Geometry and $1/N$ expansion of Matrix models

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B. Eynard, SPhT Saclay

Work, done in collaboration with N. Orantin (Saclay).

Outline:

- introduction, matrix models, counting discrete surfaces.
- Schwinger-Dyson equations $\rightarrow$ algebraic curve
- tau function of an algebraic curve
- conclusion

1 Some matrix integrals.

- 1 Matrix Model: $Z_{\rm 1MM} = \int dM\, {\rm e}^{-N\,{\rm Tr}\, V(M)}$

  $V(M) = \sum_{k=0}^{d+1} t_k M^k$

- 2 Matrix Model: $Z_{\rm 2MM} = \int dM_1 dM_2\, {\rm e}^{-N\,{\rm Tr}\,(V_1(M_1)+V_2(M_2)-M_1 M_2)}$

  $V_1(M_1) = \sum_{k=0}^{d_1+1} t_k M_1^k \quad , \quad V_2(M_2) = \sum_{k=0}^{d_2+1} \tilde{t}_k M_2^k$

- Kontsevitch integral: $Z_{\rm K} = \int dM\, {\rm e}^{-N\,{\rm Tr}\,(\frac{M^3}{3}-\Lambda^2 M)}$

  $t_k = \frac{1}{N}\,{\rm Tr}\,\Lambda^{-k}$

- Generalized Kontsevitch integral: $Z_{\rm GK} = \int dM\, {\rm e}^{-N\,{\rm Tr}\,(V(M)-Q(\Lambda)M)}$

2 Topological expansions.

In all cases: topological expansion: $\ln Z = \sum_{g=0}^{\infty} N^{2-2g} F^{(g)}$

- $F_{\rm 1MM}^{(g)} = \sum_{n_3,n_4,\dots,n_{d+1}} t_3^{n_3} \dots t_{d+1}^{n_{d+1}} \mathcal{N}_{n_1,n_2,\dots,n_{d+1}}^{(g)}$

  $\mathcal{N}_{n_1,n_2,\dots,n_{d+1}}^{(g)}$ = number of discrete surfaces of genus g, with n_3 triangles, ... n_k $k-$gones,... (we assume $t_1 = 0$ and $t_2 = \frac{1}{2}$).

- $F_{\rm K}^{(g)} = \sum_{n_1,n_2,\dots,n_{d+1}} t_1^{n_1} \dots t_{d+1}^{n_{d+1}} \mathcal{I}_{n_1,n_2,\dots,n_{d+1}}^{(g)}$

  $\mathcal{I}_{n_1,n_2,\dots,n_{d+1}}^{(g)}$ = intersection number of moduli space of surfaces of genus g, with $n = \sum_i n_i$ marked points.

- similar interpretation for $F_{\rm 2MM}^{(g)}$, $F_{\rm GK}^{(g)}$.

3 Double scaling limit.

Continuous surfaces $\leftrightarrow$ surfaces with a very large number of polygons

Example average number of triangles: $< n_3 >= t_3 \frac{\partial \ln F^{(g)}_{1\mathrm{MM}}}{\partial t_3}$

i.e. singularity of $F^{(g)}_{1\mathrm{MM}}$:

$$F^{(g)}_{1\mathrm{MM}} \sim (t_{3c} - t_3)^{\gamma_g} F^{(g)}_{\mathrm{DSL}} + \text{subleading} \qquad , \qquad F^{(g)}_{\mathrm{DSL}} = \lim (t_{3c} - t_3)^{-\gamma_g} F^{(g)}_{1\mathrm{MM}}$$

One finds: $\gamma_g = (2-2g)(1-\frac{\gamma}{2})$, set $\kappa = N(t_{3c} - t_3)^{1-\frac{\gamma}{2}}$.

Define:

$$F_{\mathrm{DSL}} = \sum_{g=0}^{\infty} \kappa^{2-2g} F^{(g)}_{\mathrm{DSL}}$$

$F^{(g)}_{\mathrm{DSL}} \sim$ generating function for counting continuous surfaces of genus g.

