discussed the first possibility. The second twist is implemented by the replacement $\breve{T}_C \to \breve{T}_C - \frac{1}{2}\partial \breve{J}_C$ and follows from the first by the substitution $\breve{J}_C \to -\breve{J}_C$. This also implies that the transformation (2.22) is now defined with $\mathcal{W} = -\oint (\phi - \chi)\breve{J}_C$ which leads to

$$\begin{aligned} T_C &= \breve{T}_C - \partial(\phi - \chi)\breve{J}_C + \frac{3}{2}(\partial\phi - \partial\chi)^2 \,, \\ J_C &= \breve{J}_C + 3\mathcal{P} \,, \end{aligned} \tag{2.30}$$

and

$$J = \partial\rho = J^{\rm gh} + J_C = \partial\sigma + 2\partial\chi - 3\partial\phi + \breve{J}_C \,. \tag{2.31}$$

With this definition, ρ still has background charge $Q_\rho = -1$. The twisted $c = 9$ SCFT is generated by $(T_C - \frac{1}{2}\partial J_C, G_C^+, G_C^-, J_C)$, where now the conformal weights of G_C^+ and G_C^- are two and one, respectively. The full right-moving supersymmetry generators for the type IIA theory are (suppressing signs and cocycle factors)

$$\mathcal{G}_R^\pm = G_R^\pm + G_{C\,R}^\mp \,. \tag{2.32}$$

The map between RNS and hybrid variables must also be modified for the latter to be neutral under J_C:

$$\begin{aligned} \theta^\alpha &= c\xi\, e^{-\frac{3}{2}\phi}\,\Sigma\, S^\alpha \,, \qquad & \bar{\theta}^{\dot\alpha} &= e^{\frac{\phi}{2}}\,\bar{\Sigma}\,\bar{S}^{\dot\alpha} \,, \\ p_\alpha &= b\eta\, e^{\frac{3}{2}\phi}\,\bar{\Sigma}\, S_\alpha \,, \qquad & \bar{p}_{\dot\alpha} &= e^{-\frac{\phi}{2}}\,\Sigma\,\bar{S}_{\dot\alpha} \,. \end{aligned} \tag{2.33}$$

In the type IIA theory this applies for the right-movers, given (2.18) for the left movers. Summarizing, the difference between type IIA and type IIB is seen in the different right-moving U(1) charge assignment to ρ_R.

2.4 Physical state conditions and $\mathcal{N} = 4$-embeddings

Having the dictionary (2.29) at hand it is simple to rephrase the standard physical state conditions of the RNS formalism in terms of hybrid variables. We refer to [4, 10] for details. Physical RNS vertex operators are in the cohomology of $Q_{\rm BRST} = \oint j_{\rm BRST}$ and $\oint \eta$:

$$j_{\rm BRST}(V^+) = 0 \,, \quad \eta(V^+) = 0 \,, \quad \delta V^+ = j_{\rm BRST}(\eta(\Lambda^-)) \,, \tag{2.34}$$

The condition imposed by $\oint \eta$ implies that V^+ is in the small RNS Hilbert space, i.e., it does not depend on the ξ zero-mode. Furthermore, V^+ has ghost number 1 with respect to (2.24) as indicated with the superscript. The charge with respect to $-(bc+\beta\gamma)$ is $1+\mathcal{P}$, where $\mathcal{P}$ is the picture (2.20). Using (2.29), the conditions (2.34) are expressed in hybrid variables as

$$\mathcal{G}^+(V^+) = 0\,, \quad \widetilde{\mathcal{G}}^+(V^+) = 0\,, \quad \delta V^+ = \mathcal{G}^+(\widetilde{\mathcal{G}}^+(\Lambda^-))\,, \tag{2.35}$$

In addition, V^+ has $\mathcal{J}$-charge 1 as indicated. Note that $\mathcal{G}^+$ and $\widetilde{\mathcal{G}}^+$ have trivial cohomologies, since

$$\mathcal{G}^+\left(\sqrt{2}e^{\rho}\bar{\theta}^2\right) = 1\,, \quad \widetilde{\mathcal{G}}^+\left(\sqrt{2}\widetilde{(e^{\rho}\bar{\theta}^2)}\right) = 1\,. \tag{2.36}$$

Therefore, one can solve, e.g., the $\widetilde{\mathcal{G}}^+$-constraint by introducing the U(1)-neutral field $\mathcal{V}$ satisfying

$$V^+ = \widetilde{\mathcal{G}}^+(\mathcal{V})\,. \tag{2.37}$$

Up to the gauge transformations $\delta\mathcal{V} = \widetilde{\mathcal{G}}^+(\widetilde{\Lambda}^-)$, $\mathcal{V}$ is determined in terms of V^+ by $\mathcal{V} = \sqrt{2}\widetilde{(e^{\rho}\theta^2)}V^+$, where we used (2.36). It follows that (2.35) can be rephrased in terms of $\mathcal{V}$ as

$$\mathcal{G}^+(\widetilde{\mathcal{G}}^+(\mathcal{V})) = 0\,, \quad \delta\mathcal{V} = \mathcal{G}^+(\Lambda^-) + \widetilde{\mathcal{G}}^+(\widetilde{\Lambda}^-)\,. \tag{2.38}$$

Using RNS variables these manipulations become much more transparent. Notice that $\sqrt{2}\widetilde{(e^{\rho}\bar{\theta}^2)} = e^{-2\rho - i\sqrt{3}H}\theta^2 = \xi$. The first equality follows from the definition (B.23), the second from (2.18) (the additional conjugation in Appendix B.2 does not affect this result). Therefore, $\mathcal{V}$ lives in the large RNS Hilbert space. Using the RNS variables it is straightforward to show that $\mathcal{G}^+(\mathcal{V}) = ZV^+ = Z\,\widetilde{\mathcal{G}}^+(\mathcal{V})$, hence $\mathcal{G}^+(\mathcal{V})$ and $\widetilde{\mathcal{G}}^+(\mathcal{V})$ are related by picture changing. This will play a role momentarily when we discuss integrated vertex operators. From now on we will often drop the bracket on expressions like $\mathcal{G}^\pm(\mathcal{V})$ when the generators $\mathcal{G}^\pm$ and alike are involved, i.e., $\mathcal{G}^\pm\mathcal{V} \equiv \mathcal{G}^\pm(\mathcal{V})$.

Following [10] we fix the gauge symmetry (2.38) by choosing a gauge condition which resembles Siegel gauge: we require the vanishing of the second-order poles in the OPE's of $\mathcal{G}^-$ and $\widetilde{\mathcal{G}}^-$ with $\mathcal{V}$. Vertex operators $\mathcal{V}$ in this gauge have conformal weight 0. For SU(2) singlets these gauge fixing conditions are equivalent to the vanishing of the second-order poles of $\mathcal{G}^-$ and $\mathcal{G}^+$ with $\mathcal{V}$. For massless fields $\mathcal{V}(x,\theta,\bar{\theta})$ that depend only on x, θ^α, and $\bar{\theta}^{\dot{\alpha}}$ but not their derivatives, there are no poles of order 3 or higher in the OPE of $\mathcal{V}(x,\theta,\bar{\theta})$ with $\mathcal{G}^\pm$. Hence for these operators the gauge fixing constraints are equivalent to the primarity constraints of the $\mathcal{N}=2$ subalgebra, which here means the vanishing of all poles of order 2 and higher in the OPE of $\mathcal{V}$ with $\mathcal{G}^\pm$. This has also been explained in [4, 1, 3] and we will use these gauge-fixing constraints in the next sections also for massless fields that depend non-trivially on the compactification. So far we have discussed the unintegrated vertex operators V^+ and $\mathcal{V}$ residing in the small and large Hilbert spaces, respectively. To construct integrated vertex operators one proceeds like in the RNS formulation: $\int b(V^+) = \int \mathcal{G}^-V^+$. Note that for this choice the integrated and the unintegrated vertex operators are in the same picture $\mathcal{P}$. To obtain different pictures one considers $\int \widetilde{\mathcal{G}}^-V^+$. As is shown in [10], this provides the integrated vertex operator in a different ghost picture, $\int \widetilde{\mathcal{G}}^-V^+ = \int \mathcal{G}^-(Z_0V^+)$. Expressing the operators V^+ in terms of $\mathcal{V}$ opens new though related possibilities: using the previous result that relates $\widetilde{\mathcal{G}}^+\mathcal{V}$ and $\mathcal{G}^+\mathcal{V}$, one concludes that the four possible integrated vertex operators, $\int \mathcal{G}^-\widetilde{\mathcal{G}}^+\mathcal{V}$, $\int \mathcal{G}^-\mathcal{G}^+\mathcal{V}$, $\int \widetilde{\mathcal{G}}^-\widetilde{\mathcal{G}}^+\mathcal{V}$, and $\int \widetilde{\mathcal{G}}^-\mathcal{G}^+\mathcal{V}$ are all related by picture changing.

In the next sections we will use the following canonical ghost pictures: we take the unintegrated NS- and R-vertex operators in the -1 and $-\frac{1}{2}$ picture, respectively, the integrated ones in the 0 and $+\frac{1}{2}$ picture. Therefore, the relevant prescription is

$$\int b(ZV^+) = \int \mathcal{G}^-\mathcal{G}^+\mathcal{V} = \int \widetilde{\mathcal{G}}^-\widetilde{\mathcal{G}}^+\mathcal{V} \tag{2.39}$$

Adding the right-moving sector, the relevant expression for the integrated vertex operators can be written as

$$\int d^2z\,\mathcal{W}(z,\bar{z}) = \int d^2z\,\left|\mathcal{G}^-\mathcal{G}^+\right|^2\mathcal{V}(z,\bar{z})\,. \tag{2.40}$$

For better readability we drop the parenthesis here and in the following when the first-order poles in OPE with the generators $\mathcal{G}^{\pm}$ and alike are meant.

It is convenient to label the fermionic generators by indices $i,j=1,2$ according to

$$\mathcal{G}_i^+ = (\mathcal{G}^+,\widetilde{\mathcal{G}}^+)\,,\quad \mathcal{G}_i^- = (\mathcal{G}^-,\widetilde{\mathcal{G}}^-)\,. \tag{2.41}$$

They satisfy the hermiticity property $(\mathcal{G}_i^+)^\dagger = \mathcal{G}_i^-$. Consider general linear combinations

$$\widehat{\mathcal{G}}_i^- = u_{ij}\mathcal{G}_j^-\,,\quad \widehat{\mathcal{G}}_i^+ = u_{ij}^*\mathcal{G}_j^+\,, \tag{2.42}$$

where the second equation follows from the first by hermitian conjugation. Requiring that $\widehat{\mathcal{G}}_i^{\pm}$ satisfy the same $\mathcal{N}=4$ relations as $\mathcal{G}_i^{\pm}$ implies that u_{ij} are SU(2) parameters: $u_{11}=u_{22}^*\equiv u_1$ and $u_{21}^*=-u_{12}\equiv u_2$ with $|u_1|^2+|u_2|^2=1$. This shows that the $\mathcal{N}=4$ algebra has an SU(2) automorphism group that rotates the fermionic generators among each other. The u_i's parameterize the different embeddings of the $\mathcal{N}=2$ subalgebras into the $\mathcal{N}=4$ algebra. More explicitly, we have

$$\begin{aligned}\widehat{\widetilde{\mathcal{G}}}{}^+ &= \widehat{\mathcal{G}}_2^+ = u_1\tilde{\mathcal{G}}^+ + u_2\mathcal{G}^+\,,\\ \widehat{\mathcal{G}}^- &= \widehat{\mathcal{G}}_1^- = u_1\mathcal{G}^- - u_2\tilde{\mathcal{G}}^-\,,\end{aligned} \tag{2.43}$$

and analogous expressions for $\widehat{\mathcal{G}}^+ = \mathcal{G}_1^+$ and $\widehat{\widetilde{\mathcal{G}}}{}^- = \widehat{\mathcal{G}}_2^-$, which involve the complex conjugate parameters u_i^*.

It is advantageous to formulate the physical state conditions for general embeddings. This generalization also plays a role in the definition of scattering amplitudes. As will become clearer in section 3, the choice of a specific embedding is related to working in a specific picture in the RNS setting. Vertex operators are therefore defined in terms of the cohomologies of the operators $\oint\widehat{\mathcal{G}}^+$ and $\oint\widehat{\widetilde{\mathcal{G}}}{}^+$ as in (2.35) and (2.38). Correspondingly, integrated vertex operators have zero total U(1)-charge and can be written in the form

$$\mathcal{U} = \int d^2z\,|\widehat{\mathcal{G}}^-\widehat{\mathcal{G}}^+|^2\mathcal{V}\,. \tag{2.44}$$

We have $\int d^2z\,|\widehat{\mathcal{G}}^-\widehat{\mathcal{G}}^+|^2\mathcal{V} = \int d^2z\,|\widehat{\mathcal{G}}^+\widehat{\mathcal{G}}^-|^2\mathcal{V}$ where one drops a total derivative under the integral. Further, if $\mathcal{V}$ is an SU(2)-singlet one has $\int d^2z\,|\widehat{\mathcal{G}}^-\widehat{\mathcal{G}}^+|^2\mathcal{V} = \int d^2z\,|\widehat{\widetilde{\mathcal{G}}}{}^-\widehat{\widetilde{\mathcal{G}}}{}^+|^2\mathcal{V}$. Therefore, as will be used later, $\widehat{\mathcal{G}}^+\mathcal{U} = \widehat{\widetilde{\mathcal{G}}}{}^+\mathcal{U} = 0$.

