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Free energy topological expansion for the 2-matrix model

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Abstract: We compute the complete topological expansion of the formal hermitian two-matrix model. For this, we refine the previously formulated diagrammatic rules for computing the $\frac{1}{N}$ expansion of the nonmixed correlation functions and give a new formulation of the spectral curve. We extend these rules obtaining a closed formula for correlation functions in all orders of topological expansion. We then integrate it to obtain the free energy in terms of residues on the associated Riemann surface.

Keywords: Matrix Models, Differential and Algebraic Geometry, Topological Strings.

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Contents

1. Introduction

Matrix models is a fascinating topic unifying many otherwise seemingly unrelated disciplines, including integrable systems, conformal theories, topological expansion, etc [33, 14]. However, only recently the understanding came that geometry itself plays a crucial role in constructing perturbative solutions to matrix models. Dijkgraaf and Vafa [15] studies in

the one-matrix model (1MM), together with subsequent observations [28, 4, 31, 32] based on the idea of [30] that topological hierarchies are encoded already in the planar limit of the solutions to the Hermitian one-matrix model with multiple-connected support of eigenvalues, led to understanding of a crucial role of the corresponding spectral curve in constructing explicit solutions.

The first attempts of computing the subleading terms in the $1/N^2$ expansion, where based on loop equations [2, 3], and allowed the authors of [3] to find the first few terms of the expansion of the free energy, only in the 1-cut case of the 1-hermitean matrix model.

In the begining of the 2000's, a geometrical approach, supported by using the master loop equation, resulted in constructing solutions in the first subleading order of the 1MM in the mutlticut case [19, 10, 20, 36].

Then, a Feynmann-like diagrammatic technique was invented in [16], which allowed to reformulate the loop equations themselves in a proper geometric way, and it became possible to compute all correlation functions of the 1-MM to all orders in the $1/N^2$ expansion [16]. However, the diagrammatic technique of [16] could not be applied directly to the expansion of the free energy. The topological expansion of the free energy to all orders was found with a refined diagrammatic technique in [11], where the key ingredient was the homogenity property of the free energy to all orders.

In parallel, the hermitian two-matrix model (2MM) solutions have been obtained, at early stages, in the planar limit of the $1/N$-expansion [34, 21]. It was however almost immediately observed [5, 18, 21] that the 2MM solution in the planar limit enjoys the same geometrical properties as the 1MM solution; only the spectral curve becomes an arbitrary algebraic curve, not just an hyperellitic curve arising in the 1MM case. The subsequent progress was however hindered by that the corresponding master loop equation in the 2MM case cannot immediately be expressed in terms of correlation functions alone. Nevertheless, using the geometrical properties of this equation, the solution in the first subleading order has been constructed [19, 20] on the base of knowledge of Bergmann tau function on Hurwitz spaces. However, the general belief arose that the 2MM case should not be very much different from the 1MM case, that is, we expect to find all ingredients of the 1MM solution in the 2MM case. Next step towards constructing the topological expansion of the 2MM was performed in [24], where the first variant of the diagrammatic technique for the correlation functions in the 2MM case was constructed. In the present paper we improve the technique of [24] (actually effectively simplifying it) to accommodate the action of the loop insertion operator. In fact, we demonstrate that the diagrammatic technique for the 2MM case is even closer to the one in the 1MM case that was before: in particular, we need only three-valent vertices, and the additional operator H we need to obtain the free energy turns out to be of the same origin as the one in the 1MM.

The paper is organized as follows. In section 2, we collect all the definitions, algebraic-geometrical notation, and facts about loop equations and filling fractions we need in what follows. In section 3, we provide a new formula for the spectral curve, which results in the new diagrammatic rules formulated in section 4. In the same section, we express the action of the loop insertion operator in terms of our diagrammatic technique, which makes the construction closed as regarding the nonmixed multipoint correlation functions. We

introduce the "integration" operator H in section 5 and, using this operator, present the diagrammatic technique that enables us to construct the complete topological expansion for the 2MM to all orders except the subleading order. But for this latter, the answer has been found in [20], so we eventually formulate a complete procedure for constructing the topological expansion in the 2MM. In appendix A, we extend our technique to calculating the first mixed correlation function; higher mixed correlation functions need further refining of this technique, which is beyond the scope of this publication. In appendix B, we prove the symmetricity of the free-energy expression w.r.t. interchanging x and y variables, which is crucial for the proper integration of the expression for the first mixed correlation function.

2. Definitions and algebraic-geometrical notation

2.1 Definition of the model

We study the formal two-matrix model [27] and compute the free energy $\mathcal{F}$ of this model in the asymptotic $\frac{1}{N}$-expansion. The partition function Z is the formal matrix integral

$$Z := \int_{H_N \times H_N} dM_1 dM_2 \, e^{-\frac{1}{\hbar} Tr(V_1(M_1)+V_2(M_2)-M_1 M_2)} = e^{-\mathcal{F}}, \tag{2.1}$$

where M_1 and M_2 are two $N \times N$ Hermitian matrices, dM_1 and dM_2 are the products of Lebesgue measures of the real components of M_1 and M_2, $\hbar = \frac{T}{N}$ is a formal expansion parameter and V_1 and V_2 are two polynomial potentials of respective degrees $d_1 + 1$ and $d_2 + 1$

$$V_1(x) = \sum_{k=1}^{d_1+1} \frac{t_k}{k} x^k \qquad , \qquad V_2(y) = \sum_{k=1}^{d_2+1} \frac{\tilde{t}_k}{k} y^k. \tag{2.2}$$

Formal integral means that it is computed order by order in powers of the t_k's (see section 2.3 or [22]). We consider polynomial potentials here only for simplicity of notations, but it is clear that the whole method presented in this paper extends to "rational potentials" (i.e. such that V_1' and V_2' are rational functions), and we expect it to extend to the whole semiclassical setting [6] including hard edges as well.

One can expand the free energy as well as all the correlation functions of this model in a $\hbar$ series assuming $\hbar$ to the order of the reciprocal matrix size; this procedure is customarily called the *topological expansion* pertaining to the fat-graph representation for formal integrals ([35, 9, 14]).

The topological expansion of the Feynman diagrams series reads, in terms of the free energy,

$$\mathcal{F} = \mathcal{F}(N, T, t_1, t_2, \ldots, t_{d_1+1}, \tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_{d_2+1}) = \sum_{h=0}^{\infty} \hbar^{2h-2} \mathcal{F}^{(h)}(T, \ldots). \tag{2.3}$$

We find a general formula for $\mathcal{F}^{(h)}$ for any positive integer h. Actually, we address the problem for $h \geq 2$; the solutions for $h = 0$ ([30, 4]) and $h = 1$ ([20]) are already known.

2.2 Notations

2.2.1 Variable sets

We consider functions of many variables $x_1, x_2, x_3, \ldots$, or of a subset of those variables. For this, we introduce the following notation:

Let K be a set of k integers:

$$K = (i_1, i_2, \ldots, i_k). \tag{2.4}$$

Let $k = |K|$ denote the length (or cardinality) of K. For any $j \leq |K|$, let K_j denote the set of all $j-$upples (i.e., subsets of length j) contained in K:

$$K_j := \{J \subset K \ , \ |J| = j\}. \tag{2.5}$$

We define the following $k-$upple of complex numbers:

$$\mathbf{x}_K := (x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \qquad , \qquad d\mathbf{x}_K := \prod_{j=1}^{k} dx_j. \tag{2.6}$$

2.2.2 Resolvents

For given integers k and l, we define the resolvents:

$$\begin{aligned}\overline{w}_{|K|,|L|}(\mathbf{x}_K, \mathbf{y}_L) &:= \hbar^{2-|K|-|L|} \left\langle \prod_{i=1}^{|K|} \mathrm{tr}\, \frac{1}{x_i - M_1} \prod_{i=1}^{|L|} \mathrm{tr}\, \frac{1}{y_i - M_2} \right\rangle_{\mathrm{conn}} \\ &= -\hbar^2 \frac{\partial}{\partial V_2(y_{|L|})} \cdots \frac{\partial}{\partial V_2(y_1)} \frac{\partial}{\partial V_1(x_{|K|})} \frac{\partial}{\partial V_1(x_{|K|-1})} \cdots \frac{\partial \mathcal{F}}{\partial V_1(x_1)} \\ &\quad + \frac{\delta_{|K|,1}\delta_{|L|,0}}{x_1} + \frac{\delta_{|K|,0}\delta_{|L|,1}}{y_1}\end{aligned} \tag{2.7}$$

with the formal loop insertion operators

$$\frac{\partial}{\partial V_1(x)} = -\sum_{j=1}^{\infty} \frac{j}{x^{j+1}} \frac{\partial}{\partial t_j} \quad \text{and} \quad \frac{\partial}{\partial V_2(x)} = -\sum_{j=1}^{\infty} \frac{j}{y^{j+1}} \frac{\partial}{\partial \tilde{t}_j}. \tag{2.8}$$

We also introduce the polynomials in y:

$$\overline{u}_k(x, y; x_{|K|}) := \hbar^{1-k} \left\langle \mathrm{tr}\, \frac{1}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \prod_{r=1}^{|K|} \mathrm{tr}\, \frac{1}{x_r - M_1} \right\rangle_{\mathrm{conn}}, \tag{2.9}$$

where the subscript *conn* denotes the connected component.

For convenience, we renormalize the two-point functions:

$$w_{|K|,|L|}(\mathbf{x}_K, \mathbf{y}_L) = \overline{w}_{|K|,|L|}(\mathbf{x}_K, \mathbf{y}_L) + \frac{\delta_{|K|,2}\delta_{|L|,0}}{(x_1 - x_2)^2} + \frac{\delta_{|K|,0}\delta_{|L|,2}}{(y_1 - y_2)^2} \tag{2.10}$$

and their polynomial correspondent:

$$u_k(x, y; x_K) := \overline{u}_k(x, y; x_K) - \delta_{k,0}(V_2'(y) - x). \tag{2.11}$$

We consider the $\hbar^2$-expansions of the above quantities:

$$w_{|K|,|L|}(\mathbf{x}_K, \mathbf{y}_L) = \sum_{h=0}^{\infty} \hbar^{2h}\, w^{(h)}_{|K|,|L|}(\mathbf{x}_K, \mathbf{y}_L), \tag{2.12}$$

$$u_k(x, y; x_K) = \sum_{h=0}^{\infty} \hbar^{2h}\, u^{(h)}_k(x, y; x_K). \tag{2.13}$$

We also need the functions closely related to the algebraic structure of the problem:

$$Y(x) := V_1'(x) - \overline{w}_1(x) \tag{2.14}$$

and

$$P(x, y) := \hbar \left\langle \operatorname{tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \right\rangle_{\text{conn}}. \tag{2.15}$$

The latter function together with all its terms of $\hbar^2$-expansion, is a polynomial of degree $d_1 - 1$ in x and $d_2 - 1$ in y.

2.2.3 The master loop equation

Among many different ways to solve matrix models [33], addressing the formal 2-matrix model problem, we choose the so-called loop equations [34, 13] encoding the matrix integral in (2.1) to be invariant under special changes of variables. They correspond to the generalization of the Virasoro constraints of the one-matrix model, i.e. the W-algebra.

Considering a particular change of variables, we come to the *master loop equation* [19]:

$$\boxed{E(x, y) = (y - Y(x))u_0(x, y) + \hbar^2 u_1(x, y; x)}, \tag{2.16}$$

where

$$E(x, y) = (V_1'(x) - y)(V_2'(y) - x) - P(x, y) + T \tag{2.17}$$

is a polynomial in x and y defining the *spectral curve*.

