

Dynamical field theory for glass-forming liquids, self-consistent resummations and time-reversal symmetry

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Dynamical field theory for glass-forming liquids, self-consistent resummations and time-reversal symmetry

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Abstract. We analyse the symmetries and the self-consistent perturbative approaches of dynamical field theories for glass-forming liquids. In particular, we focus on the time-reversal symmetry, which is crucial to obtain fluctuation–dissipation relations (FDRs). Previous field theoretical treatment violated this symmetry, whereas others pointed out that constructing symmetry-preserving perturbation theories is a crucial and open issue. In this work we solve this problem and then apply our results to the mode-coupling theory of the glass transition (MCT).

We show that in the context of dynamical field theories for glass-forming liquids time-reversal symmetry is expressed as a *nonlinear* field transformation that leaves the action invariant. Because of this nonlinearity, standard perturbation theories generically do not preserve time-reversal symmetry and in particular fluctuation–dissipation relations. We show how one can cure this problem and set up symmetry preserving perturbation theories by introducing some auxiliary fields. As an outcome we obtain Schwinger–Dyson dynamical equations that automatically preserve FDR and that serve as a basis for carrying out symmetry-preserving approximations. We apply our results to the mode-coupling theory of the glass transition, revisiting previous field theory derivations of MCT equations and showing that they generically violate FDR. We obtain symmetry-preserving mode-coupling equations and discuss their advantages and drawbacks. Furthermore, we show, contrary to previous works, that the structure of the dynamic equations is such that the ideal glass transition is not cut off at any finite order of perturbation theory, even in the presence of coupling between

current and density. The opposite results found in previous field theoretical works, such as the ones based on nonlinear fluctuating hydrodynamics, were only due to an incorrect treatment of time-reversal symmetry.

Keywords: ergodicity breaking (theory), mode coupling theory, slow dynamics and ageing (theory), structural glasses (theory)

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Introduction

Liquids, if cooled fast enough in order to avoid crystallization, generically enter a metastable phase in which remarkable dynamical phenomena occur [1, 2] upon decreasing the temperature. For instance, the structural relaxation time increases very fast, in many cases faster than an Arrhenius law; correlation functions have slower than exponential relaxations; and dynamics become strongly heterogeneous, in contrast to the simple motion which occurs at higher temperature. Typically, at a temperature that is roughly two-thirds of the melting temperature the relaxation time becomes macroscopic, of the order of minutes or hours, and the liquid freezes into an amorphous solid called glass.

The equilibrium dynamics of moderately supercooled liquids is rather well described by the mode-coupling theory (MCT) [3]–[5]. MCT is a closure scheme for the correlation function developed 20 years ago [6, 7], leading to a nonlinear integro-differential equation, which has to be solved self-consistently. It predicts a power law divergence of the relaxation time and of the viscosity at a finite temperature T_{MCT} . Although it is now clear that this is a spurious transition, several quantitative results compare well with experimental and numerical findings, for example the wavevector dependence of the Debye–Waller factor (the analogue of the Edwards–Anderson parameter for spin-glasses) see [3]–[5]. The conventional interpretation is that at T_{MCT} there is a dynamical crossover: ‘hopping or activated’ events not contained in the theory and which can be roughly neglected for $T > T_{\text{MCT}}$ prevent the existence of the transition and dominate the slowing down for $T < T_{\text{MCT}}$. Thus, although the transition is strictly speaking avoided, the dynamics for $T > T_{\text{MCT}}$ can be explained in terms of MCT. This interpretation is based on theoretical extensions of MCT [8, 9] and recent developments originating from mean-field disordered systems [10]. However, it is important to stress that the MCT dynamical crossover is not that sharp in real systems, therefore many MCT statements have to be considered with

great care. Actually, even the existence of an MCT crossover remains a long debated issue in the glass literature; see e.g. [12]. MCT was derived originally using the projection operator formalism of Mori and Zwanzig [6, 7]. A few years later, MCT was rederived starting from stochastic nonlinear hydrodynamics equations as a one-loop self-consistent theory [9]. This field theory derivation and subsequent ones have been criticized for two reasons. The first is because the mechanism behind the MCT transition was thought to be a short scale phenomenon: locally particles get jammed and jailed in cages due to the interactions with the nearest neighbouring particles in the liquid, resulting in a fast rattling motion but no structural relaxation. Stochastic nonlinear hydrodynamics is an effective theory for moderate and long length scales and therefore it was thought that many predictions were dependent on the short scale cut-off or coincided with the previous one accidentally [8].

The second criticism is more recent and is related to the fluctuation–dissipation relation (FDR) between correlation and response function (see [13] and next sections). The one-loop derivations of MCT presented in the literature assume FDR in order to get the correct mode-coupling (MC) equations but actually they are *incompatible* with FDR. As a consequence they are not consistent and are difficult to extend to more general cases.

Despite these drawbacks, the field theory approach is particularly appealing and is gaining interest for many reasons. First, within this formalism it has been shown that MCT is a dynamic critical phenomenon and it leads to strong dynamical correlations in the four-point density correlation function [16] (see [14, 15] for very important early insights based on the study of mean-field disordered systems). These results, which seem still out of reach of the projection operator formalism, have been indeed verified in simulations of model systems, and quantitative MCT predictions [16] are in rather good agreement with numerical results [17] (see also [18]). Furthermore, they provide an answer to the first criticism cited above: the MCT transition is related to growing correlations in the dynamics and *is not a short scale phenomenon*. The subtle point here is that the order parameter is the density correlator, a two-point function, whereas in many other cases³ the order parameter is a one-point function. Therefore, the MCT equations have to be interpreted as mean-field equations on the order parameter. They play the same role as Weiss mean-field equations for ferromagnets. This makes it clear that, first, they are very different from mode-coupling equations for critical phenomena. Indeed, the latter are self-consistent equations of the correlations of the order parameter, and not of the order parameter itself. Second, to obtain the diverging correlation length one has to go beyond the mean-field MCT equations and compute the fluctuations of the order parameter, i.e. a four-point density correlation function. From this perspective the fact that the order parameter, the density–density correlations, does not contain any diverging length is natural, and does not imply that the MCT is a short scale phenomenon.

The second reason which makes a field theory approach appealing is that it can be used naturally in off-equilibrium regimes. Indeed, there is a strong interest in extending the MCT equations to regimes in which a liquid is ageing or sheared. There have also been recently very interesting attempts to do this within the projection operator formalism [19]–[21].

³ But not all! A famous example is the BCS superconducting transition.

1. Summary

The aim of this paper is twofold: first we analyse field theories for the dynamics of glass-forming liquids in terms of symmetries of the action. To our knowledge this has never been done and is very important in order to preserve *physical* symmetries in approximate treatments. In particular, we will focus on time-reversal symmetry, which implies important relationships between observables. A well known example is the fluctuation–dissipation relation (FDR) between correlation and response functions. Other important examples which have been discovered recently are the Jarzynski and Crooks equality for non-equilibrium processes [22, 23]. We will focus on two types of field theories: the first is obtained from fluctuating nonlinear hydrodynamics (FNH) equations that describe in a coarse grained way the dynamics of liquids in which particles obey Newton equations [24, 9]. The second one is obtained from the stochastic equations derived by Dean [25]. They are *exact* stochastic equations governing the evolution of the density field of interacting particles evolving with Langevin dynamics. The field transformations related to time-reversal symmetry that leave the action invariant are *nonlinear* in the fields. This is at the origin of the violation of FDR (and time-reversal symmetry) by self-consistent perturbation theories that has been already noticed and discussed by Miyazaki and Reichman [13]. We shall show that introducing some auxiliary fields this problem can be solved and one can set up a self-consistent perturbation theory compatible with FDR. At this stage, we warn the reader that we will use the words ‘perturbation’ and ‘perturbative’ loosely throughout this paper, as there will be no small parameter. In our context a ‘perturbation’ is a formal series expansion, of which the radius of convergence may not be known. The second aim of this work is to analyse the field theory derivation of MCT using self-consistent one-loop approximations which preserve the fluctuation–dissipation relation between density correlations and response functions. This will clarify different issues related to the field theory derivations of MCT and to extended MCT (and the related cut-off of the transition). In particular, we shall show that the two field theories associated with FNH and Dean equations lead to essentially the same self-consistent one-loop equations. These equations are very similar to the usual MCT ones *but* but with a slightly different wavevector dependence of the vertex. This difference is clearly a drawback because the particular wavevector dependence of the MCT vertex is very important for quantitative results. Furthermore, in the case of the Dean equation, our vertex leads to an apparently unphysical divergence. Our results show that some kind of resummation has to be performed in order to get MCT-like equations that are both quantitatively successful (such as the one derived in [6]) and respect explicitly time-reversal symmetry. An interesting exact by-product of our analysis concerns extended MCT. Indeed, we obtain the exact Schwinger–Dyson equation for the non-ergodic (Edwards–Anderson) parameter of the glass phase. We find that if the time-reversal symmetry is preserved by the *perturbative* self-consistent theory the MCT transition is not cut off *even when density and currents are coupled*, contrary to the conclusions of previous works [9, 36]. In these cases approximations violating time-reversal symmetry produced a spurious cut-off of the dynamical transition. Our conclusion is that from a field theory perspective the mechanism which cuts the MCT transition off is a non-perturbative one unrelated to the presence of density–current coupling and likely the same for Brownian or Newtonian dynamics as observed recently in numerical simulations [40] and also suggested by very recent works [35, 47].

This manuscript is organized as follows. In section 2 we introduce the two field theories for glass-forming liquids we focus on. From sections 3 to 6, we focus on the field theory for the Dean equation, which provides a good illustration of the problems arriving with nonlinear symmetries. In section 3 we show the field transformations which leave the action invariant and are related to time-reversal symmetry. In section 4 we discuss how nonlinear field transformations affect perturbation theory and the origin of violation of FDR in the previous derivation of MCT. In section 5 we show how one can restore FDR in self-consistent perturbation theories introducing more auxiliary fields. On the way an exact and compact structure of the Schwinger–Dyson equation is given, where the consequences of time-reversal symmetry have been exploited to the maximum. As an illustration of the general strategy, we derive in section 6 mode-coupling equations which preserve FDR. In section 7, we present the result of the use of the strategy developed in the previous sections for fluctuating nonlinear hydrodynamics. Section 8 is devoted to a discussion of MCT-related issues, in particular the fact that the transition is not cut off even in the presence of currents. Finally, appendices deal with the most technical details.

2. Dynamical field theory for glass-forming liquids

The aim of this section is to introduce two field theories used to study glass-forming liquids such as supercooled liquids and dense colloidal systems.

2.1. Brownian dynamics and the Dean equation

In the following we derive the field theory corresponding to a system of N interacting point particles evolving under Langevin dynamics, which provides a good qualitative description of the dynamics of a dense colloidal suspension [26]. Actually one should also take into account hydrodynamic interactions due to the solvent but they will be neglected for simplicity. From a more theoretical point of view this is a limiting case in which, except for the density, there are no other conserved variables. We shall consider in the next section the field theory corresponding to Newtonian dynamics in which energy, momentum and density are conserved variables.

The starting point is the Langevin equation which describes the dynamics of N interacting particles, evolving in Euclidean three-dimensional space (coordinates are labelled by lower-case Roman letters, particles labelled by lower-case Greek letters):

$$\partial_t \mathbf{x}_\alpha = - \sum_{\alpha < \beta} \nabla V(\mathbf{x}_\alpha - \mathbf{x}_\beta) + \boldsymbol{\zeta}_\alpha, \quad (1)$$

where ζ_α is a Gaussian white noise, whose correlations are given by the Stokes–Einstein relation,

$$\langle \zeta_{\alpha,i}(\mathbf{x}, t) \zeta_{\beta,j}(\mathbf{x}', t') \rangle = 2T \delta_{\alpha\beta} \delta_{ij} \delta(\mathbf{x} - \mathbf{x}') \delta(t - t'). \quad (2)$$

Here, V is the pair potential between particles and the time is expressed in units of the microscopic diffusion constant.

The potential V is defined up to an additive constant. We shall tune this constant so that $\int d^3\mathbf{x} V(\mathbf{x}) = -T/\rho_0$, which will be convenient later⁴. Observables measured

⁴ More precisely, we put the system in a box of volume Ω , replace the potential $V(\mathbf{x})$ by $V(\mathbf{x}) - (1/\Omega)((T/\rho_0) - \int d^3\mathbf{x} V(\mathbf{x}))$ and take the limit $\Omega \rightarrow \infty$.

in experiments are functions of the local density $\rho(\mathbf{x}, t) = \sum_{\alpha} \delta(\mathbf{x} - \mathbf{x}_{\alpha}(t))$. Using Itô calculus, Dean has shown that the density of a system of particles obeying (1) obeys the following Langevin equation [25]:

$$\partial_t \rho(\mathbf{x}, t) = \nabla \cdot \left(\rho(\mathbf{x}, t) \nabla \frac{\delta \mathcal{F}}{\delta \rho(\mathbf{x}, t)} \right) + \eta(\mathbf{x}, t), \quad (3)$$

where η is a Gaussian random noise, whose correlators are

$$\langle \eta(\mathbf{x}, t) \eta(\mathbf{x}', t') \rangle = 2T \rho(\mathbf{x}, t) \nabla \cdot \nabla' \delta(\mathbf{x} - \mathbf{x}') \delta(t - t'). \quad (4)$$

The prime on the gradient means that it acts on functions of x' . The occurrence of multiplicative noise here is not surprising, as the local density does not fluctuate in empty regions. The density functional \mathcal{F} for a fluid of average density ρ_0 can be written as

$$\begin{aligned} \mathcal{F}[\rho] &= T \int d^3 \mathbf{x} \rho(\mathbf{x}) \left(\ln \frac{\rho(\mathbf{x})}{\rho_0} - 1 \right) + \frac{1}{2} \int d^3 \mathbf{x} \int d^3 \mathbf{x}' \rho(\mathbf{x}) \rho(\mathbf{x}') V(\mathbf{x} - \mathbf{x}') \\ &= -T \mathcal{S}[\rho] + \mathcal{F}_{\text{int}}[\rho]. \end{aligned} \quad (5)$$

The Langevin equation (3) is equivalent to the equation derived heuristically by Kawasaki [27], when $-\beta V(\mathbf{x})$ is replaced by the direct pair correlation function $c(\mathbf{x})$, and the functional defined in (5) becomes the Ramakrishnan–Yousouff (RY) density functional [29].

The dynamic average of an observable \mathcal{A} over thermal histories can be expressed as a functional integral over the density field:

$$\langle \mathcal{A} \rangle = \int \mathcal{D}\rho \mathcal{A}[\rho] \langle \delta [\partial_t \rho(\mathbf{x}, t) - R[\rho, \eta](\mathbf{x}, t)] \rangle_{\eta}, \quad (6)$$

where $R[\rho, \eta](\mathbf{x}, t)$ is the RHS of (3), $\langle \cdot \rangle_{\eta}$ means the average over the Gaussian noise η and $\delta[\cdot]$ is a functional delta function. This is the standard procedure, see [30], to derive field theories from stochastic equations. The only subtlety is the absence of a Jacobian. This is due to the fact that stochastic equations with multiplicative noise (3) are defined following the Itô prescription and therefore the Jacobian is a constant that can be absorbed in the definition of the functional integral. This is also related to the Markov property of the stochastic differential equation (3) (in the Itô discretization).

By using an integral representation of the functional Dirac distribution through a conjugated field $\hat{\rho}$ and averaging over the noise η , the Martin–Siggia–Rose–Janssen–de Dominicis (MSRJD) action is obtained [31]:

$$\langle \mathcal{A} \rangle = \int \mathcal{D}\rho \int \mathcal{D}\hat{\rho} \mathcal{A}[\rho] e^{S[\rho, \hat{\rho}]}, \quad (7)$$

with

$$\begin{aligned} S[\rho, \hat{\rho}] &= \int d^3 \mathbf{x} \int dt \left\{ \hat{\rho}(\mathbf{x}, t) \left[-\partial_t \rho(\mathbf{x}, t) + \nabla \cdot \left(\rho(\mathbf{x}, t) \nabla \frac{\delta \mathcal{F}[\rho]}{\delta \rho(\mathbf{x}, t)} \right) \right] \right. \\ &\quad \left. + T \rho(\mathbf{x}, t) (\nabla \hat{\rho}(\mathbf{x}, t))^2 \right\}, \end{aligned} \quad (8)$$

or explicitly

$$S[\rho, \hat{\rho}] = \int d^3\mathbf{x} \int dt \left\{ \hat{\rho}(\mathbf{x}, t) \left[-\partial_t \rho(\mathbf{x}, t) + T \nabla^2 \rho(\mathbf{x}, t) \right. \right. \\ \left. \left. + \nabla \cdot \left(\rho(\mathbf{x}, t) \int d^3\mathbf{x}' \nabla V(\mathbf{x} - \mathbf{x}') \rho(\mathbf{x}', t) \right) \right] + T \rho(\mathbf{x}, t) (\nabla \hat{\rho}(\mathbf{x}, t))^2 \right\}. \quad (9)$$

For clarity, and as the dynamical action involves one-time quantities, we shall not write the explicit time dependence of the fields in the rest of the paper, except for correlation functions involving fields at different times. For instance, $\rho(\mathbf{x}, t)$ will be written ρ_x , and $\int d^3\mathbf{x} \int dt$ will be replaced by \int_x .

2.2. Fluctuating nonlinear hydrodynamics

We shall now recall the field theory used to investigate the dynamics of compressible liquids, in particular close to the glass transition. In this case particles evolve under Newtonian dynamics and the derivation of the corresponding stochastic equations is not from first principles, in contrast to Dean's equation. The resulting equations are meant to be a generalization of hydrodynamic equations to intermediate time- and length scales, and are not expected to lead to an accurate description of the physics occurring on short time- and length scales. Hence, they describe the evolution of slow variables (associated with conserved quantities) subject to a thermal noise (corresponding to the fast degrees of freedom that have been integrated out) and they have been called fluctuating nonlinear hydrodynamics equations (FNH). A phenomenological derivation and a discussion of the equations can be found in [9, 24]. In the following we will focus on the FNH equations used by Das and Mazenko [9] to investigate the problem of the glass transition. They focused only on density and momentum as conserved variables but in principle the energy can be introduced as well. The equations of Das and Mazenko read

$$\partial_t \rho_x = \int d^3\mathbf{x}' \{ \rho_x, g_{i,x'} \} \frac{\delta \mathcal{F}}{\delta g_{i,x'}} \quad (10)$$

$$\partial_t g_{i,x} = \int d^3\mathbf{x}' \{ g_{i,x}, \rho_{x'} \} \frac{\delta \mathcal{F}}{\delta \rho_{x'}} + \int d^3\mathbf{x}' \sum_j \{ g_{i,x}, g_{j,x'} \} \frac{\delta \mathcal{F}}{\delta g_{j,x'}} \\ + \sum_j \int d^3\mathbf{x}' \Gamma_{ij}(\mathbf{x} - \mathbf{x}') \frac{\delta \mathcal{F}}{\delta g_{j,x'}} + \eta_{i,x}, \quad (11)$$

where ρ and g_i are respectively the density field and the i th component of the momentum density field and η_i is a white Gaussian noise with variance $\langle \eta_i(\mathbf{x}, t) \eta_j(\mathbf{x}', t') \rangle = 2T \Gamma_{ij}(\mathbf{x} - \mathbf{x}') \delta(t - t')$. The effective free-energy functional is $\mathcal{F} = \mathcal{F}_{\text{KIN}} + \mathcal{F}_U$, with

$$\mathcal{F}_{\text{KIN}}[\rho, \mathbf{g}] = \frac{1}{2} \int d^3\mathbf{x} \frac{\mathbf{g}^2(\mathbf{x})}{\rho(\mathbf{x})}, \\ \mathcal{F}_U[\rho, \mathbf{g}] = \frac{T}{m} \int d^3\mathbf{x} \rho(\mathbf{x}) (\log[\rho(\mathbf{x})/\rho_0] - 1) \\ - \frac{T}{2m^2} \int d^3\mathbf{x} d^3\mathbf{x}' c(\mathbf{x} - \mathbf{x}') (\rho(\mathbf{x}) - \rho_0) (\rho(\mathbf{x}') - \rho_0), \quad (12)$$

where ρ_0 is the density of the system, m is the particle mass and $c(\mathbf{x})$ is the direct correlation function. Note that the potential term, \mathcal{F}_U , coincides with the Ramakrishnan–Yousouff functional. The Poisson brackets $\{\cdot, \cdot\}$ and the Γ_{ij} are chosen so that the continuity equation for the density is verified and in such a way that the linearized equations coincide with the usual linear hydrodynamic equations:

$$\begin{aligned}\Gamma_{ij}(\mathbf{x} - \mathbf{y}) &= [-\eta_0(\frac{1}{3}\nabla_i\nabla_j + \delta_{i,j}\nabla^2) - \zeta_0\nabla_i\nabla_j]\delta(\mathbf{x} - \mathbf{y}) = L_{ij}\delta(\mathbf{x} - \mathbf{y}) \\ \{\rho(\mathbf{x}), g_i(\mathbf{x}')\} &= -\nabla_x^i\delta(\mathbf{x} - \mathbf{x}')\rho(\mathbf{x}) \\ \{g_i(\mathbf{x}), \rho(\mathbf{x}')\} &= -\rho(\mathbf{x})\nabla_x^i\delta(\mathbf{x} - \mathbf{x}') \\ \{g_i(\mathbf{x}), g_j(\mathbf{x}')\} &= -\nabla_x^j\delta(\mathbf{x} - \mathbf{x}')g_i(\mathbf{x}') - g_j(\mathbf{x})\nabla_x^i\delta(\mathbf{x} - \mathbf{x}'),\end{aligned}\tag{13}$$

and η_0 and ζ_0 are respectively the bare shear and bulk viscosity. Using the previous definition one can rewrite the equations (10), (12) in a more explicit way:

$$\partial_t\rho_x = -\nabla \cdot \mathbf{g}_x\tag{14}$$

$$\partial_t g_{i,x} = -\rho_x \nabla_i \frac{\delta\mathcal{F}_U}{\delta\rho_x} - \sum_j \nabla_j \left(\frac{g_{i,x}g_{j,x}}{\rho_x} \right) - \sum_j L_{ij} \left(\frac{g_{j,x}}{\rho_x} \right) + \eta_{i,x}.\tag{15}$$

Following the strategy of the previous sections we find that the corresponding MSRJD action is the integral of

$$\begin{aligned}s_x &= -\hat{\rho}_x \left[\partial_t\rho_x + \nabla_i \left(\rho_x \frac{\delta F}{\delta g_{i,x}} \right) \right] + T\hat{g}_{i,x}L_{ij}\hat{g}_{j,x} \\ &\quad - \hat{g}_{i,x} \left[\partial_t g_{i,x} + \rho_x \nabla_i \frac{\delta\mathcal{F}}{\delta\rho_x} + \nabla_j \left(g_{i,x} \frac{\delta\mathcal{F}}{\delta g_{j,x}} \right) + g_{j,x} \nabla_i \frac{\delta\mathcal{F}}{\delta g_{j,x}} + L_{ij} \frac{\delta\mathcal{F}}{\delta g_{j,x}} \right],\end{aligned}\tag{16}$$

where $\hat{\rho}$ and $\hat{\mathbf{g}}$ are respectively the conjugated field used to express the delta functions corresponding to equations (14) and (15), and summation over indices is implicit. In the next sections, we will focus on BDD only. We shall show later in section 7 how to generalize the results to the case of FNH.

3. Time-reversal symmetry and fluctuation–dissipation relations

Time-reversal symmetry relates the probabilities of a path and of its time-reversal counterpart in configuration space. This is a very important symmetry obeyed by systems in thermodynamic equilibrium and it has far reaching consequences. In particular, all physical correlation functions are invariant under time reversal; correlation and response functions are related by the fluctuation–dissipation relation. In the context of dynamic field theory time-reversal symmetry is related to a transformation of the fields leaving the action invariant. In the following, we first recall the standard field transformation for one particle evolving under Langevin dynamics. We will later show how this can be generalized to the more complex field theories introduced in the previous section.