2.5 Massless vertex operators

Of particular interest are the universal, compactification independent vertex operators contained in the real superfield $\mathcal{V} = \mathcal{V}(x, \theta_L, \bar{\theta}_L, \theta_R, \bar{\theta}_R)$ which was discussed in [14] (cf. appendix C for some details). It contains the degrees of freedom of $\mathcal{N} = 2$ supergravity and those of the universal tensor multiplet. It satisfies the $\mathcal{N} = 2$ primarity constraints which imply transversality constraints and linearized equations of motion for the component fields. In the amplitude computations of the next section, we will pick a certain fixed term in the u_i expansion of the integrated vertex operators (2.44), namely $\int |\mathcal{G}^+\mathcal{G}^-|^2\mathcal{V} = \int |\widetilde{\mathcal{G}}^+\widetilde{\mathcal{G}}^-|^2\mathcal{V}$. These operators satisfy the same properties listed below (2.44) as the full u_i-dependent operators (2.44). For this choice, the corresponding integrated vertex operator is obtained from the definition (2.40) and is (up to an overall numerical factor, cf. appendix A.3)

$$\mathcal{U} = \int d^2z \, \big|\bar{d}_{\dot{\alpha}} D^2 \bar{D}^{\dot{\alpha}} - d^\alpha \bar{D}^2 D_\alpha \\ -2i\Pi_{\alpha\dot{\alpha}}[D^\alpha, \bar{D}^{\dot{\alpha}}] + 8(\bar{\Pi}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}} - \Pi^\alpha D_\alpha)\big|^2 \, \mathcal{V}(z,\bar{z}) \,. \tag{2.45}$$

The integrated vertex operator contains (among other parts) the field strengths of the supergravity and universal tensor multiplets:

$$\int d^2z (d_L^\alpha d_R^\beta P_{\alpha\beta} + d_L^\alpha \bar{d}_R^{\dot{\beta}} Q_{\alpha\dot{\beta}}) + \text{h.c.}\,, \tag{2.46}$$

where $P_{\alpha\beta} = (\bar{D}^2 D_\alpha)_L (\bar{D}^2 D_\beta)_R \mathcal{V}$ and $Q_{\alpha\dot{\beta}} = (\bar{D}^2 D_\alpha)_L (D^2 \bar{D}_{\dot{\alpha}})_R \mathcal{V}$ are chiral and twisted-chiral superfields[7]. As discussed below, on-shell, these superfields describe the linearized Weyl multiplet and the derivative of the linearized field-strength multiplet of the universal tensor. For later purposes we also introduce $\mathcal{U}'$ and $\mathcal{U}''$ defined by $\mathcal{U} = |\mathcal{G}^+|^2\mathcal{U}'$ and $\mathcal{U} = |\widetilde{\mathcal{G}}^+|^2\mathcal{U}''$, i.e., $\mathcal{U}' = \int d^2z |e^\rho d^\alpha D_\alpha|^2 \mathcal{V}$ and $\mathcal{U}'' = \int d^2z |e^{-2\rho - \int J_C} \bar{d}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}}|^2 \mathcal{V}$.

The complex structure moduli are in one-to-one correspondence to elements of $H^{2,1}(CY)$ and related to primary fields Ω_c of the chiral (c,c) ring [15]. The corresponding type IIB hybrid vertex operators are[8]

$$\mathcal{V}_{cc} = |e^\rho \bar{\theta}^2|^2 M_c \Omega_c \,, \quad \mathcal{V}_{aa} = (\mathcal{V}_{cc})^\dagger = |e^{-\rho}\theta^2|^2 \bar{M}_c \bar{\Omega}_c \,, \tag{2.47}$$

where M_c is a real chiral superfield (vector multiplet). Note that in the (twisted) type IIB theory Ω_c has conformal weight $h_L = h_R = 0$, while $\bar{\Omega}_c$ has conformal weight $h_L = h_R = 1$. The complexified Kähler moduli are in one-to-one correspondence to elements of $H^{1,1}(CY)$ and related to primary fields Ω_{tc} of the twisted-chiral ring (c,a):

$$\mathcal{V}_{ca} = e^{\rho_L - \rho_R} \, \bar{\theta}_L^2 \, \theta_R^2 \, M_{tc} \Omega_{tc} \,, \quad \mathcal{V}_{ac} = (\mathcal{V}_{ca})^\dagger = e^{-\rho_L + \rho_R} \, \theta_L^2 \, \bar{\theta}_R^2 \, \bar{M}_{tc} \bar{\Omega}_{tc} \,, \tag{2.48}$$

where M_{tc} are real twisted-chiral superfields (tensor multiplets). The conformal weight of Ω_{tc} is $h_L = 0$ and $h_R = 1$. The integrated vertex operators are

$$\mathcal{U}_{cc} = \int d^2z \, M_c |G_C^-|^2 \Omega_c + \ldots \,, \quad \mathcal{U}_{ca} = \int d^2z \, M_{tc} (G_C^-)_L (G_C^+)_R \Omega_{tc} + \ldots \,, \tag{2.49}$$

[7] See appendix C.3 for details.

[8] We are suppressing the indices distinguishing between the different elements of the ring.

where we have suppressed terms involving derivatives acting on M_c and M_{tc}. These terms carry nonzero ρ-charge and will not play a role in the discussion of the amplitudes in section 4.

The vertex operators of IIA associated to elements of the (c,c) (complex structure) and (c,a) ring (Kähler) are

$$V_{cc} = e^{\rho_L - \rho_R}\,\bar\theta_L^2\,\theta_R^2\,M_{tc}\Omega_c\,,\quad V_{ca} = |e^{\rho}\bar\theta^2|^2 M_c\Omega_{tc}\,. \tag{2.50}$$

For type IIA the conformal weights of Ω_c are $h_L = 0$ and $h_R = 1$ while Ω_{tc} has weight $h_L = h_R = 0$. The integrated vertex operators involve

$$U_{cc} = \int d^2z\, M_{tc}(G_C^-)_L(G_C^-)_R\Omega_c + \ldots\,,\quad U_{ca} = \int d^2z\, M_c(G_C^-)_L(G_C^+)_R\Omega_{tc} + \ldots\,. \tag{2.51}$$

3 Amplitudes and correlation functions

In this section we review the definition of scattering amplitudes on Riemann surfaces Σ_g with genus $g \geq 2$ as given in [1]. We also collect correlation functions for chiral bosons.

3.1 Amplitudes

Scattering amplitudes of hybrid string theory are defined in [1] for $g \geq 2$ as[9]

$$F_g(u_L,u_R) = \int_{\mathcal{M}} \frac{[dm_g]}{\det(\mathrm{Im}\tau)} \prod_{i=1}^{g} \Big\langle \int d^2v_i \prod_{j=1}^{g-1} |\,\widehat{\widetilde{\mathcal{G}}}^+(v_j)|^2 |\mathcal{J}(v_g)|^2 \prod_{k=1}^{3g-3} |(\mu_k, \widehat{\mathcal{G}}^-)|^2 \prod_{l=1}^{N} \mathcal{U}_l \Big\rangle\,. \tag{3.1}$$

Since $F_g(u_L,u_R)$ is a homogeneous polynomial in both u_{iL} and u_{iR} of degree $4g-4$ (we are taking U to carry no $u_{iL,R}$ dependence as is explained in section 2.4) this definitions provides a whole set of amplitudes $F_g^{n,m}$ given by the coefficients in the $u_{i\,L,R}$-expansion:

$$F_g(u_L,u_R) = \sum_{n,m} \binom{4g-4}{2g-2-n}\binom{4g-4}{2g-2-m} F_g^{n,m} u_{1L}^{2g-2+n} u_{2L}^{2g-2-n} u_{1R}^{2g-2+m} u_{2\,R}^{2g-2-m}\,, \tag{3.2}$$

where $2-2g \leq m,n \leq 2g-2$. We focus on either the left- or right-moving sector in the following. In view of (2.43) it is clear that F_g^n involves $2g-2+n$ insertions of $\widetilde{\mathcal{G}}^+$ and $\mathcal{G}^-$ and $2g-2-n$ insertions of $\mathcal{G}^+$ and $\widetilde{\mathcal{G}}^+$. It is shown in [1] that up to contact terms all distributions of $\widetilde{\mathcal{G}}^+$'s, $\mathcal{G}^-$'s, $\mathcal{G}^+$'s, and $\widetilde{\mathcal{G}}^+$'s satisfying these constraints are equivalent. We can therefore determine $F_g^{n,m}$ (3.2) by evaluating a single amplitude with an admissible distribution of insertions.

In addition there is a selection rule that relies on the cancellation of the R-parity anomaly [1]. The R-charge is

$$R = \oint \left(\partial\rho + \frac{1}{2}\theta^\alpha d_\alpha - \frac{1}{2}\bar\theta_{\dot\alpha}\bar d^{\dot\alpha}\right)\,, \tag{3.3}$$

[9] This differs by the factor $(\det(\mathrm{Im}\tau))^{-1}$ from the expression given in [1] and [10]. We will comment on this below.

with background charge $1-g$. In the RNS formulation R coincides with the superconformal ghost-number (picture) operator, i.e., $R=\oint\mathcal{P}$. $\widetilde{\mathcal{G}}^{\pm}$ carry R-charges ∓ 1 while those of $\mathcal{G}^{\pm}$ are zero. The contribution to the R-charge of the insertions is $g-1-n$. The anomaly is therefore canceled only if the vertex operators insertions have total R-charge n. Put differently: given vertex operators $\prod_{i=1}^{N}\mathcal{U}_i$ with total R-charge n, the only non-vanishing contribution to (3.2) is F_g^n. This selection rule is completely analogous to the one that relies on picture charge in the RNS formulation.

It is convenient to rewrite (3.1) in the form

$$F_g(u_L,u_R)= \tag{3.4}$$
$$=\int_{\mathcal{M}}\frac{[dm_g]}{\det(\mathrm{Im}\tau)}\prod_{i=1}^{g}\Big\langle\int d^2v_i\prod_{j=1}^{g}|\,\widehat{\widetilde{\mathcal{G}}}^{+}(v_j)|^2\prod_{k=1}^{3g-4}|(\mu_k,\widehat{\mathcal{G}}^{-})|^2|(\mu_{3g-3},\mathcal{J}^{--})|^2\prod_{l=1}^{N}\mathcal{U}_l\Big\rangle\,.$$

This is obtained from (3.1) by contour deformation using $\widehat{\mathcal{G}}^{-}=\oint\widehat{\widetilde{\mathcal{G}}}^{+}\mathcal{J}^{--}$ and $\widehat{\widetilde{\mathcal{G}}}^{+}=-\oint\widehat{\widetilde{\mathcal{G}}}^{+}\mathcal{J}$ and the fact that $\oint\widehat{\widetilde{\mathcal{G}}}^{+}$ has no non-trivial OPE with any of the other insertions except a simple pole with $\mathcal{J}$. Consider the integrand of (3.4). As a function of, say, v_1, it has a pole only at the insertion point of $\mathcal{J}^{--}$. But the residue $\langle\prod_{i=2}^{g}\widehat{\widetilde{\mathcal{G}}}^{+}(v_i)\prod_{j=1}^{3g-3}(\mu_j,\mathcal{G}^{-})\prod_l\mathcal{U}_l\rangle$ vanishes: each of the remaining $\widehat{\widetilde{\mathcal{G}}}^{+}(v_i)$ can be written as $-\oint\widehat{\widetilde{\mathcal{G}}}^{+}\mathcal{J}(v_i)$ and $\widehat{\widetilde{\mathcal{G}}}^{+}$ has no singular OPE with any of the other insertions. Analyticity and the fact that $\widehat{\widetilde{\mathcal{G}}}^{+}$ are Grassmann odd and of weight one, fixes the v-dependence of the integrand as $\det(\omega_i(v_j))$. The ω_i are the g holomorphic one-forms on Σ_g. In (3.4) we can thus replace

$$\prod\widehat{\widetilde{\mathcal{G}}}^{+}(v_i)=\det(\omega_i(v_j))\,\frac{\prod\widehat{\widetilde{\mathcal{G}}}^{+}(\tilde{v}_l)}{\det(\omega_k(\tilde{v}_l))}\,, \tag{3.5}$$

where $\tilde{v}_k$ are g arbitrary points on Σ_g that can be chosen for convenience. Combining left- and right-movers the v-integrations can be performed with the result

$$\prod_{i=1}^{g}\int d^2v_i|\det(\omega_k(v_l))|^2\propto\det(\mathrm{Im}\,\tau)\,. \tag{3.6}$$

τ is the period matrix of Σ_g. Using similar arguments one can rewrite

$$\frac{1}{\det(\mathrm{Im}\tau)}\Big(\int_{\Sigma_g}|\widehat{\widetilde{\mathcal{G}}}^{+}|^2\Big)^g\propto\Big|\prod_{i=1}^{g}\oint_{a_i}\widehat{\widetilde{\mathcal{G}}}^{+}\Big|^2\,. \tag{3.7}$$

The reason for the insertion $\oint\widehat{\widetilde{\mathcal{G}}}^{+}$ on every a-cycle of Σ_g was presented in [1, 10]: it projects to the reduced Hilbert space formed by the physical fields of an $\mathcal{N}=2$ twisted theory. Amplitudes for these states can be calculated using the rules of $\mathcal{N}=2$ topological strings.

