Universal distribution of random matrix eigenvalues near the 'birth of a cut' transition

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Abstract. We study the eigenvalue distribution of a random matrix, at a transition where a new connected component of the eigenvalue density support appears away from other connected components. Unlike previously studied critical points, which correspond to rational singularities $\rho(x) \sim x^{p/q}$ classified by conformal minimal models and integrable hierarchies, this transition shows logarithmic and non-analytical behaviours. There is no critical exponent; instead, the power of N changes in a sawtooth behaviour.

Keywords: conformal field theory, matrix models

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Contents

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1. Introduction

Random matrix models [20, 13] have been studied in relationship with many areas of physics and mathematics. The reason for their success in most of their applications is their 'universality' property, i.e. the fact that the eigenvalue statistical distribution of a large random matrix depends only on the symmetries of the matrix ensemble, and not on the detailed Boltzmann weight (characterized by a potential). Although this universality property has been much studied for generic potentials, some universality should also hold for critical potentials. Different kinds of critical potentials have been studied, and their universality classes have been found to be in correspondence with non-linear integrable hierarchies (KdV, MKdV, KP, ...) [2, 3, 7, 21, 6, 12], and with the (p,q) rational minimal models of conformal field theory [8]. They correspond to rational singularities of the equilibrium density:

$$\rho(x) \sim (x - a)^{p/q}. \tag{1.1}$$

In the Hermitian 1-matrix model, we have only $q = 2$ and p arbitrary; thus we get only half-integer singularities (hyperelliptical curves), which are related to the KdV hierarchies, whereas a 2-matrix model allows us to have any rational singularity (p,q) [8]. The specific heat near such a rational singularity obeys a Gelfand–Dikii-type equation (Painlevé I equation for $(p,q) = (3,2)$). A well known case is the edge of the spectrum where $(p,q) = (1,2)$, which gives the Tracy–Widom law [23], and which is governed by the Painlevé II equation. Another well known case is the merging of two cuts (Bleher and Its [2, 1]), where $(p,q) = (2,1)$, which is also governed by a Painlevé II equation as well (indeed, (p,q) and (q,p) are known to be dual to each other [14]).

Here, we shall study a kind of critical point which has been mostly disregarded (because usual methods do not apply to it): 'the birth of a cut critical point'[1].

Such a critical point is characterized by the fact that when a parameter of the model (let us call it temperature) is varied, a new connected component appears in the support of the large N average eigenvalue density. When the temperature T is just above the critical temperature T_c, the number of eigenvalues in the newborn connected component is small (see figures 1 and 2), and thus, many usual large n methods do not work in that case.

Our goal is to study the eigenvalue statistics in the vicinity of the critical point, and find its universality class.

For this purpose, we shall start from the partition function, treat the eigenvalues in the other cuts with mean field approximations, and reduce the problem to an effective partition function for eigenvalues in the newborn cut only, in a method similar to that of [4].

We find that, unlike for rational critical points, the birth of a cut critical point does not correspond to power law behaviours or transcendental differential equations, but exhibits logarithmic behaviours, and discontinuous functions.

The matrix model is associated with a family of orthogonal polynomials, whose zeros lie inside the connected components of the density [20, 22, 10]. Our goal is also to study the asymptotic behaviour of the orthogonal polynomials in the vicinity of the newborn cut.

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[1] The name was suggested by P Bleher who initiated this work.

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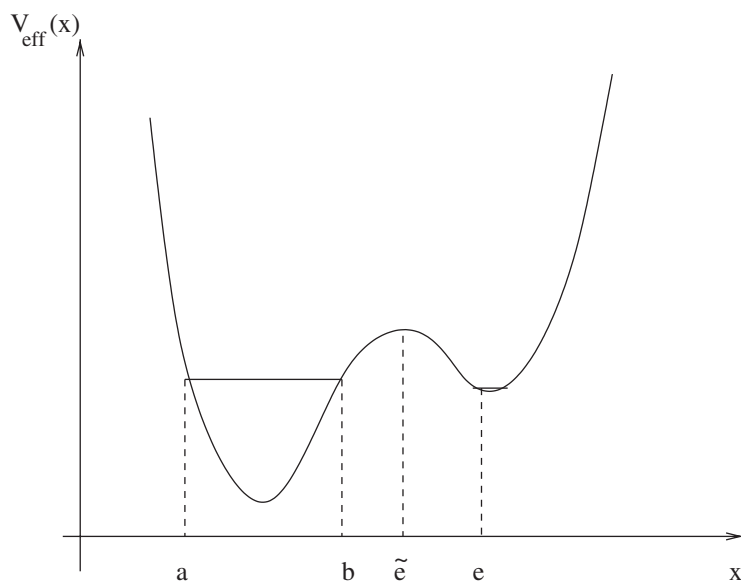
Figure 1. The 'birth of a cut' critical potential is such that one of the potential wells of the effective potential is just at the Fermi level. At a νth order critical point, the effective potential behaves like $V_{\text{eff}}(e) + (1/2\nu!)V_{\text{eff}}^{(2\nu)}(e)\,(x-e)^{2\nu} + \cdots$.
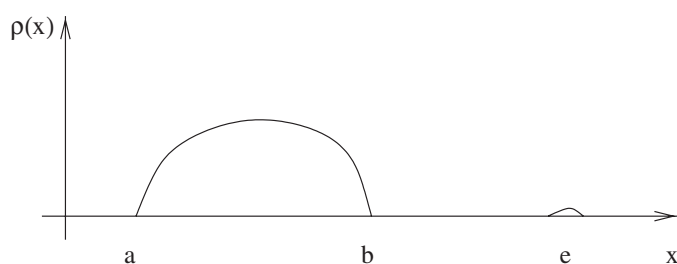
Figure 2. The 'birth of a cut' density of eigenvalues is such that one of the connected components of the support contains a very small number of eigenvalues ($\ll n$).

Outline of the article:

- In section 2 we introduce definitions and notation for orthogonal polynomials and associated quantities.

- In section 3 we recall classical results of random matrix theory: the semiclassical behaviours of free energy, density, correlation functions, orthogonal polynomials, valid away from critical points.

- In section 4 we study the analytical continuation of the previous semiclassical approximations, near the 'birth of a cut' critical point (divergencies at critical point).

- In section 5 we compute the partition function with mean field theory for eigenvalues not in the newborn cut, and derive an effective partition function for the newborn cut eigenvalues.

- In section 6 we use the results of section 5 to deduce the asymptotic behaviours of correlation functions and orthogonal polynomials in the vicinity of the critical point.
- Section 7 is the conclusion.
- In the appendix, we recall Stirling's formula and its consequences, and we recall some elliptical function basics.

2. Setting

Given an integer n (we will later consider the limit $n \to \infty$), a real polynomial $V(x)$ called the potential:

$$V(x) := g_0 + \sum_{k=1}^{d+1} \frac{g_k}{k} x^k, \qquad \deg V = d+1 \tag{2.1}$$

of even degree and positive leading coefficient ($g_{d+1} > 0$), and a temperature $T > 0$, we define the partition function:

$$Z_n(T, V) := \frac{1}{n!} \int_{\mathbf{R}^n} \mathrm{d}x_1 \dots \mathrm{d}x_n \ (\Delta(x_i))^2 \ \prod_{i=1}^{n} \mathrm{e}^{-(n/T)V(x_i)} \tag{2.2}$$

(where $\Delta(x_i) = \prod_{i<j}(x_j - x_i)$ is the Vandermonde determinant) and the free energy $F_n(T, V)$:

$$\mathrm{e}^{-(n^2/T^2)F_n(T,V)} := \frac{Z_n(T, V)}{H_n} \tag{2.3}$$

where H_n is a combinatorial normalization:

$$H_n := (2\pi)^{n/2} \, n^{-n^2/2} \, \mathrm{e}^{3/4n^2} \prod_{k=0}^{n-1} k! = 2^{n/2} \, \pi^{n^2/2} \, n^{-n^2/2} \, \frac{\mathrm{e}^{3/4n^2}}{n! U_n} \tag{2.4}$$

and U_n is the volume of the group $U(n)/U(1)^n$.

Then we define the resolvent:

$$W_n(x, T, V) := \frac{T}{n! \, Z_n(T, V)} \int \mathrm{d}x_1 \cdots \mathrm{d}x_n \, \frac{1}{x - x_1} \, \Delta^2(x_i) \, \prod_{i=1}^{n} \mathrm{e}^{-(n/T)V(x_i)} \tag{2.5}$$

and its first moment:

$$\mathcal{T}_n(T, V) := \frac{T}{n! \, Z_n(T, V)} \int \mathrm{d}x_1 \cdots \mathrm{d}x_n \, x_1 \, \Delta^2(x_i) \, \prod_{i=1}^{n} \mathrm{e}^{-(n/T)V(x_i)}$$

$$= \frac{1}{2\mathrm{i}\pi} \oint x W_n(x, T, V) \, \mathrm{d}x = \frac{\partial F_n(T, V)}{\partial g_1}. \tag{2.6}$$

Notice that more generally for $k > 0$:

$$k \frac{\partial F_n(T, V)}{\partial g_k} = \frac{1}{2\mathrm{i}\pi} \oint x^k W_n(x, T, V) \, \mathrm{d}x. \tag{2.7}$$

Then, given a temperature T_c and an integer N, we define[2]:

$$h_n := \frac{Z_{n+1}((T_c(n+1)/N), V)}{Z_n(T_c(n/N), V)}, \tag{2.8}$$

and

$$\gamma_n := \sqrt{\frac{h_n}{h_{n-1}}} = \sqrt{\frac{Z_{n+1}(T_c((n+1)/N), V)Z_{n-1}(T_c((n-1)/N), V)}{Z_n^2(T_c(n/N), V)}}, \tag{2.9}$$

and

$$\beta_n := \frac{N}{T_c}\left(\mathcal{T}_{n+1}\left(T_c\frac{n+1}{N}, V\right) - \mathcal{T}_n\left(T_c\frac{n}{N}, V\right)\right) = -\frac{T_c}{N}\frac{\partial \ln h_n}{\partial g_1}. \tag{2.10}$$

Notice that $Z_n(T, V)$, h_n and γ_n are strictly positive for all n, in particular they do not vanish.

We also introduce the functions [22, 20, 10]:

$$\pi_n(\xi) = \frac{Z_n(T_c(n/N), V(x) - (T_c/N)\ln(\xi - x))}{Z_n(T_c(n/N), V)}, \qquad \psi_n(\xi) = \frac{\pi_n(\xi)}{\sqrt{h_n}}\,e^{-(N/2T)\,V(\xi)} \tag{2.11}$$

which form an orthogonal family of monic polynomials ($\deg \pi_n = n$):

$$\int \pi_n(\xi)\pi_m(\xi)e^{-(N/T_c)V(\xi)}\,\mathrm{d}\xi = h_n\delta_{nm}, \tag{2.12}$$

and which satisfy the three-term recursion relation:

$$\xi\,\pi_n(\xi) = \pi_{n+1}(\xi) + \beta_n\,\pi_n(\xi) + \gamma_n^2\,\pi_{n-1}(\xi). \tag{2.13}$$

And we introduce the functions:

$$\hat{\pi}_n(\xi) = \frac{Z_{n+1}(T_c((n+1)/N), V(x) + (T_c/N)\ln(\xi - x))}{Z_n(T_c(n/N), V)}, \tag{2.14}$$

$$\phi_n(\xi) = \frac{\hat{\pi}_n(\xi)}{\sqrt{h_n}}\,e^{(N/2T)\,V(\xi)}$$

which are the Hilbert transforms of the $\pi_n(x)$:

$$\hat{\pi}_n(x) = \int \frac{\mathrm{d}x'}{x - x'}\pi_n(x')e^{-V(x')}, \tag{2.15}$$

and which satisfy the same three-term recursion relation, with an initial term:

$$\xi\,\hat{\pi}_n(\xi) = \hat{\pi}_{n+1}(\xi) + \beta_n\,\hat{\pi}_n(\xi) + \gamma_n^2\,\hat{\pi}_{n-1}(\xi) + \delta_{n,0}h_0. \tag{2.16}$$

We also define the kernel:

$$K_n(\xi, y) := \sum_{j=0}^{n-1}\frac{1}{h_j}\,\pi_j(\xi)\pi_j(y)\,e^{-(N/2T_c)(V(\xi)+V(y))}, \tag{2.17}$$

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[2] We introduce two integers n and N, because for instance we will be interested in Z_{N-1}, Z_N and Z_{N+1}, and it is more convenient to compute Z_n for arbitrary values of n not necessarily linked to N. n is the degree of the orthogonal polynomial under consideration, and N is the weight in the exponential measure of orthogonality $e^{-NV(x)}$.

