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J. Stat. Mech. (2006) P04004

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Abstract. We introduce generalizations of Aldous's Brownian continuous random tree as scaling limits for multicritical models of discrete trees. These discrete models involve trees with fine-tuned vertex-dependent weights ensuring a kth root singularity in their generating function. The scaling limit involves continuous trees with branching points of order up to $k + 1$. We derive explicit integral representations for the average profile of this kth order multicritical continuous random tree, as well as for its history distributions measuring multi-point correlations. The latter distributions involve non-positive universal weights at the branching points together with fractional derivative couplings. We prove universality by rederiving the same results within a purely continuous axiomatic approach based on the resolution of a set of consistency relations for the multi-point correlations. The average profile is shown to obey a fractional differential equation whose solution involves hypergeometric functions and matches the integral formula of the discrete approach.

Contents

1. Introduction

1.1. Motivations

In recent years, a number of advances have related the combinatorics of various classes of planar maps to that of decorated plane trees. More precisely, bijections based on appropriate cutting procedures were found between maps with prescribed vertex valences and so-called blossom trees [1]–[4], as well as between maps with prescribed face valences and so-called labelled mobiles [5, 6], which are simply trees made of polygonal plaquettes. In this latter formulation, the labelling of the tree may be thought of as a one-dimensional embedding of its vertices. This allows [5, 7] for relating universal statistical properties of generic large random maps, known in the physics language as two-dimensional quantum gravity (2DQG) [8]–[10], to universal properties of large random embedded trees, captured in limiting models of embedded continuous random trees such as the Brownian snake [11] and the integrated super-Brownian excursion (ISE) in one dimension [12].

A simpler version of this equivalence concerns maps describing the so-called two-dimensional Lorentzian gravity (2DLG), i.e. random maps with a regular shell structure in one direction [13, 14]. As illustrated in figures 1 and 2, these are in one-to-one correspondence with plane trees without decorations (figure 1), or with unlabelled mobiles (figure 2). The universal large scale properties of generic Lorentzian random maps are now simply captured by that of a limiting model of continuous trees, the Brownian continuous random tree (CRT) of Aldous [15].
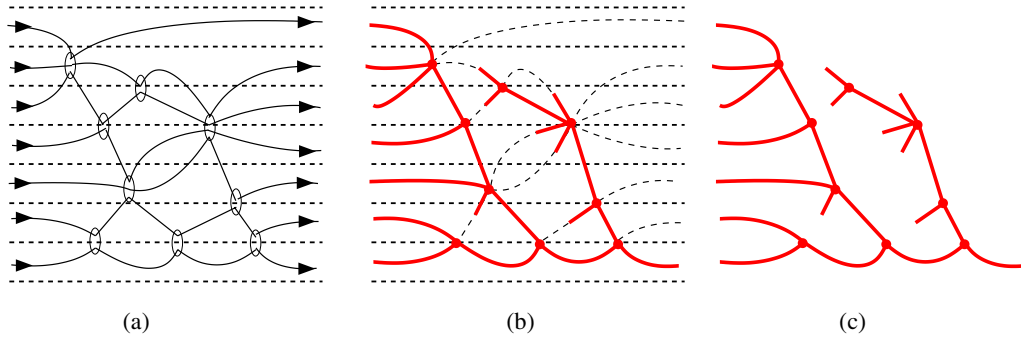
Figure 1. An example of a 2D Lorentzian map (a) with *vertices* of even valences and a vertical shell structure, with slices delimited by dashed lines. For each slice, an oriented line enters from the left and exits to the right. The lines are non-crossing but may come into (possibly multiple) contacts, giving rise to the (circled) vertices of the map. At each $2p$-valent vertex, we erase (b) the $p-1$ (dashed) upper rightmost outgoing edge halves, thus forming a tree (c) which can be planted at its lower rightmost edge.
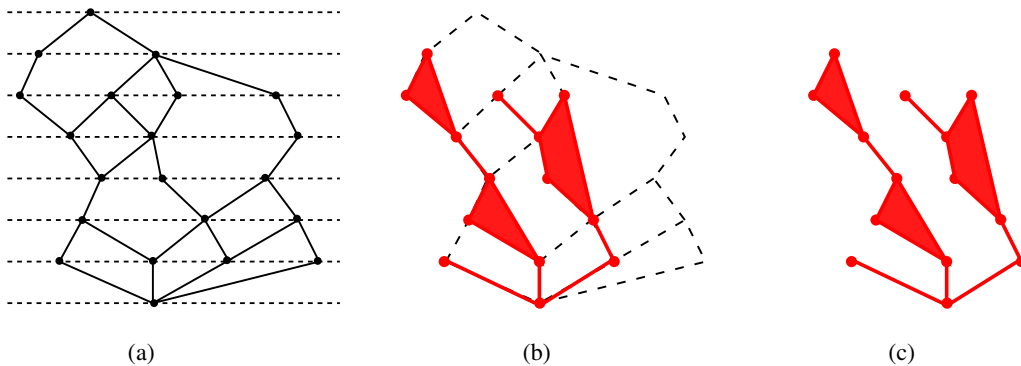

Figure 2. An example (a) of a 2D Lorentzian map (dual to that of figure 1), made of polygonal *faces* of even valences, and with a regular shell structure in the vertical direction. In each $2p$-valent face, we pick the p lower leftmost vertices and join them into a (shaded) polygonal plaquette (which reduces to a segment when $p = 2$). The collection of these plaquettes forms a mobile (c), planted at the bottom vertex.

The above equivalences may also be extended to maps with matter degrees of freedom, namely carrying configurations of statistical models such as the Ising model [16] or hard particles [17, 18]. In the language of quantum gravity, these models may be fine-tuned so as to reach new critical points at which their large scale statistical properties follow different laws to those of generic maps. These laws fall into universality classes characterized in particular by their order of multicriticality. When transposed into the tree language, this should give rise to natural multicritical generalizations of the one-dimensional Brownian snake or ISE, as well as of the CRT.

A first family of multicritical points of quantum gravity may be reached by considering maps with weights depending on the vertex valences and by fine-tuning these weights [19].

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This allows one to attain a kth order multicritical point for any integer $k \geq 2$, both in the standard 2DQG framework and in the simpler 2DLG. Via the tree correspondence, we expect similar kth order multicritical points to arise in various fine-tuned limits of large tree models. Some properties of a kth order multicritical ISE might be inferred from [7] by interpreting the multicritical two-point scaling function derived there as the average profile of a 'multicritical' continuous random tree embedded in the positive half-line. Similarly, some properties of a kth order multicritical CRT may be in principle read off from the results of [14] for multicritical 2DLG. A general understanding of multicritical ISE and CRT is however still lacking.

The aim of this paper is to provide in the simpler case of unlabelled trees an explicit construction of a limiting model of a multicritical continuous random tree, hereafter referred to as MCRT. More precisely, we will derive basic properties of a kth order multicritical continuous random tree (denoted as MCRT_k) from suitable scaling limits of discrete models of planted plane trees with properly fine-tuned vertex weights. These properties are expected to be universal and are used in a second step as guidelines for an independent self-consistent definition of the MCRT.

Beyond the above-mentioned connection with 2DLG, another interest of the MCRT is that it should also describe the universal properties of trees carrying particles subject to proper exclusion rules, such as that of [18], and tuned to be at their multicritical points (generalized Lee–Yang edge singularities). In a discrete formulation, these trees may indeed be transformed into trees without particles under a redistribution of the weights on the vertices. The critical points of such trees with particles will therefore lie in the very same universality classes of MCRTs as those constructed here.

The paper is organized as follows. In section 1.2 below, we present a short summary of the main basic properties of the CRT, following Aldous [12, 15]. Section 2 is devoted to a construction of the MCRT from discrete models of trees. These models are defined in all generality in section 2.1 and various statistical properties are introduced, such as the average profile and the so-called m-point correlation functions. We then discuss in section 2.2 the emergence of critical and multicritical points in these models. The continuum scaling limits of large trees at these multicritical points are investigated in section 2.3 where we derive the average profile of the MCRT, as well as its 'history' distribution, describing the scaling limit of m-point correlations through the branching structure of the common genealogical tree of these points (hereafter called the history tree). All these distributions are given explicitly via integral formulae which may also be interpreted in terms of fractional derivative couplings at branching points corresponding to the most recent common ancestors of the marked points. A number of properties of the MCRT are discussed in section 2.4, such as the moments of its average profile and the relative weights of the possible branching structures for history trees. Section 3 is devoted to the derivation *ab initio* of the very same properties of the MCRT without reference to a discrete model. The first step (section 3.1) consists in using invariance principles to obtain, with a minimal number of assumptions, a general form for the average profile and history distributions of the MCRT. The precise value of these quantities is derived by solving a set of consistency relations linking the correlations between m marked points to those with $m + 1$. The net result is a fractional differential equation for the average profile. This equation is transformed in section 3.2 into an ordinary differential equation, which is solved in section 3.3. The final result is shown in section 3.4 to coincide with the

explicit formula arising from the discrete approach of section 2. A few concluding remarks are gathered in section 4.

1.2. CRT: main properties

Before we proceed with our construction, it is useful to first recall a few known facts about Aldous's Brownian CRT [12, 15]. This CRT can be understood as the (weak) $N \to \infty$ limit of discrete planted plane trees conditioned to have size N (measured, say, by the number of leaves), with all edges scaled to have length $1/\sqrt{2N}$. The properties of the CRT may be encoded into its average profile $\rho(x)$, which is the limiting density of leaves at a given (rescaled) generation x in the tree[3]. The latter reads

$$\rho(x) = x\mathrm{e}^{-(x^2/2)} \tag{1.1}$$

and is universal up to a possible rescaling $x \to \Lambda x$.

Correlations between points on the tree may be measured by marking, say, m distinguished leaves chosen uniformly at random on the tree and considering the subtree spanned by the root and these leaves, i.e. their common genealogical tree formed by the union of their ancestral lines. This genealogical tree, or *history tree*, H is characterized by its *shape S*, which is a planted plane tree encoding the relative ordering of the most recent common ancestors of the marked leaves, and by its (continuous) branch lengths recording the (rescaled) number of generations between the most recent common ancestors or the marked leaves. Returning to discrete trees of size N, we note that the probability for three or more leaves to have the same most recent common ancestor tends to zero as $N \to \infty$. As a consequence, in the continuous limiting CRT, the only shapes with a non-vanishing measure are *binary* plane trees whose leaves carry the original marked leaves. An example of such an admissible history tree is depicted in figure 3. For fixed m, there are $(2m-2)!/(m-1)!$ possible shapes (including the labelling of the distinct marked leaves). An important property of the CRT is that all these shapes turn out to be equiprobable, with probability

$$w(S) = \frac{(m-1)!}{(2m-2)!}. \tag{1.2}$$

More precisely, the history probability density for a given history tree H depends only on its number $p = m - 1$ of branching points and on the *sum* of all its $2m - 1$ (rescaled) branch lengths x_j, $j = 1, \ldots, 2m - 1$ (see figure 3). It reads

$$\rho(H) = \left(\frac{1}{2}\right)^p \rho\left(\sum_j x_j\right), \tag{1.3}$$

with $\rho(x)$ as in equation (1.1). Note that the prefactor $(1/2)^p$ is an artefact of planarity and may be suppressed by considering non-planar trees, i.e. by recording in a unique shape the 2^p shapes equivalent under arbitrary interchanges of the two descending subtrees around each vertex. Note finally that $w(S)$ is recovered by integrating $\rho(H)$ over the $2m - 1$ branch lengths x_j, which amounts to evaluating the $(2m-2)$th moment of $\rho(x)$.