3.2 Correlation functions of chiral bosons

In this section we provide the correlation functions which are necessary to compute the amplitudes, cf. [16, 17, 18]. In the hybrid formulation there is no sum over spin structures and no need for a GSO projection. The correlation functions are with periodic boundary conditions around all homology cycles of the Riemann surface Σ_g.

We start with the correlators of the chiral boson H:

$$\Big\langle \prod_k e^{i\frac{q_k}{\sqrt{3}}H(z_k)}\Big\rangle = Z_1^{-1/2} F(\tfrac{1}{\sqrt{3}}\sum q_k z_k - Q_H\Delta)\prod_{i<j} E(z_i,z_j)^{\frac{1}{3}q_i q_j}\prod_l \sigma(z_l)^{\frac{1}{\sqrt{3}}Q_H q_l}\,, \tag{3.8}$$

where Z_1 is the chiral determinant of [16]. The prime forms $E(z,w)$ express the pole and zero structure of the correlation function while the σ's express the coupling to the background charge. Of the remaining part F, which is due to the zero-modes of H, only the combination in which the insertion points enter will be relevant. It is, in fact, an appropriately defined theta-function [5]. Also $F(-z) = F(z)$. In the above expression (and below), z either means a point on Σ_g or its image under the Jacobi map, i.e., $\vec{I}(z) = \int_{p_0}^{z}\vec{\omega}$, depending on the context.

The ρ-correlation functions are subtle. The field ρ is very much like the chiral boson ϕ which appears in the 'bosonization' of the superconformal (β,γ) ghost system in the RNS formulation, the only difference being the value of its background charge. In the RNS superconformal ghost system ϕ is accompanied by a fermionic spin 1 (η,ξ) system. Expressions for correlation functions of products of $e^{q_i\phi(z_i)}$ which are used in RNS amplitude calculations are always done in the context of the complete (β,γ) ghost system. Following [1] our strategy will be to combine an auxiliary fermionic spin 1 (η,ξ) system with the ρ-scalar to build a bona-fide spin 1 (β,γ) system. We then compute correlation functions as in the RNS formulation, which we divide by the contribution of the auxiliary (η,ξ)-system. Following [18], we obtain

$$\Big\langle \prod_k e^{q_k\rho(z_k)}\Big\rangle_{(\beta,\gamma)} = \frac{Z_1^{1/2}}{\theta(\sum q_k z_k - Q_\rho\Delta)}\prod_{k<l} E(z_k,z_l)^{-q_k q_l}\prod_r \sigma(z_r)^{-Q_\rho q_r} \tag{3.9}$$

with $Q_\rho = -1$. As in [18], the correlation function had to be regularized due to the fact that the zero-mode contribution of the ρ-field diverges. The regularization involved a projection of the ρ-momentum plus the momentum of the regulating (η,ξ) system in the loops to arbitrary but fixed values. These projections were accompanied by factors $\oint_{a_i}\eta$ for each a-cycle on Σ and one factor of ξ to absorb its (constant) zero mode. The contribution of (η,ξ) has to be divided out in order to obtain the regulated correlators of the ρ-system. This means that (3.9) must be divided by

$$\Big\langle \prod_{i=1}^{g}\oint_{a_i}\frac{dz_i}{2\pi i}\eta(z_i)\,\xi(w)\Big\rangle = Z_1\,. \tag{3.10}$$

Altogether we thus find

$$\Big\langle \prod_k e^{q_k\rho(z_k)}\Big\rangle_{\text{reg.}} = \frac{Z_1^{-1/2}}{\theta(\sum q_k z_k + \Delta)}\prod_{k<l} E(z_k,z_l)^{-q_k q_l}\prod_r \sigma(z_r)^{q_r}\,. \tag{3.11}$$

A useful identity is the 'bosonization formula' [16]:

$$\prod_{i=1}^{g} E(z_i,w)\sigma(w) = \frac{\prod_{i<j}E(z_i,z_j)\prod_{i=1}^{g}\sigma(z_i)}{Z_1^{3/2}\det(\omega_i(z_j))}\theta(\textstyle\sum_{i=1}^{g} z_i - w - \Delta)\,. \tag{3.12}$$

Using this identity one finds

$$\Big\langle \prod_{k=1}^{g} e^{-\rho(z_k)}\, e^{\rho(w)} \Big\rangle_{\text{reg.}} = \frac{1}{Z_1^2\, \det(\omega_k(z_l))}\,, \tag{3.13}$$

which differs by a factor of $\det(\mathrm{Im}\tau)$ from the corresponding expression used in [1].

4 Topological Amplitudes

4.1 Generalities

The expressions for F_g^n that one obtains by inserting the generators (2.43) into (3.1) in general are very involved. Certain restrictions are imposed by background charge cancellation. Since the total U(1) charge of the vertex operators is zero the insertions of $\widehat{\widehat{\mathcal{G}}}^+$ and $\widehat{\mathcal{G}}^-$ in (3.1) are precisely such that they cancel the anomaly of the total U(1) current. It is therefore sufficient to study the constraints imposed by requiring cancellation of the background charge of the ρ-field.[10] A consequence of this constraint is that if the vertex operators are not charged under $\partial\rho$ then $|n| \leq g-1$. For $|n| < g-1$ there are several possibilities how the various parts of the operators (2.43) can contribute. For $|n| = g-1$ and uncharged vertex operators there is only a single amplitude that must be considered. These cases are studied in the following. We restrict to the case with $2g$ vertex operator insertions. There are then just enough insertions of θ and p to absorb their zero modes an no nontrivial contractions occur.

4.2 R-charge $(g-1, g-1)$

This amplitude was computed in the RNS formalism in [5]. In this section we review the computation in the hybrid formalism of [1]. Imposing ρ and H background charge saturation (3.1) leads to[11]

$$\begin{aligned}\mathcal{A}_g \;=\; & \int_{\mathcal{M}} [dm_g] \frac{1}{|\det(\omega_i(\tilde{v}_j))|^2} \Big\langle \Big| \prod_{j=1}^{m} e^{\rho} G_C^{++}(\tilde{v}_j) \prod_{j=m+1}^{g} e^{-\rho} \bar{d}^2(\tilde{v}_j) \\ & \times \prod_{l=1}^{m} (\mu_l, e^{-2\rho - \int J_C} \bar{d}^2) \prod_{l=m+1}^{3g-3} (\mu_l, G_C^-) \Big|^2 \mathcal{U}' \mathcal{U}^{2g-1} \Big\rangle\,. \end{aligned} \tag{4.1}$$

We have used the fact that $\oint e^{-\rho} \bar{d}^2$, when pulled off from $\mathcal{U}'$, only gets stuck at $J(v_g)$. $0 \leq m \leq g-1$ parametrizes different ways to saturate the background charges.[12] We now use the freedom to choose $\tilde{v}_l = z_l$ for $l = 1, \ldots, g$ where z_l are the arguments of the Beltrami differentials μ_l (which are integrated over). This is possible since the OPEs which one encounters are the naive products (no poles or

[10] Since the J_C current is a linear combination of the $\partial\rho$ and the total U(1)-current, background charge cancellation for H is then automatic.

[11] Here and in the following we drop certain numerical factors and use the notation as explained below (2.43).

[12] For notational simplicity we have chosen the same m for the left- and for the right-movers.

zeros). This gives

$$\mathcal{A}_g = \tag{4.2}$$
$$= \int_{\mathcal{M}} [dm_g] \int \prod_{l=1}^{g} d^2 z_l \frac{1}{|\det(\omega_i(z_l))|^2} \Big\langle \big|(\mu_l, e^{-\rho} G_C^- \bar{d}^2(z_l))\big|^2 \prod_{k=1}^{2g-3} |(\mu_k, G_C^-)\big|^2 \mathcal{U}' \mathcal{U}^{2g-1} \Big\rangle$$

which is independent of m.[13] Its evaluation is straightforward. One easily sees that there are just enough operator insertions to absorb the p and $\bar{p}$ zero modes. θ and $\bar{\theta}$ then also only contribute with their (constant) zero modes. The p zero modes must come from the explicit d-dependence of the vertex operator. The $(p,\theta)_L$ and $(p,\theta)_R$ correlation functions contribute a factor $|Z_1|^4 (\det \mathrm{Im}\tau)^2$, where the integrals over the insertion points have already been performed. What is left is the integral over the θ zero-modes which are the Grassmann odd co-ordinates of $\mathcal{N}=2$ chiral superspace. The spinor indices arrange themselves to produce $(P_{\alpha\beta}P^{\alpha\beta})^{g-1} P_{\gamma\delta} D_L^\gamma D_R^\delta V$. The $(\bar{p},\bar{\theta})$ correlators give a term $|Z_1|^4 |\det \omega_i(z_l)|^4$, leaving only the $\bar{\theta}$ zero-mode integrations. They can be performed using $\int (d^2\bar{\theta})_L (d^2\bar{\theta})_R \Psi = \bar{D}_L^2 \bar{D}_R^2 \Psi|_{\bar{\theta}_L=\bar{\theta}_R=0}$. Since $\bar{D}_{\dot{\alpha}} P_{\beta\gamma} = 0$, the only effect of this is to convert $D_L^\alpha D_R^\beta V$ to $P^{\alpha\beta}$. Finally, the ρ-correlator gives, using (3.11) and (3.12), $(|Z_1|^4 |\det \omega_i(z_l)|^2)^{-1}$. The partition function of the x^m contributes a factor $|Z_1|^{-4} (\det \mathrm{Im}\tau)^{-2}$. To the given order of spacetime derivatives, the x^m-dependence of the vertex operators is only through its zero mode. Combining arguments we obtain

$$\mathcal{A}_g = \int (d^2\theta)_L (d^2\theta)_R (P_{\alpha\beta}P^{\alpha\beta})^g \int_{\mathcal{M}} [dm_g] \Big\langle \prod_{i=1}^{3g-3} |(\mu_i, G_C^-)|^2 \Big\rangle . \tag{4.3}$$

The last part of this expression is the string partition function of the topological B-model:

$$F_g^B = \int_{\mathcal{M}} [dm_g] \Big\langle \prod_{i=1}^{3g-3} |(\mu_i, G_C^-)|^2 \Big\rangle . \tag{4.4}$$

To determine the dependence of F_g^B on the chiral or twisted-chiral moduli one inserts the appropriate expressions (C.9) into these correlation functions. It can be shown, using the arguments of [19], that F_g^B does not depend on perturbations induced by either (c,a) or (a,c) operators. It therefore depends only on the complex structure moduli and the amplitudes calculated are therefore vector multiplet couplings (type IIB).

4.3 R-charge $(1-g, 1-g)$

Starting from (3.1) and imposing ρ and H-background charge saturation, one obtains, in close analogy to (4.1),

$$\begin{aligned} \mathcal{A}'_g \quad = \quad & \int_{\mathcal{M}} [dm_g] \frac{1}{|\det(\omega_i(\tilde{v}_j))|^2} \Big\langle \Big| \prod_{j=1}^{m} G_C^+(\tilde{v}_j) \prod_{j=m+1}^{g} e^{2\rho + \int J_C} d^2(\tilde{v}_j) \prod_{l=1}^{m} (\mu_l, e^{\rho} d^2) \\ & \times \prod_{l=m+1}^{3g-3} (\mu_l, e^{-\rho} G_C^{--}) \Big|^2 \mathcal{U}'' \mathcal{U}^{2g-1} \Big\rangle . \end{aligned} \tag{4.5}$$

[13] This shows that for this amplitude all admissible distributions of vertex operators parametrized by m indeed lead to the same result and that the only subtleties that arise from contact terms are the ones analyzed in [19, 20]. We are not aware of an argument that this is generally the case.

