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# Universal distribution of random matrix eigenvalues near the "birth of a cut" transition.

B. Eynard[1]

Service de Physique Théorique de Saclay,
F-91191 Gif-sur-Yvette Cedex, France.



**Abstract**

We study the eigenvalue distribution of a random matrix, at a transition where a new connected component of the eigenvalue density support appears away from other connected components. Unlike previously studied critical points, which correspond to rational singularities $\rho(x) \sim x^{p/q}$ classified by conformal minimal models and integrable hierarchies, this transition shows logarithmic and non-analytical behaviours. There is no critical exponent, instead, the power of $N$ changes in a saw teeth behaviour.

## 1 Introduction

Random matrix models [20, 13] have been studied in relationship with many areas of physics and mathematics. The reason of their success for most of their applications is their "universality" property, i.e. the fact that the eigenvalues statistical distribution of a large random matrix depends only on the symmetries of the matrix ensemble, and not on the detailed Boltzmann weight (characterized by a potential). Although this universality property has been much studied for generic potentials, some universality should also hold for critical potentials. Different kinds of critical potentials have been studied, and their universality classes have been found to be in correspondence with non-linear integrable hierarchies (KdV, MKdV, KP,...) [2, 3, 7, 21, 6, 12], and with the $(p, q)$ rational minimal models of conformal field theory [8]. They correspond to rational singularituies of the equilibrium density:

$$\rho(x) \sim (x-a)^{p/q} \,. \tag{1.1}$$

[1] E-mail: eynard@spht.saclay.cea.fr

In the hermitian 1-matrix model, we have only $q = 2$ and $p$ arbitrary, thus we get only half integer singularities (hyperelliptical curves), which are in relationship with the KdV hierarchies, whereas a 2-matrix model allows to have any rational singularity $(p, q)$ [8]. The specific heat near such a rational singularity obeys a Gelfand-Dikii-type equation (Painlevé I equation for $(p, q) = (3, 2)$). A well known case is the edge of the spectrum where $(p, q) = (1, 2)$, which gives Tracy-Widom law [23], and which is governed by the Painlevé II equation. Another well known case is the merging of two cuts (Bleher and Its [2, 1]), where $(p, q) = (2, 1)$, which is also governed by a Painlevé II equation as well (indeed, $(p, q)$ and $(q, p)$ are known to be dual to each other [14]).

Here, we shall study a kind of critical point which has been mostly disregarded (because usual methods don't apply to it): "the birth of a cut critical point"[2].

Such a critical point, is characterized by the fact that when a parameter of the model (let us call it temperature) is varied, a new connected component appears in the support of the large $N$ average eigenvalue density. When the temperature $T$ is just above critical temperature $T_c$, the number of eigenvalues in the newborn connected component is small (see fig.2), and thus, many usual large $n$ methods don't work in that case.

Our goal is to study the eigenvalues statictics in the vicinity of the critical point, and find its universality class.

In this purpose, we shall start from the partition function, and treat the eigenvalues in the other cuts with mean field approximations, and reduce the problem to an effective partition function for eigenvalues in the newborn cut only, in a method similar to [4].

We find that, unlike rational critical points, the birth of a cut critical point does not corrsepond to power law behaviors or transcandental differential equations, but it exhibits logarithmic behaviors, and discontinuous functions.

The matrix model is associated to a family of orthogonal polynomials, whose zeroes lie inside the connected components of the density [20, 22, 10]. Our goal is also to study the asymptotic behavior of the orthogonal polynomials in the vicinity of the newborn cut.

**Outline of the article:**

- In section 2 we introduce definitions and notations for orthogonal polynomials and associated quantities.
- In section 3 we recall classical results of random matrix theory: the semiclassical behaviors of free energy, density, correlation functions, orthogonal polynomials, valid away from critical points.

[2]Name suggested by P. Bleher who initiated this work.

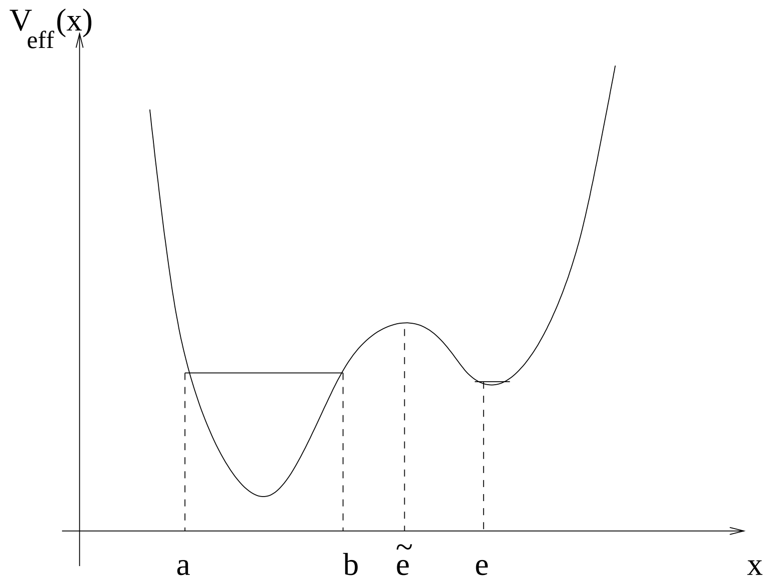


Figure 1: The "birth of a cut" critical potential is such that one of the potential wells of the effective potential is just at the Fermi level. At a $\nu^{\rm th}$ order critical point, the effective potential behaves like $V_{\rm eff}(e) + \frac{1}{2\nu!} V_{\rm eff}^{(2\nu)}(e)\,(x-e)^{2\nu} + \ldots$.

- In section 4 we study the analytical continuation of the previous semiclassical approximations, near the "birth of a cut" critical point (divergencies at critical point).
- In section 5 we compute the partition function with mean field theory for eigenvalues not in the newborn cut, and derive an effective partition function for the newborncut eigenvalues.
- In section 6 we use the results of section 5 to deduce the asymptotic behaviors of correlation functions and orthogonal polynomials in the vicinity of the critical point.
- Section 7 is the conclusion.
- In appendix, we recall Stirling's formula and its consequences, and we recall some elliptical function basics.

# 2 Setting

Given an integer $n$ (we will later consider the limit $n \to \infty$), a real polynomial $V(x)$ called the potential:

$$V(x) := g_0 + \sum_{k=1}^{d+1} \frac{g_k}{k} x^k \qquad , \qquad \deg V = d+1 \tag{2.2}$$

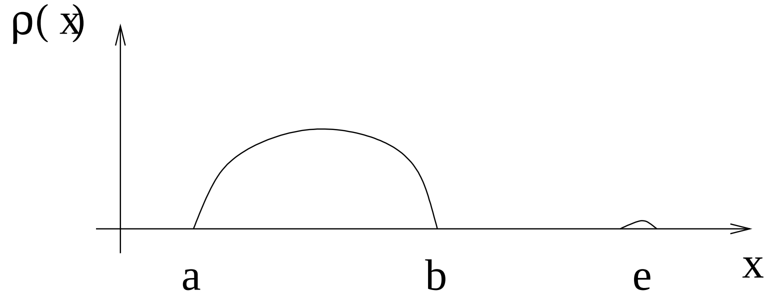


Figure 2: The "birth of a cut" density of eigenvalues is such that one of the connected components of the support contains a very small number of eigenvalues ($\ll n$).



of even degree and positive leading coefficient ($g_{d+1} > 0$), and a temperature $T > 0$, we define the partition function:

$$Z_n(T,V) := \frac{1}{n!}\int_{\mathbf{R}^n} \mathrm{d}x_1 \dots \mathrm{d}x_n \;\; (\Delta(x_i))^2 \;\prod_{i=1}^n \mathrm{e}^{-\frac{n}{T}V(x_i)} \tag{2.3}$$

(where $\Delta(x_i) = \prod_{i<j}(x_j - x_i)$ is the Vandermonde determinant) and the free energy $F_n(T,V)$:

$$\mathrm{e}^{-\frac{n^2}{T^2}F_n(T,V)} := \frac{Z_n(T,V)}{H_n} \tag{2.4}$$

where $H_n$ is a combinatorial normalization:

$$H_n := (2\pi)^{n/2}\, n^{-n^2/2}\, \mathrm{e}^{3/4n^2} \prod_{k=0}^{n-1} k! = 2^{n/2}\, \pi^{n^2/2}\, n^{-n^2/2}\, \frac{\mathrm{e}^{3/4n^2}}{n! U_n} \tag{2.5}$$

and $U_n$ is the volume of the group $U(n)/U(1)^n$.

Then we define the resolvent:

$$W_n(x,T,V) := \frac{T}{n!\, Z_n(T,V)}\int \mathrm{d}x_1 \dots \mathrm{d}x_n \; \frac{1}{x-x_1} \; \Delta^2(x_i) \; \prod_{i=1}^n \mathrm{e}^{-\frac{n}{T}V(x_i)} \tag{2.6}$$

and its first moment:

$$\begin{aligned} \mathcal{T}_n(T,V) &:= \frac{T}{n!\, Z_n(T,V)}\int \mathrm{d}x_1 \dots \mathrm{d}x_n \; x_1 \; \Delta^2(x_i) \; \prod_{i=1}^n \mathrm{e}^{-\frac{n}{T}V(x_i)} \\ &= \frac{1}{2i\pi}\oint x W_n(x,T,V) dx = \frac{\partial F_n(T,V)}{\partial g_1} \end{aligned} \tag{2.7}$$

Notice that more generally for $k > 0$:

$$k\frac{\partial F_n(T,V)}{\partial g_k} = \frac{1}{2i\pi}\oint x^k W_n(x,T,V) dx \tag{2.8}$$

Then, given a temperature $T_c$ and an integer $N$, we define:

$$h_n := \frac{Z_{n+1}(T_c \frac{n+1}{N}, V)}{Z_n(T_c \frac{n}{N}, V)}\,, \tag{2.9}$$

and

$$\gamma_n := \sqrt{\frac{h_n}{h_{n-1}}} = \sqrt{\frac{Z_{n+1}(T_c \frac{n+1}{N}, V) Z_{n-1}(T_c \frac{n-1}{N}, V)}{Z_n^2(T_c \frac{n}{N}, V)}}\,, \tag{2.10}$$

and

$$\beta_n := \frac{N}{T_c}\left(\mathcal{T}_{n+1}(T_c \frac{n+1}{N}, V) - \mathcal{T}_n(T_c \frac{n}{N}, V)\right) = -\frac{T_c}{N}\frac{\partial \ln h_n}{\partial g_1}\,. \tag{2.11}$$

Notice that $Z_n(T,V)$, $h_n$ and $\gamma_n$ are strictly positive for all $n$, in particular they don't vanish.

We also introduce the functions [22, 20, 10]:

$$\pi_n(\xi) = \frac{Z_n(T_c \frac{n}{N}, V(x) - \frac{T_c}{N}\ln(\xi - x))}{Z_n(T_c \frac{n}{N}, V)}\,, \qquad , \qquad \psi_n(\xi) = \frac{\pi_n(\xi)}{\sqrt{h_n}}\,\mathrm{e}^{-\frac{N}{2T}V(\xi)} \tag{2.12}$$

which form an orthogonal family of monic polynomials ($\deg \pi_n = n$):

$$\int \pi_n(\xi)\pi_m(\xi)\mathrm{e}^{-\frac{N}{T_c}V(\xi)}d\xi = h_n \delta_{nm}\,, \tag{2.13}$$

and which satisfy the 3-terms recursion relation:

$$\xi\,\pi_n(\xi) = \pi_{n+1}(\xi) + \beta_n\,\pi_n(\xi) + \gamma_n^2\,\pi_{n-1}(\xi)\,. \tag{2.14}$$

And we introduce the functions:

$$\hat{\pi}_n(\xi) = \frac{Z_{n+1}(T_c \frac{n+1}{N}, V(x) + \frac{T_c}{N}\ln(\xi - x))}{Z_n(T_c \frac{n}{N}, V)}\,, \qquad , \qquad \phi_n(\xi) = \frac{\hat{\pi}_n(\xi)}{\sqrt{h_n}}\,\mathrm{e}^{\frac{N}{2T}V(\xi)} \tag{2.15}$$

which are the Hilbert transforms of the $\pi_n(x)$:

$$\hat{\pi}_n(x) = \int \frac{dx'}{x - x'}\pi_n(x')\mathrm{e}^{-V(x')}\,, \tag{2.16}$$

and which satisfy the same 3-terms recursion relation, with an initial term:

$$\xi\,\hat{\pi}_n(\xi) = \hat{\pi}_{n+1}(\xi) + \beta_n\,\hat{\pi}_n(\xi) + \gamma_n^2\,\hat{\pi}_{n-1}(\xi) + \delta_{n,0}h_0\,. \tag{2.17}$$

We also define the kernel:

$$K_n(\xi, y) := \sum_{j=0}^{n-1}\frac{1}{h_j}\,\pi_j(\xi)\pi_j(y)\,\mathrm{e}^{-\frac{N}{2T_c}(V(\xi)+V(y))}\,, \tag{2.18}$$

and it is well known that all correlation functions can be expressed in term of that kernel (Dyson's theorem [15, 20]):

$$\begin{aligned}
\rho_n(x) &= \frac{1}{n} K_n(x,x) \\
\rho_n(x,y) &= \frac{1}{n^2}\left(K_n(x,x)K_n(y,y) - K_n(x,y)K_n(y,x)\right) \\
\ldots
\end{aligned} \tag{2.19}$$

and that we have the Christoffel-Darboux theorem:

$$K_n(x,y) = \frac{1}{h_{n-1}} \frac{\pi_n(x)\pi_{n-1}(y) - \pi_{n-1}(x)\pi_n(y)}{x-y} \, \mathrm{e}^{-\frac{N}{2T_c}(V(x)+V(y))} \,. \tag{2.20}$$

Our goal is to study $\gamma_n$, $\beta_n$, $\pi_n$, $\hat{\pi}_n$, in the vicinity of $N \to \infty$, and $|n-N| \ll N$, and with $T_c$ chosen such that we are at a special critical point described below.