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and it is well known that all correlation functions can be expressed in terms of that kernel (Dyson's theorem [15, 20]):

$$\rho_n(x) = \frac{1}{n} K_n(x, x)$$

$$\rho_n(x, y) = \frac{1}{n^2} \left(K_n(x, x) K_n(y, y) - K_n(x, y) K_n(y, x) \right) \tag{2.18}$$

$$\cdots$$

and that we have the Christoffel–Darboux theorem:

$$K_n(x, y) = \frac{1}{h_{n-1}} \frac{\pi_n(x) \pi_{n-1}(y) - \pi_{n-1}(x) \pi_n(y)}{x - y} \, \mathrm{e}^{-(N/2T_c)(V(x) + V(y))}. \tag{2.19}$$

Our goal is to study γ_n, β_n, π_n, $\hat{\pi}_n$, in the vicinity of $N \to \infty$, and $|n - N| \ll N$, and with T_c chosen such that we are at a special critical point described below.

For the moment, let us study the large N limits away from the critical point.

3. Classical limits

It is well known that in the semiclassical limit $N \to \infty$, and $|n - N| \ll N$, if $T \neq T_c$, the free energy has a large n limit [11, 16, 4, 19, 13]:

$$F_n(T, V) \longrightarrow F(T, V) + \mathrm{O}(1/n^2) \tag{3.1}$$

and so has the resolvent $W_n(x, T, V)$:

$$W_n(x, T, V) \longrightarrow W(x, T, V) + \mathrm{O}(1/n). \tag{3.2}$$

3.1. The large n resolvent

It is well known that, if the potential is such that $V'(x)$ is a rational fraction, with its poles outside the cuts (that assumption will become clear below), the large n resolvent $W(x, T, V)$ can be written as the solution of an hyperelliptical equation [13, 5]:

$$W(x, T, V) = \tfrac{1}{2} \left(V'(x) - M(x, T, V) \sqrt{\sigma(x, T, V)} \right) \tag{3.3}$$

where σ is a monic even degree $(2s \geq 2)$ polynomial with distinct simple zeros only:

$$\sigma(x) = \prod_{i=1}^{s} (x - a_i)(x - b_i), \qquad \cdots < a_i < b_i < a_{i+1} < \cdots \tag{3.4}$$

whose zeros are called the end points, and $\bigcup_{i=1}^{s} [a_i, b_i]$ is called the support, and M is a rational function with the same poles as V'. If one assumes that s and σ are known, $M(x, T, V)$ is determined by the condition that $W(x, T, V)$ is finite (in the physical sheet) when $x \to \infty$ and when x approaches the poles of $V'(x)$.

The large n limit of the density of eigenvalues is then:

$$\rho(x, T, V) = \frac{1}{2\pi T} M(x, T, V) \sqrt{-\sigma(x, T, V)}, \qquad x \in \bigcup_{i=1}^{s} [a_i, b_i]. \tag{3.5}$$

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We also define the effective potential:

$$V_{\text{eff}}(x,T,V) := V(x) - 2T \ln x - 2 \int_{\infty}^{x} \left(W(x,T,V) - \frac{T}{x} \right) \, \mathrm{d}x. \tag{3.6}$$

Notice that its derivative is $V'(x) - 2W(x,T,V) = M(x,T,V)\sqrt{\sigma(x,T,V)}$.

So far, we have not explained how to determine s and the polynomial σ. If one assumes that s is known, $\sigma(x,T,V)$ is determined by the conditions:

$$\begin{aligned}
W(x,T,V) &\underset{x\to\infty}{\sim} \frac{T}{x} + \mathrm{O}(1/x^2) \\
\text{if } s > 1, &\qquad \forall i = 1,\ldots,s-1, \qquad V_{\text{eff}}(b_i) = V_{\text{eff}}(a_{i+1}).
\end{aligned} \tag{3.7}$$

The large n free energy is then given by:

$$F(T,V) = \frac{1}{4\mathrm{i}\pi} \oint W(x,T,V)V(x) \, \mathrm{d}x + \frac{1}{2} T V_{\text{eff}}(b_s) \tag{3.8}$$

where the integration contour is an anticlockwise circle around ∞.

The number of end points $s = s(T,V)$ (we have $1 \le s \le d$) is determined by the condition that the free energy is minimum (one can prove that $s(T,V) \le (d+1)/2$) [9].

3.2. Derivatives with respect to T

Let us introduce:

$$\Omega(x,T,V) := \frac{\partial W(x,T,V)}{\partial T} = \frac{Q_\Omega(x,T,V)}{\sqrt{\sigma(x,T,V)}} \tag{3.9}$$

where $Q_\Omega(x,T,V)$ is a monic polynomial of degree $s-1$, determined by the conditions:

$$\text{if } s > 1, \qquad \forall i = 1,\ldots,s-1, \qquad \int_{b_i}^{a_{i+1}} \frac{Q_\Omega(x,T,V)}{\sqrt{\sigma(x,T,V)}} \, \mathrm{d}x = 0. \tag{3.10}$$

In algebraic geometry, Ω is called a 'normalized Abelian differential of the third kind' [18, 17].

We introduce the multivalued function $\Lambda(x,T,V)$:

$$\Lambda(x,T,V) := \exp \left(\int_{b_s}^{x} \Omega(x',T,V) \, \mathrm{d}x' \right) \tag{3.11}$$

and

$$\gamma(T,V) := \lim_{x\to\infty} \frac{x}{\Lambda(x,T,V)}. \tag{3.12}$$

Then we have the following derivatives:

$$\frac{\partial F}{\partial T} = V_{\text{eff}}(b_s) \tag{3.13}$$

$$\frac{\partial^2 F}{\partial T^2} = -2 \ln \gamma \tag{3.14}$$

$$\frac{\partial V_{\text{eff}}(x)}{\partial T} = -2\ln\left(\gamma\Lambda(x)\right) \tag{3.15}$$

$$\frac{\partial \mathcal{T}}{\partial T} = \frac{1}{2i\pi} \oint x\,\Omega(x)\,\mathrm{d}x \tag{3.16}$$

where $\mathcal{T} = \operatorname{Res} xW(x)$ was defined in equation (2.6).

Notice that:

$$V_{\text{eff}}(b_s) = \frac{1}{2i\pi} \oint \Omega(x)V(x)\,\mathrm{d}x - 2T\ln\gamma. \tag{3.17}$$

3.3. Poles of the potential

Assume that $V'(x)$ has a simple pole at $x = \xi$, with residue r (it may have other poles too); then we define the function:

$$H(x,\xi,T,V) := \frac{\partial W(x,T,V)}{\partial r} = \frac{1}{2\sqrt{\sigma(x)}}\left(\frac{\sqrt{\sigma(x)} - \sqrt{\sigma(\xi)}}{x-\xi} - Q_H(x,\xi)\right) \tag{3.18}$$

where $Q_H(x,\xi)$ is a monic polynomial in x, of degree $s-1$, determined by the conditions:

$$\text{if } s > 1, \qquad \forall i = 1,\ldots,s-1, \qquad \int_{b_i}^{a_{i+1}} \frac{Q_H(x,\xi,T,V) + \sqrt{\sigma(\xi)}/x - \xi}{\sqrt{\sigma(x,T,V)}}\,\mathrm{d}x = 0. \tag{3.19}$$

We also define its (multivalued) primitive:

$$\ln E(x,\xi) := \int_{\infty}^{x} H(x',\xi)\,\mathrm{d}x'. \tag{3.20}$$

Notice that it is finite near the end points and near $x = \xi$. In algebraic geometry, $(x - \xi)/E(x,\xi)$ is related to the 'prime form' [18, 17].

Then we have:

$$\frac{\partial V_{\text{eff}}(x)}{\partial r} = \ln(x - \xi) - 2\ln E(x,\xi) \tag{3.21}$$

$$\frac{\partial \mathcal{T}}{\partial T} = \frac{1}{2i\pi} \oint x\,H(x,\xi)\,\mathrm{d}x \tag{3.22}$$

$$\frac{\partial F}{\partial r} = \frac{1}{2}\left(V(x) - V_{\text{eff}}(x)\right)|_{x=\xi} \tag{3.23}$$

$$\frac{\partial^2 F}{\partial r\partial T} = \ln(\gamma\Lambda(\xi)) = \ln(\xi - b_s) - 2\ln E(b_s,\xi). \tag{3.24}$$

If $V'(x)$ has simple poles at $x = \xi_1$ with residue r_1 and at $x = \xi_2$ with residue r_2, we have:

$$\frac{\partial^2 F}{\partial r_1 \partial r_2} = \ln E(\xi_1,\xi_2) \tag{3.25}$$

and thus it is clear that $\ln E$ has some symmetry properties: $\ln E(x,y) = \ln E(y,x)$.

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3.4. One-cut case

If $W(x, T, V)$ has one cut $[a(T, V), b(T, V)]$ with $a < b$, we use Joukowski's parametrization:

$$x = \frac{b+a}{2} + \frac{b-a}{2} \cosh \phi, \tag{3.26}$$

i.e.:

$$\sqrt{\sigma(x)} = \frac{b-a}{2} \sinh \phi. \tag{3.27}$$

We have:

$$\Omega(x) = \frac{1}{\sqrt{(x-a)(x-b)}} = \frac{\partial \phi}{\partial x}, \qquad \Lambda(x) = \mathrm{e}^{\phi(x)}, \qquad \gamma = \frac{b-a}{4} \tag{3.28}$$

$$H(x, \xi) = \frac{\partial \phi(x)}{\partial x} \frac{1}{\mathrm{e}^{\phi(x)+\phi(\xi)} - 1}, \qquad E(x, \xi) = 1 - \mathrm{e}^{-(\phi(x)+\phi(\xi))} = \frac{x - \xi}{\Lambda(x) - \Lambda(\xi)} \tag{3.29}$$

$$\frac{\partial \mathcal{T}}{\partial T} = \frac{a+b}{2}, \qquad \frac{\partial \mathcal{T}}{\partial r} = \frac{\gamma}{\Lambda(\xi)}. \tag{3.30}$$

Then, it is well known [13] that we have the large n, N asymptotics (in the regime $n/N =$ finite):

$$\gamma_n \sim \frac{b(T_\mathrm{c}(n/N)) - a(T_\mathrm{c}(n/N))}{4}, \qquad \beta_n \sim \frac{b(T_\mathrm{c}(n/N)) + a(T_\mathrm{c}(n/N))}{2}. \tag{3.31}$$

3.5. Two-cut case

If $W(x, T, V)$ has two cuts, $[a(T, V), b(T, V)] \cup [c(T, V), d(T, V)]$ with $a < b < c < d$. Let m be their biratio:

$$m = \frac{(b-a)(d-c)}{(c-a)(d-b)}. \tag{3.32}$$

We parametrize:

$$x(v) = d - \frac{d-a}{1 + ((b-a)/(d-b))\mathrm{sn}^2(v, m)} \tag{3.33}$$

where sn is the elliptical sine function (see appendix B, or for instance [24]), i.e., by definition:

$$\begin{aligned} v(x) &:= -\frac{\mathrm{i}}{2} \sqrt{(d-b)(c-a)} \int_a^x \frac{\mathrm{d}y}{\sqrt{\sigma(y)}} \\ &= \int_0^{\sqrt{((d-b)/(b-a))((x-a)/(d-x))}} \frac{\mathrm{d}y}{\sqrt{(1-y^2)(1-my^2)}}. \end{aligned} \tag{3.34}$$

We have:

$$\begin{aligned} v(a) &= 0, & v(b) &= K(m), \\ v(c) &= K(m) + \mathrm{i}K'(m), & v(d) &= \mathrm{i}K'(m). \end{aligned} \tag{3.35}$$

We have:

$$\sqrt{\sigma(x)} = -\mathrm{i}(d-a)(b-a)\sqrt{\frac{c-a}{d-b}} \frac{\mathrm{sn}(v,m)\,\mathrm{cn}(v,m)\,\mathrm{dn}(v,m)}{(1+((b-a)/(d-b))\mathrm{sn}^2(v,m))^2}. \tag{3.36}$$

Let us define v_∞ such that:

$$v_\infty := \mathrm{i}\int_0^{\sqrt{(d-b)/(b-a)}} \frac{\mathrm{d}y}{\sqrt{(1+y^2)(1+my^2)}}, \tag{3.37}$$

i.e.:

$$\mathrm{sn}(v_\infty,m) = \mathrm{i}\sqrt{\frac{d-b}{b-a}}, \qquad \mathrm{cn}(v_\infty,m) = \sqrt{\frac{d-a}{b-a}}, \qquad \mathrm{dn}(v_\infty,m) = \sqrt{\frac{(d-a)}{(c-a)}}. \tag{3.38}$$