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[3] We could as well consider the limiting density of edges at a given generation x and we would find the same profile, up to a global normalization factor. We could also consider trees conditioned to have a fixed number N' of *edges* rather than leaves, in which case the same limiting density is recovered provided edge lengths are now scaled by a factor $1/\sqrt{N'}$.
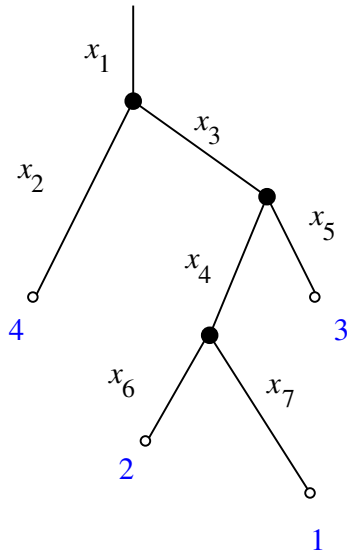
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Figure 3. An example of continuous history tree of the CRT with $m = 4$ distinguished leaves, characterized by its shape (planted binary plane tree with labelled leaves) and its $2m - 1 = 7$ branch lengths x_j. The associated probability density, as given by equation (1.3), is a function of the number $p = m - 1 = 3$ of branching points and of the total length $x = \sum_j x_j$ only.

2. MCRT from discrete models

2.1. Discrete models

The simplest approach to MCRT is via discrete models of random trees. In this section, we shall concentrate on *planted plane trees*, i.e. trees with a distinguished leaf (the *root*), and with vertices of arbitrary but uniformly bounded valences. For $i \geq 1$, each $(i + 1)$-valent vertex gives rise to i *distinguished* descendent subtrees and receives a *weight* g_i (with the convention that $g_i = 0$ for $(i + 1)$ larger than the maximal allowed vertex valence). We shall be interested in the ensemble $\mathcal{E}_N$ of trees with *fixed size*, defined as the total number N of *leaves* in the tree. Each tree $\mathcal{T}$ therefore receives a total weight $\mu(\mathcal{T})$ equal to

$$\mu(\mathcal{T}) = \frac{1}{Z_N(\{g_i\})} \prod_{\substack{i \geq 1 \\ n_i(\mathcal{T}) > 0}} g_i^{n_i(\mathcal{T})}, \tag{2.1}$$

where $n_i(\mathcal{T})$ stands for the number of $(i + 1)$-valent vertices in $\mathcal{T}$ ($i \geq 1$), and where the normalization factor, or *partition function*, $Z_N(\{g_i\})$ is chosen so that

$$\sum_{\mathcal{T} \in \mathcal{E}_N} \mu(\mathcal{T}) = 1. \tag{2.2}$$

The MCRT will be associated with various large N scaling limits of this ensemble $\mathcal{E}_N$ for suitably fine-tuned weights.

For convenience, it is also useful to consider the related, *grand canonical*, ensemble $\mathcal{E}$ of trees of arbitrary size, but now with an extra weight λ per leaf. In this new ensemble,
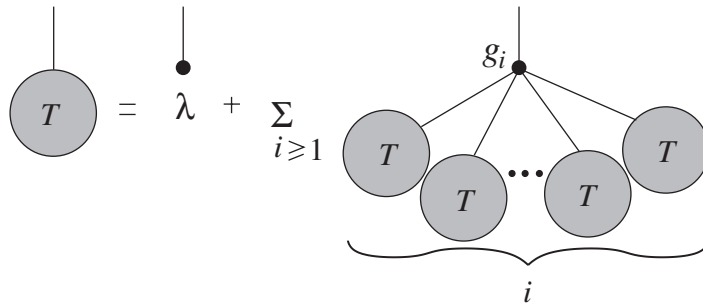
Figure 4. The recursive formula for the tree generating function T in the grand canonical ensemble $\mathcal{E}$. The trees are decomposed according to the environment of the vertex attached to the root. If this vertex is a leaf, it receives a weight λ. If the valence of this vertex is $i + 1$ $(i \geq 1)$, we get a weight g_i for the vertex and a factor T for each of the i descendent subtrees.

each tree $\mathcal{T}$ receives an unnormalized weight

$$\nu(\mathcal{T}) = \lambda^{N(\mathcal{T})} \prod_{\substack{i \geq 1 \\ n_i(\mathcal{T}) > 0}} g_i^{n_i(\mathcal{T})}, \tag{2.3}$$

with $N(\mathcal{T})$ its total number of leaves. Introducing the corresponding generating function

$$T \equiv \sum_{\mathcal{T} \in \mathcal{E}} \nu(\mathcal{T}) = \sum_{N \geq 1} \lambda^N Z_N(\{g_i\}), \tag{2.4}$$

we have the recursive formula illustrated in figure 4:

$$T = \lambda + f(T), \qquad f(T) \equiv \sum_{i \geq 1} g_i T^i, \tag{2.5}$$

obtained by inspecting the environment of the vertex attached to the root. Note that, as valences are bounded, f is polynomial in T. Equation (2.5) determines uniquely $T(\lambda)$ as a formal power series of λ whose coefficients are polynomials of the g_is, and which is the functional inverse of $\lambda(T) \equiv T - f(T)$ defined in the vicinity of $T = 0$.

The knowledge of $T(\lambda)$ gives access to the partition function Z_N via the contour integral (encircling the origin)

$$Z_N = \oint \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi \lambda^{N+1}} T(\lambda). \tag{2.6}$$

More generally, we may consider the *average profile* $\rho_N(L)$ of trees in $\mathcal{E}_N$, equal to the weighted sum over trees of size N with a marked leaf at distance $L \geq 1$ from the root. The average profile reads

$$\rho_N(L) = \frac{1}{NZ_N} \oint \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi \lambda^{N+1}} \{f'(T(\lambda))\}^{L-1} \lambda. \tag{2.7}$$

Indeed, the marked leaf defines a unique minimal path from the root passing through $L - 1$ inner vertices. At each of these $L - 1$ vertices, the choice of the outgoing edge
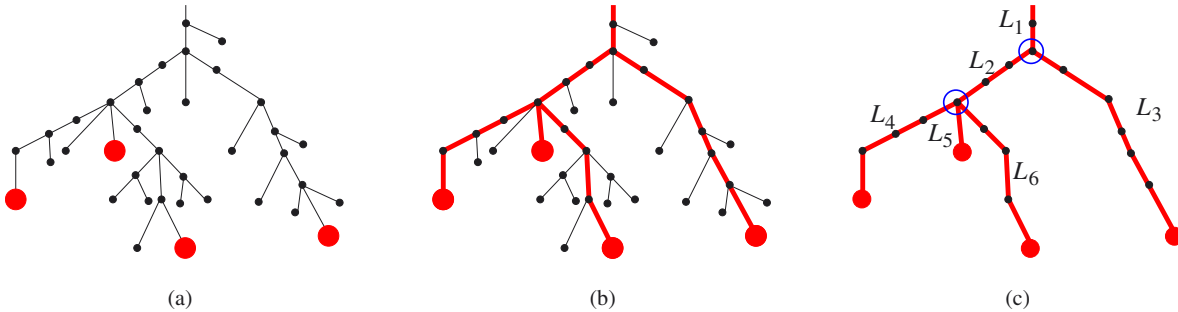
Figure 5. Starting from a marked tree (a), here with $m = 4$ marked leaves (filled circles), we form (b) the union of all minimal paths (thick edges) from the root to these leaves. Dropping all other edges and vertices, this defines a marked history tree (c) whose branching points (encircled here) correspond to pairwise most recent common ancestors. With each branch is associated a length $L_j \geq 1$. Here $L_5 = 1$, $L_1 = 2$, $L_2 = 3$, $L_4 = L_6 = 4$ and $L_3 = 6$.

produces the weight $f'(T(\lambda))$. Finally the marked leaf receives a weight λ. The prefactor $1/(NZ_N)$ ensures the correct normalization $\sum_{L \geq 1} \rho_N(L) = 1$.

We may also consider m-point correlation functions, i.e. averages over trees with a number $m \geq 2$ of marked leaves chosen uniformly and independently at random among all leaves different from the root. For each such marked tree $\mathcal{T}$, we may define its *history tree* $\mathcal{H}$ as the union of all minimal paths from the root to the m marked leaves (see figure 5). This union of *ancestral lines* indeed forms a tree whose branching points correspond to most recent common ancestors of the marked leaves. For any history tree $\mathcal{H}$, let us denote by p_i, $i \geq 2$, its number of $(i+1)$-valent (on $\mathcal{H}$) branching points and p_0 its number of leaves (other than the root). The branching points of $\mathcal{H}$ divide it into a number n of 'branches', with

$$n = p_0 + \sum_{i \geq 2} p_i = 1 + \sum_{i \geq 2} i p_i. \tag{2.8}$$

These branches have lengths $L_j \geq 1$, $j = 1, \ldots, n$. Note that the history tree must be supplemented with the data of the positions of the m marked leaves: this will be called a *marked history tree* (still denoted as $\mathcal{H}$ by a slight abuse of notation). By construction, all the p_0 leaves must be marked at least once. This implies the inequality

$$m \geq p_0 = 1 + \sum_{i \geq 2} (i-1) p_i. \tag{2.9}$$

A marked history tree will be called *non-degenerate* if no two marked leaves are equal. This corresponds to having $m = p_0$. It is easily seen that non-degenerate history trees dominate in the large N limit. The weighted sum over all marked trees sharing the same marked history $\mathcal{H}$ reads

$$\rho_N(\mathcal{H}) = \frac{1}{N^m Z_N} \oint \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi \lambda^{N+1}} \{f'(T)\}^{\sum_j (L_j - 1)} \prod_{i \geq 2} \left\{ \frac{f^{(i)}(T)}{i!} \right\}^{p_i} \lambda^{p_0} \tag{2.10}$$

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by a straightforward generalization of the above argument leading to equation (2.7). The factors $f^{(i)}(T)/i!$ account for the choices of the i descending subtrees at $(i + 1)$-valent branching points. The prefactor $1/(N^m Z_N)$ ensures the correct normalization $\sum_{\mathcal{H}} \rho_N(\mathcal{H}) = 1$, where the sum runs over all (possibly degenerate) histories with m marked leaves.

2.2. Critical and multicritical points

For fixed g_is, the large N properties of the ensemble $\mathcal{E}_N$ are related to the properties of the ensemble $\mathcal{E}$ when $\lambda \to \lambda_c$, where λ_c is the radius of convergence of $T(\lambda)$. It is obtained by first writing $\lambda'(T) = 0$, which determines $T = T_c$, and then identifying $\lambda_c = \lambda(T_c)$. This equivalently amounts to the system of equations[4]

$$T_c = \lambda_c + f(T_c) \qquad \text{and} \qquad 1 = f'(T_c). \tag{2.11}$$

For a generic choice of the g_is, we have $f''(T_c) \neq 0$, which implies, by a Taylor expansion of λ around T_c, that $(\lambda_c - \lambda) \propto (T_c - T)^2$, corresponding to a generic square root singularity of T as a function of λ. A higher multicritical behaviour of order $k > 2$ may be reached by tuning the g_is so as to enforce the relations

$$f''(T_c) = \cdots = f^{(k-1)}(T_c) = 0, \qquad f^{(k)}(T_c) \neq 0. \tag{2.12}$$

A Taylor expansion of λ around T_c now leads to the behaviour

$$\frac{\lambda_c - \lambda}{\lambda_c} \sim A \left(\frac{T_c - T}{T_c} \right)^k, \qquad A \equiv (-1)^k \frac{f^{(k)}(T_c)}{k!} \frac{T_c^k}{\lambda_c}, \tag{2.13}$$

leading to a kth root singularity of T as a function of λ. A minimal realization of conditions (2.12) above consists in taking

$$f(T) = \frac{kT - 1 + (1 - T)^k}{k} \tag{2.14}$$

in which case $T_c = 1$, $\lambda_c = 1/k$ and $A = 1$. This corresponds to the choice $g_i = g_i^*$

$$g_1^* = 0 \qquad \text{and} \qquad g_i^* = \frac{(-1)^i}{k} \binom{k}{i} \qquad \text{for } i = 2, \ldots, k \tag{2.15}$$

while $g_i^* = 0$ for $i > k$.