$0 \le m \le g-1$ parametrizes the different ways of saturating the background charges. By appropriate choices of the $\tilde{v}_j$ this amplitude can be brought to the form

$$\begin{aligned} \mathcal{A}'_g &= \int_{\mathcal{M}} [dm_g] \int \prod_{j=1}^{g} d^2 z_j \frac{1}{|\det \omega_i(z_j)|^2} \Big\langle \big| (\mu(z_j), e^{\rho} d^2 G^+_C(z_j)) \\ &\times \prod_{k=1}^{2g-3} (\mu_k, e^{-\rho} G^{--}_C) \big|^2 \mathcal{U}'' \mathcal{U}^{2g-1} \Big\rangle , \end{aligned} \tag{4.6}$$

which shows that also this amplitude is independent of m. However, its evaluation is most easily done for a different choice of the insertion points $\tilde{v}_j$. To fix them, we start from (4.5) with the choice $m = 0$ and compute the ρ and the H correlators. Their product is, using (3.8) and (3.11),

$$\begin{aligned} &\frac{1}{Z_1} \times \frac{F(\sqrt{3} \sum \tilde{v}_j - \frac{2}{\sqrt{3}} \sum z_k - \sqrt{3} w + \sqrt{3} \Delta)}{\theta(2 \sum \tilde{v}_j - \sum z_k - 2w + \Delta)} \\ &\quad \times \frac{\prod_{k<l} E(z_k, z_l)^{\frac{1}{3}} \prod_j E(\tilde{v}_j, w) \prod_k \sigma(z_k) \sigma(w)}{\prod_{i<j} E(\tilde{v}_i, \tilde{v}_j) \prod_j \sigma(\tilde{v}_j)} , \end{aligned} \tag{4.7}$$

where we have only displayed the holomorphic part. With the help of the identity (3.12) this is equal to

$$\begin{aligned} &\frac{1}{(Z_1)^{\frac{5}{2}}} \times \frac{F(\sqrt{3} \sum \tilde{v}_j - \frac{2}{\sqrt{3}} \sum z_k - \sqrt{3} w + \sqrt{3} \Delta)}{\theta(2 \sum \tilde{v}_j - \sum z_k - 2w + \Delta)} \\ &\quad \times \frac{\theta(\sum \tilde{v}_j - w - \Delta)}{\det w_i(\tilde{v}_j)} \cdot \prod_{k<l} E(z_k, z_l)^{\frac{1}{3}} \prod_k \sigma(z_k) . \end{aligned} \tag{4.8}$$

We now choose the g positions $\tilde{v}_j$ such that $\vec{I}(\sum \tilde{v}_j - w - \Delta) = \vec{I}(2 \sum \tilde{v}_j - \sum z_k - 2w + \Delta)$. Then the theta functions cancel and the remaining terms are

$$\frac{1}{(Z_1)^{\frac{5}{2}} \det \omega_i(\tilde{v}_j)} \cdot F(\tfrac{1}{\sqrt{3}} \textstyle\sum z_k - \sqrt{3} \Delta) \cdot \displaystyle\prod_{k<l} E(z_k, z_l)^{\frac{1}{3}} \prod_k \sigma(z_k) . \tag{4.9}$$

This can be written as

$$\frac{1}{Z_1^2 \det \omega_i(\tilde{v}_j)} \Big\langle \prod_{k=1}^{3g-3} e^{-\frac{i}{\sqrt{3}} H(z_k)} \Big\rangle . \tag{4.10}$$

The $p, \theta, \bar{p}$ and $\bar{\theta}$ correlators are as in the previous amplitude (with the roles on barred and unbarred variables interchanged) and one finally obtains

$$\mathcal{A}'_g = \int (d^2 \bar{\theta})_L (d^2 \bar{\theta})_R (\bar{P}_{\dot{\alpha}\dot{\beta}} \bar{P}^{\dot{\alpha}\dot{\beta}})^g \int_{\mathcal{M}} [dm_g] \Big\langle \prod_{i=1}^{3g-3} |(\mu_i, \check{G}^-_C)|^2 \Big\rangle . \tag{4.11}$$

Here $\check{G}^-_C = e^{-\frac{i}{\sqrt{3}} H} G'_C$ where G'_C is defined to be $G^+_C = e^{\frac{i}{\sqrt{3}} H} G'_C$. Note that G^-_C and $\check{G}^-_C$ both have conformal weight two. The internal amplitude multiplying the spacetime part is the complex conjugate of the B-model amplitude (4.4): this follows from the fact that the expression (4.9) can be written as

$$\frac{1}{Z_1^2 \det \omega_i(\tilde{v}_j)} \Big\langle \prod_{k=1}^{3g-3} e^{\frac{i}{\sqrt{3}} H(z_k)} \Big\rangle_{Q_H=\sqrt{3}}, \tag{4.12}$$

where we used (3.8) but with the reversed background charge as compared to (4.10). This happens if one chooses the opposite twisting in (2.6). Since the operators $\check{G}_C^-$ and G_C^+ both contain the same operator G_C', the internal part of the amplitude (4.11) is equal to

$$\Big\langle \prod_{i=1}^{3g-3} |(\mu_i, \check{G}_C^-)|^2 \Big\rangle_{++} = \Big\langle \prod_{i=1}^{3g-3} |(\mu_i, G_C^+)|^2 \Big\rangle_{--}. \tag{4.13}$$

The subscripts refer to the two possible twistings $T_C \to T_C + \frac{1}{2}\partial J_C$ and $T_C \to T_C - \frac{1}{2}\partial J_C$ for left- and right-movers. Finally, since for unitary theories $(G_C^-)^\dagger = G_C^+$, the right-hand side of (4.13) is the complex conjugate of F_g^B given in (4.4), and therefore $\mathcal{A}_g'$ defined in (4.5) is the complex conjugate of the chiral amplitude $\mathcal{A}_g$ of (4.1).

4.4 R-charges $(g-1, 1-g)$ and $(1-g, g-1)$

The 'mixed' amplitudes with R-charges $(g-1, 1-g)$ and $(1-g, g-1)$ can now be written down immediately. They are expressed as integrals over twisted chiral superspace and involve the superfields $Q_{\alpha\dot\beta}$ and $\bar{Q}_{\dot\alpha\beta}$. They are

$$\mathcal{A}_g'' = \int (d^2\theta)_L (d^2\bar\theta)_R (Q_{\alpha\dot\beta} Q^{\alpha\dot\beta})^g \int_{\mathcal{M}} [dm_g] \Big\langle \prod_{i=1}^{3g-3} (\mu_i, G_C^-)_L (\bar\mu_i, \check{G}_C^-)_R \Big\rangle + \text{c.c.} \tag{4.14}$$

By the same arguments as given before, one shows that this type IIB string amplitude only depends on deformations in the (a, c) (and (c, a) for the complex conjugate piece) ring, i.e., on Kähler moduli. In type IIB, these are in tensor multiplets. From the discussion in section 4.3 it also follows that

$$\int_{\mathcal{M}} [dm_g] \Big\langle \prod_{i=1}^{3g-3} (\mu_i, G_C^-)_L (\bar\mu_i, \check{G}_C^-)_R \Big\rangle_{++} = \int_{\mathcal{M}} [dm_g] \Big\langle \prod_{i=1}^{3g-3} (\mu_i, G_C^-)_L (\bar\mu_i, G_C^+)_R \Big\rangle_{+-} = F_g^A, \tag{4.15}$$

which is the topological A-model amplitude.

So far we have computed amplitudes of type IIB string theory. To compute type IIA amplitudes we need to twist the left- and right-moving internal SCFTs oppositely. In the amplitudes this induces the following changes: $(G_C^-)_R \to (G_C^+)_R$ and $(\check{G}_C^-)_R \to (\check{G}_C^+)_R$ where $\check{G}_C^+ = e^{\frac{i}{\sqrt{3}} H} \bar{G}_C'$. Due to the opposite twist, the conformal weights are preserved under this operation. For instance, the spacetime part of (4.3) gets combined with F_g^A, that of (4.14) with F_g^B. According to (C.9) and (C.13), F_g^A depends on the moduli contained in vector multiplets, F_g^B on those contained in tensor multiplets.

4.5 Summary of the amplitude computation

We have recomputed certain chiral and twisted-chiral couplings that involve g powers of P^2 or Q^2, respectively, using hybrid string theory. The amplitudes involve the topological string partition functions F_g^A and F_g^B. F_g^A depends on the moduli parametrizing the (c,a) ring, F_g^B on those of the (c,c) ring. In type IIA or type IIB, these are contained in spacetime chiral (vector) or twisted-chiral (tensor) multiplets, as summarized in the table. The dependence on the moduli of the complex conju-

type IIA		type IIB	
$(P^2)^g F_g^A$	(c,a): vector	$(P^2)^g F_g^B$	(c,c): vector
$(Q^2)^g F_g^B$	(c,c): tensor	$(Q^2)^g F_g^A$	(c,a): tensor

gate rings is only through the holomorphic anomaly [19]. As discussed in [14, 21], on-shell, the superfield $P_{\alpha\beta}$ describes the linearization of the Weyl multiplet. Its lowest component is the selfdual part of the graviphoton field strength, $P_{\alpha\beta}| = F_{\alpha\beta}$. The $\theta_L\theta_R$-component is the selfdual part $C_{\alpha\beta\gamma\delta}$ of the Weyl tensor. The bosonic components of $Q_{\alpha\dot\beta}$ are $Q_{\alpha\dot\beta}| = \partial_{\alpha\dot\beta}Z$, where Z is the complex R-R-scalar of the RNS formulation of the type II string; its $\theta_L\bar\theta_R$-component is $\partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta}S$. The real component of S is the dilaton, its imaginary component is dual to the antisymmetric tensor of the NS-NS-sector. These results can be obtained by explicit computation from the θ-expansion of the superfield V. After integrating (4.3) and (4.14) over chiral and twisted-chiral superspace, respectively, $2g-2$ powers of $F_{\alpha\beta}$ are coupled to two powers of $C_{\alpha\beta\gamma\delta}$, while $2g-2$ powers of ∂Z are coupled to two powers of $\partial^2 S$, with the tensorial structure discussed in [5]. In [14, 22, 23] the question is addressed how these (and other) couplings can be described in an off-shell (projective) superspace description at the non-linearized level.

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A Conventions and Notations

A.1 Spinors and superspace

Throughout this paper we use the conventions of Wess and Bagger [24]. In particular the space-time metric is $\eta^{mn} = \mathrm{diag}(-1,+1,+1,+1)$ and the spinor metric is $\epsilon^{12} = \epsilon^{\dot1\dot2} = \epsilon_{21} = \epsilon_{\dot2\dot1} = 1$. Spinor indices are raises and lowered as $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$, $\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$, and likewise for the dotted indices. Spinor indices are contracted in the following way: $\psi\chi = \psi^\alpha\chi_\alpha$, $\bar\psi\bar\chi = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha}$. Barred spinors always have dotted indices. We define $x_{\alpha\dot\alpha} = \sigma^m_{\alpha\dot\alpha}x_m$ where $\sigma^m_{\alpha\dot\alpha} = (-\mathbf{1},\vec\sigma)$ with $v^m v_m = -\frac12 v^{\alpha\dot\alpha}v_{\alpha\dot\alpha}$ and $\partial_{\alpha\dot\alpha} = \sigma^m_{\alpha\dot\alpha}\partial_m$ such that $\partial_{\alpha\dot\alpha}x^{\beta\dot\beta} = -2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta}$. Starting from the supersymmetry invariant one-forms on superspace

$$\begin{aligned} e^a &= dx^a - id\theta\sigma^a\bar\theta + i\theta\sigma^a d\bar\theta\,, \\ e^\alpha &= d\theta^\alpha\,, \\ e_{\dot\alpha} &= d\bar\theta_{\dot\alpha}\,, \end{aligned} \tag{A.1}$$

one finds their pullbacks to the world-sheet

$$\begin{aligned} \Pi^{\alpha\dot\alpha} &= \partial x^{\alpha\dot\alpha} + 2i\partial\theta^\alpha\bar\theta^{\dot\alpha} + 2i\partial\bar\theta^{\dot\alpha}\theta^\alpha\,, \\ \Pi^\alpha &= \partial\theta^\alpha\,, \\ \bar\Pi_{\dot\alpha} &= \partial\bar\theta_{\dot\alpha}\,, \end{aligned} \tag{A.2}$$

and likewise for the right-movers. Expressed in terms of the Π's, the energy momentum tensor of the x, θ, p variables is

$$T = \frac{1}{4}\Pi^{\alpha\dot\alpha}\Pi_{\alpha\dot\alpha} - \Pi^\alpha d_\alpha - \bar\Pi_{\dot\alpha}\bar d^{\dot\alpha} - \frac{1}{2}(\partial\rho)^2 + \frac{1}{2}\partial^2\rho\,, \tag{A.3}$$

with d_α and $\bar d^{\dot\alpha}$ defined by

$$\begin{aligned} d_\alpha &= p_\alpha + i\bar\theta^{\dot\alpha}\partial x_{\alpha\dot\alpha} - \bar\theta^2\partial\theta_\alpha + \frac{1}{2}\theta_\alpha\partial\bar\theta^2\,, \\ \bar d^{\dot\alpha} &= \bar p^{\dot\alpha} + i\theta_\alpha\partial x^{\alpha\dot\alpha} - \theta^2\partial\bar\theta^{\dot\alpha} + \frac{1}{2}\bar\theta^{\dot\alpha}\partial\theta^2\,. \end{aligned} \tag{A.4}$$