For the moment, let us study the large $N$-limits away from the critical point.

# 3 Classical limits

It is well known that in the semiclassical limit $N \to \infty$, and $|n-N| \ll N$, if $T \neq T_c$, the free energy has a large $n$ limit [11, 16, 4, 19, 13]:

$$F_n(T,V) \longrightarrow F(T,V) + O(1/n^2) \tag{3.21}$$

and so has the resolvent $W_n(x,T,V)$:

$$W_n(x,T,V) \longrightarrow W(x,T,V) + O(1/n) \tag{3.22}$$

## 3.1 The large $n$ resolvent

It is well known that, if the potential is such that $V'(x)$ is a rational fraction, with its poles outside the cuts (that assumption will become clear below), the large $n$ resolvent $W(x,T,V)$ can be written as the solution of an hyperelliptical equation [13, 5]:

$$W(x,T,V) = \frac{1}{2}\left(V'(x) - M(x,T,V)\sqrt{\sigma(x,T,V)}\right) \tag{3.23}$$

where $\sigma$ is a monic even degree ($2s \geq 2$) polynomial with distinct simple zeroes only:

$$\sigma(x) = \prod_{i=1}^{s}(x-a_i)(x-b_i) \qquad , \qquad \ldots < a_i < b_i < a_{i+1} < \ldots \tag{3.24}$$

whose zeroes are called the endpoints, and $\bigcup_{i=1}^{s}[a_i,b_i]$ is called the support, and $M$ is a rational function with the same poles as $V'$. If one assumes that $s$ and $\sigma$ are known,

$M(x,T,V)$ is determined by the condition that $W(x,T,V)$ is finite (in the physical sheet) when $x \to \infty$ and when $x$ approaches the poles of $V'(x)$.

The large $n$ limit of the density of eigenvalues is then:

$$\rho(x,T,V) = \frac{1}{2\pi T} M(x,T,V) \sqrt{-\sigma(x,T,V)} \qquad , \qquad x \in \bigcup_{i=1}^{s} [a_i, b_i] \tag{3.25}$$

We also define the effective potential:

$$V_{\text{eff}}(x,T,V) := V(x) - 2T \ln x - 2 \int_{\infty}^{x} \left( W(x,T,V) - \frac{T}{x} \right) dx \tag{3.26}$$

Notice that its derivative is $V'(x) - 2W(x,T,V) = M(x,T,V)\sqrt{\sigma(x,T,V)}$.

So far, we have not explained how to determine $s$ and the polynomial $\sigma$. If one assumes that $s$ is known, $\sigma(x,T,V)$ is determined by the conditions:

$$\left\{ \begin{array}{l} W(x,T,V) \underset{x\to\infty}{\sim} \frac{T}{x} + O(1/x^2) \\ \text{if } s > 1, \quad \forall i = 1, \dots, s-1, \; V_{\text{eff}}(b_i) = V_{\text{eff}}(a_{i+1}) \end{array} \right. \tag{3.27}$$

The large $n$ free energy is then given by:

$$F(T,V) = \frac{1}{4i\pi} \oint W(x,T,V) V(x) dx + \frac{1}{2} T V_{\text{eff}}(b_s) \tag{3.28}$$

where the integration contour is a counter clockwise circle around $\infty$.

The number of endpoints $s = s(T,V)$, (we have $1 \leq s \leq d$) is determined by the condition that the free energy is minimum (one can prove that $s(T,V) \leq (d+1)/2$) [9].

### 3.2 Derivatives with respect to $T$

Let us introduce:

$$\Omega(x,T,V) := \frac{\partial W(x,T,V)}{\partial T} = \frac{Q_\Omega(x,T,V)}{\sqrt{\sigma(x,T,V)}} \tag{3.29}$$

where $Q_\Omega(x,T,V)$ is a monic polynomial of degree $s-1$, determined by the conditions:

$$\text{if } s > 1, \quad \forall i = 1, \dots, s-1, \quad \int_{b_i}^{a_{i+1}} \frac{Q_\Omega(x,T,V)}{\sqrt{\sigma(x,T,V)}} dx = 0 \tag{3.30}$$

In algebraic geometry, $\Omega$ is called "normalized abelian differential of the third kind" [18, 17].

We introduce the multivalued function $\Lambda(x,T,V)$:

$$\Lambda(x,T,V) := \exp\left( \int_{b_s}^{x} \Omega(x',T,V)\, dx' \right) \tag{3.31}$$

and

$$\gamma(T,V) := \lim_{x\to\infty} \frac{x}{\Lambda(x,T,V)} \tag{3.32}$$

Then we have the following derivatives:

$$\frac{\partial F}{\partial T} = V_{\rm eff}(b_s) \tag{3.33}$$

$$\frac{\partial^2 F}{\partial T^2} = -2\ln\gamma \tag{3.34}$$

$$\frac{\partial V_{\rm eff}(x)}{\partial T} = -2\ln\left(\gamma\Lambda(x)\right) \tag{3.35}$$

$$\frac{\partial \mathcal{T}}{\partial T} = \frac{1}{2i\pi}\oint x\,\Omega(x)\,{\rm d}x \tag{3.36}$$

Notice that:

$$V_{\rm eff}(b_s) = \frac{1}{2i\pi}\oint \Omega V - 2T\ln\gamma \tag{3.37}$$

## 3.3 Poles of the potential

Assume that $V'(x)$ has a simple pole at $x=\xi$, with residue $r$ (it may have other poles too), then we define the function:

$$H(x,\xi,T,V) := \frac{\partial W(x,T,V)}{\partial r} = \frac{1}{2\sqrt{\sigma(x)}}\left(\frac{\sqrt{\sigma(x)}-\sqrt{\sigma(\xi)}}{x-\xi} - Q_H(x,\xi)\right) \tag{3.38}$$

where $Q_H(x,\xi)$ is a monic polynomial in $x$, of degree $s-1$, determined by the conditions:

$$\text{if } s>1, \quad \forall i=1,\dots,s-1, \quad \int_{b_i}^{a_{i+1}} \frac{Q_H(x,\xi,T,V)+\frac{\sqrt{\sigma(\xi)}}{x-\xi}}{\sqrt{\sigma(x,T,V)}}\,dx = 0 \tag{3.39}$$

We also define its (multivalued) primitive:

$$\ln E(x,\xi) := \int_\infty^x H(x',\xi)\,dx' \tag{3.40}$$

Notice that it is finite near the endpoints and near $x=\xi$. In algebraic geometry, $(x-\xi)/E(x,\xi)$ is related to the "prime form" [18, 17].

Then we have:

$$\frac{\partial V_{\rm eff}(x)}{\partial r} = \ln(x-\xi) - 2\ln E(x,\xi) \tag{3.41}$$

$$\frac{\partial \mathcal{T}}{\partial T} = \frac{1}{2i\pi}\oint x\,H(x,\xi)\,{\rm d}x \tag{3.42}$$

$$\frac{\partial F}{\partial r} = \frac{1}{2}\left(V(x) - V_{\rm eff}(x)\right)|_{x=\xi} \tag{3.43}$$

$$\frac{\partial^2 F}{\partial r \partial T} = \ln\left(\gamma\Lambda(\xi)\right) = \ln\left(\xi - b_s\right) - 2\ln E(b_s,\xi) \tag{3.44}$$

If $V'(x)$ has simple poles at $x=\xi_1$ with residue $r_1$ and at $x=\xi_2$ with residue $r_2$, we have:

$$\frac{\partial^2 F}{\partial r_1 \partial r_2} = \ln E(\xi_1,\xi_2) \tag{3.45}$$

and thus it is clear that $\ln E$ has some symmetry properties: $\ln E(x,y) = \ln E(y,x)$.

### 3.4 1-cut case

If $W(x,T,V)$ has one cut $[a(T,V),b(T,V)]$ with $a<b$, we use the Joukowski's parameterization:

$$x = \frac{b+a}{2} + \frac{b-a}{2}\cosh\phi \tag{3.46}$$

i.e.

$$\sqrt{\sigma(x)} = \frac{b-a}{2}\sinh\phi \tag{3.47}$$

We have:

$$\Omega(x) = \frac{1}{\sqrt{(x-a)(x-b)}} = \frac{\partial\phi}{\partial x} \quad , \quad \Lambda(x) = {\rm e}^{\phi(x)} \quad , \quad \gamma = \frac{b-a}{4} \tag{3.48}$$

$$H(x,\xi) = \frac{\partial\phi(x)}{\partial x}\,\frac{1}{{\rm e}^{\phi(x)+\phi(\xi)}-1} \quad , \quad E(x,\xi) = 1 - {\rm e}^{-(\phi(x)+\phi(\xi))} = \frac{x-\xi}{\Lambda(x)-\Lambda(\xi)} \tag{3.49}$$

$$\frac{\partial \mathcal{T}}{\partial T} = \frac{a+b}{2} \quad , \quad \frac{\partial \mathcal{T}}{\partial r} = \frac{\gamma}{\Lambda(\xi)} \tag{3.50}$$

Then, it is well known [] that we have the large $n,N$ asymptotics (in the regime $n/N$ =finite):

$$\gamma_n \sim \frac{b(T_c\frac{n}{N}) - a(T_c\frac{n}{N})}{4} \quad , \quad \beta_n \sim \frac{b(T_c\frac{n}{N}) + a(T_c\frac{n}{N})}{2} \tag{3.51}$$

### 3.5 2-cut case

If $W(x,T,V)$ has two cuts $[a(T,V),b(T,V)] \cup [c(T,V),d(T,V)]$ with $a<b<c<d$, Let $m$ be their biratio:

$$m = \frac{(b-a)(d-c)}{(c-a)(d-b)} \tag{3.52}$$

We parameterize:

$$x(u) = d - \frac{d-a}{1+\frac{b-a}{d-b}\mathrm{sn}^2(u,m)} \tag{3.53}$$

where sn is tha elliptical sine function (see appendix I, or for instance [24]), i.e., by definition:

$$u(x) := -\frac{i}{2}\sqrt{(d-b)(c-a)}\,\int_a^x \frac{\mathrm{d}y}{\sqrt{\sigma(y)}} = \int_0^{\sqrt{\frac{d-b}{b-a}\frac{x-a}{d-x}}} \frac{dy}{\sqrt{(1-y^2)(1-my^2)}} \tag{3.54}$$

We have:

$$\begin{gathered} u(a)=0, \quad u(b)=K(m), \\ u(c)=K(m)+iK'(m), \quad u(d)=iK'(m). \end{gathered} \tag{3.55}$$

We have:

$$\sqrt{\sigma(x)} = -i(d-a)(b-a)\sqrt{\frac{c-a}{d-b}}\,\frac{\mathrm{sn}(u,m)\,\mathrm{cn}(u,m)\,\mathrm{dn}(u,m)}{(1+\frac{b-a}{d-b}\mathrm{sn}^2(u,m))^2}. \tag{3.56}$$

Let us define $u_\infty$ such that:

$$u_\infty := i\int_0^{\sqrt{\frac{d-b}{b-a}}} \frac{dy}{\sqrt{(1+y^2)(1+my^2)}}, \tag{3.57}$$

i.e.