In the following, we shall consider the general case of a function $f(T)$ satisfying the kth order multicriticality conditions (2.12). However, as the values of T_c, λ_c and A are non-universal, we decide to fix them, without loss of generality, to the same values as in the minimal case (2.14) above, namely $T_c = A = 1$, $\lambda_c = 1/k$. This amounts to taking a function $f(T)$ satisfying

$$f(T) = \frac{kT - 1 + (1 - T)^k}{k} + \mathcal{O}\left((1 - T)^{k+1} \right) \tag{2.16}$$

and making sure that the radius of convergence of $T(\lambda)$ remains at $\lambda = \lambda_c = 1/k$. Writing $f(T) = \sum_i g_i T^i$, the property (2.16) is obtained by imposing $k + 1$ relations on the g_is. In particular, reaching a kth order multicritical point requires one to consider planted plane trees with at least $k - 1$ different values for the inner vertex valences strictly larger that 2. For $k > 2$, this moreover forces some of the g_is to be *negative*. The minimal choice (2.14) amounts to picking the lowest possible vertex valences, namely $3, \ldots, k + 1$.

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[4] Note that, in general, this system has more than one solution and we must select among those the actual radius of convergence λ_c.
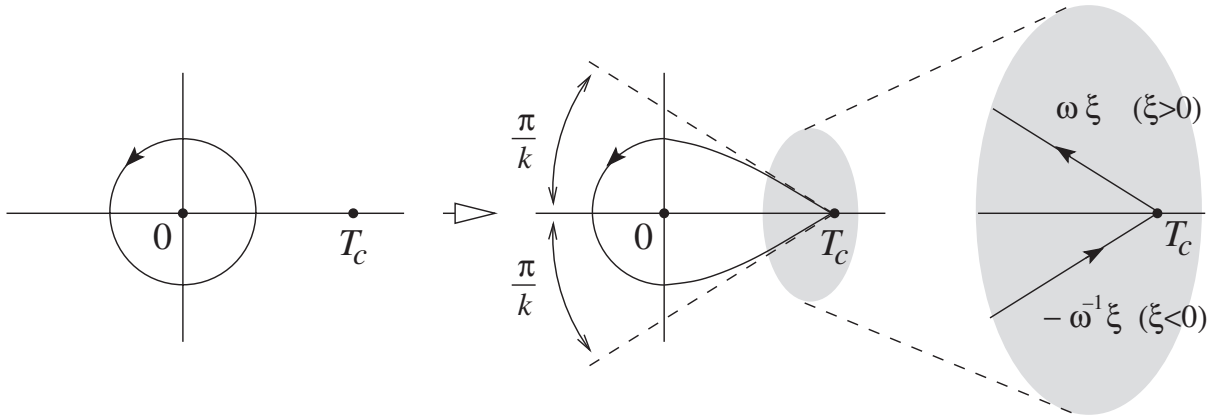
Figure 6. The contour of integration in equation (2.17) for the T variable in the complex plane. We deform the contour on the left so as to approach the large N saddle point value T_c. The approach to T_c is tangent to the lines with angles $\pm\pi/k$ corresponding to the real values of $\lambda(T)$ with a minimum at T_c. At large N, the integral is dominated by the vicinity of T_c (zoomed here) and amounts to an integral over a real ξ variable as shown, with $\omega = \mathrm{e}^{\mathrm{i}\pi((k-1)/k)}$.

2.3. Continuum limit

In the light of the discussion of previous section, we may now consider the large N behaviour of the functions Z_N, $\rho_N(L)$ and $\rho_N(\mathcal{H})$ of equations (2.6), (2.7) and (2.10) at a kth order multicritical point, i.e. for fixed values of the g_is that ensure equation (2.12). We will be led to various rescalings of the trees, eventually allowing for a continuous formulation describing a kth *order multicritical continuous random tree* of fixed size 1, hereafter referred to as the MCRT$_k$.

The integral (2.6) for Z_N may be evaluated by first changing variables from λ to T, namely using

$$Z_N = \oint \frac{\mathrm{d}T}{2\mathrm{i}\pi} T \frac{\lambda'(T)}{\lambda(T)} \exp(-N \operatorname{Log}\lambda(T)). \tag{2.17}$$

Assuming the generic form (2.16) for $f(T)$ and writing $\lambda(T) = T - f(T)$, this integral is dominated at large N by the saddle point $T = T_c = 1$ around which equation (2.13) holds with $\lambda_c = \lambda(T_c) = 1/k$. This suggests deforming the contour of integration for T as displayed in figure 6. In the vicinity of T_c, this amounts to switching to a real variable ξ such that

$$T = T_c \left(1 + \omega^{\operatorname{sgn}(\xi)} \frac{|\xi|}{(kN)^{1/k}}\right) \tag{2.18}$$

where $\omega = \mathrm{e}^{\mathrm{i}\pi((k-1)/k)}$, leading to

$$\lambda \sim \lambda_c \left(1 + A \frac{|\xi|^k}{kN}\right) \tag{2.19}$$

with $A = 1$. The precise choice of phase $\omega^{\operatorname{sgn}(\xi)}$ ensures that λ be real and minimal at T_c on the new contour, while the factors of N have been tuned so that the argument of the

exponential term in (2.17) is of order zero. The large N equivalent of Z_N is obtained by integrating ξ on the real axis, leading to

$$
\begin{aligned}
Z_N &\sim \frac{k^N}{2\mathrm{i}\pi N} \int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{sgn}(\xi)|\xi|^{k-1} \exp\left(-\frac{|\xi|^k}{k}\right) \left(1 + \omega^{\mathrm{sgn}(\xi)} \frac{|\xi|}{(kN)^{1/k}}\right) \\
&= \frac{k^{N-(1/k)}}{\pi N^{1+(1/k)}} \operatorname{Im}(\omega) \int_0^{\infty} \mathrm{d}\xi\, \xi^k \exp\left(-\frac{\xi^k}{k}\right) \\
&= \frac{k^N}{\pi N^{1+(1/k)}} \sin\left(\frac{\pi}{k}\right) \Gamma\left(1 + \frac{1}{k}\right).
\end{aligned}
\tag{2.20}
$$

Repeating the above exercise for $\rho_N(L)$ as in equation (2.7), we simply have to take into account the extra factor $(f'(T))^{L-1}$ with

$$
f'(T) \sim 1 - (1-T)^{k-1} = 1 + \tau^{\mathrm{sgn}(\xi)} \frac{|\xi|^{k-1}}{(kN)^{(k-1)/k}}
\tag{2.21}
$$

where $\tau = -(-\omega)^{k-1} = \mathrm{e}^{\mathrm{i}(\pi/k)}$. In order to get a sensible scaling limit, we must take

$$
L \sim x(kN)^{(k-1)/k}.
\tag{2.22}
$$

In our construction, the MCRT_k therefore corresponds to a limit of kth order multicritical discrete trees in which edges are scaled to have length $(kN)^{-(k-1)/k}$ rather than 1. We shall refer to $d_k = k/(k-1)$ as the *fractal dimension* of the MCRT_k ($N \sim L^{d_k}$). With the above redefinitions, the rescaled average profile

$$
\rho(x) \equiv \lim_{N \to \infty} (kN)^{(k-1)/k} \rho_N\left(x(kN)^{(k-1)/k}\right)
\tag{2.23}
$$

takes the form

$$
\begin{aligned}
\rho(x) = \frac{1}{k^{1/k}\Gamma\left((1/k)+1\right)\sin\left(\pi/k\right)} \operatorname{Im}\left\{\int_0^{\infty} \mathrm{d}\xi\, \xi^{k-1} \exp\left(-\frac{\xi^k}{k} + \tau \xi^{k-1} x\right)\right\} \\
\text{with } \tau = \mathrm{e}^{\mathrm{i}(\pi/k)}.
\end{aligned}
\tag{2.24}
$$

This function is the *universal average profile* for all ensembles of kth order multicritical trees. It is represented in figure 7 for $k = 2, 3, 4$. For $k = 2$, equation (2.24) reduces to (1.1) as expected, since the CRT is nothing but the MCRT_2.

We may alternatively obtain an integral transform of $\rho(x)$ by considering the critical continuous counterpart of the equation

$$
\sum_{N \geq 0} \lambda^N N Z_N \rho_N(L) = (f'(T(\lambda)))^{L-1} \lambda
\tag{2.25}
$$

which is the grand canonical equivalent of equation (2.7). Approaching the critical point via

$$
\lambda = \frac{1}{k}\, \mathrm{e}^{-\epsilon}, \qquad L = \frac{x}{\epsilon^{(k-1)/k}}
\tag{2.26}
$$

with $\epsilon \to 0$, we use $f'(T) \sim 1 - \epsilon^{(k-1)/k}$ and $T \to 1$ to rewrite the rhs of equation (2.25) as e^{-x}. The sum in the lhs tends to an integral over a variable $M = \epsilon\, kN$ so equation (2.25)
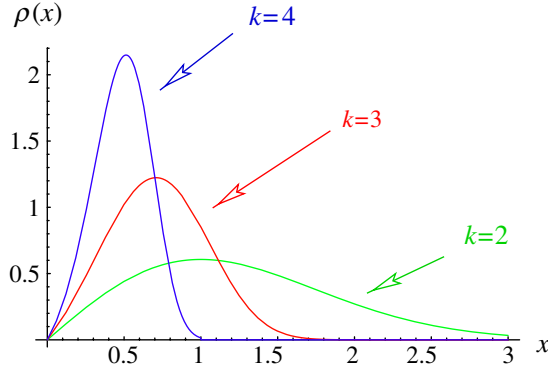
Figure 7. Plot of the universal average profile $\rho(x)$ of equation (2.24) for $k = 2, 3, 4$.

turns into

$$\frac{k^{1/k} \sin(\pi/k) \, \Gamma(1 + (1/k))}{\pi} \int_0^\infty \frac{\mathrm{d}M}{M^{1+(1/k)}} \, \mathrm{e}^{-(M/k)} \left\{ \frac{M}{M^{(k-1)/k}} \rho\left(\frac{x}{M^{(k-1)/k}}\right) \right\} = \mathrm{e}^{-x} \quad (2.27)$$

where we have used equations (2.20) and (2.23). The variable M may be viewed as the (continuous) size of the trees in the MCRT and the quantity

$$\rho_M(x) \equiv \frac{M}{M^{(k-1)/k}} \rho\left(\frac{x}{M^{(k-1)/k}}\right) \quad (2.28)$$

is nothing but the average profile for the MCRT$_k$ with fixed size M, normalized by $\int_0^\infty \rho_M(x) \, \mathrm{d}x = M$.