3.1. Langevin dynamics

Let us now focus on one particle evolving under Langevin dynamics with additive noise. Denoting the particle position $X(t)$ and the external potential V the Langevin equation reads

$$\partial_t X(t) = -\nabla V(X(t)) + \eta(t), \quad (17)$$

where η is a Gaussian white noise with zero mean and variance $2T$. After the introduction of a conjugated field \hat{X} , the dynamical action reads

$$S_0[\phi] = \int_{-\infty}^{\infty} dt \left[-\hat{X}(t) \cdot (\partial_t X(t) + \nabla V(X(t))) + T \hat{X}(t)^2 \right], \quad (18)$$

where the dynamical field is $\phi = (X, \hat{X})$. It is easy to check that S_0 is invariant under time reversal:

$$\mathcal{O} : \begin{cases} t \rightarrow -t \\ \hat{X}(t) \rightarrow \hat{X}(t) - \frac{1}{T} \partial_t X(t). \end{cases} \quad (19)$$

As a consequence correlation functions of $X(t)$ are invariant under this field transformation and therefore they are invariant under time reversal. The application of the field transformation to the response function, which can be shown to be equal to $\langle X(t) \hat{X}(t') \rangle$, leads to

$$\langle \hat{X}(t') X(t) \rangle = \langle \hat{X}(t) X(t') \rangle - \frac{1}{T} \partial_t \langle X(t) X(t') \rangle. \quad (20)$$

For $t > t'$, the response vanishes, and we get

$$\langle \hat{X}(t') X(t) \rangle = -\frac{1}{T} \partial_t \langle X(t) X(t') \rangle, \quad (21)$$

which is the FDR between correlation and response functions.

3.2. Brownian dynamics for the density field

We will now focus on the stochastic evolution of the local density field for interacting Langevin particles (in what follows we use the acronym BDD). Note that the extension of the ideas in the next paragraphs to generic multiplicative noise is straightforward. First we derive the expression of the response function in terms of field average and afterwards we show two field transformations which lead to a *physical* time-reversal symmetry and, hence, to FDR.

3.2.1. Response function. In order to establish FDR, one has to write the expression for the response of the system to an external potential. Here, as we probe density fluctuations, this force is an external potential, which is taken into account by adding an extra term $\mathcal{F}_{\text{ext}}[\rho] = -\int_x \rho_x \mu_x$ to the free energy (5). The response $R_{xx'}(t, t')$ at time t and position \mathbf{x} to an infinitesimal external force switched on at time t' and position \mathbf{x}' is defined by

$$\langle \rho(\mathbf{x}, t) \rangle_{\mu} = \langle \rho(\mathbf{x}, t) \rangle_{\mu=0} + \int d^3 \mathbf{x}'' \int_{t'}^t dt'' R_{xx'}(t, t'') \mu(\mathbf{x}'', t'') + o(\mu), \quad (22)$$

where $\langle \cdot \rangle_\mu$ is the average taken with the free energy functional $\mathcal{F} + \mathcal{F}_{\text{ext}}$. Thus, expanding the path integral to first order in μ , one gets

$$\begin{aligned} \langle \rho(\mathbf{x}, t) \rangle_\mu &= \langle \rho(\mathbf{x}, t) \rangle_{\mu=0} + \int d^3\mathbf{x}'' \int_{t'}^\infty dt'' \langle \rho(\mathbf{x}, t) \\ &\quad \times \hat{\rho}(\mathbf{x}'', t'') \nabla \cdot (\rho(\mathbf{x}'', t'') \nabla \mu(\mathbf{x}'', t'')) \rangle. \end{aligned} \quad (23)$$

However, due to causality the time t'' in the integral runs until t only, and integrating twice by parts, one obtains the expression of the response:

$$R_{xx'}(t, t') = -\langle \rho(\mathbf{x}, t) \nabla \cdot (\rho(\mathbf{x}', t') \nabla \hat{\rho}(\mathbf{x}', t')) \rangle. \quad (24)$$

This response function is the same as the one studied in [13].

3.2.2. First expression of time-reversal symmetry. The first field transformation related to time-reversal symmetry is

$$\mathcal{T} : \begin{cases} t \rightarrow -t \\ \hat{\rho}_x \rightarrow \hat{\rho}_x + f_x, \end{cases} \quad (25)$$

where f verifies

$$\nabla \cdot (\rho_x \nabla f_x) = -\frac{1}{T} \partial_t \rho_x. \quad (26)$$

The field f plays a role similar to the longitudinal part of a current for the density field. By integrating twice by parts, the variation of (9) under \mathcal{T} is found to be

$$-\frac{1}{T} \int_x \partial_t \mathcal{F}[\rho_x]. \quad (27)$$

Hence, at equilibrium the action (9) is invariant under \mathcal{T} . The density field is affected in a simple way by \mathcal{T} : $\rho(\mathbf{x}, t) \rightarrow \rho(\mathbf{x}, -t)$. This implies that any average of the type $\langle \prod_i \rho(\mathbf{x}_i, t_i) \rangle$ is invariant under time reversal. By making the change \mathcal{T} in the expression (24) of the response function and using the condition (26), we get

$$\frac{1}{T} \partial_{t'} C_{xx'}(t - t') = R_{xx'}(t - t') - R_{xx'}(t' - t), \quad (28)$$

where $C_{xx'}(t - t') = \langle \delta \rho(\mathbf{x}, t) \delta \rho(\mathbf{x}', t') \rangle$. This is the FDR, which can be easily generalized to correlators of more density fields:

$$\begin{aligned} \frac{1}{T} \partial_{t'} C_{x_1 \dots x_n x'}(t_1 \dots, t_n, t') &= R_{x_1 \dots x_n x'}(t_1 \dots, t_n, t') \\ &\quad - R_{x_1 \dots x_n x'}(-t_1 \dots, -t_n, -t'), \end{aligned} \quad (29)$$

where

$$R_{x_1 \dots x_n x'}(t_1 \dots, t_n, t') = -\left\langle \nabla \cdot (\rho(\mathbf{x}', t') \nabla \hat{\rho}(\mathbf{x}', t')) \prod_{i=1}^n \rho(\mathbf{x}_i, t_i) \right\rangle \quad (30)$$

and

$$C_{x_1 \dots x_n x'}(t_1 \dots, t_n, t') = \left\langle \rho(\mathbf{x}', t') \prod_{i=1}^n \rho(\mathbf{x}_i, t_i) \right\rangle. \quad (31)$$

3.2.3. *Second expression of time-reversal symmetry.* As said above, the time-reversal symmetry may be expressed through another transformation of the fields. Indeed, consider the following change:

$$\mathcal{U} : \begin{cases} t \rightarrow -t \\ \hat{\rho}_x \rightarrow -\hat{\rho}_x + \beta \frac{\delta \mathcal{F}}{\delta \rho_x}. \end{cases} \quad (32)$$

We remark that this transformation involves only the non-physical field $\hat{\rho}$, as does \mathcal{T} . Thus, both transforms clearly give rise to the same relations between correlation functions.

As above the dynamical action can be easily proved to be invariant under \mathcal{U} . The time-reversal symmetry of any density correlation functions follows directly. FDR can also be derived using the identity

$$\left\langle \rho(\mathbf{x}, t) \frac{\delta S}{\delta \hat{\rho}(\mathbf{x}', t')} \right\rangle = 0, \quad (33)$$

which implies

$$\begin{aligned} \langle \rho(\mathbf{x}, t) \partial_t \rho(\mathbf{x}', t') \rangle &= \left\langle \rho(\mathbf{x}, t) \nabla \cdot \left(\rho(\mathbf{x}', t') \nabla \frac{\delta \mathcal{F}}{\delta \rho(\mathbf{x}', t')} \right) \right\rangle \\ &\quad - 2T \langle \rho(\mathbf{x}, t) \nabla \cdot (\rho(\mathbf{x}', t') \nabla \hat{\rho}(\mathbf{x}', t')) \rangle. \end{aligned} \quad (34)$$

Splitting the last term in two identical parts and applying the transformation \mathcal{U} to one of them leads us to

$$\begin{aligned} \langle \rho(\mathbf{x}, t) \partial_t \rho(\mathbf{x}', t') \rangle &= \left\langle \rho(\mathbf{x}, t) \nabla \cdot \left(\rho(\mathbf{x}', t') \nabla \frac{\delta \mathcal{F}}{\delta \rho(\mathbf{x}', t')} \right) \right\rangle \\ &\quad - T \langle \rho(\mathbf{x}, t) \nabla \cdot (\rho(\mathbf{x}', t') \nabla \hat{\rho}(\mathbf{x}', t')) \rangle \\ &\quad + T \langle \rho(\mathbf{x}, -t) \nabla \cdot (\rho(\mathbf{x}', -t) \nabla \hat{\rho}(\mathbf{x}', -t')) \rangle \\ &\quad - \left\langle \rho(\mathbf{x}, -t) \nabla \cdot \left(\rho(\mathbf{x}', -t) \nabla \frac{\delta \mathcal{F}}{\delta \rho(\mathbf{x}', -t')} \right) \right\rangle. \end{aligned} \quad (35)$$

The two terms containing no $\hat{\rho}$ in the RHS cancel and one gets (28).

From \mathcal{U} one can derive another useful identity as follows. Applying the transformation to $\mathcal{G}_{xx'}(t-t') = \langle \rho(\mathbf{x}, t) \hat{\rho}(\mathbf{x}', t') \rangle$ one gets

$$\beta \left\langle \rho(\mathbf{x}, t) \frac{\delta \mathcal{F}[\rho, \hat{\rho}]}{\delta \rho(\mathbf{x}', t')} \right\rangle = \Theta(t-t') \langle \rho(\mathbf{x}, t) \hat{\rho}(\mathbf{x}', t') \rangle + \Theta(t'-t) \langle \rho(\mathbf{x}', t') \hat{\rho}(\mathbf{x}, t) \rangle. \quad (36)$$

As we shall show in section 5.6, this identity allows substantial simplifications when \mathcal{F} is quadratic. Indeed, in that case \mathcal{U} is a linear transformation and (36) then becomes a linear relation between the density–density correlator $\langle \delta \rho(\mathbf{x}, t) \delta \rho(\mathbf{x}', t') \rangle$ and the naive response $\mathcal{G}_{xx'}(t-t')$, which are both two-point functions.

4. Nonlinear symmetries and perturbation theories

As will be emphasized all through this paper, a physical symmetry is related to field transformations that leave the action invariant. In the following we shall show from a general point of view why the linearity of these transformations is essential to construct

symmetry preserving perturbative expansions. Neither of the BDD and FNH field theories has this property, namely the field transformations related to time-reversal symmetry are nonlinear. In the following we shall highlight all the complications which arise in perturbation theories because of the nonlinearity, focusing on BDD. In particular, this will make clear why mode-coupling approximations (MCAs) generically violate time-reversal symmetry and hence FDR.

4.1. General discussion

For the sake of generality, let us consider a field ϕ_α (where $\alpha = (x, t)$ in our case) and a generic field theory:

$$\int \prod_\alpha d\phi_\alpha \exp \left[-\frac{1}{2} \phi_\alpha (G_0^{-1})_{\alpha,\beta} \phi_\beta + gV(\phi) \right], \quad (37)$$

where g is the coupling constant controlling the perturbative expansion, V is the interaction and the field ϕ is assumed to have a zero mean. The bare propagator is G_0 and the bare action $S_0[\phi] = -\frac{1}{2} \phi_\alpha (G_0^{-1})_{\alpha,\beta} \phi_\beta + gV(\phi)$.

Now consider a linear transformation \mathcal{O} of the field $\phi_\alpha \rightarrow \phi'_\alpha = O_{\alpha,\beta} \phi_\beta$ which leaves S_0 invariant. We require that the integration measure is left invariant (the contrary giving rise to so-called anomalies in quantum field theory), which means that \mathcal{O} must have determinant of modulus equal to one (or at least to a constant). Then we have the following Ward–Takahashi (WT) identities:

$$\langle \phi_{\alpha_1} \cdots \phi_{\alpha_n} \rangle = O_{\alpha_1,\beta_1} \cdots O_{\alpha_n,\beta_n} \langle \phi_{\beta_1} \cdots \phi_{\beta_n} \rangle. \quad (38)$$

One immediately sees the problem when the symmetry is not linear. Consider for instance a transformation of the form $\phi_\alpha \rightarrow \phi'_\alpha = O_{\alpha,\beta} \phi_\beta^2$. The WT identities are

$$\langle \phi_{\alpha_1} \cdots \phi_{\alpha_n} \rangle = O_{\alpha_1,\beta_1} \cdots O_{\alpha_n,\beta_n} \langle \phi_{\beta_1}^2 \cdots \phi_{\beta_n}^2 \rangle. \quad (39)$$

Clearly, a nonlinear symmetry induces relations between correlation functions of different orders in the fields. A concrete example is the proof of FDR obtained in the previous sections for BDD field theory. The general idea behind the mode-coupling approximation and other self-consistent approaches is to provide closed equations for two-point functions. Hence, because the transformation mixes correlation functions of different order in the fields it seems difficult, if not impossible, to construct MC-like approximations which preserve the symmetry. In order to understand where perturbation theories fail in the case of nonlinear symmetries it is useful to recall what are the key ingredients that make them work when the symmetry is linear.

Let us first focus on standard (non-self-consistent) perturbation theory. A linear symmetry implies a relation between two-point functions; see equation (38). Why is this relation preserved order by order in perturbation theory in g ? The reason is that the potential part $V(\phi)$ is itself invariant under \mathcal{O} and thus (38) is also true when both LHS and RHS are computed at any order in g . For concreteness, let us focus on the MSRJD field theory for one particle evolving under Langevin dynamics. The dynamical action can be split into two parts, $S_{\text{QUAD}} + S_{\text{INT}}$, with

$$S_{\text{QUAD}}[\phi] = \int_{-\infty}^{\infty} dt \left[-\hat{X}(t) \cdot \partial_t X(t) + T \hat{X}(t)^2 \right] \quad (40)$$

and

$$S_{\text{INT}}[\phi] = -g \int_{-\infty}^{\infty} dt \hat{X}(t) \cdot \nabla V(X(t)). \quad (41)$$

Both parts of the action are invariant under time reversal and one can expand in powers of g :

$$\langle \hat{X}(t') X(t) \rangle = \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \langle \hat{X}(t') X(t) S_{\text{INT}}[\phi]^n \rangle_{\text{GAUSS}}, \quad (42)$$

where $\langle \cdot \rangle_{\text{GAUSS}}$ stands for the Gaussian average with action S_{QUAD} . However S_{INT} is itself invariant under the action of \mathcal{O} , thus the identity (21), FDR, is true at any order in perturbation theory.

Let us now focus on self-consistent perturbation theory. This approximation scheme consists in cutting the perturbation series at a given order and replacing the bare propagator by the dressed one in the formal expression of the self-energy (some diagrams present in bare perturbation theory have to be neglected in order to avoid double counting). Doing so one obtains self-consistent equations for the evolution of the dressed propagator, which in principle can be improved considering more diagrams. Note also that these self-consistent equations can often be shown to be exact in some large N limit, where the field has become an N component field [11].

The formal way to obtain this self-consistent perturbation theory is through the Legendre transform $\Gamma(G) = \log F(\Sigma(G), G)$ of two-point functions which generates all the two-particle irreducible diagrams [32]:

$$\exp \Gamma(G) = \int \prod_{\alpha} d\phi_{\alpha} \exp \left[-\frac{1}{2} \sum_{\alpha, \beta} \phi_{\alpha} (G_0^{-1})_{\alpha, \beta} \phi_{\beta} + gV(\phi) - \frac{1}{2} \sum_{\alpha, \beta} \Sigma_{\alpha, \beta} (\phi_{\beta} \phi_{\alpha} - G_{\beta, \alpha}) \right], \quad (43)$$

where Σ is determined by the condition $\partial F / \partial \Sigma = 0$, i.e. Σ such that the propagator of the theory equals G . De Dominicis and Martin have shown that Γ is equal to

$$-\frac{1}{2} \text{Tr} (G_0^{-1} G) + \frac{1}{2} \text{Tr} \ln G + \phi_{2\text{PI}}(G), \quad (44)$$

where $\phi_{2\text{PI}}(G)$ is the sum of all 2PI diagrams with the full propagator G used as internal lines [32]. The physical propagator of the theory is obtained by finding the G which makes Γ stationary, i.e. solving $\partial \Gamma / \partial G = 0$. Hence, it is immediate to check that the self-energy as a function of the propagator is $\Sigma(G) = 2\partial \phi_{2\text{PI}} / \partial G$. Practically, the usual self-consistent or mode-coupling approximations keep the lowest non-trivial diagrams, the bubbles, in $\phi_{2\text{PI}}$.

Let us now prove that the symmetry is preserved if the field transformation $\phi_{\alpha} \rightarrow \phi'_{\alpha} = O_{\alpha, \beta} \phi_{\beta}$ that leaves the action invariant is linear. For the same reason as before we assume $|\det O| = 1$. Under this transformation the propagator transforms into $G'_{\alpha, \beta} = M_{\alpha, \beta; \alpha', \beta'} G_{\alpha', \beta'}$, where $M_{\alpha, \beta; \alpha', \beta'} = O_{\alpha, \alpha'} O_{\beta, \beta'}$ ($|\det M| = 1$). We shall prove that this relation is preserved in the self-consistent perturbative expansion.

The first thing we want to prove is that $\Gamma(MG) = \Gamma(G)$, which in a certain sense is expected since a physical symmetry should not change the value of the functional. This is easy to prove starting from the expression for $\Gamma(MG)$ and rewriting $\phi_{\alpha} = O_{\alpha, \beta} \phi'_{\beta}$ and

changing variable in the functional integral. The measure as well as the action is invariant, so that one finds

$$\Gamma(MG) = \log \int \prod_{\alpha} d\phi_{\alpha} \exp \left[-\frac{1}{2} \sum_{\alpha\beta} \phi_{\alpha} (G^0)_{\alpha,\beta}^{-1} \phi_{\beta} + gV(\phi) - \frac{1}{2} (\Sigma M)_{\alpha,\beta} (\phi_{\alpha} \phi_{\beta} - G_{\alpha,\beta}) \right]. \quad (45)$$

Calling $\Sigma' = \Sigma M$ and noticing that $\partial\Gamma(\Sigma, MG)/\partial\Sigma = 0$ is equivalent to $\partial\Gamma(\Sigma', G)/\partial\Sigma' = 0$, we obtain that indeed $\Gamma(MG) = \Gamma(G)$. Furthermore, since this is true for any value of g , the coupling constant, then this identity is true for Γ expanded to any given order in g . The consequences of this identity are particularly useful. First, if G is a solution of $\partial\Gamma/\partial G = 0$ so is MG . Thus, if there is a unique solution—the symmetry is unbroken—then $G = MG$. In the case of time-reversal symmetry this identity leads to FDR. The fact that the solution is unique is expected in our case since time-reversal symmetry is certainly not broken at equilibrium. Since this is true for any value of g then it is true when Γ is expanded to any finite order in g , i.e. for the self-consistent equations written to any finite order in g . Finally, the last remark following from $\Gamma(MG) = \Gamma(G)$ is the symmetry relation verified by the self-energy: $\Sigma(MG) = \Sigma(G)M$.

To recap, we have shown that a symmetry related to a linear transformation in the field is a symmetry of the self-consistent equations written to any finite order in g . This is the reason why FDR is preserved for standard MSRJD Langevin field theory. Note that a very different proof can be found in [33].

4.2. Subtleties in perturbation theory

In this section we focus on BDD field theory and we highlight the difficulties and the failures of perturbation theories *vis à vis* FDR. The case of FNH is conceptually identical but practically more clumsy because of the larger number of fields. Let us start with bare perturbation theory. In order to do a perturbative analysis it is convenient to separate the Gaussian zero-mean part of the local density field from the interacting one by introducing density fluctuations in (9) $\delta\rho_x = \rho_x - \rho_0$. This gives

$$S = \int_x (s_{0,x} + s_{\text{INT},x}), \quad (46)$$

with

$$s_{0,x} = \hat{\rho}_x \left(-\partial_t \delta\rho_x + T \nabla^2 \delta\rho_x + \rho_0 \int_{x'} V(\mathbf{x} - \mathbf{x}') \nabla^2 \delta\rho_{x'} \right) + T \rho_0 (\nabla \hat{\rho}_x)^2 \quad (47)$$

$$s_{\text{INT},x} = T \delta\rho_x (\nabla \hat{\rho}_x)^2 + \hat{\rho}_x \nabla \cdot \left(\delta\rho_x \int_{x'} \nabla V(\mathbf{x} - \mathbf{x}') \delta\rho_{x'} \right).$$




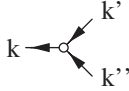
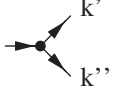
This may be written in a more compact form through the bidimensional vector field $(\delta\rho, \hat{\rho})^\dagger$. In Fourier space, the inverse of the propagator of this field is

$$\tilde{G}_0^{-1} = \begin{pmatrix} 0 & i\omega + T\mathbf{k}^2 (1 + \beta\rho_0 V(\mathbf{k})) \\ -i\omega + T\mathbf{k}^2 (1 + \beta\rho_0 V(\mathbf{k})) & -2T\rho_0 \mathbf{k}^2 \end{pmatrix}. \quad (48)$$

However, including the potential in the form (48) makes it practically impossible to write weak coupling expansions that preserve time-reversal symmetry. Hence we shall drop it out of the inverse of the propagator:

$$\begin{aligned} G_0^{-1} &= \begin{pmatrix} C_0(\mathbf{k}, \omega) & G_0(\mathbf{k}, \omega) \\ G_0^*(\mathbf{k}, \omega) & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 0 & i\omega + T\mathbf{k}^2 \\ -i\omega + T\mathbf{k}^2 & -2T\rho_0\mathbf{k}^2 \end{pmatrix}, \end{aligned} \quad (49)$$

and we shall treat the quadratic term as an insertion in a line. This gives the following Feynman rules:

- bare density correlator:  $C_0(\mathbf{k}, \omega)$
- bare naive response:  $\mathcal{G}_0(\mathbf{k}, \omega)$
- line insertion:  $-\rho_0\mathbf{k}^2V(\mathbf{k})$
- potential vertex:  $\frac{1}{2}(\mathbf{k} \cdot \mathbf{k}'V(\mathbf{k}') + \mathbf{k} \cdot \mathbf{k}''V(\mathbf{k}'')) = \Gamma(\mathbf{k}, \mathbf{k}', \mathbf{k}'')$
- noise vertex:  $-T\mathbf{k}' \cdot \mathbf{k}''$

The bare density correlator is

$$C_0(\mathbf{k}, \omega) = \frac{2T\rho_0\mathbf{k}^2}{\omega^2 + (T\mathbf{k}^2)^2}, \quad (50)$$

and the bare naive response

$$\mathcal{G}_0(\mathbf{k}, \omega) = \frac{1}{T\mathbf{k}^2 - i\omega}. \quad (51)$$

Due to the form of the vertices, diagrams with tadpoles and hence corrections to the average density vanish to all orders (the momentum at the entrance to the tadpole is zero).

In terms of density fluctuations, the response is

$$\begin{aligned} R_{xx'}(t, t') &= -\langle \delta\rho(\mathbf{x}, t) \nabla \cdot (\delta\rho(\mathbf{x}', t') \nabla \hat{\rho}(\mathbf{x}', t')) \rangle - \rho_0 \nabla^2 \langle \delta\rho(\mathbf{x}, t) \hat{\rho}(\mathbf{x}', t') \rangle \\ &= \chi_{xx'}(t, t') - \rho_0 \nabla^2 \mathcal{G}_{xx'}(t, t'), \end{aligned} \quad (52)$$

where \mathcal{G} is the naive response and χ is an ‘anomalous’ response. Having a look at (52), one sees that part of the non-triviality of the FDR arises from the anomalous response χ , which itself comes from the multiplicative aspect of the noise, or equivalently from the nonlinearity of the transformation of the fields associated with time reversal.