$$\mathrm{sn}(u_\infty,m) = i\sqrt{\frac{d-b}{b-a}},\ ,\ \mathrm{cn}(u_\infty,m) = \sqrt{\frac{d-a}{b-a}},\ ,\ \mathrm{dn}(u_\infty,m) = \sqrt{\frac{(d-a)}{(c-a)}}. \tag{3.58}$$

Then we define $x_0$:

$$x_0 = d + i\sqrt{(c-a)(d-b)}\left(E(u_\infty,m) - \left(1-\frac{E'(m)}{K'(m)}\right)u_\infty\right). \tag{3.59}$$

It satisfies:

$$\int_b^c \frac{x-x_0}{\sqrt{(x-a)(x-b)(x-c)(x-d)}}dx = 0, \tag{3.60}$$

thus we have:

$$\Omega(x) = \frac{x-x_0}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} \tag{3.61}$$

$$\Lambda(x) = \mathrm{e}^{\pi\frac{u(x)u_\infty}{KK'}}\,\frac{\theta_1((u(x)+u_\infty)/2K)}{\theta_1((u(x)-u_\infty)/2K)} \tag{3.62}$$

$$\gamma = \frac{i}{4K}\sqrt{(d-b)(c-a)}\,\mathrm{e}^{-\pi\frac{u_\infty^2}{KK'}}\,\frac{\theta_1'(0,\tau)}{\theta_1(u_\infty/K,\tau)} \tag{3.63}$$

$$E(x,\xi) = \frac{\theta_1(u(x)+u(\xi))\,\theta_1(2u_\infty)}{\theta_1(u_\infty+u(\xi))\,\theta_1(u_\infty+u(x))} \tag{3.64}$$

$$\frac{\partial \mathcal{T}}{\partial T} = \frac{a+b+c+d}{2} - x_0 \tag{3.65}$$

we have the asymptotics [11, 4]:

$$\frac{d(T_c\frac{n}{N}) - a(T_c\frac{n}{N}) - c(T_c\frac{n}{N}) + b(T_c\frac{n}{N})}{4} \leq \gamma_n \leq \frac{d(T_c\frac{n}{N}) - a(T_c\frac{n}{N}) + c(T_c\frac{n}{N}) - b(T_c\frac{n}{N})}{4} \tag{3.66}$$

$$\frac{d(T_c\frac{n}{N}) + a(T_c\frac{n}{N}) - c(T_c\frac{n}{N}) + b(T_c\frac{n}{N})}{2} \leq \beta_n \leq \frac{d(T_c\frac{n}{N}) + a(T_c\frac{n}{N}) + c(T_c\frac{n}{N}) - b(T_c\frac{n}{N})}{2} \tag{3.67}$$

Therefore, we shall now study $W(x,T,V)$ in different regimes.

# 4 The Birth of a cut critical point

Let us choose the potential $V$ and the temperature $T_c$ such that:

- for $T < T_c$ we are in a one-cut case,

$$W(x,T) = \frac{1}{2}\left(V'(x) - M_-(x,T)\sqrt{(x-a)(x-b)}\right) \tag{4.68}$$

  with $a(T) < b(T)$ and

$$M_-(x,T_c) = (x-e)^{2\nu-1}\,Q(x)\,, \tag{4.69}$$

  where $\nu \geq 1$ is an integer, and $Q(x)$ is a real polynomial whose properties are described bellow.

- for $T > T_c$ we are in a two-cuts case,

$$W(x,T) = \frac{1}{2}\left(V'(x) - M_+(x,T)\sqrt{(x-a)(x-b)(x-c)(x-d)}\right) \tag{4.70}$$

  with $a(T) < b(T) < c(T) < d(T)$ and

$$c(T_c) = d(T_c) = e \quad , \quad M_+(x,T_c) = (x-e)^{2\nu-2}\,Q(x)\,, \tag{4.71}$$

- at $T = T_c$ one cut has vanishing size $c(T_c) = d(T_c)$. With no loss of generality, we can assume that:

$$a(T_c) = -2 \qquad , \qquad b(T_c) = 2, \tag{4.72}$$

  and we write:

$$e(T_c) = 2\cosh\phi_e = c(T_c) = d(T_c) \tag{4.73}$$

The polynomial $Q(x)$ must have the following properties:

- $\deg Q = d - 2\nu$ with $d$ odd and $d > 2\nu$,
- The leading coefficient of $Q$ is positive,
- $Q$ has an odd number of zeroes in $]2, e[$,
- $Q(x) < 0$ in $[-2, 2]$,
- $Q(e) > 0$,
- $$\forall x < -2, \quad \int_x^{-2} Q(x)(x-e)^{2\nu-1}\sqrt{x^2-4}\, dx \; > 0 \tag{4.74}$$
- $$\forall x > 2, \; x \neq e, \quad \int_2^{x} Q(x)(x-e)^{2\nu-1}\sqrt{x^2-4}\, dx \; > 0 \tag{4.75}$$
- $$\int_2^{e} Q(x)(x-e)^{2\nu-1}\sqrt{x^2-4}\, dx = 0 \tag{4.76}$$
- $$V'(x) = \mathop{\mathrm{Pol}}_{x\to\infty} \left( (x-e)^{2\nu-1} Q(x) \sqrt{x^2-4} \right) \tag{4.77}$$
- $$T_c = \frac{1}{2} \mathop{\mathrm{Res}}_{\infty} (x-e)^{2\nu-1} Q(x) \sqrt{x^2-4}\, dx \tag{4.78}$$

**Remark:** notice that for all $\nu \geq 1$, it is possible to find a potential $V(x)$ and a temperature $T_c$ with such properties. Indeed, choose $e$ and $Q(x)$ with the above properties and determine $V'(x)$ and $T_c$ by 4.77 and 4.78. Notice also that it is always possible to find a polynomial $Q(x)$ which satisfies the above mentioned conditions, indeed consider any real $e > 2$, and any real polynomial $\tilde{Q}(x)$, of even degree $d - 2\nu - 1$, with positive leading coefficient, and with no real zero, then set:

$$\tilde{e} = \frac{\int_2^e x\, \tilde{Q}(x)\, (x-e)^{2\nu-1} \sqrt{x^2-4}\, dx}{\int_2^e \tilde{Q}(x)\, (x-e)^{2\nu-1} \sqrt{x^2-4}\, dx} \tag{4.79}$$

clearly, $\tilde{e} \in ]2, e[$, and then set:

$$Q(x) = (x - \tilde{e}) \tilde{Q}(x) \tag{4.80}$$

In particular, one may choose $d = 2\nu + 1$ and $\tilde{Q} = 1$.

### 4.1 Example $\nu = 1$

Let $e > 2$ be fixed. We write $e = 2\cosh\phi_e$.

We consider the following quartic potential:

$$V'(x) = \left(x^3 - (e+\tilde{e})x^2 + (e\tilde{e}-2)x + 2(e+\tilde{e})\right) \qquad , \qquad T_c = 1 + e\tilde{e} \tag{4.81}$$

where $\tilde{e}$ is given by $\int_2^e (x-e)(x-\tilde{e})\sqrt{x^2-4} = 0$, i.e. :

$$\tilde{e} = 2\frac{\phi_e \cosh\phi_e - \frac{1}{3}\sinh\phi_e(2+\cosh^2\phi_e)}{\frac{1}{3}\sinh\phi_e\cosh\phi_e(5-2\cosh^2\phi_e) - \phi_e} \tag{4.82}$$

### 4.2 At the critical point $T = T_c$:

At $T = T_c$, both formula 4.68 and 4.70 reduce to:

$$W(x, T_c) = \frac{1}{2}\left(V'(x) - (x-e)^{2\nu-1}Q(x)\sqrt{x^2-4}\right) \tag{4.83}$$

which would correspond to an average large $N$ eigenvalue density in $[-2, 2]$:

$$\rho(x) = \frac{1}{2\pi T_c}(x-e)^{2\nu-1}Q(x)\sqrt{4-x^2} \tag{4.84}$$

and one would have:

$$\gamma_N \sim 1 \qquad , \qquad \beta_N \sim 0 \tag{4.85}$$

However, this is wrong, because the semiclassical asymptotics 3.22 are valid only if $T \neq T_c$, they break down at $T = T_c$. It is the purpose of section 5, to determine the asymptotic behavior of $\gamma_n$ and $\beta_n$ near $n = N$ and $T = T_c$. For the moment, let us consider the limits of 4.68 and 4.70 near $T_c$.

### 4.3 Variations near the critical point, $T < T_c$ (one-cut)

Let us consider the limit of 4.68 near $T_c$. Write $T = T_c + t$, and $t < 0$, and:

$$W(x, T) = \frac{1}{2}\left(V'(x) - M_-(x, T)\sqrt{(x-a)(x-b)}\right) \tag{4.86}$$

At $T = T_c$ we have

$$a(T_c) = -2 \qquad , \qquad b(T_c) = 2 \qquad , \qquad M_-(x, T_c) = (x-e)^{2\nu-1}\,Q(x)\,, \tag{4.87}$$

Then, make use of formula 3.29 and 3.48, i.e.

$$-\frac{1}{2}\frac{\partial M_-(x,T)}{\partial T} + \frac{1}{4}\,M_-(x,T)\,\frac{\frac{\partial a}{\partial T}}{(x-a)} + \frac{1}{4}\,M_-(x,T)\,\frac{\frac{\partial b}{\partial T}}{(x-b)} = \frac{1}{(x-a)(x-b)} \tag{4.88}$$

matching the pole at $x = a$ gives:

$$\frac{\partial a}{\partial T} = \frac{4}{(a-b)\, M_-(a,T)} \sim -\frac{1}{(a-e)^{2\nu-1}Q(a)} \qquad (4.89)$$

which is finite at $T = T_c$, thus, we find that to first order in $t$:

$$a \sim -2 + \frac{t}{(2+e)^{2\nu-1}Q(-2)} \qquad , \qquad b \sim 2 - \frac{t}{(e-2)^{2\nu-1}Q(2)} \qquad (4.90)$$

(notice that $Q(-2) < 0$ and $Q(2) < 0$). Relation 3.48 implies:

$$\gamma \sim 1 + O(t) \qquad (4.91)$$

Then, 4.88 reduces to:

$$\frac{1}{2}\,\frac{\partial M_-(x,T)}{\partial T} = \frac{M_-(x,T) - M_-(a,T)}{M_-(a,T)(a-b)(x-a)} + \frac{M_-(x,T) - M_-(b,T)}{M_-(b,T)(b-a)(x-b)} \qquad (4.92)$$

which is finite at $T = T_c$, thus one gets the asymptotics of $M_-$:

$$\begin{aligned} 2\,\frac{\partial M_-(x,T)}{\partial T} &= -\frac{(x-e)^{2\nu-1}(Q(x)-Q(a)) + ((x-e)^{2\nu-1} - (a-e)^{2\nu-1})Q(a)}{(a-e)^{2\nu-1}\,Q(a)\,(x-a)} \\ &\quad + \frac{(x-e)^{2\nu-1}(Q(x)-Q(b)) + ((x-e)^{2\nu-1} - (b-e)^{2\nu-1})Q(b)}{(b-e)^{2\nu-1}\,Q(b)\,(x-b)} \\ &= -\frac{(x-e)^{2\nu-1}}{(a-e)^{2\nu-1}\,Q(a)}\,\frac{Q(x)-Q(a)}{x-a} - \frac{(x-e)^{2\nu-1} - (a-e)^{2\nu-1}}{(x-a)\,(a-e)^{2\nu-1}} \\ &\quad + \frac{(x-e)^{2\nu-1}}{(b-e)^{2\nu-1}\,Q(b)}\,\frac{Q(x)-Q(b)}{x-b} + \frac{(x-e)^{2\nu-1} - (b-e)^{2\nu-1}}{(x-b)\,(b-e)^{2\nu-1}} \end{aligned} \qquad (4.93)$$

in particular in the vicinity of $x = e$ one has:

$$\begin{aligned} M_-(x,T) &\sim (x-e)^{2\nu-1}Q(x) \\ &\quad + \frac{t}{2}\left[\sum_{k=0}^{2\nu-2}(x-e)^k((2-e)^{-k-1} - (-2-e)^{-k-1}) + O(x-e)^{2\nu-1}\right] \end{aligned} \qquad (4.94)$$

Note that the zeroes of $M_-(x,T)$ in the vicinity of $e$, are the $2\nu - 1^{\text{th}}$ roots of unity:

$$M_-(x, T_c + t) = 0 \quad \leftrightarrow \quad x = e + \left(\frac{2t}{Q(e)(e^2-4)}\right)^{1/2\nu-1} + O(t^{2/2\nu-1}) \qquad (4.95)$$

Using 3.51, we get:

$$\begin{aligned} &\gamma_n \sim 1 - \frac{t}{4}\left(\frac{1}{(e-2)^{2\nu-1}Q(2)} + \frac{1}{(e+2)^{2\nu-1}Q(-2)}\right) \\ &\beta_n \sim -\frac{t}{2}\left(\frac{1}{(e-2)^{2\nu-1}Q(2)} - \frac{1}{(e+2)^{2\nu-1}Q(-2)}\right) \\ &\text{where} \quad n = N(1 + t/T_c) \end{aligned} \qquad (4.96)$$

## 4.4 Variations near the critical point, $T > T_c$ (two cuts)

Let us consider the limit of 4.70 near $T_c$. Write $T = T_c + t$, and $t > 0$,and

$$W(x,T) = \frac{1}{2}\left(V'(x) - M_+(x,T)\sqrt{(x-a)(x-b)(x-c)(x-d)}\right) \tag{4.97}$$

At $T = T_c$ we have

$$\begin{aligned} a(T_c) = -2 \quad &, \quad b(T_c) = 2 \\ c(T_c) = d(T_c) = e = 2\cosh\phi_e \quad &, \quad M_+(x,T_c) = (x-e)^{2\nu-2}\, Q(x)\,, \end{aligned} \tag{4.98}$$

and at $T > T_c$, $a+2$, $b-2$, $c-e$, $d-e$, and $M_+(x) - (x-e)^{2\nu-2}Q(x)$ are small. In particular, we write:

$$M_+(x,T) = H(x,T)Q(x,T) \tag{4.99}$$

where $H(x,T)$ is a monic polynomial of degree $2\nu-2$ which contains all the roots of $M_+$ close to $e$, and $Q(x,T)$ is the remaining part. In other words, $H(x,T)-(x-e)^{2\nu-2}$ is small and $Q(x,T) - Q(x)$ is small in the small $t$ limit.