4 Schwinger–Dyson equations.

Infinitesimal change of variable: $M \to M + \epsilon\delta M$.

$$\mathrm{Jac} = 1 + \epsilon\delta J(M) + O(\epsilon^2)$$

$$\mathrm{Tr}\, V(M) \to \mathrm{Tr}\, V(M) + \epsilon\, \mathrm{Tr}\, \delta M\, V'(M) + O(\epsilon^2)$$

$\to$ Schwinger-Dyson equation: $\langle \delta J(M) \rangle = N \left\langle \mathrm{Tr}\, V'(M)\, \delta M \right\rangle$

Example: 1MM, $\delta M = \frac{1}{x-M}$ gives:

$$\frac{1}{N^2} \left\langle \mathrm{Tr}\, \frac{1}{x-M}\, \mathrm{Tr}\, \frac{1}{x-M} \right\rangle = \frac{V'(x)}{N} \left\langle \mathrm{Tr}\, \frac{1}{x-M} \right\rangle - \frac{1}{N} \left\langle \mathrm{Tr}\, \frac{V'(x)-V'(M)}{x-M} \right\rangle$$

5 Schwinger–Dyson equation of the 1MM.

$$\frac{1}{N^2}\left\langle \mathrm{Tr}\,\frac{1}{x-M}\,\mathrm{Tr}\,\frac{1}{x-M}\right\rangle = \frac{V'(x)}{N}\left\langle \mathrm{Tr}\,\frac{1}{x-M}\right\rangle - \frac{1}{N}\left\langle \mathrm{Tr}\,\frac{V'(x)-V'(M)}{x-M}\right\rangle$$

Introduce notations: Resolvent $= W(x) = \frac{1}{N}\left\langle \mathrm{Tr}\,\frac{1}{x-M}\right\rangle$,

$P_{d-1}(x) = \frac{1}{N}\left\langle \mathrm{Tr}\,\frac{V'(x)-V'(M)}{x-M}\right\rangle$ = polynomial of degree $d-1$ in x.

$W_2(x,x') = \left\langle \mathrm{Tr}\,\frac{1}{x-M}\,\mathrm{Tr}\,\frac{1}{x'-M}\right\rangle - N^2 W(x)W(x')$,

The Schwinger-Dyson equations reads:

$$W^2(x) + \frac{1}{N^2}W_2(x,x) = V'(x)W(x) - P_{d-1}(x)$$

Additional equation: fixed filling fractions:

$$\oint_{C_i} W(x)dx = -2i\pi\epsilon_i \qquad , \qquad i = 1,\dots,d-1$$

6 Classical spectral curve of the 1MM.

Drop the $1/N^2$ connected term in the Schwinger-Dyson equation:

$$W^2(x) + \frac{1}{N^2} W_2(x,x) = V'(x)W(x) - P_{d-1}(x)$$

Large N limit = algebraic equation:

$$W^2 - V'(x)W + P_{d-1}(x) = 0$$

Rename: $W = \frac{1}{2}V'(x) - y$, and get the classical spectral curve:

$$0 = E_{1\mathrm{MM}}(x,y) = y^2 - \frac{1}{4}V'^2(x) - P_{d-1}(x) \qquad , \qquad \oint_{C_i} y dx = 2i\pi\epsilon_i$$