Considering the 't Hooft expansion in orders of $\hbar^2$, for any $h \geq 1$, we have:

$$\begin{aligned} E^{(h)}(x, y) = {} & (y - Y(x))u_0^{(h)}(x, y) + w_{1,0}^{(h)}(x)u_0^{(0)}(x, y) \\ & + \sum_{m=1}^{h-1} w_{1,0}^{(m)}(x)u_0^{(h-m)}(x, y) + \frac{\partial}{\partial V_1(x)} u_0^{(h-1)}(x, y), \end{aligned} \tag{2.18}$$

where $E^{(h)}(x, y)$ is the h'th term in the $\hbar^2$-expansion of the spectral curve $E(x, y)$.

2.2.4 Algebraic geometry notations

In this section, we recall the algebraic-geometrical pattern of our problem (see more details in [24 – 26]).

To leading order, the *master loop equation* (2.18) reduces to an algebraic equation

$$E^{(0)}(x, Y(x)) = 0 \tag{2.19}$$

with $E^{(0)}(x,y)$ being a polynomial of degree d_1+1 in x and d_2+1 in y.

We parameterize the algebraic curve $E^{(0)}(x,y)=0$ implied by (2.19) by a running point p on the corresponding compact Riemann surface $\mathcal{E}$. We therefore define two analytical meromorphic functions $x(p)$ and $y(p)$ on $\mathcal{E}$ such that:

$$E^{(0)}(x,y)=0 \Leftrightarrow \exists!\, p\in\mathcal{E} \quad x=x(p)\ ,\ y=y(p). \tag{2.20}$$

The functions x and y are not bijective. Indeed, since $E^{(0)}(x,y)$ has a degree d_2+1 in y, it admits d_2+1 solutions for a given x; that is we have d_2+1 points p on $\mathcal{E}$ such that $x(p)=x$. Thus, the Riemann surface is made of d_2+1 x-sheets, or respectively, of d_1+1 y-sheets. We then denote

$$x(p) = x \Leftrightarrow p=p^{(j)}(x) \quad \text{for} \quad j=0,\dots,d_2, \tag{2.21}$$

$$y(p) = y \Leftrightarrow p=\tilde{p}^{(j)}(x) \quad \text{for} \quad j=0,\dots,d_1. \tag{2.22}$$

Among the different x-sheets (resp. y-sheet), there exists only one where $y(p)\sim_{x(p)\to\infty} V_1'(x(p))-\frac{T}{x(p)}+O(1/x^2(p))$ (resp. $x(p)\sim_{y(p)\to\infty} V_2'(y(p))-\frac{T}{y(p)}+O(1/y^2(p))$). We call it the physical sheet, and it bears the superscript 0.

Genus and cycles. The curve $\mathcal{E}$ is a compact Riemann surface of finite genus $g\leq d_1d_2-1$.[1] We do not require g to be equal to d_1d_2-1, that is, we allow double points on the corresponding Riemann surface. We choose $2g$ canonical cycles as $\mathcal{A}_i$, $\mathcal{B}_i$, $i=1,\dots,g$, such that:

$$\mathcal{A}_i\cap\mathcal{A}_j=0 \qquad , \qquad \mathcal{B}_i\cap\mathcal{B}_j=0 \qquad , \qquad \mathcal{A}_i\cap\mathcal{B}_j=\delta_{ij}. \tag{2.23}$$

Branch points. The x-branch points μ_α, $\alpha=1,\dots,d_2+1+2g$, are the zeroes of the differential dx, respectively, the y-branch points ν_β, $\beta=1,\dots,d_1+1+2g$, are the zeroes of dy. We assume here that all branch points are simple and distinct. Note also that $E^{(0)}_y(x(p),y(p))$ vanishes (simple zeroes) at the branch points (it vanishes in other points as well).

A branch point is a point where two sheets of the Riemann surface meet. If p is a point on the Riemann surface near a branch point, there is another point p^i, which we note $\overline{p}$, near the same branch point. In other words:

$$\forall\alpha \quad \exists!\overline{p}\neq p \quad \text{such that} \quad x(\overline{p})=x(p) \quad \text{and} \quad y(\overline{p})\to_{p\to\mu_\alpha} y(p) \tag{2.24}$$

Let us emphasize that the point $\overline{p}$ and the corresponding sheet, depend on which branch point we are considering.

Bergmann kernel. On the Riemann surface $\mathcal{E}$, we have a unique Abelian bilinear differential $B(p,q)$, with one double pole at $p=q$ such that

$$B(p,q)\underset{p\to q}{\sim}\frac{dx(p)dx(q)}{(x(p)-x(q))^2}+\text{finite} \quad \text{and} \quad \forall i \oint_{p\in\mathcal{A}_i} B(p,q)=0. \tag{2.25}$$

[1] This genus g must not be confused with the genus indicating the order of the topological expansion

It is symmetric,

$$B(p,q) = B(q,p), \tag{2.26}$$

it can be expressed in terms of theta-functions [26, 25], and depends only on the complex structure of $\mathcal{E}$.

Abelian differential of the third kind. On the Riemann surface $\mathcal{E}$, there exists a unique Abelian differential of the third kind $dS_{q,r}(p)$, with two simple poles at $p = q$ and at $p = r$, such that

$$\operatorname*{Res}_{p\to q} dS_{q,r}(p) = 1 = -\operatorname*{Res}_{p\to r} dS_{q,r}(p) \quad \text{and} \quad \forall i \oint_{\mathcal{A}_i} dS_{q,r}(p) = 0. \tag{2.27}$$

Notice that the Abelian differential of the third kind and the Bergmann kernel are linked by

$$dS_{q,r}(p) = \int_{\xi=r}^{q} B(\xi,p) \quad \text{and} \quad B(p,q) = d_q\left(dS_{q,r}(p)\right). \tag{2.28}$$

where the contour of integration is a line from r to q, which does not cross any $\mathcal{A}$ or $\mathcal{B}$ cycle.

Given a branch point μ_α, and a point q in the vicinity of μ_α, we introduce the following notation:

$$dE_{q,\overline{q}}(p) = \int_{\overline{q}}^{q} B(\xi,p), \tag{2.29}$$

where now, the integration path is chosen as the shortest path between q and $\overline{q}$, i.e. a path which lies in a small vicinity of μ_α. That definition differs from the one above. If the branchpoint μ_α is surrounded by contour $\mathcal{A}_i$, we have:

$$dE_{q,\overline{q}}(p) = dS_{q,\overline{q}}(p) + \oint_{\mathcal{B}_i} B(p,\xi) \tag{2.30}$$

The main property of that $dE_{q,\overline{q}}(p)$, is that it vanishes at $q = \overline{q}$, i.e. it vanishes at the branch point μ_α.

Correlation functions on the Riemann surface Given the algebraic curve, we see that we can redefine the correlation functions (2.12) and (2.13) more precisely. Indeed, they are defined only as formal series as their arguments $\mathbf{x}_K$ and $\mathbf{y}_L$ tend to infinity. The loop equations show that these formal series are in fact algebraic functions and hence have a finite radius of convergency, with cuts beyond the radius. As functions of the x and y variables, they are multivalued.

If instead of writing them as functions of x or y, we write them as functions on the Riemann surface, they become monovalued. This is the reason why we prefer to introduce the following notation for the correlation functions, as meromophic differential forms on the Riemann surface:

$$W_{|K|,|L|}(\mathbf{p}_K,\mathbf{q}_L) := w_{|K|,|L|}(x(\mathbf{p}_K),y(\mathbf{q}_L))dx(\mathbf{p}_K)dy(\mathbf{q}_L), \tag{2.31}$$

and

$$U_{|K|}(p,y;\mathbf{p}_K) := u_{|K|}(x(p),y;x(\mathbf{p}_K))dx(\mathbf{p}_K), \tag{2.32}$$

where p_i's and q_j's are points on the surface $\mathcal{E}$ whose images $x(p_i)$ or $y(q_i)$ are complex numbers. y is a complex number.

Remark 1. The interpretation of those correlation functions as generating series for the moments $< \prod_i \operatorname{Tr} M_1^{k_i} \prod_j \operatorname{Tr} M_2^{l_j} >_{\mathrm{c}}$, corresponds to the situation where all p_i's are in the x-physical sheet, in a vicinity of ∞_x, and all q_j's are in the y-physical sheet in a vicinity of ∞_y.

2 point function It is well known (see for instance [1, 5, 28 – 30, 19, 21]) that, to leading order, the 2-point function $W_{2,0}$ is the Bergmann kernel:

$$W_{2,0}^{(0)}(p,q) = B(p,q) \tag{2.33}$$

The non-renormalized 2 point function:

$$\overline{W}_{2,0}(p,q) = W_{2,0}(p,q) - \frac{dx(p)dx(q)}{(x(p)-x(q))^2} \tag{2.34}$$

is finite at $p = q$.

2.3 Loop equations and fixed filling fractions

We have showed that to leading order, the 1-point function $Y^{(0)}(x)$ obeys an algebraic equation (2.19):

$$E^{(0)}(x, Y(x)) = 0, \tag{2.35}$$

where

$$E^{(0)}(x,y) = (V_1'(x) - y)(V_2'(y) - x) - P^{(0)}(x,y) + T \tag{2.36}$$

But so far we have not discussed how to determine the polynomial $P^{(0)}(x,y)$.

As this was extensively discussed in the literature, we only briefly summarize it below.

We need $d_1 d_2 - 1$ equations to fix the $d_1 d_2 - 1$ unknown coefficients of $P^{(0)}$. Those additional $d_1 d_2 - 1$ equations do not come from any loop equation, so we need an independent hypothesis related to the precise definition of our matrix model. In a sense, loop equations express the invariance of the integral under reparameterizations, independently on the integration paths. Additional equations are those that depend on the choice of integration path.

For arbitrary integration paths, the $\hbar$ expansion may not exist. The choice of integration method we use in this paper, corresponds to the so-called *formal matrix model, with fixed filling fractions.*

We consider here the problem of a formal matrix model, i.e. the one represented by the formal power series expansion of a matrix integral, where the non-quadratic terms in the potentials V_1 and V_2 are treated as perturbations near quadratic potentials. Such a perturbative expansion can be performed only near local extrema of $V_1(x) + V_2(y) - xy$, i.e. near the points (ξ_i, η_i), $i = 1, \ldots, d_1 d_2$, such that

$$V_1'(\xi_i) = \eta_i \quad \text{and} \quad V_2'(\eta_i) = \xi_i \tag{2.37}$$

which has in general $d_1 d_2$ solutions. Therefore, perturbative expansion can be performed near matrices of the form:

$$\overline{M_1} = \mathrm{diag}\,(\overbrace{\xi_1, \ldots, \xi_1}^{n_1}, \overbrace{\xi_2, \ldots, \xi_2}^{n_2}, \ldots, \overbrace{\xi_{d_1 d_2}, \ldots, \xi_{d_1 d_2}}^{n_{d_1 d_2}}), \tag{2.38}$$

$$\overline{M_2} = \mathrm{diag}\,(\overbrace{\eta_1, \ldots, \eta_1}^{n_1}, \overbrace{\eta_2, \ldots, \eta_2}^{n_2}, \ldots, \overbrace{\eta_{d_1 d_2}, \ldots, \eta_{d_1 d_2}}^{n_{d_1 d_2}}), \tag{2.39}$$

such that $\sum_{i=1}^{d_1 d_2} n_i = N$. The perturbative integral is computed by writing $M_1 = \overline{M_1} + \delta M_1$ and $M_2 = \overline{M_2} + \delta M_2$, by expanding higher order terms (cubic and higher) in δM_1 and δM_2, and by treating quadratic terms in δM_1 and δM_2 as gaussian integrals.