We use the notations of section 3.5. The biratio 3.52 is thus:

$$m = \frac{(b-a)(d-c)}{(c-a)(d-b)} \sim \frac{4}{e^2-4}\,(d-c) \sim \frac{d-c}{\sinh^2\phi_e} \tag{4.100}$$

we see that we have to consider the limit $m \to 0$. In that limit 3.57 becomes

$$u_\infty = i\int_0^{\sqrt{\frac{d-b}{b-a}}} \frac{dy}{\sqrt{(1+y^2)(1+my^2)}} \sim i\int_0^{\sqrt{\frac{e-2}{4}}} \frac{dy}{\sqrt{1+y^2}} = i\frac{\phi_e}{2} \tag{4.101}$$

$$\begin{aligned} E(u_\infty) - u_\infty &= im\int_0^{\sqrt{\frac{d-b}{b-a}}} \frac{y^2\,dy}{\sqrt{(1+y^2)(1+my^2)}} \\ &\sim im\int_0^{\sqrt{\frac{e-2}{4}}} \frac{y^2\,dy}{\sqrt{(1+y^2)}} = im\,\frac{\sinh\phi_e - \phi_e}{4} \end{aligned} \tag{4.102}$$

And, as can be found in any handbook of classical functions [], we have the small $m$ behavior:

$$\frac{E'(m)}{K'(m)} \sim -\frac{2}{\ln m} \tag{4.103}$$

Since for small $m$ one has $\left|\frac{1}{\ln m}\right| \gg m$, 3.59 becomes:

$$\delta x_0 := x_0 - d \sim \frac{2\phi_e \sinh\phi_e}{\ln m} \tag{4.104}$$

Notice that:

$$x_0 - c = x_0 - d + d - c \sim x_0 - d + m \sinh^2 \phi_e \sim x_0 - d \sim \delta x_0 \tag{4.105}$$

From 3.29 and 3.61 we have:

$$-\frac{1}{2}\frac{\partial M_+(x,T)}{\partial T} - \frac{1}{4}\frac{M_+(x,T)}{\sigma(x,T)}\frac{\partial \sigma(x,T)}{\partial T} = \frac{x - x_0}{\sigma(x,T)} \tag{4.106}$$

Matching the pole at $x = c$ gives:

$$\frac{\partial c}{\partial t} = \frac{-4(d - c + \delta x_0)}{(c-a)(c-b)(c-d)M_+(c, T_c + t)} \sim \frac{-4\delta x_0}{(e^2-4)Q(e)}\,\frac{1}{(c-d)\,H(c)} \tag{4.107}$$

and matching the pole at $x = d$ gives:

$$\frac{\partial d}{\partial t} = \frac{-4\delta x_0}{(d-a)(d-b)(d-c)M_+(d)} \sim -\frac{4\delta x_0}{(e^2-4)Q(e)}\,\frac{1}{(d-c)\,H(d)} \tag{4.108}$$

and matching the poles close to $e$ gives:

$$\frac{\partial H(x)}{\partial t} \sim -\frac{2\delta x_0}{(e^2-4)Q(e)}\,\frac{1}{d-c}\left(\frac{H(x)-H(d)}{(x-d)H(d)} - \frac{H(x)-H(c)}{(x-c)H(c)}\right) \tag{4.109}$$

The following guess solves the 3 equations 4.107, 4.108, 4.109 to small $t$ leading order:

$$c \sim e - 2\zeta\left(-\frac{t}{\ln t}\right)^{\frac{1}{2\nu}} \quad , \quad d \sim e + 2\zeta\left(-\frac{t}{\ln t}\right)^{\frac{1}{2\nu}} \tag{4.110}$$

$$H(x) \sim \left(-\frac{t}{\ln t}\right)^{-1+\frac{1}{\nu}}\; G\left((x-e)\left(-\frac{t}{\ln t}\right)^{\frac{-1}{2\nu}}\right) \tag{4.111}$$

where $\zeta$ is a positive real number, and $G$ is a degree $2\nu - 2$ even monic polynomial, which will be determined below.

For later convenience, we also define the following positive constant:

$$C := \frac{4\nu^2\phi_e}{\sinh\phi_e\, Q(e)} > 0 \tag{4.112}$$

Using ansatz 4.110 and 4.104, we have in that limit:

$$\delta x_0 \sim \frac{4\nu\phi_e \sinh\phi_e}{\ln t} \tag{4.113}$$

Then, inserting 4.110 and 4.111 into 4.107 and 4.108, we get:

$$4\zeta^2 = \frac{C}{G(-2\zeta)} = \frac{C}{G(2\zeta)} \tag{4.114}$$

Then, setting $x = e+\xi\left(-\frac{t}{\ln t}\right)^{\frac{1}{2\nu}}$, and inserting 4.111 into 4.109 we get the following equation for $G$:

$$(2\nu-2)\,G(\xi) - \xi\,G'(\xi) = \frac{C}{4\zeta}\,\left(\frac{G(\xi)-G(2\zeta)}{(\xi-2\zeta)G(2\zeta)} - \frac{G(\xi)-G(-2\zeta)}{(\xi+2\zeta)G(-2\zeta)}\right) \tag{4.115}$$

which using 4.114 becomes:

$$(2\nu-2)\,G(\xi) - \xi\,G'(\xi) = \frac{4\zeta^2}{\xi^2-4\zeta^2}\,\left(G(\xi)-G(2\zeta)\right) \tag{4.116}$$

the solution of which is:

$$G(\xi) = \sum_{k=0}^{\nu-1} \frac{2k!}{k!k!}\,\zeta^{2k}\,\xi^{2(\nu-1-k)} = \mathrm{Pol}\,\frac{\xi^{2\nu-1}}{\sqrt{\xi^2-4\zeta^2}} \tag{4.117}$$

or:

$$G(2\zeta\cosh\psi) = \zeta^{2\nu-2}\sum_{j=0}^{\nu-1}\begin{pmatrix}2\nu-1\\ \nu+j\end{pmatrix}\frac{\sinh{(2j+1)\psi}}{\sinh\psi} \tag{4.118}$$

In particular,

$$\frac{G(2\zeta)}{\zeta^{2\nu-2}} = \frac{1}{2}\,\frac{2\nu!}{\nu-1!\nu!} = \frac{C}{4\zeta^{2\nu}} \tag{4.119}$$

i.e. the parameter $\zeta$ is determined by:

$$\zeta = \left(\frac{C}{2}\,\frac{\nu!\,\nu-1!}{2\nu!}\right)^{\frac{1}{2\nu}} = \left(\frac{2\nu^2\phi_e}{\sinh\phi_e\,Q(e)}\,\frac{\nu!\,\nu-1!}{2\nu!}\right)^{\frac{1}{2\nu}} \tag{4.120}$$

In that scaling regime, we have:

$$m \sim \frac{4\zeta}{\sinh^2\phi_e}\,\left(-\frac{t}{\ln t}\right)^{\frac{1}{2\nu}} \tag{4.121}$$

i.e. this corresponds to a torus of modulus

$$\tau = i\,\frac{K'}{K} \sim \frac{-i}{\pi}\ln m \sim \frac{-i}{2\nu\pi}\ln t \tag{4.122}$$

and, using 3.63:

$$\begin{aligned}\gamma &= \frac{i}{4K}\sqrt{(d-b)(c-a)}\,\mathrm{e}^{-\pi\frac{u_\infty^2}{KK'}}\,\frac{\theta_1'(0,\tau)}{\theta_1(u_\infty/K,\tau)} \\ &\sim \mathrm{e}^{-\pi\frac{u_\infty^2}{KK'}} \\ &\sim 1-\frac{\phi_e^2}{\ln m} \\ &\sim 1-\frac{2\nu\phi_e^2}{\ln t}\end{aligned} \tag{4.123}$$

we also find that the filling fraction in the $[c,d]$ cut is of order:

$$\begin{aligned} \epsilon &= \frac{1}{2\pi T_c}\int_c^d \rho(x)\,\mathrm{d}x \\ &\sim -\frac{t}{\ln t}\,\frac{\sinh\phi_e Q(e)}{\pi T_c}\int_{-2\zeta}^{2\zeta} G(\xi)\sqrt{4\zeta^2-\xi^2}\,\mathrm{d}\xi \\ &\sim -i\frac{t}{\ln t}\,\frac{\sinh\phi_e Q(e)}{\pi T_c}\int_{-2\zeta}^{2\zeta} \xi^{2\nu-1}\,\frac{G(\xi)\sqrt{4\zeta^2-\xi^2}}{\xi^{2\nu-1}}\,\mathrm{d}\xi \end{aligned} \tag{4.124}$$

then, integrating by parts and using 4.116, we find:

$$\begin{aligned} \epsilon &\sim i\frac{t}{T_c\,\ln t}\,\frac{\sinh\phi_e Q(e)}{2\nu\pi}\int_{-2\zeta}^{2\zeta} \xi^{2\nu}\,\frac{4\zeta^2\,\xi^{-2\nu}}{\sqrt{\xi^2-4\zeta^2}}\,G(2\zeta)\,\mathrm{d}\xi \\ &\sim i\frac{t}{T_c\,\ln t}\,\frac{\sinh\phi_e Q(e)}{2\nu\pi}\,4\zeta^2\,G(2\zeta)\int_{-2\zeta}^{2\zeta}\frac{\mathrm{d}\xi}{\sqrt{\xi^2-4\zeta^2}} \\ &\sim -\frac{t}{T_c\,\ln t}\,\frac{\sinh\phi_e Q(e)}{2\nu}\,C \end{aligned} \tag{4.125}$$

i.e.

$$\epsilon(T_c+t) \sim -\frac{t}{T_c\,\ln t}\;2\nu\phi_e \tag{4.126}$$

This means that for $n=N(1+t/T_c)$, the average number of eigenvalues located near $e$ is:

$$k \sim \frac{n-N}{\ln N}\;2\nu\phi_e \tag{4.127}$$

i.e. the eigenvalues start to explore the potential well near $e$ when $n-N\sim\ln N$.

According to 3.66 and 3.67, the coefficients $\gamma_n$ and $\beta_n$ vary between:

$$1+\zeta\left(\frac{-t}{\ln t}\right)^{\frac{1}{2\nu}} \leq \gamma_n \leq \cosh\phi_e \tag{4.128}$$

$$2\zeta\left(\frac{-t}{\ln t}\right)^{\frac{1}{2\nu}} \leq \beta_n \leq e-2 \tag{4.129}$$

$$\text{where}\quad n=N(1+t/T_c)$$

The transition takes place on a scale of order $p\sim\ln N$.