We may finally derive the scaling limit of the m-point correlation function corresponding to a given marked history tree. Rewriting equation (2.10) as

$$\rho_N(\mathcal{H}) = \frac{1}{N^m Z_N} \oint \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi \lambda^{N+1}} (f'(T))^{\sum_j (L_j - 1)} \prod_{i \geq 2} \left\{ \frac{\lambda^{i-1} f^{(i)}(T)}{i!} \right\}^{p_i} \lambda \quad (2.29)$$

and comparing with $\rho_N(L)$, we simply have to substitute $(L - 1) \to \sum_j (L_j - 1)$ and to insert extra factors of the form $\lambda^{i-1} f^{(i)}(T)/i!$. These factors scale as

$$\left\{ \frac{\lambda^{i-1} f^{(i)}(T)}{i!} \right\} \sim \frac{(-1)^i}{k^i} \binom{k}{i} (1 - T)^{k-i} = \frac{1}{k^{i((k-1)/k)}} \frac{(-1)^i}{k} \binom{k}{i} \left(-\frac{\omega^{\mathrm{sgn}(\xi)} |\xi|}{N^{1/k}} \right)^{k-i} \quad (2.30)$$

for $i = 2, \ldots, k$, while for $i > k$, they tend to constants depending on the g_is only. Again, taking the scaling $L_j \sim x_j (kN)^{(k-1)/k}$ for $j = 1, \ldots, n$ (with n as in equation (2.8)), let us define the rescaled correlation function

$$\rho(H) \equiv \lim_{N \to \infty} (kN)^{n((k-1)/k)} \rho_N(\mathcal{H}). \quad (2.31)$$

Here the notation H stands for the continuous history tree corresponding to $\mathcal{H}$ defined as follows. The tree H has the same branching structure as $\mathcal{H}$, but the n branches are replaced by segments of real lengths $x_j > 0$. Collecting the factors of N in equation (2.29),
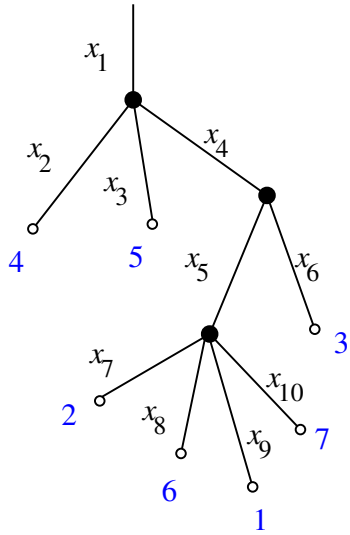
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Figure 8. A continuous history tree H is characterized by its shape, namely a planted plane tree with branching points of valence less than $k+1$ (here $k \geq 4$) and by the real lengths $x_j > 0$ of its branches. The leaves are distinguished and labelled from 1 to m (here $m = 7$).

we get an overall factor $N^{-\alpha}$ in (2.31), with

$$\alpha = m - \frac{1}{k} - n\frac{k-1}{k} + \sum_{i=2}^{k} \frac{k-i}{k} p_i = (m - p_0) + \frac{1}{k} \sum_{i>k} (i-k) p_i. \qquad (2.32)$$

The exponent α is always non-negative from equation (2.9), and vanishes only if $p_i = 0$ for all $i > k$ and $m = p_0$. This selects marked history trees which are non-degenerate and whose branching points have valences less than or equal to $k+1$. In other words, the discrete histories surviving in the continuum limit at a kth order multicritical point are those with branching points with valences $3, 4, \ldots, (k+1)$ only, and all leaves but the root marked exactly once. From now on, we shall always consider such histories. As displayed in figure 8, the associated continuous history H is entirely characterized by the data (i) of a *shape*, namely a planted plane tree with branching points of valence $3, 4, \ldots, (k+1)$ and with *distinguished* leaves labelled $1, \ldots, m$, and (ii) of a collection of real lengths x_j for its edges. We still denote by p_i, $i = 2, \ldots, k$, the number of its $(i+1)$-valent branching points. This generalizes to arbitrary values $k \geq 2$ the binary structure of the CRT discussed in section 1.2, recovered here for $k = 2$. The correlation $\rho(H)$ reads

$$\rho(H) = \frac{\prod_{i=2}^{k} (\mu_i)^{p_i}}{k^{1/k} \Gamma\left((1/k) + 1\right) \sin\left(\pi/k\right)}$$
$$\times \operatorname{Im} \left\{ \int_0^{\infty} d\xi\, \xi^{k-1} \left(\frac{\xi}{\tau}\right)^{\sum_{i=2}^{k} (k-i)p_i} \exp\left(-\frac{\xi^k}{k} + \tau \xi^{k-1} \sum_j x_j\right) \right\} \qquad (2.33)$$

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where $\tau = \mathrm{e}^{\mathrm{i}(\pi/k)}$ and

$$\mu_i \equiv \frac{(-1)^i}{k} \binom{k}{i}. \tag{2.34}$$

This gives the *universal history distribution* for the MCRT$_k$, generalizing the CRT history probability density (1.3). The formula (2.33) may be interpreted as follows. First note that $\rho(H)$ only depends on the total length $x \equiv \sum_j x_j$ of H. Furthermore, with each $(i+1)$-valent branching point of H is associated a weight μ_i, whose value is universal and given by (2.34). Note that these universal weights carry alternating signs for $k \geq 3$. Finally, in addition to these numerical weights, each $(i+1)$-valent branching point contributes a factor $(\xi/\tau)^{k-i}$ to the integrand. Note that for $i = k$, this contribution reduces to 1. In particular, in the case of a history H with only $(k+1)$-valent branching points, the history distribution $\rho(H)$ is simply proportional to $\rho(x)$. The expression (2.33) may be recast in terms of *fractional derivatives* of x as

$$\rho(H) = \prod_{i=2}^{k} \left(\mu_i(-d)^{(k-i)/(k-1)}\right)^{p_i} \rho(x) \tag{2.35}$$

with $x \equiv \sum_j x_j$ and where $\rho(x)$ is the average profile given by equation (2.24). Here the fractional derivative $(-d)^{r/(k-1)}$ for non-negative integer r acts on a function $\varphi(x)$ with the integral representation

$$\varphi(x) = \mathrm{Im}\left\{\int_0^\infty \mathrm{d}\xi\, \tilde{\varphi}(\xi) \exp\left(-\frac{\xi^k}{k} + \tau x \xi^{k-1}\right)\right\} \tag{2.36}$$

as

$$(-d)^{r/(k-1)}\varphi(x) = \mathrm{Im}\left\{\int_0^\infty \mathrm{d}\xi\, \left(\frac{\xi}{\tau}\right)^r \tilde{\varphi}(\xi) \exp\left(-\frac{\xi^k}{k} + \tau x \xi^{k-1}\right)\right\}. \tag{2.37}$$

This fractional derivative coincides with (minus) the usual derivative with respect to x when $r = (k-1)$, and satisfies the additivity property

$$(-d)^{r/(k-1)}(-d)^{s/(k-1)} = (-d)^{(r+s)/(k-1)} \tag{2.38}$$

and the generalized Leibniz formula

$$(-d)^{r/(k-1)}x\varphi(x) = x(-d)^{r/(k-1)}\varphi(x) - \frac{r}{k-1}(-d)^{(r/(k-1))-1}\varphi(x). \tag{2.39}$$

The definition also extends to negative values of r provided $\tilde{\varphi}(\xi) = \mathcal{O}(\xi^{-r})$ for $\xi \to 0$. In particular, for $r = -(k-1)$ and $\tilde{\varphi}(\xi) = \mathcal{O}(\xi^{k-1})$, we have

$$(-d)^{-1}\varphi(x) = \int_0^\infty \mathrm{d}y\, \varphi(x+y). \tag{2.40}$$

Alternatively, we may avoid using fractional derivatives at the expense of introducing $k-1$ basic (normalized) distributions σ_j, $j = 2, \ldots, k$, defined as

$$\sigma_j(x) \equiv \frac{k^{(j-1)/k}}{\Gamma\left((k-j+1)/k\right)\sin\left((k-j+1)(\pi/k)\right)}$$
$$\times \mathrm{Im}\left\{\int_0^\infty \mathrm{d}\xi\, \left(\frac{\xi}{\tau}\right)^{k-j} \xi^{k-1} \exp\left(-\frac{\xi^k}{k} + \tau \xi^{k-1} x\right)\right\}, \tag{2.41}$$
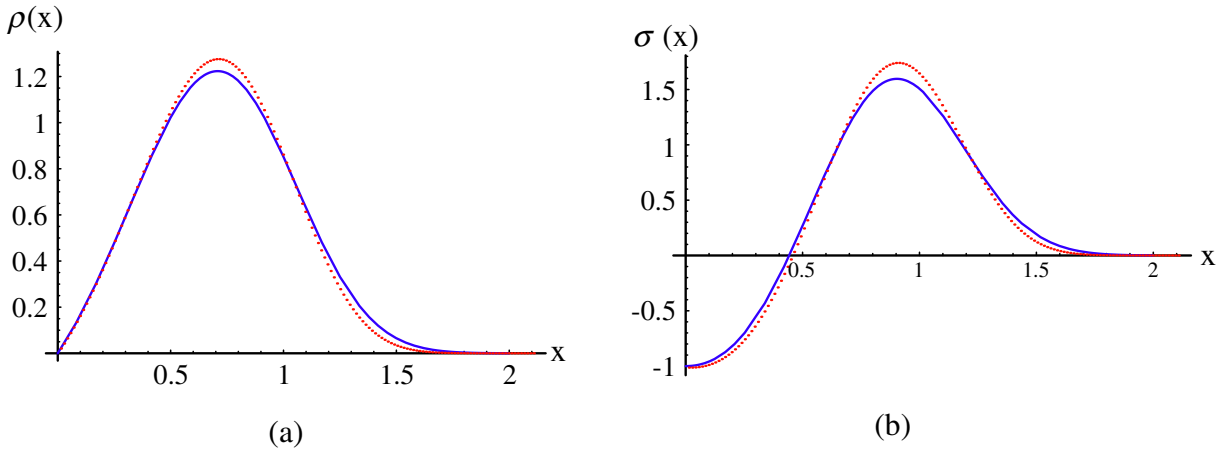
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Figure 9. A plot of the $k = 3$ fundamental distributions: (a) the average profile $\rho(x)$ and (b) the distribution $\sigma(x) \equiv \mu_2(-d)^{1/2}\rho(x)$. These distributions (solid lines) are compared with the corresponding discrete data (dotted lines) obtained from the minimal model (2.14) for a number $N = 200$ of leaves.

such that $\rho(x) = \sigma_k(x)$ and $\rho(H) \propto \mu_j \sigma_j(\sum_{i=1}^n x_i)$ if H has a unique $(j + 1)$-valent branching point. The distribution associated with any history is then proportional to $\mathrm{d}^m \sigma_j(x)/\mathrm{d}x^m$ where m and $k - j$ are the quotient and the remainder of the Euclidean division: $\sum_i (k - i)p_i = m(k - 1) + k - j$.

To conclude this section, let us mention the generalization of equation (2.27) to arbitrary histories:

$$\frac{k^{1/k} \sin(\pi/k)\, \Gamma(1 + (1/k))}{\pi} \int_0^\infty \frac{\mathrm{d}M}{M^{1+(1/k)}} \mathrm{e}^{-(M/k)} \left\{ \frac{M^m}{M^{n((k-1)/k)}} \rho\left(\frac{H}{M^{(k-1)/k}}\right) \right\}$$

$$= \mathrm{e}^{-x} \prod_{i=2}^k (\mu_i)^{p_i} \tag{2.42}$$

where $x = \sum_i x_i$ and where the notation H/Λ means that all the x_is of H are replaced by x_i/Λ.

Finally, let us illustrate the case $k = 3$ by considering the two fundamental distributions: the average profile $\rho(x)$ and the distribution $\sigma(x) \equiv \mu_2(-d)^{1/2}\rho(x)$ corresponding to $m = 2$ marked leaves and a total length $x = x_1 + x_2 + x_3$. These functions are plotted in figure 9 against their properly rescaled discrete counterparts (see equations (2.23) and (2.31)) in the particular case of the minimal model, namely with trivalent and tetravalent vertices weighted respectively by 1 and $-1/3$, according to equation (2.15). Note that $\sigma(x)$ is negative for small x, which shows that the MCRT corresponds to a signed measure. This is a generic property for all $k > 2$.