In other words, a formal integral can be computed as soon as we have chosen a vacuum around which to make a perturbative expansion. The choice of a vacuum is equivalent to choosing a partition of N into $d_1 d_2$ parts:

$$\sum_{i=1}^{d_1 d_2} n_i = N. \tag{2.40}$$

It is easy to see that if we truncate the perturbative expansion to any finite order, we have:

$$\frac{1}{2i\pi} \oint_{\mathcal{C}_i} \left\langle \mathrm{tr}\, \frac{1}{x - M_1} \right\rangle dx = \frac{1}{2i\pi} \oint_{\mathcal{C}_i} \mathrm{tr}\, \frac{1}{x - \overline{M_1}} dx = -n_i, \tag{2.41}$$

where $\mathcal{C}_i$ is a small direct circle around the point ξ_i in the complex plane. And thus:

$$\frac{1}{2i\pi} \oint_{\mathcal{C}_i} w_{1,0}(x) dx = -n_i \hbar := -\epsilon_i. \tag{2.42}$$

If we do not truncate to a finite order, all functions become algebraic, with cuts, and the contours $\mathcal{C}_i$ are then enhanced to finite contours around the cuts, and up to a redefinition of the contours, the *filling fractions* are the $\mathcal{A}$-cycle integrals. We define:

$$\frac{1}{2i\pi} \oint_{\mathcal{A}_i} y dx = -\frac{1}{2i\pi} \oint_{\mathcal{A}_i} x dy = \epsilon_i \tag{2.43}$$

We call ϵ_i's the filling fractions, and they are given new parameters (moduli) of the model. In what follows, we consider them to be independent of the potential or on any other parameter. Let us notice that the such defined ϵ_i's are not necessarily of the form n_i/N with n_i a positive integer, but they can be arbitrary complex numbers, provided that:

$$\sum_i \epsilon_i = t = N\hbar \tag{2.44}$$

In particular, since all correlation functions $w_{k,0}(x_1, \ldots, x_k)$ are obtained by derivation of $w_{1,0}$ with respect to the potential V_1, we have:

$$\frac{1}{2i\pi} \oint_{\mathcal{A}_i} w_{k,0}(x_1, \ldots, x_k) dx_1 = 0. \tag{2.45}$$

Eq. (2.43) together with the large x and y behaviors, suffices for determining completely all the coefficients of the polynomial $E^{(0)}(x, y)$, and thus the leading large N resolvents $w_{1,0}(x)$ and $w_{0,1}(y)$.

Remark 2. It is easy to see that if we truncate the perturbative gaussian integral to any finite order, the result is a polynomial in $1/N^2$, and thus, formal matrix models always have a $1/N^2$ expansion. This comes from the fact that, to any finite order, the perturbative gaussian integral is the generating function for counting discrete surfaces made of a finite bounded number of polygons, thus having a maximal Euler characteristic (power of N).

Remark 3. We emphasize again the formality of our model: in principle, the $\mathcal{A}$-cycles do not necessarily lie in the physical sheet, so we would need additional assumptions to segregate physically meaningful models. Recall that the ϵ_i's can be arbitrary complex numbers.

Remark 4. We have $d_1 d_2$ non vanishing contour integrals, but they are not all independent. Their sum can be deformed as a contour around ∞. Thus, we only have $d_1 d_2 - 1$ independent non vanishing contour integrals, and thus the maximal genus of our curve is $d_1 d_2 - 1$. The number of independent filling fractions is $d_1 d_2 - 1$, due to the constraint on their sum.

Double points The Riemann surface $\mathcal{E}$ may have double points.

A point $p \in \mathcal{E}$ is a double point iff:

$$\exists q \neq p, \;\; x(p) = x(q) \text{ and } \; y(p) = y(q). \tag{2.46}$$

Double points are such that both derivatives E_x and E_y vanish, and thus the curve has two tangents at such points. The differential forms dx and dy do not vanish at double points.

We need an extra asumption to deal with double points.

We may consider that in the framework of the perturbative gaussian matrix integral seen above, double points correspond to vanishing filling fractions $\epsilon_i = 0$. Thus:

$$\oint_{\mathcal{C}} w_{1,0}(x) dx = 0 \tag{2.47}$$

where $\mathcal{C}$ is a contour which encircles the double point. In other words, we make the assumption that the resolvent has no residue at double points, to all orders in the $\hbar$ expansion. We will see below that this vanishing residue condition is sufficient to ensure that correlation functions never have poles at double points.

Another reason to make that assumption is based on the Krichever formula [30], which involves summation only over branch-points and not over double points.

3. A new formula for the spectral curve

The 't Hooft expansion of the master loop equation (2.18), links correlation functions of genus h to correlation functions of lower genus. The knowledge of the l.h.s. $E(x, y)$ would then give a way of deriving the whole topological expansion of the $W_{|K|,0}$.

From the definition of the surface $\mathcal{E}$, one derives the leading term as:

$$\begin{aligned} E^{(0)}(x(p),y(q)) &= -t_{d_1+1} \times \prod_{i=0}^{d_1}(x(p)-x(\tilde{q}^{(i)}(y))) \\ &= -\tilde{t}_{d_2+1} \times \prod_{i=0}^{d_2}(y(q)-y(p^{(i)}(x))) \end{aligned} \tag{3.1}$$

Actually, not only the leading term, but the whole function admits such a product structure:

Theorem 1. *The functions $E(x,y)$ and $U_0(p,y)$ can be written:*

$$\boxed{E(x(p),y) = -\tilde{t}_{d_2+1}\text{"}\left\langle \prod_{i=0}^{d_2}(y - V_1'(x(p)) + \hbar\,\mathrm{Tr}\,\frac{1}{x(p^i)-M_1})\right\rangle\text{"}} \tag{3.2}$$

and

$$U_0(p,y) = -\tilde{t}_{d_2+1}\text{"}\left\langle \prod_{i=1}^{d_2}(y - V_1'(x(p)) + \hbar\,\mathrm{Tr}\,\frac{1}{x(p^i)-M_1})\right\rangle\text{"}. \tag{3.3}$$

where the quotes " $\langle . \rangle$ " mean that when we expand into cumulants, each time we find a 2-point function $\overline{w}_{2,0}$, we replace it by $w_{2,0} = \overline{w}_{2,0} + \frac{1}{(x_1-x_2)^2}$ as in (2.10).

In other words, $E(x,y)$ is the $(d_2+1)-$point correlation function, with all points corresponding to the same x in all the d_2+1 sheets, and $U_0(p,y)$ is the d_2-point correlation function, with all points corresponding to the same $x(p)$ in all sheets but the same as p.

Proof. We prove this theorem by showing that the two sides of the equalities (3.2) and (3.3) are defined by the same recursion relations.

We let $\widetilde{E}(x(p),y)$ and $\tilde{U}_0(p,y)$ be defined by the respective r.h.s. of (3.2) and (3.3). Consider their topological expansions:

$$\widetilde{E}(x,y) = \sum_{h=0}^{\infty}\hbar^{2h}\widetilde{E}^{(h)}(x,y) \quad \text{and} \quad \tilde{U}_0(p,y) = \sum_{h=0}^{\infty}\hbar^{2h}\widetilde{U}_0^{(h)}(x(p),y) \tag{3.4}$$

By expanding in cumulants (connected parts), one can observe that:

$$\begin{aligned} \widetilde{E}^{(h)}(x(p),y) &= (y-y(p))\widetilde{U}_0^{(h)}(p,y) + W_{1,0}^{(h)}(p)\widetilde{U}_0^{(0)}(p,y) \\ &\quad + \sum_{m=1}^{h-1} W_{1,0}^{(m)}(p)\widetilde{U}_0^{(h-m)}(p,y) + \frac{\partial}{\partial V_1(x(p))}\widetilde{U}_0^{(h-1)}(p,y), \end{aligned} \tag{3.5}$$

which has exactly the same form as (2.18).

Obviously we have $E^{(0)}(x,y) = \widetilde{E}^{(0)}(x,y)$ and $U^{(0)}(p,y) = \widetilde{U}^{(0)}(p,y)$.

Let us now show that, given $E^{(0)}(x,y)$, (2.18) admits unique solution. For this, assume that we know $E^{(h-1)}(x,y)$ and $U_0^{(m)}(p,y)$ for all $m \leq h-1$ and prove that (2.18) allows computing $E^{(h)}(x,y)$ and $U_0^{(h)}(p,y)$.

Consider (2.18) for $p = q^i$ and $y = y(p)$ for any $i = 0 \dots d_2$. It reads

$$E^{(h)}(x(p), y(p^i)) = \sum_{m=1}^{h} W_{1,0}^{(m)}(p^i) U_0^{(h-m)}(p^i, y(p^i)) + \frac{\partial}{\partial V_1(p^i)} U_0^{(h-1)}(p^i, y(p^i); p^i). \quad (3.6)$$

By the recursion hypothesis, one knows the r.h.s. totally, so one knows the l.h.s. . This gives the value of $E^{(h)}(x, y)$ for $d_2 + 1$ values of y. Because $E^{(h)}(x, y)$ is a polynomial in y of degree d_2, an interpolation formula gives its value for any y. It remains to compute $U_0^{(h)}(p, y)$ which is straightforward due to (2.18).

Thus, because $E^{(h)}(x, y)$ and $\widetilde{E}^{(h)}(x, y)$ obey the same equations with the same initial condition, we conclude that $E^{(h)}(x, y) = \widetilde{E}^{(h)}(x, y)$ for any h.

4. Diagrammatic rules for the correlation functions: new insight.

Using loop equations, two of the authors have derived two sets of diagrammatic rules allowing to compute any $W_{|K|,0}$ as residues on the Riemann surface [24]. One of these theories involved only cubic vertices but presented the disadvantage of using explicitly the auxiliary functions $U_{|K|}$, whereas the second one involved only the $W_{|K|,0}$'s but expressed it through vertices of valence up to d_2.

In this section, we present new diagrammatic rules composed of cubic vertices and involving only W's whose arguments are in the vicinity of the branch points. They are much simpler than all the preceding ones.

From (3.2) and the definition (2.17) of $E(x, y)$, we obtain the following equation:

$$\begin{aligned} -\tilde{t}_{d_2+1} \text{"} \left\langle \prod_{i=0}^{d_2} (y - V_1'(x(p)) + \hbar \operatorname{Tr} \tfrac{1}{x(p^i) - M_1}) \right\rangle \text{"} = {} & (V_2'(y) - x(p))(V_1'(x(p)) - y) \\ & - P(x(p), y) + 1. \end{aligned} \quad (4.1)$$

Recall that $P(x, y)$ is a polynomial both in y (of degree $d_2 - 1$) *and* in x.

The large y expansion of the l.h.s. reads

$$\begin{aligned} & -\tilde{t}_{d_2+1}\, y^{d_2+1} - \tilde{t}_{d_2+1} \sum_{i=0}^{d_2} \left\langle \hbar \operatorname{Tr} \frac{1}{x(p^i) - M_1} - V_1'(x(p)) \right\rangle y^{d_2} \\ & -\tilde{t}_{d_2+1} \sum_{i=0}^{d_2} \sum_{j \neq i} \text{"} \left\langle \left(\hbar \operatorname{Tr} \frac{1}{x(p^i) - M_1} - V_1'(x(p)) \right) \times \right. \\ & \qquad \left. \times \left(\hbar \operatorname{Tr} \frac{1}{x(p^j) - M_1} - V_1'(x(p)) \right) \right\rangle \text{"} y^{d_2-1} + O(y^{d_2-2}), \end{aligned} \quad (4.2)$$

and the large y expansion of the r.h.s. yields:

$$-\tilde{t}_{d_2+1}\, y^{d_2+1} - y^{d_2}(\tilde{t}_{d_2} - \tilde{t}_{d_2+1} V_1'(x(p))) - \tilde{t}_{d_2+1} Q(x(p)) y^{d_2-1} + O(y^{d_2-2}), \quad (4.3)$$

where $Q(x(p))$ is a polynomial in x of degree $d_1 - 1$.