## 4.5 Order of the transition

From 3.34 and 4.91 we have below $T_c$, i.e. for $t<0$:

$$\frac{\partial^2 F}{\partial t^2}(T_c+t) = -2\ln\gamma \sim \frac{t}{2}\left(\frac{1}{(e-2)^{2\nu-1}Q(2)}+\frac{1}{(e+2)^{2\nu-1}Q(-2)}\right) \tag{4.130}$$

and above $T_c$, i.e. for $t > 0$, we have from 4.123:

$$\frac{\partial^2 F}{\partial t^2}(T_c + t) = -2\ln\gamma \sim \frac{4\nu\phi_e^2}{\ln t} \tag{4.131}$$

The second derivative of the free energy is continuous, but the third derivative is not. therefore we have a third order transition, with logarithmic divergency, cf fig.3.

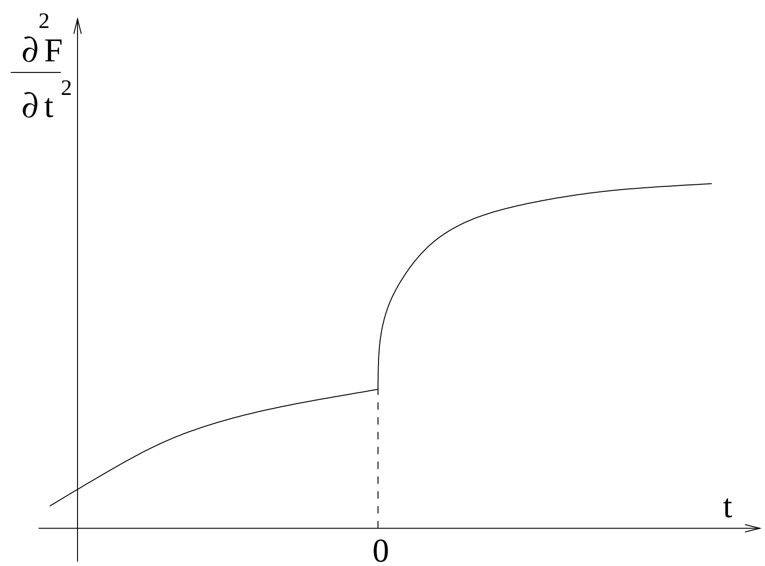


Figure 3: Behavior of the second derivative of the classical free energy with respect to $t$ near the transition.

# 5 Mean field asymptotics for the partition function

We compute the partition function for a potential:

$$V_{r_0}(x) = V(x) + r_0 \ln(x_0 - x) \tag{5.132}$$

where $r_0$ is assumed of order $1/N$:

$$r_0 = -a\frac{T_c}{N} \tag{5.133}$$

## 5.1 Mean Field theory

We use the same idea as in [4]. We split the 2-cuts integral into 1-cut integrals. Let us say that there are $k$ eigenvalues in the new cut (near $e$), and $n-k$ in the old cut $[a, b]$:

$$n!\, Z_n(\frac{n}{N}T_c, V_{r_0})$$

$$\sim \sum_{k=0}^{n} \binom{n}{k} \int_{x_i>\tilde{e}} \mathrm{d}x_1 \ldots \mathrm{d}x_k\, \Delta^2(x_i) \prod_{i=1}^{k} (x_0 - x_i)^a\, \mathrm{e}^{-\frac{N}{T_c} V(x_i)}$$
$$(n-k)!\, \overline{Z}_{n-k}(T_c \frac{n-k}{N}, \mathcal{V}_{r_0,r_j}) \tag{5.134}$$

where

$$\mathcal{V}_{r_j}(x) = \sum_{j=0}^{k} r_j \ln{(x_j - x)} \quad , \quad r_0 = -\frac{aT_c}{N} \quad , \quad r_1 = \ldots = r_k = -\frac{2T_c}{N} \tag{5.135}$$

and $\overline{Z}_{n-k}(T_c \frac{n-k}{N}, \mathcal{V}_{r_j})$is a one-cut integral:

$$\begin{aligned} \overline{Z}_{n-k}(T_c \frac{n-k}{N}, \mathcal{V}_{r_j}) &:= \frac{1}{n-k!} \int_{<\tilde{e}} \mathrm{d}x_{k+1} \ldots \mathrm{d}x_n\, \Delta^2(x_i) \prod_{i=k+1}^{n} \mathrm{e}^{-\frac{N}{T_c} \mathcal{V}_{r_j}(x_i)} \\ &= H_{n-k}\, \mathrm{e}^{-\frac{N^2}{T_c^2} \overline{F}_{n-k}(T_c \frac{n-k}{N}, \mathcal{V}_{r_j})} \end{aligned} \tag{5.136}$$

That gives:

$$\begin{aligned} & Z_n(\frac{n}{N} T_c, V_{r_0}) \\ \sim\ & \sum_{k=0}^{n} \frac{H_{n-k}}{k!} \int_{x_i>\tilde{e}} \mathrm{d}x_1 \ldots \mathrm{d}x_k\, \Delta^2(x_i) \prod_{i=1}^{k} (x_0 - x_i)^a\, \mathrm{e}^{-\frac{N}{T_c} V(x_i)}\ \mathrm{e}^{-\frac{N^2}{T_c^2} \overline{F}_{n-k}(T_c \frac{n-k}{N}, \mathcal{V}_{r_j})} \end{aligned} \tag{5.137}$$

In other words, we integrate out $n-k$ eigenvalues, and consider the integral over $k$ eigenvalues only. The $k$ remaining eigenvalues are submitted to the potential $V$, as well as their mutual Coulomb repulsion, and the mean field of the exterior $n-k$ eigenvalues.

Since $\overline{F}_n(T, \mathcal{V}_{r_j})$ corresponds to a one-cut distribution, it can be evaluated with standard semiclassical technique (see section 3), and in particular, it has a large $n$ expansion:

$$\overline{F}_n(T, \mathcal{V}_{r_j}) \sim \overline{F}(T, \mathcal{V}_{r_j}) + \frac{T}{n} \overline{F}^{(1/2)}(T, \mathcal{V}_{r_j}) + \frac{T^2}{n^2} \overline{F}^{(1)}(T, \mathcal{V}_{r_j}) + O(\frac{1}{n^3}) \tag{5.138}$$

and each term of the expansion is analytical in $\mathcal{V}_{r_j}$.

We want to evaluate $\overline{F}(T, \mathcal{V}_{r_j})$ in a regime where $T - T_c$ is "small" (we make that more precise below) and $r_0$ and the $r_j$'s are of order $O(1/N)$.

we first do a Taylor expansion in $T - T_c$ and the $r_j$'s:

$$\begin{aligned} \overline{F}(T, \mathcal{V}_{r_j}) \ \sim\ & \overline{F}(T_c, V) + \sum_{i=0}^{k} r_i \frac{\partial \overline{F}}{\partial r_i} + \frac{1}{2} \sum_{i,j} r_i r_j \frac{\partial^2 \overline{F}}{\partial r_i \partial r_j} + o(\frac{k^2}{N^2}) \\ & + \sum_j (T - T_c) r_j \frac{\partial^2 \overline{F}}{\partial T \partial r_j} \end{aligned}$$

$$+(T-T_c)\frac{\partial \overline{F}}{\partial T}+\frac{1}{2}(T-T_c)^2\frac{\partial^2 \overline{F}}{\partial T^2}+o(\frac{1}{N^2}) \tag{5.139}$$

where all derivatives are computed at $T = T_c$ and $r_i = 0$ (see section 3.2 and 3.3). That expansion is valid only if $k \ll N$.

Thus we have:

$$\begin{aligned}\frac{1}{H_n}\, Z_n(\frac{n}{N}T_c, V_{r_0}) \quad \sim \quad & \mathrm{e}^{-\frac{N^2}{T_c^2}\overline{F}(T_c,V)}\mathrm{e}^{-\overline{F}^{(1)}(T_c,V)}\mathrm{e}^{a\,\frac{N}{T_c}\overline{F}_{r_0}}\mathrm{e}^{-\frac{a^2}{2}\overline{F}_{r_0,r_0}} \\ & \sum_{k=0}^{n}\frac{1}{k!}\frac{H_{n-k}}{H_n}\,\mathrm{e}^{-\frac{N}{T_c}(n-N-k)\overline{F}_T}\mathrm{e}^{a(n-N-k)\overline{F}_{T,r_0}}\mathrm{e}^{-\frac{(n-N-k)^2}{2}\overline{F}_{T,T}} \\ & \int_{x_i>\tilde{e}}\mathrm{d}x_1\ldots\mathrm{d}x_k\;\Delta^2(x_i)\prod_i(x_0-x_i)^a\,\mathrm{e}^{-\frac{N}{T_c}V(x_i)} \\ & \prod_{j=1}^{k}\mathrm{e}^{2\frac{N}{T_c}\overline{F}_{r_j}}\,\mathrm{e}^{2(n-N-k)\overline{F}_{T,r_j}}\,\mathrm{e}^{-2a\overline{F}_{r_0,r_j}}\prod_{j,l\geq 1}\mathrm{e}^{-2\overline{F}_{r_l,r_j}}\end{aligned} \tag{5.140}$$

Notice that:

$$\frac{H_{n-k}}{H_n}\sim(2\pi)^{-k}(1-\frac{k}{n})^{-1/12}(1+O(1/n)) \tag{5.141}$$

## 5.2 Computation of the derivatives

Now, use formula given in section 3.2 and 3.3 in the one-cut case, and get (derivatives taken at $T = T_c$ and $r_i = 0$, and take into account that $V_{\text{eff}}(b) = V_{\text{eff}}(e)$):

$$\frac{\partial}{\partial r_i}\overline{F}=-\frac{1}{2}\left(V_{\text{eff}}(x_i)-V(x_i)\right)\qquad,\qquad\frac{\partial}{\partial T}\overline{F}=V_{\text{eff}}(e) \tag{5.142}$$

$$\frac{\partial^2}{\partial r_i\partial r_j}\overline{F}=\ln\frac{x_i-x_j}{\Lambda(x_i)-\Lambda(x_j)}\qquad,\qquad\frac{\partial^2}{\partial r_i^2}\overline{F}=-\ln\Lambda'(x_i) \tag{5.143}$$

$$\frac{\partial^2}{\partial T\partial r_i}\overline{F}=\ln\Lambda(x_i)\qquad,\qquad\frac{\partial^2}{\partial T^2}\overline{F}=0 \tag{5.144}$$

Moreover:

$$\frac{\partial\mathcal{T}(T,V)}{\partial T}=\frac{a+b}{2}=0\qquad,\qquad\frac{\partial\mathcal{T}(T,V)}{\partial r_j}=\frac{1}{\Lambda(x_j)} \tag{5.145}$$

The effective potential behaves in the vicinity of $e$ as:

$$V_{\text{eff}}(x)\underset{x\to e}{\sim}V_{\text{eff}}(e)+\frac{V_{\text{eff}}^{(2\nu)}(e)}{2\nu!}\,(x-e)^{2\nu} \tag{5.146}$$

$$\frac{V_{\text{eff}}^{(2\nu)}(e)}{2\nu!}=\frac{2\sinh\phi_e\,Q(e)}{2\nu} \tag{5.147}$$

In the limit where the $x_i$'s are close to $e$, we have:

$$\frac{x_i - x_j}{\Lambda(x_i) - \Lambda(x_j)} \mathop{\sim}_{x_i, x_j \to e} \frac{1}{\Lambda'(x_i)} \sim 2\sinh\phi_e\, \mathrm{e}^{-\phi_e} \tag{5.148}$$

and

$$\Lambda(x_i) \mathop{\sim}_{x_i \to e} \mathrm{e}^{\phi_e} \tag{5.149}$$

## 5.3 Result

write $n = N + p$:

$$\begin{aligned} Z_{N+p}(\frac{N+p}{N}T_c, V_{r_0}) \quad \sim \quad & \mathrm{e}^{-\frac{N^2}{T_c^2}\overline{F}(T_c,V)} \mathrm{e}^{-\overline{F}^{(1)}(T_c,V)} \\ & \left(\frac{\mathrm{e}^{\phi_e}}{2\sinh\phi_e}\right)^{\frac{a^2}{2}} \mathrm{e}^{-\frac{aN}{2T_c}(V_{\mathrm{eff}}(x_0)-V(x_0))} \\ & \sum_{k=0}^{N+p} \frac{H_{N+p-k}}{k!}\, \mathrm{e}^{-\frac{N}{T_c}(p-k)V_{\mathrm{eff}}(e)}\, \mathrm{e}^{a(p-k)\phi_e} \\ & \int_{x_i > \tilde{e}} \mathrm{d}x_1 \ldots \mathrm{d}x_k\ \Delta^2(x_i) \prod_{i=1}^{k} (x_0 - x_i)^a\, \mathrm{e}^{-\frac{N}{T_c}V(x_i)} \\ & \prod_{i=1}^{k} \mathrm{e}^{-\frac{N}{T_c}(V_{\mathrm{eff}}(x_i)-V(x_i))}\, \mathrm{e}^{2k(p-k)\phi_e} \left(\frac{\mathrm{e}^{\phi_e}}{2\sinh\phi_e}\right)^{2ak+2k^2} \\ \sim \quad & H_N\, \mathrm{e}^{-\frac{N^2}{T_c^2}\overline{F}(T_c,V)} \mathrm{e}^{-\overline{F}^{(1)}(T_c,V)}\, \mathrm{e}^{-\frac{aN}{2T_c}(V_{\mathrm{eff}}(x_0)-V(x_0))}\, \mathrm{e}^{-p\frac{N}{T_c}V_{\mathrm{eff}}(e)} \\ & \sum_{k=0}^{N+p} \frac{(2\pi)^{p-k}}{k!}\, \mathrm{e}^{(2k+a)(p-k)\phi_e} \left(\frac{\mathrm{e}^{\phi_e}}{2\sinh\phi_e}\right)^{2(k+\frac{a}{2})^2} \\ & \int_{x_i > \tilde{e}} \mathrm{d}x_1 \ldots \mathrm{d}x_k\ \Delta^2(x_i) \prod_{i=1}^{k} (x_0 - x_i)^a\, \mathrm{e}^{-\frac{N}{T_c}(V_{\mathrm{eff}}(x_i)-V_{\mathrm{eff}}(e))} \end{aligned} \tag{5.150}$$

