Integrability of graph combinatorics via random walks and heaps of dimers

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Abstract

We investigate the integrability of the discrete non-linear equation governing the dependence on geodesic distance of planar graphs with inner vertices of even valences. This equation follows from a bijection between graphs and blossom trees and is expressed in terms of generating functions for random walks. We construct explicitly an infinite set of conserved quantities for this equation, also involving suitable combinations of random walk generating functions. The proof of their conservation, i.e. their eventual independence on the geodesic distance, relies on the connection between random walks and heaps of dimers. The values of the conserved quantities are identified with generating functions for graphs with fixed numbers of external legs. Alternative equivalent choices for the set of conserved quantities are also discussed and some applications are presented.

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1. Introduction

Graph combinatorics seems to provide a remarkable source for (discrete) integrable systems. So far, the underlying integrability seemed to be intimately related to the existence of a matrix model formulation for the counting of these graphs [1,2]. In the context of matrix model solutions to 2D Quantum Gravity [3], the generating functions for possibly decorated graphs of arbitrary genus may indeed be interpreted as tau-functions of various integrable hierarchies [4].

More recently, an alternative purely combinatorial and bijective approach to the enumeration of *planar* graphs was developed, based on the transformation of graphs into decorated trees [5-7]. In this context, another remarkable, apparently unrelated integrable structure was observed, now involving the *geodesic distance* on the graphs [8].

More precisely, in Ref.[8], special attention was paid to the case of planar graphs with vertices of even valence only, and to their generating function R_n with two distinguished points at geodesic distance at most n, with n a non-negative integer. The latter was shown to obey a self-consistent *master equation* turning into a recursion relation on n for graphs with bounded valences. This non-linear recursion relation turned out to be exactly solvable in terms of discrete soliton-like tau-functions. The precise solution involved integration constants which may be rephrased into *conserved quantities*, thus displaying explicitly the integrability of the master equation. The knowledge of these conserved quantities provides a powerful tool for generating explicit expressions for various generating functions of interest in the graph combinatorial framework. In particular, exploiting the conserved quantities allows to bypass the quite tedious use of the exact solutions of Ref.[8].

The aim of this paper is to construct *ab initio* a set of conserved quantities for the master equation in a purely combinatorial setting, and to interpret them in the language of graphs. As will be recalled below, the master equation has a compact expression in terms of discrete random walks. Our construction relies therefore on properties of partition functions for such random walks, as well as for "hard dimers" on a line, both with suitable weights involving the R_n's. Both objects are known [9,11] to be related via some boson/fermion type inversion relation. The origin of this relation is best understood by relating random walks to so-called "heaps of dimers", a boson-like counterpart of the mutually excluding, thus fermion-like, hard dimers.

The paper is organized as follows. In Section 2, we first recall the existing bijection between on the one hand planar graphs with two external legs and inner vertices of even valence, and on the other hand planted trees with two kinds of leaves, the so-called blossom trees (Sect.2.1). This bijection is used to re-derive the master equation for R_n in Sect.2.2. Other possible applications of the master equation are discussed in Sect.2.3, as well as its integrable nature in Sect.2.4. Section 3 is devoted to the study of generating functions for random walks, heaps of dimers, and one-dimensional hard dimer configurations and to their relationships. In Section 3.1, we first present a useful fundamental relation for the generating function of random walks, which is then identified with that of a particular class of heaps of dimers called pyramids. Details of this identification are gathered in Appendix A. This identification allows to derive a crucial inversion relation involving the partition function for hard dimers on a segment. In Section 4, we first give explicit formulas for the conserved quantities $\Gamma_{2i}(n)$, $i = 1, 2, \ldots$ in terms of partition functions for random walks (Sect.4.1). The actual conservation of these quantities, i.e. their independence of n whenever the master equation is satisfied, is proved in Sect.4.2 by extensive use of the fundamental and inversion formulas of Section 3. Section 5 concerns the graph interpretation of the conserved quantities $\Gamma_{2i}(n)$ in terms of $2i$-point functions, i.e. generating functions for graphs with $2i$ external legs (Sect.5.1). In the case of graphs with bounded valences, say up to $2m$, the infinite set of conserved quantities is reducible to the finite set of the $m-1$ first ones. The corresponding (linear) relations between their constant values are derived in Sect.5.2. Finally, Sect.5.3 presents an alternative bijective proof for the conservation of $\Gamma_2(n)$, interpreted as the generating function for graphs with the two legs in the same face. Possible extensions of this bijective proof to higher order conserved quantities are also discussed, in view of particularly suggestive identities equivalent to the conservation of Γ_{2i} for $i = 2, 3, 4$. In Section 6, we first present two equivalent alternative sets of conserved quantities, one respecting the "time-reversal" ($n \to -n$) symmetry of the master equation (Sect.6.1) and the other compacted, i.e. involving the least possible number of R_n's (Sect.6.2). Proofs of their conservation rely on generalized inversion formulas, all displayed in Appendix B. To illustrate the interest of using conserved quantities, we give in Sect.6.3 an example of application to the statistics of neighbors of the external face in pure tetravalent and pure hexavalent graphs. We gather a few concluding remarks in Section 7.

2. Generating functions for planar graphs: master equation

2.1. From planar graphs to blossom trees

Let us first recall the bijection between planar graphs and blossom trees. More precisely, by planar graphs, we mean here graphs with planar topology, with inner vertices of *even valences* only and with two univalent endpoints (see Fig.1-(a) for an example). We shall refer to the endpoints as *legs* and to the graphs as *two-leg diagrams*. The two legs are distinguished into an incoming and an outcoming leg and need not be adjacent to the same face. We call *geodesic distance* between the two legs the minimal number of edges to be crossed in a path joining both legs in the plane. In the planar representation, we choose to have the incoming leg adjacent to the external face.

On the other hand, by blossom trees, we mean planted plane trees satisfying

(B1) all the inner vertices of the tree have even valences;
(B2) the endpoints of the tree are of two types: buds and leaves;
(B3) each $2k$-valent inner vertex is adjacent to exactly $k-1$ buds;

(see Fig.1-(e) for an example). In this characterization, it is assumed that the tree has at least one inner vertex but, by convention, the tree reduced to a single leaf attached to the root is also considered as a blossom tree. Note that from property (B3), we deduce that the total number of leaves in a blossom tree is equal to the total number of buds plus one.

The bijection may be obtained by cutting each two-leg diagram into a blossom tree as follows. We first visit successively each edge bordering the external face in counterclockwise direction around the diagram (see Fig.1-(b)), starting from the incoming leg. Any visited

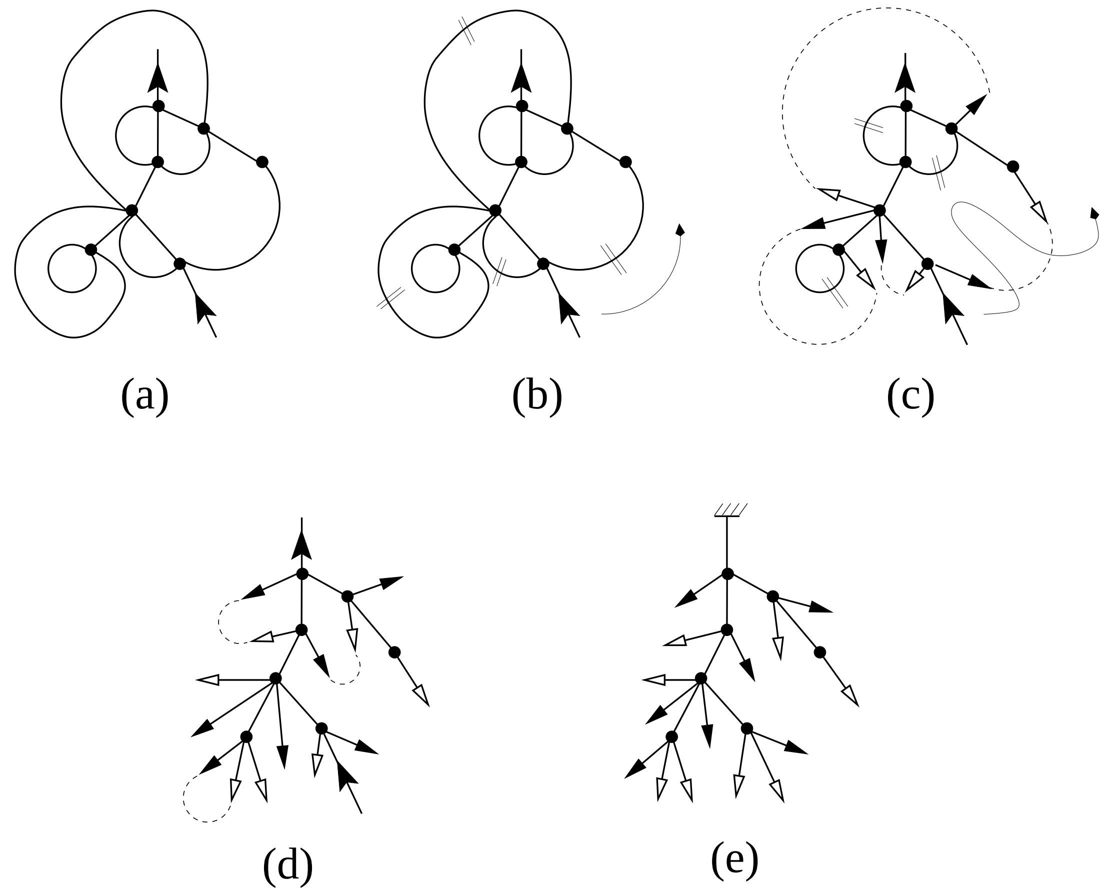

Fig. 1: A two leg-diagram with distinguished incoming and outcoming legs (a). The incoming leg is adjacent to the external face. All inner vertices have even valence. Here the geodesic distance between the legs is 1. This diagram is transformed into a blossom tree (e) by cutting edges into bud-leaf pairs as explained in the text. Buds (resp. leaves) are represented by black (resp. white) arrows. Here the cutting procedure requires performing two turns around the graph: (b)→(c) and (c)→(d). The outcoming leg serves as root for the blossom tree while the incoming one is replaced by a leaf. In the blossom tree, each inner vertex of valence $2k$ is adjacent to exactly $k-1$ buds.

edge is cut into a bud-leaf pair if and only if this operation does not disconnect the diagram. After one turn, the net result has been to merge a number of internal faces with the external one. We repeat the process on this newly obtained diagram (see Fig.1-(c)) until all faces are merged, resulting into a tree (see Fig.1-(d)). We finally replace the incoming leg by a leaf and plant the tree at the outcoming one (see Fig.1-(e)). As shown in Ref.[5], the resulting tree is a blossom tree and the above construction establishes a bijection between, on the one hand two-leg diagrams and, on the other hand blossom trees. Note that the inner vertices of the obtained tree are clearly in one-to-one correspondence with those of the corresponding diagram and that their valence is preserved. This allows to keep track of vertex valences in both pictures.

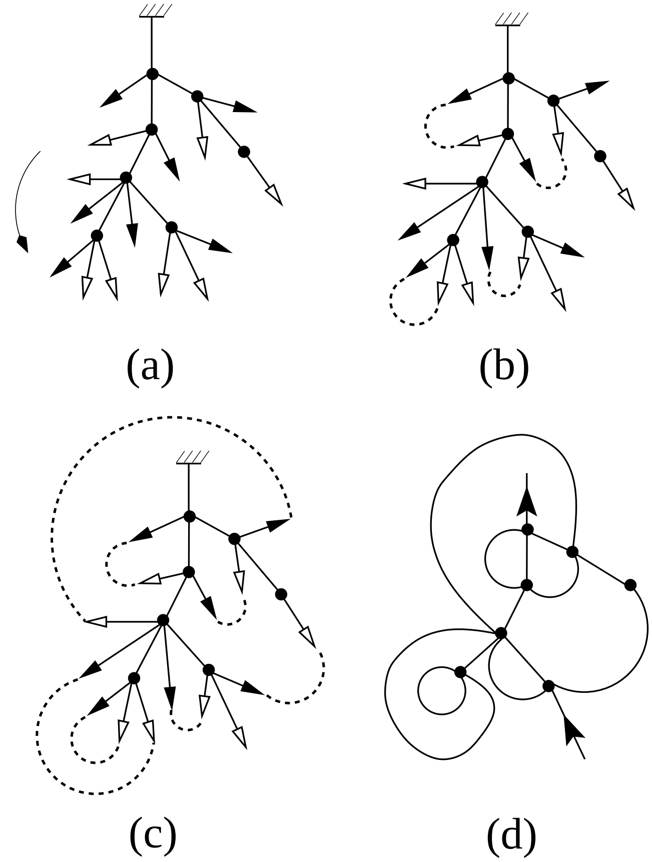

Fig. 2: The closing procedure from a blossom tree (a) to a two-leg diagram (d). Each pair of a bud immediately followed counterclockwise by a leaf is glued into an edge (b). This is iterated (c) until all leaves but one are glued. This remaining leaf is replaced by an incoming leg while the root becomes an outcoming one. The closing procedure here encircles the root once, hence the geodesic distance between the legs is 1.

The inverse of the cutting procedure consists in reading the sequence of buds and leaves in counterclockwise order around the tree (see Fig.2-(a)) and connecting each pair of a bud immediately followed by a leaf into an edge (see Fig.2-(b)) The process is repeated until only one leaf is left (see Fig.2-(c)), which is chosen as the incoming leg while the root becomes the outcoming one (see Fig.2-(d)). Note that this root is usually encircled by a number of edges separating it from the outer face. This number is nothing but the geodesic distance between the legs. A simple way to measure it is to forbid to encircle the root in the closing process. The geodesic distance is then given by the number of *excess* buds, i.e. those buds which remain unmatched. Alternatively, we may define the *contour walk* of a blossom tree by reading the sequence of buds and leaves clockwise around the tree starting from the root and performing a down (resp. up) step for each encountered bud (resp. leaf) (see Fig.3). Starting from height 0 and making steps of ±1, the contour walk ends at height 1 and the geodesic distance between the legs of the corresponding diagram

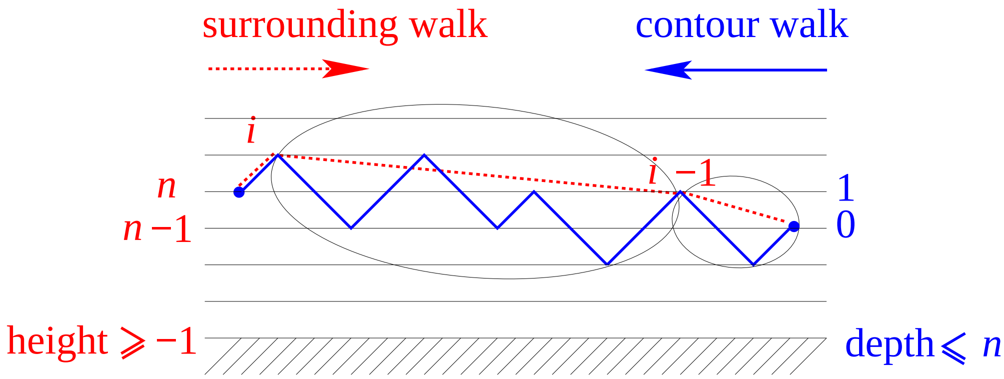

Fig. 3: The contour and surrounding walks of the blossom tree of Fig.1(e), or equivalently of Fig.2(a). The contour walk, represented in solid line, is here to be read from right to left and records the succession of buds and leaves clockwise around the tree from the root. Starting from height 0, it makes an ascending step for each encountered leaf and a descending one for each bud, hence ends at height 1. Its depth is here 1, equal to the absolute value of the minimum visited height. The surrounding walk, represented in dotted line, is to be read from left to right and records to sequence of buds and blossom subtrees counterclockwise around the root vertex from the root. To check whether the tree at hand contributes to R_n, we let the surrounding walk start from height n and make an ascending step for each encountered bud and a descending one for each blossom subtree. Each descending step may be viewed as the net contribution of the subtree within the contour walk. Here each blob corresponds to such a subtree. The depth of the contour walk will remain less or equal to n if the depth of each (vertically shifted) contour walk in a blob starting at height i remains less or equal to i.

is given by the *depth* of the walk, defined as the absolute value of the minimum height reached by the walk.