A.2 Hybrid variables and $\mathcal{N}=2$ algebra

The singular parts of the operator products of the hybrid variables are

$$\begin{aligned} x^m(z,\bar z)x^n(w,\bar w) &\sim -\eta^{mn}\ln|z-w|^2\,, \\ \theta_\alpha(z)p^\beta(w) &\sim \frac{\delta_\alpha^{\ \beta}}{(z-w)}\,, \\ \bar\theta^{\dot\alpha}(z)\bar p_{\dot\beta}(w) &\sim \frac{\delta^{\dot\alpha}_{\ \dot\beta}}{(z-w)}\,, \\ \rho(z)\rho(w) &\sim -\ln(z-w)\,, \end{aligned} \tag{A.5}$$

Both (A.3) and (A.5) follow from the action (2.1). We also note that

$$d_\alpha(z)\bar d_{\dot\alpha}(w) \sim \frac{2i\Pi_{\alpha\dot\alpha}(w)}{(z-w)}\,, \tag{A.6}$$

while dd and $\bar d\bar d$ are finite. The action of d and $\bar d$ on a generic superfield M is

$$\begin{aligned} d_\alpha(z)M(w) &\sim -\frac{D_\alpha M(w)}{(z-w)} \qquad \text{where} \qquad D_\alpha \equiv \partial_\alpha + i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha} \\ d_{\dot\alpha}(z)M(w) &\sim -\frac{\bar D_{\dot\alpha} M(w)}{(z-w)} \qquad \text{where} \qquad \bar D_{\dot\alpha} \equiv -\bar\partial_{\dot\alpha} - i\theta^\alpha\partial_{\alpha\dot\alpha} \end{aligned} \tag{A.7}$$

with $\{D_\alpha, \bar D_{\dot\alpha}\} = -2i\partial_{\alpha\dot\alpha}$. For later purposes we note the useful identities

$$[D_\alpha, \bar D^2] = -4i\partial_{\alpha\dot\alpha}\bar D^{\dot\alpha} \qquad \frac{1}{16}[\bar D^2, D^2] = \partial^m\partial_m + \frac{i}{2}\partial_{\alpha\dot\alpha}D^\alpha\bar D^{\dot\alpha} \tag{A.8}$$

One defines the space-time supercharges

$$Q_\alpha = \oint \left(p_\alpha - i\bar\theta^{\dot\alpha}\partial x_{\alpha\dot\alpha} + \frac{1}{2}\bar\theta^2\partial\theta_\alpha\right)$$
$$\bar Q^{\dot\alpha} = \oint \left(\bar p^{\dot\alpha} - i\theta_\alpha\partial x^{\alpha\dot\alpha} + \frac{1}{2}\theta^2\partial\bar\theta^{\dot\alpha}\right) \tag{A.9}$$

such as to satisfy

$$\{Q_\alpha, \bar Q_{\dot\alpha}\} = -2i\oint \partial x_{\alpha\dot\alpha}$$
$$\{Q_\alpha, Q_\beta\} = \{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0 \tag{A.10}$$

and to (anti)commute with the d's and Π's. In deriving these relations we have dropped total derivatives involving fermion bilinears. Note that $\{Q_\alpha, \bar Q_{\dot\alpha}\}\phi = 2i\sigma^m_{\alpha\dot\alpha}\partial_m\phi$, as expected.

Vertex operators for physical states are required to be primary. By definition, a primary state is annihilated by all positive modes of the generators of the superconformal algebra. For a superfield $\mathcal{U}$ which is independent of ρ and of the internal CFT this leads to the on-shell conditions $\partial_m\partial^m\mathcal{U} = D^2\mathcal{U} = \bar D^2\mathcal{U} = 0$.

A.3 The integrated vertex operator

We compute [25]

$$\mathcal{W}(z) = G^- G^+ \mathcal{V}(z) \tag{A.11}$$

where $\mathcal{V}(z)$ is primary (assumed to be bosonic) and depends on $x, \theta, \bar\theta$ but is independent of ρ. The first commutator is straightforward to compute. With the help of (A.7) one finds

$$\sqrt{32}G^+\mathcal{V}(w) = \oint_{C_w} dz\, e^{-\rho(z)}\left(-\frac{\bar D^2\mathcal{V}(w)}{(z-w)^2} + \frac{2\bar d\bar D\mathcal{V}(w)}{z-w}\right) = 2e^{-\rho(w)}\bar d_{\dot\alpha}\bar D^{\dot\alpha}\mathcal{V}(w) \tag{A.12}$$

where $\bar D^2\mathcal{V} = 0$ has been used. The computation of the second commutator is more involved. Applying the rules stated in [26] one finds

$$\begin{aligned}&-32\, G^- G^+\, \mathcal{V}(w) = \\ &16\partial\bar\theta_{\dot\alpha}\bar D^{\dot\alpha}\Phi(w) + 8id^\alpha\partial_{\alpha\dot\alpha}\bar D^{\dot\alpha}\mathcal{V}(w) - 4i\Pi_{\alpha\dot\alpha}D^\alpha\bar D^{\dot\alpha}\mathcal{V}(w) + \bar d_{\dot\alpha}D^2\bar D^{\dot\alpha}\mathcal{V}(w)\end{aligned} \tag{A.13}$$

where normal ordering is implied in all terms. A term $-4i\partial\rho\,\partial_{\alpha\dot\alpha}D^\alpha\bar D^{\dot\alpha}\mathcal{V}(w)$ has been dropped; using (A.8) it can be shown to vanish if $\mathcal{V}$ is primary, as we have assumed. One can cast (A.13) into a more symmetric form if one adds a total derivative[14] of $\mathcal{V}$:

$$\begin{aligned}&-32G^- G^+\,\mathcal{V}(w) + 8\partial\mathcal{V}(w) \\ &= -8(\Pi^\alpha D_\alpha - \bar\Pi_{\dot\alpha}\bar D^{\dot\alpha})\mathcal{V}(w) - 2i\Pi_{\alpha\dot\alpha}[D^\alpha, \bar D^{\dot\alpha}]\mathcal{V}(w) + (\bar d_{\dot\alpha}D^2\bar D^{\dot\alpha} - d^\alpha\bar D^2 D_\alpha)\mathcal{V}(w)\end{aligned} \tag{A.14}$$

Here we have used the notation defined in (A.2).

[14]This total derivative could contribute to boundary terms when two vertex operators collide and might thus play an important role in amplitude computation.

B Mapping the RNS to the hybrid variables

In this appendix we give the details of the field mapping suppressed in section 2.3 which relate the RNS and the hybrid variables, following closely [4]. It is easiest to split this map into a part involving a field redefinition and one involving a similarity transformation. The field redefinition defines a set of Green-Schwarz-like variables in terms of the RNS variables. These are then related to the hybrid variables by a similarity transformation.

B.1 Field redefinition from RNS to chiral GS variables

From the RNS variables one first forms a set of variables according to (2.18) and (2.25). Following [2, 1], these are called the "chiral GS-variables". In [2, 1] these variables were denoted collectively by $\tilde{\Phi}$, whereas in [4] they were labeled with the superscript "old". In this section we label the chiral GS-variables with the subscript "GS" for clarity, while this is suppressed in the main text.

In order to achieve the correct normalization (2.29) we must perform the following rescaling of RNS variables:

$$b \to 2\sqrt{2}b\,, \quad c \to (2\sqrt{2})^{-1}c\,, \quad \eta \to 2\sqrt{2}\eta\,, \quad \xi \to (2\sqrt{2})^{-1}\xi\,, \quad e^{-\phi} \to 2\sqrt{2}e^{-\phi}\,. \tag{B.1}$$

These rescalings preserve all the OPEs. We use the rescaled RNS variables in this section.

B.2 Similarity transformation relating chiral GS to hybrid variables

The chiral GS-variables $\Phi_{\rm GS}$, including those of the internal SCFT, are related to the hybrid ones Φ by the similarity transformation

$$(\Phi)_{\rm GS} = e^{\mathcal{M}+\mathcal{M}_C^-}\,(\Phi)\,e^{-(\mathcal{M}+\mathcal{M}_C^-)}\,. \tag{B.2}$$

We have defined

$$\mathcal{M} = \oint i\theta^\alpha\bar{\theta}^{\dot{\alpha}}\partial x_{\alpha\dot{\alpha}} + \frac{1}{4}(\theta^2\partial\bar{\theta}^2 - \bar{\theta}^2\partial\theta^2)\,, \tag{B.3}$$

and

$$\mathcal{M}_C^- = -\sqrt{2}c_-\oint e^{-\rho}\theta^2 G_C^-\,, \quad \mathcal{M}_C^+ = \sqrt{2}c_+\oint e^{\rho}\bar{\theta}^2 G_C^+\,, \tag{B.4}$$

with $\left[\mathcal{M}_C^\pm, \mathcal{M}\right] = 0$. One way to see that this is indeed the correct transformation is to verify that

$$e^{\mathcal{M}}p_\alpha e^{-\mathcal{M}} = d_\alpha\,, \tag{B.5}$$

from which

$$e^{\mathcal{M}+\mathcal{M}_C^-}\left(\frac{1}{\sqrt{32}}e^{\rho}p^\alpha p_\alpha\right)e^{-(\mathcal{M}+\mathcal{M}_C^-)} = \frac{1}{\sqrt{32}}e^{\rho}d^\alpha d_\alpha + c_-G_C^- = \mathcal{G}^- \tag{B.6}$$

follows. By definition (B.2), the l.h.s. of this expression equals $(\frac{1}{\sqrt{32}}e^{-\rho}p^\alpha p_\alpha)_{\rm GS}$, which, according to (2.18) and (B.1), equals the RNS ghost field b. One therefore concludes

$$b = (\frac{1}{\sqrt{32}}e^{-\rho}p^\alpha p_\alpha)_{\rm GS} = \frac{1}{\sqrt{32}}e^{-\rho}d^\alpha d_\alpha + c_-G_C^- = \mathcal{G}^- \tag{B.7}$$

as stated in (2.26).

It can be verified that for the generators $\mathcal{J} = \mathcal{J}_{\rm GS}$, $\mathcal{T} = \mathcal{T}_{\rm GS}$, $\mathcal{J}^{\pm\pm} = \mathcal{J}^{\pm\pm}_{\rm GS}$. They are therefore not affected by (B.2). These results were used in section 2.4.

B.3 Hermitian conjugation of the hybrid variables

Hermitian conjugation acts on the hybrid variables as

$$(x^m)^\dagger = x^m \,, \quad (\theta^\alpha)^\dagger = \bar\theta^{\dot\alpha} \,, \quad (p_\alpha)^\dagger = -\bar p_{\dot\alpha} \,. \tag{B.8}$$

From these properties one concludes that $(\partial x^m)^\dagger = -\partial x^m$ and $(\partial\theta^\alpha)^\dagger = -\partial\bar\theta^{\dot\alpha}$. In addition, we define

$$\rho^\dagger = -\rho - \ln z + i\pi \,, \quad H^\dagger = H - i\sqrt{3}\ln z + \pi\sqrt{3} \,. \tag{B.9}$$

Some comments are in order here. The $\ln z$ terms are due to the background charges of currents $J = \partial\rho$ and $J_C = i\sqrt{3}\partial H$. In the presence of a (real) background charge Q, the operator product of the energy-momentum tensor and a generic (hermitian) current is modified to $T(z)j(w) \sim \frac{Q}{(z-w)^3} + \frac{j(w)}{(z-w)^2} + \frac{\partial j(w)}{(z-w)}$. In terms of the modes this reads $[L_n, j_m] = \frac{1}{2}Qn(n+1)\delta_{n+m} - mj_{n+m}$ and implies $j_n^\dagger = j_{-n} - Q\delta_{n,0}$ and $L_n^\dagger = L_{-n} - Q(n-1)j_{-n}$. These results are to be applied for the currents $j = -\partial\rho = -J$ and $j = J_C$ with the background charges -1 and -3, respectively. This implies that $(p_\rho)^\dagger = p_\rho - 1$ and $(p_H)^\dagger = p_H + \sqrt{3}$ such that the cocycle factors introduced in section 2.1 satisfy $(c_+)^\dagger = c_-$. The constant shifts $+i\pi$ and $+\pi\sqrt{3}$ seem to be needed in order to obtain the correct hermiticity relations between various $\mathcal{N} = 4$ generators and the correct algebra. It is consistent with the fact that ρ and H are compact bosons with periodicity $2\pi i$ and $2\pi\sqrt{3}$, respectively. Using the general CFT rule, $[\phi(z)]^\dagger = \phi^\dagger(\frac{1}{\bar z})\bar z^{-2h}$, valid for a primary field ϕ of dimension h, one shows $\exp(q\rho)^\dagger = (-1)^q \exp(-q\rho)$ and $\exp(\frac{iq}{\sqrt{3}H})^\dagger = (-1)^q \exp(-\frac{iq}{\sqrt{3}}H)$. This, together with $(G_C^\pm)^\dagger = G_C^\mp$, completes the discussion of hermitian conjugation of the hybrid variables.