Then we rescale:

$$x_i = e + N^{\frac{-1}{2\nu}} \left(\frac{2\sinh\phi_e\, Q(e)}{T_c}\right)^{\frac{-1}{2\nu}} y_i\ ,\ \ x_0 = e + N^{\frac{-1}{2\nu}} \left(\frac{2\sinh\phi_e\, Q(e)}{T_c}\right)^{\frac{-1}{2\nu}} y \tag{5.151}$$

and we get:

$$\begin{aligned} & Z_{N+p}(\frac{N+p}{N}T_c, V_{r_0})\, \mathrm{e}^{-\frac{aN}{2T_c}V(x_0)} \\ \sim\ & H_N\, \mathrm{e}^{-\frac{N^2}{T_c^2}\overline{F}(T_c,V)} \mathrm{e}^{-\overline{F}^{(1)}(T_c,V)}\, \mathrm{e}^{-(p+a/2)\frac{N}{T_c}V_{\mathrm{eff}}(e)}\, \mathrm{e}^{-a\frac{y^{2\nu}}{4\nu}} \\ & N^{\frac{a^2}{8\nu}}\, (2\sinh\phi_e)^{\frac{-a^2}{2}}\, (2\pi)^p \\ & \sum_{k=0}^{N+p} N^{-\frac{(k+a/2)^2}{2\nu}}\, \mathrm{e}^{(k+a/2)2p\phi_e}\, \mathrm{e}^{(k+a/2)a\phi_e}\, A^{-(k^2+ak)}\, (2\pi)^{-k} \end{aligned}$$

$$\frac{1}{k!}\int \mathrm{d}y_1\ldots \mathrm{d}y_k\ \Delta^2(y_i)\prod_{i=1}^k (y-y_i)^a\, \mathrm{e}^{-\frac{y_i^{2\nu}}{2\nu}}$$
$$(1+O(N^{-\frac{1}{2\nu}})) \qquad (5.152)$$

where we have defined:

$$A := (2\sinh\phi_e)^2\left(\frac{2\sinh\phi_e\, Q(e)}{T_c}\right)^{\frac{1}{2\nu}} \qquad (5.153)$$

Eq. (5.152) is valid only up to $O(N^{-\frac{1}{2\nu}})$ because Eq. (5.146), Eq. (5.148) and Eq. (5.149) are valid only to that order in the regime of Eq. (5.151).

## 5.4 The effective matrix model

Let us define the matrix model in the potential $\frac{y^{2\nu}}{2\nu}$.

Let $\zeta_{k,\nu}$ be the partition function of the $k\times k$ matrix model in the potential $\frac{y^{2\nu}}{2\nu}$:

$$\zeta_{k,\nu} := \frac{1}{k!}\int \mathrm{d}x_1\ldots \mathrm{d}x_k\ \Delta^2(x_i)\ \prod_i \mathrm{e}^{-\frac{x_i^{2\nu}}{2\nu}} \qquad (5.154)$$

Notice that for $\nu=1$, this is the gaussian matrix model, and we have:

$$\ln\zeta_{k,1} = \frac{k}{2}\ln 2\pi + \ln(\prod_{j=0}^{k-1} j!) = \ln H_k + \frac{k^2}{2}\ln k - \frac{3}{4}k^2 \qquad (5.155)$$

We define the amplitude:

$$A_k := A^{-k^2}\,(2\pi)^{-k}\,\zeta_{k,\nu} \qquad (5.156)$$

We also introduce:

$$h_{k,\nu} := \frac{\zeta_{k+1,\nu}}{\zeta_{k,\nu}} = 2\pi\, A^{2k+1}\,\frac{A_{k+1}}{A_k} \qquad (5.157)$$

$$\gamma_{k,\nu} := \sqrt{\frac{h_{k,\nu}}{h_{k-1,\nu}}} = A\,\sqrt{\frac{A_{k+1}A_{k-1}}{A_k^2}} \qquad (5.158)$$

and the associated orthogonal polynomials

$$P_k(y) := \frac{\int dy_1\ldots dy_k \Delta(y_i)^2\ \prod_{i=1}^k (y-y_i)\, \mathrm{e}^{-\frac{y_i^{2\nu}}{2\nu}}}{\int dy_1\ldots dy_k \Delta(y_i)^2\ \prod_{i=1}^k\ \mathrm{e}^{-\frac{y_i^{2\nu}}{2\nu}}} \qquad (5.159)$$

$$\psi_{k,\nu}(y) := \sqrt{\frac{\zeta_{k,\nu}}{\zeta_{k+1,\nu}}}\, P_k(y)\, \mathrm{e}^{-\frac{y^{2\nu}}{4\nu}} \qquad (5.160)$$

and their Hilbert transforms:

$$\hat{P}_{k-1}(y) := \frac{\int dx_1\ldots dx_k \Delta(x_i)^2\ \prod_{i=1}^k \frac{1}{y-x_i}\, \mathrm{e}^{-\frac{x_i^{2\nu}}{2\nu}}}{\int dx_1\ldots dx_k \Delta(x_i)^2\ \prod_{i=1}^k\ \mathrm{e}^{-\frac{x_i^{2\nu}}{2\nu}}} \qquad (5.161)$$

$$\phi_{k,\nu}(y) := \sqrt{h_{k,\nu}}\,\hat{P}_k(y)\,\mathrm{e}^{\frac{y^{2\nu}}{4\nu}} \qquad (5.162)$$

## 5.5 Partition function

Thus we have for $a = 0$:

$$Z_{N+p}(\frac{N+p}{N}T_c, V)$$

$$\sim \qquad H_N \,\mathrm{e}^{-\frac{N^2}{T_c^2}\overline{F}(T_c,V)}\mathrm{e}^{-\overline{F}^{(1)}(T_c,V)}\,\mathrm{e}^{-p\frac{N}{T_c}V_{\mathrm{eff}}(e)}\,(2\pi)^p\sum_{k=0}^{N+p} N^{-\frac{k^2}{2\nu}}\,\mathrm{e}^{2kp\phi_e}A_k(1+O(N^{-\frac{1}{2\nu}}))$$

(5.163)

$$h_{N+p} \sim 2\pi\,\mathrm{e}^{-\frac{N}{T_c}V_{\mathrm{eff}}(e)}\,\frac{\sum_{k=0}^{N+p+1} N^{-\frac{k^2}{2\nu}}\,\mathrm{e}^{2kp\phi_e}\,\mathrm{e}^{2k\phi_e}\,A_k}{\sum_{k=0}^{N+p} N^{-\frac{k^2}{2\nu}}\,\mathrm{e}^{2kp\phi_e}\,A_k}(1+O(N^{-\frac{1}{2\nu}})) \tag{5.164}$$

$$\gamma_{N+p}^2 \;\sim\; \frac{\left(\sum_{k=0}^{N+p+1} N^{-\frac{k^2}{2\nu}}\,\mathrm{e}^{2kp\phi_e}\,\mathrm{e}^{2k\phi_e}\,A_k\right)\left(\sum_{k=0}^{N+p-1} N^{-\frac{k^2}{2\nu}}\,\mathrm{e}^{2kp\phi_e}\,\mathrm{e}^{-2k\phi_e}\,A_k\right)}{\left(\sum_{k=0}^{N+p} N^{-\frac{k^2}{2\nu}}\,\mathrm{e}^{2kp\phi_e}\,A_k\right)^2}$$
$$(1+O(N^{-\frac{1}{2\nu}})) \tag{5.165}$$

## 5.6 Orthogonal polynomial

According to Heine's formula (cf Eq. (2.12)), we have:

$$\psi_n = \frac{Z_n(T_c\frac{n}{N}, V(x)-\frac{T_c}{N}\ln(\xi-x))}{\sqrt{Z_n(T_c\frac{n}{N},V)\,Z_{n+1}(T_c\frac{n+1}{N},V)}}\,\mathrm{e}^{-\frac{N}{2T}V(\xi)}$$

$$\phi_{n-1}(\xi) = \frac{Z_n(T_c\frac{n}{N}, V(x)+\frac{T_c}{N}\ln(\xi-x))}{\sqrt{Z_n(T_c\frac{n}{N},V)\,Z_{n-1}(T_c\frac{n-1}{N},V)}}\,\mathrm{e}^{\frac{N}{2T}V(\xi)}$$

$$\psi_{N+p}(x_0) = \frac{Z_{N+p}(\frac{N+p}{N}T_c, V_{r_0})\,\mathrm{e}^{-\frac{N}{2T}V(x_0)}}{\sqrt{Z_{N+p}(\frac{N+p}{N}T_c,V)\,Z_{N+p+1}(\frac{N+p+1}{N}T_c,V)}}$$
$$r_0 = -\frac{T_c}{N} \qquad , \qquad a=1 \tag{5.166}$$

thus, in the regime:

$$x_0 = e + N^{-\frac{1}{2\nu}}\,\frac{4\sinh^2\phi_e}{A}\,y \tag{5.167}$$

using Eq. (5.152) with $a = 1$ we get:

$$\psi_{N+p}(x_0) \;\sim\; N^{\frac{1}{8\nu}}\sqrt{\frac{A}{2\sinh\phi_e}}$$
$$\frac{\sum_{k=0}^{N+p} N^{-\frac{(k+1/2)^2}{2\nu}}\,\mathrm{e}^{(k+1/2)2p\phi_e}\,\mathrm{e}^{(k+1/2)\phi_e}\,\sqrt{A_kA_{k+1}}\;\psi_{k,\nu}(y)}{\sqrt{(\sum_{k=0}^{N+p+1} N^{-\frac{k^2}{2\nu}}\,\mathrm{e}^{2kp\phi_e}\,\mathrm{e}^{2k\phi_e}\,A_k)(\sum_{k=0}^{N+p} N^{-\frac{k^2}{2\nu}}\,\mathrm{e}^{2kp\phi_e}\,A_k)}}$$
$$(1+O(N^{-\frac{1}{2\nu}})) \tag{5.168}$$

## 5.7 Hilbert transforms

According to Eq. (2.15), we have:

$$\phi_{N+p-1}(x_0) = \frac{Z_{N+p}(\frac{N+p}{N}T_c, V_{r_0})\, \mathrm{e}^{\frac{N}{2T}V(x_0)}}{\sqrt{Z_{N+p-1}(\frac{N+p-1}{N}T_c, V)\, Z_{N+p}(\frac{N+p}{N}T_c, V)}}$$
$$r_0 = \frac{T_c}{N} \quad , \quad a = -1 \tag{5.169}$$

thus, in the regime:

$$x_0 = e + N^{-\frac{1}{2\nu}}\, \frac{4\sinh^2\phi_e}{A}\, y \tag{5.170}$$

using Eq. (5.152) with $a = -1$ we get:

$$\phi_{N+p-1}(x_0) \sim N^{\frac{1}{8\nu}} \sqrt{\frac{A}{2\sinh\phi_e}} \frac{\sum_{k=0}^{N+p} N^{-\frac{(k-1/2)^2}{2\nu}}\, \mathrm{e}^{(k-1/2)2p\phi_e}\, \mathrm{e}^{-(k-1/2)\phi_e}\, \sqrt{A_{k-1}A_k}\, \phi_{k-1,\nu}(y)}{\sqrt{(\sum_{k=0}^{N+p} N^{-\frac{k^2}{2\nu}}\, \mathrm{e}^{2kp\phi_e}\, A_k)(\sum_{k=0}^{N+p-1} N^{-\frac{k^2}{2\nu}}\, \mathrm{e}^{2kp\phi_e}\, \mathrm{e}^{-2k\phi_e}\, A_k)}} (1 + O(N^{-\frac{1}{2\nu}})) \tag{5.171}$$