Then we define x_0:

$$x_0 = d + \mathrm{i}\sqrt{(c-a)(d-b)}\left(E(v_\infty,m) - \left(1 - \frac{E'(m)}{K'(m)}\right)v_\infty\right). \tag{3.39}$$

It satisfies:

$$\int_b^c \frac{x-x_0}{\sqrt{(x-a)(x-b)(x-c)(x-d)}}\,\mathrm{d}x = 0, \tag{3.40}$$

and thus we have:

$$\Omega(x) = \frac{x-x_0}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} \tag{3.41}$$

$$\Lambda(x) = \exp\left(\pi\frac{v(x)v_\infty}{KK'}\right)\frac{\theta_1((v(x)+v_\infty)/2K)}{\theta_1((v(x)-v_\infty)/2K)} \tag{3.42}$$

$$\gamma = \frac{\mathrm{i}}{4K}\sqrt{(d-b)(c-a)}\exp\left(-\pi\frac{v_\infty^2}{KK'}\right)\frac{\theta_1'(0,\tau)}{\theta_1(v_\infty/K,\tau)} \tag{3.43}$$

$$E(x,\xi) = \frac{\theta_1(v(x)+v(\xi))\,\theta_1(2v_\infty)}{\theta_1(v_\infty+v(\xi))\,\theta_1(v_\infty+v(x))} \tag{3.44}$$

$$\frac{\partial \mathcal{T}}{\partial T} = \frac{a+b+c+d}{2} - x_0. \tag{3.45}$$

We have the asymptotics [11, 4]:

$$\frac{d(T_\mathrm{c}(n/N)) - a(T_\mathrm{c}(n/N)) - c(T_\mathrm{c}(n/N)) + b(T_\mathrm{c}(n/N))}{4} \leq \gamma_n$$
$$\leq \frac{d(T_\mathrm{c}(n/N)) - a(T_\mathrm{c}(n/N)) + c(T_\mathrm{c}(n/N)) - b(T_\mathrm{c}(n/N))}{4} \tag{3.46}$$

$$\frac{d(T_\mathrm{c}(n/N)) + a(T_\mathrm{c}(n/N)) - c(T_\mathrm{c}(n/N)) + b(T_\mathrm{c}(n/N))}{2} \leq \beta_n$$
$$\leq \frac{d(T_\mathrm{c}(n/N)) + a(T_\mathrm{c}(n/N)) + c(T_\mathrm{c}(n/N)) - b(T_\mathrm{c}(n/N))}{2}. \tag{3.47}$$

Therefore, we shall now study $W(x,T,V)$ in different regimes.

4. The birth of a cut critical point

Let us choose the potential V and the temperature T_c such that:

- for $T < T_c$ we are in a one-cut case,

$$W(x, T) = \tfrac{1}{2} \left(V'(x) - M_-(x, T)\sqrt{(x-a)(x-b)} \right) \tag{4.1}$$

  with $a(T) < b(T)$ and

$$M_-(x, T_c) = (x - e)^{2\nu - 1} Q(x), \tag{4.2}$$

  where $\nu \geq 1$ is an integer, and $Q(x)$ is a real polynomial whose properties are described below;

- for $T > T_c$ we are in a two-cut case,

$$W(x, T) = \tfrac{1}{2} \left(V'(x) - M_+(x, T)\sqrt{(x-a)(x-b)(x-c)(x-d)} \right) \tag{4.3}$$

  with $a(T) < b(T) < c(T) < d(T)$ and

$$c(T_c) = d(T_c) = e, \qquad M_+(x, T_c) = (x - e)^{2\nu - 2} Q(x), \tag{4.4}$$

- at $T = T_c$ one cut has vanishing size $c(T_c) = d(T_c)$; with no loss of generality, we can assume that:

$$a(T_c) = -2, \qquad b(T_c) = 2, \tag{4.5}$$

  and we write:

$$e(T_c) = 2 \cosh \phi_e = c(T_c) = d(T_c). \tag{4.6}$$

The polynomial $Q(x)$ must have the following properties:

- $\deg Q = d - 2\nu$ with d odd and $d > 2\nu$,
- the leading coefficient of Q is positive,
- Q has an odd number of zeros in $]2, e[$,
- $Q(x) < 0$ in $[-2, 2]$,
- $Q(e) > 0$,
-

$$\forall x < -2, \qquad \int_x^{-2} Q(x)(x-e)^{2\nu-1}\sqrt{x^2-4}\,\mathrm{d}x > 0, \tag{4.7}$$

-

$$\forall x > 2, \qquad x \neq e, \qquad \int_2^x Q(x)(x-e)^{2\nu-1}\sqrt{x^2-4}\,\mathrm{d}x > 0, \tag{4.8}$$

-

$$\int_2^e Q(x)(x-e)^{2\nu-1}\sqrt{x^2-4}\,\mathrm{d}x = 0, \tag{4.9}$$

-

$$V'(x) = \Pol_{x \to \infty} \left((x-e)^{2\nu-1} Q(x) \sqrt{x^2-4} \right), \tag{4.10}$$

-

$$T_c = \tfrac{1}{2} \Res_{\infty} (x-e)^{2\nu-1} Q(x) \sqrt{x^2-4}. \tag{4.11}$$

Remark. Notice that for all $\nu \geq 1$, it is possible to find a potential $V(x)$ and a temperature T_c with such properties. Indeed, choose e and $Q(x)$ with the above properties and determine $V'(x)$ and T_c from (4.10) and (4.11). Notice also that it is always possible to find a polynomial $Q(x)$ which satisfies the above mentioned conditions; indeed consider any real $e > 2$, and any real polynomial $\tilde{Q}(x)$, of even degree $d - 2\nu - 1$, with positive leading coefficient, and with no real zero, then set:

$$\tilde{e} = \frac{\int_2^e x \, \tilde{Q}(x) \, (x-e)^{2\nu-1} \sqrt{x^2-4} \, dx}{\int_2^e \tilde{Q}(x) \, (x-e)^{2\nu-1} \sqrt{x^2-4} \, dx} \tag{4.12}$$

clearly, $\tilde{e} \in \,]2, e[$; then set:

$$Q(x) = (x - \tilde{e})\tilde{Q}(x). \tag{4.13}$$

In particular, one may choose $d = 2\nu + 1$ and $\tilde{Q} = 1$.

4.1. Example $\nu = 1$

Let $e > 2$ be fixed. We write $e = 2 \cosh \phi_e$.

We consider the following quartic potential (see figure 1):

$$V'(x) = \left(x^3 - (e + \tilde{e})x^2 + (e\tilde{e} - 2)x + 2(e + \tilde{e}) \right), \qquad T_c = 1 + e\tilde{e} \tag{4.14}$$

where $\tilde{e}$ is given by $\int_2^e (x-e)(x-\tilde{e})\sqrt{x^2-4} = 0$, i.e.:

$$\tilde{e} = 2 \frac{\phi_e \cosh \phi_e - (1/3) \sinh \phi_e (2 + \cosh^2 \phi_e)}{(1/3) \sinh \phi_e \cosh \phi_e (5 - 2 \cosh^2 \phi_e) - \phi_e}. \tag{4.15}$$

4.2. At the critical point $T = T_c$

At $T = T_c$, both formulae (4.1) and (4.3) reduce to:

$$W(x, T_c) = \tfrac{1}{2} \left(V'(x) - (x-e)^{2\nu-1} Q(x) \sqrt{x^2-4} \right) \tag{4.16}$$

which would correspond to an average large N eigenvalue density in $[-2, 2]$:

$$\rho(x) = \frac{1}{2\pi T_c} (x-e)^{2\nu-1} Q(x) \sqrt{4-x^2} \tag{4.17}$$

and one would have:

$$\gamma_N \sim 1, \qquad \beta_N \sim 0. \tag{4.18}$$

However, this is wrong, because the semiclassical asymptotics (3.2) are valid only if $T \neq T_c$; they break down at $T = T_c$. It is the purpose of section 5, to determine the asymptotic behaviour of γ_n and β_n near $n = N$ and $T = T_c$. For the moment, let us consider the limits of (4.1) and (4.3) near T_c.

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4.3. Variations near the critical point, $T < T_{\mathrm{c}}$ (one cut)

Let us consider the limit of (4.1) near T_{c}. Write $T = T_{\mathrm{c}} + t$, and $t < 0$, and:

$$W(x,T) = \tfrac{1}{2}\left(V'(x) - M_-(x,T)\sqrt{(x-a)(x-b)}\right). \tag{4.19}$$

At $T = T_{\mathrm{c}}$ we have

$$a(T_{\mathrm{c}}) = -2, \qquad b(T_{\mathrm{c}}) = 2, \qquad M_-(x,T_{\mathrm{c}}) = (x-e)^{2\nu-1}Q(x), \tag{4.20}$$

Then, make use of formula (3.9) and (3.28), i.e.:

$$-\frac{1}{2}\frac{\partial M_-(x,T)}{\partial T} + \frac{1}{4}M_-(x,T)\frac{\partial a/\partial T}{(x-a)} + \frac{1}{4}M_-(x,T)\frac{\partial b/\partial T}{(x-b)} = \frac{1}{(x-a)(x-b)} \tag{4.21}$$

matching the pole at $x = a$ gives:

$$\frac{\partial a}{\partial T} = \frac{4}{(a-b)\,M_-(a,T)} \sim -\frac{1}{(a-e)^{2\nu-1}Q(a)} \tag{4.22}$$

which is finite at $T = T_{\mathrm{c}}$; thus, we find that to first order in t:

$$a \sim -2 + \frac{t}{(2+e)^{2\nu-1}Q(-2)}, \qquad b \sim 2 - \frac{t}{(e-2)^{2\nu-1}Q(2)} \tag{4.23}$$

(notice that $Q(-2) < 0$ and $Q(2) < 0$). Relation (3.28) implies:

$$\gamma \sim 1 + \mathrm{O}(t). \tag{4.24}$$

Then, (4.21) reduces to:

$$\frac{1}{2}\frac{\partial M_-(x,T)}{\partial T} = \frac{M_-(x,T)-M_-(a,T)}{M_-(a,T)(a-b)(x-a)} + \frac{M_-(x,T)-M_-(b,T)}{M_-(b,T)(b-a)(x-b)} \tag{4.25}$$

which is finite at $T = T_{\mathrm{c}}$; thus one gets the asymptotics of M_-:

$$
\begin{aligned}
2\frac{\partial M_-(x,T)}{\partial T} &= -\frac{(x-e)^{2\nu-1}(Q(x)-Q(a)) + ((x-e)^{2\nu-1}-(a-e)^{2\nu-1})Q(a)}{(a-e)^{2\nu-1}Q(a)\,(x-a)} \\
&\quad + \frac{(x-e)^{2\nu-1}(Q(x)-Q(b)) + ((x-e)^{2\nu-1}-(b-e)^{2\nu-1})Q(b)}{(b-e)^{2\nu-1}Q(b)\,(x-b)} \\
&= -\frac{(x-e)^{2\nu-1}}{(a-e)^{2\nu-1}Q(a)}\frac{Q(x)-Q(a)}{x-a} - \frac{(x-e)^{2\nu-1}-(a-e)^{2\nu-1}}{(x-a)\,(a-e)^{2\nu-1}} \\
&\quad + \frac{(x-e)^{2\nu-1}}{(b-e)^{2\nu-1}Q(b)}\frac{Q(x)-Q(b)}{x-b} + \frac{(x-e)^{2\nu-1}-(b-e)^{2\nu-1}}{(x-b)\,(b-e)^{2\nu-1}}.
\end{aligned} \tag{4.26}
$$

In particular in the vicinity of $x = e$ one has:

$$
\begin{aligned}
M_-(x,T) &\sim (x-e)^{2\nu-1}Q(x) \\
&\quad + \frac{t}{2}\left[\sum_{k=0}^{2\nu-2}(x-e)^k((2-e)^{-k-1}-(-2-e)^{-k-1}) + \mathrm{O}(x-e)^{2\nu-1}\right]. \tag{4.27}
\end{aligned}
$$

Note that the zeros of $M_-(x, T)$ in the vicinity of e, are the $2\nu - 1$th roots of unity:

$$M_-(x, T_c + t) = 0 \leftrightarrow x = e + \left(\frac{2t}{Q(e)(e^2 - 4)} \right)^{1/2\nu - 1} + O(t^{2/2\nu - 1}). \quad (4.28)$$

Using (3.31), we get:

$$\gamma_n \sim 1 - \frac{t}{4} \left(\frac{1}{(e-2)^{2\nu-1}Q(2)} + \frac{1}{(e+2)^{2\nu-1}Q(-2)} \right)$$

$$\beta_n \sim -\frac{t}{2} \left(\frac{1}{(e-2)^{2\nu-1}Q(2)} - \frac{1}{(e+2)^{2\nu-1}Q(-2)} \right) \quad (4.29)$$

$$\text{where} \quad n = N(1 + t/T_c).$$

4.4. Variations near the critical point, $T > T_c$ (two cuts)

Let us consider the limit of (4.3) near T_c. Write $T = T_c + t$, and $t > 0$, and

$$W(x, T) = \tfrac{1}{2} \left(V'(x) - M_+(x, T) \sqrt{(x-a)(x-b)(x-c)(x-d)} \right). \quad (4.30)$$

At $T = T_c$ we have

$$a(T_c) = -2, \qquad b(T_c) = 2$$

$$c(T_c) = d(T_c) = e = 2\cosh\phi_e, \qquad M_+(x, T_c) = (x - e)^{2\nu-2} Q(x), \quad (4.31)$$

and at $T > T_c$, $a + 2$, $b - 2$, $c - e$, $d - e$, and $M_+(x) - (x-e)^{2\nu-2}Q(x)$ are small. In particular, we write:

$$M_+(x, T) = H(x, T)Q(x, T) \quad (4.32)$$

where $H(x, T)$ is a monic polynomial of degree $2\nu - 2$ which contains all the roots of M_+ close to e, and $Q(x, T)$ is the remaining part. In other words, $H(x, T) - (x-e)^{2\nu-2}$ is small and $Q(x, T) - Q(x)$ is small in the small t limit.