7 Classical spectral curve of the 2MM.

Changes of variables $\delta M_1 = \frac{1}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}$, $\delta M_2 = \frac{1}{x-M_1}$ give Schwinger Dyson equations:

$$E_{2\mathrm{MM}}(x, Y(x)) = \frac{1}{N^2} U_1(x, Y(x), x)$$

where $Y(x) = V_1'(x) - \frac{1}{N}\left\langle \mathrm{Tr}\, \frac{1}{x-M_1}\right\rangle$,
$P_{d_1-1,d_2-1}(x,y) = \frac{1}{N}\left\langle \mathrm{Tr}\, \frac{V_1'(x)-V_1'(M_1)}{x-M_1}\, \frac{V_2'(y)-V_2'(M_2)}{y-M_2}\right\rangle$,
$U_1(x,y,x') = \left\langle \mathrm{Tr}\, \frac{1}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\ \mathrm{Tr}\, \frac{1}{x'-M_1}\right\rangle_{\mathrm{c}}$, and:

$$E_{2\mathrm{MM}}(x,y) = (V_1'(x)-y)(V_2'(y)-x) - P_{d_1-1,d_2-1}(x,y) + 1$$

Classical spectral curve:

$$E_{2\mathrm{MM}}(x,y) = 0 \qquad , \qquad \oint_{C_i} y\,dx = 2i\pi\epsilon_i$$

8 Classical spectral curve of G-Kontsevitch.

Let $S(y) = \det(y - Q(\Lambda))$, the change of variables $\delta M = \frac{1}{x-M} \frac{S(y)-S(Q(\Lambda))}{y-Q(\Lambda)}$ gives:

$$E_{\mathrm{GK}}(x,y) = (V'(x) - y)S(y) - \frac{1}{N} \left\langle \mathrm{Tr}\, \frac{V'(x) - V'(M)}{x - M} \frac{S(y) - S(Q(\Lambda))}{y - Q(\Lambda)} \right\rangle$$

- Kontsevitch case, $V(M) = \frac{M^3}{3}$, $Q(\Lambda) = \Lambda^2$ gives:

$$E_{\mathrm{K}}(x,y) = (x^2 - y)S(y) - \frac{1}{N} \left\langle \mathrm{Tr}\, (x + M) \frac{S(y) - S(\Lambda^2)}{y - \Lambda^2} \right\rangle$$

Rational parametrization:

$$\begin{cases} y(z) = z^2 \\ x(z) = z + \frac{1}{2N} \mathrm{Tr}\, \frac{1}{\Lambda} \frac{1}{z-\Lambda} = z - \frac{1}{2} \sum_{k=0}^{\infty} t_{k+2} z^k \end{cases}$$

9 Classical spectral curve of DSL.

$F_{1\mathrm{MM}}^{(g)}$ is singular when the classical curve $E_{1\mathrm{MM}}(x, y) = 0$ is singular.
Consider a p/q singularity at $t_3 = t_{3c}$:

$$y - y_0 \sim (x - x_0)^{p/q}$$

Consider $\delta = (t_{3c} - t_3)$ small but non zero, i.e. the singularity is smoothed:

$$\begin{cases} x(z) = x_0 + \delta^{\alpha q}\; T_q(z/\delta^\alpha) + \dots \\ y(z) = y_0 + \delta^{\alpha p}\; T_p(z/\delta^\alpha) + \dots \end{cases}$$

where $\alpha = \frac{1}{p+q-1}$, T_p and T_q are polynomials of degree p and q, which obey (Poisson):

$$q\, T_p'(z)\, T_q(z) - p\, T_q'(z)\, T_p(z) = 1$$

Define: $E_{\mathrm{DSL}(p,q)}(x, y) = \quad \mathrm{Resultent}(x - T_q(z), y - T_p(z)) \quad = \begin{cases} x = T_q(z) \\ y = T_p(z) \end{cases}$

Example: if $p = q + 1$, then T_p, T_q =Tchebychev's polynomials:
$E_{\mathrm{DSL}(p,q)}(x, y) = T_p(x) - T_q(y)$.