2.4. A few properties of the MCRT$_k$

Having the exact form of the average profile and history distributions (2.24) and (2.33), we may infer a number of properties of the MCRT$_k$. A first question of interest is the

large x asymptotics of $\rho(x)$. A saddle point approximation shows that

$$\rho(x) \sim \gamma(k) x^{k/2} \exp\left(-\frac{(k-1)^{k-1}}{k} x^k\right) \tag{2.43}$$

for some constant $\gamma(k)$. We recognize Fisher's law relating the exponent δ of the large x exponential decay $\rho(x) \sim \mathrm{e}^{-ax^\delta}$ to the fractal dimension via $\delta = d_k/(d_k - 1) = k$. In particular, this behaviour guarantees the existence of all moments of $\rho(x)$:

$$\langle x^b \rangle_\rho \equiv \int_0^\infty \mathrm{d}x \, x^b \rho(x). \tag{2.44}$$

These may be readily computed from equation (2.27) upon multiplication by x^b and integrating over x. We find immediately

$$\frac{k^{1/k} \sin(\pi/k) \, \Gamma(1+(1/k))}{\pi} \int_0^\infty \frac{\mathrm{d}M}{M} M^{((k-1)/k)(b+1)} \exp\left(-\frac{M}{k}\right) \langle x^b \rangle_\rho = b! \tag{2.45}$$

and hence

$$\langle x^b \rangle_\rho = \frac{b!}{k^{b((k-1)/k)}} \frac{\Gamma((k-1)/k)}{\Gamma((b+1)((k-1)/k))}. \tag{2.46}$$

Beyond the properties of the average profile, we may compare the relative contributions of various histories as follows. Recall that a continuous history H is given by two types of data: (i) its shape S recording its branching structure, characterized by the numbers p_i, and (ii) the lengths x_j of its branches. For a given number m of marked points, we have a number of admissible shapes, namely all planted plane trees with valences $3, \ldots, k+1$ and with this number m of labelled leaves. Such shapes obey the sum rule

$$m = 1 + \sum_{i=2}^k (i-1) p_i. \tag{2.47}$$

We can then compute the relative weight $w(S)$ of a given shape S among the admissible ones by considering all continuous history trees H sharing this shape and integrating the distribution $\rho(H)$ over the lengths x_j, in number $n = 1 + \sum_{i=2}^k i p_i$. We find

$$w(S) = \int_0^\infty \prod_{i=1}^n \mathrm{d}x_i \, \rho(H) = \int_0^\infty \mathrm{d}x \, \frac{x^{n-1}}{(n-1)!} \rho(H), \tag{2.48}$$

where the variable x stands for the total length of H. This may be readily evaluated by multiplying both sides of equation (2.42) by $x^{n-1}/(n-1)!$ and by integrating over x, leading to

$$\frac{k^{1/k} \sin(\pi/k) \, \Gamma(1+(1/k))}{\pi} \int_0^\infty \frac{\mathrm{d}M}{M} M^{m-(1/k)} \exp\left(-\frac{M}{k}\right) w(S) = \prod_{i=2}^k (\mu_i)^{p_i} \tag{2.49}$$

where the n-dependent terms have cancelled out; hence

$$w(S) = \frac{(-1)^{m-1}}{k^m} \frac{\Gamma((1/k)+1-m)}{\Gamma((1/k)+1)} \prod_{i=2}^k (\mu_i)^{p_i}. \tag{2.50}$$

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Note that, as the number of leaves m is fixed, we may rewrite

$$w(S) = \frac{1}{z_m} \prod_{i=2}^{k} (\mu_i)^{p_i}, \tag{2.51}$$

displaying the universal weight of a shape in the MCRT$_k$ as the product of the universal weights μ_i of equation (2.34) at its branching points. The global prefactor

$$z_m = (-1)^{m-1} k^m \frac{\Gamma\left((1/k) + 1\right)}{\Gamma\left((1/k) + 1 - m\right)} \tag{2.52}$$

ensures the proper normalization $\sum_S w(S) = 1$, where the sum runs over all shapes with m leaves. This may be checked as follows: let Y denote the (grand canonical) generating function for *unmarked* shapes, i.e. shapes whose leaves are not labelled, and with weights μ_i per $(i+1)$-valent branching point $(i = 2, \ldots, k)$ and μ_0 per leaf. We have the relation

$$Y = \mu_0 + \sum_{i=2}^{k} \mu_i Y^i = \mu_0 + \frac{kY - 1 + (1-Y)^k}{k}, \tag{2.53}$$

where we used the explicit values (2.34) for μ_i with $i \geq 2$. This relation may be inverted into

$$Y = 1 - (1 - k\mu_0)^{1/k}, \tag{2.54}$$

leading to the desired sum rule

$$\sum_{\substack{S \text{ with} \\ m \text{ leaves}}} \prod_{i=2}^{k} (\mu_i)^{p_i(S)} = m! \left. Y \right|_{\mu_0^m} = (-1)^{m-1} k^m \frac{\Gamma\left((1/k)+1\right)}{\Gamma\left((1/k)+1-m\right)} = z_m, \tag{2.55}$$

where the factor $m!$ accounts for the possible labellings of the m leaves.

For illustration, we have represented in figure 10 the list of all possible shapes with $m = 4$ when $k = 4$, together with the associated universal weights. Note that for $k = 2$, as $p_2 = m - 1$ is entirely fixed, the law (2.51) is uniform over all admissible shapes, in agreement with equation (1.2).

3. MCRT: direct universal formulation

3.1. Axiomatic approach

In this section, we propose an alternative derivation of the MCRT$_k$ average profile and history distributions directly in a continuous formulation, without having recourse to discrete models, but with a minimal number of reasonable hypotheses. The MCRT$_k$ describes a (possibly signed) measure over an ensemble of abstract trees with a *fixed total size*, say 1, and we can characterize its properties by the induced measure on continuous marked history trees obtained by first picking leaves independently and uniformly at random in the abstract trees and then extracting the subtree spanned by the root and these (marked) leaves. The MCRT$_k$ will be constructed so that it allows for histories with up to $(k+1)$-valent branching points while histories including $(k+2)$-valent branching points and more have a vanishing measure.
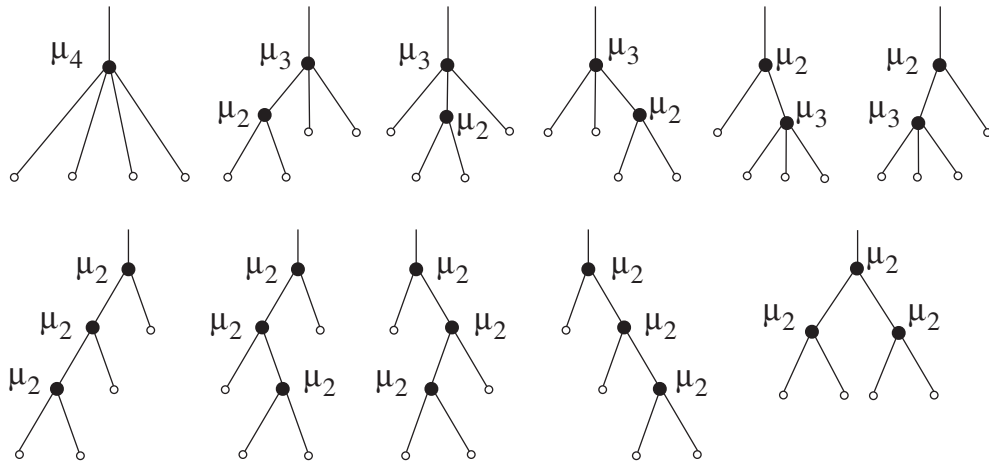
Figure 10. The 11 possible shapes with $m = 4$ leaves when $k = 4$. Each shape should be supplemented by a labelling of its leaves, with 4! possible labellings per shape. The universal weight of a shape is the product of the μ_i factors at its branching points, as displayed, times a global factor $1/z_4$. Here we have $z_4 = 231$, as given by equation (2.52) for $k = m = 4$, while $\mu_2 = 3/2$, $\mu_3 = -1$ and $\mu_4 = 1/4$ from equation (2.34). We easily check the sum rule $(\mu_4 + 5\mu_2\mu_3 + 5\mu_2^3) \times (4!/z_4) = 1$, as expected.

Here we wish to define directly this measure, still denoted by $\rho(H)$, where continuous marked history trees H are as before characterized by a shape S (taken to be a planted plane tree with labelled leaves and with numbers p_i of $(i + 1)$-valent branching points) endowed with branch lengths x_j.

As just mentioned, the first requirement for the MCRT$_k$ is that the only possible valences for branching points in history trees belong to $\{3, 4, \ldots, k + 1\}$.

Secondly, we assume that the measure is invariant under *rearrangements* of the trees, inspired by a similar property for discrete trees, and defined as follows. For a given continuous history tree H, we may distribute a number of (oriented) markers on its branches and then cut the tree at these markers (see figure 11 for an illustration). The resulting pieces are then rearranged in such a way as to preserve the orientation of markers and to produce another connected history tree H'. The above invariance means that $\rho(H) = \rho(H')$. The consequences of this invariance are twofold. For a fixed shape, the measure of histories sharing that shape depends only on the *total* length $x \equiv \sum_j x_j$ of H. This simply uses the possibility within the above rearrangements of redistributing, at fixed shape, internal segments within branches. Moreover, the dependence on the shape is only through its numbers p_i of $(i + 1)$-valent branching points. This uses the possibility, as part of the above rearrangements, of permuting the branching points by appropriate cutting of branches. With the above hypotheses, we may write $\rho(H)$ in the form

$$\rho(H) = \phi(x; \{p_2, p_3, \ldots, p_k\}) \tag{3.1}$$

for all H with fixed values of p_i, $i = 2, \ldots, k$, and fixed total length x.

We may now write an infinite set of consistency equations relating the ϕs at different values of their arguments as follows. Given a history H with m leaves, we may consider all histories H' obtained by adding to H one extra leaf and express that $\rho(H)$ is the
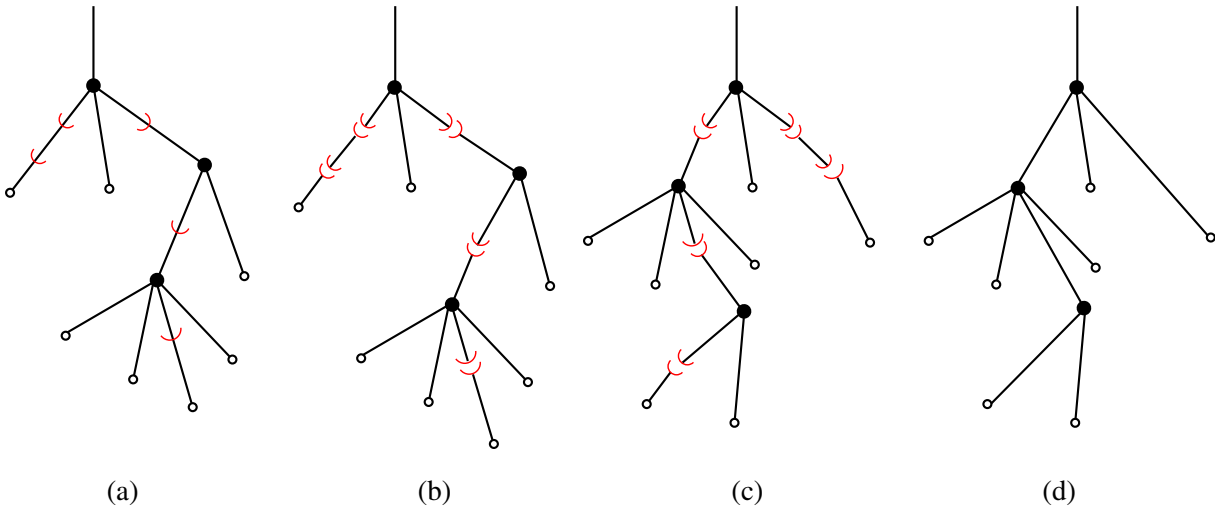
Figure 11. A example of rearrangement of a continuous tree, obtained by (a) placing oriented markers on the branches of the tree, (b) cutting at these markers, (c) exchanging the pieces obtained and finally (d) re-gluing them so as to build a new tree. Note that the lengths of the various segments of branches are preserved at each step of the rearrangement.
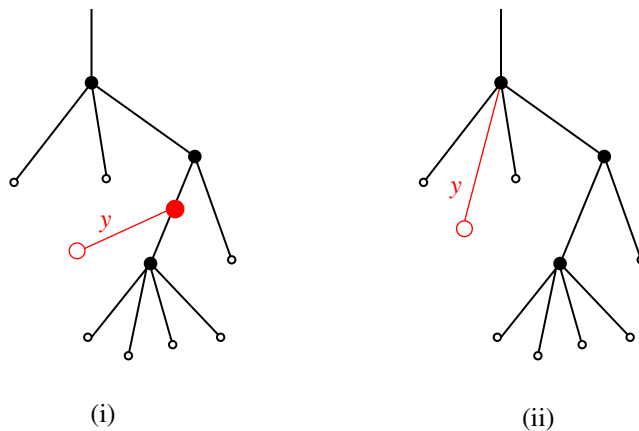