B.4 Hermitian conjugation of the RNS variables

Going through the sequence of similarity transformations and field redefinitions outlined in Appendices B.1 and B.2, the hybrid conjugation rules induce a hermitian conjugation for the RNS variables. This conjugation is not the standard one. We discuss this in detail below and obtain, as a side-product, the a justification for the complete dictionary given in (2.29).

Using $\mathcal{M}^\dagger = \mathcal{M}$ and $(\mathcal{M}_C^-)^\dagger = \mathcal{M}_C^+$, hermitian conjugation of (B.6) and (B.7) (or direct computation) yields

$$b^\dagger = e^{-(\mathcal{M}+\mathcal{M}_C^+)}\left(-\frac{1}{\sqrt{32}}e^{-\rho}\bar p_{\dot\alpha}\bar p^{\dot\alpha}\right)e^{\mathcal{M}+\mathcal{M}_C^+} = -\frac{1}{\sqrt{32}}e^{-\rho}\bar d_{\dot\alpha}\bar d^{\dot\alpha} - c_+G_C^+ = \mathcal{G}^+ \,. \tag{B.10}$$

As is argued below, this expression equals the current $j_{\rm BRST}$ in accordance with (2.26). The hybrid hermiticity properties therefore imply in particular that $b^\dagger = j_{\rm BRST}$. We work this out in more detail: one first remarks that (B.10) is not the similarity transformation (B.2), since latter involves the charge $\mathcal{M}+\mathcal{M}_C^-$. In fact, under this transformation $-\frac{1}{\sqrt{32}}e^{-\rho}\bar{p}^2$ is mapped to

$$e^{\mathcal{M}+\mathcal{M}_C^-}\left(-\frac{1}{\sqrt{32}}e^{-\rho}\bar{p}_{\dot\alpha}\bar{p}^{\dot\alpha}\right)e^{-(\mathcal{M}+\mathcal{M}_C^-)}=\left(-\frac{1}{\sqrt{32}}e^{-\rho}\bar{p}_{\dot\alpha}\bar{p}^{\dot\alpha}\right)_{\rm GS}=-b\gamma^2\,. \quad \text{(B.11)}$$

The first equality is just the definition (B.2), while the second one is a consequence of the field redefinition (2.18) and (B.1). Inverting this relation and inserting the result in (B.10) one finds

$$b^\dagger = e^{-\mathcal{R}}(-b\gamma^2)e^{\mathcal{R}}. \quad \text{(B.12)}$$

The claim is that the r.h.s. is $j_{\rm BRST}$. We have defined

$$e^{\mathcal{R}}=e^{\mathcal{M}+\mathcal{M}_C^-}e^{\mathcal{M}+\mathcal{M}_C^+}=e^{2\mathcal{M}+\mathcal{M}_C^-+\mathcal{M}_C^++\frac{1}{2}[\mathcal{M}_C^-,\mathcal{M}_C^+]}\,. \quad \text{(B.13)}$$

While the first equality is the definition, the second one holds whenever one has $[\mathcal{M}_C^-,[\mathcal{M}_C^-,\mathcal{M}_C^+]]=[\mathcal{M}_C^+,[\mathcal{M}_C^+,\mathcal{M}_C^-]]=0$. That this is indeed the case as can be seen by calculating the commutator

$$[\mathcal{M}_C^-,\mathcal{M}_C^+]=2\oint\left[\theta^2\bar\theta^2\left(J_C+\frac{c}{3}\partial\rho\right)-\frac{c}{3}\bar\theta^2\partial\theta^2\right]\,. \quad \text{(B.14)}$$

We used the normalization

$$G_C^-(z)G_C^+(w)\sim\frac{\frac{c}{3}}{(z-w)^3}-\frac{J_C(w)}{(z-w)^2}+\frac{T_C(w)-\partial J_C(w)}{z-w}\,, \quad \text{(B.15)}$$

which follows from (2.5). Since $G_C^\pm$ has only a simple pole with J_C, $\mathcal{M}_C^\pm$ commute with this commutator and (B.13) is established. The explicit expression for $\mathcal{R}$ in terms of hybrid variables is

$$\begin{aligned}\mathcal{R}=\oint\Big[&2i\theta^\alpha\bar\theta^{\dot\alpha}\partial x_{\alpha\dot\alpha}-\sqrt{2}c_-e^{-\rho}\theta^2G_C^-+\sqrt{2}c_+e^{\rho}\bar\theta^2G_C^+\\&+\theta^2\bar\theta^2(J_C+\frac{c}{3}\partial\rho)+(1+\frac{c}{3})\theta^2\partial\bar\theta^2\Big]\,.\end{aligned} \quad \text{(B.16)}$$

In order to evaluate the r.h.s. of (B.12), we re-express this operator in terms of RNS variables. Thereby one must bear in mind that the field map (B.2) affects all fields, including the generators of the internal SCFT. In particular, one finds that under (B.2)

$$\begin{aligned}(\partial x_{\alpha\dot\alpha})_{\rm GS}&=\partial x_{\alpha\dot\alpha}+2i\partial(\theta_\alpha\bar\theta_{\dot\alpha})\,,\\(c_+e^{\rho}\bar\theta^2G_C^+)_{\rm GS}&=c_+e^{\rho}\bar\theta^2G_C^++\sqrt{2}\left[\theta^2\bar\theta^2\left(J_C+\frac{c}{3}\partial\rho\right)-\frac{c}{3}\bar\theta^2\partial\theta^2\right]\,,\end{aligned} \quad \text{(B.17)}$$

while θ^α, $\bar\theta_{\dot\alpha}$, and the combinations $\theta^2\bar\theta^2J_C$ and $\theta^2\bar\theta^2\partial\rho$ remain unaffected. Using (2.21), (2.23), and (2.25) and dropping total derivatives one finds that $\mathcal{R}$ is the following simple expression in RNS variables:[15]

$$\mathcal{R}=\oint\left[c\xi e^{-\phi}T_F+\frac{1}{2}e^{-2\phi}c\partial c\xi\partial\xi(\partial\phi+\frac{c-9}{3}\partial\sigma)\right]\,. \quad \text{(B.18)}$$

[15]In order to obtain this result, one must take special care of the overall signs for the RNS expressions of θ^2, $\bar\theta^2$, $\theta^2\partial\bar\theta^2$, and alike. We suppress these details in this note.

The terms in (B.16) involving the current $\breve{J}_C$ have canceled and the last term in (B.18) vanishes for $c = 9$. We have defined $T_F = T_F^{x,\psi} + \breve{G}_C^+ + \breve{G}_C^-$, where $T_F^{x,\psi}$ is the supercurrent of the space-time matter sector. It is normalized as $T_F(z)T_F(w) \sim \frac{2}{3}(c^{x,\psi} + c)(z-w)^{-3} + \ldots$, with $c^{x,\psi} = 6$ [see also (B.15)].

Using the conventions of [13] one can verify that (B.12) with $\mathcal{R}$ given in (B.18) indeed produces the BRST current (this current differs from the usual current by addition of total derivative terms),

$$\begin{aligned} e^{-\mathcal{R}}(-b\gamma^2)e^{\mathcal{R}} = j_{\text{BRST}} = & \, c\left(T - b\partial c - \frac{1}{2}(\partial\phi)^2 - \partial^2\phi + \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}\partial^2\chi\right) \\ & + \gamma\, T_F - b\,\gamma^2 + \partial^2 c + \partial(c\partial\chi)\,. \end{aligned} \tag{B.19}$$

where $T = T^{x,\psi} + \breve{T}_C$. Details of this computation can be found in [27]. The BRST charge is $Q_{\text{BRST}} = \oint \left[cT + e^{\phi}\eta\, T_F + bc\partial c + be^{2\phi}\eta\partial\eta + c(\partial\xi\eta - \frac{1}{2}(\partial\phi)^2 - \partial^2\phi)\right]$ and coincides with the charge which follows from the BRST current in [28] after bosonization. From this one derives an expression for the picture-changing operator in bosonized form,

$$Z = \{Q_{\text{BRST}}, \xi\} = e^{\phi}T_F + c\partial\xi - b\partial\eta e^{2\phi} - \partial(b\eta^{2\phi})\,, \tag{B.20}$$

which enters in (2.28).

It does not seem possible to give a closed formula for the hermitian conjugation of a generic RNS field. If, however, an RNS field Φ_{RNS} is expressible in terms of chiral GS-variables, $\Phi_{\text{GS}} = \Phi_{\text{RNS}}$, one can use the same argument as above and deduce the rule:

$$(\Phi_{\text{RNS}})^\dagger = e^{-\mathcal{R}}\,\Psi_{\text{RNS}}\,e^{\mathcal{R}}\,, \quad \text{with} \quad \Psi_{\text{RNS}} := (\Phi_{\text{GS}})^\dagger\,, \tag{B.21}$$

where $(\Phi_{\text{GS}})^\dagger$ is calculated the same way as the corresponding expression in hybrid variables. For instance, in above argument, $\Phi_{\text{RNS}} = b = \left(\frac{1}{\sqrt{32}}e^{-\rho}p^2\right)_{\text{GS}} = \Phi_{\text{GS}}$, and $\Psi_{\text{RNS}} = (\Phi_{\text{GS}})^\dagger = \left(-\frac{1}{\sqrt{32}}e^{\rho}\bar{p}^2\right)_{\text{GS}} = -b\gamma^2$, which leads to (B.12).

Some clarifying remarks on the hermitian conjugation rule of RNS variables are in place here. The conformal weights of Φ_{RNS} and Ψ_{RNS} generally differ when evaluated w.r.t. T_{RNS}. The reason for this is the following: as we have reviewed above, in the presence of a background charge, $T^\dagger = T - Q\partial j$. If $\mathcal{O}$ is an operator with U(1) charge q, it can be written in the form $\mathcal{O} = \exp(q\int^z j)\mathcal{O}'$ with $\mathcal{O}'$ neutral under j. (Here we have normalized the current according to $j(z)j(w) \sim \frac{1}{(z-w)^2}$). The hermitian conjugate operator is $\mathcal{O}^\dagger = \exp(-q\int^z j)(\mathcal{O}')^\dagger$. Its conformal weight measured with $T^\dagger$ is the same as that of $\mathcal{O}$ measured with T. One defines the operator $\widetilde{\mathcal{O}} = \exp(q\int^z j)(\mathcal{O}')^\dagger$ which has the same U(1) charge and weight (w.r.t. T) as $\mathcal{O}$.

For the case of interest, this means that $T = -\frac{1}{2}(\partial\rho)^2 - \frac{1}{2}(\partial H)^2 + \frac{1}{2}\partial^2\rho + \frac{i}{2}\sqrt{3}\partial^2 H$ becomes $T^\dagger = T - \partial^2\rho - i\sqrt{3}\partial^2 H = T - \partial\mathcal{J}$. The conformal weight of Ψ_{RNS} w.r.t. $T^\dagger$ is then the same as that of Φ_{RNS} w.r.t. T. It is now straightforward to find that the Ψ_{RNS} corresponding to $\Phi_{GS} = e^{\sigma}$, e^{χ}, and e^{ϕ}, for example are given (up to overall signs and rescalings) by $e^{\sigma+2\chi-2\phi}$, $e^{2\sigma+\chi-2\phi}$, and $e^{2\sigma+2\chi-3\phi}$, respectively. Finally, we define the operator conjugation $\mathcal{O} \to \widetilde{\mathcal{O}}$ for the case at hand. We write any operator with ρ-charge p (with respect to the current $\partial\rho$) and $\text{U}(1)_C$-charge q as

$$\mathcal{O} = e^{-p\rho + \frac{iq}{\sqrt{3}}H}\,\mathcal{O}' = e^{\frac{1}{2}(p+q)\int^z \mathcal{J}}\,e^{-\frac{1}{2}(3p+q)(\rho + \frac{i}{\sqrt{3}}H)}\,\mathcal{O}'\,. \tag{B.22}$$

Then an operator conjugation preserving the conformal w.r.t. to T is defined by

$$\widetilde{\mathcal{O}} = e^{\frac{1}{2}(p+q)\int^z \mathcal{J}}\, e^{\frac{1}{2}(3p+q)(\rho+\frac{i}{\sqrt{3}}H)}\, (\mathcal{O}')^\dagger = e^{(2p+q)\rho+(3p+2q)\frac{i}{\sqrt{3}}H}\, (\mathcal{O}')^\dagger\,. \tag{B.23}$$

This is the conjugation used in section 2.1.