A perturbative expansion in powers of the potential preserves the symmetry. Indeed, the dynamical average of any function $\mathcal{A}[\rho, \hat{\rho}]$ may be written as

$$\begin{aligned} \langle \mathcal{A}[\rho, \hat{\rho}] \rangle &= \int \mathcal{D}\rho \int \mathcal{D}\hat{\rho} \mathcal{A}[\rho, \hat{\rho}] e^{S_{\text{FREE}}[\rho, \hat{\rho}] + S_V[\rho, \hat{\rho}]} \\ &= \sum_{p=1}^{\infty} \int \mathcal{D}\rho \int \mathcal{D}\hat{\rho} \mathcal{A}[\rho, \hat{\rho}] \frac{(S_V[\rho, \hat{\rho}])^p}{p!} e^{S_{\text{FREE}}[\rho, \hat{\rho}]}, \end{aligned} \quad (53)$$

where S_{FREE} contains the terms of the action which do not contain the potential. The key point is that both S_{FREE} and S_V are invariant under \mathcal{T} , and thus FDR is verified to any finite order in the expansion in power series of S_V . However, S_{FREE} is not quadratic in the fields. The nonlinearity of the field transformation relates the noise cubic term to quadratic terms. Thus this vertex must be taken into account non-perturbatively. In other words, to compute the correlators at a given order in the power series expansion in S_V , one has to include contributions at all orders in the noise vertex. This is a difficult but not impossible task because, at a given order p in S_V , the diagrammatic expansion contains a finite number of diagrams, due to the absence of the propagator connecting two $\hat{\rho}$. Indeed, if one considers a diagram with p potential vertices and q noise vertices contributing to a correlation function of r ρ and s $\hat{\rho}$, one must have $q + s \leq p + r$. Hence such a diagram must have less than $p + r - s$ noise vertices.

Thus bare perturbation theory which preserves the nonlinear symmetry can be set up, but is considerably more complicated than the usual one. Finally, this discussion makes it clear that any approximation which drops a part of the diagrams of the full expansion in terms of the noise vertex is expected to be in contradiction with the FDR. This is indeed what happens in self-consistent approximations, as we shall show below.

4.3. Violation of fluctuation–dissipation relations in self-consistent perturbation theory

Let us now focus on self-consistent perturbation theory, in particular on the mode-coupling approximation introduced by Kawasaki [34] that consists in neglecting vertex renormalization. First, we write the Schwinger–Dyson (SD) equations

$$G_0^{-1} \cdot G = 1 + \Sigma \cdot G, \quad (54)$$

where Σ is the self-energy,

$$\Sigma(\mathbf{k}, \omega) = \begin{pmatrix} \Sigma_{\rho\rho}(\mathbf{k}, \omega) & \Sigma_{\rho\hat{\rho}}(\mathbf{k}, \omega) \\ \Sigma_{\hat{\rho}\rho}(\mathbf{k}, \omega) & \Sigma_{\hat{\rho}\hat{\rho}}(\mathbf{k}, \omega) \end{pmatrix}, \quad (55)$$

and the associative product \cdot is defined as follows:

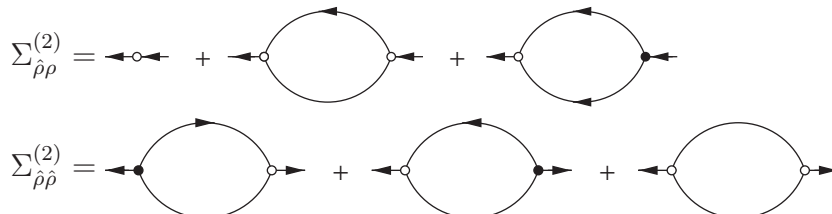
$$(A \cdot B)(\mathbf{k}, t) = \int_{-\infty}^{\infty} dt' A(\mathbf{k}, t - t')B(\mathbf{k}, t'). \quad (56)$$

Causality and reality of the density auto-correlator imply that the self-energies verify

$$\Sigma_{\rho\hat{\rho}}(\mathbf{k}, t) = \Sigma_{\hat{\rho}\rho}(\mathbf{k}, -t) \quad (57)$$

$$\Sigma_{\rho\rho}(\mathbf{k}, t) = 0. \quad (58)$$

The first diagrams contributing to the self-energies are



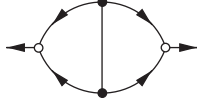


Figure 1. Example of a diagram which contributes at order V^2 to FDR but is absent from the MC equations.

Diagrams of higher orders all contain vertex renormalization. Hence, if one neglects renormalization of both vertices, the SD equations (54) become MC equations for (3) (for $t > 0$):

$$\begin{aligned}\partial_t \mathcal{G}(\mathbf{k}, t) &= -\rho_0 T \mathbf{k}^2 (1 + \beta V(\mathbf{k})) \mathcal{G}(\mathbf{k}, t) + \int_{-\infty}^t dt' \Sigma_{\hat{\rho}\rho}(\mathbf{k}, t - t') C(\mathbf{k}, t') \\ \partial_t C(\mathbf{k}, t) &= -\rho_0 T \mathbf{k}^2 (1 + \beta V(\mathbf{k})) C(\mathbf{k}, t) + \int_0^{\infty} dt' \Sigma_{\hat{\rho}\hat{\rho}}(\mathbf{k}, t - t') \mathcal{G}(\mathbf{k}, t') \\ &\quad + \int_{-\infty}^t dt' \Sigma_{\hat{\rho}\rho}(\mathbf{k}, t - t') C(\mathbf{k}, t'),\end{aligned}\tag{59}$$

with

$$\begin{aligned}\Sigma_{\hat{\rho}\rho}(\mathbf{k}, t) &= 4 \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \mathcal{G}(\mathbf{k}, t) C(\mathbf{k} - \mathbf{q}, t) \Gamma(\mathbf{q}, \mathbf{k}, \mathbf{k} - \mathbf{q}) \Gamma(\mathbf{k}, \mathbf{q}, \mathbf{k} - \mathbf{q}) \\ &\quad - 2 \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \mathcal{G}(\mathbf{k}, t) \mathcal{G}(\mathbf{k} - \mathbf{q}, t) \mathbf{q} \cdot (\mathbf{k} - \mathbf{q}) \Gamma(\mathbf{q}, \mathbf{k}, \mathbf{k} - \mathbf{q}) \\ \Sigma_{\hat{\rho}\hat{\rho}}(\mathbf{k}, t) &= 2 \int \frac{d^3 \mathbf{q}}{(2\pi)^3} C(\mathbf{k}, t) C(\mathbf{k} - \mathbf{q}, t)^2 \\ &\quad - 8 \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \text{Re} \mathcal{G}(\mathbf{k}, t) C(\mathbf{k} - \mathbf{q}, t) \mathbf{q} \cdot \mathbf{k} \Gamma(\mathbf{q}, \mathbf{k}, \mathbf{k} - \mathbf{q}).\end{aligned}\tag{60}$$

These equations are not compatible with FDR, as can be seen from the solution of SD equations at low orders in the potential. FDR is trivially verified at order zero. At order one the MC equations are exact and hence they are automatically compatible with FDR. Incompatibilities appear at order two, where diagrams such as those shown in figure 1, the first diagrams contributing to vertex renormalization, have to be taken into account in the non-self-consistent perturbation theory and therefore also in the self-consistent one (as discussed previously, in order to preserve FDR one has always to take into account that the contributions at all orders in the noise vertex contribute to FDR). This suggests that if one wants to improve the approximation by keeping for instance the first vertex corrections, one has to include at least all the diagrams of order two (in powers of the potential) in the self-energy. In that case, the incompatibility with FDR would be an effect of order three. However, nothing guarantees that the violation of FDR by the self-consistent approximation is attenuated when the order of the approximation is increased. Another consequence of practical importance is violation of causality in (59), where times in the integrals are not restricted in $[0, t]$. In contrast, when time reversal—and thus FDR—is preserved, times in the integrals run from 0 to t . To conclude this paragraph, we

remark that incompatibilities with FDR have arisen from an explicit breaking of the time-reversal symmetry, which is due to the nonlinearity of the field transformation related to time-reversal symmetry.

5. Restoration of time-reversal symmetry in perturbative expansions

In this section, by introducing some extra fields, we construct a generalization of the field theory described previously, in which time-reversal symmetry corresponds to a field transformation which is *linear*. As a consequence, all the problems described in the previous section are eliminated. In particular, this allows us to set up self-consistent perturbative equations which preserve FDR. We explain in detail our procedure for the BDD field theory; the result of this strategy for the FNH field theory will be given in section 7.

5.1. Introducing extra fields

As already discussed, in order to overcome the violation of time-reversal symmetry in perturbative expansions we need to make the related field transformation linear. This can be achieved by introducing some auxiliary fields. Furthermore, we also introduce a *response field* such that the response function becomes explicitly a two-field correlation function. We have found two different field transformations related to the same physical symmetry (time reversal). As a consequence, we can make each one or both of the symmetries linear, which leads to three different field theories. In this section we shall treat the case in which only either the \mathcal{U} or \mathcal{T} transformation is linearized. The case of the completely linearized theory (both symmetries are made linear) is considered in appendix E.

Let us consider the time-reversal transformation \mathcal{T} first: we start from the identity

$$\langle \mathcal{A} \rangle = \int \mathcal{D}\rho \mathcal{A}[\rho] \langle \delta (\partial_t \rho(\mathbf{x}, t) - R[\rho, \eta](\mathbf{x}, t)) \rangle, \quad (61)$$

where $R[\rho, \eta](\mathbf{x}, t)$ is the RHS of (3). We now plug into the functional integral the representation of the identity

$$\int \mathcal{D}f \prod_{x,t} \delta \left(\nabla \cdot (\rho_x \nabla f_x) + \frac{1}{T} \partial_t \rho_x \right) (\det [-\nabla \cdot (\rho_x \nabla)]) = 1. \quad (62)$$

We put a minus sign in such a way that the operator inside the determinant is positive definite. Thus we do not need to take the absolute value of the determinant. We exponentiate the delta function using an auxiliary field \hat{f} and we introduce fermionic fields ϕ and $\bar{\phi}$ to exponentiate the determinant. As a consequence, there is a new term to add to the previous action that reads

$$\int_x \hat{f}_x \left(\nabla \cdot (\rho_x \nabla f_x) + \frac{1}{T} \partial_t \rho_x \right) - \int_x \rho_x \nabla \phi_x \cdot \nabla \bar{\phi}_x. \quad (63)$$

Furthermore, we also introduce the field $\psi = \nabla \cdot (\rho \nabla \hat{\rho})$, which allows for the usual two-field correlator representation of the response function. This leads us to introduce a conjugated field $\hat{\psi}$ for the Fourier representation of the delta function related to ψ . The final action

is the integral of

$$s_x = -\hat{\rho}_x \partial_t \rho_x + \psi_x \frac{\delta \mathcal{F}}{\delta \rho_x} - T \hat{\rho}_x \psi_x + \hat{\psi}_x (\psi_x - \nabla_x \cdot (\rho_x \nabla_x \hat{\rho}_x)) \\ + \hat{f}_x \left(\frac{1}{T} \partial_t \rho_x + \nabla_x \cdot (\rho_x \nabla_x f_x) \right) - \rho_x \nabla_x \cdot (\phi_x \nabla_x \bar{\phi}_x). \quad (64)$$

This action now remains invariant up to the boundary terms under the following linear transformation \mathcal{T}_1 . First invert the time $t \rightarrow -t$ then change the fields in sequential order as follows:

$$\mathcal{T}_1 : \begin{cases} \hat{\rho}_x \rightarrow \hat{\rho}_x + f_x \\ \psi_x \rightarrow \psi_x + \frac{1}{T} \partial_t \rho_x \\ \hat{\psi}_x \rightarrow \hat{\psi}_x + T f_x \\ \hat{f}_x \rightarrow -\hat{f}_x + T f_x + \hat{\psi}_x + T \hat{\rho}_x \\ f_x \rightarrow -f_x. \end{cases} \quad (65)$$

Formally, we write this as $\tilde{\phi} = O \cdot \phi$, where $\phi = (\rho, \hat{\rho}, \psi, \hat{\psi}, f, \hat{f}, \phi, \bar{\phi})^T$ and $\tilde{\phi}(\mathbf{x}, t) = \phi(-\mathbf{x}, t)$. This implies an identity for correlators:

$$\tilde{G} = O \cdot G \cdot O^T. \quad (66)$$

The transformation has a determinant of modulus one, as a product of simple transformations with this property.

Let us now show that this transformation implies FDR. Consider $R_{xx'}(t' - t) = -\langle \rho(\mathbf{x}, -t) \psi(\mathbf{x}', -t') \rangle$. Under the transformation \mathcal{T}_1 this transforms into $R_{xx'}(t - t') - (1/T) \partial_{t'} C_{xx'}(t - t')$. Thus the equality (66) implies in particular $R_{xx'}(t - t') = R_{xx'}(t' - t) + (1/T) \partial_{t'} C_{xx'}(t - t')$, which is the fluctuation–dissipation relation.

We now show how to linearize the second transformation \mathcal{U} . We introduce the field $\theta = \delta \mathcal{F} / \delta \rho$ and the conjugate one $\hat{\theta}$ to exponentiate the delta function⁵.

The action S is then transformed into an integral of

$$s_x = -\hat{\rho}_x \partial_t \rho_x + T \rho_x (\nabla_x \hat{\rho}_x)^2 + \hat{\theta}_x \left(\theta_x - \frac{\delta \mathcal{F}}{\delta \rho_x} \right) - \rho_x (\nabla_x \hat{\rho}_x) (\nabla_x \theta_x). \quad (67)$$

The associated linear transformation \mathcal{U}_1 is

$$\mathcal{U}_1 : \begin{cases} \hat{\rho}_x \rightarrow -\hat{\rho}_x + \frac{1}{T} \theta_x \\ \hat{\theta}_x \rightarrow \hat{\theta}_x - \frac{1}{T} \partial_t \rho_x. \end{cases} \quad (68)$$

As before we write it as $\tilde{\phi} = O \cdot \phi$, where $\phi = (\rho, \hat{\rho}, \theta, \hat{\theta})^T$ and $\tilde{\phi}(\mathbf{x}, t) = \phi(-\mathbf{x}, t)$. The same identity (66) holds for correlators. Again, the transformation has a determinant of modulus one. The response function can be written as $R_{xx'}(t - t') = \langle \rho(\mathbf{x}, t) \hat{\theta}(\mathbf{x}', t') \rangle$. Using the transformation \mathcal{U}_1 we find again that $\langle \rho(\mathbf{x}, -t) \hat{\theta}(\mathbf{x}', -t') \rangle$ equals $R_{xx'}(t - t') -$

⁵ The usefulness of introducing these two fields when dealing with the BDD field theory was noticed by Chamon and Cugliandolo from a slightly different perspective [28].

$(1/T)\partial_{t'}C_{xx'}(t-t')$, hence FDR. Note that this second field theory is considerably simpler than the previous one because it has fewer fields.

One might wonder how the final result depends on the choice of the linearized field theory. As far as self-consistent perturbation theory and MCT is concerned, we have written the dynamical equations obtained by using

- (a) the completely linearized theory where both symmetries are rendered linear,
- (b) only the fields involved in the transformation \mathcal{T}_1 in addition to ρ ,
- (c) only the fields involved in the transformation \mathcal{U}_1 in addition to ρ and
- (d) writing the terms of the action such that the potential V is in one of the vertices (with the goal of making the link with standard MCT).

At the order of one loop, we have found the same sets of equations for correlation and response functions at long times in all cases. This is not surprising, since the different transformations do not affect the physical fields and change the response fields in the same way. We thus expect this to be valid at all orders. Indeed, as we will show below, one gets closed equations for the dynamical evolutions of correlators involving only the fields ρ and θ . In addition, this tends to confirm that FDR makes the results robust with respect to the choice of extra dynamical variables. Thus in the following we will focus on the simplest theory written above in terms of ρ , $\hat{\rho}$, θ and $\hat{\theta}$ only, the choice of the fields to work with being merely a matter of taste. We refer the reader interested in the theory with all fields introduced above to appendix E.

5.2. Minimal theory preserving fluctuation–dissipation relations and its basic properties

In order to avoid cumbersome calculations, we shall from now on describe a minimal (in the sense of the number of fields) theory for which the symmetry associated with FDR is linear. It is the theory produced by \mathcal{U}_1 . In the following we shall describe the WT relations for correlation functions and self-energies due to time-reversal symmetry. These are particularly useful because they make clear that there are only three independent correlation functions or self-energies out of 16. In the next section we shall write down the general form of the Schwinger–Dyson equations. Using the WT relations we find that these reduce to only three independent integro-differential equations.

Henceforth, we use the density of action (67) and the symmetry \mathcal{U}_1 . We split the density of action into a Gaussian part s_0 and the interaction part s_{INT} :

$$s_{0,x} = -\hat{\rho}_x \partial_t \delta \rho_x - T \rho_0 \hat{\rho}_x \nabla_x^2 \hat{\rho}_x + \hat{\theta}_x \theta_x - \hat{\theta}_x (W \star \delta \rho)_x + \rho_0 \hat{\rho}_x \nabla^2 \theta_x \quad (69)$$

$$s_{\text{INT},x} = T \delta \rho_x (\nabla \hat{\rho}_x)^2 - \delta \rho_x (\nabla_x \hat{\rho}_x) (\nabla_x \theta_x) + T \hat{\theta}_x \sum_{n>1} \frac{1}{n} \left[-\frac{\delta \rho_x}{\rho_0} \right]^n, \quad (70)$$

where

$$(W \star \rho)_x = \int_y W(\mathbf{x} - \mathbf{y}) \rho_y \quad (71)$$

and

$$W(\mathbf{x}) = V(\mathbf{x}) + \frac{T}{\rho_0} \delta(\mathbf{x}). \quad (72)$$

It is useful to remark here that each term of the expansion in n is independently invariant under \mathcal{U}_1 .

One gets the following relations for correlators from identity (66) applied to the case of \mathcal{U}_1 (the limit $t, t' \rightarrow \infty, \tau = t - t'$ is taken):

$$C_{\rho\hat{\rho},xx'}(\tau) = \frac{\Theta(\tau)}{T} C_{\rho\theta,xx'}(\tau) \quad (73)$$

$$C_{\theta\hat{\rho},xx}(\tau) = \frac{\Theta(\tau)}{T} C_{\theta\theta,xx'}(\tau) \quad (74)$$

$$C_{\theta\hat{\theta},xx'}(\tau) = -\frac{1}{T} \frac{\partial}{\partial \tau} (\Theta(\tau) C_{\rho\theta,xx'}(\tau)) \quad (75)$$

$$C_{\rho\hat{\theta},xx'}(\tau) = -\frac{\Theta(\tau)}{T} \frac{\partial}{\partial \tau} C_{\rho\rho,xx'}(\tau) = R_{xx'}(\tau), \quad (76)$$

with the obvious notation $C_{ab,xx'}(t - t') = \langle a_x(t)b_{x'}(t') \rangle$. These identities imply only three independent two-field correlators, namely $C_{\rho\rho,xx'}$, $C_{\rho\theta,xx'}$ and $C_{\theta\theta,xx'}$. Moreover, the causality of the theory, which is insured by the Itô discretization, is explicit. A perturbative proof of causality can be found in appendix B and comes from the causality of the bare propagator, which in Fourier space is

$$\begin{pmatrix} \frac{2T\rho_0k^2}{(\omega^2 + \rho_0^2k^4W(\mathbf{k})^2)} & -\frac{1}{i\omega + \rho_0k^2W(\mathbf{k})} & \frac{2T\rho_0k^2W(\mathbf{k})}{(\omega^2 + \rho_0^2k^4W(\mathbf{k})^2)} & \frac{\rho_0k^2}{i\omega + \rho_0k^2W(\mathbf{k})} \\ -\frac{1}{-i\omega + \rho_0k^2W(\mathbf{k})} & 0 & -\frac{W(\mathbf{k})}{-i\omega + \rho_0k^2W(\mathbf{k})} & 0 \\ -\frac{2T\rho_0k^2W(\mathbf{k})}{(\omega^2 + \rho_0^2k^4W(\mathbf{k})^2)} & -\frac{W(\mathbf{k})}{i\omega + \rho_0k^2W(\mathbf{k})} & -\frac{2T\rho_0k^2W(\mathbf{k})^2}{(\omega^2 + \rho_0^2k^4W(\mathbf{k})^2)} & -\frac{i\omega}{i\omega + \rho_0k^2W(\mathbf{k})} \\ \frac{\rho_0k^2}{-i\omega + \rho_0k^2W(\mathbf{k})} & 0 & \frac{i\omega}{i\omega + \rho_0k^2W(\mathbf{k})} & 0 \end{pmatrix}. \quad (77)$$

The bare propagator also helps to understand the anomaly in (75) where the time derivative of the Heaviside function is present. One can easily see that the bare propagator (77) has the form

$$C_{0,\theta\hat{\theta}}(\mathbf{k}, \omega) = -1 + \text{function of } (\mathbf{k}, \omega). \quad (78)$$

It is easy to prove using (69) and (70) that there are no diagrammatic corrections to the constant part of the above correlator; that is, in the above equation the constant part persists perturbatively and we can write

$$C_{\theta\hat{\theta},xx'}(\tau) = -\frac{\Theta(\tau)}{T} \frac{\partial}{\partial \tau} C_{\rho\theta,xx'}(\tau) - \delta(\tau). \quad (79)$$

One can find (see appendix C) that $C_{\rho\theta,xx'}(0) = T$. The latter, together with (79), allows us to write (75).

Recall that the Schwinger–Dyson equations have the form $G_0^{-1} \cdot G = 1 + D$, with $D = \Sigma \cdot G$. Thus, Σ transforms under \mathcal{U}_1 in the way

$$\tilde{\Sigma} = O^{-T} \cdot \Sigma \cdot O^{-1}, \quad (80)$$

enforcing the following constraints on self-energies:

$$\Sigma_{\hat{\rho}\rho}(\mathbf{k}, \tau) = \frac{\partial}{\partial \tau} \Sigma_{\hat{\theta}\theta}(\mathbf{k}, \tau) \quad (81)$$

$$\Sigma_{\hat{\theta}\rho}(\mathbf{k}, \tau) = -\frac{\Theta(\tau)}{T} \frac{\partial}{\partial \tau} \Sigma_{\hat{\theta}\theta}(\mathbf{k}, \tau) \quad (82)$$

$$\Sigma_{\hat{\rho}\theta}(\mathbf{k}, \tau) = -\frac{\Theta(\tau)}{T} \Sigma_{\hat{\rho}\rho}(\mathbf{k}, \tau) \quad (83)$$

$$\Sigma_{\hat{\rho}\hat{\theta}}(\mathbf{k}, \tau) = T\Theta(-\tau)\Sigma_{\hat{\theta}\theta}(\mathbf{k}, -\tau) - T\Theta(\tau)\Sigma_{\hat{\theta}\theta}(\mathbf{k}, \tau). \quad (84)$$

The other elements ($\Sigma_{\rho\rho}$, $\Sigma_{\theta\rho}$ and $\Sigma_{\theta\theta}$) vanish. One can write the equation for $\Sigma_{\hat{\rho}\rho}(\mathbf{k}, \tau)$ in the following form:

$$\Sigma_{\hat{\rho}\rho}(\mathbf{k}, \tau) = -\frac{1}{T} \frac{\partial}{\partial \tau} \left[\Theta(\tau) \Sigma_{\hat{\rho}\hat{\theta}}(\mathbf{k}, \tau) \right]. \quad (85)$$

As before, we start with the diagrammatic analysis, recovering a set of diagrams which contribute to the δ -function term. Their direct resummation is cumbersome. We used the Schwinger–Dyson equations to identify the δ -function term. Note that as for correlation functions there are only three independent self-energy terms. All the others are either zero or related through WT identities.

5.3. Dynamical equations

In the following section we transcribe the full dynamical equations and leave their derivation for appendix C. There are only three independent equations. Using the transformation laws under \mathcal{U}_1 it is easy to see that there are not more than four independent equations. The proof that one of these equations is trivially verified once the other three are verified is more tricky and is done in appendix D.

Let us first choose $\tau > 0$ and write the equations which will give the time evolution of the correlators at strictly positive time difference. The values at $\tau = 0$ of the correlators will be obtained from the study of the singularities (remember there is a $\delta(\tau)$ in the RHS of (54)).