Figure 12. The two possible ways of adding an extra leaf to a given history (see the text).

integrated measure over these histories H'. More precisely, as illustrated in figure 12, there are two ways of adding an extra leaf:

(i) By creating a new 3-valent branching point along an existing branch of H, splitting that branch into two pieces and creating a new branch, say of length y, connected to the extra leaf. Integrating over the position of this new branching point produces a factor $2x$ equal to the external perimeter of H. Note that in H', the numbers of branching points read $p'_i = p_i$ for $i > 2$ and $p'_2 = p_2 + 1$.

(ii) By connecting the extra leaf to an existing $(j+1)$-valent branching point (with $j < k$) via a new branch of length y, thus replacing it by a $(j+2)$-valent one. As we are

dealing with plane trees, there are $j + 1$ inequivalent ways of doing this. In H', the numbers of branching points read $p'_i = p_i$ for $i \neq j, j + 1$, while $p'_j = p_j - 1$ and $p'_{j+1} = p_{j+1} + 1$.

Integrating over the length of the extra branch y and using the form (3.1) for $\rho(H)$, we deduce the consistency relations

$$\phi(x; \{p_2, \ldots, p_k\}) = 2x \int_0^\infty \mathrm{d}y \, \phi(x + y; \{p_2 + 1, p_3, \ldots, p_k\})$$
$$+ \sum_{j=2}^{k-1} (j + 1)p_j \int_0^\infty \mathrm{d}y \, \phi(x + y; \{p_2, \ldots, p_{j-1}, p_j - 1, p_{j+1} + 1, p_{j+2}, \ldots, p_k\}).$$

(3.2)

Note that, when all p_is are equal to 0, ϕ is nothing but the average profile, namely

$$\rho(x) = \phi(x; \{0, \ldots, 0\}). \tag{3.3}$$

This function is assumed to be sufficiently regular, and *exponentially decreasing* at large x so that, for instance, all its moments are finite.

To proceed further, we rely on a *factorization assumption* and write all ϕs as

$$\phi(x; \{p_2, \ldots, p_k\}) = \prod_{i=2}^{k} (O_i)^{p_i} \rho(x), \tag{3.4}$$

stating that each $(i + 1)$-valent branching point contributes via the action of an operator O_i on $\rho(x)$, where these operators commute for different indices. The relation (3.4) should be universal in the sense that it should remain the same for all ensembles E_M of abstract trees with fixed total size M rather than 1. For trees in E_M, the history distributions and average profile are rescaled into

$$\phi_M(x, \{p_2, \ldots, p_k\}) = \frac{M^m}{M^{n \nu_k}} \phi\left(\frac{x}{M^{\nu_k}}; \{p_2, \ldots, p_k\}\right), \qquad \rho_M(x) = \frac{M}{M^{\nu_k}} \rho\left(\frac{x}{M^{\nu_k}}\right) \tag{3.5}$$

where $n = 1 + \sum_{i=2}^{k} i p_i$ is the number of branches, $m = 1 + \sum_{i=2}^{k} (i - 1)p_i$ the number of marked leaves, and where ν_k is the inverse of the fractal dimension d_k of the MCRT, yet to be computed. To ensure that (3.4) still holds with the substitutions $\phi \to \phi_M$ and $\rho \to \rho_M$, we need that the operator O_i have a non-trivial *scaling dimension* α_i, namely satisfy

$$g(x) = f(\Lambda x) \Rightarrow [O_i g](x) = \Lambda^{\alpha_i} [O_i f](\Lambda x) \tag{3.6}$$

for some real α_i. Rewriting equation (3.4) for the ϕ_Ms, we also get the consistency relation

$$m - n\nu_k = 1 - \nu_k \left(1 + \sum_{i=2}^{k} \alpha_i p_i\right) \tag{3.7}$$

which must hold for all values of the p_is. This yields the scaling dimensions

$$\alpha_i = 1 + \frac{\nu_k - 1}{\nu_k}(i - 1). \tag{3.8}$$

We now look for a solution to the consistency relations (3.2) in the form (3.4), and make a simple educated guess for the operators O_i, namely that

$$O_i = \mu_i \, (-d)^{\alpha_i} \tag{3.9}$$

involving a scalar multiplicative factor μ_i and a power α_i of (minus) the derivative operator $(-d)$ with respect to x, with α_i as in equation (3.8). At this stage, this particular form is only an ansatz inspired from the discrete approach of section 2 and which, as we shall see, turns out to yield a fully consistent solution to equation (3.2) for a specific value of the scaling exponent ν_k in equation (3.8). We believe that this solution is actually unique provided all ϕs are exponentially decreasing at large x.

It is known that there are various (inequivalent) definitions of the pseudo-derivative operators $(-d)^\alpha$ but here we are led to a very natural choice by noting that we let them act on a function $\rho(x)$ such that all its derivatives vanish when $x \to \infty$. We define the fractional integrals and derivatives as

$$\begin{aligned}
(-d)^{-\beta} f(x) &\equiv \frac{1}{\Gamma(\beta)} \int_0^\infty \mathrm{d}t \, t^{\beta-1} \mathrm{e}^{t\,d} f(x) \\
&= \frac{1}{\Gamma(\beta)} \int_x^\infty \mathrm{d}u \, (u-x)^{\beta-1} f(u)
\end{aligned} \tag{3.10}$$

for positive β, and, for positive α,

$$(-d)^\alpha = (-d)^{[\alpha]+1}(-d)^{-\beta} \qquad \text{with } \beta \equiv [\alpha] + 1 - \alpha \tag{3.11}$$

where $[\alpha]$ stands for the integer part of α, so β lies in $(0,1]$. In equation (3.10), we have used the standard shift operator $\mathrm{e}^{t\,d} f(x) = f(x+t)$. The above definitions coincide with the so-called Weyl fractional integral [20].

Apart from the desired scaling property (3.6) easily checked from our definition (3.11) with $\alpha = \alpha_i$, the pseudo-derivative operators obey the additivity property and generalized Leibniz formula

$$(-d)^\alpha (-d)^\beta = (-d)^{\alpha+\beta}, \qquad (-d)^\alpha x = x(-d)^\alpha - \alpha(-d)^{\alpha-1} \tag{3.12}$$

when acting on functions which are exponentially decreasing at infinity. Note that the derivations of these properties involve integrations by parts and it is crucial that all boundary terms at infinity vanish, as a consequence of the exponential decrease. We shall also use

$$(-d)^{-1} \phi(x) = \int_0^\infty \mathrm{d}y \, \phi(x+y) \tag{3.13}$$

to replace the integrals in equation (3.2) with pseudo-derivative operators. As we shall see below, the values of α_i will be fixed to be rationals of the form $r/(k-1)$; hence the reader may alternatively think of $\rho(x)$ as having an integral representation of the form (2.36) with some $\tilde{\rho}(\xi)$ to be determined, in which case the action of the pseudo-derivative operators above is equivalently defined via equation (2.37).

The solution is further fixed by replacing the form (3.4) with O_i as in (3.9) in the consistency relations (3.2). The latter now read

$$\mathcal{D}\rho = 2x\mu_2 \mathcal{D}(-d)^{\alpha_2-1}\rho + \sum_{j=2}^{k-1} (j+1)p_j \frac{\mu_{j+1}}{\mu_j} \mathcal{D}(-d)^{\alpha_{j+1}-\alpha_j-1}\rho \tag{3.14}$$

where $\mathcal{D} = (-d)^{\sum_i \alpha_i p_i}$. Using the Leibniz formula in (3.12) to commute x through the

operator $\mathcal{D}$, we deduce

$$\mathcal{D}\Biggl\{(\rho - 2x\mu_2(-d)^{\alpha_2-1}\rho) \\ - \Biggl(\sum_{j=2}^{k-1} p_j\Biggl(2\mu_2\alpha_j + (j+1)\frac{\mu_{j+1}}{\mu_j}\Biggr) + 2\mu_2\alpha_k p_k\Biggr)(-d)^{\alpha_2-2}\rho\Biggr\} = 0 \qquad (3.15)$$

where we have used $\alpha_{j+1} - \alpha_j - 1 = \alpha_2 - 2$ from equation (3.8). This must hold for all values of the p_is whose coefficients must therefore all vanish. The p_k term yields $\alpha_k = 0$, from which we deduce

$$\nu_k = \frac{k-1}{k} \qquad (3.16)$$

and the scaling dimensions

$$\alpha_i = \frac{k-i}{k-1}. \qquad (3.17)$$

The p_j terms for $j = 2, \ldots, k-1$ fix the relations

$$\frac{\mu_{j+1}}{\mu_j} = -\frac{2\mu_2}{j+1}\frac{k-j}{k-1} \qquad (3.18)$$

and hence

$$\mu_j \equiv \frac{(-1)^j}{k}\binom{k}{j}\left(\frac{2\mu_2}{k-1}\right)^{j-1}. \qquad (3.19)$$

Finally, we are left with a set of relations for ρ valid for all p_is and implied by the unique relation obtained when all p_is are equal to 0, in which case $\mathcal{D} = 1$:

$$\rho(x) = 2\mu_2 x(-d)^{-(1/(k-1))}\rho(x). \qquad (3.20)$$

The factor μ_2 in equations (3.19) and (3.20) is arbitrary and may be reabsorbed into a rescaling of the length $x \to \Lambda x$. We decide to fix $\mu_2 = (k-1)/2$, in which case equation (3.19) matches (2.34) of the discrete approach and we end up with the fractional differential equation for the average profile $\rho(x)$:

$$\rho(x) = (k-1)x(-d)^{-(1/(k-1))}\rho(x) \\ = \frac{(k-1)x}{\Gamma(1/(k-1))}\int_x^\infty \mathrm{d}u\, (u-x)^{(1/(k-1))-1}\rho(u). \qquad (3.21)$$

As this stage, we note that if we interpret the fractional derivative according to equations (2.36) and (2.37), we immediately see that the solution $\rho(x)$ given by (2.24) in the discrete approach does indeed satisfy the equation (3.21). As we shall see below, we need not have recourse to any integral representation to prove that equation (3.21) has a unique solution with exponential decay at large x, and that moreover, this solution matches that of the discrete approach. This is the aim of the next three subsections.