C Vertex operators

C.1 Massless RNS vertex operators

The field redefinition between the RNS and hybrid variables presented in section 2.3 induces a map of the vertex operators of the RNS formulation to those of the hybrid formulation. We first discuss the unintegrated vertex operators for massless states. The field redefinition (2.18) [or (2.33) for the right-moving sector of the type IIA string] relates the RNS vertex operators in the large Hilbert space to operators expressed in terms of chiral GS-variables. To obtain the vertex operators in the hybrid variables one needs to perform the additional map (B.2). It can be shown, however, that this map does not affect any of the expressions discussed below. The reason is that at the massless level the unintegrated vertex operators do not depend on p or $\bar{p}$. Furthermore, they contain at least two powers of θ and $\bar{\theta}$ such that the map is trivial as long as the internal part of the vertex operators are primary.

The vertex operators of the bosonic components of the space-time $\mathcal{N}=2$ multiplets descend from the NSNS and RR sectors of the 10d superstring. The vertex operators in the large Hilbert space are of the general form

$$\mathcal{V}^{(q,\tilde{q})} = |c\xi e^{\alpha\phi}|^2\, \breve{\Phi}^{(q,\tilde{q})}\, W\,. \tag{C.1}$$

As explained in subsection 2.4, $\mathcal{V}$ has conformal weight 0 and ghost number 0 with respect to (2.24). W is the space-time part of the vertex operator; $\Phi^{(q,\tilde{q})}$ are primary fields of $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ charge $(q,\tilde{q})$ of the internal $c=9$, $\mathcal{N}=(2,2)$ SCFT. In what follows, we mainly concentrate on the left-moving part of the vertex operators, which we denote by $\mathcal{V}^{(q)}$. Notice that the charge of $\breve{\Phi}^{(q)}$ with respect to $\breve{J}_C$ is the same as the one of $\Phi^{(q)}$ defined by (2.22) with respect to J_C of (2.21). According to (2.25), the charge of the vertex operators $\mathcal{V}^{(q)}$ under $J=\partial\rho$ is $-q+3\,(1+\alpha)$ and must therefore carry a factor of $e^{[q-3(1+\alpha)]\rho}$ when expressed in hybrid variables. Since the ghost number $J_{\mathrm{gh}} = \mathcal{J} = \partial\rho + J_C$ of the vertex operator $\mathcal{V}^{(q)}$ is zero, the ρ-charge is minus the J_C-charge. Therefore the internal part of $\mathcal{V}^{(q)}$ in hybrid variables must involve $\Phi^{(q-3[1+\alpha])}$. Given any RNS vertex operator, this rule fixes the form of the vertex operator in the hybrid formulation up to the spacetime part. The latter is determined by (2.18) from which one derives, for example,[16]

$$e^{-\rho}\,\theta^2 = c\,, \qquad e^{\rho}\,\bar{\theta}^2 = c\,e^{-2(\phi-\chi)}\,. \tag{C.2}$$

The RNS vertex operators are restricted by the requirement that their operator product with the spacetime gravitino $e^{-\frac{1}{2}\phi}S^\alpha\Sigma$ is local. This applies to the left-moving part. For the right-moving part of type IIB (IIA) locality with $(e^{-\frac{1}{2}\phi}S^\alpha\Sigma)_R$ $((e^{-\frac{1}{2}\phi}S^\alpha\bar{\Sigma})_R)$ is required. Given the OPE $\Sigma(z)\breve{\Phi}^{(q)}(w) \sim (z-w)^{\frac{q}{2}}\breve{\Phi}^{(q+\frac{3}{2})}(w)+\ldots$ this implies restrictions on q.

[16] Here and in what follows we suppress overall signs and numerical factors.

It is now straightforward to find the following maps of vertex operators in the NS sector in the canonical ghost picture ($\alpha = -1$):[17]

$$
\begin{aligned}
\mathcal{V}_{\rm NS}^{(0)} &= c\,\xi\, e^{-\phi}\, \psi^m = (\theta\sigma^m\bar\theta)\,,\\
\mathcal{V}_{\rm NS}^{(+1)} &= c\,\xi\, e^{-\phi}\, \breve\Phi^{(+1)} = e^{\rho}\,\bar\theta^2\,\Phi^{(+1)}\,,\\
\mathcal{V}_{\rm NS}^{(-1)} &= c\,\xi\, e^{-\phi}\, \breve\Phi^{(-1)} = e^{-\rho}\,\theta^2\,\Phi^{(-1)}\,.
\end{aligned}
\tag{C.3}
$$

In the R sector in the canonical ghost picture ($\alpha = -\frac{1}{2}$) one finds, for example,

$$
\begin{aligned}
\mathcal{V}_{\rm R}^{(+\frac{3}{2})} &= c\,\xi\, e^{-\frac{\phi}{2}}\, S^\alpha\Sigma = \theta^\alpha\,\bar\theta^2\,,\\
\mathcal{V}_{\rm R}^{(+\frac{1}{2})} &= c\,\xi\, e^{-\frac{\phi}{2}}\, \bar S^{\dot\alpha}\,\breve\Phi^{(+\frac{1}{2})} = e^{-\rho}\,\bar\theta^{\dot\alpha}\,\theta^2\,\Phi^{(-1)}\,,\\
\mathcal{V}_{\rm R}^{(-\frac{1}{2})} &= c\,\xi\, e^{-\frac{\phi}{2}}\, S^{\alpha}\,\breve\Phi^{(-\frac{1}{2})} = e^{-2\rho}\,\theta^\alpha\,\partial\theta^2\,\Phi^{(-2)}\,.
\end{aligned}
\tag{C.4}
$$

These expressions illustrate that RNS vertex operators, which are (RNS) hermitian conjugates of each other, are generally not mapped to operators which are hermitian conjugates in the hybrid sense (cf. Appendix B). Conversely, two hermitian conjugate hybrid operators are related to RNS operators in different ghost pictures. For instance, $\bar\theta^{\dot\alpha}\theta^2(w) = c\partial c\xi\partial\xi e^{-\frac{5}{2}\phi}\bar S^{\dot\alpha}\bar\Sigma(w) = \lim_{z\to w} Y(z)\, c\xi e^{-\frac{\phi}{2}}\,\bar S^{\dot\alpha}\bar\Sigma(w)$ where $Y = c\partial\xi e^{-2\phi}$ is the inverse picture changing operator.

For type IIB compactifications (C.2), (C.3) and (C.4) are the same for both the left- and right-moving sectors. For type IIA compactifications the expressions for the right-movers are different; they can be obtained from above relations by reversing the signs of all explicit charge labels and replacing $\Sigma \leftrightarrow \bar\Sigma$.

C.2 Universal massless multiplets

The vertex operators for the universal sector of type II strings on CY_3 [29, 30] are associated to the identity $\Phi^{(0,0)} = \mathbb{1}$ and the states in the RR sector connected to the identity by spectral flow. They are grouped in to the real superfield $\mathcal{U}(x, \theta_{L,R}, \bar\theta_{L,R})$, which was constructed in [3], and contains the $24+24$ degrees of freedom of supergravity multiplet and the $8+8$ degrees of freedom of the universal tensor multiplet (which can be dualized to the universal dilaton multiplet). In the Wess-Zumino gauge, the metric, the antisymmetric tensor, and the dilaton appear at the lowest non-vanishing order of the θ-expansion of $\mathcal{U}$. Other fields, such as the (anti)selfdual part of the graviphoton field strength $F_{\alpha\beta}$ ($F_{\dot\alpha\dot\beta}$) and the derivative of the complex RR-scalar Z, to which we referred to in the main text, appear at higher orders in the θ-expansion:

$$
\begin{aligned}
\mathcal{U} =& \,\zeta_{mn}(\theta\sigma^m\bar\theta)_L(\theta\sigma^n\bar\theta)_R + \Big[F_{\alpha\beta}\theta_L^\alpha\theta_R^\beta|\bar\theta^2|^2 + \text{h.c.}\Big]\\
&+\Big[(\partial_{\alpha\dot\beta}Z + \ldots)\theta_L^\alpha\bar\theta_L^2\bar\theta_R^{\dot\beta}\theta_R^2 + \text{h.c.}\Big] + \ldots
\end{aligned}
\tag{C.5}
$$

[17] We do not display the $e^{i\,k\cdot X}$ factors which must be included in the complete expression for each vertex.

The full expansion can be found in [3]. Using the expressions (C.3) and (C.4) these operators are identified as the RNS vertex operators. One finds, for instance,

$$\begin{aligned}\mathcal{U}_\zeta &= \zeta_{mn}\,\psi_L^m\psi_R^n\,\left|c\,\xi e^{-\phi}\right|^2 \\ \mathcal{U}_{\partial Z} &= \partial_m Z\,(S_L\sigma^m\bar{S}_R)\,\left|c\,\xi\,e^{-\frac{1}{2}\phi}\Sigma\right|^2 \\ \mathcal{U}'_{\partial\bar{Z}} &= \partial_m \bar{Z}\,(\bar{S}_L\sigma^m S_R)\,\left|c\,\xi\,e^{-\frac{1}{2}\phi}\bar{\Sigma}\right|^2\end{aligned} \tag{C.6}$$

As explained in the previous section, it is not $\mathcal{U}'_{\partial\bar{Z}}$ that is mapped directly to the hybrid vertex operator, but the picture changed operator $Y\mathcal{U}'_{\partial\bar{Z}}$.

C.3 Compactification dependent massless multiplets

The spacetime parts of the massless vertex operators, the presence of which depends on the particular choice of Calabi-Yau compactification, can be grouped into real chiral or twisted-chiral multiplets as described in sec. 2.5.

Chiral superfields M_c satisfy $\bar{D}_{\dot\alpha L}M_c = 0 = \bar{D}_{\dot\alpha R}M_c$. Real chiral superfields (vector multiplets) satisfy in addition $D_L^2 M_c = \bar{D}_R^2\bar{M}_c$ and comprise $8+8$ components. The chirality constraint means that M_c is a function of $y^m = x^m + i(\theta\sigma^m\bar\theta)_L + i(\theta\sigma^m\bar\theta)_R$, θ_L and θ_R where y^m satisfies $D_L y^m = \bar{D}_R y^m = 0$. Parts of the θ-expansion are

$$M_c(y^m,\theta_L,\theta_R) = t + \cdots + f_{\alpha\beta}\theta_L^\alpha\theta_R^\beta + \cdots + |\theta^2|^2\partial^m\partial_m\bar{t}'\,, \tag{C.7}$$

where the complex scalars t and t' and the selfdual two-tensor $f_{\alpha\beta}$ are functions of y^m. The reality constraint implies in particular that $t = t'$ and that the two-tensor satisfies the Bianchi constraints, which are solved by writing it as a vector field strength. The complete expansion can be found in [3].

Twisted-chiral superfields M_{tc} satisfy $\bar{D}_{\dot\alpha L}M_{tc} = 0 = D_{\alpha R}M_{tc}$. Real twisted-chiral superfields (tensor multiplets) satisfy in addition $D_L^2 M_{tc} = D_R^2\bar{M}_{tc}$ and comprise $8+8$ components. The relevant parts of its expansion are

$$M_{tc}(z,\theta_L,\bar\theta_R) = l_{++} + \cdots + v_{\alpha\dot\beta}\,\theta_L^\alpha\bar\theta_R^{\dot\beta} + \cdots + \theta_L^2\,\bar\theta_R^2\,\partial^m\partial_m l_{--}\,. \tag{C.8}$$

where $v_{\alpha\dot\beta} = v_m\sigma^m_{\alpha\dot\beta}$ is a complex vector and $l_{\pm\pm}$ complex scalars. All component fields are functions of $z^m = x^m + i(\theta\sigma^m\bar\theta)_L - i(\theta\sigma^m\bar\theta)_R$ with $\bar{D}_L z^m = D_R z^m = 0$. The reality condition implies $\bar{l}_{++} = l_{--}$. Its real part requires $\partial_m v_n - \partial_n v_m = 0$ while for its imaginary part we need $\partial^m v_m = 0$. These conditions are solved for $v_m = \partial_m l_{+-} + i\epsilon_{mnpq}H^{npq}$ with $H = dB$. The three scalars $(l_{+-}, l_{--} = \bar{l}_{++})$ form a SU(2) triplet. The antisymmetric tensor with field strength H can be dualized to a fourth scalar which can be combined with l_{+-} to a complex scalar. The complete expansion of this field can again be found in [3].

C.3.1 Kähler moduli

The $h^{1,1}$ complexified Kähler deformations are in one-to-one correspondence to elements of $H^{1,1}(CY_3)$. In the CFT description they are described by twisted-chiral primaries Ω_{tc} in the (c,a) ring of charge $q_L = -q_R = 1$ and conformal weight $h_L = h_R = \frac{1}{2}$ (in the untwisted theory). They are obtained, via spectral flow, from

RR ground states with $q_L = -q_R = -\frac{1}{2}$ and $h_L = h_R = \frac{3}{8}$. In type IIA these deformations are associated with the complex scalars of vector multiplets and for type IIB with the NSNS-scalars of hypermultiplets (or tensor multiplets).