We use the notation of section 3.5. The biratio (3.32) is thus:

$$m = \frac{(b-a)(d-c)}{(c-a)(d-b)} \sim \frac{4}{e^2 - 4}(d-c) \sim \frac{d-c}{\sinh^2\phi_e}. \quad (4.33)$$

We see that we have to consider the limit $m \to 0$. In that limit (3.37) becomes

$$v_\infty = i \int_0^{\sqrt{(d-b)/(b-a)}} \frac{dy}{\sqrt{(1+y^2)(1+my^2)}} \sim i \int_0^{\sqrt{(e-2)/4}} \frac{dy}{\sqrt{1+y^2}} = i\frac{\phi_e}{2} \quad (4.34)$$

$$E(v_\infty) - v_\infty = im \int_0^{\sqrt{(d-b)/(b-a)}} \frac{y^2\, dy}{\sqrt{(1+y^2)(1+my^2)}}$$

$$\sim im \int_0^{\sqrt{(e-2)/4}} \frac{y^2\, dy}{\sqrt{(1+y^2)}} = im\frac{\sinh\phi_e - \phi_e}{4}. \quad (4.35)$$

And, as can be found in any handbook of classical functions [24], we have the small m behaviour:

$$\frac{E'(m)}{K'(m)} \sim -\frac{2}{\ln m}. \quad (4.36)$$

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Since for small m one has $|1/\ln m| \gg m$, (3.39) becomes:

$$\delta x_0 := x_0 - d \sim \frac{2\phi_e \sinh \phi_e}{\ln m}. \tag{4.37}$$

Notice that:

$$x_0 - c = x_0 - d + d - c \sim x_0 - d + m \sinh^2 \phi_e \sim x_0 - d \sim \delta x_0. \tag{4.38}$$

From (3.9) and (3.41) we have:

$$-\frac{1}{2} \frac{\partial M_+(x,T)}{\partial T} - \frac{1}{4} \frac{M_+(x,T)}{\sigma(x,T)} \frac{\partial \sigma(x,T)}{\partial T} = \frac{x - x_0}{\sigma(x,T)}. \tag{4.39}$$

Matching the pole at $x = c$ gives:

$$\frac{\partial c}{\partial t} = \frac{-4(d - c + \delta x_0)}{(c-a)(c-b)(c-d)M_+(c, T_c + t)} \sim \frac{-4\delta x_0}{(e^2 - 4)Q(e)} \frac{1}{(c-d)\,H(c)} \tag{4.40}$$

and matching the pole at $x = d$ gives:

$$\frac{\partial d}{\partial t} = \frac{-4\delta x_0}{(d-a)(d-b)(d-c)M_+(d)} \sim -\frac{4\delta x_0}{(e^2 - 4)Q(e)} \frac{1}{(d-c)\,H(d)} \tag{4.41}$$

and matching the poles close to e gives:

$$\frac{\partial H(x)}{\partial t} \sim -\frac{2\delta x_0}{(e^2 - 4)Q(e)} \frac{1}{d - c} \left(\frac{H(x) - H(d)}{(x-d)H(d)} - \frac{H(x) - H(c)}{(x-c)H(c)} \right). \tag{4.42}$$

The following guess solves the three equations (4.40), (4.41), (4.42) to small t leading order:

$$c \sim e - 2\zeta \left(-\frac{t}{\ln t} \right)^{1/2\nu}, \qquad d \sim e + 2\zeta \left(-\frac{t}{\ln t} \right)^{1/2\nu} \tag{4.43}$$

$$H(x) \sim \left(-\frac{t}{\ln t} \right)^{-1+(1/\nu)} G\left((x - e) \left(-\frac{t}{\ln t} \right)^{-1/2\nu} \right) \tag{4.44}$$

where ζ is a positive real number, and G is a degree $2\nu - 2$ even monic polynomial, which will be determined below.

For later convenience, we also define the following positive constant:

$$C := \frac{4\nu^2 \phi_e}{\sinh \phi_e \, Q(e)} > 0. \tag{4.45}$$

Using ansatz (4.43) and (4.37), we have in that limit:

$$\delta x_0 \sim \frac{4\nu \phi_e \sinh \phi_e}{\ln t}. \tag{4.46}$$

Then, inserting (4.43) and (4.44) into (4.40) and (4.41), we get:

$$4\zeta^2 = \frac{C}{G(-2\zeta)} = \frac{C}{G(2\zeta)}. \tag{4.47}$$

Then, setting $x = e + \xi \left(-t/\ln t\right)^{1/2\nu}$, and inserting (4.44) into (4.42) we get the following equation for G:

$$(2\nu - 2)\, G(\xi) - \xi\, G'(\xi) = \frac{C}{4\zeta}\, \left(\frac{G(\xi) - G(2\zeta)}{(\xi - 2\zeta)G(2\zeta)} - \frac{G(\xi) - G(-2\zeta)}{(\xi + 2\zeta)G(-2\zeta)}\right) \quad (4.48)$$

which using (4.47) becomes:

$$(2\nu - 2)\, G(\xi) - \xi\, G'(\xi) = \frac{4\zeta^2}{\xi^2 - 4\zeta^2}\, (G(\xi) - G(2\zeta)) \quad (4.49)$$

the solution of which is:

$$G(\xi) = \sum_{k=0}^{\nu-1} \frac{(2k)!}{k!k!}\, \zeta^{2k}\, \xi^{2(\nu-1-k)} = \mathrm{Pol}\, \frac{\xi^{2\nu-1}}{\sqrt{\xi^2 - 4\zeta^2}} \quad (4.50)$$

or:

$$G(2\zeta \cosh \psi) = \zeta^{2\nu-2} \sum_{j=0}^{\nu-1} \binom{2\nu-1}{\nu+j} \frac{\sinh(2j+1)\psi}{\sinh \psi}. \quad (4.51)$$

In particular,

$$\frac{G(2\zeta)}{\zeta^{2\nu-2}} = \frac{1}{2}\, \frac{(2\nu)!}{(\nu-1)!\nu!} = \frac{C}{4\zeta^{2\nu}} \quad (4.52)$$

i.e. the parameter ζ is determined by:

$$\zeta = \left(\frac{C}{2}\, \frac{\nu!\,(\nu-1)!}{(2\nu)!}\right)^{1/2\nu} = \left(\frac{2\nu^2\phi_e}{\sinh \phi_e\, Q(e)}\, \frac{\nu!\,(\nu-1)!}{(2\nu)!}\right)^{1/2\nu}. \quad (4.53)$$

In that scaling regime, we have:

$$m \sim \frac{4\zeta}{\sinh^2 \phi_e}\, \left(-\frac{t}{\ln t}\right)^{1/2\nu}; \quad (4.54)$$

i.e. this corresponds to a torus of modulus:

$$\tau = \mathrm{i}\, \frac{K'}{K} \sim \frac{-\mathrm{i}}{\pi}\, \ln m \sim \frac{-\mathrm{i}}{2\nu\pi}\, \ln t \quad (4.55)$$

and, using (3.43):

$$\gamma = \frac{\mathrm{i}}{4K}\, \sqrt{(d-b)(c-a)}\, \mathrm{e}^{-\pi(u_\infty^2/KK')}\, \frac{\theta_1'(0,\tau)}{\theta_1(u_\infty/K,\tau)}$$

$$\sim \mathrm{e}^{-\pi(u_\infty^2/KK')}$$

$$\sim 1 - \frac{\phi_e^2}{\ln m}$$

$$\sim 1 - \frac{2\nu\phi_e^2}{\ln t}. \quad (4.56)$$

We also find that the filling fraction in the $[c, d]$ cut is of order:

$$
\begin{aligned}
\epsilon &= \frac{1}{2\pi T_c} \int_c^d \rho(x)\, \mathrm{d}x \\
&\sim -\frac{t}{\ln t} \frac{\sinh\phi_e Q(e)}{\pi T_c} \int_{-2\zeta}^{2\zeta} G(\xi)\sqrt{4\zeta^2 - \xi^2}\, \mathrm{d}\xi \\
&\sim -\mathrm{i}\frac{t}{\ln t} \frac{\sinh\phi_e Q(e)}{\pi T_c} \int_{-2\zeta}^{2\zeta} \xi^{2\nu-1} \frac{G(\xi)\sqrt{\xi^2 - 4\zeta^2}}{\xi^{2\nu-1}}\, \mathrm{d}\xi
\end{aligned}
\tag{4.57}
$$

and then, integrating by parts and using (4.49), we find:

$$
\begin{aligned}
\epsilon &\sim \mathrm{i}\frac{t}{T_c \ln t} \frac{\sinh\phi_e Q(e)}{2\nu\pi} \int_{-2\zeta}^{2\zeta} \xi^{2\nu} \frac{4\zeta^2\, \xi^{-2\nu}}{\sqrt{\xi^2 - 4\zeta^2}}\, G(2\zeta)\, \mathrm{d}\xi \\
&\sim \mathrm{i}\frac{t}{T_c \ln t} \frac{\sinh\phi_e Q(e)}{2\nu\pi} 4\zeta^2\, G(2\zeta) \int_{-2\zeta}^{2\zeta} \frac{\mathrm{d}\xi}{\sqrt{\xi^2 - 4\zeta^2}} \\
&\sim -\frac{t}{T_c \ln t} \frac{\sinh\phi_e Q(e)}{2\nu}\, C,
\end{aligned}
\tag{4.58}
$$

i.e.:

$$
\epsilon(T_c + t) \sim -\frac{t}{T_c \ln t}\, 2\nu\phi_e.
\tag{4.59}
$$

This means that for $n = N(1 + t/T_c)$, the average number of eigenvalues located near e is:

$$
k \sim \frac{n - N}{\ln N}\, 2\nu\phi_e,
\tag{4.60}
$$

i.e. the eigenvalues start to explore the potential well near e when $n - N \sim \ln N$.

According to (3.46) and (3.47), the coefficients γ_n and β_n vary between:

$$
1 + \zeta \left(\frac{-t}{\ln t}\right)^{1/2\nu} \leq \gamma_n \leq \cosh\phi_e
\tag{4.61}
$$

$$
2\zeta \left(\frac{-t}{\ln t}\right)^{1/2\nu} \leq \beta_n \leq e - 2
\tag{4.62}
$$

where $\quad n = N(1 + t/T_c)$.

The transition takes place on a scale of order $\ln N$.