2.2. Master equation

We may now define the generating function R_n for two-leg diagrams with weight g_k per $2k$-valent vertex and with legs at a geodesic distance *less or equal* to n. From the above bijection, R_n is also the generating function for blossom trees with weight g_k per $2k$-valent inner vertex and whose contour walk has depth less or equal to n or equivalently with at most n excess buds in the closing process that forbids encircling the root.

It is now easy to write down a *master equation* for R_n by decomposing each blossom tree according to the environment of the inner vertex attached to the root (root vertex). This environment may be decomposed into a sequence of buds and *descendent subtrees*. All these subtrees are themselves blossom trees either made of a single leaf or satisfying the above characterizations (B1-B3). Note that the contour walk of the original tree is the concatenation of down steps for the buds and of the (vertically shifted) contour walks of the blossom subtrees. Hence, the constraint of depth less than n translates into constraints on the depth reached by the contour walk of each blossom subtree. This turns into a closed relation between the R_n's.

More precisely, the environment of the root vertex is best described by yet another walk, the *surrounding walk*, now obtained by reading the sequence of buds and subtrees in the opposite, counterclockwise direction around the root vertex. Starting from height n at the root, we make a step $+1$ for each encountered bud and -1 for each encountered blossom subtree (see Fig.3). Each -1 step may be seen as the net contribution of the blossom subtree at hand within the total (reversed) contour walk. To ensure that the depth of the entire contour walk remains less or equal to n, we must require that each subtree encountered at height i in the surrounding walk has a contour walk with depth less or equal to i. This leads us to attach a weight R_i to each -1 step from height i to height $(i-1)$.

The description of the root vertex environment by its surrounding walk and the attached weights are efficiently encoded into a "transfer matrix" Q acting on a formal orthonormal basis $|i\rangle$ $(i \in \mathbb{Z})$ as

$$Q|i\rangle = |i+1\rangle + R_i\,|i-1\rangle \tag{2.1}$$

The generating function $Z_{a,b}(k)$ for walks of k steps going from, say height a to height b, and with weight R_i for each descent $i \to (i-1)$ is easily expressed in terms of Q as

$$Z_{a,b}(k) \equiv \langle b|Q^k|a\rangle \tag{2.2}$$

Note that $Z_{a,b}(k)$ is non zero only for $(b-a) = k \bmod 2$. From the above analysis, the quantity $g_k\, Z_{n,n-1}(2k-1)$ is the generating function for blossom trees with a $2k$-valent root vertex and with contour walk of depth less or equal to n. To incorporate arbitrary (even) valences, we introduce the quantities

$$V'(Q) = \sum_{k\geq 1} g_k Q^{2k-1} \tag{2.3}$$

and

$$V'_{a,b} \equiv \langle b|V'(Q)|a\rangle = \sum_{k\geq 1} g_k Z_{a,b}(2k-1) \tag{2.4}$$

(which is non zero only for $(b-a)$ odd). The quantity $V'_{a,b}$ may be viewed as a "grand-canonical" generating function for walks from a to b and with arbitrary odd length.

With these notations, we may finally write the master equation

$$\boxed{R_n = 1 + V'_{n,n-1}} \tag{2.5}$$

where the first contribution (1) corresponds to having a single leaf connected to the root. This relation is valid for all $n \geq 0$ with the convention that $R_i = 0$ for all $i < 0$. This

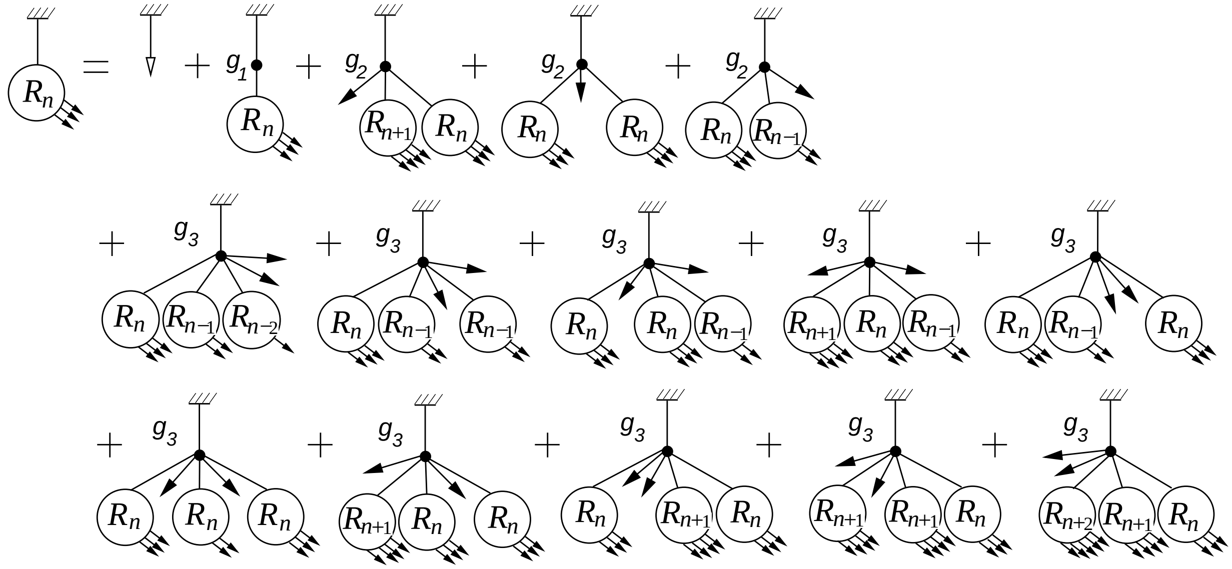

Fig. 4: A schematic representation of the truncated master equation (2.6). The generating function R_n is decomposed according to the environment of the root vertex, with 0, 1, 2 buds and 1, 2, 3 blossom subtrees according to its valence 2, 4, 6 respectively. To ensure that the depth of the contour walk does not exceed n, we have to impose bounds on the maximal number of excess buds in each subtree. We have represented (here for $n = 3$) these excess buds in maximal number for each subtree. The index of R for each subtree is nothing but this maximal number.

equation determines all the R_n's as formal power series of the g_k's upon imposing that $R_n = 1 + \mathcal{O}(\{g_k\})$ for all $n \geq 0$.
For illustration, let us consider the simple case of diagram with only bi-, tetra- and hexavalent inner vertices, i.e. with $g_k = 0$ for all $k \geq 4$. The master equation then reads

$$\begin{aligned} R_n &= 1 + g_1 R_n + g_2 R_n (R_{n-1} + R_n + R_{n+1}) \\ &\quad + g_3 R_n \big(R_{n-2} R_{n-1} + R_{n-1}^2 + R_n^2 + 2R_n (R_{n-1} + R_{n+1}) \\ &\qquad\qquad + R_{n-1} R_{n+1} + R_{n+1}^2 + R_{n+1} R_{n+2} \big) \end{aligned} \tag{2.6}$$

This relation is illustrated in Fig.4. In the following, we will also make use of the generating function R for two-leg diagrams without restriction on the geodesic distance. This may be recovered as the limit $n \to \infty$ of R_n, hence R satisfies the relation

$$R = 1 + \sum_{k \geq 1} g_k \binom{2k-1}{k} R^k \tag{2.7}$$

obtained by counting walks of length $2k - 1$ with global height difference equal to -1, thus having k descending steps each weighted by R. In the truncated case above, Eq.(2.7) reduces to

$$R = 1 + g_1 R + 3 g_2 R^2 + 10 g_3 R^3 \tag{2.8}$$

2.3. Other applications of the master equation

The master equation may be rewritten as

$$R_n = \frac{1}{1 - V'_{n-1,n}} \tag{2.9}$$

by noting that $V'_{n,n-1} = R_n V'_{n-1,n}$. The latter is a direct consequence of the general reflection relation

$$Z_{a,a-1}(2k-1) = R_a Z_{a-1,a}(2k-1) \tag{2.10}$$

obtained by first transferring the weights R_i to the ascending steps $(i-1) \to i$ and then reflecting the walks, thus interpreting them as walks from $(a-1) \to a$. Note that there are as many descending steps $i \to (i-1)$ as ascending steps $(i-1) \to i$ except for $i = a$ where there is one more descent (before reflection), hence the extra factor R_a needed to reproduce the correct weight.

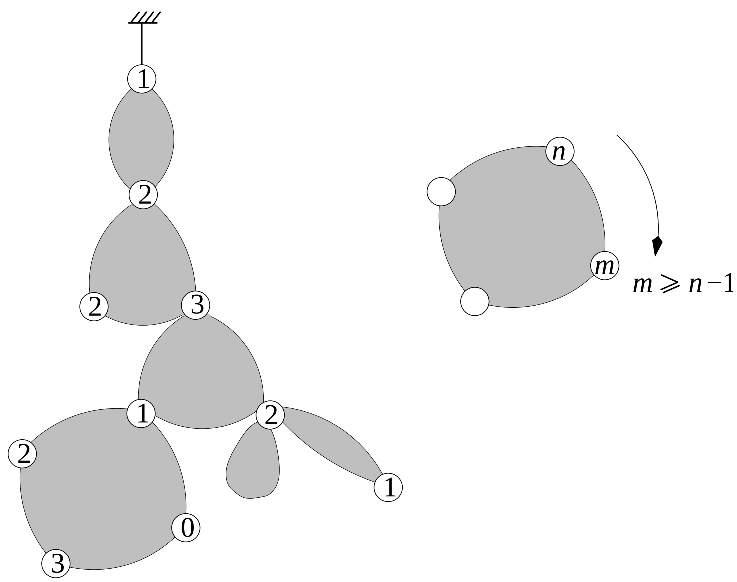

Fig. 5: An example of labeled mobile (left) made of (inflated) polygons attached by their vertices into a tree-like structure. The vertex labels around each polygon obey the rule displayed on the right that they either decrease by 1 or increase weakly clockwise.

In its alternative form (2.9), the master equation appeared in a slightly different enumeration problem, that of so-called *labeled mobiles* [12], generalizing the so-called well labeled trees of Ref.[13]. These mobiles are made of rigid polygons, with weight g_k per k-gon, glued by their vertices into a rooted tree-like object (see Fig.5 for an example). The labels are subject to local rules around each k-gon: going clockwise around a k-gon, the labels must either decrease by 1 or increase weakly. This rule again is efficiently described by the transfer matrix Q of Eq.(2.1) by reading the labels around k-gons from the vertex at which they are suspended and translating the sequence of labels into heights forming a closed walk of k steps around the k-gon. This walk is then transformed into a closed *boundary walk* of length $2k$ by transforming each non-negative step $(+p)$ into a descent followed by $p+1$ unit ascending steps. Denoting by R_n the generating function for planted mobiles with root labeled n and inspecting the possible environments of the root, we have the relation

$$R_n = \frac{1}{1 - \sum\limits_{k\geq 1} g_k M_n(k)} \tag{2.11}$$

where $M_n(k)$ is the generating function for k-gons rooted at a vertex labelled n and with $k-1$ sub-mobiles dangling at the other vertices. In $M_n(k)$, we have to assign a weight R_i to each vertex of the k-gon labelled i, except for the root vertex. This amounts equivalently to assign a weight R_i to each -1 step from height i in the boundary walk, except for the -1 step from the root. This leads to

$$M_n(k) = Z_{n-1,n}(2k-1) = \langle n|Q^{2k-1}|n-1 > \tag{2.12}$$

Eqs.(2.11) and (2.12) boil down to Eq.(2.9). Again this equation must be supplemented with some initial conditions, for instance that $R_i = 0$ for $i < 0$ if we demand that the labels remain non-negative.

Returning to graph enumeration problems, it was shown that mobiles with non-negative labels are in one to one correspondence with planar graphs with *faces* of even valences only and with a *distinguished origin vertex* [12,13]. The vertices of the mobile are in correspondence with vertices of the associated graph, and the labels represent the geodesic distance (on the associated graph) of these vertices from the origin vertex. In this framework, R_n is now interpreted as the generating function for the graphs with a *distinguished edge* at geodesic distance less than n from the origin vertex.

Finally, a third domain of application of the master equation comes from the study of spatially extended branching processes, i.e. probabilistic models for the evolution and spreading of a population. In this language, the index n stands for the (discrete one-dimensional) position of individuals and the blossom trees or mobiles are interpreted as the genealogical structure of families. As shown in Ref.[14], the generating function R_n is then related to the probability for a population with germ at position n never to spread up to position 0.

2.4. Integrability of the master equation

A remarkable feature of the master equation is that it may be solved exactly. More precisely, we will consider truncated versions of Eq.(2.5) by considering only diagrams with valences up to, say $2m$. This amounts to set $g_k = 0$ for $k > m$ in which case $V'(Q)$ is an odd polynomial of degree $2m-1$. As explained in Ref.[8], the general solution of Eq.(2.5), now with arbitrary initial conditions but with the requirement that it converges at $n \to \infty$ takes the surprisingly simple form

$$R_n = R\,\frac{u_{n+1}u_{n+4}}{u_{n+2}u_{n+3}} \tag{2.13}$$

where u_n has a multi-soliton structure involving $m-1$ integration constants $\lambda_1, \ldots, \lambda_{m-1}$. Those constants may be fixed by the initial conditions. This structure is characteristic of (discrete) integrable systems and implies the existence of conserved quantities. More precisely, by inverting the system $R_i = R_i(\{\lambda\})$ for any set $i = n, n+1, \ldots, n+m-2$, one may *in principle* construct $m-1$ conserved quantities for Eq.(2.5) in the form $\lambda_j = \Lambda_j(R_n . R_{n+1}, \ldots, R_{n+m-2}) =$ const. independently of n. In practice, this construction is however difficult to implement. The remainder of the paper is devoted to the explicit construction of conserved quantities *ab initio*, i.e. without reference to the above solution (2.13). This construction will be carried out by use of simple combinatorial tools only.