- Equating the coefficient of y^{d_2} gives

$$\boxed{\sum_{i=0}^{d_2} Y(p^i) = V_1'(x(p)) - \frac{\tilde{t}_{d_2}}{\tilde{t}_{d_2+1}}}, \tag{4.4}$$

which is a polynomial in $x(p)$. Expanded to order h, this means that

$$\sum_{i=0}^{d_2} W_{1,0}^{(h)}(p^i) = 0 \quad \text{for}\, h \geq 1. \tag{4.5}$$

Applying the loop insertion operator to (4.4) we get:

$$\sum_{i=0}^{d_2} W_{2,0}^{(h)}(p^i, q) = \delta_{h,0}\, \frac{dx(p)dx(q)}{(x(p) - x(q))^2}, \tag{4.6}$$

which can also be written:

$$\overline{W}_{2,0}(p^i, q) + \sum_{j \neq i} W_{2,0}(p^j, q) = 0. \tag{4.7}$$

- Equating the coefficient of y^{d_2-1} gives

$$\frac{1}{2} \sum_{i \neq j = 0}^{d_2} Y(p^i)\, Y(p^j) + \hbar^2 w_{2,0}(p^i, p^j) = V_1'(x(p)) \frac{\tilde{t}_{d_2}}{\tilde{t}_{d_2+1}} + \frac{\tilde{t}_{d_2-1}}{\tilde{t}_{d_2+1}} + Q(x(p)) \tag{4.8}$$

where

$$Q(x) = \hbar \left\langle \operatorname{Tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \right\rangle \tag{4.9}$$

is a polynomial in x of degree $d_1 - 1$. Using (4.4) and (4.7) we get:

$$\sum_{i=0}^{d_2} Y(p^i)^2 + \hbar^2 \overline{w}_{2,0}(p^i, p^i) = V_1'(x(p))^2 - 2\frac{\tilde{t}_{d_2-1}}{\tilde{t}_{d_2+1}} + \left(\frac{\tilde{t}_{d_2}}{\tilde{t}_{d_2+1}}\right)^2 - 2Q(x(p)) \tag{4.10}$$

Expanding this equation in $\hbar$, we obtain for $h \geq 1$

$$\boxed{\begin{aligned} 2\textstyle\sum_{i=0}^{d_2} y(p^i) W_{1,0}^{(h)}(p^i) dx(p) \;=\; & \textstyle\sum_{i=0}^{d_2} \sum_{m=1}^{h-1} W_{1,0}^{(m)}(p^i) W_{1,0}^{(h-m)}(p^i) \\ & + \textstyle\sum_{i=0}^{d_2} \overline{W}_{2,0}^{(h-1)}(p^i, p^i) + 2Q^{(h)}(x(p)) dx(p)^2. \end{aligned}} \tag{4.11}$$

We consider this equation when p is near a branch point μ_α, multiply it by $\frac{1}{2}\frac{dE_{p,\overline{p}}(q)}{y(p)-y(\overline{p})}$, take the residues when $p \to \mu_\alpha$ and sum over all the branch points. Notice that $\frac{1}{2}\frac{dE_{p,\overline{p}}(q)}{y(p)-y(\overline{p})}$ is finite at $p = \overline{p}$, i.e. it has no poles at branch points.

Let us first compute the l.h.s. :

$$
\begin{aligned}
&\sum_\alpha \operatorname*{Res}_{p\to\mu_\alpha} \frac{dE_{p,\overline{p}}(q)\sum_{i=0}^{d_2} y(p^i)W_{1,0}^{(h)}(p^i)}{y(p)-y(\overline{p})} \\
&= \sum_\alpha \operatorname*{Res}_{p\to\mu_\alpha} \frac{dE_{p,\overline{p}}(q)\left[y(p)W_{1,0}^{(h)}(p)+y(\overline{p})W_{1,0}^{(h)}(\overline{p})+\sum_{p^i\neq p,\overline{p}} y(p^i)W_{1,0}^{(h)}(p^i)\right]}{y(p)-y(\overline{p})} \\
&= \sum_\alpha \operatorname*{Res}_{p\to\mu_\alpha} \frac{dE_{p,\overline{p}}(q)\left[y(p)W_{1,0}^{(h)}(p)+y(\overline{p})W_{1,0}^{(h)}(\overline{p})\right]}{y(p)-y(\overline{p})} \\
&= \sum_\alpha \operatorname*{Res}_{p\to\mu_\alpha} \frac{dE_{p,\overline{p}}(q)\left[y(p)W_{1,0}^{(h)}(p)-y(\overline{p})(W_{1,0}^{(h)}(p)+\sum_{p^i\neq p,\overline{p}} W_{1,0}^{(h)}(p^i))\right]}{y(p)-y(\overline{p})} \\
&= \sum_\alpha \operatorname*{Res}_{p\to\mu_\alpha} \frac{dE_{p,\overline{p}}(q)\left[y(p)W_{1,0}^{(h)}(p)-y(\overline{p})W_{1,0}^{(h)}(p)\right]}{y(p)-y(\overline{p})} \\
&= \sum_\alpha \operatorname*{Res}_{p\to\mu_\alpha} dE_{p,\overline{p}}(q)\, W_{1,0}^{(h)}(p) \\
&= \sum_\alpha \operatorname*{Res}_{p\to\mu_\alpha} dS_{p,o}(q)\, W_{1,0}^{(h)}(p) - dS_{\overline{p},o}(q)\, W_{1,0}^{(h)}(p) \\
&= \sum_\alpha \operatorname*{Res}_{p\to\mu_\alpha} dS_{p,o}(q)\, (W_{1,0}^{(h)}(p) - W_{1,0}^{(h)}(\overline{p})) \\
&= \sum_\alpha \operatorname*{Res}_{p\to\mu_\alpha} dS_{p,o}(q)\, (2W_{1,0}^{(h)}(p) + \sum_{p^i\neq p,\overline{p}} W_{1,0}^{(h)}(p^i)) \\
&= 2\sum_\alpha \operatorname*{Res}_{p\to\mu_\alpha} dS_{p,o}(q)\, W_{1,0}^{(h)}(p) \\
&= -2 \operatorname*{Res}_{p\to q} dS_{p,o}(q)\, W_{1,0}^{(h)}(p) \\
&\qquad + \frac{1}{2i\pi}\sum_i \left(\oint_{\mathcal{A}_i} W_{1,0}^{(h)}(p) \oint_{\mathcal{B}_i} B(p,q) + \oint_{\mathcal{B}_i} W_{1,0}^{(h)}(p) \oint_{\mathcal{A}_i} B(p,q)\right) \\
&= -2 \operatorname*{Res}_{p\to q} dS_{p,o}(q)\, W_{1,0}^{(h)}(p) \\
&= 2W_{1,0}^{(h)}(q)
\end{aligned}
\tag{4.12}
$$

Let us compute the r.h.s. in a similar way. It reads

$$
\begin{aligned}
&\sum_\alpha \operatorname*{Res}_{p\to\mu_\alpha} \frac{\frac{1}{2}dE_{p,\overline{p}}(q)}{(y(p)-y(\overline{p}))dx(p)}\Big[\sum_{i=0}^{d_2}\Big(\overline{W}_{2,0}^{(h-1)}(p^i,p^i) + \sum_{m=1}^{h-1} W_{1,0}^{(m)}(p^i)W_{1,0}^{(h-m)}(p^i)\Big) \\
&\qquad\qquad + 2Q^{(h)}(x(p))dx(p)^2\Big] \\
&= \sum_\alpha \operatorname*{Res}_{p\to\mu_\alpha} \frac{\frac{1}{2}dE_{p,\overline{p}}(q)}{(y(p)-y(\overline{p}))dx(p)}\Big[\overline{W}_{2,0}^{(h-1)}(p,p) + \sum_{m=1}^{h-1} W_{1,0}^{(m)}(p)W_{1,0}^{(h-m)}(p) \\
&\qquad\qquad + \overline{W}_{2,0}^{(h-1)}(\overline{p},\overline{p}) + \sum_{m=1}^{h-1} W_{1,0}^{(m)}(\overline{p})W_{1,0}^{(h-m)}(\overline{p})\Big] \\
&= 2\sum_\alpha \operatorname*{Res}_{p\to\mu_\alpha} \frac{\frac{1}{2}dE_{p,\overline{p}}(q)}{(y(p)-y(\overline{p}))dx(p)}\Big[\overline{W}_{2,0}^{(h-1)}(p,p) + \sum_{m=1}^{h-1} W_{1,0}^{(m)}(p)W_{1,0}^{(h-m)}(p)\Big]
\end{aligned}
\tag{4.13}
$$

Thus we have, for $h \geq 1$:

$$W_{1,0}^{(h)}(q) = \sum_{\alpha} \operatorname*{Res}_{p \to \mu_\alpha} \frac{\frac{1}{2} dE_{p,\overline{p}}(q)(\overline{W}_{2,0}^{(h-1)}(p,p) + \sum_{m=1}^{h-1} W_{1,0}^{(m)}(p) W_{1,0}^{(h-m)}(p))}{(y(p) - y(\overline{p}))dx(p)} \tag{4.14}$$

Using properties (4.5) and (4.7), this last equation can be changed into (we add terms which don't have poles at branch points, and thus whose resiudes vanish):

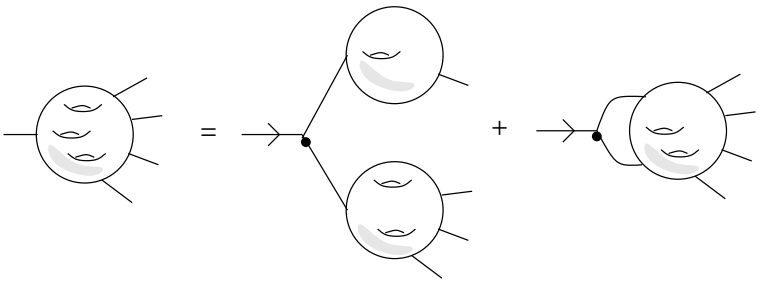
$$W_{1,0}^{(h)}(q) = -\sum_{\alpha} \operatorname*{Res}_{p \to \mu_\alpha} \frac{\frac{1}{2} dE_{p,\overline{p}}(q)(W_{2,0}^{(h-1)}(p,\overline{p}) + \sum_{m=1}^{h-1} W_{1,0}^{(m)}(p) W_{1,0}^{(h-m)}(\overline{p}))}{(y(p) - y(\overline{p}))dx(p)} \tag{4.15}$$

This is the case $k = 0$ of the following theorem (proved below by recursively applying the loop insertion operator $\partial/\partial V_1$):

Theorem 2.

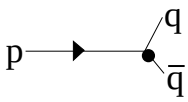
$$\boxed{\begin{aligned} W_{k+1,0}^{(h)}(q, p_K) = & -\textstyle\sum_\alpha \operatorname{Res}_{p \to \mu_\alpha} \frac{\frac{1}{2} dE_{p,\overline{p}}(q)}{(y(p) - y(\overline{p}))\, dx(p)} \Big(W_{k+1,0}^{(h-1)}(p, \overline{p}, p_K) + \\ & + \textstyle\sum_{j,m} W_{j+1,0}^{(m)}(p, p_J)\, W_{k+1-j,0}^{(h-m)}(\overline{p}, p_{K-J}) \Big), \end{aligned}} \tag{4.16}$$

which can be diagrammatically represented by



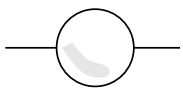
(4.17)

where a sphere with h holes and k legs is the k-point function to order h $W_k^{(h)}$, and the arrow means the following integration:



$$= \sum_{\alpha} \operatorname*{Res}_{p \to \mu_\alpha} \frac{-\frac{1}{2} dE_{q,\overline{q}}(p)}{(y(q) - y(\overline{q}))\, dx(q)} \tag{4.18}$$

This equation is a recursion relation, whose initial condition is given by (2.33)*, i.e.*



$$= \mathsf{p} \text{———} \mathsf{q} = B(p, q). \tag{4.19}$$

Proof. We have already proved theorem (4.16) for $k = 0$. Let us now show that the rule (4.18) is compatible with the derivation wrt the potential V_1.