4.5. Order of the transition

From (3.14) and (4.24) we have below T_c, i.e. for $t < 0$:

$$
\frac{\partial^2 F}{\partial t^2}(T_c + t) = -2\ln\gamma \sim \frac{t}{2} \left(\frac{1}{(e-2)^{2\nu-1}Q(2)} + \frac{1}{(e+2)^{2\nu-1}Q(-2)} \right)
\tag{4.63}
$$

and above T_c, i.e. for $t > 0$, we have from (4.56):

$$
\frac{\partial^2 F}{\partial t^2}(T_c + t) = -2\ln\gamma \sim \frac{4\nu\phi_e^2}{\ln t}.
\tag{4.64}
$$

The second derivative of the free energy is continuous, but the third derivative is not. Therefore we have a third order transition, with logarithmic divergency; cf figure 3.
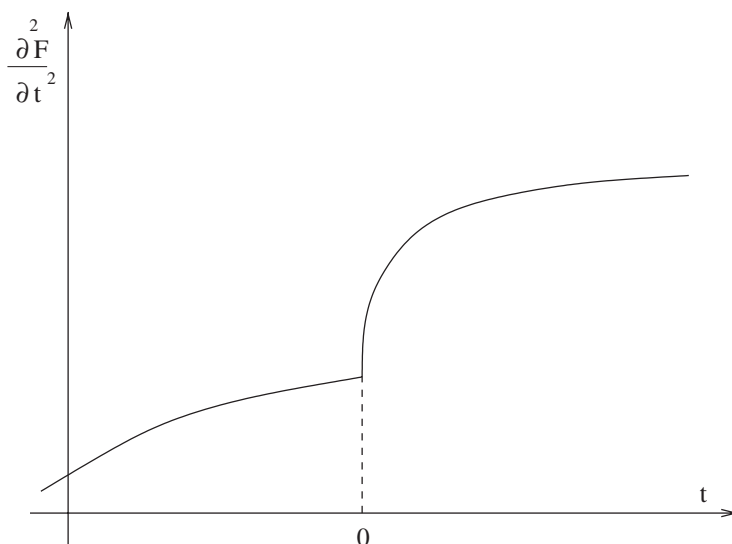
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Figure 3. Behaviour of the second derivative of the classical free energy with respect to t near the transition.

5. Mean field asymptotics for the partition function

We compute the partition function for a potential:

$$V_{r_0}(x) = V(x) + r_0 \ln(x_0 - x) \tag{5.1}$$

where r_0 is assumed of order $1/N$:

$$r_0 = -a\frac{T_c}{N}. \tag{5.2}$$

5.1. Mean field theory

We use the same idea as in [4]. We split the two-cut integral into one-cut integrals. Let us say that there are k eigenvalues in the new cut (near e), and $n-k$ in the old cut $[a, b]$:

$$n! \, Z_n\left(\frac{n}{N}T_c, V_{r_0}\right)$$

$$\sim \sum_{k=0}^{n} \binom{n}{k} \int_{x_i > \tilde{e}} \mathrm{d}x_1 \cdots \mathrm{d}x_k \, \Delta^2(x_i) \prod_{i=1}^{k} (x_0 - x_i)^a \, \mathrm{e}^{-(N/T_c)V(x_i)}$$

$$\times (n-k)! \, \bar{Z}_{n-k}\left(T_c\frac{n-k}{N}, \mathcal{V}_{r_j}\right) \tag{5.3}$$

where

$$\mathcal{V}_{r_j}(x) = V(x) + \sum_{j=0}^{k} r_j \ln(x_j - x), \qquad r_0 = -\frac{aT_c}{N}, \qquad r_1 = \cdots = r_k = -\frac{2T_c}{N} \tag{5.4}$$

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and $\bar{Z}_{n-k}(T_c((n-k)/N), \mathcal{V}_{r_j})$ is a one-cut integral:

$$\bar{Z}_{n-k}\left(T_c\frac{n-k}{N}, \mathcal{V}_{r_j}\right) := \frac{1}{(n-k)!} \int_{x_{k+i+1}<\tilde{e}} \mathrm{d}x_{k+1}\cdots\mathrm{d}x_n \, \Delta^2(x_i) \prod_{i=k+1}^{n} \mathrm{e}^{-(N/T_c)\mathcal{V}_{r_j}(x_i)}$$

$$= H_{n-k} \exp\left(-\frac{N^2}{T_c^2}\bar{F}_{n-k}\left(T_c\frac{n-k}{N}, \mathcal{V}_{r_j}\right)\right). \tag{5.5}$$

That gives:

$$Z_n\left(\frac{n}{N}T_c, V_{r_0}\right) \sim \sum_{k=0}^{n} \frac{H_{n-k}}{k!} \int_{x_i>\tilde{e}} \mathrm{d}x_1\cdots\mathrm{d}x_k \, \Delta^2(x_i) \prod_{i=1}^{k}(x_0-x_i)^a$$

$$\times \exp\left(-\frac{N}{T_c}V(x_i)\right)\exp\left(-\frac{N^2}{T_c^2}\bar{F}_{n-k}\left(T_c\frac{n-k}{N}, \mathcal{V}_{r_j}\right)\right). \tag{5.6}$$

In other words, we integrate out $n-k$ eigenvalues, and consider the integral over k eigenvalues only. The k remaining eigenvalues are subjected to the potential V, as well as their mutual Coulomb repulsion, and the mean field of the exterior $n-k$ eigenvalues.

Since $\bar{F}_n(T, \mathcal{V}_{r_j})$ corresponds to a one-cut distribution, it can be evaluated with the standard semiclassical technique (see section 3), and in particular, it has a large n expansion:

$$\bar{F}_n(T, \mathcal{V}_{r_j}) \sim \bar{F}(T, \mathcal{V}_{r_j}) + \frac{T}{n}\bar{F}^{(1/2)}(T, \mathcal{V}_{r_j}) + \frac{T^2}{n^2}\bar{F}^{(1)}(T, \mathcal{V}_{r_j}) + \mathrm{O}\left(\frac{1}{n^3}\right) \tag{5.7}$$

and each term of the expansion is analytical in $\mathcal{V}_{r_j}$.

We want to evaluate $\bar{F}(T, \mathcal{V}_{r_j})$ in a regime where $T-T_c$ is 'small' (we make that more precise below) and r_0 and the r_j's are of order $\mathrm{O}(1/N)$.

We first do a Taylor expansion in $T-T_c$ and the r_j's:

$$\bar{F}(T, \mathcal{V}_{r_j}) \sim \bar{F}(T_c, V) + \sum_{i=0}^{k} r_i\frac{\partial\bar{F}}{\partial r_i} + \frac{1}{2}\sum_{i,j} r_i r_j\frac{\partial^2\bar{F}}{\partial r_i\partial r_j} + \mathrm{o}\left(\frac{k^2}{N^2}\right)$$

$$+ \sum_j (T-T_c)r_j\frac{\partial^2\bar{F}}{\partial T\partial r_j}$$

$$+ (T-T_c)\frac{\partial\bar{F}}{\partial T} + \frac{1}{2}(T-T_c)^2\frac{\partial^2\bar{F}}{\partial T^2} + \mathrm{o}\left(\frac{1}{N^2}\right) \tag{5.8}$$

where all derivatives are computed at $T=T_c$ and $r_i=0$ (see sections 3.2 and 3.3). That expansion is valid only if $k \ll N$.

Thus we have:

$$\frac{1}{H_n} Z_n \left(\frac{n}{N} T_{\rm c}, V_{r_0}\right) \sim {\rm e}^{-(N^2/T_{\rm c}^2)\bar{F}(T_{\rm c},V)} {\rm e}^{-\bar{F}^{(1)}(T_{\rm c},V)} {\rm e}^{a\,(N/T_{\rm c})\bar{F}_{r_0}} {\rm e}^{-(a^2/2)\bar{F}_{r_0,r_0}}$$

$$\times \sum_{k=0}^{n} \frac{1}{k!} \frac{H_{n-k}}{H_n} \, {\rm e}^{-(N/T_{\rm c})(n-N-k)\bar{F}_T} {\rm e}^{a(n-N-k)\bar{F}_{T,r_0}} {\rm e}^{-((n-N-k)^2/2)\bar{F}_{T,T}}$$

$$\times \int_{x_i>\tilde{e}} {\rm d}x_1 \cdots {\rm d}x_k \, \Delta^2(x_i) \prod_i (x_0-x_i)^a \, {\rm e}^{-(N/T_{\rm c})V(x_i)}$$

$$\times \prod_{j=1}^{k} {\rm e}^{2(N/T_{\rm c})\bar{F}_{r_j}} \, {\rm e}^{2(n-N-k)\bar{F}_{T,r_j}} \, {\rm e}^{-2a\bar{F}_{r_0,r_j}} \prod_{j,l\geq 1} {\rm e}^{-2\bar{F}_{r_l,r_j}}. \quad (5.9)$$

Notice that:

$$\frac{H_{n-k}}{H_n} \sim (2\pi)^{-k} \left(1-\frac{k}{n}\right)^{-1/12} (1+{\rm O}(1/n)). \quad (5.10)$$

5.2. Computation of the derivatives

Now, use the formula given in sections 3.2 and 3.3 in the one-cut case, and get (derivatives taken at $T=T_{\rm c}$ and $r_i=0$, and taking into account that $V_{\rm eff}(b)=V_{\rm eff}(e)$):

$$\frac{\partial}{\partial r_i}\bar{F} = -\frac{1}{2}\left(V_{\rm eff}(x_i)-V(x_i)\right), \qquad \frac{\partial}{\partial T}\bar{F} = V_{\rm eff}(e) \quad (5.11)$$

$$\frac{\partial^2}{\partial r_i \partial r_j}\bar{F} = \ln\frac{x_i-x_j}{\Lambda(x_i)-\Lambda(x_j)}, \qquad \frac{\partial^2}{\partial r_i^2}\bar{F} = -\ln|\Lambda'(x_i)| \quad (5.12)$$

$$\frac{\partial^2}{\partial T \partial r_i}\bar{F} = \ln\Lambda(x_i), \qquad \frac{\partial^2}{\partial T^2}\bar{F} = 0. \quad (5.13)$$

Moreover:

$$\frac{\partial \mathcal{T}(T,V)}{\partial T} = \frac{a+b}{2} = 0, \qquad \frac{\partial \mathcal{T}(T,V)}{\partial r_j} = \frac{1}{\Lambda(x_j)}. \quad (5.14)$$

The effective potential behaves in the vicinity of e as:

$$V_{\rm eff}(x) \underset{x\to e}{\sim} V_{\rm eff}(e) + \frac{V_{\rm eff}^{(2\nu)}(e)}{2\nu!}(x-e)^{2\nu} \quad (5.15)$$

$$\frac{V_{\rm eff}^{(2\nu)}(e)}{2\nu!} = \frac{2\sinh\phi_e\,Q(e)}{2\nu}. \quad (5.16)$$

In the limit where the x_i's are close to e, we have:

$$\frac{x_i-x_j}{\Lambda(x_i)-\Lambda(x_j)} \underset{x_i,x_j\to e}{\sim} \frac{1}{\Lambda'(x_i)} \sim 2\sinh\phi_e\,{\rm e}^{-\phi_e} \quad (5.17)$$

and

$$\Lambda(x_i) \underset{x_i\to e}{\sim} {\rm e}^{\phi_e}. \quad (5.18)$$

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5.3. Result

We write $n = N + p$:

$$
Z_{N+p}\left(\frac{N+p}{N}T_{\mathrm{c}}, V_{r_0}\right)
$$

$$
\sim \mathrm{e}^{-(N^2/T_{\mathrm{c}}^2)\bar{F}(T_{\mathrm{c}},V)}\mathrm{e}^{-\bar{F}^{(1)}(T_{\mathrm{c}},V)}
$$

$$
\times \left(\frac{\mathrm{e}^{\phi_e}}{2\sinh\phi_e}\right)^{a^2/2} \mathrm{e}^{-(aN/2T_{\mathrm{c}})(V_{\mathrm{eff}}(x_0)-V(x_0))}
$$

$$
\times \sum_{k=0}^{N+p}\frac{H_{N+p-k}}{k!}\,\mathrm{e}^{-(N/T_{\mathrm{c}})(p-k)V_{\mathrm{eff}}(e)}\,\mathrm{e}^{a(p-k)\phi_e}
$$

$$
\times \int_{x_i>\tilde{e}}\mathrm{d}x_1\cdots\mathrm{d}x_k\,\Delta^2(x_i)\prod_{i=1}^{k}(x_0-x_i)^a\,\mathrm{e}^{-(N/T_{\mathrm{c}})V(x_i)}
$$

$$
\times \prod_{i=1}^{k}\mathrm{e}^{-(N/T_{\mathrm{c}})(V_{\mathrm{eff}}(x_i)-V(x_i))}\,\mathrm{e}^{2k(p-k)\phi_e}\left(\frac{\mathrm{e}^{\phi_e}}{2\sinh\phi_e}\right)^{2ak+2k^2}
$$

$$
\sim H_N\,\mathrm{e}^{-(N^2/T_{\mathrm{c}}^2)\bar{F}(T_{\mathrm{c}},V)}\mathrm{e}^{-\bar{F}^{(1)}(T_{\mathrm{c}},V)}\,\mathrm{e}^{-(aN/2T_{\mathrm{c}})(V_{\mathrm{eff}}(x_0)-V(x_0))}\,\mathrm{e}^{-p(N/T_{\mathrm{c}})V_{\mathrm{eff}}(e)}
$$

$$
\times \sum_{k=0}^{N+p}\frac{(2\pi)^{p-k}}{k!}\,\mathrm{e}^{(2k+a)(p-k)\phi_e}\left(\frac{\mathrm{e}^{\phi_e}}{2\sinh\phi_e}\right)^{2(k+(a/2))^2}
$$

$$
\times \int_{x_i>\tilde{e}}\mathrm{d}x_1\cdots\mathrm{d}x_k\,\Delta^2(x_i)\prod_{i=1}^{k}(x_0-x_i)^a\,\mathrm{e}^{-(N/T_{\mathrm{c}})(V_{\mathrm{eff}}(x_i)-V_{\mathrm{eff}}(e))}. \tag{5.19}
$$

