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Ultraviolet properties of Maximal Supergravity

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Abstract

In a recent paper we suggested that the dualities of M-theory might imply that the four-graviton amplitude of four-dimensional $N = 8$ supergravity is ultraviolet finite in four dimensions. In this paper we argue that even without invoking duality conjectures, but by making direct use of string perturbation theory as a covariant ultraviolet regulator, the four-graviton amplitude of four-dimensional $N = 8$ supergravity should be ultraviolet finite up to at least eight loops. We further argue that the h-loop M-graviton amplitude should be finite for $h < 7 + M/2$.

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Maximally extended supergravity has for a long time held a privileged position among supersymmetric field theories. Its four-dimensional incarnation as $N = 8$ supergravity [1] initially raised the hope of a perturbatively finite quantum theory of gravity while its origin in $N = 1$ eleven-dimensional supergravity [2] provided the impetus for the subsequent development of M-theory, or the non-perturbative completion of string theory. Maximal supergravities in various dimensions arise as special limits within type II string theory, which is free of ultraviolet divergences. However, the fact that higher-dimensional maximal supergravity is not renormalizable means that it cannot be quantized in any conventional manner in isolation from string theory.

Another well-known problem with theories such as type II supergravity, is that it is notoriously difficult to analyze the constraints implied by maximal supersymmetry in a fully covariant manner since there is no practical off-shell formalism that makes manifest the full supersymmetry. For example, it has not yet been possible to determine the extent to which supersymmetry protects operators against perturbative corrections. However, a number of indirect arguments suggest that such protection may be far greater than earlier estimates might suggest. For example, a certain amount of evidence has accumulated over the past few years that the sum of the Feynman diagrams of maximal supergravity at a given number of loops may be less ultraviolet divergent than expected [3]. This is largely based on uncovering fascinating connections with the diagrams of $N = 4$ Yang–Mills, which are known to be ultraviolet finite in four dimensions. A different approach is to study implications of string/M-theory duality for the scattering amplitudes of type II supergravity. In [4] we considered the L-loop Feynman diagrams of the four-graviton scattering amplitude in eleven-dimensional supergravity compactified on a two-torus and its string theory interpretation. The lack of information about the short-distance structure of M-theory is reflected by the nonrenormalizability of supergravity and this ignorance was parameterized by an unlimited number of unknown coefficients of counterterms. Nevertheless, requiring the structure of the amplitude to be consistent with string theory led to interesting constraints. Among these were strong nonrenormalization conditions in the ten-dimensional type IIA string theory limit – where the eleven-dimensional theory is compactified on a circle of radius R_{11}. This condition followed from dimensional analysis together with the fact that the string coupling constant is given by $e^{\phi} = R_{11}^{3/2}$, where ϕ is the dilaton. The result has important implications for the ultraviolet properties of maximally extended supergravity, suggesting that four-dimensional $N = 8$ supergravity might have no ultraviolet divergences.

However, the arguments of [4] were rather indirect. Here we will proceed more directly and more conservatively by using perturbative string theory as a regulator of the ultraviolet divergences of supergravity. We will see that the nonrenormalization conditions of perturbative type II string theory [5], which are weaker than those proposed in [4], lead to the result that the four-graviton amplitude in four-dimensional $N = 8$ supergravity amplitude has no ultraviolet divergences up to at least eight loops.

We begin by noting that the h-loop contribution to the four-graviton amplitude in ten-dimensional string theory has the form

$$A_4^h = \alpha'^{\beta_h - 1}\, e^{2(h-1)\phi} \sum_i \mathcal{S}_i^{(\beta_h)}\, I_i^{(h)}(\alpha' s, \alpha' t, \alpha' u)\, R^4\,, \tag{1}$$

where R is the Weyl curvature and $\mathcal{S}_i^{(\beta_h)}$ are monomials of power β_h in the Mandelstam invariants, s, t and u. Explicit one-loop [6] and two-loop [3] calculations show that $\beta_1 = 0$ and $\beta_2 = 2$. In fact, recent perturbative superstring calculations [5] determine that $\beta_h = h$ up to five loops ($h = 5$). The less direct arguments of [4] make use of string/M-theory dualities to argue that $\beta_h = h$ might hold to all orders. In the following we will not specify the value of β_h until we need to. The functions $I_i^{(h)}$ are given by integrals over the moduli space of the h-loop string world-sheet. The number of such terms depends on the genus. For example, at two loops ($h = 2$) there are three terms,