We first consider $(G_0^{-1} \cdot G - \Sigma \cdot G)_{\hat{\rho}\rho}(\mathbf{k}, \tau) = 0$. The corresponding equation is

$$\begin{aligned} \partial_\tau C_{\rho\rho}(\mathbf{k}, \tau) + \rho_0 k^2 C_{\rho\theta}(\mathbf{k}, \tau) &= \int_0^\tau dt \Sigma_{\hat{\rho}\theta}(\mathbf{k}, \tau - t) C_{\rho\theta}(\mathbf{k}, t) \\ &+ \int_0^\tau dt \Sigma_{\hat{\theta}\theta}(\mathbf{k}, \tau - t) \partial_t C_{\rho\rho}(\mathbf{k}, t). \end{aligned} \quad (86)$$

Now consider $(G_0^{-1} \cdot G - \Sigma \cdot G)_{\hat{\rho}\theta}(\mathbf{k}, \tau) = 0$. The corresponding equation is

$$\begin{aligned} \partial_\tau C_{\rho\theta}(\mathbf{k}, \tau) + \rho_0 k^2 C_{\theta\theta}(\mathbf{k}, \tau) &= -\Sigma_{\hat{\theta}\theta}(\mathbf{k}, \tau) C_{\rho\theta}(\mathbf{k}, 0) \\ &+ \int_0^\tau dt \Sigma_{\hat{\rho}\theta}(\mathbf{k}, \tau - t) C_{\theta\theta}(\mathbf{k}, t) + \int_0^\tau dt \Sigma_{\hat{\theta}\theta}(\mathbf{k}, \tau - t) \partial_t C_{\rho\theta}(\mathbf{k}, t). \end{aligned} \quad (87)$$

Finally, consider $(G_0^{-1} \cdot G - \Sigma \cdot G)_{\hat{\theta}\theta}(\mathbf{k}, \tau) = 0$. The corresponding equation is

$$W(\mathbf{k})C_{\rho\rho}(\mathbf{k}, \tau) - C_{\rho\theta}(\mathbf{k}, \tau) = \frac{1}{T}\Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, 0)C_{\rho\rho}(\mathbf{k}, \tau) + \int_0^\tau dt \Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, \tau - t)C_{\rho\theta}(\mathbf{k}, t) - \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, \tau - t)\partial_t C_{\rho\rho}(\mathbf{k}, t). \quad (88)$$

As discussed at the beginning of this section, there is an extra equation that at first sight might seem independent from the first three (but in fact is not, see appendix D). It comes from $(G_0^{-1} \cdot G - \Sigma \cdot G)_{\hat{\theta}\theta}(\mathbf{k}, \tau) = 0$ and reads

$$W(\mathbf{k})C_{\rho\theta}(\mathbf{k}, \tau) - C_{\theta\theta}(\mathbf{k}, \tau) = \frac{1}{T}\Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, 0)C_{\rho\theta}(\mathbf{k}, \tau) - \frac{1}{T}\Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, \tau)C_{\rho\theta}(\mathbf{k}, 0) + \int_0^\tau dt \Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, \tau - t)C_{\theta\theta}(\mathbf{k}, t) - \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, \tau - t)\partial_t C_{\rho\theta}(\mathbf{k}, t). \quad (89)$$

We stress here that obtaining such consistent structure—which is the *exact one*—for the dynamical equations is far from trivial. Approximations which violate FDR generally lead to a different structure. In such cases, the resulting conclusions about the existence of the glass transition are very suspicious because they might be due to the violation of this fundamental structure.

5.4. Static limit

The static equations are obtained by taking $\tau = 0$ in the SD equations. This is an advantage of the present field theory approach, in which the statics is included in the dynamics, compared to derivations of dynamical equations based on Mori–Zwanzig formalism. We remark that the closed set of static equations cannot be obtained from the evolution equations as written above, as the singularities at $\tau = 0$ have been excluded from the latter. The correct derivation is given in appendix C. The correlators and their derivatives at initial time are

$$C_{\rho\theta}(\mathbf{k}, 0) = T \quad (90)$$

$$\dot{C}_{\rho\rho}(\mathbf{k}, 0^+) = -\rho_0 \mathbf{k}^2 C_{\rho\theta}(\mathbf{k}, 0) = -T\rho_0 \mathbf{k}^2 \quad (91)$$

$$C_{\theta\theta}(\mathbf{k}, 0) = W(\mathbf{k})C_{\rho\theta}(\mathbf{k}, 0) = TW(\mathbf{k}) \quad (92)$$

$$\dot{C}_{\rho\theta}(\mathbf{k}, 0^+) = -W(\mathbf{k})T\rho_0 \mathbf{k}^2 - T\Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, 0) \quad (93)$$

$$C_{\rho\rho}(\mathbf{k}, 0) = \frac{T}{W(\mathbf{k})} + \frac{1}{TW(\mathbf{k})}\Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, 0)C_{\rho\rho}(\mathbf{k}, 0), \quad (94)$$

where \dot{C} stands for $\partial_\tau C$. Equation (94) is obtained from (88) and (107).

5.5. Equation for the non-ergodicity parameter

In MCT, the glass phase is characterized by a non-zero value of the so-called non-ergodicity parameter [37, 38], which signals the existence of infinite-time correlations. The non-ergodicity parameter $f(\mathbf{k})$ is defined as follows:

$$f(\mathbf{k}) = \lim_{t \rightarrow \infty} \frac{C_{\rho\rho}(\mathbf{k}, t)}{S(\mathbf{k})}. \quad (95)$$

In MCT, $f(\mathbf{k})$ is zero in liquid phase and jumps to a non-zero value at the glass transition. Here the equation for the non-ergodicity parameter can be obtained by taking the limit $\tau \rightarrow \infty$ in (86)–(89).

We first give an argument in support of the vanishing of $C_{\rho\theta}(\mathbf{k}, \infty)$ and $C_{\theta\theta}(\mathbf{k}, \infty)$. The physical interpretation is the following. We label by α each ergodic component into which the system may break (if no ergodic to non-ergodic transition occurs then there is only one such component). Then in each component the system decorrelates completely at long times, and we can write

$$\left\langle \rho(\mathbf{k}, 0) \frac{\delta\mathcal{F}}{\delta\rho(-\mathbf{k}, \infty)} \right\rangle = \sum_{\alpha} \mathcal{W}_{\alpha} \langle \rho \rangle_{\alpha} \left\langle \frac{\delta\mathcal{F}}{\delta\rho} \right\rangle_{\alpha}, \quad (96)$$

where each component is characterized by its weight \mathcal{W}_{α} and the static average $\langle \cdot \rangle_{\alpha}$ inside it. However, the average force $\langle \delta\mathcal{F}/\delta\rho \rangle_{\alpha}$ vanishes, and hence $C_{\rho\theta}(\mathbf{k}, \infty)$ too. This also applies to $C_{\theta\theta}(\mathbf{k}, \infty)$. We stress here that the possibility to express infinite time averages in term of static quantities is intimately related to the fact that the dynamical equations are consistent with the dynamical symmetries at equilibrium. Indeed, this guarantees that the asymptotic measure is Gibbsian. Let us add that $C_{\rho\theta}(\mathbf{k}, \infty) = 0$ may also be seen as a direct consequence of $C_{\rho\hat{\rho}}(\mathbf{k}, \infty) = 0$. Furthermore, equation (86) gives $\Sigma_{\hat{\theta}\theta}(\mathbf{k}, \infty) = 0$, and putting (94) into (94) leads to the *exact equation*:

$$\frac{f(\mathbf{k})}{1 - f(\mathbf{k})} = \frac{1}{T^2} S(\mathbf{k}) \Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, \infty). \quad (97)$$

Of course, this equation is too difficult to analyse directly because $\Sigma_{\hat{\theta}\hat{\theta}}$ contains an infinite number of terms that cannot be resummed. The same occurs in the Mori–Zwanzig formalism, where $\Sigma_{\hat{\theta}\hat{\theta}}$ is related to the long time limit of the memory kernel. One has to resort to approximation schemes, the simplest one being to replace $\Sigma_{\hat{\theta}\hat{\theta}}$ by its one-loop expression. This will be discussed in the following.

5.6. Quadratic density functionals

In this section we discuss the particular case in which the density functional is taken to be quadratic in the fields. This corresponds to making a simple quadratic expansion of the entropic part \mathcal{S} and replacing in the FNH case the kinetic term by $\mathbf{g}^2/2\rho_0$, where ρ_0 is the average density. The former case has been already studied in [13] and the latter in [36]. In these works, especially in [13], the issue of preserving FDR has already been investigated and it has been shown that one-loop mode-coupling equations preserve FDR in this case (contrary to the general case).

From our perspective, the reason for this preservation is simple: in these cases the field transformations become linear even without introducing extra fields. Let us consider in detail the BDD case studied by Miyazaki and Reichman [13].

When expanded at order two, the entropic part is written as

$$\mathcal{S}[\rho] \approx \mathcal{S}[\rho_0] - T \int_x \frac{\delta \rho_x^2}{2\rho_0}. \quad (98)$$

Thus the free energy functional $\mathcal{F}[\rho]$ becomes quadratic:

$$\mathcal{F}[\rho] = \frac{1}{2} \int_{x,x'} \rho_x \rho_{x'} W(\mathbf{x} - \mathbf{x}'). \quad (99)$$

The equilibrium measure from (99) is Gaussian:

$$\mathcal{P}[\rho] \propto \exp\left(-\frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\rho(\mathbf{k})\rho(-\mathbf{k})}{S(\mathbf{k})}\right) \quad (100)$$

$$S(\mathbf{k}) = \langle \delta\rho(\mathbf{k})\delta\rho(-\mathbf{k}) \rangle = \frac{T}{W(\mathbf{k})},$$

where we have used $\int d^3\mathbf{x} W(\mathbf{x}) = 0$. This Gaussian form considerably simplifies the structure of the resulting field theory. Indeed, the transformation \mathcal{U} defined earlier becomes linear:

$$\mathcal{U} : \begin{cases} t \rightarrow -t \\ \hat{\rho}_x \rightarrow -\hat{\rho}_x + \beta \int_{x'} W(\mathbf{x} - \mathbf{x}') \rho_{x'}. \end{cases} \quad (101)$$

As stated previously, the use of \mathcal{U} makes it possible to derive an identity between the density–density correlators and the naive response. One can write

$$\langle \rho(\mathbf{x}, t) \hat{\rho}(\mathbf{x}', t') \rangle = -\langle \rho(\mathbf{x}, -t) \hat{\rho}(\mathbf{x}', -t') \rangle + \beta \int d^3\mathbf{x}' W(\mathbf{x} - \mathbf{x}') \langle \rho(\mathbf{x}, -t) \rho(\mathbf{x}', -t') \rangle, \quad (102)$$

or equivalently in Fourier space

$$\mathcal{G}(\mathbf{k}, \omega) + \mathcal{G}(\mathbf{k}, -\omega) = \frac{C(\mathbf{k}, \omega)}{S(\mathbf{k})}. \quad (103)$$

The symmetry \mathcal{U} is linear and thus does not mix correlators of n fields with correlators of $n + 1$ fields, unlike \mathcal{T} . This makes the identity (103) valid at any order in perturbation—self-consistent or not—as explained in section 2. In appendix A, in order to give a concrete view of how diagrams have to be put together to lead to (103), a diagrammatic perturbative proof is given, as well as the derivation of a similar relation between self-energies:

$$\Sigma_{\hat{\rho}\hat{\rho}}(\mathbf{k}, t) = -\frac{1}{S(\mathbf{k})} \Sigma_{\hat{\rho}\rho}(\mathbf{k}, t), \quad t > 0. \quad (104)$$

This identity makes (103) compatible with the SD equations (54). Using these identities, it is possible to show that FDR is preserved by self-consistent one-loop theory and also to all orders in self-consistent perturbation theory as explained in appendix A.

A final remark about the quadratic case is that from (97) it follows that $f(\mathbf{k}) = 0$, as there is no vertex involving $\hat{\theta}$ and hence $\Sigma_{\hat{\theta}\hat{\theta}}$ must vanish on general grounds.

6. Mode-coupling approximation preserving fluctuation dissipation relations

Now, we will carry out the MCA for the BDD field theory that we studied in the previous sections. The corresponding analysis of fluctuating nonlinear hydrodynamics follows the same guidelines (see section 7). It is important to stress the difference between our and previous approaches. We set up a self-consistent diagrammatic expansion that automatically preserves time-reversal symmetry and FDR. As a consequence, the corresponding MCA equations, or any other approximated expression of the self-energies, will do this too.

6.1. Mode-coupling approximation

We first focus on the term $\int_x \hat{\theta}_x (\delta \mathcal{F}[\rho] / \delta \rho_x)$. The entropic part of \mathcal{F} gives a contribution $\int_x \hat{\theta}_x \log(1 + (\delta \rho_x / \rho_0))$ to this term. In order to compute the dynamical partition function one may expand the logarithm in powers of $\delta \rho_x / \rho_0$. This gives an infinite number of vertices of the type $T((-1)^p / p) \hat{\theta}_x (\delta \rho_x / \rho_0)^p$. A crucial point is that, as stressed above, all the terms in the action coming from different powers in the series expansion are independently invariant under the transformation \mathcal{U}_1 . We also can put some couplings in front of the other vertices and carry out truncated expansions at different orders for different vertices. However, as we focus on the one-loop theory, and for simplicity, we shall treat these vertices as if they were all of order T . Then there are two ways of dealing with the vertices arising from the expansion of the logarithm. On one hand one can take into account all these vertices. However, this leads to the sum of an infinite number of terms, whose meaning is not clear, due to the presence of an infinite number of tadpoles which contribute to static vertex renormalization. On the other hand one can truncate the series expansion of the logarithm. However, one needs to go beyond first order in order to take into account nonlinearities. For simplicity we cut the series at order two, but the calculation can be in principle extended to any order.

6.2. Expression of the self-energies

Within the approximation described in the previous paragraph, the self-energies read

$$\Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, t) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{(\mathbf{k} \cdot \mathbf{q})}{\rho_0^2} C_{\rho\theta}(\mathbf{q}, t) C_{\rho\rho}(\mathbf{k} - \mathbf{q}, t) \quad (105)$$

$$\begin{aligned} \Sigma_{\hat{\rho}\hat{\theta}}(\mathbf{k}, t) = \frac{1}{2T} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \{ & (\mathbf{k} \cdot \mathbf{q})^2 C_{\theta\theta}(\mathbf{q}, t) C_{\rho\rho}(\mathbf{k} - \mathbf{q}, t) \\ & + (\mathbf{k} \cdot \mathbf{q})[\mathbf{k} \cdot (\mathbf{k} - \mathbf{q})] C_{\rho\theta}(\mathbf{q}, t) C_{\rho\theta}(\mathbf{k} - \mathbf{q}, t) \} \end{aligned} \quad (106)$$

$$\Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, t) = \frac{T^2}{4\rho_0^4} \int \frac{d^3\mathbf{q}}{(2\pi)^3} C_{\rho\rho}(\mathbf{q}, t) C_{\rho\rho}(\mathbf{k} - \mathbf{q}, t). \quad (107)$$

It is instructive at this stage to see how the MC equations derived by Miyazaki and Reichman [13] in the case of the quadratic expansion can be obtained from the above equations. The simplifications arising from the linearity of \mathcal{U}_1 will then become clear. Using the method used to derive (102), one obtains

$$W(\mathbf{k}) C_{\rho\theta}(\mathbf{k}, \tau) = T C_{\theta\theta}(\mathbf{k}, \tau), \quad (108)$$

and (102) can be simply written as

$$W(\mathbf{k}) C_{\rho\rho}(\mathbf{k}, \tau) = T\Theta(\tau) C_{\rho\theta}(\mathbf{k}, \tau^+) + T\Theta(-\tau) C_{\rho\theta}(\mathbf{k}, (-\tau)^-). \quad (109)$$

Using these two identities one can eliminate $C_{\rho\theta}$ and $C_{\theta\theta}$ in (86). In addition, when the entropy is expanded at quadratic order, all self-energies except $\Sigma_{\rho\theta}$ vanish, and the equation which remains is the one derived and discussed in [13].

6.3. Static limit

The static equations are obtained by taking $\tau = 0$ in equations (105)–(107). Remarkably, it is identical with the one which would be obtained by doing the same MCA on the static theory involving the density functional $\mathcal{F}[\rho]$. Indeed, after truncation of the entropy at cubic order, the density auto-correlator is

$$C_{\rho\rho,yy'}(t=0) = \frac{\int \mathcal{D}\delta\rho \rho(\mathbf{y})\rho(\mathbf{y}')e^{-\beta\mathcal{F}_3[\delta\rho]}}{\int \mathcal{D}\delta\rho e^{-\beta\mathcal{F}_3[\delta\rho]}}, \quad (110)$$

with

$$\mathcal{F}_3[\delta\rho] = \frac{1}{6} \int d^3\mathbf{x} \frac{\delta\rho(\mathbf{x})^3}{\rho_0^3} - \beta \int d^3\mathbf{x} \int d^3\mathbf{x}' W(\mathbf{x} - \mathbf{x}') \delta\rho(\mathbf{x})\delta\rho(\mathbf{x}'), \quad (111)$$

where we have used the fact that $\int d^3\mathbf{x} W(\mathbf{x}) = 0$. It can be easily verified that using the MCA as described above for the correlator defined by (110) gives again (94).

6.4. Equation for the non-ergodicity parameter

Using the one-loop expression (107) in (97), one gets the equation for the non-ergodicity parameter:

$$\frac{f(\mathbf{k})}{1 - f(\mathbf{k})} = \frac{1}{2\rho_0^4} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q})f(\mathbf{k} - \mathbf{q})S(\mathbf{q})S(\mathbf{k} - \mathbf{q})S(\mathbf{k}). \quad (112)$$

Writing this in the form

$$f(\mathbf{k}) = 1 - \frac{1}{m(\mathbf{k})}, \quad (113)$$

one sees that $f(\mathbf{k}) = 0$ and $f(\mathbf{k}) = 1$ are two solutions of (112). However, the second solution, which in addition makes the integral in $m(\mathbf{k})$ diverge, implies $C_{\rho\rho}(\mathbf{k}, t) = 1$ for all t and thus has to be rejected. We stress that in contrast with standard MCT the q -dependence of the vertex (in particular, the absence of the term $(\mathbf{k} \cdot \mathbf{q}c(\mathbf{q}) + \mathbf{k} \cdot (\mathbf{k} - \mathbf{q})c(\mathbf{k} - \mathbf{q}))^2$) does not enforce the convergence of the integral in (112). However, equation (94), which gives the structure factor, is also ill defined, as the integral over \mathbf{q} diverges too. We have solved these equations numerically, putting a cut-off for large values of $|\mathbf{k}|$. We have found a $|\mathbf{k}|$ dependence of $f(\mathbf{k})$ very similar to the usual one of MCT. However, as the cut-off goes to infinity our numerical solution seems to converge toward the solution $f(\mathbf{k}) = 1$ for all \mathbf{k} . However, although there is a clear *physical* cut-off to the description in terms of Langevin equations, the cut-off dependence we found is clearly unphysical. It might be that convergence is only obtained via a further resummation of diagrams that renormalize the vertex. We leave this problem for a future work. Ignoring this cut-off problem (which is in a sense solved in the FNH case) for the time being, our MC equations have the following properties.

- As in the standard MCT, one can derive a schematic theory [37] by assuming that the structure factor is dominated by a single mode $S(\mathbf{k}) = 1 + A\delta(|\mathbf{k}| - k_0)$. Then the schematic equation for the non-ergodicity parameter becomes identical to that of the schematic MCT.
- We have also checked that the dynamical critical properties are the same as for standard MCT.
- The static equations can be reduced to an equation involving the density field only, which is identical to that we can get from the equilibrium density functional by making equivalent approximations. The way the theory has been written thus makes it possible to treat the dynamics in a similar way to the statics.

7. Fluctuating nonlinear hydrodynamics

In this section, we describe the case of FNH. As the derivations follow closely those of BDD, we just give the results.

7.1. Time-reversal symmetry and fluctuation–dissipation relations

This time there are four response functions produced by the extra term $\mathcal{F}_{\text{ext}} = -\int_x (\rho_x \mu_x + \mathbf{g}_x \cdot \mathbf{p}_x)$

$$\begin{aligned} \langle \rho(\mathbf{x}, t) \rangle_\mu &= \langle \rho(\mathbf{x}, t) \rangle_{\mu=0} + \int d^3y \int_s^t dt'' R_{\rho\rho,xx'}(t-t'')\mu(\mathbf{y}, t'') \\ &\quad + \int d^3y \int_s^t dt'' R_{\rho g,xx'}^k(t-t'')p_k(\mathbf{y}, t'') + o(\mu, \mathbf{p}) \end{aligned} \quad (114)$$

$$\begin{aligned} \langle g_i(\mathbf{x}, t) \rangle_\mu &= \langle g_i(\mathbf{x}, t) \rangle_{\mu=0} + \int d^3y \int_s^t dt'' R_{gg,xx'}^{ij}(t-t'')p_j(\mathbf{y}, t'') \\ &\quad + \int d^3y \int_s^t dt'' R_{g\rho,xx'}^i(t-t'')\mu(\mathbf{y}, t'') + o(\mu, \mathbf{p}), \end{aligned} \quad (115)$$

which gives

$$R_{\rho\rho,xx'}(t-, t') = \langle \rho(\mathbf{x}, t) \nabla \cdot (\rho \hat{\mathbf{g}})(\mathbf{x}', t') \rangle \quad (116)$$

$$R_{g\rho,xx'}^k(t-t') = \langle g_k(\mathbf{x}, t) \nabla \cdot (\rho \hat{\mathbf{g}})(\mathbf{x}', t') \rangle \quad (117)$$

$$\begin{aligned} R_{\rho g,xx'}^k(t-t') &= \langle \rho(\mathbf{x}, t) \rho(\mathbf{x}', t') \nabla_k \hat{\rho}(\mathbf{x}', t') \rangle + \langle \rho(\mathbf{x}, t) g_i(\mathbf{x}', t') \nabla_k \hat{g}_i(\mathbf{x}', t') \rangle \\ &\quad + \langle \rho(\mathbf{x}, t) \nabla_i (g_k \hat{g}_i)(\mathbf{x}', t') \rangle - \langle \rho(\mathbf{x}, t) L_{ki} \hat{g}_i(\mathbf{x}', t') \rangle \end{aligned} \quad (118)$$

$$\begin{aligned} R_{gg,xx'}^{kl}(t-t') &= \langle g_k(\mathbf{x}, t) \rho(\mathbf{x}, t) \nabla_l \hat{\rho}(\mathbf{x}', t') \rangle + \langle g_k(\mathbf{x}, t) g_i(\mathbf{x}', t') \nabla_l \hat{g}_i(\mathbf{x}', t') \rangle \\ &\quad + \langle g_k(\mathbf{x}, t) \nabla_i (g_l \hat{g}_i)(\mathbf{x}', t') \rangle - \langle g_k(\mathbf{x}, t) L_{li} \hat{g}_i(\mathbf{x}', t') \rangle. \end{aligned} \quad (119)$$

The transformation of the fields associated with time-reversal is now

$$\mathcal{V} : \begin{cases} t \rightarrow -t \\ \mathbf{g}_x \rightarrow -\mathbf{g}_x \\ \hat{\rho}_x \rightarrow -\hat{\rho}_x + \frac{1}{T} \frac{\delta \mathcal{F}}{\delta \rho_x} \\ \hat{\mathbf{g}}_x \rightarrow \hat{\mathbf{g}}_x - \frac{1}{T} \frac{\delta \mathcal{F}}{\delta \mathbf{g}_x}. \end{cases} \quad (120)$$

This transformation leaves the action invariant up to boundary terms:

$$\int_x \left[\frac{\delta \mathcal{F}}{\delta \mathbf{g}_x} \cdot \partial_t \mathbf{g}_x + \frac{\delta \mathcal{F}}{\delta \rho_x} \partial_t \rho_x + \nabla_i \left(g_{j,x} \frac{\delta \mathcal{F}}{\delta g_{i,x}} \frac{\delta \mathcal{F}}{\delta g_{j,x}} \right) \right]. \quad (121)$$

Following the procedure used for the BDD field theory one can see that density correlation functions are invariant under time-reversal and also derive FDR. We stress again that the naive self-consistent perturbation theory for the FNH violates time-reversal symmetry in the same way it does in the case of the BDD. An extension of the original model is needed providing linear time-reversal symmetry in order to satisfy the symmetry in the perturbation expansion.