One preliminary needed formula is the action of the loop insertion operator on the function $y(p)$. It is well known (see for exemple [19]) that:

$$\frac{\partial y(p)}{\partial V_1(x(r))}\,dx(r) = -\frac{B(p,r)}{dx(p)} \tag{4.20}$$

First, the action of the loop insertion operator on the Bergmann kernel gives [24]:

$$\begin{aligned}\frac{\partial B(p,q)}{\partial V_1(x(r))}\,dx(r) &= \sum_\alpha \operatorname*{Res}_{\xi\to\mu_\alpha} \frac{dE_{\xi,\overline{\xi}}(q)B(p,\overline{\xi})B(\xi,r)}{\big(y(\overline{\xi})-y(\xi)\big)\,dx(\xi)} \\ &= \sum_\alpha \operatorname*{Res}_{\xi\to\mu_\alpha} \frac{1}{2}\frac{dE_{\xi,\overline{\xi}}(q)\left[B(p,\overline{\xi})B(\xi,r)+B(r,\overline{\xi})B(\xi,p)\right]}{\big(y(\overline{\xi})-y(\xi)\big)\,dx(\xi)}.\end{aligned} \tag{4.21}$$

Note that this corresponds exactly to the conjecture for $k=1$ and $h=0$.

By integrating this expression with respect to p, we obtain the action on the arrowed edges:

$$\frac{\partial dE_{p,\overline{p}}(q)}{\partial V_1(x(r))}\,dx(r) = \sum_\alpha \operatorname*{Res}_{\xi\to\mu_\alpha} \frac{1}{2}\frac{dE_{\xi,\overline{\xi}}(q)\left[dE_{p,\overline{p}}(\overline{\xi})B(\xi,r)+B(r,\overline{\xi})dE_{p,\overline{p}}(\xi)\right]}{\big(y(\overline{\xi})-y(\xi)\big)\,dx(\xi)}, \tag{4.22}$$

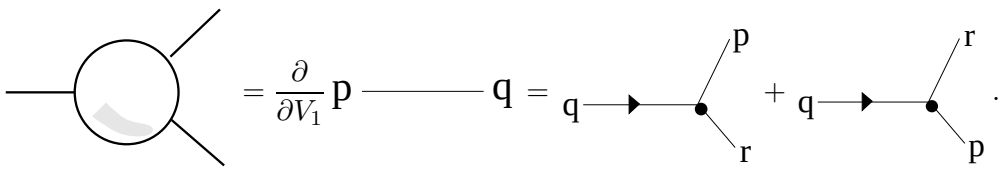
where q, p and r are outside the integration contour.

There is one last quantity to evaluate, corresponding to the vertices:

$$\frac{\partial}{\partial V_1(r)}\left(\frac{1}{y(p)-y(\overline{p})}\right)dx(r) = \frac{B(p,r)-B(\overline{p},r)}{(y(p)-y(\overline{p}))^2 dx(p)} \tag{4.23}$$

Let us now interpret diagrammatically these relations.

Eq. (4.21) represents the action of the loop insertion operator on the non arrowed edges:



(4.24)

In order to interpret (4.22), one has to move the point p inside the integration contour for ξ. For this purpose, let us consider the action of the loop operator on the vertex, $dx(r)\,\frac{\partial}{\partial V_1(x(r))}\sum_\alpha \operatorname{Res}_{q\to\mu_\alpha}\frac{dE_{q,\overline{q}}(p)}{2}\frac{1}{(y(\overline{q})-y(q))dx(q)}$. It gives the contribution

$$\begin{aligned}&\textstyle\sum_\alpha \operatorname{Res}_{q\to\mu_\alpha}\frac{dE_{q,\overline{q}}(p)}{2}\frac{B(\overline{q},r)-B(q,r)}{(y(\overline{q})-y(q))^2dx(q)^2} \\ &\quad+\textstyle\sum_\alpha \operatorname{Res}_{q\to\mu_\alpha}\operatorname{Res}_{\xi\to\mu_\alpha}\frac{dE_{\xi,\overline{\xi}}(p)}{2(y(\overline{\xi})-y(\xi))dx(\xi)}\frac{dE_{q,\overline{q}}(\overline{\xi})B(\xi,r)+dE_{q,\overline{q}}(\xi)B(\overline{\xi},r)}{2(y(\overline{q})-y(q))dx(q)}\end{aligned} \tag{4.25}$$

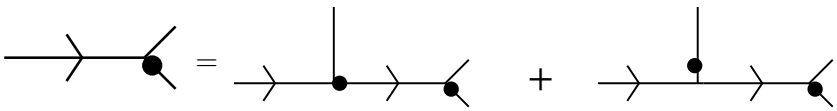
where q lies outside the integration contour for ξ. We move the integration contour for ξ in the second term so that q lies inside. One has poles only when $\xi\to q$ and $\xi\to\overline{q}$ which can be written

$$\operatorname*{Res}_{q\to\mu_\alpha}\operatorname*{Res}_{\xi\to\mu_\alpha} = \operatorname*{Res}_{\xi\to\mu_\alpha}\operatorname*{Res}_{q\to\mu_\alpha} - \operatorname*{Res}_{q\to\mu_\alpha}\operatorname*{Res}_{\xi\to q} - \operatorname*{Res}_{q\to\mu_\alpha}\operatorname*{Res}_{\xi\to\overline{q}}. \tag{4.26}$$

These contributions cancel totally the first term in (4.25). Thus one finally obtains:

$$dx(r)\,\frac{\partial}{\partial V_1(x(r))}\sum_\alpha \operatorname*{Res}_{q\to\mu_\alpha}\frac{dE_{q,\overline{q}}(p)}{2}\frac{1}{(y(\overline{q})-y(q))dx(q)} =$$
$$= \sum_\alpha \operatorname*{Res}_{\xi\to\mu_\alpha}\operatorname*{Res}_{q\to\mu_\alpha}\frac{dE_{\xi,\overline{\xi}}(p)}{2(y(\overline{\xi})-y(\xi))dx(\xi)}\frac{dE_{q,\overline{q}}(\overline{\xi})B(\xi,r)+dE_{q,\overline{q}}(\xi)B(\overline{\xi},r)}{2(y(\overline{q})-y(q))dx(q)}, \quad (4.27)$$

which corresponds to adding one leg to the vertex:

$dx(r)\,\frac{\partial}{\partial V_1(x(r))}$ 

(4.28)

This ensures that the theorem is true.

Therefore we have the following diagrammatic rules:

Theorem 3. *The $k+1$ point function to order h, $W^{(h)}_{k+1}$ is the sum over all possible diagrams obtained as follows:*

- *choose one of the variables, say p_1 as the root.*
- *draw all rooted skeleton binary trees (i.e. trees with vertices of valence ≤ 3) with $k-1+2h$ edges. Draw arrows going from root to leaves, that puts a partial ordering on vertices. A vertex V_1 preceeds V_2 if there is an oriented path from V_1 to V_2, and a vertex always preceeds itself.*
- *add, in all possible ways, k non arrowed edges ending at the point $p_2,\ldots,p_{k+1}$, so that each vertex has valence ≤ 3.*
- *complete the diagrams in all possible ways with h non arrowed edges joining 2 comparable (with the partial ordering of the tree) vertices, so as to get diagrams with valence 3 only.*
- *At each vertex, mark one leg with a dot, in all possible inequivalent ways. In general there are 2 possibilities at each vertex, except if the 2 branches coming out of the vertex are symetric (either 1 non-oriented edge, or 2 identical subgraphs with no external legs). Each inequivalent diagram is counted exactly once (this is the standard way of computing symmetry factors).*
- *assign to each such diagram a value obtained as follows: each non-arrowed edge is a Bergmann kernel, a residue of the form (4.18) is computed at each vertex, where the $\overline{q}$ variable corresponds to the marked edge. The ordering for computing residues is given by the arrows, from leaves to root.*

Notice that those rules are extremely similar to those first found for the 1-matrix model in [16, 24, 11]. The 1-MM is merely the reduction of those rules to the case of an hyperelliptical surfaces.

5. Topological expansion of the free energy

In order to find the topological expansion of the free energy, one has to "integrate wrt the potential V_1" the expansion of the one point function. First let us define the H_x and H_y operators corresponding respectively to the "inverse" of the loop insertion operator $\frac{\partial}{\partial V_1}$ and $\frac{\partial}{\partial V_2}$.

5.1 The H operators

We define the operators H_x and H_y such that, for any meromorphic differential form ϕ, one has

$$H_x.\phi := \operatorname*{Res}_{\infty_x} V_1(x)\,\phi - \operatorname*{Res}_{\infty_y} (V_2(y) - xy)\,\phi + T\int_{\infty_x}^{\infty_y} \phi + \sum_i \epsilon_i \oint_{\mathcal{B}_i} \phi \tag{5.1}$$

and

$$H_y.\phi := \operatorname*{Res}_{\infty_y} V_2(y)\,\phi - \operatorname*{Res}_{\infty_x} (V_1(x) - xy)\,\phi + T\int_{\infty_y}^{\infty_x} \phi - \sum_i \epsilon_i \oint_{\mathcal{B}_i} \phi. \tag{5.2}$$

Note that

$$(H_x + H_y).\phi = \operatorname*{Res}_{\infty_x,\infty_y} xy.\phi \tag{5.3}$$

Note that if $\phi = df$ is an exact differential, we have:

$$H_x.df = -\operatorname*{Res}_{\infty_x,\infty_y} f\,ydx \qquad , \qquad H_y.df = -\operatorname*{Res}_{\infty_x,\infty_y} f\,xdy. \tag{5.4}$$

We now compute the action of H_x on the two-point function on the Bergmann kernel $B(p,q)$ (in this computation H_x acts on the variable p), that is

$$\begin{aligned} H_x.B(p,q) \;=\; & \operatorname*{Res}_{\infty_x} V_1(x(p))\,B(p,q) - \operatorname*{Res}_{\infty_y} (V_2(y(p)) - xy)\,B(p,q) \\ & + T\int_{\infty_x}^{\infty_y} B(p,q) + \sum_i \epsilon_i \oint_{\mathcal{B}_i} B(p,q). \end{aligned} \tag{5.5}$$

We integrate by parts the residues at infinities. Provided that $B(p,q) = d_p dS_{p,o}(q)$, that $V_1'(x(p)) - y(p) \sim \frac{T}{x(p)}$ as $p \to \infty_x$ and that $V_2'(y(p)) - x(p) \sim \frac{T}{y(p)}$ as $p \to \infty_y$, we can write

$$\begin{aligned} H_x.B(p,q) \;=\; & -\operatorname*{Res}_{\infty_x} \frac{T}{x(p)} dx(p)\,dS_{p,o}(q) - \operatorname*{Res}_{\infty_x,\infty_y} y(p)dx(p)\,dS_{p,o}(q) \\ & + \operatorname*{Res}_{\infty_y} \frac{T}{y(p)} dy(p)\,dS_{p,o}(q) + TdE_{\infty_y,\infty_x}(q) + \sum_i \epsilon_i \oint_{\mathcal{B}_i} B(p,q). \end{aligned} \tag{5.6}$$