$$\begin{aligned}\sum_{i=1}^{3} \quad \mathcal{S}_i^{(2)}\, I_i^{(2)}(\alpha' s, \alpha' t, \alpha' u) &= s^2 I_1^{(2)}(\alpha' s, \alpha' t, \alpha' u) \\ &+ t^2 I_2^{(2)}(\alpha' s, \alpha' t, \alpha' u) + u^2 I_3^{(2)}(\alpha' s, \alpha' t, \alpha' u)\,.\end{aligned} \tag{2}$$

The detailed evaluation of the functions $I_i^{(h)}$ for $h > 2$ is a daunting task but here we will only be concerned with the general structure of the amplitude.

We wish to consider the low-energy field theory limit of A_4^h obtained by expanding $I_i^{(h)}$ in the limit $\alpha' \to 0$ while holding the ten-dimensional Newton coupling, $\kappa_{(10)}^2 = \alpha'^4\, e^{2\phi}$, fixed. Since each $I_i^{(h)}$ is an integral over world-sheet moduli and cannot have poles in s, t, u [5] its low-energy expansion starts with a constant or logarithmic term. At low energies the amplitude (1) therefore takes the symbolic form

$$A_4^h \sim \kappa_{(10)}^{2(h-1)}\, \alpha'^{3-4h+\beta_h}\, \mathcal{S}^{(\beta_h)}\, (1 + O(\alpha' s))\, R^4\,, \tag{3}$$

where we have not kept track of possible factors that are logarithmic in the Mandelstam invariants but which are, in principle, defined precisely by evaluating the amplitude. The symbol $\mathcal{S}^{(\beta_h)}$ represents a symmetric monomial of the Mandelstam invariants of dimension $2\beta_h$. The expression (1) is finite in ten-dimensional string theory due to the presence of the string length that provides an ultraviolet cutoff. Indeed, the divergence of the expression (3) in the low-energy limit, $\alpha' \to 0$, translates into the ultraviolet divergence of the sum of all the contributions to the supergravity h-loop amplitude. After interpreting the inverse string length as an ultraviolet momentum cutoff, $\Lambda \sim \sqrt{\alpha'^{-1}}$, (3) becomes

$$A_4^h \sim \kappa_{(10)}^{2(h-1)}\, \Lambda^{8h-6-2\beta_h}\, \mathcal{S}^{(\beta_h)}\, (1 + O(\alpha' s))\, R^4\,. \tag{4}$$

So the presence of the prefactor $\mathcal{S}^{(\beta_h)}\, R^4$ means that the leading divergences, Λ^{8h+2}, of individual h-loop Feynman diagrams cancel and the ultraviolet divergence of the sum of diagrams is reduced by a factor of $\Lambda^{-8-2\beta_h}$.

We are interested in the maximal supergravity limit in lower dimensions so we will consider compactifying the string loop amplitude on a $(10-d)$-torus (with the external momenta and polarizations oriented in the d non-compact directions). Now consider the low energy limit $\alpha' \to 0$ with the radii of the torus proportional to $\sqrt{\alpha'}$, so that all the massive Kaluza–Klein states, winding-number states and excited string states decouple. This leads to an expression for the h-loop supergravity amplitude in d dimensions with cut-off Λ_d. For example, for a square torus with all radii equal to $r\sqrt{\alpha'}$, the expression for A_4^h behaves as $\kappa_{(d)}^{2(h-1)}\, \Lambda_d^{(d-2)h-2\beta_h-6}$, where the d-dimensional Newton constant, given by $\kappa_{(d)}^2 = \alpha'^{(d-2)/2}\, e^{2\phi}\, r^{d-10}$, is held fixed. Therefore ultraviolet divergences are absent in dimensions satisfying the bound