7.2. Restoration of time-reversal symmetry in perturbative expansions

In order to make the transformation \mathcal{V} linear we introduce the two additional fields $\theta = \delta \mathcal{F} / \delta \rho$ and $\mathbf{v} = \delta \mathcal{F} / \delta \mathbf{g}$. We are led to add

$$- \int_x \hat{\theta}_x \left[\theta_x - \frac{\delta \mathcal{F}}{\delta \rho_x} \right] - \int_x \hat{\mathbf{v}}_x \cdot \left[\mathbf{v}_x - \frac{\delta \mathcal{F}}{\delta \mathbf{g}_x} \right] \quad (122)$$

to the action, which becomes the integral of

$$s_x = \left\{ -\hat{\rho}_x [\partial_t \rho_x + \nabla_i (\rho_x v_i)] + T \hat{g}_{i,x} L_{ij} \hat{g}_{j,x} \right. \\ \left. - \hat{g}_{i,x} [\partial_t g_{i,x} + \rho_x \nabla_i \theta + \nabla_j (g_{i,x} v_j) + g_{j,x} \nabla_i v_j + L_{ij} v_j] \right. \\ \left. - \hat{\theta}_x \left[\theta_x - \frac{\delta \mathcal{F}}{\delta \rho_x} \right] - \hat{\mathbf{v}}_x \cdot \left[\mathbf{v}_x - \frac{\delta \mathcal{F}}{\delta \mathbf{g}_x} \right] \right\}. \quad (123)$$

The corresponding linear time-reversal transformation is

$$\mathcal{V}_1 : \begin{cases} t \rightarrow -t \\ \mathbf{g}_x \rightarrow -\mathbf{g}_x \\ \mathbf{v}_x \rightarrow -\mathbf{v}_x \\ \hat{\rho}_x \rightarrow -\hat{\rho}_x + \frac{1}{T} \theta_x \\ \hat{\mathbf{g}}_x \rightarrow \hat{\mathbf{g}}_x - \frac{1}{T} \mathbf{v}_x \\ \hat{\theta}_x \rightarrow \hat{\theta}_x + \frac{1}{T} \partial_t \rho_x \\ \hat{\mathbf{v}}_x \rightarrow -\hat{\mathbf{v}}_x - \frac{1}{T} \partial_t \mathbf{g}_x. \end{cases} \quad (124)$$

As before, the action is conveniently split into Gaussian and interaction parts: we expand all the terms in powers of $\delta\rho/\rho_0$.

$$s_{0,x} = -\hat{\rho}_x(\partial_t\delta\rho_x + \rho_0\nabla\cdot\mathbf{v}_x) - \hat{g}_{i,x}[\partial_t g_{i,x} + \rho_0\nabla_i\theta_x + L_{ij}v_{j,x}] \\ + T\hat{g}_{i,x}L_{ij}\hat{g}_{j,x} - \hat{\theta}_x\theta_x + \hat{\theta}_x(W^{\text{FNH}}\star\delta\rho)_x - \hat{\mathbf{v}}_x\cdot\mathbf{v}_x + \frac{1}{\rho_0}\hat{\mathbf{v}}_x\cdot\mathbf{g}_x \quad (125)$$

$$s_{\text{INT},x} = -\hat{\rho}_x\nabla\cdot(\delta\rho_x\mathbf{v}_x) - \delta\rho_x\hat{\mathbf{g}}_x\cdot\nabla\theta_x - \hat{g}_{i,x}\nabla_j(g_{i,x}v_{j,x}) \\ - \hat{g}_{i,x}g_{j,x}\nabla_iv_{j,x} - \sum_{p>1}\frac{(-1)^p T}{p} \frac{\hat{\theta}_x}{m} \frac{\delta\rho_x^p}{\rho_0^p} \\ + \sum_{n>0}(-1)^n \left[(\hat{\mathbf{v}}_x\cdot\mathbf{g}_x)\delta\rho_x + n\hat{\theta}_x \frac{\mathbf{g}_x^2}{2} \right] \frac{\delta\rho_x^{n-1}}{\rho_0^{n+1}}, \quad (126)$$

with

$$W^{\text{FNH}}(\mathbf{x}) = \frac{T}{m} \left[\frac{1}{\rho_0}\delta(\mathbf{x}) - \frac{c(\mathbf{x})}{m} \right]. \quad (127)$$

This expansion in powers produces two series of vertices, each being independently invariant (also order by order) under the field transformation \mathcal{V}_1 .

Different equalities between correlators (and self-energies) result as consequences of the use of \mathcal{V}_1 as for the BDD field theory. In particular, using that the response function is the correlation between ρ and $\hat{\theta}$ one gets FDR between correlation and response. The set of all relations and the simplified dynamical equations are presented in appendix F.

7.3. Static limit

The analysis of singularities of the SD equations at short time difference gives

$$C_{\rho\theta}(\mathbf{k}, 0) = T \quad (128)$$

$$C_{gv}(\mathbf{k}, 0) = T \quad (129)$$

$$W(\mathbf{k})C_{\rho\rho}(\mathbf{k}, 0) = T + \frac{1}{T}\Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, 0)C_{\rho\rho}(\mathbf{k}, 0) \quad (130)$$

$$C_{\theta\theta}(\mathbf{k}, 0) = TW(\mathbf{k}) \quad (131)$$

$$\frac{1}{\rho_0}C_{gg}(\mathbf{k}, 0) = T + \frac{1}{T}\Sigma_{\hat{v}\hat{v}}(\mathbf{k}, 0)C_{gg}(\mathbf{k}, 0) \quad (132)$$

$$C_{vv}(\mathbf{k}, 0) = \frac{T}{\rho_0}. \quad (133)$$

7.4. Equation for the non-ergodicity parameter

We now focus on the equation for the non-ergodicity parameter. As for BDD, due to long time decorrelation inside ergodic components, only $C_{\rho\rho}$, $C_{\rho g^\perp}$ and $C_{g^\perp g^\perp}$ (where \mathbf{g}^\perp is the transverse current) can have a non-zero limit when $\tau \rightarrow \infty$. This is confirmed by the analysis of the dynamical equations given in appendix F. We however do not expect frozen currents, and thus it is quite reasonable to assume that $C_{\rho g}(\mathbf{k}, \infty)$ and $C_{g g}(\mathbf{k}, \infty)$ do vanish. Since at least one of these appear in any diagram of $\Sigma_{\hat{\rho}\hat{\rho}}(\mathbf{k}, \infty)$, $\Sigma_{\hat{\rho}\hat{v}}(\mathbf{k}, \infty)$, $\Sigma_{\hat{g}\hat{g}}(\mathbf{k}, \infty)$, $\Sigma_{\hat{g}\hat{v}}(\mathbf{k}, \infty)$ and $\Sigma_{\hat{v}\hat{v}}(\mathbf{k}, \infty)$, these self-energies also vanish. As a consequence, one obtains the non-perturbative equation for the non-ergodicity parameter using (F.54) and its static limit (130):

$$\frac{f(\mathbf{k})}{1 - f(\mathbf{k})} = \frac{1}{T^2} S(\mathbf{k}) \Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, \infty). \quad (134)$$

This structure is identical to the one found for BDD and is *an exact result*. Any general approximation (one loop, two loops, etc) for the self-energy on the right-hand side leads to a nonlinear equation on $f(\mathbf{k})$. As we will discuss in detail later, previous works have obtained very different structures because in those cases time-reversal symmetry was violated. This may be very dangerous because it can generate spurious results, as is indeed the case for the cut-off of the transition. For example, in the analysis of [9] this strongly modifies the general structure of the Schwinger–Dyson equations and implies that the non-ergodic parameter has to vanish. However, this has nothing to do with the physical mechanism that cut off the MCT transition and is just an artefact of having violated the time-reversal symmetry.

7.5. Mode-coupling approximation

We restrict ourselves in this section to an approximation similar to the one used for BDD, truncating the vertex series in $\delta\rho/\rho_0$ at the lowest order:

$$\begin{aligned} s_{\text{INT},x} = & -\hat{\rho}_x \nabla \cdot (\delta\rho_x \mathbf{v}_x) - \delta\rho_x \hat{\mathbf{g}}_x \cdot \nabla \theta_x - \hat{g}_{i,x} \nabla_j (g_{i,x} v_{j,x}) \\ & - \hat{g}_{i,x} g_{j,x} \nabla_i v_{j,x} - \hat{\mathbf{v}}_x \cdot \mathbf{g}_x \frac{\delta\rho_x}{\rho_0^2} - \hat{\theta}_x \frac{\mathbf{g}_x^2}{2\rho_0^2} - \frac{T}{2m} \hat{\theta}_x \frac{\delta\rho_x^2}{\rho_0^2}. \end{aligned} \quad (135)$$

We do not write the set of all equations at the one-loop level. We just remark that as in the BDD case the static correlation functions can be obtained from the dynamical equations. They coincide with the ones obtained making the analogous approximation on the static theory. At the level of one loop, the expression of $\Sigma_{\hat{\theta}\hat{\theta}}$ reduces to the one previously obtained for BDD (up to a multiplication by the mass). We want to stress that it is a coincidence which is absent when higher order schemes are considered. We believe that this is due to the fact that the static effective free energies are different in the two field theories.

As a consequence, the MCA equation for the non-ergodic parameter is the same as the one obtained for BDD and the previous remarks also apply to this case. We have also checked that the critical long-time behaviour of the correlation function close to the MCT transition is the standard MCT one and coincides with the one obtained in the BDD case. Note that in contrast to BDD there is now a cut-off to regularize the integral over \mathbf{q} because FNH is not valid on a short length scale.

8. Relation with previous works and issues related to mode-coupling theory

In this section we would like to put our work in the context of field theory derivations of MCT and discuss what can be learned from the non-perturbative structure of the equations which we derived, and from the resulting mode-coupling equations.

The first issue that we want to discuss is the field theory derivation of mode-coupling equations. One can find in the literature [39, 9] different field theory derivations of the original full k -dependent mode-coupling equations introduced and studied by Götze *et al* [4, 5]. All these derivations are not consistent because they assume some identities (related to time-reversal symmetry) that are incompatible with the same self-consistent equations as used to derive MCT. Indeed, the Kawasaki derivation of MCT using field theory [39] starts from a BDD field theory (in which the potential is replaced by a term proportional to the static direct correlation function). Kawasaki computed $\Sigma_{\hat{\rho}\hat{\rho}}$ for the original (without extra fields) BDD theory and *assumed* that $\Sigma_{\hat{\rho}\hat{\rho}}$ was related to it by a symmetry transformation similar to FDR for correlation functions:

$$\Sigma_{\hat{\rho}\hat{\rho}}(\tau) = -\frac{1}{T}\partial_\tau\Sigma_{\hat{\rho}\hat{\rho}}(\tau). \quad (136)$$

Looking at the whole set of one-loop equations it is easy to see that this relation is not verified. Actually, the structure of the Schwinger–Dyson equations is such that if this relation was verified there would have been a fluctuation–dissipation relation between the correlation function and $C_{\rho\hat{\rho}}(t)$, which *is not* the response function as discussed previously.

We now focus on the Das–Mazenko [9] field theory derivation of MCT. In the original papers and all subsequent ones there is a linear relationship that has been assumed to hold between $C_{\rho\rho}$ and $C_{\rho\hat{\rho}}$ (see equation (6.62) of [9]). This relation, that has been used to close the equations on $C_{\rho\rho}$, is true only in the hydrodynamic limit (long time- and length scales) or for a purely quadratic free energy density functional. Instead, time-reversal symmetry generically implies a more complicated identity that is the generalization of (36) to the FNH case:

$$\beta \left\langle \rho(\mathbf{x}, t) \frac{\delta \mathcal{F}[\rho, \mathbf{g}]}{\delta \rho(\mathbf{x}', t')} \right\rangle = \Theta(t - t') \langle \rho(\mathbf{x}, t) \hat{\rho}(\mathbf{x}', t') \rangle + \Theta(t' - t) \langle \rho(\mathbf{x}', t') \hat{\rho}(\mathbf{x}, t) \rangle, \quad (137)$$

where \mathcal{F} is the free energy functional introduced for the FNH field theory. Forcing the relation between $C_{\rho\rho}$ and $C_{\rho\hat{\rho}}$ is useful to close the equations, but (1) it is very dangerous as already discussed in section 7.4 and as will be explained later and (2) it is inconsistent with time-reversal symmetry, that imposes a different relation, equation (137).

In contrast to these two approaches, our derivation of MC equations preserves time-reversal symmetry because it is constructed upon a self-consistent expansion that automatically preserves this symmetry. This is important for the self-consistency of the theory and it is crucial in order to study off-equilibrium dynamics as discussed in the introduction. At one loop, it leads to a vertex with a different k -dependence than usual MCT. This is problematic for two reasons: first the good quantitative results of MCT depend crucially on the vertex k -dependence, and second our vertex leads to a strange and cut-off-dependent behaviour or the mode-coupling equations. Our conclusion is that more refined (e.g. higher loops) self-consistent approximations have to be explored in such a way as to obtain MCA that preserves time reversal symmetry and is at the same time

quantitatively successful such as the one developed in [6]. The investigation of this issue is left for future studies.

An interesting consequence of our derivation and our equations concerns extended MCT and the cut-off of the transition. Using field theory [9] and also projection operator formalism [6], equations beyond standard MCT have been obtained. These are called extended MCT equations and have been conjectured to contain the ‘hopping or activated’ effects that destroy the MCT transition. A striking result is that these ‘hopping or activated’ effects are present only if there is a density–current coupling. As a direct consequence, they would be expected for particles evolving under Newtonian dynamics (and thus described by FNH) and not for particles evolving under Brownian dynamics (and thus described by BDD). Recent numerical simulations have shown [40] that quite the contrary happens: the same effects cutting off the transition are present for Brownian or Newtonian particles. It has also been believed for some time that ideal MCT transition might occur (i.e. is not avoided) for Brownian dynamics, but it is now clear that it is impossible since there are mathematical theorems proving [41] that the self-diffusion coefficient of hard spheres evolving under Brownian dynamics never vanishes at any finite temperature or chemical potential.

As a conclusion the mode coupling transition is expected to be always replaced by a cross-over. The crucial question is what is the physical mechanism that cut off the transition, how to capture it in an analytical theory and whether it is the same or not for Newtonian and Brownian dynamics. In our derivation we find at long times the same transition, and at one-loop the same equations, in the Newtonian and Brownian cases, hence there is no sign that the cut-off of the transition is due to the presence of coupling to currents. This conclusion remains valid at any finite order in the self-consistent perturbation theory (one loop, two loops, etc). In practice, any approximation will provide an expression for $\Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, \infty)$ as a function of $f(\mathbf{k})$ that has to be plugged into equation (134) or its BDD counterpart. There is no general argument implying that the resulting nonlinear equation on $f(\mathbf{k})$ cannot have a non-vanishing solution, and indeed one expects that the transition present at one loop carries on at higher loops. This has indeed been verified in toy models [42] such as a Langevin particle inside a double-well potential. Our conclusion is therefore that the cut-off mechanism is due to non-perturbative effects that cannot be captured at any finite order of the self-consistent perturbation theory.

This is in clear contradiction with previous field theoretical works that found a cut-off mechanism for one-loop self-consistent expansion. In the following we unveil, using exact results, that that mechanism was only due to the approximations that were used and that violated explicitly violation of time-reversal symmetry.

The evidence for an avoided transition in the field theory derivation comes mainly from two works. Das and Mazenko [9] found that when all one-loop diagrams are considered the transition is cut off. However, this was due to the relation they assumed between the two correlation functions $C_{\rho\rho}$ and $C_{\rho\hat{\rho}}$. In fact, as we have explained earlier, $C_{\rho\hat{\rho}}$ cannot have a plateau. Therefore, by forcing this relation one kills artificially any possibility of having a glass transition. Another way to put it, as discussed in section 7.4, is that this relationship alters completely the non-perturbative structure of Schwinger–Dyson equations. Whereas it is not possible to conclude anything about the cut-off of the transition just looking at the general equation for the non-ergodic parameter, see section 7.4, the relationship assumed in [9] between $C_{\rho\rho}$ and $C_{\rho\hat{\rho}}$ plus the general form of the Schwinger–Dyson equations

imply that the non-ergodic parameter has to be zero. Indeed, consider one of the exact Schwinger–Dyson equations derived in appendix F,

$$\begin{aligned} \partial_\tau C_{\rho\rho}(\mathbf{k}, \tau) - i\rho_0 \mathbf{k} C_{v\rho}(\mathbf{k}, \tau) &= \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{\rho}\hat{\theta}}(\mathbf{k}, \tau - t) \partial_t C_{\rho\rho}(\mathbf{k}, t) \\ &+ \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{\rho}\hat{v}}(\mathbf{k}, \tau - t) \partial_t C_{g\rho}(\mathbf{k}, t) - \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{\rho}\hat{\rho}}(\mathbf{k}, \tau - t) C_{\theta\rho}(\mathbf{k}, t) \\ &- \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{\rho}\hat{g}}(\mathbf{k}, \tau - t) C_{v\rho}(\mathbf{k}, t) \end{aligned} \quad (138)$$

and the exact fluctuation–dissipation relation derived in appendix F,

$$C_{\rho\hat{\rho}}(\mathbf{k}, \tau) = \frac{\Theta(\tau)}{T} C_{\rho\theta}(\mathbf{k}, \tau). \quad (139)$$

The linear relationship assumed in [9] between $C_{\rho\rho}$ and $C_{\rho\hat{\rho}}$ implies that if $C_{\rho\rho}$ has an infinite plateau at the transition so does $C_{\rho\hat{\rho}}$ and, using the FDR relation (139), $C_{\rho\theta}$. However, a plateau of $C_{\rho\theta}$ is incompatible with the exact equation (138). Indeed, integrating equation (138) over τ between zero and infinity, we would get an infinite right-hand side and a finite right-hand side⁶. Therefore, one would conclude that the first hypothesis, i.e. an infinite plateau for $C_{\rho\rho}$, has to be wrong. However, this is *only due* to having forced a relation between $C_{\rho\rho}$ and $C_{\rho\hat{\rho}}$! Thus, the cut-off of the transition found in [9] is spurious. The relation between $C_{\rho\rho}$ and $C_{\rho\hat{\rho}}$ is valid only on hydrodynamic length and time-scales.

We finally remark that other arguments complementary to ours have been given recently [35], in favour of rejection of the cut-off mechanism derived within the extended MCT provided in [9].

The other work that is cited as supporting evidence for the cut-off of the transition is the one of Schmitz *et al* [36]. However, they considered a purely quadratic free energy functional. We have seen in our derivation that the transition comes from the density nonlinearity term in the free-energy functional and therefore it is absent in the quadratic case. In [36] FDR is respected and the corresponding equations are a particular case of the ones we derived. The fact that there is no transition is the natural consequence that theories with quadratic free energy functional are *too simple* to lead to any MCT transition in the FNH as well as in the BDD case. In fact, in contrast to the case of non-quadratic functionals, there is no transition at any finite order in the self-consistent perturbation theory simply because the self-energy $\Sigma_{\hat{\theta}\hat{\theta}}$ in equation (134) is identically zero! Note that in the Brownian dynamic case this has been already found, although in a different way, by Miyazaki and Reichman [13].

9. Conclusion

In this paper we analysed field theories for the dynamics of glass forming liquids focusing on their symmetry properties. In particular, we have shown that straightforward perturbation theories generically do not preserve time-reversal symmetry. We have found

⁶ Only the third term on the right-hand side would give an infinite contribution since $C_{v\rho}$ cannot have a plateau. This can be found by inspection studying the other equations or just on physical grounds, because at the glass transition amorphous circulating currents are not expected.

that time-reversal symmetry is related to a *nonlinear* field transformation that leaves the action invariant. The nonlinearity is the source of the problems. Introducing some auxiliary fields we have shown how to set up perturbation theories that preserve time-reversal symmetry and hence FDR.

Furthermore, we have critically revisited the field theory derivations of MCT showing that they always assumed some relations which are actually incompatible with the self-consistent equations. Our derivation is completely consistent and preserves FDR but leads to a different vertex than in the usual MCT. This leads to similar qualitative results but clearly different quantitative results. Furthermore, strange and spurious divergences appear if we do not put an ultraviolet cut-off on momentum integration. We leave for future work an accurate investigation of this puzzle that will certainly need the introduction of some kind of diagram resummation.

We have also reconsidered the evidence for the cut-off of the MCT transition when a coupling between density and current is present. We have shown using exact results that there is no obvious cut-off mechanism which acts order by order in self-consistent perturbation theory. From this perspective there is no difference between Brownian and Newtonian dynamics. Whether or not density–current couplings are present, we have found the same formal structure for the equations on the non-ergodic parameter in the glass phase. Actually, at one loop the corresponding equations for the non-ergodic parameter are identical for BDD and FNH. This structure is fundamentally different from the one previously obtained [9] that suggested a cut-off mechanism only for Newtonian dynamics; the latter was due to the assumption of a linear relationship between correlation functions, that although valid in the hydrodynamic limit is not verified in general, and that, via the general structure of the Schwinger–Dyson equations, forces the non-ergodic parameter to be zero. The correct relation, equation (137), has no influence on the existence of the MCT transition. We conclude that the MCT transition has to be cut off by non-perturbative mechanisms. A very recent work [47] shows by a schematic approximation that once any factorization approximation is avoided, i.e. at a non-perturbative level, the MCT transition indeed disappears.

Despite the problems—due to excessive simplicity of approximations such as the MCA considered here—related to our vertex we think that our approach is promising since it automatically connects statics to dynamics in a precise way. It remains the issue of finding appropriate approximations for the self-energies. However, one has not to worry without about compatibility with time-reversal symmetry, which is guaranteed by the form (86)–(89) of the Schwinger–Dyson equations. This step was missing in the previous attempts to derive MCT equations from field theory, where approximations were made too early. Thus, it opens the way to dynamical equivalents of the self-consistent schemes developed in liquid theory [43, 44]. This was already initiated in a pioneering work [45], where the relationship between dynamic and static theory for the glass transition was investigated.

Finally, we remark that the identification of field transformations associated with time-reversal symmetry may help developing other non-perturbative approaches. For instance, in the non-perturbative renormalization group [46] approach, an ansatz for the effective dynamical action has to be made. The choice among different possible ansätze is very wide but can be drastically reduced by the use of symmetries. We hope that the analysis of the symmetries performed in this work may be helpful in applying NPRG techniques to the glass transition problem.

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Appendix A. Perturbative proofs for the quadratic theory

A.1. Proof of (103)

In this appendix a perturbative proof of (104) will be given (in the quadratic expansion of the free energy). The identity (103) will follow from the relation between self-energy and connected two-point functions:

$$G^{-1} - G_0^{-1} = \Sigma. \quad (\text{A.1})$$

Here the order n in perturbation theory consists of diagrams with n vertices $B[\rho, \hat{\rho}]$, the vertex including two parts $B = B_1 + B_2$, B_1 and B_2 being the noise and W vertices respectively.