Finally, the Riemann bilinear identities [25, 26] associated to the fixed filling fractions conditions give

$$\begin{aligned}
H_X.B(p,q) &= TdS_{\infty_x,o}(q) - \operatorname*{Res}_{\infty_x} y(p)dx(p)\, dS_{p,o}(q) - \operatorname*{Res}_{\infty_y} y(p)dx(p)\, dS_{p,o}(q) \\
&\quad -T\, dS_{\infty_y,o}(q) + TdS_{\infty_y,\infty_x}(q) + \sum_i \epsilon_i \oint_{\mathcal{B}_i} B(p,q) \\
&= \operatorname*{Res}_q y(p)dx\, dS_{p,o}(q) - \frac{1}{2i\pi}\sum_i \oint_{\mathcal{A}_i} y(p)dx(p) \oint_{\mathcal{B}_i} B(p,q) \\
&\quad + \frac{1}{2i\pi}\sum_i \oint_{\mathcal{B}_i} y(p)dx(p) \oint_{\mathcal{A}_i} B(p,q) + \sum_i \epsilon_i \oint_{\mathcal{B}_i} B(p,q) \\
&= -y(q)dx(q). \qquad (5.7)
\end{aligned}$$

That is:

$$H_x.B(.,q) = -y(q)dx(q). \qquad (5.8)$$

Remark 5. The leading order of the free energy is already known [4]:

$$\begin{aligned}
2\mathcal{F}^{(0)} &= \operatorname*{Res}_{\infty_x} (V_1(x) + V_2(y) - xy)\, ydx + T \fint_{\infty_x}^{\infty_y} ydx + \sum_i \epsilon_i \oint_{\mathcal{B}_i} ydx \\
&= \operatorname*{Res}_{\infty_y} (V_1(x) + V_2(y) - xy)\, xdy + T \fint_{\infty_y}^{\infty_x} xdy - \sum_i \epsilon_i \oint_{\mathcal{B}_i} xdy. \qquad (5.9)
\end{aligned}$$

In other terms, one has

$$2\mathcal{F}^{(0)} = H_x(ydx) \qquad (5.10)$$

5.2 Finding the free energy

Theorem 4. *For any h we have:*

$$(1-2h)W_{1,0}^{(h)} = H_x.W_{2,0}^{(h)} \qquad (5.11)$$

Notice that this is already proven for $h = 0$.

Proof. Let us study how H_x acts on the diagrams composing $W_{2,0}^{(h)}$. By symmetry, one can consider that H_x acts on the leaf of the diagram.[2] Two different configurations can occur.

If the leaf is linked to an innermost vertex, one has to compute

$$\begin{aligned}
\sum_\alpha \operatorname*{Res}_{r\to\mu_\alpha} \frac{dE_{r,\overline{r}}(q)H_x.B(r,.)B(r,p)f(p)}{2((y(\overline{r})-y(r))dx(r)} &= -\sum_\alpha \operatorname*{Res}_{r\to\mu_\alpha} \frac{dE_{r,\overline{r}}(q)y(r)B(r,p)f(p)}{2((y(\overline{r})-y(r))} \\
&= 0. \qquad (5.12)
\end{aligned}$$

Otherwise, H_x acts on $B(r,.)$ where there is an arrow going out of the vertex corresponding to the integration of r:

$$A = \sum_{\alpha'} \operatorname*{Res}_{r\to\mu_{\alpha'}} \sum_\alpha \operatorname*{Res}_{p\to\mu_\alpha} \frac{dE_{r,\overline{r}}(q)dE_{p,\overline{p}}(r)y(\overline{r})f(p)}{2(y(r)-y(\overline{r}))(y(p)-y(\overline{p}))dx(r)dx(p)}. \qquad (5.13)$$

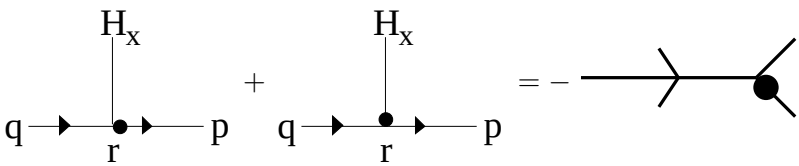
[2] The diagrams composing $W_{2,0}^{(h)}$ have two external legs, one root with an arrow and one leaf which is a Bergmann kernel.

By moving the integration contours so that r lies inside the contour of p, one keeps only the contributions when $r \to p$ and $r \to \overline{p}$, that is

$$\begin{aligned} A &= \sum_\alpha \operatorname*{Res}_{p\to\mu_\alpha} \operatorname*{Res}_{r\to p} \frac{dE_{r,\overline{r}}(q)dE_{p,\overline{p}}(r)y(\overline{r})f(p)}{2(y(r)-y(\overline{r}))(y(p)-y(\overline{p}))dx(r)dx(p)} \\ &\quad + \sum_\alpha \operatorname*{Res}_{p\to\mu_\alpha} \operatorname*{Res}_{r\to \overline{p}} \frac{dE_{r,\overline{r}}(q)dE_{p,\overline{p}}(r)y(\overline{r})f(p)}{2(y(r)-y(\overline{r}))(y(p)-y(\overline{p}))dx(r)dx(p)} \\ &= \sum_\alpha \operatorname*{Res}_{p\to\mu_\alpha} \frac{dE_{p,\overline{p}}(q)f(p)}{2(y(p)-y(\overline{p}))dx(p)}. \end{aligned} \tag{5.14}$$

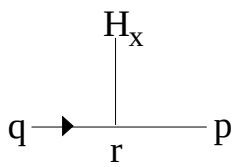
The same diagram with H_x acting on $B(\overline{r},.)$ gives the same contribution and cancels the $\frac{1}{2}$ factor.

Diagrammatically, this reads



(5.15)

and



$= 0.$ (5.16)

Thus, because there are $2h-1$ arrowed edges composing any graph contributing to $W_1^{(h)}$, one has:

$$\begin{aligned} H_{x_q}.W_{2,0}^{(h)}(p,q) &= H_{x_q}.\frac{\partial}{\partial V_1(q)} W_{1,0}^{(h)}(p) \\ &= -(2h-1)W_{1,0}^{(h)}(p). \end{aligned} \tag{5.17}$$

Beside, we have the following property:

Lemma 1. *For any $h \neq 1$:*

$$H_x.W_{1,0}^{(h)}(q) - H_y.W_{0,1}^{(h)}(q) = 0 \tag{5.18}$$

whose rather technical proof is found in appendix B.

Corollary 1. *One easily derive from this theorem that for any $h \neq 1$, the free energy can be written as*

$$\boxed{(2-2h)\mathcal{F}^{(h)} = -H_x W_{1,0}^{(h)} = -H_y W_{0,1}^{(h)},} \tag{5.19}$$

up to an integration constant which does not depend on V_1 or V_2.

5.3 Dependence on other parameters

Now, with the explicit dependence of the free energy on the potentials V_1 and V_2 in hands, one is interested in the dependence in the other momenta, i.e. the filling fractions ϵ_i and the temperature T.

The definitions of the filling fractions implies that

$$\frac{\partial y(p)dx(p)}{\partial \epsilon_i} = 2i\pi du_i = \oint_{\mathcal{B}_i} B(p,q), \tag{5.20}$$

where the du_i's denote the normalized holomorphic differential on cycles $\mathcal{A}_i$. This allows to check that the derivation wrt the filling fractions is compatible with our diagrammatic rules through

$$\frac{\partial B(p,q)}{\partial \epsilon_i} = \oint_{\mathcal{B}_i} \frac{\partial}{\partial V_1(x(p))} B(q,r)dx(q), \tag{5.21}$$

which generalizes to

$$\frac{\partial W_{1,0}^{(h)}(p)}{\partial \epsilon_i} = \oint_{\mathcal{B}_i} W_{2,0}^{(h)}(p,q). \tag{5.22}$$

In the same fashion, one derives

$$\frac{\partial W_{1,0}^{(h)}(p)}{\partial T} = \int_{\infty_x}^{\infty_y} W_{2,0}^{(h)}(p,q). \tag{5.23}$$

5.4 Scaling equation

For the sake of completeness and in order to interpret the formula found for the free energy, one has to consider a slightly more general model, in which the coupling constant between the two matrices is not fixed, i.e. we consider the partition function

$$Z := \int_{H_N \times H_N} dM_1 dM_2\, e^{-\frac{1}{\hbar}\operatorname{Tr}\left(V_1(M_1)+V_2(M_2)-\kappa M_1 M_2\right)} = e^{-\mathcal{F}}. \tag{5.24}$$

where κ is an arbitrary non vanishing coupling constant.

One can easily obtain this model by rescaling our momenta:

$$t_k \to \frac{t_k}{\kappa}\ ,\ \tilde{t}_k \to \frac{\tilde{t}_k}{\kappa} \text{ and } T \to \frac{T}{\kappa}. \tag{5.25}$$

After this redefinition of the parameters, all the results presented in this paper remain valid[3] and one obtains

$$(2-2h)\mathcal{F}^{(h)} = -H_x W_{1,0}^{(h)} + f(T,\epsilon_i,\kappa), \tag{5.26}$$

with

$$H_x.\phi := \operatorname*{Res}_{\infty_x} V_1(x)\,\phi - \operatorname*{Res}_{\infty_y}\left(V_2(y)-\kappa xy\right)\phi + T\int_{\infty_x}^{\infty_y}\phi + \sum_i \epsilon_i \oint_{\mathcal{B}_i}\phi. \tag{5.27}$$

[3]Note that the diagrammatic rules remain the same, only the Riemann surface is affected by this rescaling.

Now, define

$$\forall h \neq 1 \;\; \tilde{F}^{(h)} = -\frac{H_x W_{1,0}^{(h)}(p)}{2-2h}. \tag{5.28}$$

Eq. (5.22) and (5.23) give

$$\begin{aligned}(2-2h)\frac{\partial \tilde{F}^{(h)}}{\partial \epsilon_i} &= -\frac{\partial H_x W_{1,0}^{(h)}(p)}{\partial \epsilon_i} \\ &= -\left(\frac{\partial H_x}{\partial \epsilon_i}\right) W_{1,0}^{(h)}(p) - H_x \frac{\partial W_{1,0}^{(h)}(p)}{\partial \epsilon_i} \\ &= -(2-2h)\oint_{\mathcal{B}_i} W_{1,0}^{(h)}(p).\end{aligned} \tag{5.29}$$

That is to say

$$\frac{\partial \tilde{F}^{(h)}}{\partial \epsilon_i} = -\oint_{\mathcal{B}_i} W_{1,0}^{(h)}(p) \quad \text{and} \quad \frac{\partial \tilde{F}^{(h)}}{\partial T} = -\int_{\infty_x}^{\infty_y} W_{1,0}^{(h)}(p). \tag{5.30}$$

On the other hand, by definition

$$\sum_{k=1}^{d_1} t_k \frac{\partial \tilde{F}^{(h)}}{\partial t_k} = \operatorname*{Res}_{\infty_x} V_1(x(p)) \frac{\partial \tilde{F}^{(h)}}{\partial V_1(x(p))}. \tag{5.31}$$

Which leads to

$$\sum_{k=1}^{d_1} t_k \frac{\partial \tilde{F}^{(h)}}{\partial t_k} = -\operatorname*{Res}_{\infty_x} V_1(x(p)) W_{1,0}^{(h)}(p). \tag{5.32}$$

One can also derive

$$\begin{aligned}\sum_{k=1}^{d_2} \tilde{t}_k \frac{\partial \tilde{F}^{(h)}}{\partial \tilde{t}_k} &= -\operatorname*{Res}_{\infty_y} \frac{V_2(y(p))}{(2-2h)} \frac{\partial H_x . W_{1,0}^{(h)}(.)}{\partial V_2(y(p))} \\ &= -\operatorname*{Res}_{\infty_y} \frac{V_2(y(p))}{(2-2h)} \frac{\partial (H_x + H_y) . W_{1,0}^{(h)}(.)}{\partial V_2(y(p))} \\ &\quad + \operatorname*{Res}_{\infty_y} \frac{V_2(y(p))}{(2-2h)} \frac{\partial H_y . W_{1,0}^{(h)}(.)}{\partial V_2(y(p))} \\ &= \operatorname*{Res}_{\infty_y} V_2(y(p)) W_{1,0}^{(h)}(p).\end{aligned} \tag{5.33}$$

The last identity uses the property (5.3) of the H operators.