3. Random walks and heaps of dimers

3.1. From random walks to heaps of dimers

In this Section, we shall investigate a number of properties of the partition function of random walks $Z_{a,b}(k)$ above. In particular, we will emphasize the connection between random walks and so-called *heaps of dimers*, leading us eventually to an inversion relation involving partition functions for hard dimers on a segment. All the present relations will be instrumental for proving the conservation statements of Section 4 below.

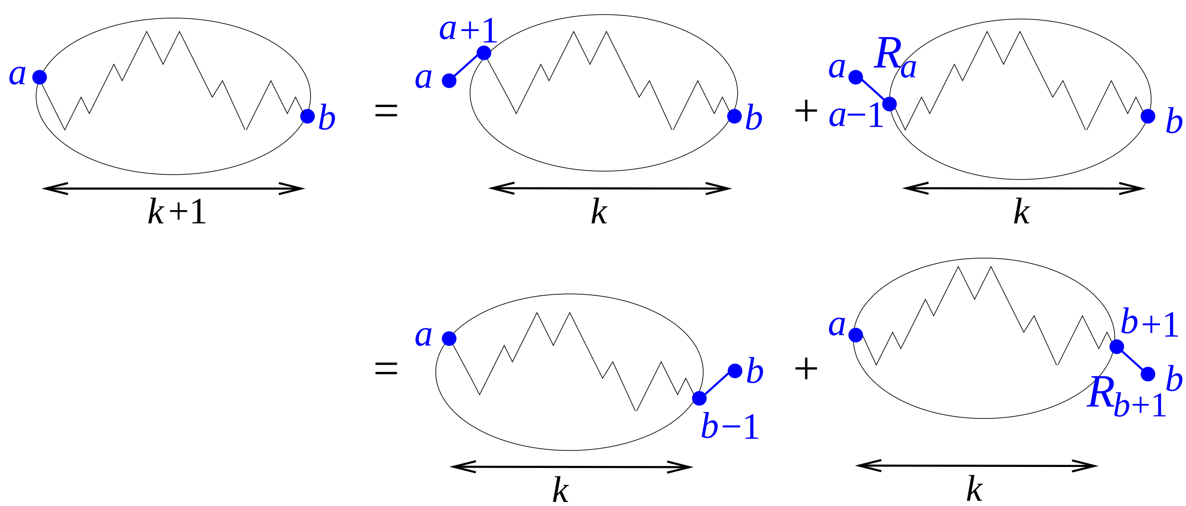

Fig. 6: A schematic representation of the fundamental identity (3.1) for walks. The walks are decomposed according to either their first step (first line) or their last one (second line).

A first fundamental identity is obtained by expressing the generating function for paths of length $k+1$ in two different ways: (i) as the concatenation of a path of length 1 (first step) and a path of length k, or (ii) as the concatenation of a path of length k and a path of length 1 (last step). According to whether the first (resp. the last) step is up or down, we have respectively

$$Z_{a,b}(k+1) = Z_{a+1,b}(k) + R_a Z_{a-1,b}(k) = Z_{a,b-1}(k) + R_{b+1} Z_{a,b+1}(k) \tag{3.1}$$

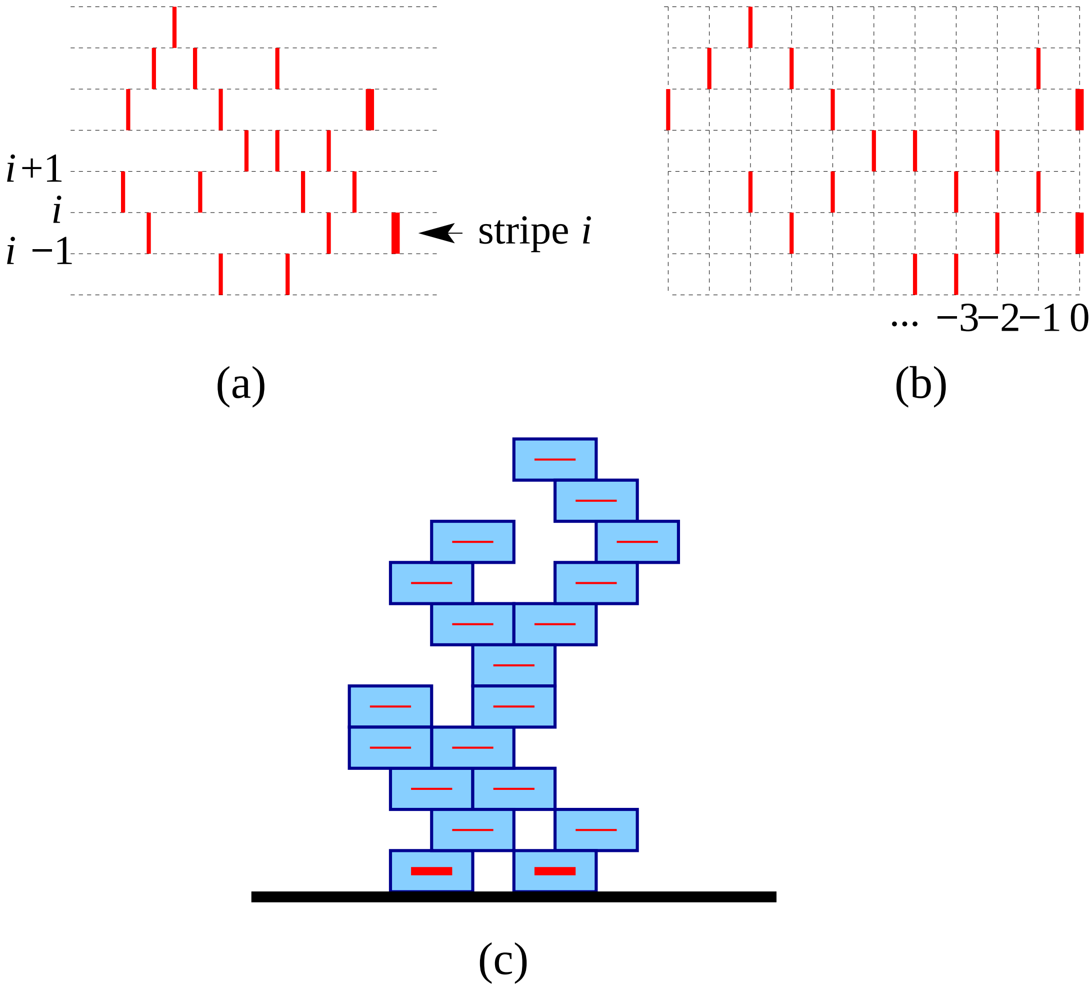

Fig. 7: An example of a heap of dimers (a) and its canonical representation (b). The thickened dimers correspond to the right projection of the heap, forming in (b) a hard dimer configuration on the vertical line $x = 0$. By rotating the picture (b) and extending each dimer into a 2×1 rectangle, we obtain a piling up of these rectangles (c) whose bottom row is formed by the former right projection.

as illustrated in Fig.6.

A second important property is that we may interpret $Z_{a,b}(k)$ as a generating function for heaps of dimers, a particular case of the so-called heaps of pieces introduced in Ref.[9]. These heaps of dimers also occur in relation to lattice animals [10] and so-called Lorentzian gravity [11] and may be defined as follows. Let us decompose the plane into parallel (say horizontal) stripes of width 1 by drawing horizontal lines at integer vertical positions. The stripes are labeled by the position of their top boundary. We then place within the stripes a number of dimers, i.e. vertical segments of length 1 (see Fig.7-(a)). All dimers may freely slide within their stripe provided they do not cross a dimer within the same stripe or within the stripe immediately above and below. A heap of dimers is defined modulo this sliding freedom and only records the *relative* positions of the dimers. In a canonical representation, we may place all the dimers at integer horizontal coordinates within their respective stripes with the requirement that these coordinates be negative or zero and *maximal* (see Fig.7-(b)). This is realized by pushing all dimers as much to the right as possible while staying at integer horizontal positions in the left half plane. This in turn allows upon rotating the picture by 90° clockwise to view a configuration as a piling up of rectangles of size 2×1 (see Fig.7-(c)) deposited on the $x = 0$ line. For each heap of dimers, we may define its *right projection* as the configuration made of those dimers which may slide freely all the way to infinity to the right. In the canonical representation, these are those dimers with horizontal position 0. Clearly, the right projection of any heap of dimers is a configuration of *hard dimers* on a (vertical) line, i.e. a set of dimers occupying (vertical) segments $[i-1, i]$ with the restriction that no two dimers may come in contact (the occupied segments must be disjoint).

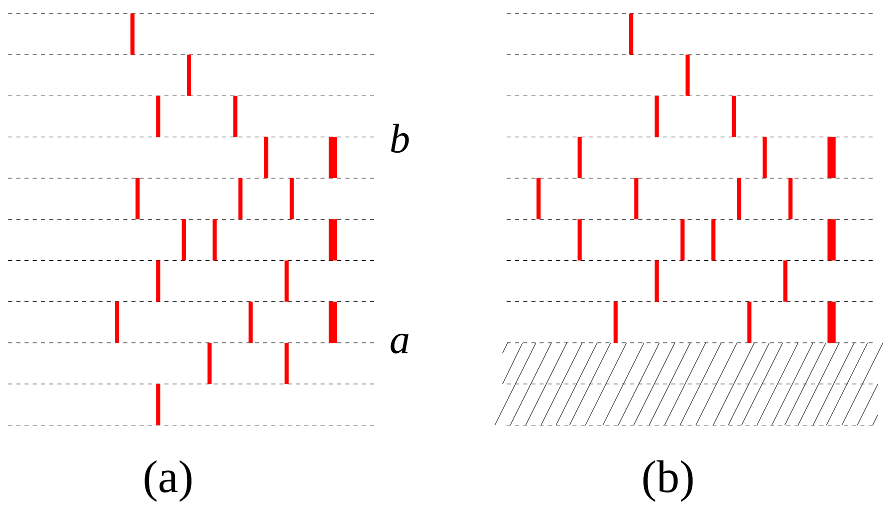

Fig. 8: An example of a pyramid (a) and a half-pyramid (b), both with base $[a, b]$.

For $(b-a)$ odd and positive, we may consider the maximally occupied hard dimer configuration of the segment $[a, b]$. It is made of $(b-a+1)/2$ dimers occupying the elementary segments $[a+2i, a+2i+1]$, $i = 0, \ldots, (b-a-1)/2$. Any heap of dimer having this maximally occupied segment as right projection will be called a *pyramid* of base $[a, b]$ (see Fig.8-(a) for an example). We then define the generating function $H_{a,b}(k)$ for pyramids of base $[a, b]$ made of a *total* number k of dimers, with weight R_i per dimer in the stripe labeled i, *except* for the dimers of the right projection. Note that $H_{a,b}(k)$ is non zero only if $k \geq (b-a+1)/2$ and that $H_{a,b}((b-a+1)/2) = 1$.

We these definitions, we have the remarkable relation

$$Z_{a,b}(2k-1) = H_{a,b}(k) \tag{3.2}$$

valid for $(b-a)$ odd and positive, and $k \geq 1$. This relation is illustrated in Fig.9-(a). It

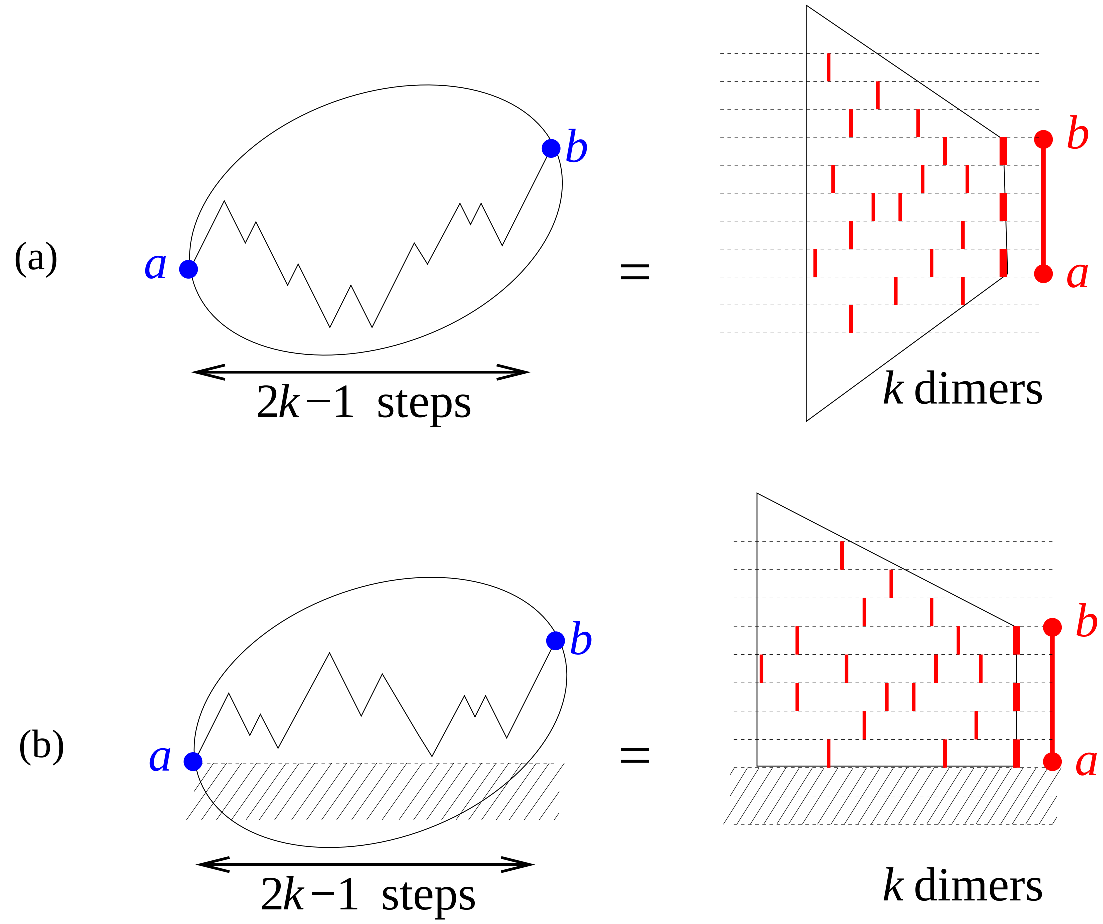

Fig. 9: A schematic representation (first line) of the equality (3.2) between generating functions $Z_{a,b}(2k-1)$ for walks from a to b with $2k-1$ steps and $H_{a,b}(k)$ for pyramids of base $[a,b]$ with a total of k dimers. The similar representation (second line) for the relation (3.3) between the generating function $Z^+_{a,b}(2k-1)$ of positive walks and that $H^+_{a,b}((k)$ of half-pyramids.

follows from a general bijection between walks and heaps of pieces given in Ref.[9]. This bijection is recalled in detail in Appendix A below for the particular case at hand.