Then we rescale:

$$
x_i = e + N^{-1/2\nu}\left(\frac{2\sinh\phi_e\,Q(e)}{T_{\mathrm{c}}}\right)^{-1/2\nu}y_i,\qquad x_0 = e + N^{-1/2\nu}\left(\frac{2\sinh\phi_e\,Q(e)}{T_{\mathrm{c}}}\right)^{-1/2\nu}y \tag{5.20}
$$

and we get:

$$
Z_{N+p}\left(\frac{N+p}{N}T_{\mathrm{c}}, V_{r_0}\right)\mathrm{e}^{-(aN/2T_{\mathrm{c}})V(x_0)} \sim H_N\,\mathrm{e}^{-(N^2/T_{\mathrm{c}}^2)\bar{F}(T_{\mathrm{c}},V)}\mathrm{e}^{-\bar{F}^{(1)}(T_{\mathrm{c}},V)}
$$

$$
\times \mathrm{e}^{-(p+a/2)(N/T_{\mathrm{c}})V_{\mathrm{eff}}(e)}\,\mathrm{e}^{-a(y^{2\nu}/4\nu)}
$$

$$
\times N^{a^2/8\nu}\,(2\sinh\phi_e)^{-a^2/2}\,(2\pi)^p
$$

$$
\times \sum_{k=0}^{N+p}N^{-(k+a/2)^2/2\nu}\,\mathrm{e}^{(k+a/2)2p\phi_e}\,\mathrm{e}^{(k+a/2)a\phi_e}\,A^{-(k^2+ak)}\,(2\pi)^{-k}
$$

$$
\times \frac{1}{k!}\int\mathrm{d}y_1\cdots\mathrm{d}y_k\,\Delta^2(y_i)\prod_{i=1}^{k}(y-y_i)^a\,\mathrm{e}^{-y_i^{2\nu}/2\nu}\left(1+\mathrm{O}(N^{-1/2\nu})\right) \tag{5.21}
$$

where we have defined:

$$
A := (2\sinh\phi_e)^2\left(\frac{2\sinh\phi_e\,Q(e)}{T_{\mathrm{c}}}\right)^{1/2\nu}. \tag{5.22}
$$

Equation (5.21) is valid only up to $\mathrm{O}(N^{-1/2\nu})$ because equation (5.15), equations (5.17) and (5.18) are valid only to that order in the regime of equation (5.20).

5.4. The effective matrix model

Let us define the matrix model in the potential $y^{2\nu}/2\nu$.

Let $\zeta_{k,\nu}$ be the partition function of the $k \times k$ matrix model in the potential $y^{2\nu}/2\nu$:

$$\zeta_{k,\nu} := \frac{1}{k!} \int \mathrm{d}x_1 \cdots \mathrm{d}x_k \ \Delta^2(x_i) \ \prod_i \mathrm{e}^{-x_i^{2\nu}/2\nu}. \tag{5.23}$$

Notice that for $\nu = 1$, this is the Gaussian matrix model, and we have:

$$\ln \zeta_{k,1} = \frac{k}{2} \ln 2\pi + \ln \left(\prod_{j=0}^{k-1} j! \right) = \ln H_k + \frac{k^2}{2} \ln k - \frac{3}{4} k^2. \tag{5.24}$$

We define the amplitude:

$$A_k := A^{-k^2} (2\pi)^{-k} \zeta_{k,\nu}. \tag{5.25}$$

We also introduce:

$$h_{k,\nu} := \frac{\zeta_{k+1,\nu}}{\zeta_{k,\nu}} = 2\pi \, A^{2k+1} \frac{A_{k+1}}{A_k} \tag{5.26}$$

$$\gamma_{k,\nu} := \sqrt{\frac{h_{k,\nu}}{h_{k-1,\nu}}} = A \sqrt{\frac{A_{k+1} A_{k-1}}{A_k^2}} \tag{5.27}$$

and the associated orthogonal polynomials

$$P_k(y) := \frac{\int \mathrm{d}y_1 \cdots \mathrm{d}y_k \, \Delta(y_i)^2 \prod_{i=1}^k (y - y_i) \, \mathrm{e}^{-y_i^{2\nu}/2\nu}}{\int \mathrm{d}y_1 \cdots \mathrm{d}y_k \, \Delta(y_i)^2 \prod_{i=1}^k \, \mathrm{e}^{-y_i^{2\nu}/2\nu}} \tag{5.28}$$

$$\psi_{k,\nu}(y) := \sqrt{\frac{\zeta_{k,\nu}}{\zeta_{k+1,\nu}}} \, P_k(y) \, \mathrm{e}^{-(y^{2\nu}/4\nu)} \tag{5.29}$$

and their Hilbert transforms:

$$\hat{P}_{k-1}(y) := \frac{\int \mathrm{d}x_1 \cdots \mathrm{d}x_k \, \Delta(x_i)^2 \prod_{i=1}^k (1/(y - x_i)) \, \mathrm{e}^{-x_i^{2\nu}/2\nu}}{\int \mathrm{d}x_1 \cdots \mathrm{d}x_k \, \Delta(x_i)^2 \prod_{i=1}^k \, \mathrm{e}^{-x_i^{2\nu}/2\nu}} \tag{5.30}$$

$$\phi_{k,\nu}(y) := \sqrt{h_{k,\nu}} \, \hat{P}_k(y) \, \mathrm{e}^{(y^{2\nu}/4\nu)}. \tag{5.31}$$

5.5. Partition function

Thus we have for $a = 0$:

$$Z_{N+p}\left(\frac{N+p}{N}T_c, V\right) \sim H_N\, e^{-(N^2/T_c^2)\bar{F}(T_c,V)} e^{-\bar{F}^{(1)}(T_c,V)}\, e^{-p(N/T_c)V_{\text{eff}}(e)}\,(2\pi)^p$$

$$\times \sum_{k=0}^{N+p} N^{-k^2/2\nu}\, e^{2kp\phi_e} A_k(1 + O(N^{-1/2\nu})) \tag{5.32}$$

$$h_{N+p} \sim 2\pi\, e^{-(N/T_c)V_{\text{eff}}(e)} \frac{\sum_{k=0}^{N+p+1} N^{-k^2/2\nu}\, e^{2kp\phi_e}\, e^{2k\phi_e}\, A_k}{\sum_{k=0}^{N+p} N^{-k^2/2\nu}\, e^{2kp\phi_e}\, A_k}(1 + O(N^{-1/2\nu})) \tag{5.33}$$

$$\gamma_{N+p}^2 \sim \frac{\left(\sum_{k=0}^{N+p+1} N^{-k^2/2\nu}\, e^{2kp\phi_e}\, e^{2k\phi_e}\, A_k\right)\left(\sum_{k=0}^{N+p-1} N^{-k^2/2\nu}\, e^{2kp\phi_e}\, e^{-2k\phi_e}\, A_k\right)}{\left(\sum_{k=0}^{N+p} N^{-k^2/2\nu}\, e^{2kp\phi_e}\, A_k\right)^2}$$

$$\times (1 + O(N^{-1/2\nu})). \tag{5.34}$$

5.6. Orthogonal polynomial

According to Heine's formula (cf equation (2.11)), we have:

$$\psi_n = \frac{Z_n(T_c(n/N), V(x) - (T_c/N)\ln(\xi - x))}{\sqrt{Z_n(T_c(n/N), V)\, Z_{n+1}(T_c((n+1)/N), V)}}\, e^{-(N/2T)\,V(\xi)}$$

$$\phi_{n-1}(\xi) = \frac{Z_n(T_c(n/N), V(x) + (T_c/N)\ln(\xi - x))}{\sqrt{Z_n(T_c(n/N), V)\, Z_{n-1}(T_c((n-1)/N), V)}}\, e^{(N/2T)\,V(\xi)} \tag{5.35}$$

$$\psi_{N+p}(x_0) = \frac{Z_{N+p}((N+p/N))T_c, V_{r_0}\, e^{-(N/2T)V(x_0)}}{\sqrt{Z_{N+p}(((N+p)/N)T_c, V)\, Z_{N+p+1}(((N+p+1)/N)T_c, V)}}$$

$$r_0 = -\frac{T_c}{N}, \qquad a = 1.$$

Thus, in the regime:

$$x_0 = e + N^{-1/2\nu}\frac{4\sinh^2\phi_e}{A}\, y \tag{5.36}$$

using equation (5.21) with $a = 1$ we get:

$$\psi_{N+p}(x_0) \sim N^{1/8\nu}\sqrt{\frac{A}{2\sinh\phi_e}}$$

$$\times \frac{\sum_{k=0}^{N+p} N^{-((k+1/2)^2/2\nu)}\, e^{(k+1/2)2p\phi_e}\, e^{(k+1/2)\phi_e}\, \sqrt{A_k A_{k+1}}\, \psi_{k,\nu}(y)}{\sqrt{(\sum_{k=0}^{N+p+1} N^{-k^2/2\nu}\, e^{2kp\phi_e}\, e^{2k\phi_e}\, A_k)(\sum_{k=0}^{N+p} N^{-k^2/2\nu}\, e^{2kp\phi_e}\, A_k)}}$$

$$\times (1 + O(N^{-1/2\nu})). \tag{5.37}$$

5.7. Hilbert transforms

According to equation (2.14), we have:

$$\phi_{N+p-1}(x_0) = \frac{Z_{N+p}(((N+p)/N)T_{\rm c}, V_{r_0})\,{\rm e}^{(N/2T)V(x_0)}}{\sqrt{Z_{N+p-1}(((N+p-1)/N)T_{\rm c}, V)\,Z_{N+p}(((N+p)/N)T_{\rm c}, V)}} \tag{5.38}$$
$$r_0 = \frac{T_{\rm c}}{N}, \qquad a = -1.$$

Thus, in the regime:

$$x_0 = e + N^{-1/2\nu}\,\frac{4\sinh^2\phi_e}{A}\,y \tag{5.39}$$

using equation (5.21) with $a = -1$ we get:

$$\phi_{N+p-1}(x_0) \sim N^{1/8\nu}\sqrt{\frac{A}{2\sinh\phi_e}}$$
$$\times\frac{\sum_{k=0}^{N+p} N^{-((k-1/2)^2/2\nu)}\,{\rm e}^{(k-1/2)2p\phi_e}\,{\rm e}^{-(k-1/2)\phi_e}\,\sqrt{A_{k-1}A_k}\,\phi_{k-1,\nu}(y)}{\sqrt{(\sum_{k=0}^{N+p} N^{-k^2/2\nu}\,{\rm e}^{2kp\phi_e}\,A_k)(\sum_{k=0}^{N+p-1} N^{-k^2/2\nu}\,{\rm e}^{2kp\phi_e}\,{\rm e}^{-2k\phi_e}\,A_k)}}$$
$$\times\,(1 + {\rm O}(N^{-1/2\nu})). \tag{5.40}$$