10 Basics of algebraic geometry.

Consider an arbitrary embedded algebraic curve given by its equation:

$$E(x,y) = 0 \qquad\qquad E = \text{polynomial}$$

For this talk only assume the curve has genus zero $\to$ rational parametrization:

$$\begin{cases} x(p) = \text{rational function of } p \\ y(p) = \text{rational function of } p \end{cases}$$

Branchpoints a_i = solutions of $x'(a_i) = 0$ (vertical tangent).
Assume the curve is non-singular, i.e. x' has only simple zeroes and $y'(a_i) \neq 0$.
• If p lies near the branch-point a_i, there exists a unique other point $\bar{p} \neq p$, which is also in the vicinity of a_i, such that:

$$x(p) = x(\bar{p})$$

Notice that $\bar{p}$ may depend on i.

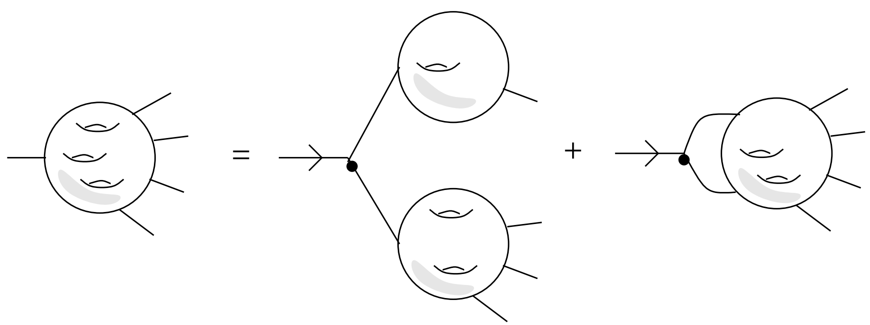
11 Definition of correlation functions.

Define the following functions:

$$W_1^{(0)}(p) = 0 \qquad , \qquad W_2^{(0)}(p,q) = \frac{1}{(p-q)^2}$$

$$W_{k+1}^{(g)}(p,\{p_K\}) = -\frac{1}{2}\sum_i \operatorname*{Res}_{q\to a_i} \frac{(q-\overline{q})dq}{(p-q)(p-\overline{q})(y(q)-y(\overline{q}))x'(\overline{q})} \Big[W_{k+2}^{(g-1)}(q,\overline{q},\{p_K\}) + \sum_{h=0}^{g}\sum_{J\subset K} W_{|J|+1}^{(h)}(q,\{p_J\})W_{k-|J|+1}^{(g-h)}(\overline{q},\{p_{K/J}\}) \Big]$$

where $K = \{i_1, i_2, \ldots, i_k\}$ and $\{p_K\} = \{p_{i_1}, p_{i_1}, \ldots, p_{i_k}\}$. i.e.



12 Definition of Free energies.

Similarly, define for $g > 1$:

$$F^{(g)}(E) = \frac{1}{2-2g} \sum_i \frac{1}{2} \mathop{\mathrm{Res}}_{q\to a_i} \frac{dq\, \int_{\overline{q}}^{q} y dx}{(y(q)-y(\overline{q}))x'(\overline{q})} \Big[W_2^{(g-1)}(q,\overline{q}) + \sum_{h=0}^{g} W_1^{(h)}(q) W_1^{(g-h)}(\overline{q}) \Big]$$

There is also a similar algebro-geometric definition for $F^{(0)}$ and $F^{(1)}$.

$$F^{(1)} = -\frac{1}{24} \ln\left(\Delta(x(a_i))^2 \prod_i y_1(a_i) \right)$$

13 Topological expansions.

Theorem:

- $F^{(g)}_{\mathrm{1MM}} = F^{(g)}(E_{\mathrm{1MM}})$
- $F^{(g)}_{\mathrm{2MM}} = F^{(g)}(E_{\mathrm{2MM}})$
- $F^{(g)}_{\mathrm{K}} = F^{(g)}(E_{\mathrm{K}})$
- $F^{(g)}_{\mathrm{GK}} = F^{(g)}(E_{\mathrm{GK}})$
- $F^{(g)}_{\mathrm{DSL}} = F^{(g)}(E_{\mathrm{DSL}})$