In the next section, we shall also make use of the generating function $Z^+_{a,b}(k)$ for walks of k steps going from height a to height b, with weight R_i per descent $i \to (i-1)$, and *which stay at or above height* a. In the following, we shall refer to these walks as *positive walks*. The generating function $Z^+_{a,b}(k)$ may be obtained from $Z_{a,b}(k)$ by simply taking $R_i \to 0$ for all labels $i \leq a$. In particular, we deduce that, for $(b-a)$ odd and positive and $k \geq 1$

$$Z^+_{a,b}(2k-1) = H^+_{a,b}(k) \tag{3.3}$$

where $H^+_{a,b}(k)$ denotes the generating function for *half-pyramids* of base $[a,b]$ with a total of k dimers, i.e. pyramids with dimers only in stripes with labels $i > a$ (see Fig.8-(b) for an example). This relation (3.3) is illustrated in Fig.9-(b).

3.2. Inversion relation

Viewing hard dimers on a line as mutually excluding and fermionic objects (with at most one dimer in a unit segment), heaps of dimers appear as their interacting bosonic counterpart in which arbitrarily many dimers may be piled up in each stripe. It is therefore natural to expect some boson/fermion inversion relations to hold between their respective generating functions. An example of such inversion relation in a "grand-canonical" ensemble, i.e. for arbitrarily many dimers, may be found in Refs.[9,11]. In this Section, we shall derive another inversion formula, now in the canonical ensemble, i.e. with fixed numbers of dimers.

Introducing the generating function $\Pi_{a,b}(k)$ for *configurations of* k *hard dimers* in the segment $[a,b]$ and with weight R_i per dimer in the segment $[i-1,i]$, we have the remarkable inversion relation

$$\sum_{\ell=0}^{k}(-1)^{k-\ell}\Pi_{a,a+2k-1}(k-\ell)Z^+_{a,a+2j}(2\ell)=\delta_{k,j} \tag{3.4}$$

for $k,j\geq 0$, with the convention that $\Pi_{a,a-1}(0)=1$. This relation may be written in a more compact matrix form by defining a lower-triangular, semi-infinite matrix $\mathbf{Z}(a)$ with entries

$$\mathbf{Z}(a)_{i,j}\equiv Z^+_{a,a+2j}(2i) \tag{3.5}$$

for $0\leq j\leq i$, and a lower-triangular, semi-infinite matrix $\mathbf{D}(a)$ with entries

$$\mathbf{D}(a)_{k,i}=(-1)^{k-i}\Pi_{a,a+2k-1}(k-i) \tag{3.6}$$

for $0\leq i\leq k$ while all other entries vanish. The relation (3.4) now reads

$$\mathbf{D}(a)\mathbf{Z}(a)=\mathbf{I} \tag{3.7}$$

with $\mathbf{I}$ the (semi-infinite) identity matrix. To prove the relation (3.4), let us consider *pairs* $\mathcal{P}$ made of

(i) a hard dimer configuration in the segment $[a,a+2k-1]$
(ii) a half-pyramid with projection $[a,a+2j-1]$

with a total number of k dimers. In the half-pyramid, each dimer in the stripe i (including the dimers of the right projection) receives a weight R_i. On the contrary, in the hard dimer configuration, each dimer in the segment $[i-1,i]$ receives a weight $-R_i$. Denoting by ℓ the number of dimers in the half-pyramid, the generating function $P_a(j,k)$ for the above pairs reads

$$P_a(j,k)=\Pi_{a,a+2j-1}(j)\sum_{\ell=0}^{k}(-1)^{k-\ell}\Pi_{a,a+2k-1}(k-\ell)H^+_{a,a+2j-1}(\ell) \tag{3.8}$$

Note that $P_a(j,k)$ is non zero only if $k\geq j$. For each such pair $\mathcal{P}$, we may absorb all dimers of the hard dimer configuration into a larger heap $\mathcal{H}(\mathcal{P})$ of k dimers by simply adding them

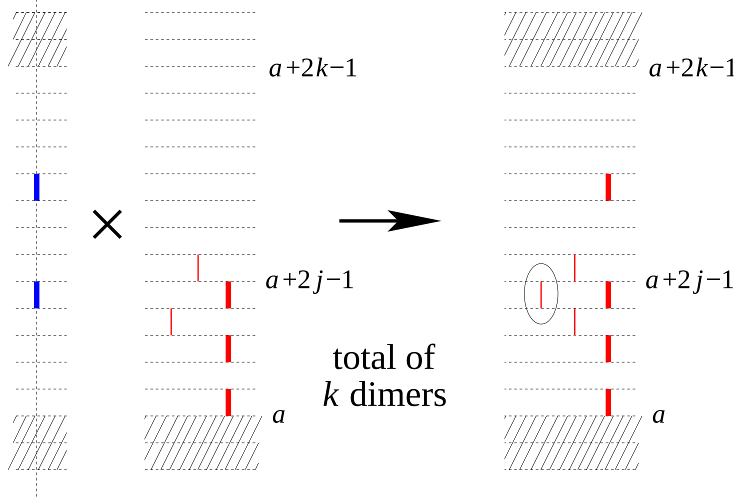

Fig. 10: An example (left) of pair $\mathcal{P}$ of a hard-dimer configuration in the segment $[a, a+2k-1]$ and a half-pyramid of base $[a, a+2j-1]$ (with $j \leq k$), with a total number k of dimers (here $k=7$). These are concatenated (right) into a larger heap $\mathcal{H}$. The equivalence class $\mathcal{C}(\mathcal{H})$ of those pairs leading to the same heap $\mathcal{H}$ is constructed by considering in $\mathcal{H}$ the dimers which belong to the left projection of the heap but do not belong to the right projection. In the present example, there is exactly one such (encircled) dimer. The equivalence class $\mathcal{C}(\mathcal{H})$ is generated by distributing in all possible ways these dimers either in the hard-dimer configuration or in the half-pyramid. If we assign opposite weights to dimers in the hard-dimer configuration and in the half-pyramid, this results in a vanishing net contribution of the class $\mathcal{C}(\mathcal{H})$ unless the left and right projections of $\mathcal{H}$ are identical. This happens only when $\mathcal{H}$ is the maximally occupied hard-dimer configuration made of k dimers in the segment $[a, a+2k-1]$.

to the left of the original half-pyramid (see Fig.10). We can then regroup all pairs leading to the same larger heap $\mathcal{H}$ into an equivalence class $\mathcal{C}(\mathcal{H})$. Conversely, given $\mathcal{H} = \mathcal{H}(\mathcal{P})$, we may reconstruct all pairs in the class $\mathcal{C}(\mathcal{H})$ by considering the set of dimers *which belong to the left projection* of $\mathcal{H}$ (those dimers which may be pushed all the way to infinity to the left) but *which do not belong to the right projection*. If this set contains at least one dimer, this dimer may be incorporated either in the hard dimer configuration of the pair or in its half-pyramid, contributing with opposite weights. This causes the contribution to $P_a(j,k)$ of the class $\mathcal{C}(\mathcal{H})$ to vanish. The only case where no such vanishing takes place is when the left projection of $\mathcal{H}$ is identical to its right projection, in which case $\mathcal{H}$ itself is a hard dimer configuration with k dimers on the segment $[a, a+2k-1]$, therefore the unique maximally occupied hard dimer configuration, with contribution $(-1)^{k-j}\Pi_{a,a+2k-1}(k)$ as the j lower dimers with label less than $a+2j-1$ belong to the half-pyramid and the $(k-j)$ upper ones belong to the hard dimer configuration. We immediately deduce that

$$P_a(j,k) = (-1)^{k-j}\Pi_{a,a+2k-1}(k)\ \delta_{k\geq j} \tag{3.9}$$

with $\delta_{k\geq j} = 1$ for $k \geq j$ and zero otherwise. Using $\delta_{k,j} = \delta_{k\geq j} - \delta_{k\geq j+1}$, we deduce

$$\begin{aligned}\delta_{k,j}\ \Pi_{a,a+2k-1}(k) &= P_a(j,k) + P_a(j+1,k) \\ &= \sum_{\ell=0}^{k} (-1)^{k-\ell}\Pi_{a,a+2k-1}(k-\ell) \\ &\quad \times \left(H^+_{a,a+2j-1}(\ell)\Pi_{a,a+2j-1}(j) + H^+_{a,a+2j+1}(\ell)\Pi_{a,a+2j+1}(j+1)\right)\end{aligned} \tag{3.10}$$

Using $H^+_{a,a+2j-1}(\ell) = Z^+_{a,a+2j-1}(2\ell-1)$, $H^+_{a,a+2j+1}(\ell) = Z^+_{a,a+2j+1}(2\ell-1)$ from Eq.(3.3) and the relation $\Pi_{a,a+2j+1}(j+1) = R_{a+2j+1}\Pi_{a,a+2j-1}(j)$ for maximally occupied segments, we obtain

$$\sum_{\ell=0}^{k} (-1)^{k-\ell}\Pi_{a,a+2k-1}(k-\ell)\left(Z^+_{a,a+2j-1}(2\ell-1) + R_{a+2j+1}Z^+_{a,a+2j+1}(2\ell-1)\right) = \delta_{k,j} \tag{3.11}$$

which, upon using

$$Z^+_{a,a+2j-1}(2\ell-1) + R_{a+2j+1}Z^+_{a,a+2j+1}(2\ell-1) = Z^+_{a,a+2j}(2\ell) \tag{3.12}$$

from Eq.(3.1), reduces to the desired inversion relation (3.4).

Note that, as $\mathbf{D}(a)$ and $\mathbf{Z}(a)$ are lower-triangular, we may decide to truncate them to $m \times m$ matrices $\mathbf{D}_m(a)$ and $\mathbf{Z}_m(a)$ with indices i strictly less than m. These matrices clearly satisfy the inversion relation $\mathbf{D}_m(a)\mathbf{Z}_m(a) = I_m$, with I_m the $m\times m$ identity matrix. For illustration, taking $m = 3$, we have

$$\begin{aligned}\mathbf{D}_3(a) &= \begin{pmatrix} 1 & 0 & 0 \\ -R_{a+1} & 1 & 0 \\ R_{a+1}R_{a+3} & -(R_{a+1}+R_{a+2}+R_{a+3}) & 1 \end{pmatrix} \\ \mathbf{Z}_3(a) &= \begin{pmatrix} 1 & 0 & 0 \\ R_{a+1} & 1 & 0 \\ R_{a+1}(R_{a+1}+R_{a+2}) & (R_{a+1}+R_{a+2}+R_{a+3}) & 1 \end{pmatrix}\end{aligned} \tag{3.13}$$

which are inverse of one-another.

4. Conserved quantities

4.1. Definition of the conserved quantities

We now present compact expressions for a particular "basis" of conserved quantities of the master equation (2.5). More precisely, we shall define below a set of quantities $\{\Gamma_{2i}(n)\}$ for $i \geq 1$ and $n \geq 0$ satisfying $\Gamma_{2i}(n)$= const. independently of n. Of course, any functions of the Γ_{2i}'s are conserved as well and the precise choice of basis below has been done in regard of its combinatorial nature. Other, alternative choices will be discussed in Sects.6.1 and 6.2 below.

Using the generating function $Z^+_{a,b}(k)$ of previous section for positive walks and the grand-canonical generating function $V'_{a,b}$ for walks of odd length, we define the quantity $\Gamma_{2i}(n)$ by

$$\boxed{\Gamma_{2i}(n) = Z^+_{n-1,n-1}(2i)\Gamma_0(n) - \sum_{j=1}^{i} Z^+_{n-1,n-1+2j}(2i)V'_{n+2j-1,n-2}} \tag{4.1}$$

for $i \geq 1$ and $n \geq 0$, with $\Gamma_0(n)$ given by

$$\Gamma_0(n) = R_{n-1} - V'_{n-1,n-2} + \delta_{n,0} \tag{4.2}$$

for $n \geq 0$. The quantity $\Gamma_0(n)$ itself is not *stricto sensu* a conserved quantity. However, the master equation precisely ensures that $\Gamma_0(n) = 1$ for $n \geq 1$ while $\Gamma_0(0) = 1$ by definition (since $R_{-1} = V'_{-1,-2} = 0$), hence the conservation of Γ_0 is a tautology. This in turn allows to simplify the expression (4.1) for $\Gamma_{2i}(n)$ by substituting $\Gamma_0(n) = 1$. The proof of the conservation of $\Gamma_{2i}(n)$ however is made much simpler by keeping as such the slightly more involved definition (4.1). Introducing the vectors $\vec{\Gamma}(n)$ and $\vec{V}'(n)$ with components

$$\vec{\Gamma}(n)_i = \Gamma_{2i}(n) \quad \text{and} \quad \vec{V}'(n)_j = (R_{n-1} + \delta_{n,0})\delta_{j,0} - V'_{n+2j-1,n-2} \tag{4.3}$$

for $i, j \geq 0$, the above relations (4.1) and (4.2) read simply

$$\vec{\Gamma}(n) = \mathbf{Z}(n-1)\ \vec{V}'(n) \tag{4.4}$$

with $\mathbf{Z}(a)$ defined as in (3.5). It will prove useful to invert this relation. This is readily performed by use of the inversion relation (3.4) or (3.7), with the result

$$\vec{V}'(n) = \mathbf{D}(n-1)\vec{\Gamma}(n) \tag{4.5}$$

with $\mathbf{D}(a)$ defined in (3.6), namely

$$V'_{n+2j-1,n-2} = \sum_{i=0}^{j}(-1)^{i-1}\Pi_{n-1,n+2j-2}(i)\Gamma_{2j-2i}(n) + \delta_{j,0}(R_{n-1} + \delta_{n,0}) \tag{4.6}$$

with again the convention that $\Pi_{n-1,n-2}(0) = 1$.