There is one additive momentum that one has to consider now, which gives

$$\frac{\partial \tilde{F}^{(h)}}{\partial \kappa} = -\operatorname*{Res}_{\infty_y} x(p) y(p) W_{0,1}(p) \tag{5.34}$$

Thus, putting everything together, for any set of filling fractions ϵ_i, any coupling constant κ and any temperature T, $\tilde{F}^{(h)}$ fulfills the homogenous equation

$$\boxed{(2-2h)\tilde{F}^{(h)} = \sum_{k=1}^{d_1+1} t_k \frac{\partial \tilde{F}^{(h)}}{\partial t_k} + \sum_{k=1}^{d_2+1} \tilde{t}_k \frac{\partial \tilde{F}^{(h)}}{\partial \tilde{t}_k} + T\frac{\partial \tilde{F}^{(h)}}{\partial T} + \kappa \frac{\partial \tilde{F}^{(h)}}{\partial \kappa} + \sum_i \epsilon_i \frac{\partial \tilde{F}^{(h)}}{\partial \epsilon_i}.} \tag{5.35}$$

Hence, $\tilde{F}^{(h)}$ is our final answer for the free energy:

$$\tilde{F}^{(h)} = \mathcal{F}^{(h)} = \frac{1}{2h-2} H_x . W^{(h)}_{(1,0)} = \frac{1}{2h-2} H_y . W^{(h)}_{(0,1)}. \tag{5.36}$$

Remark 6. One can note that (5.35) is nothing but saying that rescaling the expansion parameter $\hbar$ is equivalent to rescaling all the other momenta in the same way:

$$0 = \hbar \frac{\partial \tilde{F}}{\partial \hbar} + \sum_{k=1}^{d_1+1} t_k \frac{\partial \tilde{F}}{\partial t_k} + \sum_{k=1}^{d_2+1} \tilde{t}_k \frac{\partial \tilde{F}}{\partial \tilde{t}_k} + T \frac{\partial \tilde{F}}{\partial T} + \kappa \frac{\partial \tilde{F}}{\partial \kappa} + \sum_i \epsilon_i \frac{\partial \tilde{F}}{\partial \epsilon_i}. \tag{5.37}$$

5.5 Explicit computation of the free energy: diagrammatic rules

The equation (5.36) allows to compute any term in the expansion of the free energy, but the first correction to the leading order, using diagrammatic rules. For this purpose, one just introduce a new bi-valent vertex corresponding to the action of the H_x operator on the root of the diagrams composing the one point functions. One represent it,

$$\mathrm{q} \text{———}\circ\!\!-\!\!\bullet\text{———} \overline{\mathrm{q}} = \sum_\alpha \operatorname*{Res}_{q \to \mu_\alpha} \frac{-\frac{1}{2}\int_{\overline{q}}^{q} y(\xi) dx(\xi)}{(y(q) - y(\overline{q})) dx(q)} \tag{5.38}$$

Using this new vertex, one obtains the h'th order correction to the free energy by considering the sum of the diagrams contributing to $W^{(h)}_{1,0}$ where the ingoing arrow is replaced by this new bi-valent vertex.

Remark 7. We must remind that this technic does not give the first order correction. Nevertheless this particular correction has already been computed in the literature [20].

6. Conclusion

In this paper, we have obtained a closed expression for the complete expansion of the free energy of the formal hermitian two-matrix model as residues on the spectral curve giving an answer to one of the problems left undone in [24]. On the way, we also refined the diagrammatic rules used to compute non-mixed correlation functions. We particularly exhibit that these functions actually only depend on what happens near the branch points. The link with the diagrammatic technique of [16] for the 1MM looks also more evident.

Nevertheless, there still remain many different problems one should address from this starting point. In this paper, we begin the computation of correlation functions mixing traces in M_1 and traces in M_2, but the general case does not seem as simple for the moment. In the same way, using the result of [17], one would like to compute the expansion of mixed traces.

This natural generalization of diagrammatic techniques from the 1MM to the 2MM also points out that the study of the chain of matrices [21] would give a more general knowledge of the link between algebraic geometry and matrix models.

Let us also mention that we have considered here only polynomial potentials, but it is clear that the whole method should extend easily to the more general class of semi-classical

potentials [6, 7], whose loop equations are very similar [23]. The 1-MM with hard edges was already treated in [12].

Another extension of this problem is to compute the $1/N$ expansion of random matrix models for non-hermitian matrices. This problem is already partialy addressed in [36 – 38], but a diagrammatic technique has not been found yet.

Finally, as we already mentioned it, solving the formal model is different from solving the physical problem corresponding to a convergent integral and constrained filling fractions lowering the energy. This problem, which is expected to be similar to [8] is not addressed for the moment and should give a natural complement to this work.

Acknowledgments

We are grateful to A. ZAbrodin for discussions. L. Chekhov thanks RFBR grant No 05-01-00498, grant for Support Scientific Schools 2052.2003.1, the program Mathematical Methods of Nonlinear Dynamics. All the authors thank the Enigma european network MRT-CT-2004-5652, and the ANR project Géométrie et intégrabilité en physique mathématique ANR-05-BLAN-0029-01.

A. Computation of $W^{(h)}_{k,1}$.

In this section, we compute a new set of correlation functions including one trace in M_2. In other words, we compute all the $W^{(h)}_{k,1}$'s. More precisely, we explain how the diagrammatic rules are modified when one changes one trace

$$\mathrm{Tr}\,\frac{1}{x(p)-M_1} \;\rightarrow\; \mathrm{Tr}\,\frac{1}{y(p)-M_2}. \tag{A.1}$$

We do not give explicit formulas for all the $W^{(h)}_{k,l}$ with $l > 1$. This problem will be addressed in another work.

A.1 Examples

In order to give some idea of the result, let us review some already known functions [19, 5, 30].

A.1.1 2-point functions

$$W^{(0)}_{2,0}(p,q) = W^{(0)}_{0,2}(p,q) = -W^{(0)}_{1,1}(p,q) = B(p,q) \tag{A.2}$$

A.1.2 3-point functions

$$W^{(0)}_{3,0}(p_1,p_2,p_3) = \sum_{\text{alpha}} \operatorname*{Res}_{q\to\mu_\alpha} \frac{B(p_1,q)B(p_2,q)B(p_3,q)}{dx(q)dy(q)} \tag{A.3}$$

$$W^{(0)}_{2,1}(p_1,p_2,p_3) = -\sum_{\alpha} \operatorname*{Res}_{q\to\mu_\alpha} \frac{B(p_1,q)B(p_2,q)B(p_3,q)}{dx(q)dy(q)} - \operatorname*{Res}_{q\to p_3} \frac{B(p_1,q)B(p_2,q)B(p_3,q)}{dx(q)dy(q)} \tag{A.4}$$

Proof.

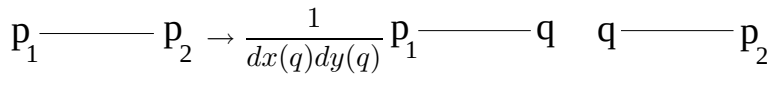
$$
\begin{aligned}
W_{2,1}(p_1,p_2,q) &= \frac{\partial}{\partial V_1(x(p_2))}(w_{1,1}(p_1,q))dy(q)dx(p_1)dx(p_2) \\
&= -\frac{\partial}{\partial V_1(x(p_2))}(w_{2,0}(p_1,q)\frac{dx(q)}{dy(q)})dy(q)dx(p_1)dx(p_2) \\
&= -W_{3,0}(p_1,p_2,q) + \frac{W_{2,0}(p_1,q)}{dx(q)}d_q\left(\frac{W_{1,1}(p_2,q)}{dy(q)}\right) \\
&\quad + \frac{W_{1,1}(p_2,q)}{dy(q)}d_q\left(\frac{W_{2,0}(p_1,q)}{dx(q)}\right) \\
&= -W_{3,0}(p_1,p_2,q) + d_q\left(\frac{W_{2,0}(p_1,q)W_{1,1}(p_2,q)}{dx(q)dy(q)}\right)
\end{aligned}
\tag{A.5}
$$

A.2 General case

Theorem 5. *For any* $|K|+h>1$

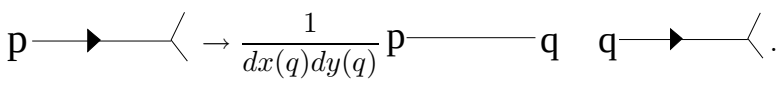
$$
W^{(h)}_{|K|,1}(p_1,\dots,p_k,q) + W^{(h)}_{|K|+1,0}(p_1,\dots,p_k,q) = d_q f^{(h)}(p_1,\dots,p_k,q) \tag{A.6}
$$

where $f^{(h)}(p_1,\dots,p_k,q)$ *is obtained from the diagrams composing* $W^{(h)}_{|K|,0}(p_1,\dots,p_k)$ *by cutting any of its propagator as follows:*



(A.7)

and



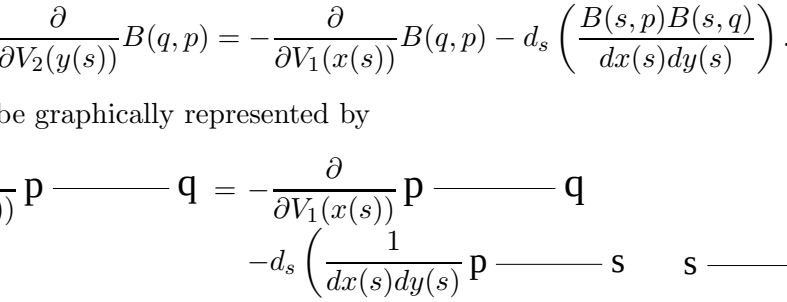
(A.8)

Proof. $W^{(h)}_{|K|,1}(p_K,q)$ can be obtained by differentiating $W^{(h)}_{|K|,0}(p_K)$ wrt $V_2(y(q))$. So one has to know the action of $\frac{\partial}{\partial V_2}$ on the diagrammatic rules.

One can reexpress the 3-point function of (A.4) as

$$
\frac{\partial}{\partial V_2(y(s))}B(q,p) = -\frac{\partial}{\partial V_1(x(s))}B(q,p) - d_s\left(\frac{B(s,p)B(s,q)}{dx(s)dy(s)}\right). \tag{A.9}
$$

This can be graphically represented by



(A.10)

On the other hand, by integration, this implies that

$$
\begin{aligned}
\frac{\partial}{\partial V_2(y(s))}dE_{p,r}(q) &= -\sum_\alpha \operatorname*{Res}_{a\to\mu_\alpha} \frac{dE_{a,\overline{a}}(q)\left[dE_{p,r}(\overline{a})B(a,s) + B(s,\overline{a})dE_{p,r}(a)\right]}{2(y(\overline{a})-y(a))dx(a)} \\
&\quad - \sum_\alpha \operatorname*{Res}_{a\to\mu_\alpha}\operatorname*{Res}_{b\to a} \frac{dE_{a,b}(q)dE_{p,r}(b)B(a,s)}{(y(b)-y(a))(x(b)-x(a))}.
\end{aligned}
\tag{A.11}
$$

Thus the derivative wrt V_2 of the first vertex, $\frac{\partial}{\partial V_2(y(s))}\sum_\alpha \operatorname{Res}_{p\to\mu_\alpha}\frac{dE_{p,\overline{p}}(q)}{2(y(\overline{p})-y(p))dx(p)}$ gives:

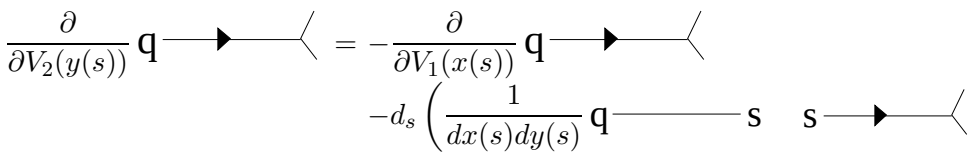
$$\begin{aligned}
&-\sum_\alpha\sum_\beta \operatorname{Res}_{p\to\mu_\alpha}\operatorname{Res}_{a\to\mu_\beta}\frac{dE_{a,\overline{a}}(q)\left[dE_{p,\overline{p}}(\overline{a})B(a,s)+B(s,\overline{a})dE_{p,\overline{p}}(a)\right]}{4(y(\overline{a})-y(a))dx(a)(y(\overline{p})-y(p))dx(p)}\\
&-\sum_\alpha \operatorname{Res}_{p\to\mu_\alpha}\operatorname{Res}_{a\to s}\operatorname{Res}_{b\to a}\frac{dE_{a,b}(q)dE_{p,\overline{p}}(b)B(a,s)}{2(y(b)-y(a))(x(b)-x(a))(y(\overline{p})-y(p))dx(p)}\,,\\
&+\sum_\alpha \operatorname{Res}_{p\to\mu_\alpha}\frac{dE_{p,\overline{p}}(q)}{2}\frac{\partial}{\partial V_2(s)}\frac{1}{(y(\overline{p})-y(p))dx(p)}
\end{aligned}\tag{A.12}$$

where p lies outside the integration contour for a. Note that all the derivations have been performed by keeping the x's fixed.