In this appendix, we will always have $t > 0$. As already said above, the splitting of ρ into ρ_0 and $\delta\rho$ leads to the splitting of both noise and interaction vertices into two parts in the dynamical action. The first one, linear in ρ_0 , is quadratic and can be seen as a 'mass' insertion, whereas the second one is cubic and gives the vertices used in the perturbative expansions. However, it turns out that both parts (the one with ρ_0 and the other) are independently invariant under the symmetry \mathcal{U} . Thus there are two strategies to carry out perturbative expansions. One consists in keeping the quadratic term proportional to ρ_0 in the bare propagator, while the other consists of treating it as a mass insertion. For the purpose of this appendix, both strategies are almost equivalent, and thus we will follow the first one, which generates fewer diagrams. In this case, the bare propagator is given by (48). At the bare level (103) is verified, that is

$$\mathcal{G}_0(\mathbf{k}, t) + \mathcal{G}_0(\mathbf{k}, -t) = \beta W(\mathbf{k}) C_0(\mathbf{k}, t). \quad (\text{A.2})$$

Causality implies that for $t > t'$ the terms of order $n + 2$ ($n \geq 0$) in the self-energies may be written formally as $\Sigma = \Theta_{\text{1PI}}$, with

$$\begin{aligned} \Theta_{\hat{\rho}\hat{\rho}}^{(n+2)}(\mathbf{k}, t - t') &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \int \frac{d^3\mathbf{q}'}{(2\pi)^3} \delta(\mathbf{p} + \mathbf{p}' - \mathbf{k}) \delta(\mathbf{q} + \mathbf{q}' - \mathbf{k}) \\ &\times \Gamma(\mathbf{p}, \mathbf{k}, \mathbf{p}') \{ 4\Gamma(\mathbf{q}, \mathbf{k}, \mathbf{q}') \langle \delta\rho(\mathbf{p}, t) \delta\rho(\mathbf{p}', t) B[\delta\rho, \hat{\rho}]^n \hat{\rho}(\mathbf{q}, t') \delta\rho(\mathbf{q}', t') \rangle \\ &- 2T\mathbf{q} \cdot \mathbf{q}' \langle \delta\rho(\mathbf{p}, t) \delta\rho(\mathbf{p}', t) B[\delta\rho, \hat{\rho}]^n \hat{\rho}(\mathbf{q}, t') \hat{\rho}(\mathbf{q}', t') \rangle \} \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} \Theta_{\hat{\rho}\hat{\rho}}^{(n+2)}(\mathbf{k}, t-t') &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \int \frac{d^3\mathbf{q}'}{(2\pi)^3} \delta(\mathbf{p} + \mathbf{p}' - \mathbf{k}) \delta(\mathbf{q} + \mathbf{q}' - \mathbf{k}) \\ &\times \Gamma(\mathbf{p}, \mathbf{k}, \mathbf{p}') \{ 2\Gamma(\mathbf{k}, \mathbf{q}, \mathbf{q}') \langle \delta\rho(\mathbf{p}, t) \delta\rho(\mathbf{p}', t) B[\delta\rho, \hat{\rho}]^n \delta\rho(\mathbf{q}, t') \delta\rho(\mathbf{q}', t') \rangle \\ &- 4T\mathbf{k} \cdot \mathbf{q} \langle \delta\rho(\mathbf{p}, t) \delta\rho(\mathbf{p}', t) B[\delta\rho, \hat{\rho}]^n \hat{\rho}(\mathbf{q}, t') \delta\rho(\mathbf{q}', t') \rangle \}. \end{aligned} \quad (\text{A.4})$$

The subscript 1PI indicates that only one-particle irreducible diagrams are considered. In term of diagrams

$$\begin{aligned} \Theta_{\hat{\rho}\hat{\rho}} &= \text{Diagram 1} + \text{Diagram 2} \\ \Theta_{\hat{\rho}\hat{\rho}} &= \text{Diagram 3} + \text{Diagram 4} \end{aligned}$$

where the D_i are obtained by doing all possible contractions in the averages of (A.3) and (A.4). The set of diagrams with m vertices will be denoted $\Omega^{(m)}$. The exponent $^{(m)}$ will in general design classes of diagram of order m exactly, while the subscript $^{(m)}$ will include all orders until m . It is convenient here to consider the subset of 1PI diagrams of order m as the subset of all diagrams of order m , minus the classes of connected but not 1PI diagrams (subset $A^{(m)}$) and of disconnected diagrams (subset $B^{(m)}$). We will then proceed as follow. First we will prove that the components of Θ verify an identity similar to (104), and that the same holds when restricted to class $B^{(n)}$. Then we will prove simultaneously that (104) is true at order m and also holds when restricted to class $A^{(m)}$. The sum of all diagrams belonging to class $A^{(m)}$ is noted $\Lambda^{(m)}$.

Applying \mathcal{U} in all terms with $\hat{\rho}$ explicitly written in (A.3) and (A.4), one gets

$$\begin{aligned} \Theta_{\hat{\rho}\hat{\rho}}^{(n+2)}(\mathbf{k}, t-t') &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \int \frac{d^3\mathbf{q}'}{(2\pi)^3} \delta(\mathbf{p} + \mathbf{p}' - \mathbf{k}) \delta(\mathbf{q} + \mathbf{q}' - \mathbf{k}) \\ &\times \Gamma(\mathbf{p}, \mathbf{k}, \mathbf{p}') \langle \delta\rho(\mathbf{p}, -t) \delta\rho(\mathbf{p}', -t) B[\delta\rho, \hat{\rho}]^n \delta\rho(\mathbf{q}, -t') \delta\rho(\mathbf{q}', -t') \rangle \\ &\times \left(4\beta \frac{\mathbf{q} \cdot \mathbf{k} W(\mathbf{k}) + \mathbf{q} \cdot \mathbf{q}' W(\mathbf{q}')}{2} W(\mathbf{q}) - 2\beta \mathbf{q} \cdot \mathbf{q}' W(\mathbf{q}) W(\mathbf{q}') \right) \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} \Theta_{\hat{\rho}\hat{\rho}}^{(n+2)}(\mathbf{k}, t-t') &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \int \frac{d^3\mathbf{q}'}{(2\pi)^3} \delta(\mathbf{p} + \mathbf{p}' - \mathbf{k}) \delta(\mathbf{q} + \mathbf{q}' - \mathbf{k}) \\ &\times \Gamma(\mathbf{p}, \mathbf{k}, \mathbf{p}') \left(2 \frac{\mathbf{k} \cdot \mathbf{q} W(\mathbf{q}) + \mathbf{k} \cdot \mathbf{q}' W(\mathbf{q}')}{2} \right. \\ &\times \langle \delta\rho(\mathbf{p}, t) \delta\rho(\mathbf{p}', t) B[\delta\rho, \hat{\rho}]^n \delta\rho(\mathbf{q}, t') \delta\rho(\mathbf{q}', t') \rangle \\ &\left. - 4\mathbf{k} \cdot \mathbf{q} W(\mathbf{q}) \langle \delta\rho(\mathbf{p}, -t) \delta\rho(\mathbf{p}', -t) B[\delta\rho, \hat{\rho}]^n \delta\rho(\mathbf{q}, -t') \delta\rho(\mathbf{q}', -t') \rangle \right). \end{aligned} \quad (\text{A.6})$$

All correlation functions involving $\hat{\rho}(\mathbf{q}, -t')$ or $\hat{\rho}(\mathbf{q}', -t')$ have vanished due to causality. Indeed, they contribute to $\Theta_{\hat{\rho}\hat{\rho}}(\mathbf{k}, t' - t) = \Theta_{\hat{\rho}\hat{\rho}}(\mathbf{k}, t - t')$, which vanishes for $t > t'$. Finally, using the fact that correlators involving only $B[\delta\rho, \hat{\rho}]^n$ and ρ are invariant under time reversal, the identity (104) is found at order $m = n + 2$.

The same steps can be followed straightforwardly for disconnected diagrams, as it amounts to splitting $B[\delta\rho, \hat{\rho}]^n$ into several powers of $B[\delta\rho, \hat{\rho}]$, which contribute to different connected components. The crucial point here is the invariance of each term $B[\delta\rho, \hat{\rho}]$ under \mathcal{U} .

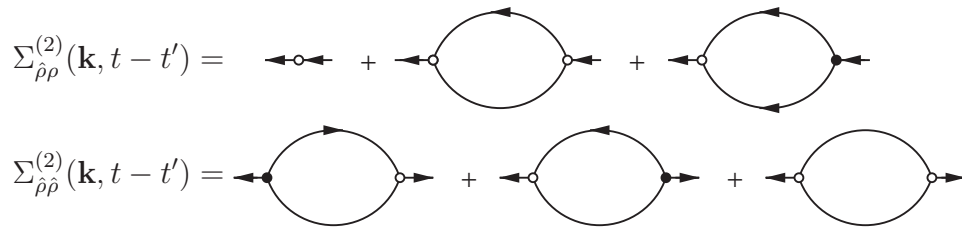
Now we prove that diagrams of order n verify that the identity (104) holds at order n and also within class $A^{(n)}$. To do so, we proceed by induction. Assuming that $\Sigma^{(p)}$ and $\Lambda^{(p)}$ verify (104) for all $p \leq n - 2$, we will show that $\Sigma^{(n)}$ and $\Lambda^{(n)}$ verify (104).

The starting point is however $n = 2$. At this order,

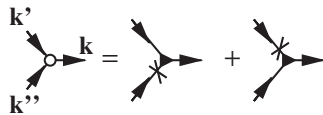
$$\Theta^{(2)}(\mathbf{k}, t - t') = \Sigma^{(2)}(\mathbf{k}, t - t') \tag{A.7}$$

$$\Lambda^{(2)}(\mathbf{k}, t - t') = 0. \tag{A.8}$$

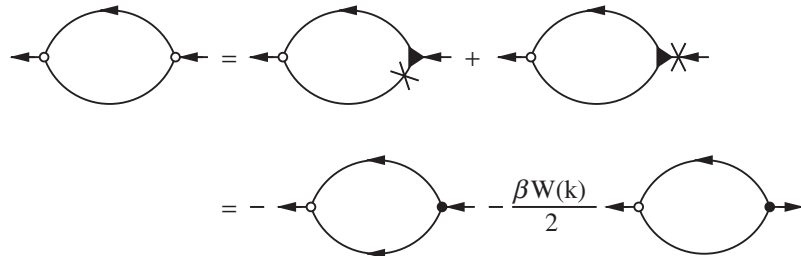
The corresponding diagrams are



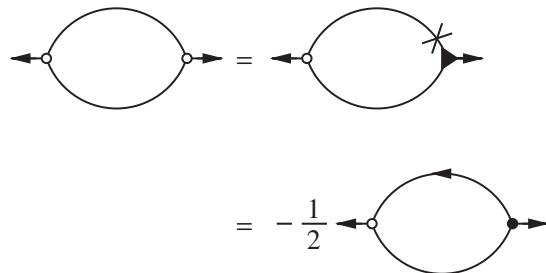
For clarity, let us split the vertex made with W into two parts:



The cross on a line with impulsion \mathbf{q} stands for $W(\mathbf{q})$ and the black triangle stands for $\mathbf{k} \cdot \mathbf{q}$, \mathbf{k} being the momentum of the outgoing arrow. Then we have



and



Putting this all together leads to

$$\Sigma_{(p)\hat{\rho}\rho}(\mathbf{k}, t - t') = -\beta W(\mathbf{k}) \Sigma_{(p)\hat{\rho}\rho}(\mathbf{k}, t - t') \tag{A.9}$$

for $p = 2$.

Now let us assume that $n \geq 4$ is even and (A.9) holds for any even p less than $n - 2$. We have in frequency

$$\Lambda^{(n)}(\mathbf{k}, \omega) = \sum_{q=2}^{n-2} (\Sigma^{(q)} G_0 \Sigma^{(n-q)}) (\mathbf{k}, \omega) + (\Sigma^{(q)} G_0 \Lambda^{(n-q)}) (\mathbf{k}, \omega), \tag{A.10}$$

where we have set $\Sigma^{(r)} = \Lambda^{(r)} = 0$ for odd r due to the absence of tadpoles. Diagrammatically,

$$\Lambda = \Sigma - \Sigma + \Sigma - \Lambda$$

Let us consider $U = A \mathcal{G}_0 B$, where $A = \Sigma^{(q)}$ and $B = \Sigma^{(n-q)}$ or $B = \Lambda^{(n-q)}$. We will show that U verifies (A.9), which will be enough to show that $\Lambda^{(n)}$ also verifies it due to (A.10). We have

$$\begin{aligned} U_{\hat{\rho}\rho}(\mathbf{k}, \omega) &= A_{\hat{\rho}\rho}(\mathbf{k}, \omega) R_0(\mathbf{k}, \omega) B_{\hat{\rho}\rho}(\mathbf{k}, \omega) \\ U_{\hat{\rho}\hat{\rho}}(\mathbf{k}, \omega) &= A_{\hat{\rho}\hat{\rho}}(\mathbf{k}, \omega) R_0^*(\mathbf{k}, \omega) B_{\rho\hat{\rho}}(\mathbf{k}, \omega) + A_{\hat{\rho}\rho}(\mathbf{k}, \omega) C_0(\mathbf{k}, \omega) B_{\rho\hat{\rho}}(\mathbf{k}, \omega) \\ &\quad + A_{\hat{\rho}\rho}(\mathbf{k}, \omega) R_0^*(\mathbf{k}, \omega) B_{\hat{\rho}\hat{\rho}}(\mathbf{k}, \omega). \end{aligned} \tag{A.11}$$

However, using the hypothesis made at order $n - 2$, A and B verify (A.9). In addition, \mathcal{G}_0 verifies (A.2) (note the absence of the minus sign compared to (104)). Thus, using (A.2), one can express the terms with indices $\hat{\rho}\hat{\rho}$ in the right-hand side of (A.11) in terms of those with indices $\hat{\rho}\rho$ and $\rho\hat{\rho}$. Expressing C_0 in terms of G_0 and G_0^* , (A.11) gives

$$U_{\hat{\rho}\hat{\rho}}(\mathbf{k}, \omega) = -\frac{U_{\rho\hat{\rho}}(\mathbf{k}, \omega) + U_{\hat{\rho}\rho}(\mathbf{k}, \omega)}{\beta W(\mathbf{k})}. \tag{A.12}$$

Let us rewrite the above induction in such a way that the link with similar diagrammatic proofs in the case of additive noise is clearer (see [33] for a diagrammatic proof of FDT in this case). The formal solution of (A.10) is

$$\Lambda_{\hat{\rho}\rho}^{(n)}(\mathbf{k}, \omega) = \sum_{r=1}^{n-2} \sum_{i_1+\dots+i_r=n} \Sigma_{\hat{\rho}\rho}^{(i_1)}(\mathbf{k}, \omega) \prod_{s=2}^r \left(R_0(\mathbf{k}, \omega) \Sigma_{\rho\hat{\rho}}^{(i_s)}(\mathbf{k}, \omega) \right). \tag{A.13}$$

Now we use the useful identity

$$\text{Re}(z_1 z_2 (\dots) z_n) = \text{Re}(z_1 z_2^* (\dots) z_n^* - z_1 \text{Re}(z_2) z_3^* (\dots) z_n^* + z_1 z_2 \text{Re}(z_3) (\dots) z_n^* - \dots)$$

to get the following diagrammatic identity:

$$\begin{aligned} R \left[\left(\text{Diagram with } n \text{ circles and arrows pointing left} \right) \right] &= R \left[\left(\text{Diagram with } n \text{ circles and arrows pointing right, some with asterisks} \right) \right] \\ &\quad + \left(\text{Diagram with } n \text{ circles and arrows pointing right, some with asterisks} \right) R \left[\left(\text{Diagram with } n \text{ circles and arrows pointing left, some with asterisks} \right) \right] + \dots \end{aligned}$$

Again using the hypothesis made at order $p \leq n - 2$ and (A.2) the result is obtained at order n . The importance of the minus sign in (104) compared to (103) is clear here. As said above, this establishes (104) for class $A^{(n)}$. Thus, by subtracting diagrams of classes $A^{(n)}$ and $B^{(n)}$ from diagrams of $\Omega^{(n)}$, (104) is established at order n .

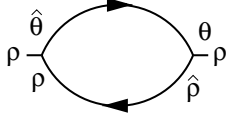


Figure B.1. One-loop diagram contributing to $\Sigma_{\rho\rho}$, for which vanishing is less obvious.

However, due to the Heaviside functions the integrals can be restricted to $[-\epsilon, \epsilon]$, with $\epsilon > 0$. In addition

$$\begin{aligned}
 & \int \frac{d^3\mathbf{q}}{(2\pi)^3} \mathbf{k} \cdot (\mathbf{k} - \mathbf{q}) \int_{-\epsilon}^{\epsilon} dt \partial_t [\Theta(t)C_{\rho\theta}(\mathbf{q}, t)] \Theta(-t)C_{\rho\theta}(\mathbf{q} - \mathbf{k}, -t)C_{\rho\hat{\rho}}(\mathbf{k}, \tau - t) \\
 & \approx \int \frac{d^3\mathbf{q}}{(2\pi)^3} \mathbf{k} \cdot (\mathbf{k} - \mathbf{q}) \int_{-\epsilon}^{0^-} dt \partial_t [\Theta(t)C_{\rho\theta}(\mathbf{q}, t)] C_{\rho\theta}(\mathbf{q} - \mathbf{k}, 0^+) C_{\rho\hat{\rho}}(\mathbf{k}, \tau) \\
 & = \int \frac{d^3\mathbf{q}}{(2\pi)^3} \mathbf{k} \cdot (\mathbf{k} - \mathbf{q}) C_{\rho\theta}(\mathbf{q} - \mathbf{k}, 0^+) C_{\rho\hat{\rho}}(\mathbf{k}, \tau) [\Theta(t)C_{\rho\theta}(\mathbf{q}, t)]_{t=-\epsilon}^{t=0^-} \\
 & = 0.
 \end{aligned} \tag{B.2}$$

Appendix C. Derivation of dynamical and static equations

In this appendix we sketch the derivation of equations (87) and (93) as an example. Other dynamical or static equations can be obtained following the same routes. We start from the SD equation

$$(G_0^{-1} \cdot G - \Sigma \cdot G)_{\hat{\rho}\hat{\rho}}(\mathbf{k}, \tau) = \delta(\tau) \tag{C.1}$$

for any value of τ . We have

$$(\Sigma \cdot G)_{\hat{\rho}\hat{\rho}}(\mathbf{k}, \tau) = (\Sigma_{\hat{\rho}\rho} \cdot C_{\rho\hat{\rho}})(\mathbf{k}, \tau) + (\Sigma_{\hat{\rho}\theta} \cdot C_{\theta\hat{\rho}})(\mathbf{k}, \tau). \tag{C.2}$$

Indeed, $C_{\rho\hat{\rho}}$ and $C_{\theta\hat{\rho}}$ vanish by causality. We then get

$$(\Sigma \cdot G)_{\hat{\rho}\hat{\rho}}(\mathbf{k}, \tau) = \frac{\Theta(\tau)}{T} \int_0^{\tau} dt [\partial_{\tau} \Sigma_{\hat{\theta}\theta}(\mathbf{k}, \tau - t) C_{\rho\theta}(\mathbf{k}, t) + \Sigma_{\hat{\rho}\theta}(\mathbf{k}, \tau - t) C_{\theta\theta}(\mathbf{k}, t)]. \tag{C.3}$$

Integrating by parts, one gets

$$\begin{aligned}
 (\Sigma \cdot G)_{\hat{\rho}\hat{\rho}}(\mathbf{k}, \tau) &= \frac{\Sigma_{\hat{\theta}\theta}(\mathbf{k}, \tau)}{T} C_{\rho\theta}(\mathbf{k}, t=0) \\
 &+ \frac{\Theta(\tau)}{T} \int_0^{\tau} dt [\Sigma_{\hat{\theta}\theta}(\mathbf{k}, \tau - t) \partial_t C_{\rho\theta}(\mathbf{k}, t) + \Sigma_{\hat{\rho}\theta}(\mathbf{k}, \tau - t) C_{\theta\theta}(\mathbf{k}, t)].
 \end{aligned} \tag{C.4}$$

In addition we have

$$\begin{aligned}
 & (G_0^{-1} \cdot G)_{\hat{\rho}\hat{\rho}}(\mathbf{k}, \tau) \partial_{\tau} C_{\rho\hat{\rho}}(\mathbf{k}, \tau) + \rho_0 \mathbf{k}^2 C_{\theta\hat{\rho}}(\mathbf{k}, \tau) \\
 &= \frac{1}{T} \partial_{\tau} (\Theta(\tau^+) C_{\rho\theta}(\mathbf{k}, \tau)) + \frac{\rho_0 \mathbf{k}^2}{T} C_{\theta\theta}(\mathbf{k}, \tau).
 \end{aligned} \tag{C.5}$$

Equating the terms proportional to $\delta(\tau)$ in (C.1) one gets

$$\frac{1}{T}C_{\rho\theta}(\mathbf{k}, 0) = 1, \quad (\text{C.6})$$

and taking the limit $\tau \rightarrow 0^+$ gives (93). Finally, (87) is obtained by taking $\tau > 0$. All other equations for the dynamical evolutions and the statics can be derived in the same way. When causality is not enough to restrict explicitly time integrals between 0 and τ , one can verify that in all cases FDT makes it possible to combine together different contributions of the same equations to finally end up with integrals between 0 and τ .

Let us add here that careful analysis of the self-energies shows that $\Sigma_{\hat{\theta}\rho}$ has a tadpole contribution. However, this tadpole can be eliminated by adding a linear term $-A \int d^3\mathbf{x} \delta\rho(\mathbf{x})$ to the entropic part of the free energy and A to the potential, with a suitable value of the constant A .

Appendix D. Proof of the linear dependence of the Schwinger–Dyson equations

As we have already mentioned, it may appear unnatural to describe the evolution of three correlators with four dynamical equations. A series expansions at low $\tau > 0$ of these equations makes this clearer. When expanded in series, they become a cascade of equations for the successive derivatives of the correlators at zero time difference. We start by expanding at first order, from which it is easy to guess what is going on at higher orders. At order 1 in τ , (86), (87), (88) and (89) read respectively

$$\begin{aligned} \dot{C}_{\rho\rho}(\mathbf{k}, 0) + \rho_0 \mathbf{k}^2 C_{\rho\theta}(\mathbf{k}, 0) + \tau \left[\ddot{C}_{\rho\rho}(\mathbf{k}, 0) + \rho_0 \mathbf{k}^2 \dot{C}_{\rho\theta}(\mathbf{k}, 0) \right] \\ = \tau \left[\Sigma_{\hat{\theta}\theta}(\mathbf{k}, 0^+) \dot{C}_{\rho\rho}(\mathbf{k}, 0) + \Sigma_{\hat{\rho}\theta}(\mathbf{k}, 0^+) C_{\rho\theta}(\mathbf{k}, 0) \right] \end{aligned} \quad (\text{D.1})$$

$$\begin{aligned} \dot{C}_{\rho\theta}(\mathbf{k}, 0) + \rho_0 \mathbf{k}^2 C_{\theta\theta}(\mathbf{k}, 0) + \tau \left[\ddot{C}_{\rho\theta}(\mathbf{k}, 0) + \rho_0 \mathbf{k}^2 \dot{C}_{\theta\theta}(\mathbf{k}, 0) \right] - \Sigma_{\hat{\theta}\theta}(\mathbf{k}, 0^+) C_{\rho\theta}(\mathbf{k}, 0) \\ + \tau \left[\Sigma_{\hat{\rho}\theta}(\mathbf{k}, 0) C_{\theta\theta}(\mathbf{k}, 0) - \dot{\Sigma}_{\hat{\theta}\theta}(\mathbf{k}, 0^+) C_{\rho\theta}(\mathbf{k}, 0) \right] \end{aligned} \quad (\text{D.2})$$

$$\begin{aligned} W(\mathbf{k}) C_{\rho\rho}(\mathbf{k}, 0) - C_{\rho\theta}(\mathbf{k}, 0) + \tau \left[W(\mathbf{k}) \dot{C}_{\rho\rho}(\mathbf{k}, 0) - \dot{C}_{\rho\theta}(\mathbf{k}, 0) \right] \\ = \frac{1}{T} \Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, 0) C_{\rho\rho}(\mathbf{k}, 0) + \tau \Sigma_{\hat{\theta}\theta}(\mathbf{k}, 0^+) C_{\rho\theta}(\mathbf{k}, 0) \end{aligned} \quad (\text{D.3})$$

$$\begin{aligned} W(\mathbf{k}) C_{\rho\theta}(\mathbf{k}, 0) - C_{\theta\theta}(\mathbf{k}, 0) + \tau \left[W(\mathbf{k}) \dot{C}_{\rho\theta}(\mathbf{k}, 0) - \dot{C}_{\theta\theta}(\mathbf{k}, 0) \right] \\ = \tau \left[\Sigma_{\hat{\theta}\theta}(\mathbf{k}, 0^+) C_{\theta\theta}(\mathbf{k}, 0) - \frac{1}{T} \dot{\Sigma}_{\hat{\theta}\hat{\theta}}(\mathbf{k}, 0) C_{\rho\theta}(\mathbf{k}, 0) \right]. \end{aligned} \quad (\text{D.4})$$

In addition, the SD equations have an apparent singularity at $\tau = 0$ which comes from the $\delta(\tau)$ in the RHS of (54). This gives an initial condition $C_{\rho\theta}(\mathbf{k}, 0^+) = T$. Thus there are five equations at order 0, which fix the values of $C_{\rho\rho}(\mathbf{k}, 0)$, $C_{\rho\theta}(\mathbf{k}, 0)$, $C_{\theta\theta}(\mathbf{k}, 0)$, $\dot{C}_{\rho\rho}(\mathbf{k}, 0)$ and $\dot{C}_{\rho\theta}(\mathbf{k}, 0)$. At order 1, there are four equations but only three quantities to be determined, namely $\dot{C}_{\theta\theta}(\mathbf{k}, 0)$, $\ddot{C}_{\rho\rho}(\mathbf{k}, 0)$ and $\ddot{C}_{\rho\theta}(\mathbf{k}, 0)$. Remark that the self-energies and their first derivatives appear in the equations. However, as they can be expressed in terms of the correlators, it can be checked that the successive derivatives of the self-energies can be expressed in terms of the quantities already computed at previous orders. This guarantees

that at every order self-energies do not give extra variables to be determined. Now we show that in fact one of the equations obtained by identifying the terms of order τ is trivially satisfied by the solution of the equations at order 0. We focus on the term proportional to τ in the LHS of (D.3). We then express this term by using a linear combination of the terms of order 0 of (D.1) and (D.2):

$$W(\mathbf{k})\dot{C}_{\rho\rho}(\mathbf{k}, 0) - \dot{C}_{\rho\theta}(\mathbf{k}, 0) = -\rho_0\mathbf{k}^2 [W(\mathbf{k})C_{\rho\rho}(\mathbf{k}, 0) - C_{\rho\theta}(\mathbf{k}, 0)] + \Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, 0^+)C_{\theta\theta}(\mathbf{k}, 0). \quad (\text{D.5})$$

From order 0 of (D.3), the terms in brackets vanish, and then we get

$$W(\mathbf{k})\dot{C}_{\rho\rho}(\mathbf{k}, t) - \dot{C}_{\rho\theta}(\mathbf{k}, t)\Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, 0^+)C_{\rho\theta}(\mathbf{k}, 0), \quad (\text{D.6})$$

which corresponds to the terms proportional to τ in (D.3). Therefore, the number of equations obtained at order τ is equal to the number of variables to be determined at this order.