Notice that shifting $k \to k+1$ we have:

$$\phi_{N+p-1}(x_0) \sim N^{1/8\nu}\sqrt{\frac{A}{2\sinh\phi_e}}$$
$$\times\frac{\sum_{k=-1}^{N+p-1} N^{-(k+1/2)^2/2\nu}\,{\rm e}^{(k+1/2)2p\phi_e}\,{\rm e}^{-(k+1/2)\phi_e}\,\sqrt{A_{k+1}A_k}\,\phi_{k,\nu}(y)}{\sqrt{(\sum_{k=0}^{N+p} N^{-k^2/2\nu}\,{\rm e}^{2kp\phi_e}\,A_k)(\sum_{k=0}^{N+p-1} N^{-k^2/2\nu}\,{\rm e}^{2kp\phi_e}\,{\rm e}^{-2k\phi_e}\,A_k)}}. \tag{5.41}$$

5.8. Computation of β_{N+p}

We start from equation (2.10):

$$\frac{N}{T_{\rm c}}\,\mathcal{T}_n\left(\frac{n}{N}T_{\rm c}, V\right)$$
$$\sim \frac{1}{n!\,Z_n((n/N)T_{\rm c}, V)}\int_{x_i>\tilde{e}}{\rm d}x_1\cdots{\rm d}x_k\,{\rm e}^{-(N/T_{\rm c})V(x_i)}\Delta^2(x_i)$$
$$\times\left(\sum_{j=1}^{k}x_j + \frac{N}{T_{\rm c}}\,\mathcal{T}_{n-k}\left(T_{\rm c}\frac{n-k}{N}, \mathcal{V}_{r_j}\right)\right)\,{\rm e}^{-(N^2/T_{\rm c}^2)\bar{F}_{n-k}(T_{\rm c}((n-k)/N),\mathcal{V}_{r_j})} \tag{5.42}$$

and where $\mathcal{V}_{r_j}$ was defined in equation (5.4). We have:

$$\mathcal{T}_{n-k}(T, \mathcal{V}_{r_j}) \sim \mathcal{T}(T_{\rm c}, V) + (T - T_{\rm c})\frac{\partial\mathcal{T}}{\partial T}(T_{\rm c}, V) + \sum_j r_j\frac{\partial\mathcal{T}}{\partial r_j}(T_{\rm c}, V) + {\rm O}(1/N^2)$$
$$\sim -\frac{2T_{\rm c}}{N}\sum_j \Lambda(x_j)^{-1} + {\rm O}(1/N^2). \tag{5.43}$$

Therefore:

$$\frac{N}{T_c} \mathcal{T}_n\left(\frac{n}{N}T_c, V\right)$$

$$\sim \frac{1}{Z_n((n/N)T_c, V)} \sum_{k=0}^{n} \frac{H_{n-k}}{k!} \int_{x_i > \tilde{e}} dx_1 \cdots dx_k \ e^{-(N/T_c)V(x_i)} \Delta^2(x_i)$$

$$\times \left(\sum_{j=1}^{k} x_j - 2\Lambda_j^{-1}\right) e^{-(N^2/T_c^2)\bar{F}_{n-k}(T_c((n-k)/N), \mathcal{V})}$$

$$\sim \frac{2\sinh\phi_e}{Z_n((n/N)T_c, V)} \sum_{k=0}^{n} \frac{H_{n-k}}{k!} \, k \int_{x_i > \tilde{e}} dx_1 \cdots dx_k \ e^{-(N/T_c)V(x_i)} \Delta^2(x_i)$$

$$\times e^{-(N^2/T_c^2)\bar{F}_{n-k}(T_c(n-k/N), \mathcal{V})}$$

$$\sim 2\sinh\phi_e \frac{\sum_k k \, e^{2pk\phi_e} N^{-k^2/2\nu} A_k}{\sum_k e^{2pk\phi_e} N^{-k^2/2\nu} A_k} \tag{5.44}$$

and:

$$\beta_{N+p} \sim 2\sinh\phi_e \left(\frac{\sum_k k \, e^{2(p+1)k\phi_e} N^{-k^2/2\nu} A_k}{\sum_k e^{2(p+1)k\phi_e} N^{-k^2/2\nu} A_k} - \frac{\sum_k k \, e^{2pk\phi_e} N^{-k^2/2\nu} A_k}{\sum_k e^{2pk\phi_e} N^{-k^2/2\nu} A_k}\right). \tag{5.45}$$

6. Asymptotic regimes

Consider p of order $\ln N$:

$$p = \frac{u}{2\nu\phi_e} \ln N, \qquad u \text{ finite.} \tag{6.1}$$

6.1. Possible asymptotic regimes for the partition function

Then equation (5.32) becomes:

$$Z_{N+p}\left(\frac{N+p}{N}T_c, V\right) \sim H_N \, e^{-(N^2/T_c^2)\bar{F}(T_c, V)} e^{-\bar{F}^{(1)}(T_c, V)}$$

$$\times e^{-p(N/T_c)V_{\text{eff}}(e)} (2\pi)^p \sum_{k=0}^{N+p} N^{((2ku-k^2)/2)\nu} A_k. \tag{6.2}$$

It is clear that the sum over k is dominated by the values of k for which the exponent of N is maximal, i.e. for which $2uk - k^2$ is maximal. This means that for $u < 0$, the sum is dominated by the vicinity of $k = 0$, and for $u \geq 0$, the sum is dominated by the vicinity of $k = u$. The sum is then well approximated by a few largest terms.

Let us denote $\bar{u}$ the positive integer closest to u:

$$\bar{u} := \begin{cases} [u + 1/2] & \text{if } u \geq 0 \\ 0 & \text{if } u \leq 0 \end{cases} \tag{6.3}$$
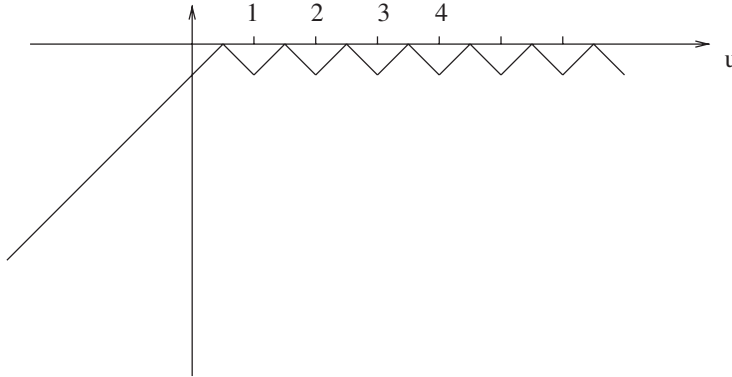
where $[\cdot]$ denotes the integer part.

Figure 4. Behaviour of $\epsilon_u(u - \bar{u}) - \frac{1}{2}$.

Define also:

$$\epsilon_u := \begin{cases} \text{sgn}(u - \bar{u}) & \text{if } u > 0 \\ 1 & \text{if } u \leq 0. \end{cases} \tag{6.4}$$

We always have:

$$\epsilon_u(u - \bar{u}) \leq \tfrac{1}{2}. \tag{6.5}$$

The largest value of $2uk - k^2$ is obtained for $k = \bar{u}$, and the second largest value is obtained for $k = \bar{u} + \epsilon_u$. The difference is (see figure 4):

$$(2u\bar{u} - \bar{u}^2) - (2u(\bar{u} + \epsilon_u) - (\bar{u} + \epsilon_u)^2) = 1 - 2\epsilon_u(u - \bar{u}). \tag{6.6}$$

Remember that our asymptotics for Z_n are valid only up to order $O(N^{-1/2\nu})$, i.e. we want to have:

$$\epsilon_u(u - \bar{u}) - \tfrac{1}{2} \geq -\tfrac{1}{2} \tag{6.7}$$

which implies that our asymptotics for Z_n are valid only if $u > 0$ and $u \notin \mathbf{N}$.

6.2. Possible asymptotic regimes for the orthogonal polynomials

We have similar considerations for the asymptotics of orthogonal polynomials, except that the sum over k is now shifted by $1/2$. This gives different regimes.

Equation (5.37) becomes in that regime

$$\psi_{N+p}(x_0) \sim N^{1/8\nu} \sqrt{\frac{A}{2 \sinh \phi_e}}$$

$$\times \frac{\sum_{k=0}^{N+p} N^{((2(k+1/2)u - (k+1/2)^2)/2\nu)} e^{(k+1/2)\phi_e} \sqrt{A_k A_{k+1}} \, \psi_{k,\nu}(y)}{\sqrt{\left(\sum_{k=0}^{N+p+1} N^{-k^2/2\nu} e^{2kp\phi_e} e^{2k\phi_e} A_k\right)\left(\sum_{k=0}^{N+p} N^{-k^2/2\nu} e^{2kp\phi_e} A_k\right)}}$$

$$\times (1 + O(N^{-1/2\nu})). \tag{6.8}$$

The sum in the numerator is dominated by the largest values of

$$2(k + 1/2)u - (k + 1/2)^2, \tag{6.9}$$

i.e. by $k = [u] = \bar{u} + (\epsilon_u - 1)/2$.

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The second largest term is obtained for $k = \bar{u} - (\epsilon_u + 1)/2$. Notice that if $u \geq \frac{1}{2}$, the two largest terms are always $\bar{u}$ and $\bar{u} - 1$. The difference is

$$(2(\bar{u} + 1/2)u - (\bar{u} + 1/2)^2) - (2(\bar{u} - 1/2)u - (\bar{u} - 1/2)^2) = 2(u - \bar{u}). \tag{6.10}$$

Since our asymptotics are valid only up to order $\mathrm{O}(N^{-1/2\nu})$, the subleading term should be discarded if $|u - \bar{u}| \geq \frac{1}{2}$, i.e. if $u \leq \frac{1}{2}$ or if u is half-integer.

This implies that our asymptotics for ψ_n are valid only if $u > \frac{1}{2}$ and $u \notin \mathbf{N} + \frac{1}{2}$.

6.3. Asymptotics in the regime $u > 0$, and u not integer or half-integer

From now on, we write:

$$n = N + p, \qquad p = \frac{u}{2\nu\phi_e} \ln N. \tag{6.11}$$

In this section, we assume that $u > 0$ and u not integer or half-integer. The sum over k in equation (6.2) is dominated by the terms $k = \bar{u}$ and $k = \bar{u} + \epsilon_u$.

6.3.1. Coefficient γ_n.   Thus we have:

$$
\begin{aligned}
Z_{N+p}&\left(\frac{N+p}{N}T_{\mathrm{c}}\right) \\
&\sim H_N\, \mathrm{e}^{-(N^2/T_{\mathrm{c}}^2)\bar{F}(T_{\mathrm{c}},V)}\mathrm{e}^{-\bar{F}^{(1)}(T_{\mathrm{c}},V)} \quad (2\pi)^p\, N^p\, \mathrm{e}^{-(Np/T_{\mathrm{c}})V_{\mathrm{eff}}(e)} \\
&\quad \times\, N^{(2u\bar{u}-\bar{u}^2)/2\nu}\,\left(A_{\bar{u}} + N^{(|u-\bar{u}|-(1/2))/\nu}\, A_{\bar{u}+\epsilon_u} + \mathrm{O}(N^{-1/2\nu})\right).
\end{aligned} \tag{6.12}
$$

We also obtain:

$$h_{N+p} \sim 2\pi\, \mathrm{e}^{-(N/T_{\mathrm{c}})V_{\mathrm{eff}}(e)}\, \mathrm{e}^{2\bar{u}\phi_e}\left(1 + 2\epsilon_u\, \mathrm{e}^{\epsilon_u\phi_e}\sinh\phi_e\, N^{(|u-\bar{u}|-(1/2))\nu}\,\frac{A_{\bar{u}+\epsilon_u}}{A_{\bar{u}}} + \mathrm{O}(N^{-(1/2)\nu})\right) \tag{6.13}$$

and:

$$\gamma_{N+p} \sim 1 + 2\sinh^2\phi_e\, N^{(|u-\bar{u}|-(1/2))/\nu}\,\frac{A_{\bar{u}+\epsilon_u}}{A_{\bar{u}}} + \mathrm{O}(N^{-1/2\nu}). \tag{6.14}$$

γ_{N+p} is nearly periodic, with period $\ln N/2\nu\phi_e$. The amplitude is minimal of order $N^{-1/2\nu}$ for u integer, and is maximal of order 1 for u half-integer; cf figure 5.