We are now ready to prove that the quantities $\Gamma_{2i}(n)$ of Eq.(4.1) are conserved quantities of the master equation (2.5). To this end, we shall now use the fundamental equation for V'

$$V'_{a+1,b} + R_a V'_{a-1,b} = V'_{a,b-1} + R_{b+1} V'_{a,b+1} \tag{4.7}$$

directly inherited from the fundamental equation (3.1) for $Z_{a,b}(k)$. Taking $a = n + 2j - 1$ and $b = n - 1$, we get the identity

$$\begin{aligned} V'_{(n+1)+2j-1,(n+1)-2} &+ R_{n+2j-1} V'_{(n+1)+2(j-1)-1,(n+1)-2} \\ &= V'_{n+2j-1,n-2} + R_n V'_{(n+2)+2(j-1)-1,(n+2)-2} \end{aligned} \tag{4.8}$$

For $j \geq 1$ and $n \geq 0$, we may substitute Eq.(4.6) into Eq.(4.8), leading to

$$\begin{aligned} &\big(-\Gamma_{2j}(n+1) + \sum_{i=1}^{j} (-1)^{i-1} \Pi_{n,(n+1)+2j-2}(i) \Gamma_{2j-2i}(n+1) \big) \\ &\quad + R_{n+2j-1} \big(\sum_{i=0}^{j-1} (-1)^{i-1} \Pi_{n,(n+1)+2(j-1)-2}(i) \Gamma_{2(j-1)-2i}(n+1) + \delta_{j-1,0} R_n \big) \\ &= \big(-\Gamma_{2j}(n) + \sum_{i=1}^{j} (-1)^{i-1} \Pi_{n-1,n+2j-2}(i) \Gamma_{2j-2i}(n) \big) \\ &\quad + R_n \big(\sum_{i=0}^{j-1} (-1)^{i-1} \Pi_{n+1,(n+2)+2(j-1)-2}(i) \Gamma_{2(j-1)-2i}(n+2) + \delta_{j-1,0} R_{n+1} \big) \end{aligned} \tag{4.9}$$

where we have used $\Pi_{n-1,n+2j-2}(0) = \Pi_{n,n+2j-1}(0) = 1$ for all $j \geq 0$ (unique empty dimer configuration). Noting that the boundary terms on both sides cancel, we now change

$i \to (i-1)$ in the last sum of each side and rearrange the factors of Γ, leading us to

$$
\begin{aligned}
&\Gamma_{2j}(n+1) - \Gamma_{2j}(n) \\
&\qquad = \sum_{i=1}^{j} (-1)^{i-1} \big\{ \Pi_{n,n+2j-1}(i) - R_{n+2j-1}\Pi_{n,n+2j-3}(i-1) \big\} \Gamma_{2j-2i}(n+1) \\
&\qquad\quad - \sum_{i=1}^{j} (-1)^{i-1} \big\{ \Pi_{n-1,n+2j-2}(i) - R_n \Pi_{n+1,n+2j-2}(i-1) \big\} \Gamma_{2j-2i}(n) \\
&\qquad\quad - \sum_{i=1}^{j} (-1)^{i} R_n \Pi_{n+1,n+2j-2}(i-1) \big(\Gamma_{2j-2i}(n+2) - \Gamma_{2j-2i}(n) \big) \\
&\qquad = \sum_{i=1}^{j} (-1)^{i-1} \Pi_{n,n+2j-2}(i) \big(\Gamma_{2j-2i}(n+1) - \Gamma_{2j-2i}(n) \big) \\
&\qquad\quad - \sum_{i=1}^{j} (-1)^{i} R_n \Pi_{n+1,n+2j-2}(i-1) \big(\Gamma_{2j-2i}(n+2) - \Gamma_{2j-2i}(n) \big)
\end{aligned}
\tag{4.10}
$$

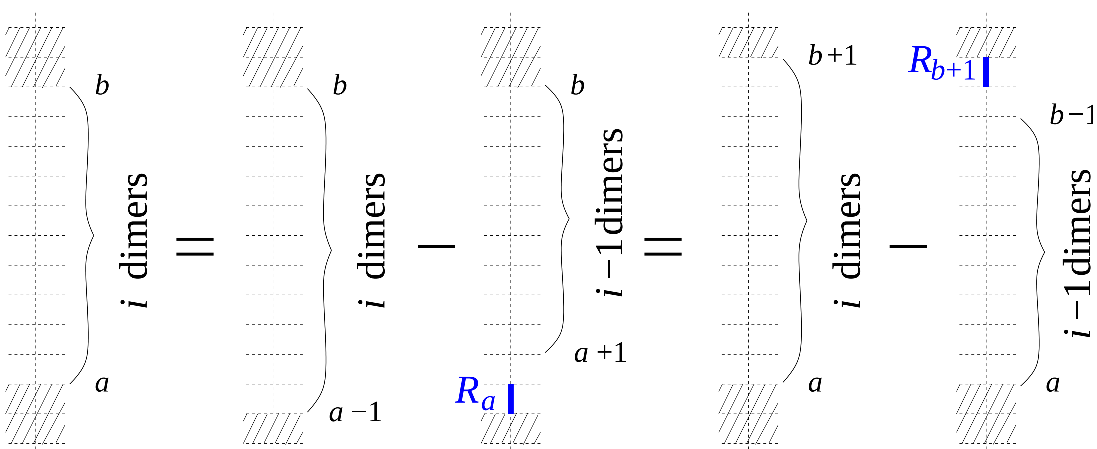

Fig. 11: A schematic representation of the fundamental identity (4.12). A configuration of hard dimers on the segment $[a, b]$ may be viewed either as a configuration on the segment $[a-1, b]$ where the unit segment $[a-1, a]$ is empty (hence the subtraction) or as a configuration on the segment $[a, b+1]$ where the unit segment $[b, b+1]$ is empty.

In the last equality, we have used the property

$$
\begin{aligned}
&\Pi_{n,n+2j-1}(i) - R_{n+2j-1}\Pi_{n,n+2j-3}(i-1) \\
&\qquad = \Pi_{n-1,n+2j-2}(i) - R_n \Pi_{n+1,n+2j-2}(i-1) = \Pi_{n,n+2j-2}(i)
\end{aligned}
\tag{4.11}
$$

which is a particular instance ($a = n$, $b = n + 2j - 2$) of the fundamental identity

$$\Pi_{a,b}(i) = \Pi_{a-1,b}(i) - R_a \Pi_{a+1,b}(i-1) = \Pi_{a,b+1}(i) - R_{b+1}\Pi_{a,b-1}(i-1) \tag{4.12}$$

satisfied by generating functions for hard dimers. This identity is illustrated in Fig.11 and may be viewed as the hard dimer counterpart of Eq.(3.1) for random walks.

From Eq.(4.10), we immediately deduce by induction on j that

$$\Gamma_{2j}(n+1) = \Gamma_{2j}(n) \tag{4.13}$$

for all $j \geq 1$ and $n \geq 0$ provided that $\Gamma_0(n+1) = \Gamma_0(n)$ for all $n \geq 0$. As already mentioned, this last requirement is precisely guaranteed by the master equation (2.5). The Γ_{2i}'s therefore form a set of conserved quantities for this equation.

For illustration, we list below the first two conserved quantities for the truncated case of graphs with valences up to $2m = 6$, corresponding to the truncated master equation (2.6):

$$\begin{aligned}
\Gamma_2(n) &= R_n - V'_{n+1,n-2} \\
\Gamma_4(n) &= R_n(R_n + R_{n+1}) - (R_n + R_{n+1} + R_{n+2})V'_{n+1,n-2} - V'_{n+3,n-2}
\end{aligned} \tag{4.14}$$

with

$$\begin{aligned}
V'_{n+1,n-2} &= R_{n+1}R_nR_{n-1}\big(g_2 + g_3(R_{n+2} + R_{n+1} + R_n + R_{n-1} + R_{n-2})\big) \\
V'_{n+3,n-2} &= g_3 R_{n+3}R_{n+2}R_{n+1}R_nR_{n-1}
\end{aligned} \tag{4.15}$$

where we explicitly substituted $\Gamma_0(n) = 1$ into Eq.(4.1).

5. Graph interpretation

5.1. Conserved quantities as multi-point correlation functions

The constant value of the conserved quantities has a nice combinatorial interpretation as a multi-point correlation function. More precisely, let us define by $G_{2i}(\{g_k\})$ the so-called (disconnected) $2i$-point function, i.e. the generating function for possibly disconnected $2i$-leg diagrams, namely graphs with inner vertices of even valences (weighted g_k per $2k$-valent vertex) and with $2i$ legs, i.e. univalent vertices, adjacent to the same (external) face. These legs are distinguished and labeled $1, 2, \ldots, 2i$ counterclockwise. We have the identification

$$\Gamma_{2i}(n) = G_{2i} \tag{5.1}$$

for all $n \geq 0$ and all $i \geq 0$ with the convention that $G_0 = 1$. To prove this, we simply evaluate $\Gamma_{2i}(n)$ at $n = 0$. Noting that $V'_{2j-1,-2} = 0$ as it is proportional to $R_{-1} = 0$, we are left with

$$\Gamma_{2i}(0) = Z^+_{-1,-1}(2i) = Z_{-1,-1}(2i) = Z_{0,-1}(2i-1) \tag{5.2}$$

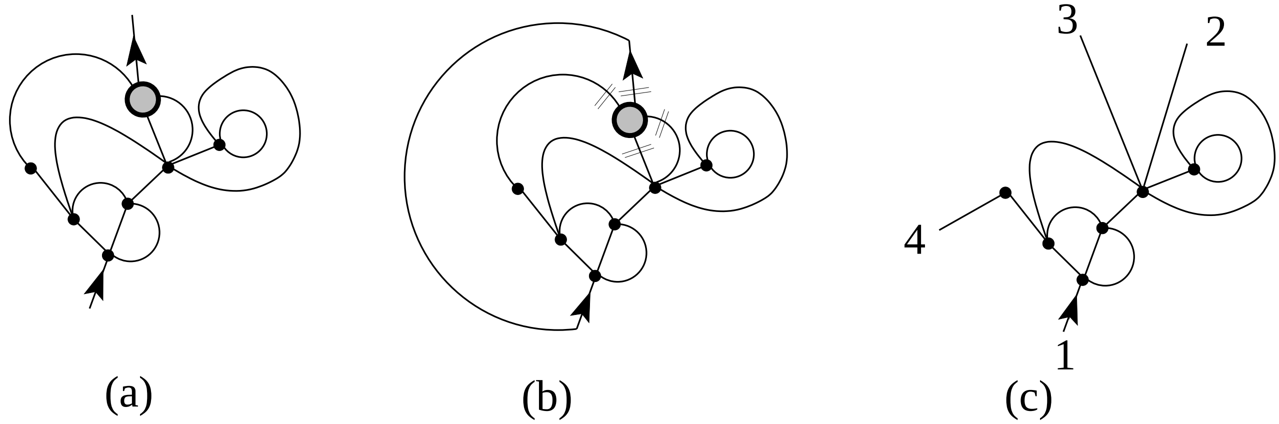

Fig. 12: Bijection between (a) two-leg diagrams with both legs in the same face and whose outcoming leg is adjacent to a $2i$-valent vertex and (c) $2i$-leg diagrams. In the intermediate step (b), we have glued the two legs into a rooted edge and cut out the $2i$-valent vertex. The legs are labeled counterclockwise by $1, 2, \ldots 2i$ with the former incoming leg labeled 1. Note that the $2i$-leg diagrams need not be connected in general.

where we first note that $R_{-1} = 0$ automatically selects positive walks and we then remove the first (ascending) step with weight 1.

As explained in Sect.2.2, the quantity $Z_{0,-1}(2i-1)$ is the generating function for blossom trees with a $2i$-valent root vertex and a contour walk with 0 depth. Returning to the graph interpretation, it is identified via the bijection of Sect.2.1 as the generating function for two-leg diagrams with an outcoming leg attached to a $2i$-valent vertex and adjacent to the *same* (external) face as the incoming leg. The $2i$-valent vertex receives a weight 1 as opposed to all other inner vertices weighted by the usual factors g_k if they are $2k$-valent. Such two-leg diagrams are in bijection with the $2i$-leg diagrams defined above. Indeed, starting from these two-leg diagrams and gluing their two legs into a marked edge, we simply erase the unweighted $2i$-valent vertex and obtain the desired $2i$-leg diagrams with a marked leg which receives the label 1 (see Fig.12 for an example). Note that $2i$-leg diagrams need not be connected except for $i = 1$ and may split into connected $2j$-leg diagrams with some j's summing to i. Eq.(5.1) follows.

An outcome of this result is that we obtain a host of explicit expressions for G_{2i} in terms of the generating functions R_i of blossom trees corresponding to various choices of n in Eq.(5.1). The first expression is nothing but $G_{2i} = Z_{0,-1}(2i-1)$ corresponding to $n = 0$. For illustration, for $i = 1, 2$ and 3, we have

$$\begin{aligned} G_2 &= R_0 \\ G_4 &= R_0(R_0 + R_1) \\ G_6 &= R_0(R_0^2 + 2R_0R_1 + R_1^2 + R_1R_2) \end{aligned} \tag{5.3}$$

Another particularly simple choice consists in using Eq.(5.1) in the limit $n \to \infty$. This gives explicit formulas for G_{2i} in terms of the function R solution of Eq.(2.7) (recall that

R is simply the generating function for two-leg diagrams with arbitrary geodesic distance between the legs). Letting $R_a \to R$ for all a, we first find that

$$
\begin{aligned}
Z^{+}_{n-1,n-1+2j}(2i) &\to \left(\binom{2i}{i-j} - \binom{2i}{i-j-1} \right) R^{i-j} \\
V'_{n+2j-1,n-2} &\to \sum_{k\geq j+1} g_k \binom{2k-1}{k+j} R^{k+j}
\end{aligned}
\tag{5.4}
$$

which finally yields from Eq.(4.1)

$$
\begin{aligned}
G_{2i} = &\left(\binom{2i}{i} - \binom{2i}{i-1} \right) R^i \\
&- \sum_{k\geq 2} g_k R^{i+k} \sum_{j=1}^{\min(i,k-1)} \left(\binom{2i}{i-j} - \binom{2i}{i-j-1} \right) \binom{2k-1}{k+j}
\end{aligned}
\tag{5.5}
$$

These expressions are determined equivalently as the solutions of the inverse relation (4.6) at large n, which reads

$$
\sum_{k\geq j+1} g_k \binom{2k-1}{k+j} R^{k+j} = \sum_{i=0}^{j} (-1)^{i-1} \binom{2j-i}{i} R^i G_{2j-2i}
\tag{5.6}
$$

where $\binom{2j-i}{i}$ is the number of hard dimer configurations of i dimers on a segment of length $2j-1$

For illustration in the truncated case with valences up to $2m = 6$, we find from Eq.(5.5) that

$$
\begin{aligned}
G_2 &= R - g_2 R^3 - 5 g_3 R^4 \\
G_4 &= 2R^2 - 3 g_2 R^4 - 16 g_3 R^5 \\
G_6 &= 5R^3 - 9 g_2 R^5 - 50 g_3 R^6
\end{aligned}
\tag{5.7}
$$

where R is now determined by Eq.(2.8).