$$d < 2 + \frac{2\beta_h + 6}{h}\,. \tag{5}$$

In dimensions that satisfy this bound the expression (4) is ill-defined since it contains a negative power of the cut-off, which vanishes, whereas finite and infrared divergent terms that are non-vanishing are not exhibited. In this case the negative power of Λ_d is replaced by a negative power of s, t and u of dimension $[s]^{(d-2)h/2-\beta_h-3}$, together with possible logarithmic factors. Infrared divergences arise for dimensions $d \leq 4$ and presumably sum up in the usual fashion to cancel the divergences due to multiple soft graviton emission [7]. These features of the field theory limit are seen explicitly in the compactified one-loop ($h=1$) amplitude, which has $\beta_1 = 0$ [8]. In that case the amplitude has the form of a prefactor of R^4 multiplying a φ^3 scalar field theory box diagram. This is ultraviolet divergent when $d \geq 8$, and finite for $4 < d < 8$. The low-energy limit of the two-loop ($h = 2$) string theory expression reduces to the supergravity two-loop amplitude which was considered in detail in [3, 9]. There it was shown that a power of s^2, t^2 or u^2 factors out of the sum of all supergravity Feynman diagrams, so that $\beta_2 = 2$. In this case the sum of Feynman diagrams has the form of prefactor $s^2 R^4$ multiplying the sum of planar and non-planar s-channel double box diagrams of φ^3 scalar field theory, together with corresponding t-channel and u-channel terms. More recently the fact that $\beta_2 = 2$ has been confirmed by explicit two-loop calculations in string theory [10]. In this case ultraviolet divergences arise when $d \geq 7$ and the amplitude is finite for $4 < d < 7$.

The value of β_h for $h > 2$ has not been established from direct supergravity calculations beyond two loops, but it is known that $\beta_h \geq 2$ for $h > 2$. Furthermore, in contrast to the $h = 1$ and $h = 2$ cases, the sum of Feynman diagrams for $h > 2$ is unlikely to reduce to a prefactor multiplying diagrams of φ^3 scalar field theory. That would require $\beta_h = 2h$, which would lead to finiteness in $d < 6$ dimensions (for $h = 3$ it would also contradict the presence of a three-loop term in $D^6 R^4$ found in [13]). Using the value $\beta_h = 2$ (the least possible value) in (5) leads to the absence of ultraviolet divergences when

$$d < 2 + 10/h\,, \qquad\qquad h > 1\,, \tag{6}$$

which appeared in [3]. This shows that the first ultraviolet divergence in four dimensions cannot arise until at least five loops.

The full extent to which the four-graviton amplitude is protected from ultraviolet divergences should become clearer with a more complete understanding of the constraints implied by maximal supersymmetry. These are difficult to establish in the absence of an off-shell supersymmetric formalism. In theories with less supersymmetry such protection is typically afforded to F-terms, which can be expressed as integrals over a subspace of the complete superspace. One might estimate the extent to which the derivative expansion of the four-graviton amplitude is protected by using on-shell superfield arguments. This is explicit in the pure spinor formalism of the superstring developed by Berkovits [5], in which terms of the form $\mathcal{S}^{(k)} R^4$ are F-terms for $k \leq 5$ and get vanishing contributions from $h > k$ loops. Such terms arise from integration over a subset of the 32 components of the left-moving and right-moving Grassmann spinor world-sheet coordinates, θ_L and θ_R. As a result, the low energy limits of both type IIA and IIB superstring theories at h loops have $\beta_h = h$ for $h = 2, 3, 4, 5$ and $\beta_h \geq 6$ for $h \geq 6$. Since $\mathcal{S}^{(h)}\, R^4$ terms are protected for $h \leq 5$ the $\mathcal{S}^{(6)}\, R^4$ interaction can only arise for $h \geq 6$. Substituting into (5) we see that (4) has no ultraviolet divergences if

$$d < 2 + 18/h\,, \qquad\qquad h > 5 \tag{7}$$

and

$$d < 4 + 6/h \qquad\qquad h = 2, \ldots, 5\,. \tag{8}$$

We therefore see that the nonrenormalization conditions in type IIA string theory [5] lead to the ultraviolet finiteness of the four-dimensional $N = 8$ supergravity four-graviton amplitude up to at least eight loops ($h = 8$). Note that although type IIA and IIB four-graviton string theory amplitudes are equal only up to four loops ($h = 4$) in ten dimensions [5], they are identical at all loops in the d-dimensional low-energy supergravity limit.