Here, we focus on type IIA, but the generalization to type IIB is straightforward. The corresponding hybrid vertex operators were given in (2.50).

$$\mathcal{U}_{ca} = |e^{\rho}\bar{\theta}|^2 M_c \Omega_{tc} \,. \tag{C.9}$$

Note that for the twisted type IIA theory Ω_{tc} has conformal weight $h_L = h_R = 0$ (while $\bar{\Omega}_{tc}$ has conformal weight $h_L = h_R = 1$) such that M_c indeed describes massless states. In the large volume limit the twisted-chiral primary operators can be written as[18]

$$\Omega_{tc} = h_{i\bar{\jmath}}\, \chi^i_L \chi^{\bar{\jmath}}_R \,, \quad \bar{\Omega}_{tc} = h_{i\bar{\jmath}} \lambda^{\bar{\jmath}}_L \lambda^i_R \,. \tag{C.10}$$

Here $h_{i\bar{\jmath}}$ is an element of $H^{1,1}(CY_3)$. For notational simplicity we drop the additional index which distinguishes between the $h^{1,1}$ different elements.

The bosonic degrees of freedom of the $h^{1,1}$ vector multiplets are found by expanding M_c in powers of θ_L and θ_R,

$$\begin{aligned} \mathcal{U}_t &= t\, |e^{\rho}\bar{\theta}^2|^2\, \Omega_{tc} \,, \\ \mathcal{U}_f &= f_{\alpha\beta}\, \theta^{\alpha}_L \theta^{\beta}_R \, |e^{\rho}\bar{\theta}^2|^2\, \Omega_{tc} \,. \end{aligned} \tag{C.11}$$

For each Kähler modulus there is a complex polarization t. Its real (imaginary) part is the space-time scalar field corresponding to fluctuations of the internal NSNS B-field $B_{i\bar{\jmath}}$ (mixed components of the CY metric $g_{i\bar{\jmath}}$). $f_{\alpha\beta}$ is the selfdual part of a field strength of the vector multiplet's gauge field which arises from the reduction of the three-form potential. This can be seen by relating these expressions to the vertex operators in the RNS formulation. Using (C.3) and (C.4) one finds

$$\begin{aligned} \mathcal{U}_t &= t\, |c\,\xi e^{-\phi}|^2\, \breve{\Omega}_{tc} \,, \\ (\mathcal{U}_t)^{\dagger} &= \bar{t}\, |c\,\xi e^{-\phi}|^2\, \breve{\bar{\Omega}}_{tc} \,, \\ (\mathcal{U}_f)^{\dagger} &= f_{\dot{\alpha}\dot{\beta}} \bar{S}^{\dot{\alpha}}_L\, \bar{S}^{\dot{\beta}}_R |c\,\xi e^{-\frac{1}{2}\phi}|^2\, \breve{\bar{\Omega}}_{tc} \,. \end{aligned} \tag{C.12}$$

Incidentally, the choice in (2.18) is such that $(\mathcal{U}_f)^{\dagger}$ is mapped to a simple RNS vertex operator in the canonical ghost picture while $\mathcal{U}_f$ is mapped to a RNS operator in another ghost picture.

C.3.2 Complex structure moduli

The $h^{2,1}$ complex structure deformations are related to chiral primary fields Ω_c in the chiral (c,c) ring of charge $q_L = q_R = 1$ and conformal weight $h_L = h_R = \frac{1}{2}$ (in the untwisted theory). These are related to operators describing RR ground states with charges $q_L = q_R = \pm\frac{1}{2}$ and $h_L = h_R = \frac{3}{8}$ by spectral flow. Again, we focus on the type IIA string, in which case they correspond to the NSNS scalars in hypermultiplets, other than the universal one which contains the dilaton. In the

[18] The left-moving $(\psi^i_L, \psi^{\bar{\imath}}_L)$ are twisted to $(\chi^i_L, \lambda^{\bar{\imath}}_L)$ with conformal weights $(0,1)$. For the type IIA twist the right-movers $(\psi^i_R, \psi^{\bar{\imath}}_R)$ are twisted to $(\lambda^i_R, \chi^{\bar{\imath}}_R)$ with conformal weight $(1,0)$.

hybrid formalism the space-time part of these states is described by a real twisted-chiral multiplet M_{tc} with the field content of a tensor multiplet. The vertex operators are contained in the potentials given in (2.50)

$$\mathcal{U}_{cc} = e^{\rho_L - \rho_R}\, \bar\theta_L^2\, \theta_R^2\, M_{tc}\Omega_c\,. \tag{C.13}$$

The chiral primary field Ω_c has conformal weight $h_L = 0$ and $h_R = 1$ (while $\bar\Omega_c$ has weights $h_L = 1$ and $h_R = 0$) such that M_{tc} indeed describes massless states. In the large volume limit one has

$$\Omega_c = h_{ij}\chi_L^i\lambda_R^j\,, \quad \bar\Omega_c = h_{\bar\imath\bar\jmath}\lambda_L^{\bar\imath}\chi_R^{\bar\jmath}\,, \tag{C.14}$$

where $h_{ij} = g_{j\bar\jmath}h_i{}^{\bar\jmath}$, and $h_i{}^{\bar\jmath}$ is related to elements $Y_{ij\bar k} = h_i{}^{\bar\imath}\bar\Omega_{\bar\imath\bar\jmath\bar k}$ of $H^{1,2}(CY_3)$. Again, we suppress the index that distinguishes between these $h^{2,1}$ different elements.

The vertex operators contained in this multiplet can be extracted by expanding M_{tc} in powers of θ_L and $\bar\theta_R$. The lowest components are

$$\begin{aligned} \mathcal{U}_{l_{++}} &= l_{++}e^{\rho_L-\rho_R}\,\bar\theta_L^2\,\theta_R^2\,\Omega_c\,,\\ \mathcal{U}_v &= v_{\alpha\dot\beta}\,\theta_L^\alpha\,\bar\theta_R^{\dot\beta}\,e^{\rho_L-\rho_R}\,\bar\theta_L^2\,\theta_R^2\,\Omega_c\,. \end{aligned} \tag{C.15}$$

The scalar l_{++} parameterizes the fluctuations $h_{ij} = g_{j\bar\jmath}h_i^{\bar\jmath}$ of the pure components of the internal graviton g_{ij}. The complex polarization v_m is related to the internal components of the RR 3-form as $C_{ij\bar k} = C\,Y_{ij\bar k}$. The complex scalar C can be expressed by the real scalar l_{+-} and the dual of a real two-form field strength H_{mnp} such that $v_m = \partial_m C = (\partial_m l_{+-} + i\epsilon_{mnpq}H^{npq})$. Reducing the RNS vertex operators for the type IIA RR three-form and of the internal graviton one finds, in agreement with (C.3) and (C.4),

$$\begin{aligned} \mathcal{U}_{l_{++}} &= l_{++}\,|c\,\xi e^{-\phi}|^2\,\breve\Omega_c\,,\\ (\mathcal{U}_v)^\dagger &= \bar v_{\dot\alpha\beta}\,\bar S_L^{\dot\alpha}\,S_R^\beta\,|c\,\xi\,e^{-\frac{\phi}{2}}|^2\,\bar{\breve\Omega}_c\,. \end{aligned} \tag{C.16}$$

As for the Kähler moduli, the operator $\mathcal{U}_v$ maps to a RNS vertex operator in a non-canonical ghost-picture.

References

[1] N. Berkovits and C. Vafa, *N=4 topological strings,* Nucl. Phys. B **433**, 123 (1995) [arXiv:hep-th/9407190].

[2] N. Berkovits, *Covariant Quantization of the Green-Schwarz Superstring in a Calabi-Yau Background,* Nucl. Phys. B **431**, 258 (1995) [arXiv:hep-th/9404162]

[3] N. Berkovits and W. Siegel, *Superspace Effective Actions for 4D Compactifications of Heterotic and Type II Superstrings*, Nucl. Phys. B **462**, 213 (1996) [arXiv:hep-th/9510106].

[4] N. Berkovits, *A new description of the superstring*, arXiv:hep-th/9604123.

[5] I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, *Topological amplitudes in string theory,* Nucl. Phys. B **413**, 162 (1994) [arXiv:hep-th/9307158].

[6] C. Vafa, *Superstrings and topological strings at large N,* J. Math. Phys. **42** (2001) 2798 [arXiv:hep-th/0008142].

[7] A. Lawrence and J. McGreevy, *Local string models of soft supersymmetry breaking,* JHEP **0406** (2004) 007 [arXiv:hep-th/0401034].

[8] W. D. . Linch and B. C. Vallilo, *Hybrid formalism, supersymmetry reduction, and Ramond-Ramond fluxes,* arXiv:hep-th/0607122.

[9] N. Berkovits, *Calculation of Green-Schwarz superstring amplitudes using the N=2 twistor string formalism,* Nucl. Phys. B **395** (1993) 77 [arXiv:hep-th/9208035].

[10] N. Berkovits, C. Vafa and E. Witten, *Conformal field theory of AdS background with Ramond-Ramond flux,* JHEP **9903**, 018 (1999) [arXiv:hep-th/9902098].

[11] M. Ademollo *et al.*, *Supersymmetric Strings And Color Confinement,* Phys. Lett. B **62** (1976) 105.
Dual String Models With Nonabelian Color And Flavor Symmetries, Nucl. Phys. B **114** (1976) 297.

[12] M. Yu, *The unitary representations of the* $N = 4$ SU(2) *extended superconformal algebras,* Nucl. Phys. B **294** (1987) 890.

[13] D. Friedan, E. J. Martinec and S. H. Shenker, *Conformal Invariance, Supersymmetry And String Theory,* Nucl. Phys. B **271**, 93 (1986).

[14] N. Berkovits and W. Siegel, *Superspace Effective Actions for 4D Compactifications of Heterotic and Type II Superstrings*, Nucl. Phys. B **462**, 213 (1996) [arXiv:hep-th/9510106].

[15] W. Lerche, C. Vafa and N. P. Warner, Nucl. Phys. B **324** (1989) 427.

[16] E. P. Verlinde and H. L. Verlinde, *Chiral Bosonization, Determinants And The String Partition Function*, Nucl. Phys. B **288**, 357 (1987); *Multiloop Calculations In Covariant Superstring Theory,* Phys. Lett. B **192** (1987) 95; *Lectures On String Perturbation Theory,* IASSNS-HEP-88-52

[17] T. Eguchi and H. Ooguri, *Chiral Bosonization On A Riemann Surface,* Phys. Lett. B **187**, 127 (1987).

[18] U. Carow-Watamura, Z. F. Ezawa, K. Harada, A. Tezuka and S. Watamura, *Chiral Bosonization Of Superconformal Ghosts On Riemann Surface And Path Integral Measure*, Phys. Lett. B **227**, 73 (1989).

[19] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes,* Commun. Math. Phys. **165**, 311 (1994) [arXiv:hep-th/9309140].

[20] H. Ooguri and C. Vafa, *All loop N=2 string amplitudes*, Nucl. Phys. B **451**, 121 (1995) [arXiv:hep-th/9505183].

[21] N. Berkovits and B. C. Vallilo, *One loop N-point superstring amplitudes with manifest* $d = 4$ *supersymmetry*, Nucl. Phys. B **624**, 45 (2002) [arXiv:hep-th/0110168].

[22] N. Berkovits, *Conformal compensators and manifest type IIB S-duality,* Phys. Lett. B **423**, 265 (1998) [arXiv:hep-th/9801009].

[23] M. Rocek, C. Vafa and S. Vandoren, *Hypermultiplets and topological strings,* JHEP **0602**, 062 (2006) [arXiv:hep-th/0512206].

[24] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton University Press 1983; I. Buchbinder and S. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity*, IOP 1998.

[25] J. de Boer and K. Skenderis, *Covariant computation of the low energy effective action of the heterotic superstring,* Nucl. Phys. B **481**, 129 (1996) [arXiv:hep-th/9608078].
J. de Boer and K. Skenderis, *Target space supersymmetric sigma model techniques,* arXiv:hep-th/9611066.

[26] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory*, Springer 1997.

[27] J. N. Acosta, N. Berkovits and O. Chandia, *A note on the superstring BRST operator,* Phys. Lett. B **454**, 247 (1999) [arXiv:hep-th/9902178].

[28] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond,*

[29] S. Cecotti, S. Ferrara and L. Girardello, *Geometry Of Type II Superstrings And The Moduli Of Superconformal Field Theories*, Int. J. Mod. Phys. A **4**, 2475 (1989).

[30] I. Antoniadis, R. Minasian, S. Theisen and P. Vanhove, *String loop corrections to the universal hypermultiplet*, Class. Quant. Grav. **20**, 5079 (2003) [arXiv:hep-th/0307268].