The explicit expressions (5.5) or (5.6) are to be compared with other known expressions for multi-point correlation functions. The $2i$-point function above may indeed be computed alternatively either by use of the planar limit of the one matrix model or by the so-called loop equations. The planar solution of the one-matrix integral may be obtained via saddle point techniques and reads in the so-called one-cut case:

$$
\begin{aligned}
\sum_{\ell=0}^{i} G_{2i-2\ell} \binom{2\ell}{\ell} R^\ell &= \lim_{n\to\infty} \langle n | Q^{2i} (Q - V'(Q)) | n-1 \rangle \\
&= \binom{2i+1}{i} R^i - \sum_{k\geq 2} g_k R^{i+k} \sum_{j=1}^{\min(i,k-1)} \binom{2i+1}{i-j} \binom{2k-1}{k+j}
\end{aligned}
\tag{5.8}
$$

This expression is readily equivalent to Eq.(5.5) by simply noting that

$$\sum_{\ell=0}^{i}\binom{2\ell}{\ell}\left(\binom{2i-2\ell}{i-\ell-j}-\binom{2i-2\ell}{i-\ell-j-1}\right)=\binom{2i+1}{i-j} \tag{5.9}$$

obtained by cutting any walk of length $2i+1$ from, say 0 to height $2j+1$ at the level of its last $0\to 1$ step, resulting into a first walk of length, say 2ℓ from 0 to 0 and a positive walk of length $2(i-\ell)$ from height 1 to height $2j+1$.

On the other hand, the loop equations simply express the (disconnected) $2i+2$-point function as a sum of either a $2i+2k$-point function if the first leg is connected to a $2k$-valent vertex or a product of two lower order correlations if the first leg is connected directly to the $(2j+2)$-th leg, namely:

$$G_{2i+2}=\sum_{k\geq 1}g_k G_{2i+2k}+\sum_{j=0}^{i}G_{2i-2j}G_{2j} \tag{5.10}$$

That Eq.(5.5) solves this loop equation may be seen as a necessary consistency of the planar limit of the one-matrix model in the one-cut case which implies that Eqs.(5.10) and (5.8) are compatible.

5.2. Relations between conserved quantities for bounded valences

In the truncated case just above of valences up to $2m=6$, Eqs.(5.7) imply the relation $g_1G_2+g_2G_4+g_3G_6+1=G_2$ provided Eq.(2.8) is satisfied. This may be seen as a particular case of the loop equation (5.10) for $i=0$ above in its truncated form. More generally, the truncated loop equations for graphs with valences up to $2m$ allow for expressing all G_{2i} for $i\geq m$ as polynomials of the first $m-1$ values $G_2,G_4,\ldots,G_{2(m-1)}$. This gives a set of polynomial relations between the *values* of the conserved quantities $\Gamma_{2i}(n)$.

Using the results of previous Section, this interdependence may be rephrased into *linear* relations between the G_{2i}'s by writing Eq.(5.6) for $j\geq m$ in which case the l.h.s. vanishes. This results in

$$0=\sum_{i=0}^{j}(-1)^{i-1}\binom{2j-i}{i}R^iG_{2j-2i} \tag{5.11}$$

where R is determined by the polynomial truncation of Eq.(2.7) with $g_k=0$ for $k>m$. For illustration, for $m=3$, we have the first two relations

$$\begin{aligned}G_6&=5RG_4-6R^2G_2+R^3\\ G_8&=7RG_6-15R^2G_4+10R^3G_2-R^4\end{aligned} \tag{5.12}$$

and similar relations for G_{2i}'s with higher indices.

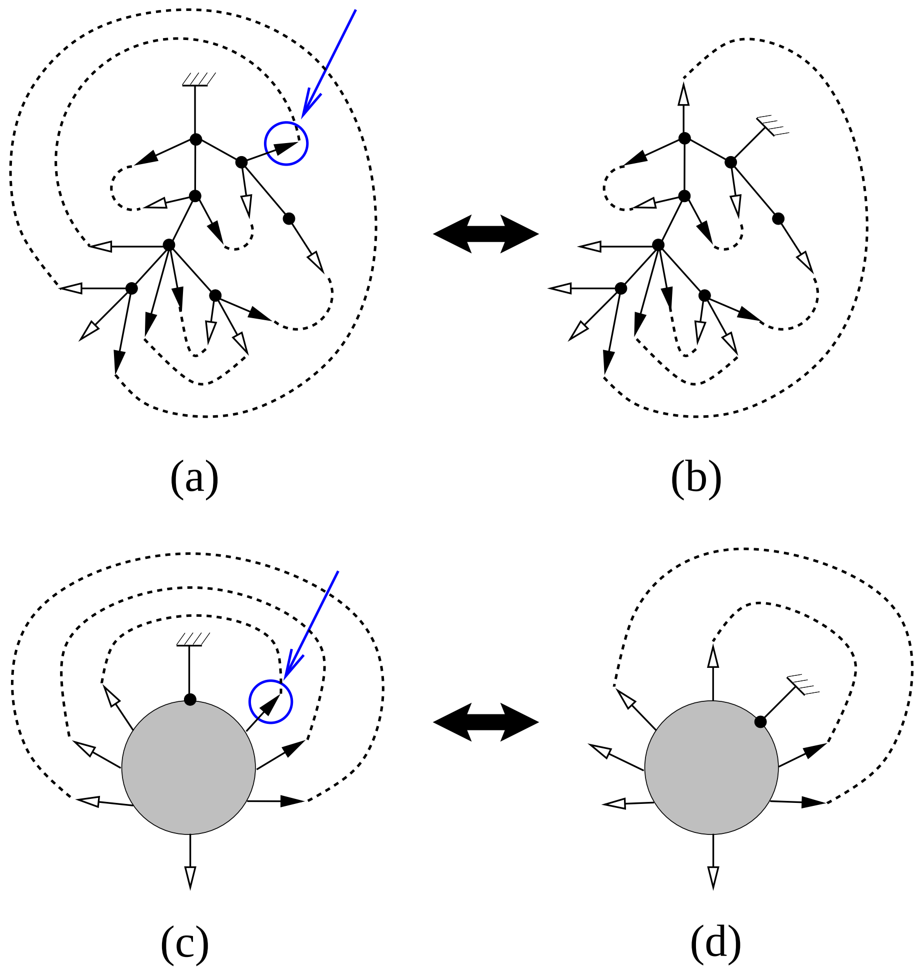

Fig. 13: Any blossom tree (a) whose root is encircled by $i \geq 1$ edges in the closing process of Sect.2.1 may be rerooted at the first excess bud clockwise from the root, while this original root is replaced by a leaf (b). This results in a rooted tree with a, say $2k$-valent root vertex adjacent to $k-2$ buds and $k+1$ blossom subtrees (with necessarily $k \geq 2$). The root of this new tree is encircled in the closing process by $i-1$ edges. In (c) and (d) we focus on the rearrangement of bud-leaf pairs in a general case. The excess buds in (d) are those of (c) except the first one, now promoted to root. This leaves three unmatched leaves.

5.3. Combinatorial interpretation of $\Gamma_2(n)$

The conservation of $\Gamma_2(n)$ has a nice combinatorial explanation in the language of blossom trees. Indeed, assuming $\Gamma_0(n) = 1$ for all $n \geq 0$, the equality $\Gamma_2(n) = \Gamma_2(0)$ for all $n \geq 1$, with $\Gamma_2(n)$ defined by Eq.(4.1), may be rewritten as

$$(R_n - R_0) = V'_{n+1,n-2} \tag{5.13}$$

The quantity $R_n - R_0$ is the generating function for blossom trees with contour walk of depth i with $0 < i \leq n$. In other words, in the closing process of these trees into two-leg diagrams, their root is encircled by at least one and at most n edges separating it from the external face. Picking the bud from which the deepest encircling edge originates, we may reroot the tree at this bud and replace the original root by a leaf (see Fig.13). The resulting object is a tree satisfying (B1), (B2) and (B3) except for the root vertex which now has exactly $(k-2)$ buds if it is $2k$-valent (note that by construction, this vertex has valence at least 4 as it originally carried a bud). In particular, all the proper subtrees of this tree are ordinary blossom trees. Finally, the contour walk of this new tree (now stepping from height 0 to height +3) has clearly depth $(i-1) \leq (n-1)$. The desired generating function may again be obtained by reading the sequence of buds and subtrees counterclockwise around the root vertex and reads $Z_{n+1,n-2}(2k-1)$ if the new root vertex is $2k$-valent. Summing over all possible valences, we get the generating function $V'_{n+1,n-2}$. As the above re-rooting procedure clearly establishes a bijection between the two types of trees at hand, this provides a *bijective proof* of the equality (5.13). As this equality is valid for all $n \geq 1$, this in turn proves the conservation of $\Gamma_2(n)$.

It would be nice to have similar bijective proofs for the conservation of $\Gamma_{2i}(n)$ for $i \geq 2$ as well. For $2i = 4$, it is easily seen that, upon using $\Gamma_0(n) = 1$ and $\Gamma_2(n+2) = \Gamma_2(n) = R_0$, the equality $\Gamma_4(n) = \Gamma_4(0)$ may be rewritten as

$$R_0(R_{n+1} - R_1) = V'_{n+3,n-2} + V'_{n+3,n} V'_{n+1,n-2} \tag{5.14}$$

The l.h.s. of this identity is nothing but the generating function for *pairs* made of a blossom tree with contour walk of depth 0 and a blossom tree with contour walk of depth i with $1 < i \leq n+1$. The r.h.s. may be interpreted as the generating function for the collection of two types of objects

(1) rooted trees satisfying (B1), (B2) and (B3) except at the root vertex which carries exactly $(k-3)$ buds if it is $2k$-valent (with necessarily $k \geq 3$). The contour walk of this tree moreover has depth at most $n-1$;

(2) pairs of rooted trees satisfying (B1), (B2) and (B3) except at their root vertex which carries exactly $(k-2)$ buds if it is $2k$-valent (with necessarily $k \geq 2$). The first of these trees has a contour walk of depth at most $n+1$ and the other a contour walk of depth at most $n-1$.

This suggests the existence of a bijection between the two collections of objects above. If such a bijection could be exhibited, it would provide an alternative bijective proof of the conservation of $\Gamma_4(n)$.

For larger i, the conservation of $\Gamma_{2i}(n)$ yields more involved relations of the type above which may suggest higher order bijections as well. For instance for $2i = 6$, we get the relation

$$\begin{aligned} &R_0 R_1 (R_{n+2} - R_2) - R_0(R_{n+1} - R_1)(R_{n+3} - R_1) = \\ &\qquad V'_{n+5,n-2} + V'_{n+5,n} V'_{n+1,n-2} + V'_{n+5,n+2} V'_{n+3,n-2} + V'_{n+5,n+2} V'_{n+3,n} V'_{n+1,n-2} \end{aligned} \tag{5.15}$$

while for $2i = 8$, we have

$$
\begin{aligned}
&R_0R_1R_2(R_{n+3}-R_3) - R_0R_1\Big((R_{n+1}-R_1)(R_{n+4}-R_2) + (R_{n+2}-R_2)(R_{n+4}-R_1)\\
&+ (R_{n+2}-R_2)(R_{n+5}-R_2)\Big) + R_0\Big((R_{n+1}-R_1)(R_{n+3}-R_1)(R_{n+5}-R_1)\Big)\\
&= V'_{n+7,n-2} + V'_{n+7,n}V'_{n+1,n-2} + V'_{n+7,n+4}V'_{n+5,n-2}\\
&+ V'_{n+7,n+2}V'_{n+3,n}V'_{n+1,n-2} + V'_{n+7,n+4}V'_{n+5,n}V'_{n+1,n-2} + V'_{n+7,n+4}V'_{n+5,n+2}V'_{n+3,n-2}\\
&+ V'_{n+7,n+2}V'_{n+3,n-2} + V'_{n+7,n+4}V'_{n+5,n+2}V'_{n+3,n}V'_{n+1,n-2}
\end{aligned}
\tag{5.16}
$$

These relations still await a bijective proof.

6. Discussion

6.1. Alternative expressions for the conserved quantities

As already mentioned, the definition (4.1) corresponds to a particular choice of conserved quantities involving positive walks. This choice however does not respect the "time reversal" symmetry of the master equation, namely under $R_{n-j} \leftrightarrow R_{n+j}$ for all j (see Eq.(2.6) for illustration). In this Section, we present another choice of conserved quantities that respect this symmetry. More precisely, we define

$$
\boxed{
\begin{aligned}
\tilde{\Gamma}_{2i}(n) = &\left\{Z_{n,n-1}(2i-1) - R_{n-1}R_{n+1}Z_{n-2,n+1}(2i-1)\right\}\tilde{\Gamma}_0(n)\\
&- \sum_{j=1}^{i}\left\{Z_{n-j,n+j-1}(2i-1) - R_{n-j-1}R_{n+j+1}Z_{n-j-2,n+j+1}(2i-1)\right\}V'_{n+j,n-j-1}
\end{aligned}
}
\tag{6.1}
$$

for $i \geq 1$ and $n \geq 0$, with $\tilde{\Gamma}_0(n)$ given by

$$
\tilde{\Gamma}_0(n) = R_n - V'_{n,n-1} \tag{6.2}
$$

for $n \geq 0$. Note that in both Z's and V''s, the indices lead to symmetric expressions.

By techniques similar to those of Sect.4.1, one can invert Eqs. (6.1) and (6.2) into

$$
V'_{n+j,n-j-1} = \sum_{i=0}^{j}(-1)^{i-1}\Pi_{n-j,n+j-1}(i)\tilde{\Gamma}_{2j-2i}(n) + \delta_{j,0}R_n \tag{6.3}
$$

with the convention $\Pi_{n,n-1}(0) = 1$. The passage from Eq.(6.1) to Eq.(6.3) makes use of an inversion relation similar to Eq.(3.4). This relation is explicited in Appendix B.