6.3.2. Coefficient β_n.   The sum in equation (5.45) is dominated by $k = \bar{u}$ and $k = \bar{u} + \epsilon_u$, i.e.:

$$\beta_{N+p} \sim 4\sinh^2\phi_e\, N^{(2|u-\bar{u}|-1)/2\nu}\, \mathrm{e}^{\epsilon_u\phi_e}\,\frac{A_{(\bar{u}+\epsilon_u)}}{A_{\bar{u}}}. \tag{6.15}$$
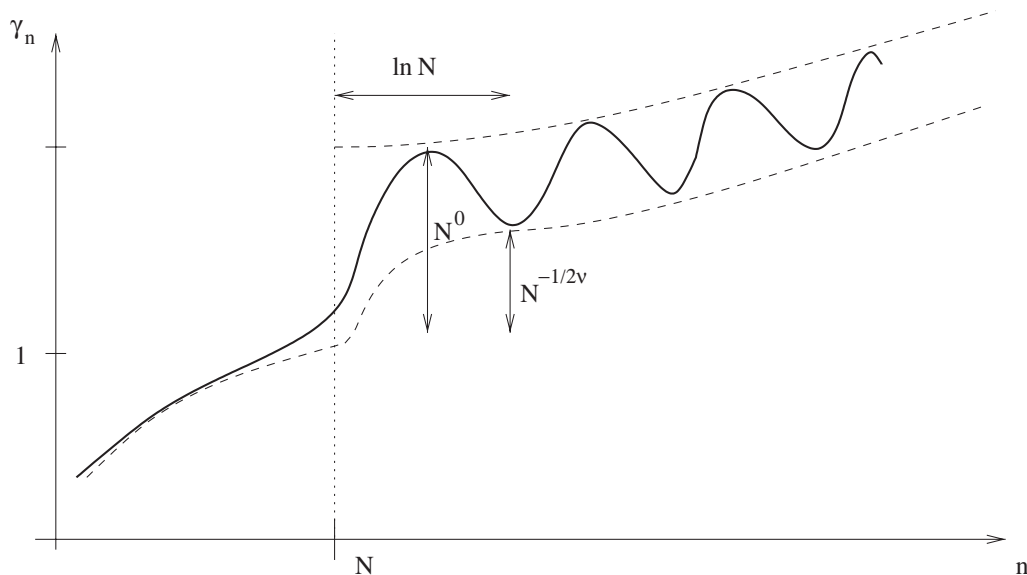
Figure 5. Behaviour of γ_n.

6.3.3. Orthogonal polynomials. The two largest terms in the numerator of equation (5.37) correspond to $k = \bar{u}$ and $k = \bar{u} - 1$ (not necessarily in this order); thus:

$$\psi_{N+p}(x_0) \sim \sqrt{\frac{A}{2 \sinh \phi_e}}$$
$$\times \frac{N^{(u-\bar{u})/2\nu} \, \mathrm{e}^{\phi_e/2} \, \sqrt{A_{\bar{u}+1}/A_{\bar{u}}} \, \psi_{\bar{u},\nu}(y) + N^{(\bar{u}-u)/2\nu} \, \mathrm{e}^{-\phi_e/2} \, \sqrt{A_{\bar{u}-1}/A_{\bar{u}}} \, \psi_{\bar{u}-1,\nu}(y)}{1 + \cosh \phi_e \, N^{(|u-\bar{u}|-(1/2))/\nu} \, \mathrm{e}^{\epsilon_u \phi_e} \, (A_{\bar{u}+\epsilon_u}/A_{\bar{u}})}$$
$$\times (1 + \mathrm{O}(N^{-1/2\nu})). \tag{6.16}$$

We also have:

$$\psi_{N+p-1}(x_0) \sim \sqrt{\frac{A}{2 \sinh \phi_e}} \Big[N^{(u-\bar{u})/2\nu} \, \mathrm{e}^{-\phi_e/2} \, \sqrt{A_{\bar{u}+1}/A_{\bar{u}}} \, \psi_{\bar{u},\nu}(y)$$
$$+ N^{(\bar{u}-u)/2\nu} \, \mathrm{e}^{\phi_e/2} \, \sqrt{A_{\bar{u}-1}/A_{\bar{u}}} \, \psi_{\bar{u}-1,\nu}(y) \Big]$$
$$\times [1 + \cosh \phi_e \, N^{(|u-\bar{u}|-(1/2))/\nu} \, \mathrm{e}^{-\epsilon_u \phi_e} \, (A_{\bar{u}+\epsilon_u})/A_{\bar{u}}]^{-1} (1 + \mathrm{O}(N^{-1/2\nu})). \tag{6.17}$$

6.3.4. Hilbert transforms. Similarly, using equation (5.41), we find that the Hilbert transform of the orthogonal polynomial π_n are asymptotically given by:

$$\phi_{N+p-1}(x_0) \sim \sqrt{\frac{A}{2 \sinh \phi_e}} [N^{(u-\bar{u})/2\nu} \, \mathrm{e}^{-\phi_e/2} \, \sqrt{(A_{\bar{u}+1}/A_{\bar{u}})} \, \phi_{\bar{u},\nu}(y)$$
$$+ N^{-(u-\bar{u})/2\nu} \, \mathrm{e}^{\phi_e/2} \, \sqrt{(A_{\bar{u}-1}/A_{\bar{u}})} \, \phi_{\bar{u}-1,\nu}(y)]$$
$$\times [1 + \cosh \phi_e \, N^{(2|u-\bar{u}|-1)/2\nu} \, \mathrm{e}^{-\epsilon_u \phi_e} \, (A_{\bar{u}+\epsilon_u}/A_{\bar{u}})]^{-1} \tag{6.18}$$

and:

$$
\begin{aligned}
\phi_{N+p}(x_0) \sim \sqrt{\frac{A}{2\sinh\phi_e}} [& N^{(u-\bar{u})/2\nu}\, \mathrm{e}^{\phi_e/2}\, \sqrt{(A_{\bar{u}+1}/A_{\bar{u}})}\, \phi_{\bar{u},\nu}(y) \\
& + N^{-(u-\bar{u})/2\nu}\, \mathrm{e}^{-\phi_e/2}\, \sqrt{(A_{\bar{u}-1}/A_{\bar{u}})}\, \phi_{\bar{u}-1,\nu}(y)] \\
& \times [1 + \cosh\phi_e\, N^{(2|u-\bar{u}|-1)/2\nu}\, \mathrm{e}^{\epsilon_u\phi_e}\, (A_{\bar{u}+\epsilon_u}/A_{\bar{u}})]^{-1}.
\end{aligned} \tag{6.19}
$$

6.3.5. Matrix form. The matrix:

$$
\Psi_n(x) = \begin{pmatrix} \psi_{n-1}(x) & \phi_{n-1}(x) \\ \psi_n(x) & \phi_n(x) \end{pmatrix} \tag{6.20}
$$

is in that regime ($u \geq 0$):

$$
\Psi_n(x) \sim \sqrt{\frac{A}{2\sinh\phi_e}}\, L^{-1} \begin{pmatrix} \mathrm{e}^{(1/2)\phi_e} & \mathrm{e}^{-(1/2)\phi_e} \\ \mathrm{e}^{-(1/2)\phi_e} & \mathrm{e}^{(1/2)\phi_e} \end{pmatrix} R \begin{pmatrix} \psi_{\bar{u}-1,\nu}(y) & \phi_{\bar{u}-1,\nu}(y) \\ \psi_{\bar{u},\nu}(y) & \phi_{\bar{u},\nu}(y) \end{pmatrix} \tag{6.21}
$$

where:

$$
R = \mathrm{diag}\left(N^{-(u-\bar{u})/2\nu}\sqrt{\frac{A_{\bar{u}-1}}{A_{\bar{u}}}}, N^{(u-\bar{u})/2\nu}\sqrt{\frac{A_{\bar{u}+1}}{A_{\bar{u}}}} \right) \tag{6.22}
$$

$$
L = 1 + \cosh\phi_e\, N^{(2|u-\bar{u}|-1)/2\nu}\, \frac{A_{\bar{u}+\epsilon_u}}{A_{\bar{u}}} \begin{pmatrix} \mathrm{e}^{-\epsilon_u\phi_e} & 0 \\ 0 & \mathrm{e}^{\epsilon_u\phi_e} \end{pmatrix}. \tag{6.23}
$$

6.3.6. Kernel. The kernel $K_n(x,x')$ is given by the Christoffel–Darboux formula:

$$
K_n(x,x') = \gamma_n\, \frac{\psi_n(x)\psi_{n-1}(x') - \psi_n(x')\psi_{n-1}(x)}{x - x'}. \tag{6.24}
$$

We find:

$$
K_n(x,x') \sim \frac{N^{1/2\nu}A}{4\sinh^2\phi_e}\, \gamma_{\bar{u},\nu}\, \frac{\psi_{\bar{u},\nu}(y)\psi_{\bar{u}-1,\nu}(y') - \psi_{\bar{u}-1,\nu}(y)\psi_{\bar{u},\nu}(y')}{(y - y')} \tag{6.25}
$$

i.e.:

$$
K_n(x,x') \sim K_{\bar{u},\nu}(y,y')\, \frac{\mathrm{d}y}{\mathrm{d}x}. \tag{6.26}
$$

6.3.7. Large u limit. Using Stirling's formula equation (A.4) (cf appendix A), we find that for large u, all those asymptotics match with the classical limit of section 4.4.

7. Conclusion

We have computed the asymptotics of orthogonal polynomials in the birth of a cut critical limit. This corresponds to the appearance of a new connected component for the support of the eigenvalue density, away from other cuts.

We have found some universal behaviour, which depends only on the degree ν of vanishing of the density at the new cut, and $2\cosh\phi_e$ which parametrizes the distance between the new cut and the old cut. The parametrix near the new cut is simply the system corresponding to a model matrix model in the potential $x^{2\nu}$, and with $\bar{u} = [\frac{1}{2} + 2\nu\phi_e((n-N)/\ln N)]$ eigenvalues.

This new universal behaviour does not seem to correspond to a conformal field theory (unlike previously known critical behaviours), because the exponent of N is not constant.

It would be interesting to complete the 'physicist's proof' presented here with a mathematical one, for instance using Riemann–Hilbert methods as in [2, 11].

Acknowledgments

Appendix A: The Stirling formula and other asymptotics

The Stirling formula is:

$$n! \sim n^n \mathrm{e}^{-n} \sqrt{2\pi n} \left(1 + \frac{1}{12n} + \cdots\right) \tag{A.1}$$

from which we deduce:

$$\ln\left(1\ldots(n-1)!\right) \sim \frac{1}{2}n^2 \ln n - \frac{3}{4}n^2 + \frac{n}{2}\ln 2\pi - \frac{1}{12}\ln n + \mathrm{O}(1) \tag{A.2}$$

and:

$$\ln H_n \sim n\ln 2\pi - \frac{\ln n}{12} + \cdots. \tag{A.3}$$

Asymptotics of $\zeta_{k,\nu}$. For large k we have:

$$\ln \zeta_{k,\nu} \sim (k^2/2\nu)\ln k - \frac{3}{4\nu}k^2 + \frac{k}{\nu}\ln k + \mathrm{O}(k). \tag{A.4}$$

Appendix B: Elliptical functions

We introduce a few definitions concerning elliptical functions [24]: the elliptical sine function $\mathrm{sn}(v, m)$ is defined by the following identity:

$$v = \int_0^{\mathrm{sn}(v,m)} \frac{\mathrm{d}y}{\sqrt{(1-y^2)(1-my^2)}}. \tag{B.1}$$

The complete integrals are defined by:

$$K(m) := \int_0^1 \frac{\mathrm{d}y}{\sqrt{(1-y^2)(1-my^2)}} \tag{B.2}$$

$$K'(m) := K(1-m) = \int_0^\infty \frac{\mathrm{d}y}{\sqrt{(1+y^2)(1+my^2)}} \tag{B.3}$$

$$E(v, m) := \int_0^{\mathrm{sn}(v,m)} \sqrt{\frac{1-my^2}{1-y^2}}\, \mathrm{d}y \tag{B.4}$$

$$E(m) := \int_0^1 \sqrt{\frac{1-my^2}{1-y^2}}\, \mathrm{d}y, \qquad E'(m) := E(1-m). \tag{B.5}$$

When $m \to 0$ one has:

$$
\begin{aligned}
K &\sim \frac{\pi}{2} \left(1 + \frac{m}{4} + \frac{9m^2}{64} + \cdots + \mathrm{O}(m^3) \right) \\
K' &\sim \ln \frac{1}{\sqrt{m}} \left(1 + \frac{m}{4} + \frac{9m^2}{64} + \cdots + \mathrm{O}(m^3) \right) \\
E &\sim \frac{\pi}{2} \left(1 - \frac{m}{4} - \frac{3m^2}{64} + \cdots + \mathrm{O}(m^3) \right) \\
E' &\sim 1 - \frac{m}{2} \ln \frac{1}{\sqrt{m}} + \mathrm{O}(m^2).
\end{aligned}
\tag{B.6}
$$

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