Remark: $F^{(g)}(E)$ commutes with the double scaling limit: $\lim F^{(g)}(E) = F^{(g)}(\lim E)$

14 Properties of $F^{(g)}(E)$

Theorem of symplectic invariance:

$\forall g$, $F^{(g)}(E)$ is invariant under the following changes of curves E:

- $y \to y + R(x)$ where $R(x)$ = any rational function of x.
- $y \to \alpha y$, $x \to \frac{1}{\alpha} x$, $\alpha \in \mathbf{C}^*$.
- $y \to -y$, $x \to x$.
- $y \leftrightarrow x$. e.g. $(p,q) = (q,p)$ duality.

= transformations which conserve the symplectic form $\pm\, dx \wedge dy$.

- $\tau_N(E) = \exp\left(\sum_{g=0}^{\infty} N^{2-2g} F^{(g)}(E)\right)$ satisfies some Hirota equation.
- Modular invariance: $\tau_N(E)$ changes in a way similar to a θ function.

15 Conclusion

- We have found the general solution of Virasoro constraints.
- We have some explicit formulae for the topological expansion of matrix models and their DSL. The same formula works for all cases.
- To every algebraic curve $E(x, y) = 0$, we can associate a sequence of symplectic invariants $F^{(g)}(E)$ and a τ-function $\tau_N(E)$.
- From the classical spectral curve $E(x, y) = 0$, we reconstruct perturbatively the full quantum integrable system $E_N(x, y) = 0 = \sum_{g=0}^{\infty} N^{-2g} E^{(g)}(x, y)$.
- There is probably an underlying quantum field theory, for which those diagrams are the Feynman diagrams. Chiral supersymmetric Chern Simons theory ?
- What are the $F^{(g)}(E)$ for other algebraic curves E ?

16 More references

- *Hermitean matrix model free energy: Feynman graph technique for all genera*, (L. Chekhov, B. E.), JHEP/009P/0206/5, hep-th/0504116.
- *Topological expansion of the 2-matrix model correlation functions: diagrammatic rules for a residue formula*, (B. E., N. Orantin), JHEP 0512 (2005) 034, math-ph/0504058.
- *Free energy topological expansion for the 2-matrix model.*, (L. Chekhov, B. E., N. Orantin), math-ph/0603003.
- *Genus one contribution to free energy in hermitian two-matrix model*, (B.E., D. Korotkin, A. Kokotov), Nucl.Phys. B694 (2004) 443-472, hep-th/0403072.
- *Breakdown of universality in multi-cut matrix models*, (G. Bonnet, F. David, B.E.), J.Phys. A33 6739-6768 (2000).

17 Virasoro constraints.

Partition function

$$Z := \mathrm{e}^{-N^2 F} := \int dM\, \mathrm{e}^{-N\,\mathrm{tr}\, V(M)} := \int dM\, \mathrm{e}^{-N \sum_j t_j\, \mathrm{Tr}\, M^j}$$

satisfies the Virasoro constraints:

$$D_k.Z = 0 \qquad \text{for } k \geq -1$$

with the Virasoro generators

$$D_k = \frac{1}{N^2} \sum_{j=0}^{k} \frac{\partial}{\partial t_j} \frac{\partial}{\partial t_{k-j}} + \sum_{j=1}^{\deg V} j t_j \frac{\partial}{\partial t_{k+j}}$$

$$[D_k, D_j] = (k-j) D_{k+j}$$

Schwinger–Dyson equations:

$$\frac{1}{N^2} D_k.Z = \frac{1}{N^2} \left\langle \sum_{j=0}^{k} \mathrm{Tr}\, M^j\, \mathrm{Tr}\, M^{k-j} \right\rangle - \frac{1}{N} \left\langle\, \mathrm{Tr}\, M^{k+1}\, V'(M) \right\rangle = 0$$