The conservation of $\tilde{\Gamma}_{2i}(n)$ may be derived along the same lines as in Sect.4.2. Using again Eq.(4.7) now with $a = n+j$ and $b = n-j$, we have the relation

$$
\begin{aligned}
&V'_{(n+1)+j,(n+1)-j-1} + R_{n+j}V'_{n+(j-1),n-(j-1)-1}\\
&\qquad = V'_{n+j,n-j-1} + R_{n-j+1}V'_{(n+1)+(j-1),(n+1)-(j-1)-1}
\end{aligned}
\tag{6.4}
$$

For $j \geq 1$ and $n \geq 0$, substituting Eq.(6.3) into this relation yields

$$
\begin{aligned}
&\big(- \tilde{\Gamma}_{2j}(n+1) + \sum_{i=1}^{j} (-1)^{i-1} \Pi_{(n+1)-j,(n+1)+j-1}(i) \tilde{\Gamma}_{2j-2i}(n+1) \big) \\
&\qquad + R_{n+j} \big(\sum_{i=0}^{j-1} (-1)^{i-1} \Pi_{n-(j-1),n+(j-1)-1}(i) \tilde{\Gamma}_{2(j-1)-2i}(n) + \delta_{(j-1),0} R_n \big) \\
&= \big(- \tilde{\Gamma}_{2j}(n) + \sum_{i=1}^{j} (-1)^{i-1} \Pi_{n-j,n+j-1}(i) \tilde{\Gamma}_{2j-2i}(n) \big) \\
&+ R_{n-j+1} \big(\sum_{i=0}^{j-1} (-1)^{i-1} \Pi_{(n+1)-(j-1),(n+1)+(j-1)-1}(i) \tilde{\Gamma}_{2(j-1)-2i}(n+1) + \delta_{(j-1),0} R_{n+1} \big)
\end{aligned}
\tag{6.5}
$$

As the boundary terms on both sides cancel, upon setting $i \to (i-1)$ in the last sum of each side, we may rewrite this identity as

$$
\begin{aligned}
&\tilde{\Gamma}_{2j}(n+1) - \tilde{\Gamma}_{2j}(n) \\
&\qquad = \sum_{i=1}^{j} (-1)^{i-1} \left\{ \Pi_{n-j+1,n+j}(i) + R_{n-j+1} \Pi_{n-j+2,n+j-1}(i-1) \right\} \tilde{\Gamma}_{2j-2i}(n+1) \\
&\qquad\quad - \sum_{i=1}^{j} (-1)^{i-1} \left\{ \Pi_{n-j,n+j-1}(i) + R_{n+j} \Pi_{n-j+1,n+j-2}(i-1) \right\} \tilde{\Gamma}_{2j-2i}(n) \\
&\qquad = \sum_{i=1}^{j} (-1)^{i-1} \left\{ \Sigma_{n-j+1,n+j-1}(i) \right\} \big(\tilde{\Gamma}_{2j-2i}(n+1) - \tilde{\Gamma}_{2j-2i}(n) \big)
\end{aligned}
\tag{6.6}
$$

where we have defined

$$
\Sigma_{a,b}(i) \equiv \Pi_{a,b+1}(i) + \Pi_{a-1,b}(i) - \Pi_{a,b}(i) \tag{6.7}
$$

In the last line of Eq.(6.6) we have used the relation $\Pi_{a,b+1}(i) + R_a \Pi_{a+1,b}(i-1) = \Pi_{a-1,b}(i) + R_{b+1} \Pi_{a,b-1}(i-1) = \Sigma_{a,b}(i)$ which is a direct consequence of Eq.(4.12). We immediately deduce from Eq.(6.6) that all $\tilde{\Gamma}_{2j}(n)$ for $j \geq 1$ are conserved provided $\tilde{\Gamma}_0(n)$ is independent of n. This is the case when the master equation (2.5) is satisfied, as it reads precisely $\tilde{\Gamma}_0(n) = 1$ for all $n \geq 0$. This completes the proof of conservation of $\tilde{\Gamma}_{2j}$.

For illustration, we list below the two independent conserved quantities for the truncated case with up to $2m = 6$-valent graph corresponding to the truncated master equation (2.6):

$$
\begin{aligned}
\tilde{\Gamma}_2(n) &= R_n - V'_{n+1,n-2} \\
\tilde{\Gamma}_4(n) &= R_n (R_{n-1} + R_n + R_{n+1}) - R_{n-1} R_{n+1} \\
&\qquad - (R_{n-1} + R_n + R_{n+1}) V'_{n+1,n-2} - V'_{n+2,n-3}
\end{aligned}
\tag{6.8}
$$

with

$$\begin{aligned} V'_{n+1,n-2} &= R_{n+1}R_nR_{n-1}\big(g_2+g_3(R_{n+2}+R_{n+1}+R_n+R_{n-1}+R_{n-2})\big) \\ V'_{n+2,n-3} &= g_3R_{n+2}R_{n+1}R_nR_{n-1}R_{n-2} \end{aligned} \tag{6.9}$$

where we explicitly substituted $\tilde{\Gamma}_0(n)=1$ into Eq.(6.1).

When compared with Eq.(4.14), we see that $\tilde{\Gamma}_2(n)=\Gamma_2(n)$ and $\tilde{\Gamma}_4(n)=\Gamma_4(n-1)+(R_{n+1}+R_n+R_{n-1})(\Gamma_2(n)-\Gamma_2(n-1))$. This shows that the new set of conserved quantities is not independent from that of Section 4, as expected.

This interdependence can be made even more explicit by noting that, for $n=0$, we again have $\tilde{\Gamma}_{2i}(0)=Z_{0,-1}(2i-1)=\Gamma_{2i}(0)$. When Eq.(2.5) is satisfied, the constant value of $\tilde{\Gamma}_{2i}(n)$ is therefore equal to G_{2i}. This is also apparent from the large n limit of Eq.(6.3) which is identical to the large n limit of Eq.(4.6). As a last remark, we note that the conservation of the Γ's is granted by that of the $\tilde{\Gamma}$'s as we may then replace $\tilde{\Gamma}_{2j-2i}(n)$ in the r.h.s. of Eq.(6.3) by a shifted value $\tilde{\Gamma}_{2j-2i}(n-j+1)$ which, upon a global shift of $n\to n+j-1$, show that the $\tilde{\Gamma}_{2i}$'s also obey Eq.(4.6) for $j\geq 1$. Together with the master equation which guarantees that $\tilde{\Gamma}_0(n)=\Gamma_0(n)=1$, this implies that, when the $\tilde{\Gamma}$'s are conserved, we necessarily have $\Gamma_{2i}(n)=\tilde{\Gamma}_{2i}(n)$ for all n, hence these are conserved as well and take the same constant values.

6.2. Compacted conserved quantities

Let us again restrict ourselves to the truncated case of valences up to $2m$ and concentrate on the $(m-1)$ first conserved quantities $\tilde{\Gamma}_{2i}(n)$ for $i=1,\ldots,m-1$. From the definition (6.1), we note that the largest index carried by an R_a comes from the contributions $g_mZ_{n+j,n-j-1}(2m-1)$ in the term $V'_{n+j,n-j-1}$ appearing in the r.h.s.. This index is equal to $a=n+m-1$ and reached for each j by a unique path of length $2m-1$ starting with $m-j-1$ up steps followed by $m+j$ down steps. This term reads precisely $-g_m\prod_{\ell=1-m}^{j}R_{n-\ell}$. Similarly, the largest index carried by an R_a in $\tilde{\Gamma}_0(n)$ comes from the contribution $-g_mZ_{n,n-1}(2m-1)$ in the term $-V'_{n,n-1}$ and reads $-g_m\prod_{\ell=1-m}^{0}R_{n-\ell}$.

We have the remarkable identity

$$\begin{aligned} Z_{n-1,n-1}(2i) &= \{Z_{n,n-1}(2i-1)-R_{n-1}R_{n+1}Z_{n-2,n+1}(2i-1)\} \\ &\quad+\sum_{j=1}^{i}\{Z_{n-j,n+j-1}(2i-1)-R_{n-j-1}R_{n+j+1}Z_{n-j-2,n+j+1}(2i-1)\}\prod_{\ell=1}^{j}R_{n-\ell} \end{aligned} \tag{6.10}$$

which may be inverted into

$$-\prod_{\ell=1}^{j}R_{n-\ell}=\sum_{i=0}^{j}(-1)^{i-1}\Pi_{n-j,n+j-1}(i)Z_{n-1,n-1}(2j-2i) \tag{6.11}$$

These identities may be proved with arguments similar to that of Sect.3.2 and follow from general inversion formulas listed in Appendix B. We deduce that in the combination

$$\boxed{\Theta_{2i}(n) = \tilde{\Gamma}_{2i}(n) + \Big(1 - \tilde{\Gamma}_0(n)\Big)\, Z_{n-1,n-1}(2i)} \tag{6.12}$$

all terms containing R_{n+m-1} are cancelled. Note that the definition (6.12) does not involve the particular value of m and therefore gives a universal recipe for compacting the conserved quantities, irrespectively of the precise (finite) order of truncation. The quantities $\Theta_{2i}(n)$ form clearly an alternative set of conserved quantities that take the *same* constant values G_{2i}, but that involve one less R_a than the original Γ_{2i} or $\tilde{\Gamma}_{2i}$.

For illustration, for $m = 3$, we have

$$\begin{aligned}
\Theta_2(n) = R_n + R_{n-1} - R_n R_{n-1}\Big[1 - g_1 - g_2(R_n + R_{n-1})& \\
- g_3\left(R_{n+1}R_n + (R_n + R_{n-1})^2 + R_{n-1}R_{n-2} - R_{n+1}R_{n-2}\right)\Big]& \\
\Theta_4(n) = R_{n+1}R_n + R_{n-1}R_{n-2} + (R_n + R_{n-1})^2& \\
- R_n R_{n-1}\Big[(1 - g_1)(R_{n+1} + R_n + R_{n-1} + R_{n-2})& \\
- g_2(R_{n+1} + R_n + R_{n-1})(R_n + R_{n-1} + R_{n-2})& \\
- g_3(R_{n+1} + R_n + R_{n-1} + R_{n-2})\left(R_{n+1}R_n + (R_n + R_{n-1})^2 + R_{n-1}R_{n-2}\right)\Big]&
\end{aligned} \tag{6.13}$$

The value of $\Theta_2(n)$ at $g_1 = g_3 = 0$ was already obtained in Ref.[14].

6.3. Application to tetra- and hexavalent graphs

In the truncated case of graphs with valences up to $2m$, the truncated master equation is a recursion relation on n allowing for the recursive determination of all R_n's with $n \geq m-1$ from the knowledge of $R_0, R_1, \ldots, R_{m-2}$. These initial values are completely determined by using the $m-1$ first conserved quantities in the form $\Gamma_{2i}(0) = \Gamma_{2i}(\infty) = G_{2i}$ (or similar equalities for $\tilde{\Gamma}_{2i}$ or Θ_{2i}) with $i = 1, \ldots, m-1$. For instance, in the case $m = 3$ of graphs with bi-, tetra- and hexavalent vertices only, this yields, by use of the first two lines of (5.3) and (5.7)

$$\begin{aligned}
R_0 &= R - g_2 R^3 - 5 g_3 R^4 \\
R_1 &= \frac{R(1 - 6g_3R^3 - g_2^2R^4 - 25g_3^2R^6 - g_2(R^2 + 10g_3R^5))}{1 - g_2R^2 - 5g_3R^3}
\end{aligned} \tag{6.14}$$

with R solution of Eq.(2.8). This result is in agreement with the expressions obtained in Ref.[8] via the exact solution (2.13).

Another possible application concerns local environments within graphs. For instance, we have access to the distribution of the *number of faces adjacent to the external face* in rooted planar graphs, namely graphs with a marked oriented edge, with the external face adjacent to this rooted edge and lying on its right. These rooted graphs are in bijection with two-leg diagrams with both legs in the same (external) face, as readily seen by gluing the two legs counterclockwise around the diagram into a rooted edge oriented from the outcoming leg to the incoming one. The rooted graphs are therefore enumerated by R_0. We may then assign an extra weight x to each face adjacent to the external one, i.e. a weight x^p whenever the external face has p adjacent faces. The corresponding generating function, still denoted $R_0 \equiv R_0(x)$, is determined by the master equation (2.5) for all $n \geq 1$ and by a modified equation at $n = 0$, namely

$$R_0 = x + V'_{0,-1} \tag{6.15}$$

In particular, all conserved quantities are valid for $n \geq 1$ and their constant value remains unchanged, given by G_{2i}.

In the truncated case of valences up to $2m$, we obtain a closed algebraic system involving $R_0, R_1, \ldots, R_{m-1}$ by supplementing Eq.(6.15) by the $m-1$ first conservation laws $\Theta_{2i}(1) = G_{2i}$ for $i = 1, \ldots m-1$. Upon eliminating all R_n's with $n > 0$, we are left with a single algebraic equation determining $R_0(x)$. Let us now illustrate this in the cases of pure tetravalent and pure hexavalent graphs.

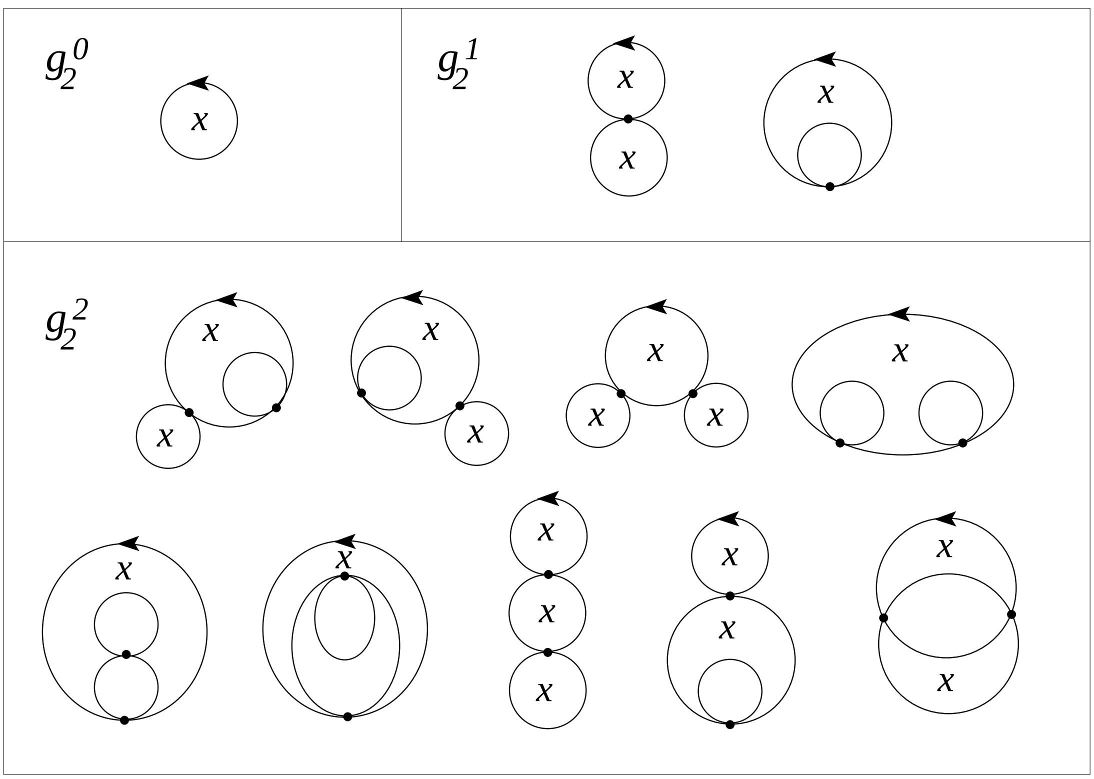

Fig. 14: The twelve rooted tetravalent planar graphs with up to 2 vertices, hence with weights g_2^p, $p = 0, 1, 2$. Assigning a weight x to each face adjacent to the external one, we reproduce the first three terms of the expansion (6.17)