The non-perturbative generalization of the previous approach comes from the following remark: the SD equations form a linear system of equations for which the unknown variables are the correlators and the coefficients are the components of G_0^{-1} and Σ . The solution of this system of equations is found easily using the Laplace transform. The SD equations read in the Laplace transform

$$C_{\rho\rho}(\mathbf{k}, 0^+) \left(1 + \frac{\hat{\Sigma}_{\hat{\rho}\hat{\theta}}(\mathbf{k}, z)}{T} \right) z \left(1 + \frac{\hat{\Sigma}_{\hat{\rho}\hat{\theta}}(\mathbf{k}, z)}{T} \right) \hat{C}_{\rho\rho}(\mathbf{k}, z) + \left(\rho_0\mathbf{k}^2 + \frac{\hat{\Sigma}_{\hat{\rho}\hat{\rho}}(\mathbf{k}, z)}{T} \right) \hat{C}_{\rho\theta}(\mathbf{k}, z), \quad (\text{D.7})$$

$$z \left(1 + \frac{\hat{\Sigma}_{\hat{\rho}\hat{\theta}}(\mathbf{k}, z)}{T} \right) \hat{C}_{\rho\theta}(\mathbf{k}, z) + \left(\rho_0\mathbf{k}^2 + \frac{\hat{\Sigma}_{\hat{\rho}\hat{\rho}}(\mathbf{k}, z)}{T} \right) \hat{C}_{\theta\theta}(\mathbf{k}, z) = T \quad (\text{D.8})$$

$$\frac{1}{T}\hat{\Sigma}_{\hat{\theta}\hat{\theta}}(\mathbf{k}, z)C_{\rho\rho}(\mathbf{k}, 0^+) = - \left(1 + \frac{\hat{\Sigma}_{\hat{\rho}\hat{\theta}}(\mathbf{k}, z)}{T} \right) \hat{C}_{\rho\theta}(\mathbf{k}, z) + \left(W(\mathbf{k}) + \frac{z\hat{\Sigma}_{\hat{\theta}\hat{\theta}}(\mathbf{k}, z) - \Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, 0^+)}{T} \right) \hat{C}_{\rho\rho}(\mathbf{k}, z) \quad (\text{D.9})$$

$$W(\mathbf{k})\hat{C}_{\rho\theta}(\mathbf{k}, z) = \left(1 + \frac{\hat{\Sigma}_{\hat{\rho}\hat{\theta}}(\mathbf{k}, z)}{T} \right) \hat{C}_{\theta\theta}(\mathbf{k}, z) - \frac{z\hat{\Sigma}_{\hat{\theta}\hat{\theta}}(\mathbf{k}, z) - \Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, \tau = 0)}{T} \hat{C}_{\rho\theta}(\mathbf{k}, z). \quad (\text{D.10})$$

For better clarity, we write this system formally as follows:

$$AC_{\rho\rho}(\mathbf{k}, 0) = zA\hat{C}_{\rho\rho}(\mathbf{k}, z) + B\hat{C}_{\rho\theta}(\mathbf{k}, z) \quad (\text{D.11})$$

$$T = zA\hat{C}_{\rho\theta}(\mathbf{k}, z) + B\hat{C}_{\theta\theta}(\mathbf{k}, z) \quad (\text{D.12})$$

$$DC_{\rho\rho}(\mathbf{k}, 0) = E\hat{C}_{\rho\rho}(\mathbf{k}, z) - A\hat{C}_{\rho\theta}(\mathbf{k}, z) \quad (\text{D.13})$$

$$0 = E\hat{C}_{\rho\theta}(\mathbf{k}, z) - A\hat{C}_{\theta\theta}(\mathbf{k}, z). \quad (\text{D.14})$$

The identity $E(\text{RHS})_1 - A(\text{RHS})_2 - zA(\text{RHS})_3 - B(\text{RHS})_4 = 0$, where $(\text{RHS})_i$ stands for the RHS of the i th equation above, is trivially verified. It remains to prove that the LHSs are linked by the same relation. Gathering the terms of $E(\text{LHS})_1 - A(\text{LHS})_2 - zA(\text{LHS})_3 - B(\text{LHS})_4 = 0$ (with obvious notation), one gets

$$\left[W(\mathbf{k}) - \frac{1}{T}\Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, 0^+) \right] C_{\rho\rho}(\mathbf{k}, 0) = T. \quad (\text{D.15})$$

This is precisely the static equation (94), and the proof is complete.

Appendix E. General field transformation and field theory in the case of Brownian dynamics for the density field

We have seen in section 5.1 that one can make each of the two symmetries of the system linear independently. One can also render them linear simultaneously using the same method: introducing additional fields.

The starting point is the action (64): the field transformation is given by \mathcal{T}_1 . We introduce four additional fields to linearize the symmetry \mathcal{U} : $\theta = \delta\mathcal{F}/\delta\rho$ and $\eta = \nabla \cdot (\rho\nabla\theta)$, as well as the conjugated fields $\hat{\theta}$ and $\hat{\eta}$. The final transformation is

$$\mathcal{T}_1 : \begin{cases} \hat{\rho}_x \rightarrow \hat{\rho}_x + f_x \\ \psi_x \rightarrow \psi_x + \frac{1}{T}\partial_t\rho_x \\ \hat{\psi}_x \rightarrow \hat{\psi}_x + T f_x \\ \hat{f}_x \rightarrow -\hat{f}_x + T f_x + \hat{\psi}_x + T\hat{\rho}_x \\ f_x \rightarrow -f_x \end{cases} \quad (\text{E.1})$$

$$\mathcal{U}_1 : \begin{cases} \hat{\rho}_x \rightarrow -\hat{\rho}_x + \frac{1}{T}\theta_x \\ \psi_x \rightarrow -\psi_x + \frac{1}{T}\eta_x \\ \hat{\psi}_x \rightarrow -\hat{\psi}_x + \theta_x \\ \hat{\eta}_x \rightarrow \hat{\eta}_x + \frac{1}{T}\theta_x - \hat{\rho}_x - \frac{1}{T}\hat{\psi}_x \\ \hat{\theta}_x \rightarrow \hat{\theta}_x - \frac{1}{T}\partial_t\rho_x \\ \hat{f}_x \rightarrow -\hat{f}_x \\ f_x \rightarrow -f_x. \end{cases} \quad (\text{E.2})$$

The action is equal to the integral over \mathbf{x}, t of

$$\begin{aligned}
 & -\hat{\rho}_x \partial_t \rho_x + \psi_x \theta_x - T \psi_x \hat{\rho}_x + \hat{f}_x \left(\nabla \cdot (\rho_x \nabla f_x) + \frac{1}{T} \partial_t \rho_x \right) - \rho_x \nabla \phi_x \cdot \nabla \bar{\phi}_x \\
 & + \hat{\psi}_x (\psi_x - \nabla \cdot (\rho_x \nabla \hat{\rho}_x)) + \hat{\theta}_x \left(\theta_x - \frac{\delta \mathcal{F}}{\delta \rho_x} \right) + \hat{\eta}_x (\eta_x - \nabla \cdot (\rho_x \nabla \theta_x)). \quad (\text{E.3})
 \end{aligned}$$

The structure of the two-field correlator matrix for $\tau > 0$ is schematically

$$\begin{bmatrix}
 C_{\rho\rho} & \frac{C_{\rho f}}{T} & \frac{\dot{C}_{\rho\rho}}{T} & 0 & C_{\rho f} & 0 & C_{\rho f} & -\frac{\dot{C}_{\rho\rho}}{T} & \dot{C}_{\rho\rho} & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 C_{\rho f} & TC_{ff} & \frac{\dot{C}_{\rho f}}{T} & 0 & TC_{ff} & 0 & T^2 C_{ff} & -\frac{\dot{C}_{\rho f}}{T} & \dot{C}_{\rho f} & 0 \\
 C_{\rho f} & -C_{ff} & -\dot{C}_{\rho f} & 0 & C_{ff} & C_{f\hat{f}} & -TC_{ff} & \frac{\dot{C}_{\rho f}}{T} & -\dot{C}_{\rho f} & 0 \\
 0 & 0 & 0 & 0 & C_{f\hat{f}} & 0 & 0 & 0 & 0 & 0 \\
 C_{\rho f} & TC_{ff} & \frac{\dot{C}_{\rho f}}{T} & 0 & TC_{ff} & 0 & T^2 C_{ff} & -\frac{\dot{C}_{\rho f}}{T} & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \dot{C}_{\rho\rho} & \frac{\dot{C}_{\rho f}}{T} & \frac{\ddot{C}_{\rho\rho}}{T} & 0 & \dot{C}_{\rho f} & 0 & \dot{C}_{\rho f} & -\frac{\ddot{C}_{\rho\rho}}{T} & \ddot{C}_{\rho\rho} & C_{\eta\hat{\eta}} \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{\eta\hat{\eta}} & 0
 \end{bmatrix} \quad (\text{E.4})$$

where the ordering of the fields is $(\rho, \hat{\rho}, \psi, \hat{\psi}, f, \hat{f}, \theta, \hat{\theta}, \eta, \hat{\eta})$. The fermionic fields ϕ and $\bar{\phi}$, for which correlators with other fields vanish, have not been included. The role of these fermionic fields will be explained later on a practical example. For completeness, the bare propagator (still without the fermionic fields) is given in figure E.1. Some of the correlators defined above are singular at $\tau = 0$. Indeed, for $\tau \in \mathbb{R}$, we have $C_{ff}(\mathbf{k}, \tau) = a(\mathbf{k}) \delta(\tau) + C_{ff}^{\text{reg}}(\mathbf{k}, \tau)$ and $C_{f\hat{f}} = b(\mathbf{k}) \delta(\tau)$, where C_{ff}^{reg} is regular (i.e. continuous by parts) and even in time. We also have $C_{\hat{\psi}\psi}(\mathbf{k}, \tau) = (1/T) \partial_\tau (\Theta(\tau) C_{\rho f}(\mathbf{k}, \tau)) = -C_{\hat{\psi}\hat{\theta}}(\mathbf{k}, \tau)$, $C_{\hat{\psi}\hat{f}}(\mathbf{k}, \tau) = -C_{\hat{\rho}f}(\mathbf{k}, \tau) = c(\mathbf{k}) \delta(\tau) + TC_{ff}^{\text{reg}}(\mathbf{k}, \tau)$ and $C_{\eta\hat{\eta}}(\mathbf{k}, \tau) = -\delta(\tau)$.

We conclude this appendix with a discussion about causality. One can show using the way Σ transforms under $\{\mathcal{T}_1, \mathcal{U}_1\}$ that $\Sigma_{\rho\rho}$ vanishes, which guarantees causality to hold non-perturbatively. Furthermore, it also holds perturbatively, as a consequence of the form of the bare propagator and of the Itô discretization. We first show how it holds at one-loop order, the generalization to higher orders being straightforward.

At one-loop, $\Sigma_{\rho\rho}$ has three non-trivially vanishing contributions, shown in figure E.2. The diagram involving the ghost loop is identical to the one involving the loop made with propagators $C_{f\hat{f}}$, with opposite sign and with $C_{\phi\bar{\phi}}$ instead of $C_{f\hat{f}}^0$. However, the bare propagators $C_{\phi\bar{\phi}}^0$ and $C_{f\hat{f}}^0$ are identical and the equations giving the renormalization of $C_{\phi\bar{\phi}}$ and $C_{f\hat{f}}$ are also identical, thus the contributions of both loops in the self-energy—which are divergent—cancel each other exactly. As usual in the Fadeev–Popov method, the role of the fermionic fields is to remove the volume of the ‘gauge’ ensemble, which is infinite. One can easily check that loops of ghost propagators annihilate at all orders with corresponding loops of propagators $C_{f\hat{f}}$. Following the steps of appendix B, the remaining diagram in figure E.2 may be shown to give a vanishing contribution when involved in the dynamic equations.

$$\left[\begin{array}{cccccccccccc}
 \frac{2k^2\rho_0 T}{w^2+k^4\rho_0^2 W^2} & -\frac{i}{w-ik^2\rho_0 W} & -\frac{k^2\rho_0}{iw+k^2\rho_0 W} & \frac{iT}{w+ik^2\rho_0 W} & -\frac{2iw}{w^2+k^4\rho_0^2 W^2} & 0 & 0 & \frac{k^2\rho_0}{iw+k^2\rho_0 W} & -\frac{2k^4\rho_0^2 TW}{w^2+k^4\rho_0^2 W^2} & 0 & -\frac{2k^4\rho_0^2 TW}{w^2+k^4\rho_0^2 W^2} & 0 \\
 \frac{i}{w+ik^2\rho_0 W} & 0 & 0 & \frac{W}{-iw+k^2\rho_0 W} & \frac{w}{k^2\rho_0 T(w+ik^2\rho_0 W)} & 0 & 0 & 0 & -\frac{k^2\rho_0 W}{-iw+k^2\rho_0 W} & 0 & -\frac{k^2\rho_0 W}{-iw+k^2\rho_0 W} & 0 \\
 -\frac{k^2\rho_0}{-iw+k^2\rho_0 W} & 0 & 0 & \frac{w}{w+ik^2\rho_0 W} & -\frac{w}{T(w+ik^2\rho_0 W)} & 0 & 0 & 0 & -\frac{k^4\rho_0^2 W}{-iw+k^2\rho_0 W} & 0 & -\frac{k^4\rho_0^2 W}{-iw+k^2\rho_0 W} & 0 \\
 -\frac{iT}{w-ik^2\rho_0 W} & \frac{W}{iw+k^2\rho_0 W} & \frac{w}{w-ik^2\rho_0 W} & 0 & -\frac{iw}{k^2\rho_0(iw+k^2\rho_0 W)} & 0 & 0 & -\frac{w}{w-ik^2\rho_0 W} & \frac{ik^2\rho_0 TW}{w-ik^2\rho_0 W} & 0 & -\frac{k^2\rho_0 TW}{iw+k^2\rho_0 W} & 0 \\
 \frac{2iw}{w^2+k^4\rho_0^2 W^2} & \frac{iw}{k^2\rho_0 T(iw+k^2\rho_0 W)} & -\frac{w}{T(w-ik^2\rho_0 W)} & \frac{iw}{k^2\rho_0(-iw+k^2\rho_0 W)} & \frac{2w^2}{k^2\rho_0 T(w^2+k^4\rho_0^2 W^2)} & \frac{1}{k^2\rho_0} & 0 & \frac{w}{T(w-ik^2\rho_0 W)} & -\frac{2ik^2\rho_0 wW}{w^2+k^4\rho_0^2 W^2} & 0 & -\frac{2ik^2\rho_0 wW}{w^2+k^4\rho_0^2 W^2} & 0 \\
 0 & 0 & 0 & 0 & \frac{1}{k^2\rho_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
 \frac{k^2\rho_0}{-iw+k^2\rho_0 W} & 0 & 0 & \frac{iw}{-iw+k^2\rho_0 W} & -\frac{iw}{T(-iw+k^2\rho_0 W)} & 0 & -1 & 0 & -\frac{ik^2\rho_0 w}{-iw+k^2\rho_0 W} & 0 & -\frac{ik^4\rho_0^2 W}{w+ik^2\rho_0 W} & 0 \\
 -\frac{2k^4\rho_0^2 TW}{w^2+k^4\rho_0^2 W^2} & -\frac{k^2\rho_0 W}{-iw+k^2\rho_0 W} & \frac{k^4\rho_0^2 W}{iw+k^2\rho_0 W} & -\frac{k^2\rho_0 TW}{-iw+k^2\rho_0 W} & \frac{2ik^2\rho_0 wW}{w^2+k^4\rho_0^2 W^2} & 0 & 0 & \frac{ik^2\rho_0 w}{iw+k^2\rho_0 W} & \frac{2k^6\rho_0^3 TW^2}{w^2+k^4\rho_0^2 W^2} & -1 & \frac{2k^6\rho_0^3 TW^2}{w^2+k^4\rho_0^2 W^2} & -1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
 -\frac{2k^4\rho_0^2 TW}{w^2+k^4\rho_0^2 W^2} & -\frac{k^2\rho_0 W}{-iw+k^2\rho_0 W} & \frac{k^4\rho_0^2 W}{iw+k^2\rho_0 W} & -\frac{k^2\rho_0 TW}{-iw+k^2\rho_0 W} & \frac{2ik^2\rho_0 wW}{w^2+k^4\rho_0^2 W^2} & 0 & 0 & -\frac{k^4\rho_0^2 W}{-iw+k^2\rho_0 W} & \frac{2k^6\rho_0^3 TW^2}{w^2+k^4\rho_0^2 W^2} & 0 & \frac{2k^6\rho_0^3 TW^2}{w^2+k^4\rho_0^2 W^2} & -1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \end{array} \right]$$

Figure E.1. Bare propagator obtained from the action (E.3), where W stands for $W(\mathbf{k})$.

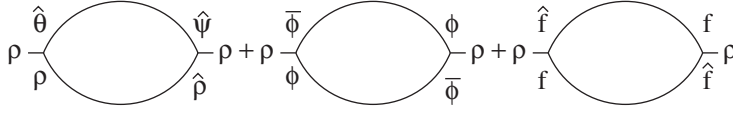


Figure E.2. Diagrams contributing to $\Sigma_{\rho\rho}$ at one loop. The fields involved in the Wick theorem are indicated around the vertices.

Appendix F. Dynamical equations for fluctuating nonlinear hydrodynamics

In this appendix we give the derivation of the dynamic equations for fluctuating nonlinear hydrodynamics. The calculus and the underlying ideas are the same as the corresponding ones for BDD although somewhat more cumbersome due to a larger number of fields. We start with the Schwinger–Dyson equations and use time reversal to simplify them.

This time (66) applied to the transformation \mathcal{U}_1 gives the following equations for correlators:

$$C_{\rho\hat{\rho}}(\mathbf{k}, \tau) = \frac{\Theta(\tau)}{T} C_{\rho\theta}(\mathbf{k}, \tau) \quad (\text{F.1})$$

$$C_{\rho\hat{g}}(\mathbf{k}, \tau) = \frac{\Theta(\tau)}{T} C_{\rho v}(\mathbf{k}, \tau) \quad (\text{F.2})$$

$$C_{\rho\hat{\theta}}(\mathbf{k}, \tau) = \frac{\Theta(\tau)}{T} \partial_\tau C_{\rho\rho}(\mathbf{k}, \tau) \quad (\text{F.3})$$

$$C_{\rho\hat{v}}(\mathbf{k}, \tau) = \frac{\Theta(\tau)}{T} \partial_\tau C_{\rho g}(\mathbf{k}, \tau) \quad (\text{F.4})$$

$$C_{g\hat{\rho}}(\mathbf{k}, \tau) = \frac{\Theta(\tau)}{T} C_{g\theta}(\mathbf{k}, \tau) \quad (\text{F.5})$$

$$C_{g\hat{g}}(\mathbf{k}, \tau) = \frac{\Theta(\tau)}{T} C_{g v}(\mathbf{k}, \tau) \quad (\text{F.6})$$

$$C_{g\hat{\theta}}(\mathbf{k}, \tau) = \frac{\Theta(\tau)}{T} \partial_\tau C_{g\rho}(\mathbf{k}, \tau) \quad (\text{F.7})$$

$$C_{g\hat{v}}(\mathbf{k}, \tau) = \frac{\Theta(\tau)}{T} \partial_\tau C_{g g}(\mathbf{k}, \tau) \quad (\text{F.8})$$

$$C_{\theta\hat{\rho}}(\mathbf{k}, \tau) = \frac{\Theta(\tau)}{T} C_{\theta\theta}(\mathbf{k}, \tau) \quad (\text{F.9})$$

$$C_{\theta\hat{g}}(\mathbf{k}, \tau) = \frac{\Theta(\tau)}{T} C_{\theta v}(\mathbf{k}, \tau) \quad (\text{F.10})$$

$$C_{\theta\hat{\theta}}(\mathbf{k}, \tau) = \frac{\Theta(\tau)}{T} \partial_\tau C_{\theta\rho}(\mathbf{k}, \tau) \quad (\text{F.11})$$

$$C_{\theta\hat{v}}(\mathbf{k}, \tau) = \frac{\Theta(\tau)}{T} \partial_\tau C_{\theta g}(\mathbf{k}, \tau) \quad (\text{F.12})$$

$$C_{v\hat{\rho}}(\mathbf{k}, \tau) = \frac{\Theta(\tau)}{T} C_{v\theta}(\mathbf{k}, \tau) \quad (\text{F.13})$$

$$C_{v\hat{g}}(\mathbf{k}, \tau) = \frac{\Theta(\tau)}{T} C_{v v}(\mathbf{k}, \tau) \quad (\text{F.14})$$

$$C_{v\hat{\theta}}(\mathbf{k}, \tau) = \frac{\Theta(\tau)}{T} \partial_\tau C_{v\rho}(\mathbf{k}, \tau) \quad (\text{F.15})$$

$$C_{v\hat{v}}(\mathbf{k}, \tau) = \frac{\Theta(\tau)}{T} \partial_\tau C_{vg}(\mathbf{k}, \tau) \quad (\text{F.16})$$

and (80) yields the following identities for self-energies:

$$\Sigma_{\hat{\rho}\rho}(\mathbf{k}, \tau) = \frac{1}{T} \partial_\tau [\Theta(\tau) \Sigma_{\hat{\rho}\hat{\theta}}(\mathbf{k}, \tau)] \quad (\text{F.17})$$