Notice that shifting $k \to k+1$ we have:

$$\phi_{N+p-1}(x_0) \sim N^{\frac{1}{8\nu}} \sqrt{\frac{A}{2\sinh\phi_e}} \frac{\sum_{k=-1}^{N+p-1} N^{-\frac{(k+1/2)^2}{2\nu}}\, \mathrm{e}^{(k+1/2)2p\phi_e}\, \mathrm{e}^{-(k+1/2)\phi_e}\, \sqrt{A_{k+1}A_k}\, \phi_{k,\nu}(y)}{\sqrt{(\sum_{k=0}^{N+p} N^{-\frac{k^2}{2\nu}}\, \mathrm{e}^{2kp\phi_e}\, A_k)(\sum_{k=0}^{N+p-1} N^{-\frac{k^2}{2\nu}}\, \mathrm{e}^{2kp\phi_e}\, \mathrm{e}^{-2k\phi_e}\, A_k)}} \tag{5.172}$$

## 5.8 Computation of $\beta_{N+p}$

We start from Eq. (2.11)

$$\frac{N}{T_c}\, \mathcal{T}_n(\frac{n}{N}T_c, V) \sim \frac{1}{n!\, Z_n(\frac{n}{N}T_c, V)} \int_{x_i > \tilde{e}} \mathrm{d}x_1 \ldots \mathrm{d}x_k\, \mathrm{e}^{-\frac{N}{T_c}V(x_i)} \Delta^2(x_i) (\sum_{j=1}^{k} x_j + \frac{N}{T_c}\, \mathcal{T}_{n-k}(T_c\frac{n-k}{N}, \mathcal{V}))\, \mathrm{e}^{-\frac{N^2}{T_c^2}\overline{F}_{n-k}(T_c\frac{n-k}{N}, \mathcal{V})} \tag{5.173}$$

We have:

$$\mathcal{T}_{n-k}(T, \mathcal{V}) \sim \mathcal{T}(T_c, V) + (T - T_c)\frac{\partial \mathcal{T}}{\partial T}(T_c, V) + \sum_j r_j \frac{\partial \mathcal{T}}{\partial r_j}(T_c, V) + O(1/N^2)$$

$$\sim \quad -\frac{2T_c}{N}\sum_j \Lambda(x_j)^{-1} + O(1/N^2) \tag{5.174}$$

Therefore:

$$\begin{aligned}
\frac{N}{T_c}\,\mathcal{T}_n(\frac{n}{N}T_c,V) &\sim \frac{1}{Z_n(\frac{n}{N}T_c,V)}\sum_{k=0}^{n}\frac{H_{n-k}}{k!}\int_{x_i>\tilde{e}} dx_1\ldots dx_k\ e^{-\frac{N}{T_c}V(x_i)}\Delta^2(x_i) \\
&\quad (\sum_{j=1}^{k} x_j - 2\Lambda_j^{-1})\ e^{-\frac{N^2}{T_c^2}\overline{F}_{n-k}(T_c\frac{n-k}{N},\mathcal{V})} \\
&\sim \frac{2\sinh\phi_e}{Z_n(\frac{n}{N}T_c,V)}\sum_{k=0}^{n}\frac{H_{n-k}}{k!}\,k\int_{x_i>\tilde{e}} dx_1\ldots dx_k\ e^{-\frac{N}{T_c}V(x_i)}\Delta^2(x_i) \\
&\quad e^{-\frac{N^2}{T_c^2}\overline{F}_{n-k}(T_c\frac{n-k}{N},\mathcal{V})} \\
&\sim 2\sinh\phi_e\,\frac{\sum_k k\,e^{2pk\phi_e}\,N^{-\frac{k^2}{2\nu}}\,A_k}{\sum_k e^{2pk\phi_e}\,N^{-\frac{k^2}{2\nu}}\,A_k}
\end{aligned} \tag{5.175}$$

and:

$$\beta_{N+p} \sim 2\sinh\phi_e\left(\frac{\sum_k k\,e^{2(p+1)k\phi_e}\,N^{-\frac{k^2}{2\nu}}\,A_k}{\sum_k e^{2(p+1)k\phi_e}\,N^{-\frac{k^2}{2\nu}}\,A_k} - \frac{\sum_k k\,e^{2pk\phi_e}\,N^{-\frac{k^2}{2\nu}}\,A_k}{\sum_k e^{2pk\phi_e}\,N^{-\frac{k^2}{2\nu}}\,A_k}\right) \tag{5.176}$$

# 6 Asymptotic regimes

Consider $p$ of order $\ln N$:

$$p = \frac{u}{2\nu\phi_e}\ln N \qquad , \qquad u \text{ finite.} \tag{6.177}$$

## 6.1 Possible asymptotic regimes for the partition function

Then Eq. (5.163) becomes:

$$\boxed{Z_{N+p}(\frac{N+p}{N}T_c) \sim H_N\,e^{-\frac{N^2}{T_c^2}\overline{F}(T_c,V)}e^{-\overline{F}^{(1)}(T_c,V)}\,e^{-p\frac{N}{T_c}V_{\rm eff}(e)}\,(2\pi)^p\sum_{k=0}^{N+p}N^{\frac{2ku-k^2}{2\nu}}\,A_k} \tag{6.178}$$

It is clear that the sum over $k$ is dominated by the values of $k$ for which the exponent of $N$ is maximal, i.e. for which $2uk - k^2$ is maximal. This means that for $u < 0$, the sum is dominated by the vicinity of $k = 0$, and for $u \geq 0$, the sum is dominated by the vicinity of $k = u$. The sum is then well approximated by a few largest terms.

Let us denote $\overline{u}$ the positive integer closest to $u$:

$$\begin{cases} \overline{u} := [u + 1/2] & \text{if } u \geq 0 \\ \overline{u} := 0 & \text{if } u \leq 0 \end{cases} \tag{6.179}$$

where [.] denotes the integer part.

Define also:

$$\begin{array}{ll} \epsilon_u := \mathrm{sgn}(u-\overline{u}) & \text{if } u>0 \\ \epsilon_u := 1 & \text{if } u\leq 0 \end{array} \tag{6.180}$$

We always have:

$$\epsilon_u(u-\overline{u}) \leq \frac{1}{2} \tag{6.181}$$

The largest value of $2uk-k^2$ is obtained for $k=\overline{u}$, and the second largest value is obtained for $k=\overline{u}+\epsilon_u$. The difference is:

$$(2u\overline{u}-\overline{u}^2)-(2u(\overline{u}+\epsilon_u)-(\overline{u}+\epsilon_u)^2) = 1-2\epsilon_u(u-\overline{u}) \tag{6.182}$$

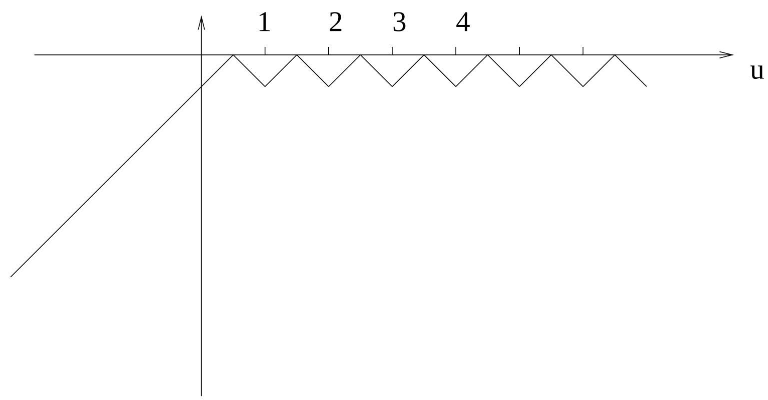


Figure 4: Behavior of $\epsilon_u(u-\overline{u})-\frac{1}{2}$.

Remember that our asymptotics for $Z_n$ are valid only up to order $O(N^{-1/2\nu}$, i.e. we want to have:

$$\epsilon_u(u-\overline{u})-\frac{1}{2} \geq -\frac{1}{2} \tag{6.183}$$

which implies that our asymptotics for $Z_n$ are valid only if $u>0$ and $u\notin \mathbf{N}$.

## 6.2 Possible asymptotic regimes for the orthogonal polynomials

We have similar considerations for the asymptotics of orthogonal polynomials, excpet that the sum over $k$ is now shifted by 1/2. This gives different regimes.

Eq. (5.168) becomes in that regime

$$\psi_{N+p}(x_0) \quad \sim \quad N^{\frac{1}{8\nu}}\sqrt{\frac{A}{2\sinh\phi_e}}$$

$$\frac{\sum_{k=0}^{N+p} N^{\frac{2(k+1/2)u-(k+1/2)^2}{2\nu}}\, e^{(k+1/2)\phi_e}\, \sqrt{A_k A_{k+1}}\ \psi_{k,\nu}(y)}{\sqrt{(\sum_{k=0}^{N+p+1} N^{-\frac{k^2}{2\nu}}\, e^{2kp\phi_e}\, e^{2k\phi_e}\, A_k)(\sum_{k=0}^{N+p} N^{-\frac{k^2}{2\nu}}\, e^{2kp\phi_e}\, A_k)}}$$
$$(1+O(N^{-\frac{1}{2\nu}})) \tag{6.184}$$

The sum in the numerator is dominated by the largest values of

$$2(k+1/2)u-(k+1/2)^2, \tag{6.185}$$

i.e. by $k=[u]=\overline{u}+\frac{\epsilon_u-1}{2}$.

The second largest term is obtained for $k=\overline{u}-\frac{\epsilon_u+1}{2}$. Notice that if $u\geq\frac{1}{2}$, the two largest terms are always $\overline{u}$ and $\overline{u}-1$. The difference

$$(2(\overline{u}+1/2)u-(\overline{u}+1/2)^2)-(2(\overline{u}-1/2)u-(\overline{u}-1/2)^2)=2(u-\overline{u}) \tag{6.186}$$

Since our asymptotics are valid only up to order $O(N^{\frac{-1}{2\nu}})$, the subleading term should be discarded if $|u-\overline{u}|\geq\frac{1}{2}$, i.e. if $u\leq\frac{1}{2}$ or if $u$ is half-integer.

This implies that our asymptotics for $\psi_n$ are valid only if $u>\frac{1}{2}$ and $u\notin\mathbf{N}+\frac{1}{2}$.

## 6.3 Asymptotics in the regime $u>0$, and $u$ not integer or half-integer

From now on, we write:

$$n=N+p \qquad , \qquad p=\frac{u}{2\nu\phi_e}\ln N. \tag{6.187}$$

In this section, we assume that $u>0$ and $u$ not integer or half integer. The sum over $k$ in Eq. (6.178) is dominated by the terms $k=\overline{u}$ and $k=\overline{u}+\epsilon_u$.

### 6.3.1 coefficient $\gamma_n$

Thus we have:

$$Z_{N+p}(\frac{N+p}{N}T_c) \sim H_N\, e^{-\frac{N^2}{T_c^2}\overline{F}(T_c,V)} e^{-\overline{F}^{(1)}(T_c,V)}\quad (2\pi)^p\, N^p\, e^{-\frac{Np}{T_c}V_{\rm eff}(e)}$$
$$N^{\frac{2u\overline{u}-\overline{u}^2}{2\nu}}\left(A_{\overline{u}}+N^{\frac{|u-\overline{u}|-\frac{1}{2}}{\nu}}\, A_{\overline{u}+\epsilon_u}+O(N^{-\frac{1}{2\nu}})\right) \tag{6.188}$$

We also obtain:

$$h_{N+p} \sim 2\pi\, e^{-\frac{N}{T_c}V_{\rm eff}(e)}\, e^{2\overline{u}\phi_e}\left(1+2\epsilon_u\, e^{\epsilon_u\phi_e}\sinh\phi_e\, N^{\frac{|u-\overline{u}|-\frac{1}{2}}{\nu}}\,\frac{A_{\overline{u}+\epsilon_u}}{A_{\overline{u}}}+O(N^{-\frac{1}{2\nu}})\right) \tag{6.189}$$

and:

$$\gamma_{N+p} \quad \sim \quad 1 + 2\,\sinh^2\phi_e\, N^{\frac{|u-\overline{u}|-\frac12}{\nu}}\, \frac{A_{\overline{u}+\epsilon_u}}{A_{\overline{u}}} + O(N^{-\frac{1}{2\nu}}) \tag{6.190}$$

$\gamma_{N+p}$ is nearly periodic, with period $\frac{\ln N}{2\nu\phi_e}$. The amplitude is minimal of order $N^{-\frac{1}{2\nu}}$ for $u$ integer, and is maximal of order 1 for $u$ half-integer, cf fig. 6.3.1.