3.2. Ordinary differential equation for $\rho(x)$

In this subsection, we derive an *ordinary* differential equation for the average profile $\rho(x)$ on the basis of manipulations of fractional derivatives as follows. We start from the fractional differential equation in the first line of (3.21) and apply the operator $-(-d)^{k/(k-1)}$ on both sides. Using the generalized Leibniz formula (3.12) for $\alpha = k/(k-1)$, we get the relation

$$-(-d)^{k/(k-1)}\rho = (k-1)x\rho' + k\rho \tag{3.22}$$

which, upon introducing the operators

$$\delta \equiv -(-d)^{k/(k-1)} \qquad \text{and} \qquad P \equiv (k-1)xd + k, \tag{3.23}$$

may be recast into

$$\delta\rho = P\rho. \tag{3.24}$$

The two operators δ and P obey the following commutation relation:

$$[\delta, P] = k\delta \tag{3.25}$$

which is a direct consequence of the generalized Leibniz formula (3.12) for $\alpha = k/(k-1)$. We may then show by induction that, for all $j = 1, 2, \ldots$,

$$\delta^j \rho = (P + (j-1)k)(P + (j-2)k)\cdots(P + k)P\rho. \tag{3.26}$$

This relation holds clearly for $j = 1$ from equation (3.24) and is simply proved by recursion upon noticing that, from equation (3.25), $\delta(P + a) = (P + a + k)\delta$ for any constant a. Taking $j = k-1$ and using $\delta^{k-1} = -d^k$, this finally yields an ordinary differential equation of order k for ρ, namely

$$-d^k\rho = \prod_{j=0}^{k-2}(P + kj)\rho. \tag{3.27}$$

For illustration, when $k = 2, 3, 4$, these equations read respectively

$$
\begin{aligned}
0 &= 2\rho + x\rho' + \rho'' \\
0 &= 18\rho + 22x\rho' + 4x^2\rho'' + \rho''' \\
0 &= 384\rho + 771x\rho' + 297x^2\rho'' + 27x^3\rho''' + \rho^{(4)}.
\end{aligned}
\tag{3.28}
$$

3.3. Explicit solution

We will now determine the function $\rho(x)$ as the suitably normalized solution of equation (3.27) with exponential decay at large x. As we shall see, this latter condition turns out for being sufficient to select a unique solution.

Let us first look for the general solution of equation (3.27) in the form of a series expansion

$$\rho(x) = \sum_{m \geq 0} \rho_m x^m. \tag{3.29}$$

Equation (3.27) translates into recursion relations on the coefficients, namely

$$\rho_{m+k} = -\frac{m!}{(m+k)!}\rho_m \prod_{j=0}^{k-2}(m(k-1)+k+kj) \tag{3.30}$$

which determines all coefficients in terms of the first k ones $\rho_0, \rho_1, \ldots, \rho_{k-1}$. We immediately get the general solution:

$$\rho(x) = \sum_{p=0}^{k-1}\rho_p x^p \ _{k-1}F_{k-1}\left(\{a_i(p)\}, \{b_i(p)\}; -\frac{(k-1)^{k-1}}{k}x^k\right) \tag{3.31}$$

in terms of k generalized hypergeometric series $_{k-1}F_{k-1}$ with arguments

$$a_i(p) = \begin{cases} \dfrac{p}{k} + \dfrac{i}{k-1} & \text{for } i = 1, 2, \ldots, k-2 \\ 1 & \text{for } i = k-1 \end{cases} \tag{3.32}$$

$$b_i(p) = \frac{p}{k} + \frac{i}{k} \qquad \text{for } i = 1, 2, \ldots, k-1$$

for $p = 0, 1, \ldots, k-1$. Recall that the generalized hypergeometric series are defined via

$$_pF_q\left(\{a_i\}, \{b_j\}; z\right) \equiv \sum_{m \geq 0}\frac{\prod_{i=1}^{p}(a_i)_m}{\prod_{j=1}^{q}(b_j)_m}\frac{z^m}{m!} \tag{3.33}$$

with $(a)_m \equiv a(a+1)\cdots(a+m-1)$. For simplicity, we rewrite the general solution (3.31) as the linear combination

$$\rho(x) = \sum_{p=0}^{k-1}C_p F_p(x) \tag{3.34}$$

with k new arbitrary coefficients C_p, and where we define the suitably normalized functions

$$\begin{aligned}
F_p(x) &\equiv \left(\frac{(k-1)^{k-1}}{k}\right)^{p/k} x^p \prod_{i=1}^{k-1}\frac{\Gamma(a_i(p))}{\Gamma(b_i(p))} \ _{k-1}F_{k-1}\left(\{a_i(p)\}, \{b_i(p)\}; -\frac{(k-1)^{k-1}}{k}x^k\right) \\
&= \sqrt{\frac{k}{2\pi(k-1)}}k^{((k-1)/k)p}\Gamma\left(\frac{k-1}{k}p+1\right)\frac{x^p}{p!} \\
&\quad \times \ _{k-1}F_{k-1}\left(\{a_i(p)\}, \{b_i(p)\}; -\frac{(k-1)^{k-1}}{k}x^k\right).
\end{aligned} \tag{3.35}$$

Here we have used the simplification

$$\begin{aligned}
\prod_{i=1}^{k-1}\frac{\Gamma(a_i(p))}{\Gamma(b_i(p))} &= \frac{\prod_{i=1}^{k-2}\Gamma\left((p/k)+(i/(k-1))\right)}{\prod_{i=1}^{k-1}\Gamma\left((p/k)+(i/k)\right)} \\
&= \sqrt{\frac{k}{2\pi(k-1)}}\frac{k^p}{(k-1)^{((k-1)/k)p}}\frac{\Gamma\left(((k-1)/k)p+1\right)}{\Gamma(p+1)}
\end{aligned} \tag{3.36}$$

obtained as a consequence of the Gamma product theorem, stating that

$$\Gamma(nx) = (2\pi)^{(1-n)/2}n^{nx-(1/2)}\prod_{i=0}^{n-1}\Gamma\left(x+\frac{i}{n}\right). \tag{3.37}$$

Remarkably, the coefficients C_p may be completely fixed by simply requiring that

(i) $\rho(x)$ has an exponential decay for large (positive) x and

(ii) $\rho(x)$ is normalized.

To implement condition (i) above, we use the following asymptotics [21] for hypergeometric series of argument $(-z)$ at large positive values of z:

$$\prod_{i=1}^{k-1} \frac{\Gamma(a_i)}{\Gamma(b_i)} \,_{k-1}F_{k-1}\left(\{a_i\}, \{b_i\}; -z\right)$$

$$= \sum_{j=1}^{k-1} \frac{\Gamma(a_j)}{z^{a_j}} \frac{\prod_{\ell \neq j} \Gamma(a_\ell - a_j)}{\prod_\ell \Gamma(b_\ell - a_j)} \,_kF_{k-2}\left(\{a_i^{(j)}\}, \{b_i^{(j)}\}; \frac{1}{z}\right) + \mathcal{O}(z^\chi e^{-z}) \tag{3.38}$$

where $\chi = \sum_i (a_i - b_i)$ and with

$$a_i^{(j)} = \begin{cases} a_j - b_i + 1 & \text{for } i = 1, 2, \ldots, k-1 \\ a_j & \text{for } i = k \end{cases}$$

$$b_i^{(j)} = \begin{cases} a_j - a_i + 1 & \text{for } i = 1, 2, \ldots, j-1 \\ a_j - a_{i+1} + 1 & \text{for } i = j, j+1, \ldots k-2. \end{cases} \tag{3.39}$$

These asymptotics may be applied to the list of arguments $\{a_i = a_i(p)\}$ and $\{b_i = b_i(p)\}$ of equation (3.32) with $p = 0, 1, \ldots, k-1$, giving rise to sets $\{a_i^{(j)}(p)\}$ and $\{b_i^{(j)}(p)\}$. First, we notice that $a_k^{(j)}(p) = b_{k-2}^{(j)}(p) = a_j(p)$; hence their respective contributions to the $_kF_{k-2}$ hypergeometric series cancel out. Moreover, for $j \leq k-2$, the remaining arguments $\{a_i^{(j)}(p)\}_{i \leq k-1}$ and $\{b_i^{(j)}(p)\}_{i \leq k-3}$ all turn out to be *independent* of p, and hence the functions

$$B_j(z) \equiv \,_kF_{k-2}\left(\{a_i^{(j)}(p)\}, \{b_i^{(j)}(p)\}; \frac{1}{z}\right) \tag{3.40}$$

for $j = 1, \ldots, k-2$ are independent of p. As regards the $j = k-1$ term in (3.38), we note that, as soon as $p \geq 1$, as both $a_{k-1}(p)$ and $b_{k-p}(p)$ are equal to 1, the corresponding prefactor in equation (3.38) vanishes identically. The $j = k-1$ term only contributes when $p = 0$ and involves a function

$$B_{k-1}(z) \equiv \,_kF_{k-2}\left(\{a_i^{(k-1)}(0)\}, \{b_i^{(k-1)}(0)\}; \frac{1}{z}\right). \tag{3.41}$$

The large x asymptotics of ρ read therefore

$$\rho(x) = \sum_{j=1}^{k-2} B_j(z) \sum_{p=0}^{k-1} C_p z^{(p/k) - a_j(p)} \Gamma(a_j(p)) \frac{\prod_{\ell \neq j} \Gamma(a_\ell(p) - a_j(p))}{\prod_\ell \Gamma(b_\ell(p) - a_j(p))}$$

$$+ B_{k-1}(z) C_0 z^{-1} \frac{\prod_{\ell \neq k-1} \Gamma(a_\ell(0) - 1)}{\prod_\ell \Gamma(b_\ell(0) - 1)} + \mathcal{O}(z^{1/2} e^{-z})$$

$$
\begin{aligned}
= &\sum_{j=1}^{k-2} B_j(z) z^{-(j/(k-1))} \frac{\prod_{\substack{\ell=1 \\ \ell \neq j}}^{k-2} \Gamma\left((\ell/(k-1)) - (j/(k-1))\right)}{\prod_{\ell=1}^{k-1} \Gamma\left((\ell/k) - (j/(k-1))\right)} \\
&\times \left\{ \sum_{p=0}^{k-1} C_p \Gamma(a_j(p)) \Gamma(1 - a_j(p)) \right\} \\
&+ B_{k-1}(z) z^{-1} \frac{\prod_{\ell=1}^{k-2} \Gamma\left((\ell/(k-1)) - 1\right)}{\prod_{\ell=1}^{k-1} \Gamma\left((\ell/k) - 1\right)} C_0 + \mathcal{O}(z^{1/2} \mathrm{e}^{-z})
\end{aligned}
\tag{3.42}
$$

with $z = x^k (k-1)^{k-1}/k$. Note that, to get the precise form of the exponentially decreasing correction, we have made use of the relation $p/k + \chi = p/k + \sum(a_i(p) - b_i(p)) = 1/2$ irrespectively of p.

Requiring an exponential decay for $\rho(x)$ at large x imposes that $C_0 = 0$ and

$$
\sum_{p=1}^{k-1} \frac{C_p}{\sin\left(\pi\left((p/k) + (j/(k-1))\right)\right)} = 0
\tag{3.43}
$$

for $j = 1, \ldots, k-2$ by use of the identity $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$. This fixes all C_p for $p = 1, \ldots, k-1$ up to a global factor C, namely

$$
C_p = C \sin\left(\pi \frac{p}{k}\right).
\tag{3.44}
$$

This may be checked by use of the identity

$$
\sum_{p=1}^{k-1} \frac{\sin\left(\pi(p/k)\right)}{\sin\left(\pi(p/k) + \alpha\right)} = k \frac{\sin((k-1)\alpha)}{\sin(k\alpha)}
\tag{3.45}
$$

which precisely vanishes at $\alpha = \pi j/(k-1)$ for $j = 1, 2, \ldots, k-2$, as wanted. Picking the values (3.44) for C_p, we are left with the leading behaviour $\rho(x) \propto z^{1/2} \mathrm{e}^{-z}$, in agreement with equation (2.43).