One of the benefits of the superfield description of F-terms is that it includes all terms related by supersymmetry. For example, interactions involving higher powers of the curvature tensor of the form $\mathcal{S}^{(k)}\, R^M$, are F-terms if $2k + M < 16$ [5] (where $\mathcal{S}^{(k)}$ is a monomial made of Mandelstam invariants of the M-particle amplitude). In this case an extension of the fermionic mode counting in [5] shows that there are no corrections beyond $h = k + M - 4$ loops. Generalizing our earlier analysis, this means that the h-loop M-graviton amplitude behaves as $\mathcal{S}^{(h+4-M)}\, R^M$ for $M \leq h + 4$. It follows that the cut-off dependence of this amplitude is $\Lambda_d^{(d-4)h-6}$ for $h < 4 + M/2$ and $\Lambda_d^{(d-2)h-14-M}$ for $h \geq 4 + M/2$. This implies that the M-graviton amplitude is finite in four dimensions ($d = 4$) if $h < 7 + M/2$. Therefore, amplitudes with $M > 4$ have better ultraviolet behaviour than in the $M = 4$ (four-graviton) case. It is notable, in particular, that these arguments show that there is no divergence associated with the seven-loop counterterm [15].

Finally, we return to the suggestion [4] that $\beta_h = h$. This was motivated by an indirect argument based on considerations of M-theory duality rather than direct string calculations and is therefore less well established. In particular, it is not yet apparent how this condition can be motivated by supersymmetry. In this case there is an extra power of s, t or u for every additional loop and the divergence of the h-loop integral is markedly reduced. Substituting $\beta_h = h$ in (5) it follows that ultraviolet divergences are absent when

$$d < 4 + 6/h \qquad\qquad h \geq 2\,. \tag{9}$$

If correct, this would imply that ultraviolet divergences are absent to all orders in the four-graviton amplitude of four-dimensional maximal supergravity. Finiteness of the four-graviton amplitude suggests finiteness of all M-point functions since they are interconnected by unitarity. Indeed, the arguments in [4] have an obvious extension to multi-graviton amplitudes, which suggests that the $\mathcal{S}^{(k)}\, R^M$ interactions again have a dependence on the cut-off of the form $\Lambda_d^{(d-4)h-6}$. This leads to the same condition, (9), for ultraviolet finiteness of M-graviton amplitudes as in the four-graviton case.

The bound (9) is the same as the condition for the absence of ultraviolet divergences in maximally supersymmetric Yang–Mills theory, which is known to be finite in four dimensions. Indeed, the work of [3] points to connections between loop amplitudes of maximal supergravity and those of maximal super-Yang–Mills motivated in part by the Kawai–Lewellyn–Tye relations [16] that connect tree-level open and closed string theory. This suggests that $N = 8$ supergravity may be more finite than previously expected [17, 18].

To summarize, we have shown that recently discovered nonrenormalization properties of higher-genus contributions to the four-graviton amplitude in type II superstring theory [5] lead to the absence of ultraviolet divergences in the four-graviton amplitude of $N = 8$ supergravity up to at least eight loops. This was achieved by using the pure spinor formulation of string perturbation theory that efficiently embodies space-time supersymmetry in the ultraviolet regulated field theory. In this formalism the nonrenormalization conditions follow from the fact that $\mathcal{S}^{(h)}\, R^4$ is an F-term and gets no corrections beyond h loops if $h \leq 5$. We also showed that the M-graviton amplitude is ultraviolet finite when $h < 7 + M/2$. It would obviously be of interest if this understanding could be extended to derive the all-orders non-renormalization conditions proposed in [4]. Interestingly, there are similar situations in highly supersymmetric theories in which an infinite number of higher-dimension operators are protected from renormalization even though a naive application of supersymmetry would suggest that only a finite number should be [19, 20].

A priori, finiteness of $N = 8$ seems very unlikely and cries out for a natural explanation. One possible framework for such an explanation might be a variant of twistor string theory [21], which naturally describes $N = 4$ Yang–Mills coupled to superconformal gravity [22, 23]. Perhaps one of the proposals for a $N = 8$ twistor string theory in [24] is on the right track.

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