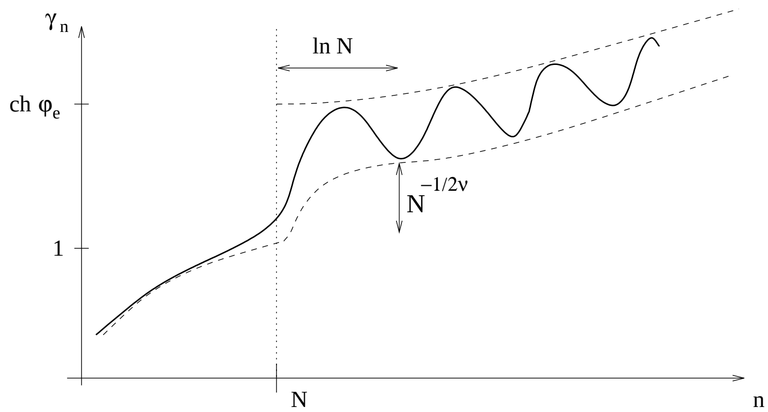


Figure 5: Behavior of $\gamma_n$.

### 6.3.2 coefficient $\beta_n$

The sum in Eq. (5.176) is dominated by $k=\overline{u}$ and $k=\overline{u}+\epsilon_u$, i.e.

$$\beta_{N+p} \sim 4\sinh^2\phi_e\;\; N^{\frac{2|u-\overline{u}|-1}{2\nu}}\, e^{\epsilon_u\phi_e}\, \frac{A_{(\overline{u}+\epsilon_u)}}{A_{\overline{u}}} \tag{6.191}$$

### 6.3.3 Orthogonal polynomials

The two largest terms in the numerator of Eq. (5.168) correspond to $k=\overline{u}$ and $k=\overline{u}-1$ (not necessarily in this order), thus:

$$\psi_{N+p}(x_0) \;\sim\; \sqrt{\frac{A}{2\sinh\phi_e}}\; \frac{N^{\frac{u-\overline{u}}{2\nu}}\, e^{\phi_e/2}\sqrt{\frac{A_{\overline{u}+1}}{A_{\overline{u}}}}\;\psi_{\overline{u},\nu}(y) + N^{\frac{\overline{u}-u}{2\nu}}\, e^{-\phi_e/2}\sqrt{\frac{A_{\overline{u}-1}}{A_{\overline{u}}}}\;\psi_{\overline{u}-1,\nu}(y)}{1+\cosh\phi_e\, N^{\frac{|u-\overline{u}|-\frac12}{\nu}}\, e^{\epsilon_u\phi_e}\,\frac{A_{\overline{u}+\epsilon_u}}{A_{\overline{u}}}}\, (1+O(N^{-\frac{1}{2\nu}})) \tag{6.192}$$

we also have:

$$\psi_{N+p-1}(x_0) \;\sim\; \sqrt{\frac{A}{2\sinh\phi_e}}$$

$$\frac{N^{\frac{u-\overline{u}}{2\nu}}\,\mathrm{e}^{-\phi_e/2}\,\sqrt{\frac{A_{\overline{u}+1}}{A_{\overline{u}}}}\;\psi_{\overline{u},\nu}(y)+N^{\frac{\overline{u}-u}{2\nu}}\,\mathrm{e}^{\phi_e/2}\,\sqrt{\frac{A_{\overline{u}-1}}{A_{\overline{u}}}}\;\psi_{\overline{u}-1,\nu}(y)}{1+\cosh\phi_e\,N^{\frac{|u-\overline{u}|-\frac{1}{2}}{\nu}}\,\mathrm{e}^{-\epsilon_u\phi_e}\,\frac{A_{\overline{u}+\epsilon_u}}{A_{\overline{u}}}}$$
$$(1+O(N^{-\frac{1}{2\nu}})) \tag{6.193}$$

### 6.3.4 Hilbert transforms

Similarly, using Eq. (5.172), we find that the Hilbert transform of the orthogonal polynomial $\pi_n$, are asymptoticaly given by:

$$\phi_{N+p-1}(x_0) \sim \sqrt{\frac{A}{2\sinh\phi_e}}\,\frac{N^{\frac{u-\overline{u}}{2\nu}}\,\mathrm{e}^{-\phi_e/2}\,\sqrt{\frac{A_{\overline{u}+1}}{A_{\overline{u}}}}\,\phi_{\overline{u},\nu}(y)+N^{-\frac{u-\overline{u}}{2\nu}}\,\mathrm{e}^{\phi_e/2}\,\sqrt{\frac{A_{\overline{u}-1}}{A_{\overline{u}}}}\,\phi_{\overline{u}-1,\nu}(y)}{1+\cosh\phi_e\,N^{\frac{2|u-\overline{u}|-1}{2\nu}}\,\mathrm{e}^{-\epsilon_u\phi_e}\,\frac{A_{\overline{u}+\epsilon_u}}{A_{\overline{u}}}} \tag{6.194}$$

and

$$\phi_{N+p}(x_0) \sim \sqrt{\frac{A}{2\sinh\phi_e}}\,\frac{N^{\frac{u-\overline{u}}{2\nu}}\,\mathrm{e}^{\phi_e/2}\,\sqrt{\frac{A_{\overline{u}+1}}{A_{\overline{u}}}}\,\phi_{\overline{u},\nu}(y)+N^{-\frac{u-\overline{u}}{2\nu}}\,\mathrm{e}^{-\phi_e/2}\,\sqrt{\frac{A_{\overline{u}-1}}{A_{\overline{u}}}}\,\phi_{\overline{u}-1,\nu}(y)}{1+\cosh\phi_e\,N^{\frac{2|u-\overline{u}|-1}{2\nu}}\,\mathrm{e}^{\epsilon_u\phi_e}\,\frac{A_{\overline{u}+\epsilon_u}}{A_{\overline{u}}}} \tag{6.195}$$

### 6.3.5 Matrix form

The matrix:

$$\Psi_n(x)=\begin{pmatrix}\psi_{n-1}(x) & \phi_{n-1}(x)\\ \psi_n(x) & \phi_n(x)\end{pmatrix} \tag{6.196}$$

is in that regime ($u\geq 0$):

$$\boxed{\Psi_n(x)\sim\sqrt{\frac{A}{2\sinh\phi_e}}\;L^{-1}\begin{pmatrix}\mathrm{e}^{\frac{1}{2}\phi_e} & \mathrm{e}^{-\frac{1}{2}\phi_e}\\ \mathrm{e}^{-\frac{1}{2}\phi_e} & \mathrm{e}^{\frac{1}{2}\phi_e}\end{pmatrix}R\begin{pmatrix}\psi_{\overline{u}-1,\nu}(y) & \phi_{\overline{u}-1,\nu}(y)\\ \psi_{\overline{u},\nu}(y) & \phi_{\overline{u},\nu}(y)\end{pmatrix}} \tag{6.197}$$

where

$$R=\mathrm{diag}\left(N^{-\frac{u-\overline{u}}{2\nu}}\sqrt{\frac{A_{\overline{u}-1}}{A_{\overline{u}}}},N^{\frac{u-\overline{u}}{2\nu}}\sqrt{\frac{A_{\overline{u}+1}}{A_{\overline{u}}}}\right) \tag{6.198}$$

$$L=1+\cosh\phi_e\,N^{\frac{2|u-\overline{u}|-1}{2\nu}}\,\frac{A_{\overline{u}+\epsilon_u}}{A_{\overline{u}}}\begin{pmatrix}\mathrm{e}^{-\epsilon_u\phi_e} & 0\\ 0 & \mathrm{e}^{\epsilon_u\phi_e}\end{pmatrix} \tag{6.199}$$

#### 6.3.6 Kernel

The kernel $K_n(x,x')$ is given by the Christioffel–Darboux formula:

$$K_n(x,x') = \gamma_n \, \frac{\psi_n(x)\psi_{n-1}(x') - \psi_n(x')\psi_{n-1}(x)}{x-x'} \tag{6.200}$$

We find:

$$K_n(x,x') \quad \sim \quad \frac{N^{\frac{1}{2\nu}}\, A}{4\sinh^2\phi_e}\, \gamma_{\overline{u},\nu} \, \frac{\psi_{\overline{u},\nu}(y)\psi_{\overline{u}-1,\nu}(y') - \psi_{\overline{u}-1,\nu}(y)\psi_{\overline{u},\nu}(y')}{(y-y')} \tag{6.201}$$

i.e.

$$K_n(x,x') \sim K_{\overline{u},\nu}(y,y')\, \frac{dy}{dx} \tag{6.202}$$

#### 6.3.7 Large $u$ limit

Using Stirling's formula (cf appendix H), we find that for large $u$, all those asymptotics match with the classical limit of section 4.4.

# 7 Conclusion

We have computed the asymptotics of orthogonal polynomials in the birth of a cut critical limit. This corresponds to the appearance of a new connected component for the support of the eigenvalue density, away from other cuts.

We have found some universal behaviour, which depends only on the degree $\nu$ of vanishing of the density at the new cut, and $2\cosh\phi_e$ which parametrizes the distance between the new cut and the old cut. The parametrix near the new cut, is simply the system corresponding to a model matrix model in the potential $x^{2\nu}$, and with $\overline{u} = [\frac{1}{2} + 2\nu\phi_e \frac{n-N}{\ln N}]$ eigenvalues.

This new universal behaviour does not seem to correspond to a conformal field theory (unlike previously known critical behaviours), because the exponent of $N$ is not constant.

It would be interesting to complete the "physicist's proof" presented here, with a mathematical one, for instance using Riemann-Hilbert metods as in [2, 11].

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# Appendix H

thechapterStirling formula and other asymptotics

Stirling formula:

$$n! \sim n^n e^{-n}\sqrt{2\pi n}(1 + \frac{1}{12n} + \ldots) \tag{H.1}$$

from which we deduce:

$$\ln(1\ldots(n-1)!) \sim \frac{1}{2}n^2\ln n - \frac{3}{4}n^2 + \frac{n}{2}\ln 2\pi - \frac{1}{12}\ln n + O(1) \tag{H.2}$$

and

$$\ln H_n \sim n\ln 2\pi - \frac{\ln n}{12} + \ldots \tag{H.3}$$

Asymptotics of $\zeta_{k,\nu}$. For large $k$ we have:

$$\ln\zeta_{k,\nu} \sim \frac{k^2}{2\nu}\ln k - \frac{3}{4\nu}k^2 + \frac{k}{\nu}\ln k + O(k) \tag{H.4}$$

# Appendix I

thechapterElliptical functions

We introduce a few definitions about elliptical functions [24]: The elliptical sine function $\mathrm{sn}(u,m)$ is defined by the following identity:

$$u = \int_0^{\mathrm{sn}(u,m)} \frac{dy}{\sqrt{(1-y^2)(1-my^2)}} \tag{I.1}$$

The complete integrals are defined by:

$$K(m) := \int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-my^2)}} \tag{I.2}$$

$$K'(m) := K(1-m) = \int_0^\infty \frac{dy}{\sqrt{(1+y^2)(1+my^2)}} \tag{I.3}$$

$$E(u,m) := \int_0^{\mathrm{sn}(u,m)} \sqrt{\frac{1-my^2}{1-y^2}}\,\mathrm{d}y \tag{I.4}$$

$$E(m) := \int_0^1 \sqrt{\frac{1-my^2}{1-y^2}}\,\mathrm{d}y \quad , \quad E'(m) := E(1-m) \tag{I.5}$$

When $m \to 0$ one has:

$$\begin{aligned} K &\sim \frac{\pi}{2}\left(1 + \frac{m}{4} + \frac{9m^2}{64} + \ldots + O(m^3)\right) \\ K' &\sim \ln\frac{1}{\sqrt{m}}\left(1 + \frac{m}{4} + \frac{9m^2}{64} + \ldots + O(m^3)\right) \end{aligned}$$

$$
\begin{aligned}
E &\sim \frac{\pi}{2}\left(1-\frac{m}{4}-\frac{3m^2}{64}+\ldots+O(m^3)\right) \\
E' &\sim 1-\frac{m}{2}\ln\frac{1}{\sqrt{m}}+O(m^2)
\end{aligned}
$$

$(I.6)$

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