Finally the remaining constant C is fixed by the normalization condition (ii) above. This is best seen by integrating the differential equation (3.27) on $[0, \infty)$, leading to

$$
\rho^{(k-1)}(0) = \prod_{j=0}^{k-2} (1 + kj) \int_0^\infty \rho(x) \, \mathrm{d}x = \prod_{j=0}^{k-2} (1 + kj),
\tag{3.46}
$$

where we have used the fact that $\int_0^\infty P f = \int_0^\infty f$ for any exponentially decaying differentiable function f. This is to be compared with

$$
\begin{aligned}
\rho^{(k-1)}(0) &= C_{k-1} \, F_{k-1}^{(k-1)}(0) \\
&= C \sin\left(\pi \frac{k-1}{k}\right) \sqrt{\frac{k}{2\pi(k-1)}} \, k^{(1/k)+k-2} \Gamma\left(\frac{1}{k} + k - 1\right)
\end{aligned}
\tag{3.47}
$$

obtained from equations (3.34) and (3.35) where only the $p = k-1$ term contributes. This immediately leads to

$$
C = \sqrt{\frac{2\pi(k-1)}{k}} \frac{1}{k^{1/k} \Gamma\left((1/k) + 1\right) \sin\left(\pi/k\right)}
\tag{3.48}
$$

and, upon substitution into equation (3.34) of the values of the coefficients C_p read off

from equation (3.44), we get the final explicit formula:

$$\rho(x) = \sum_{p=1}^{k-1} \frac{\sin\left(\pi(p/k)\right)}{\sin\left(\pi/k\right)} \frac{k^{((k-1)/k)p}\Gamma\left(((k-1)/k)p+1\right)}{k^{1/k}\Gamma\left((1/k)+1\right)} \frac{x^p}{p!}$$

$$\times\;{}_{k-1}F_{k-1}\left(\{a_i(p)\}, \{b_i(p)\}; -\frac{(k-1)^{k-1}}{k}x^k\right) \tag{3.49}$$

where the ${}_{k-1}F_{k-1}$ functions in practice reduce to ${}_{k-2}F_{k-2}$ functions as $a_{k-1}(p) = b_{k-p}(p) = 1$ for all $p = 1, \dots, k-1$. For illustration, when $k = 2, 3, 4$ this yields

$$k = \begin{cases} 2: & \rho(x) = x\mathrm{e}^{-(x^2/2)} \\[4pt] 3: & \rho(x) = 3^{1/3}x\frac{\Gamma(5/3)}{\Gamma(4/3)}\,{}_1F_1\left(\frac{5}{6}, \frac{2}{3}, -\frac{4}{3}x^3\right) + 2x^2\,{}_1F_1\left(\frac{7}{6}, \frac{4}{3}, -\frac{4}{3}x^3\right) \\[4pt] 4: & \rho(x) = 2x\frac{\Gamma(7/4)}{\Gamma(5/4)}\,{}_2F_2\left(\left\{\frac{7}{12}, \frac{11}{12}\right\}, \left\{\frac{1}{2}, \frac{3}{4}\right\}, -\frac{27}{4}x^4\right) \\[4pt] & \quad + 4x^2\frac{\Gamma(5/2)}{\Gamma(5/4)}\,{}_2F_2\left(\left\{\frac{5}{6}, \frac{7}{6}\right\}, \left\{\frac{3}{4}, \frac{5}{4}\right\}, -\frac{27}{4}x^4\right) \\[4pt] & \quad + \frac{15}{2}x^3\,{}_2F_2\left(\left\{\frac{13}{12}, \frac{17}{12}\right\}, \left\{\frac{5}{4}, \frac{3}{2}\right\}, -\frac{27}{4}x^4\right). \end{cases} \tag{3.50}$$

For $k = 2$, we recover the expression (1.1), as expected. As we shall see in the next section, the above expressions match the integral formulae used for the plot of figure 7.

3.4. Compatibility with the integral formulae derived from discrete models

Let us now compare the result (3.49) to the integral formula (2.24) of section 2.3. Expanding this integral formula in x, we use

$$\mathrm{Im}\left\{\mathrm{e}^{\tau\xi^{k-1}x}\right\} = \sum_{\ell=0}^{\infty} \sin\left(\pi\frac{\ell}{k}\right)\xi^{(k-1)\ell}\frac{x^\ell}{\ell!}$$

$$= \sum_{p=0}^{k-1} \sin\left(\pi\frac{p}{k}\right)x^p \sum_{m=0}^{\infty}(-1)^m\xi^{(k-1)(km+p)}\frac{x^{km}}{(km+p)!} \tag{3.51}$$

to rewrite

$$\rho(x) = \frac{1}{k^{1/k}\Gamma\left((1/k)+1\right)\sin\left(\pi/k\right)} \sum_{p=1}^{k-1} \sin\left(\pi\frac{p}{k}\right)x^p A_p\left(-\frac{(k-1)^{k-1}}{k}x^k\right) \tag{3.52}$$

with

$$A_p(z) = \sum_{m=0}^{\infty}\left(\frac{k}{(k-1)^{k-1}}\right)^m\frac{z^m}{(km+p)!}\int_0^\infty \mathrm{d}\xi\,\xi^{k-1}\exp\left(-\frac{\xi^k}{k}\right)\xi^{(k-1)(km+p)}$$

$$= \sum_{m=0}^{\infty}\frac{k^{km+((k-1)/k)p}}{(k-1)^{(k-1)m}}\frac{\Gamma\left((k-1)\left(m+(p/k)\right)+1\right)}{\Gamma\left(k\left(m+(p/k)\right)+1\right)}z^m$$

$$= \sum_{m=0}^{\infty}\sqrt{\frac{2\pi(k-1)}{k}}\left(\frac{(k-1)^{k-1}}{k}\right)^{p/k}\frac{\prod_{i=1}^{k-2}\Gamma\left(m+(p/k)+(i/(k-1))\right)}{\prod_{i=1}^{k-1}\Gamma\left(m+(p/k)+(i/k)\right)}z^m \tag{3.53}$$

by use of the Gamma product theorem (3.37). Using

$$\frac{\prod_{i=1}^{k-2} \Gamma\left(m + (p/k) + (i/(k-1))\right)}{\prod_{i=1}^{k-1} \Gamma\left(m + (p/k) + (i/k)\right)} = \frac{1}{m!} \prod_{i=1}^{k-1} \frac{\Gamma\left(a_i(p)\right)}{\Gamma\left(b_i(p)\right)} \prod_{i=1}^{k-1} \frac{(a_i(p))_m}{(b_i(p))_m} \qquad (3.54)$$

with $a_i(p)$ and $b_i(p)$ given by (3.32), we identify

$$A_p(z) = \sqrt{\frac{2\pi(k-1)}{k}} F_p(z) \qquad (3.55)$$

with F_p as in equation (3.35). Equation (3.52) then boils down to $\rho(x) = \sum_{p=1}^{k-1} C_p F_p(z)$ with C_p as in (3.44) and C as in (3.48). The two formulae (2.24) and (3.49) therefore *define the same function* $\rho(x)$, as wanted. Finally, all history distributions of section 3.1, obtained by acting on $\rho(x)$ via equation (3.4) with the particular choices (3.9) and (3.17) do coincide with the integral expressions (2.33) of section 2, as seen from equation (2.35). We conclude that the axiomatic approach developed here fully agrees with the discrete results of section 2.

4. Conclusion

In this paper, we have introduced a generalization of Aldous's CRT describing the continuum scaling limit of multicritical ensembles of discrete trees. Restricting to k-multicritical ensembles leads to a unique universal limit, the MCRT_k, characterized by continuous history trees with branching points of valence 3 to $k+1$. We have obtained the average profile and history distributions of the MCRT_k in the form of either integral representations or combinations of hypergeometric series. The history distributions are expressed in terms of universal weights attached to the branching points, combined with fractional derivative operators acting on the average profile. Except for the usual CRT ($k = 2$), the universal weights carry signs; hence the measure on trees is not positive. This phenomenon parallels a similar property for the scaling limits of 2DQG as obtained from a one-matrix model [19] and is the signature of the non-unitary character of the underlying conformal field theory [22]. Likewise, the solutions of 2DLG display the same alternating signs to ensure multicriticality [14].

Our approach was twofold: we first derived the MCRT profile and history distributions as scaling limits of explicit expressions obtained in discrete models, and then recovered the same results from a purely continuous axiomatic approach, thereby corroborating the universal nature of the MCRT. A more abstract definition of the MCRT itself remains to be found. A possible route would consist in using random walks as in the CRT. Indeed, from [14], we already know that k-multicritical 2DLG can be rephrased in terms of random walks with properly weighted ascending steps of various lengths. Presumably, a continuous limit of these walks will provide the generalization of Brownian motion needed for the definition of the MCRT.

An important ingredient in our construction was the use of ensembles of trees with fixed numbers of leaves. While in the Brownian CRT, the use of ensembles with fixed numbers of edges turns out to lead to equivalent results, this is not the case for the MCRT with $k > 3$. For instance, we may consider trees with a fixed number N' of edges and with inner vertices weighted according to the minimal prescription (2.15). It is then easily

seen that the partition function $Z_{N'}$ of this new ensemble has the sign $(-1)^{N'-1}$ due to the fact that g_i^* has the sign $(-1)^i$. Up to this global sign, $Z_{N'}$ is identical to the partition function of the ensemble with $g_i = |g_i^*|$, and hence with only positive weights. The large N' continuum limit is therefore governed by the usual (non-multicritical) Brownian CRT.

So far, we have concentrated on a particular class of (non-unitary) multicritical points. We know however in the language of 2DQG how to reach unitary multicritical points corresponding to unitary conformal field theories coupled to 2DQG [23, 24]. Examples are the critical point of the Ising model on large random maps [16] and multicritical points of hard objects with generalized exclusion rules on large bipartite random maps [18]. All these models happen to have reformulations in terms of decorated discrete trees and one could hope that a limiting process could produce continuous random trees in a new universality class, one hopes now with a positive measure.

Finally, we have limited our study to unlabelled trees but a generalization to embedded MCRTs, i.e. multicritical equivalents of the Brownian snake, also seems reachable by the same techniques. From the exact results of [7] for discrete maps labelled by the geodesic distance to an origin, we expect that the associated tree model should have both the underlying tree and the labels simultaneously multicritical. This may be seen as follows: the fractal dimension $2k$ of these maps relates the size N of the map to the typical value D of the geodesic distance between vertices in the map via $N \sim D^{2k}$ [7]. In the tree language, N is still the size of the tree, now related to the typical value L of the generation via $N \sim L^{d_k}$, while D is the typical value of the labels. We deduce that $L \sim D^{2k/d_k} = D^{2(k-1)}$, and hence the scaling of labels with generation displays a non-trivial fractal dimension $2(k-1)$ instead of 2 for usual diffusion. In other words, the embedding of the associated MCRT should involve a multicritical diffusion process. A similar phenomenon occurs in 2DLG when passing from the notion of distance L that we used here, defined as the natural distance on the associated tree (see figure 1(c)), to the notion of 'time lapse' t, namely the number of slices separating two points (see figure 1(a)). As the fractal 'time' dimension of k-multicritical 2DLG is k [14], we necessarily have the anomalous scaling $L \sim t^{k-1}$.

Acknowledgments

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