$$\Sigma_{\hat{\rho}g}(\mathbf{k}, \tau) = \frac{1}{T} \partial_\tau [\Theta(\tau) \Sigma_{\hat{\rho}\hat{v}}(\mathbf{k}, \tau)] \quad (\text{F.18})$$

$$\Sigma_{\hat{\rho}\theta}(\mathbf{k}, \tau) = -\frac{1}{T} \Theta(\tau) \Sigma_{\hat{\rho}\hat{\rho}}(\mathbf{k}, \tau) \quad (\text{F.19})$$

$$\Sigma_{\hat{\rho}v}(\mathbf{k}, \tau) = -\frac{1}{T} \Theta(\tau) \Sigma_{\hat{\rho}\hat{g}}(\mathbf{k}, \tau) \quad (\text{F.20})$$

$$\Sigma_{\hat{g}\rho}(\mathbf{k}, \tau) = \frac{1}{T} \partial_\tau [\Theta(\tau) \Sigma_{\hat{g}\hat{\theta}}(\mathbf{k}, \tau)] \quad (\text{F.21})$$

$$\Sigma_{\hat{g}g}(\mathbf{k}, \tau) = \frac{1}{T} \partial_\tau [\Theta(\tau) \Sigma_{\hat{g}\hat{v}}(\mathbf{k}, \tau)] \quad (\text{F.22})$$

$$\Sigma_{\hat{g}\theta}(\mathbf{k}, \tau) = -\frac{1}{T} \Theta(\tau) \Sigma_{\hat{g}\hat{\rho}}(\mathbf{k}, \tau) \quad (\text{F.23})$$

$$\Sigma_{\hat{g}v}(\mathbf{k}, \tau) = -\frac{1}{T} \Theta(\tau) \Sigma_{\hat{g}\hat{g}}(\mathbf{k}, \tau) \quad (\text{F.24})$$

$$\Sigma_{\hat{\theta}\rho}(\mathbf{k}, \tau) = \frac{1}{T} \Theta(\tau) \partial_\tau \Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, \tau) \quad (\text{F.25})$$

$$\Sigma_{\hat{\theta}g}(\mathbf{k}, \tau) = \frac{1}{T} \Theta(\tau) \partial_\tau \Sigma_{\hat{\theta}\hat{v}}(\mathbf{k}, \tau) \quad (\text{F.26})$$

$$\Sigma_{\hat{\theta}\theta}(\mathbf{k}, \tau) = -\frac{1}{T} \Theta(\tau) \Sigma_{\hat{\theta}\hat{\rho}}(\mathbf{k}, \tau) \quad (\text{F.27})$$

$$\Sigma_{\hat{\theta}v}(\mathbf{k}, \tau) = -\frac{1}{T} \Theta(\tau) \Sigma_{\hat{\theta}\hat{g}}(\mathbf{k}, \tau) \quad (\text{F.28})$$

$$\Sigma_{\hat{v}\rho}(\mathbf{k}, \tau) = \frac{1}{T} \Theta(\tau) \partial_\tau \Sigma_{\hat{v}\hat{\theta}}(\mathbf{k}, \tau) \quad (\text{F.29})$$

$$\Sigma_{\hat{v}g}(\mathbf{k}, \tau) = \frac{1}{T} \Theta(\tau) \partial_\tau \Sigma_{\hat{v}\hat{v}}(\mathbf{k}, \tau) \quad (\text{F.30})$$

$$\Sigma_{\hat{v}\theta}(\mathbf{k}, \tau) = -\frac{1}{T} \Theta(\tau) \Sigma_{\hat{v}\hat{\rho}}(\mathbf{k}, \tau) \quad (\text{F.31})$$

$$\Sigma_{\hat{v}v}(\mathbf{k}, \tau) = -\frac{1}{T} \Theta(\tau) \Sigma_{\hat{v}\hat{g}}(\mathbf{k}, \tau). \quad (\text{F.32})$$

One can get some additional identities,

$$\Sigma_{\hat{g}\hat{\rho}}(\mathbf{k}, \tau) = \Sigma_{\hat{\rho}\hat{g}}(\mathbf{k}, \tau) \quad (\text{F.33})$$

$$\Sigma_{\hat{\theta}\hat{\rho}}(\mathbf{k}, \tau) = -\Sigma_{\hat{\rho}\hat{\theta}}(\mathbf{k}, \tau) \quad (\text{F.34})$$

$$\Sigma_{\hat{\theta}\hat{g}}(\mathbf{k}, \tau) = -\Sigma_{\hat{g}\hat{\theta}}(\mathbf{k}, \tau) \quad (\text{F.35})$$

$$\Sigma_{\hat{v}\hat{\rho}}(\mathbf{k}, \tau) = -\Sigma_{\hat{\rho}\hat{v}}(\mathbf{k}, \tau) \quad (\text{F.36})$$

$$\Sigma_{\hat{v}\hat{g}}(\mathbf{k}, \tau) = -\Sigma_{\hat{g}\hat{v}}(\mathbf{k}, \tau) \quad (\text{F.37})$$

$$\Sigma_{\hat{v}\hat{\theta}}(\mathbf{k}, \tau) = \Sigma_{\hat{\theta}\hat{v}}(\mathbf{k}, \tau), \quad (\text{F.38})$$

and similar ones for correlators,

$$C_{g\rho}(\mathbf{k}; t, s) = C_{\rho g}(\mathbf{k}; t, s) \quad (\text{F.39})$$

$$C_{\theta\rho}(\mathbf{k}; t, s) = C_{\rho\theta}(\mathbf{k}; t, s) \quad (\text{F.40})$$

$$C_{\theta g}(\mathbf{k}; t, s) = C_{g\theta}(\mathbf{k}; t, s) \quad (\text{F.41})$$

$$C_{v\rho}(\mathbf{k}; t, s) = C_{\rho v}(\mathbf{k}; t, s) \quad (\text{F.42})$$

$$C_{vg}(\mathbf{k}; t, s) = C_{gv}(\mathbf{k}; t, s) \quad (\text{F.43})$$

$$C_{v\theta}(\mathbf{k}; t, s) = C_{\theta v}(\mathbf{k}; t, s). \quad (\text{F.44})$$

All these identities reduce the number of independent correlators to ten, which are $C_{\rho\rho}$, $C_{\rho g}$, $C_{\rho\theta}$, $C_{\rho v}$, $C_{g g}$, $C_{g\theta}$, $C_{g v}$, $C_{\theta\theta}$, $C_{\theta v}$ and $C_{v v}$.

In the case of FNH, there are in principle 64 Schwinger–Dyson equation. We write 16 of these equations, the others being trivially linear dependent on these:

$$\begin{aligned} \partial_\tau C_{\rho\rho}(\mathbf{k}, \tau) - i\rho_0 \mathbf{k} C_{v\rho}(\mathbf{k}, \tau) &= \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{\rho}\hat{\theta}}(\mathbf{k}, \tau - t) \partial_t C_{\rho\rho}(\mathbf{k}, t) \\ &+ \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{\rho}\hat{v}}(\mathbf{k}, \tau - t) \partial_t C_{g\rho}(\mathbf{k}, t) - \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{\rho}\hat{\rho}}(\mathbf{k}, \tau - t) C_{\theta\rho}(\mathbf{k}, t) \\ &- \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{\rho}\hat{g}}(\mathbf{k}, \tau - t) C_{v\rho}(\mathbf{k}, t) \end{aligned} \quad (\text{F.45})$$

$$\begin{aligned} \partial_\tau C_{\rho g}(\mathbf{k}, \tau) - i\rho_0 \mathbf{k} C_{vg}(\mathbf{k}, \tau) &= \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{\rho}\hat{\theta}}(\mathbf{k}, \tau - t) \partial_t C_{\rho g}(\mathbf{k}, t) \\ &+ \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{\rho}\hat{v}}(\mathbf{k}, \tau - t) \partial_t C_{g g}(\mathbf{k}, t) - \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{\rho}\hat{\rho}}(\mathbf{k}, \tau - t) C_{\theta g}(\mathbf{k}, t) \\ &- \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{\rho}\hat{g}}(\mathbf{k}, \tau - t) C_{vg}(\mathbf{k}, t) \end{aligned} \quad (\text{F.46})$$

$$\begin{aligned} \partial_\tau C_{\rho\theta}(\mathbf{k}, \tau) - i\rho_0 \mathbf{k} C_{v\theta}(\mathbf{k}, \tau) &= \Sigma_{\hat{\rho}\hat{v}}(\mathbf{k}, \tau) \\ &+ \frac{1}{T} \int_0^\tau dt \left[\Sigma_{\hat{\rho}\hat{\theta}}(\mathbf{k}, \tau - t) \partial_t C_{\rho\theta}(\mathbf{k}, t) + \Sigma_{\hat{\rho}\hat{v}}(\mathbf{k}, \tau - t) \partial_t C_{g\theta}(\mathbf{k}, t) \right. \\ &\left. - \Sigma_{\hat{\rho}\hat{\rho}}(\mathbf{k}, \tau - t) C_{\theta\theta}(\mathbf{k}, t) - \Sigma_{\hat{\rho}\hat{g}}(\mathbf{k}, \tau - t) C_{v\theta}(\mathbf{k}, t) \right] \end{aligned} \quad (\text{F.47})$$

$$\begin{aligned} \partial_\tau C_{\rho v}(\mathbf{k}, \tau) - i\rho_0 \mathbf{k} C_{vv}(\mathbf{k}, \tau) &= \Sigma_{\hat{\rho}\hat{v}}(\mathbf{k}, \tau) \\ &+ \frac{1}{T} \int_0^\tau dt \left[\Sigma_{\hat{\rho}\hat{\theta}}(\mathbf{k}, \tau - t) \partial_t C_{\rho v}(\mathbf{k}, t) + \Sigma_{\hat{\rho}\hat{v}}(\mathbf{k}, \tau - t) \partial_t C_{gv}(\mathbf{k}, t) \right. \\ &\left. - \Sigma_{\hat{\rho}\hat{\rho}}(\mathbf{k}, \tau - t) C_{\theta v}(\mathbf{k}, t) - \Sigma_{\hat{\rho}\hat{g}}(\mathbf{k}, \tau - t) C_{vv}(\mathbf{k}, t) \right] \end{aligned} \quad (\text{F.48})$$

$$\begin{aligned} \partial_\tau C_{g\rho}(\mathbf{k}, \tau) - i\rho_0 \mathbf{k} C_{\theta\rho}(\mathbf{k}, \tau) + LC_{v\rho}(\mathbf{k}, \tau) &= \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{g}\hat{\theta}}(\mathbf{k}, \tau - t) \partial_t C_{\rho\rho}(\mathbf{k}, t) \\ &+ \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{g}\hat{v}}(\mathbf{k}, \tau - t) \partial_t C_{g\rho}(\mathbf{k}, t) - \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{g}\hat{\rho}}(\mathbf{k}, \tau - t) C_{\theta\rho}(\mathbf{k}, t) \\ &- \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{g}\hat{g}}(\mathbf{k}, \tau - t) C_{v\rho}(\mathbf{k}, t) \end{aligned} \quad (\text{F.49})$$

$$\begin{aligned} \partial_\tau C_{gg}(\mathbf{k}, \tau) - i\rho_0 \mathbf{k} C_{\theta g}(\mathbf{k}, \tau) + LC_{vg}(\mathbf{k}, \tau) &= \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{g}\hat{\theta}}(\mathbf{k}, \tau - t) \partial_t C_{\rho g}(\mathbf{k}, t) \\ &+ \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{g}\hat{v}}(\mathbf{k}, \tau - t) \partial_t C_{gg}(\mathbf{k}, t) - \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{g}\hat{\rho}}(\mathbf{k}, \tau - t) C_{\theta g}(\mathbf{k}, t) \\ &- \frac{1}{T} \int_0^\tau dt \Sigma_{\hat{g}\hat{g}}(\mathbf{k}, \tau - t) C_{vg}(\mathbf{k}, t) \end{aligned} \quad (\text{F.50})$$

$$\begin{aligned} \partial_\tau C_{g\theta}(\mathbf{k}, \tau) - i\rho_0 \mathbf{k} C_{\theta\theta}(\mathbf{k}, \tau) + LC_{v\theta}(\mathbf{k}, \tau) &= \Sigma_{\hat{g}\hat{\theta}}(\mathbf{k}, \tau) \\ &+ \frac{1}{T} \int_0^\tau dt \left[\Sigma_{\hat{g}\hat{\theta}}(\mathbf{k}, \tau - t) \partial_t C_{\rho\theta}(\mathbf{k}, t) - \Sigma_{\hat{g}\hat{v}}(\mathbf{k}, \tau - t) \partial_t C_{g\theta}(\mathbf{k}, t) \right] \end{aligned} \quad (\text{F.51})$$

$$- \left[\Sigma_{\hat{g}\hat{\rho}}(\mathbf{k}, \tau - t) C_{\theta\theta}(\mathbf{k}, t) + \Sigma_{\hat{g}\hat{g}}(\mathbf{k}, \tau - t) C_{v\theta}(\mathbf{k}, t) \right] \quad (\text{F.52})$$

$$\begin{aligned} \partial_\tau C_{gv}(\mathbf{k}, \tau) - i\rho_0 \mathbf{k} C_{\theta v}(\mathbf{k}, \tau) + LC_{vv}(\mathbf{k}, \tau) &= \Sigma_{\hat{g}\hat{v}}(\mathbf{k}, \tau) \\ &+ \frac{1}{T} \int_0^\tau dt \left[\Sigma_{\hat{g}\hat{\theta}}(\mathbf{k}, \tau - t) \partial_t C_{\rho v}(\mathbf{k}, t) + \Sigma_{\hat{g}\hat{v}}(\mathbf{k}, \tau - t) \partial_t C_{gv}(\mathbf{k}, t) \right. \\ &\left. - \Sigma_{\hat{g}\hat{\rho}}(\mathbf{k}, \tau - t) C_{\theta v}(\mathbf{k}, t) - \Sigma_{\hat{g}\hat{g}}(\mathbf{k}, \tau - t) C_{vv}(\mathbf{k}, t) \right] \end{aligned} \quad (\text{F.53})$$

$$\begin{aligned} C_{\theta\rho}(\mathbf{k}, \tau) - W(\mathbf{k}) C_{\rho\rho}(\mathbf{k}, \tau) &= -\frac{1}{T} \left[\Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, 0) C_{\rho\rho}(\mathbf{k}, \tau) + \Sigma_{\hat{\theta}\hat{v}}(\mathbf{k}, 0) C_{g\rho}(\mathbf{k}, \tau) \right] \\ &+ \frac{1}{T} \int_0^\tau dt \left[\Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, \tau - t) \partial_t C_{\rho\rho}(\mathbf{k}, t) + \Sigma_{\hat{\theta}\hat{v}}(\mathbf{k}, \tau - t) \partial_t C_{g\rho}(\mathbf{k}, t) \right. \\ &\left. - \Sigma_{\hat{\theta}\hat{\rho}}(\mathbf{k}, \tau - t) C_{\theta\rho}(\mathbf{k}, t) - \Sigma_{\hat{\theta}\hat{g}}(\mathbf{k}, \tau - t) C_{v\rho}(\mathbf{k}, t) \right] \end{aligned} \quad (\text{F.54})$$

$$\begin{aligned} C_{\theta g}(\mathbf{k}, \tau) - W(\mathbf{k}) C_{\rho g}(\mathbf{k}, \tau) &= -\frac{1}{T} \left[\Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, 0) C_{\rho g}(\mathbf{k}, \tau) + \Sigma_{\hat{\theta}\hat{v}}(\mathbf{k}, 0) C_{gg}(\mathbf{k}, \tau) \right] \\ &+ \frac{1}{T} \int_0^\tau dt \left[\Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, \tau - t) \partial_t C_{\rho g}(\mathbf{k}, t) + \Sigma_{\hat{\theta}\hat{v}}(\mathbf{k}, \tau - t) \partial_t C_{gg}(\mathbf{k}, t) \right. \\ &\left. - \Sigma_{\hat{\theta}\hat{\rho}}(\mathbf{k}, \tau - t) C_{\theta g}(\mathbf{k}, t) - \Sigma_{\hat{\theta}\hat{g}}(\mathbf{k}, \tau - t) C_{vg}(\mathbf{k}, t) \right] \end{aligned} \quad (\text{F.55})$$

$$\begin{aligned} C_{\theta\theta}(\mathbf{k}, \tau) - W(\mathbf{k}) C_{\rho\theta}(\mathbf{k}, \tau) &= \frac{1}{T} \Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, \tau) \\ &- \frac{1}{T} \left[\Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, 0) C_{\rho\theta}(\mathbf{k}, \tau) + \Sigma_{\hat{\theta}\hat{v}}(\mathbf{k}, 0) C_{g\theta}(\mathbf{k}, \tau) \right] \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{T} \int_0^\tau dt \left[\Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, \tau - t) \partial_t C_{\rho\theta}(\mathbf{k}, t) + \Sigma_{\hat{\theta}\hat{v}}(\mathbf{k}, \tau - t) \partial_t C_{g\theta}(\mathbf{k}, t) \right. \\
 & \left. - \Sigma_{\hat{\theta}\hat{\rho}}(\mathbf{k}, \tau - t) C_{\theta\theta}(\mathbf{k}, t) - \Sigma_{\hat{\theta}\hat{g}}(\mathbf{k}, \tau - t) C_{v\theta}(\mathbf{k}, t) \right] \quad (\text{F.56})
 \end{aligned}$$

$$\begin{aligned}
 C_{\theta v}(\mathbf{k}, \tau) - W(\mathbf{k}) C_{\rho v}(\mathbf{k}, \tau) & = \frac{1}{T} \Sigma_{\hat{\theta}\hat{v}}(\mathbf{k}, \tau) \\
 & - \frac{1}{T} [\Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, 0) C_{\rho v}(\mathbf{k}, \tau) + \Sigma_{\hat{\theta}\hat{v}}(\mathbf{k}, 0) C_{g v}(\mathbf{k}, \tau)] \\
 & + \frac{1}{T} \int_0^\tau dt \left[\Sigma_{\hat{\theta}\hat{\theta}}(\mathbf{k}, \tau - t) \partial_t C_{\rho v}(\mathbf{k}, t) + \Sigma_{\hat{\theta}\hat{v}}(\mathbf{k}, \tau - t) \partial_t C_{g v}(\mathbf{k}, t) \right. \\
 & \left. - \Sigma_{\hat{\theta}\hat{\rho}}(\mathbf{k}, \tau - t) C_{\theta v}(\mathbf{k}, t) - \Sigma_{\hat{\theta}\hat{g}}(\mathbf{k}, \tau - t) C_{v v}(\mathbf{k}, t) \right] \quad (\text{F.57})
 \end{aligned}$$

$$\begin{aligned}
 C_{v\rho}(\mathbf{k}, \tau) - \frac{1}{\rho_0} C_{g\rho}(\mathbf{k}, \tau) & = -\frac{1}{T} [\Sigma_{\hat{v}\hat{\theta}}(\mathbf{k}, 0) C_{\rho\rho}(\mathbf{k}, \tau) + \Sigma_{\hat{v}\hat{v}}(\mathbf{k}, 0) C_{g\rho}(\mathbf{k}, \tau)] \\
 & + \frac{1}{T} \int_0^\tau dt \left[\Sigma_{\hat{v}\hat{\theta}}(\mathbf{k}, \tau - t) \partial_t C_{\rho\rho}(\mathbf{k}, t) + \Sigma_{\hat{v}\hat{v}}(\mathbf{k}, \tau - t) \partial_t C_{g\rho}(\mathbf{k}, t) \right. \\
 & \left. - \Sigma_{\hat{v}\hat{\rho}}(\mathbf{k}, \tau - t) C_{\theta\rho}(\mathbf{k}, t) - \Sigma_{\hat{v}\hat{g}}(\mathbf{k}, \tau - t) C_{v\rho}(\mathbf{k}, t) \right] \quad (\text{F.58})
 \end{aligned}$$

$$\begin{aligned}
 C_{vg}(\mathbf{k}, \tau) - \frac{1}{\rho_0} C_{gg}(\mathbf{k}, \tau) & = -\frac{1}{T} [\Sigma_{\hat{v}\hat{\theta}}(\mathbf{k}, 0) C_{\rho g}(\mathbf{k}, \tau) + \Sigma_{\hat{v}\hat{v}}(\mathbf{k}, 0) C_{gg}(\mathbf{k}, \tau)] \\
 & + \frac{1}{T} \int_0^\tau dt \left[\Sigma_{\hat{v}\hat{\theta}}(\mathbf{k}, \tau - t) \partial_t C_{\rho g}(\mathbf{k}, t) + \Sigma_{\hat{v}\hat{v}}(\mathbf{k}, \tau - t) \partial_t C_{gg}(\mathbf{k}, t) \right. \\
 & \left. - \Sigma_{\hat{v}\hat{\rho}}(\mathbf{k}, \tau - t) C_{\theta\rho}(\mathbf{k}, t) - \Sigma_{\hat{v}\hat{g}}(\mathbf{k}, \tau - t) C_{vg}(\mathbf{k}, t) \right] \quad (\text{F.59})
 \end{aligned}$$

$$\begin{aligned}
 C_{v\theta}(\mathbf{k}, \tau) - \frac{1}{\rho_0} C_{g\theta}(\mathbf{k}, \tau) & = \frac{1}{T} \Sigma_{\hat{v}\hat{\theta}}(\mathbf{k}, \tau) \\
 & - \frac{1}{T} [\Sigma_{\hat{v}\hat{\theta}}(\mathbf{k}, 0) C_{\rho\hat{\theta}}(\mathbf{k}, \tau) + \Sigma_{\hat{v}\hat{v}}(\mathbf{k}, 0) C_{g\theta}(\mathbf{k}, \tau)] \\
 & + \frac{1}{T} \int_0^\tau dt \left[\Sigma_{\hat{v}\hat{\theta}}(\mathbf{k}, \tau - t) \partial_t C_{\rho\theta}(\mathbf{k}, t) - \Sigma_{\hat{v}\hat{v}}(\mathbf{k}, \tau - t) \partial_t C_{g\theta}(\mathbf{k}, t) \right. \\
 & \left. - \Sigma_{\hat{v}\hat{\rho}}(\mathbf{k}, \tau - t) C_{\theta\rho}(\mathbf{k}, t) + \Sigma_{\hat{v}\hat{g}}(\mathbf{k}, \tau - t) C_{v\theta}(\mathbf{k}, t) \right] \quad (\text{F.60})
 \end{aligned}$$

$$\begin{aligned}
 C_{vv}(\mathbf{k}, \tau) - \frac{1}{\rho_0} C_{gv}(\mathbf{k}, \tau) & = \frac{1}{T} \Sigma_{\hat{v}\hat{v}}(\mathbf{k}, \tau) \\
 & - \frac{1}{T} [\Sigma_{\hat{v}\hat{\theta}}(\mathbf{k}, 0) C_{\rho v}(\mathbf{k}, \tau) + \Sigma_{\hat{v}\hat{v}}(\mathbf{k}, 0) C_{g v}(\mathbf{k}, \tau)] \\
 & + \frac{1}{T} \int_0^\tau dt \left[\Sigma_{\hat{v}\hat{\theta}}(\mathbf{k}, \tau - t) \partial_t C_{\rho v}(\mathbf{k}, t) - \Sigma_{\hat{v}\hat{v}}(\mathbf{k}, \tau - t) \partial_t C_{g v}(\mathbf{k}, t) \right. \\
 & \left. - \Sigma_{\hat{v}\hat{\rho}}(\mathbf{k}, \tau - t) C_{\theta\rho}(\mathbf{k}, t) + \Sigma_{\hat{v}\hat{g}}(\mathbf{k}, \tau - t) C_{v v}(\mathbf{k}, t) \right]. \quad (\text{F.61})
 \end{aligned}$$

As for BDD, the number of independent correlators is smaller than the number of equations, and there are here six redundant equations. The extension of the proof of appendix D to the present equations is straightforward but very painful.

These are the exact non-perturbative dynamical equations preserving FDT. One can then use different approximation schemes for self-energies to concretize the equations. It

is worth noting that whatever the approximation is the FDT is always verified due to the way the equations were derived.

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