Moving the integration contours so that p lies inside gives a contribution that cancels the last terms and one obtains

$$\begin{aligned}
\frac{\partial}{\partial V_2(y(s))}\sum_\alpha \operatorname{Res}_{p\to\mu_\alpha}\frac{dE_{p,\overline{p}}(q)}{2(y(\overline{p})-y(p))dx(p)} &= -\frac{\partial}{\partial V_1(x(s))}\sum_\alpha \operatorname{Res}_{p\to\mu_\alpha}\frac{dE_{p,\overline{p}}(q)}{2(y(\overline{p})-y(p))dx(p)}\\
&\quad -d_s\left(\frac{B(s,q)}{dx(s)dy(s)}\sum_\alpha \operatorname{Res}_{p\to\mu_\alpha}\frac{dE_{p,\overline{p}}(s)}{2(y(\overline{p})-y(p))dx(p)}\right).
\end{aligned}\tag{A.13}$$

This graphically reads:



(A.14)

These two relations prove straightforwardly the theorem.

Remark 8. For $|K|=1$, one can go further in the description of $W^{(h)}_{1,1}(p_1,p_2)+W^{(h)}_{2,0}(p_1,p_2)$. Indeed, the graphs obtained by cutting the propagators of $W^{(h)}_{1,0}(p_1)$ have no pole when $p_1\to p_2$ except when one cuts the root of the graph, in which case it gives the contribution $\frac{B(p_1,p_2)}{dx(p_2)dy(p_2)}W^{(h)}_{1,0}(p_2)$, that is to say:

$$W^{(h)}_{1,1}(p_1,p_2)+W^{(h)}_{2,0}(p_1,p_2)=-d_{p_2}\left\{\frac{B(p_1,p_2)}{dx(p_2)dy(p_2)}W^{(h)}_{1,0}(p_2)+f(p_1,p_2)\right\},\tag{A.15}$$

where $f(p_1,p_2)$ has no pole as $p_1\to p_2$.

B. Symmetry of the free energy under the exchange of x and y

For $h\neq 1$

$$H_{x_q}.W^{(h)}_{1,0}(q)-H_{y_q}.W^{(h)}_{0,1}(q)=\left(H_{x_{p_1}}H_{y_{p_2}}-H_{y_{p_2}}H_{x_{p_1}}\right)W^{(h)}_{1,1}(p_1,p_2)\tag{B.1}$$

The only terms that contributes in the exchange of H_x and H_y come from the integration contours around ∞_x and ∞_y, i.e.

$$\begin{aligned}
\mathcal{A} &= \left(H_{x_{p_1}}H_{y_{p_2}}-H_{y_{p_2}}H_{x_{p_1}}\right)W^{(h)}_{1,1}(p_1,p_2)\\
&= -\operatorname*{Res}_{p_1\to\infty_x}\operatorname*{Res}_{p_2\to p_1}V_1(x(p_1))(V_1(x(p_2))-x(p_2)y(p_2))W^{(h)}_{1,1}(p_1,p_2)-\\
&\quad -\operatorname*{Res}_{p_1\to\infty_y}\operatorname*{Res}_{p_2\to p_1}V_2(y(p_2))(V_2(y(p_1))-x(p_1)y(p_1))W^{(h)}_{1,1}(p_1,p_2).
\end{aligned}\tag{B.2}$$

On the other hand, one knows from remark (A.15), that

$$W_{1,1}^{(h)}(p_1,p_2) = -W_{2,0}^{(h)}(p_1,p_2) - d_{p_2}\left\{\frac{B(p_1,p_2)}{dx(p_2)dy(p_2)}W_{1,0}^{(h)}(p_2) + f(p_1,p_2)\right\}. \tag{B.3}$$

where $f(p_1,p_2)$ has no pole when $p_2 \to p_1$. Note that $W_{2,0}^{(h)}(p_1,p_2)$ do not have pole when $p_2 \to p_1$ either.

Thus, the only non vanishing terms in (B.2) come from $d_{p_2}\left\{\frac{B(p_1,p_2)}{dx(p_2)dy(p_2)}W_{1,0}^{(h)}(p_2)\right\}$. This reads

$$\begin{aligned}&\operatorname{Res}_{p_1\to\infty_x}\operatorname{Res}_{p_2\to p_1} V_1(x(p_1))(V_1(x(p_2)) - x(p_2)y(p_2))d_{p_2}\left\{\tfrac{B(p_1,p_2)}{dx(p_2)dy(p_2)}W_{1,0}^{(h)}(p_2)\right\}\\ &+\operatorname{Res}_{p_1\to\infty_y}\operatorname{Res}_{p_2\to p_1} V_2(y(p_2))(V_2(y(p_1)) - x(p_1)y(p_1))d_{p_2}\left\{\tfrac{B(p_1,p_2)}{dx(p_2)dy(p_2)}W_{1,0}^{(h)}(p_2)\right\}.\end{aligned} \tag{B.4}$$

We now evaluate these residues by part and obtain that

$$\begin{aligned}\mathcal{A} = &-\operatorname*{Res}_{p\to\infty_x} V_1(x(p))d_p\left[\frac{(V_1'(x(p)) - y(p))dx(p) - x(p)dy(p)}{dx(p)dy(p)}W_{1,0}^{(h)}(p)\right]\\ &-\operatorname*{Res}_{p\to\infty_y}(V_2(y(p)) - x(p)y(p))d_p\left[\frac{V_2'(y(p))W_{1,0}^{(h)}(p)}{dx(p)}\right].\end{aligned} \tag{B.5}$$

Another integration by part can be written

$$\begin{aligned}\mathcal{A} = &+\operatorname*{Res}_{p\to\infty_x} V_1'(x(p))\frac{(V_1'(x(p)) - y(p))dx(p) - x(p)dy(p)}{dy(p)}W_{1,0}^{(h)}(p)\\ &+\operatorname*{Res}_{p\to\infty_y}(V_2'(y(p))dy(p) - x(p)dy(p) - y(p)dx(p))\frac{V_2'(y(p))W_{1,0}^{(h)}(p)}{dx(p)}.\end{aligned} \tag{B.6}$$

Knowing that $V_1'(x(p)) - y(p) \sim \frac{T}{x(p)}$ when $p \to \infty_x$ and $V_2'(y(p)) - x(p) \sim \frac{T}{y(p)}$ when $p \to \infty_y$ and that the other factors do not have poles at infinities, one can finally write that

$$\left(H_{x_{p_1}}H_{y_{p_2}} - H_{y_{p_2}}H_{x_{p_1}}\right)W_{1,1}^{(h)}(p_1,p_2) = -\operatorname*{Res}_{q\to\infty_x,\,\infty_y} x(q)y(q)W_{1,0}^{(h)}(q). \tag{B.7}$$

Let us now show that the r.h.s. vanishes by looking precisely at the form of $W_{1,0}^{(h)}(q)$. Because it comes from a vertex, (4.16), with moving the integration contour, implies that

$$\begin{aligned}2\operatorname*{Res}_{\infty_x,\,\infty_y} x(q)y(q)W_{1,0}^{(h)}(q) &= -2\sum_\alpha \operatorname*{Res}_{q\to\mu_\alpha} x(q)y(q)W_{1,0}^{(h)}(q)\\ &= -\sum_\alpha\sum_\beta \operatorname*{Res}_{q\to\mu_\alpha}\operatorname*{Res}_{p\to\mu_\beta}\frac{x(q)y(q)dE_{p,\overline{p}}(q)}{(y(p) - y(\overline{p}))dx(p)}\times\\ &\quad\times\left[W_{2,0}^{(h-1)}(p,p) + \sum_{m=1}^{h-1}W_{1,0}^{(m)}(p)W_{1,0}^{(h-m)}(p)\right].\end{aligned} \tag{B.8}$$

Now we move one more time the integration contour and clearly segregate the case when p and q are near the same branch point.

$$
\begin{aligned}
2 \operatorname*{Res}_{\infty_x\,,\,\infty_y} x(q)y(q)W_{1,0}^{(h)}(q) = &- \sum_{\alpha\neq beta} \operatorname*{Res}_{p\to\mu_\alpha} \operatorname*{Res}_{q\to\mu_\beta} \frac{x(q)y(q)dE_{p,\overline{p}}(q)}{(y(p)-y(\overline{p}))dx(p)} \\
&\times \left[W_{2,0}^{(h-1)}(p,p) + \sum_{m=1}^{h-1} W_{1,0}^{(m)}(p)W_{1,0}^{(h-m)}(p) \right] - \\
&- \sum_{\alpha} \left[\operatorname*{Res}_{p\to\mu_\alpha} \operatorname*{Res}_{q\to\mu_\alpha} - \operatorname*{Res}_{q\to\mu_\alpha} \operatorname*{Res}_{p\to q\,,\,\overline{q}} \right] \frac{x(q)y(q)dE_{p,\overline{p}}(q)}{(y(p)-y(\overline{p}))dx(p)} \\
&\times \left[W_{2,0}^{(h-1)}(p,p) + \sum_{m=1}^{h-1} W_{1,0}^{(m)}(p)W_{1,0}^{(h-m)}(p) \right]. \qquad \text{(B.9)}
\end{aligned}
$$

One can see that the integrand has no pole as q approaches a branch point. Thus one keeps only

$$
\sum_{\alpha} \operatorname*{Res}_{q\to\mu_\alpha} \operatorname*{Res}_{p\to q\,,\,\overline{q}} \frac{x(q)y(q)dE_{p,\overline{p}}(q)}{(y(p)-y(\overline{p}))dx(p)} \left[W_{2,0}^{(h-1)}(p,p) + \sum_{m=1}^{h-1} W_{1,0}^{(m)}(p)W_{1,0}^{(h-m)}(p) \right]. \qquad \text{(B.10)}
$$

We finally perform the integration and get

$$
\begin{aligned}
2 \operatorname*{Res}_{\infty_x\,,\,\infty_y} x(q)y(q)W_{1,0}^{(h)}(q) &= \sum_{\alpha} \operatorname*{Res}_{q\to\mu_\alpha} \frac{x(q)}{dx(q)} \left[W_{2,0}^{(h-1)}(p,p) + \sum_{m=1}^{h-1} W_{1,0}^{(m)}(p)W_{1,0}^{(h-m)}(p) \right] \\
&= 0. \qquad \text{(B.11)}
\end{aligned}
$$

QED.

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