For pure tetravalent graphs, we obtain the equation

$$x(x-1) + (1 - x + g_2 R(R-2) - 2g_2^2 R^3)R_0 + x g_2 R_0^2 = 0 \tag{6.16}$$

with R solution of $R = 1 + 3g_2 R^2$. This leads to the expansion

$$R_0 = x + g_2(x + x^2) + g_2^2(3x + 4x^2 + 2x^3) + g_2^3(14x + 20x^2 + 15x^3 + 5x^4) + \ldots \tag{6.17}$$

The corresponding rooted graphs up to order g_2^2 are displayed in Fig.14 for illustration. From the knowledge of $R_0(x)$, we may extract the distribution of the number of faces adjacent to the external face in *large tetravalent rooted graphs* by considering the limit

$$\Delta(x) \equiv \sum_{p=1}^{\infty} x^p \mathcal{P}(p) = \lim_{N\to\infty} \frac{R_0(x)|_{g_2^N}}{R_0(1)|_{g_2^N}} \tag{6.18}$$

Here $\mathcal{P}(p)$ denotes the probability for the external face of infinitely large tetravalent rooted graphs to have p adjacent faces. The large N limits above are governed by the approach to the critical point $g_2 \to g_2^* = 1/12$ where R and R_0 become singular. Expanding in powers of $\epsilon = \sqrt{(g_2^* - g_2)/g_2^*}$, the above ratio giving $\Delta(x)$ is obtained as the ratio of the coefficients of ϵ^3 in the expansions of $R_0(x)$ and $R_0(1)$ respectively. Using Eq.(6.16), we get the following algebraic equation for $\Delta(x)$

$$27x(x-1) + 4(4-3x)^3 \Delta(1 - x\Delta) = 0 \tag{6.19}$$

hence

$$\Delta(x) = \frac{1}{2x}\left(1 - \frac{8-9x}{(4-3x)^{\frac{3}{2}}}\right) \tag{6.20}$$

and

$$\mathcal{P}(p) = \left(\frac{3}{16}\right)^{p+1} \frac{(2p+1)!}{(p+1)!(p-1)!} \tag{6.21}$$

For pure hexavalent graphs, we obtain the equation

$$x(x-1)^2 - (x-1)(x-1+2g_3R^2(3-2R)+24g_3^2R^5)R_0 + Rxg_3(R-2-5g_3R^3)R_0^2 + x^2 g_3 R_0^3 = 0 \tag{6.22}$$

with R solution of $R = 1 + 10g_3R^3$. This now leads to the expansion

$$R_0 = x + g_3(2x + 2x^2 + x^3) + g_3^2(28x + 32x^2 + 25x^3 + 12x^4 + 3x^5) + \ldots \tag{6.23}$$

From $R_0(x)$, we may again extract the distribution of the number of faces adjacent to the external face in *large hexavalent rooted graphs* by considering the same limiting ratio as in (6.18) with $g_2 \to g_3$ and where $\mathcal{P}(p)$ now denotes the probability for the external face of infinitely large hexavalent rooted graphs to have p adjacent faces. The corresponding large N limits are now governed by the approach to the critical point $g_3 \to g_3^* = 2/135$

where R and R_0 become singular. Performing a similar expansion as before, we arrive at the following algebraic equation for $\Delta(x)$

$$125x(x-1)^2 - 9(x-1)(125x^3+475x^2+200x-96)\Delta - 27x(x+4)(7-5x)^3\Delta^2(1-x\Delta) = 0 \tag{6.24}$$

from which we deduce the probabilities

$$\mathcal{P}(1) = \frac{125}{864}, \quad \mathcal{P}(2) = \frac{1625}{10368}, \quad \mathcal{P}(3) = \frac{865625}{5971968} \tag{6.25}$$

for the external face to have $1, 2$ or 3 neighboring faces.

7. Conclusion

In this paper, we have explicitly constructed various sets of conserved quantities for the (integrable) master equation which determines the generating function R_n for two-leg diagrams with vertices of even valences. These conserved quantities are constructed in terms of generating functions for random walks, whose relations with heaps of dimers and hard dimers were instrumental to grant the conservation.

The precise properties that we used, namely some boson/fermion correspondences and their associated inversion relations, as well as some fundamental relations satisfied by either walks or hard dimers, are very general and presumably can be adapted to other cases of interest. Indeed, in the planar graph enumeration framework, it was already recognized that many other classes of (decorated) graphs have generating functions which satisfy integrable systems of algebraic master equations. These include graphs with arbitrary even or odd valences, as well as the so-called p-constellations considered in Ref.[15]. In both cases, the two-leg diagram generating functions display a form similar to that of Eq.(2.13) [8,16]. In a particular case of 3-constellations, namely that of graphs with bicolored trivalent vertices (so-called bicubic maps), a conserved quantity was already identified in Ref.[14]. We suspect that the above methods can be extended to the even more general case of planar bipartite graphs, also enumerated (without geodesic distance) by two-matrix models.

Beside the dependence on the geodesic distance, we know from matrix-model analysis that another integrable structure underlies the topological expansion for the generating functions of graphs with arbitrary genus. It would be desirable to relate the two directions of integrability. A general theory of discrete integrable equations was built in Ref.[17] with (Hirota bilinear) master equations involving more than one integer index n (see also [18] for the theory of discrete Painlevé equations). This suggests to look for a larger integrable structure that would include both the dependence on (possibly several) geodesic distances as well as that on topology.

As explained in Sect.5.3, the conservation properties have a strong "bijective flavor" in the language of blossom trees. A bijective explanation for the conservation would be very instructive and seems to be within reach in view if the relations (5.15) and (5.16). If such a proof exists, it must involve only generic properties of blossom trees, or equivalently of planar graphs. This could allow to trace back the somewhat mysterious generic appearance of conserved quantities in the context of planar graph enumeration problems to these generic properties.

Finally, we may hope that the conservation properties have a natural interpretation of their own both in the language of labeled mobiles and in that of spatially extended branching processes. This is yet to be found. Note in this context the recent appearance of yet another integrable master equation governing the so-called "embedded binary trees" studied in Ref.[19].

To conclude, the general case of graphs with vertices of even valence is known to give access to multicritical transitions points of 2D Quantum Gravity by proper fine tuning of the parameters g_i. As we have now a complete picture of the conserved quantities, we may use them to describe these multicritical points. In the scaling limit of large graphs, the master equation is known to reduce at such points to an integrable differential equation related to the KdV hierarchy [8]. Our conserved quantities provide explicit discrete counterparts of the integrals of motion for these equations. These integrals are indeed recovered from our expressions in the scaling limit.

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Appendix A. Relation between heaps and walks

We consider the generating functions $Z_{a,b}(2k-1)$ for random walks of $2k-1$ steps from height a to height b with weight R_i per descent $i \to (i-1)$, and its truncated version $Z^+_{a,b}(2k-1)$ restricted to "positive" walks with heights larger or equal to a. Similarly, we consider the generating function $H_{a,b}(k)$ for pyramids of k dimers with base $[a,b]$ (with $(b-a)$ odd and positive) with weight R_i per dimer in the stripe i except for those in the right projection. We also consider its truncated half-pyramid version $H^+_{a,b}(k)$ with dimers only in stripes $i > a$. In both cases, the truncation simply amounts to taking $R_i \to 0$ for all $i \leq a$.

We wish to prove that

$$Z_{a,b}(2k-1) = H_{a,b}(k), \qquad Z^+_{a,b}(2k-1) = H^+_{a,b}(k) \tag{A.1}$$

for all $k \geq 1$ and $(b-a)$ odd and positive.

These relations are a consequence of the following bijection, borrowed from Ref.[9], between on the one hand random walks of $2k-1$ steps from height a to height $b > a$ and on the other hand pyramids with base $[a,b]$ and a total of k dimers. Starting from a walk, as illustrated in Fig.15-(a), viewed as the time (horizontal) evolution of a walker on the integer (vertical) line, the walker acts as the Little Thumb by dropping small pebbles at each passage in a unit segment unless a pebble is already present in which case he picks it up. As shown in Fig.15-(b), putting dimers at the times/locations of the pick-up events

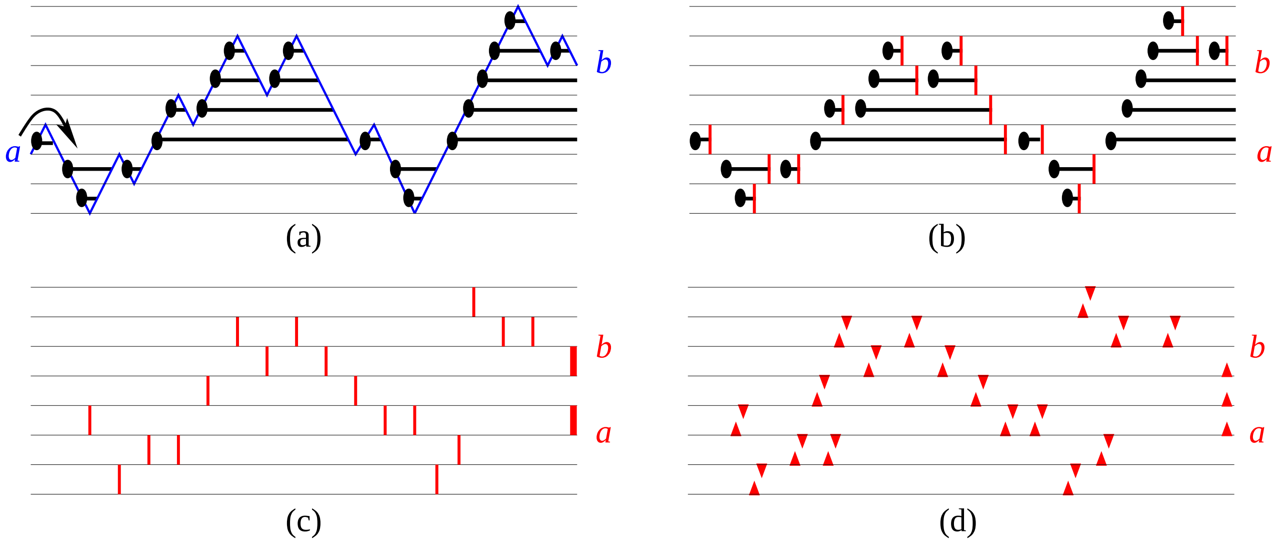

Fig. 15: Bijection between walks from a to $b > a$ with $2k - 1$ steps and pyramids of base $[a, b]$ with k dimers, as explained in the text. In (a), the times/locations of pebble droppings are indicated by filled black ellipses. Each pebble stays in place during some time (indicated by a black horizontal segment) until it is possibly picked up. In (b), we put a dimer at each pick-up time/location. This configuration is completed into a pyramid of base [a,b] in (c). The inverse mapping makes use of the splitting of dimers into up- or down-pointing triangles (d), whose sequence along each line determines the steps of the walk.

results in a heap of dimers. Moreover, if the walker goes from a to $b > a$, there are $b - a$ pebbles left behind. By completing the heap on the right by the maximally occupied hard dimer configuration on the segment $[a, b]$, we obtain a pyramid of base $[a, b]$ and with a total of k dimers if the walk has $2k - 1$ steps, as illustrated in Fig.15-(c). This establishes a bijection whose inverse goes as follows. We split each dimer of the pyramid not in the right projection into a pair of up- and down-pointing triangles, as shown in Fig.15-(d). Each such triangle has its horizontal edge on a line separating stripes. We also add on the right a succession of $b - a$ up-pointing triangles adjacent to the lines $y = a, a + 1, \ldots, b - 1$. For each line $y = i$, we record the sequence of up- and down-pointing triangles. The walk is reconstructed by starting from height a and using successively the triangles in the above sequences. More precisely, at each position $y = i$, the walker makes an up- or down-step according to the up-or down-pointing nature of the first unused triangle along the line $y = i$.

In the above bijection, the descending steps of the walk from height i to height $i - 1$ are in one-to-one correspondence with the dimers in the stripe i, except for those of the right projection. Note that these descending steps may correspond either to the dropping or to the pick-up of a pebble. In particular, weighting each descent of the walk from height i to height $i - 1$ by R_i (resp. 0 if $i \leq a$) amounts to weight each dimer in the stripe i by R_i (resp. 0 if $i \leq a$), except for those dimers in the right projection of the pyramid (resp.

half-pyramid). This immediately yields the relations (A.1).

Appendix B. Useful identities

We have the following useful identities, proved by arguments similar to that of Sect.3.2. For $j \geq 1$ and $k \geq 0$:

$$\sum_{i=0}^{j} (-1)^{j-i} \Pi_{n-j,n+j-1}(j-i) \Big(\prod_{\ell=1}^{k} R_{n+k+1-2\ell} \Big) Z_{n-k,n+k-1}(2i-1) \\ = \Pi_{n-j,n+j-1}(j) \times \delta_{j \geq k} \times \delta_{j=k \bmod 2} \tag{B.1}$$

We immediately deduce from this identity the inversion relation

$$\sum_{i=0}^{j} (-1)^{j-i} \Pi_{n-j,n+j-1}(j-i) \big(Z_{n-k,n+k-1}(2i-1) \\ - R_{n-k-1} R_{n+k+1} Z_{n-k-2,n+k+1}(2i-1) \big) = \delta_{j,k} \tag{B.2}$$

The inversion of Eq.(6.1) into Eq.(6.3) follows from this identity.
For $j \geq 1$:

$$\sum_{i=0}^{j} (-1)^{j-i} \Pi_{n-j,n+j-1}(j-i) R_{n-1} Z_{n-2,n-1}(2i-1) \\ = -\Pi_{n-j,n+j-1}(j) \times \delta_{j=0 \bmod 2} + \prod_{\ell=1}^{j} R_{n-\ell} \tag{B.3}$$

This relation, together with Eq.(B.1) for $k=0$, allows to prove Eqs. (6.10) and (6.11) by noting that $Z_{n-1,n-1}(2i) = Z_{n,n-1}(2i-1) + R_{n-1} Z_{n-2,n-1}(2i-1)$.

References

[1] E. Brézin, C. Itzykson, G. Parisi and J.-B. Zuber, *Planar Diagrams*, Comm. Math. Phys. **59** (1978) 35-51.

[2] A. Fokas, A. Its and A. Kitaev, *Discrete Painlevé equations and their appearance in quantum gravity*, Comm. Math. Phys. **142** (1991) 313-344.

[3] V. Kazakov, *Bilocal regularization of models of random surfaces*, Phys. Lett. **B150** (1985) 282-284; F. David, *Planar diagrams, two-dimensional lattice gravity and surface models*, Nucl. Phys. **B257** (1985) 45-58; J. Ambjorn, B. Durhuus and J. Fröhlich, *Diseases of triangulated random surface models and possible cures*, Nucl. Phys. **B257**(1985) 433-449; V. Kazakov, I. Kostov and A. Migdal *Critical properties of randomly triangulated planar random surfaces*, Phys. Lett. **B157** (1985) 295-300.

[4] See for instance: P. Di Francesco, P. Ginsparg and J. Zinn–Justin, *2D Gravity and Random Matrices*, Physics Reports **254** (1995) 1-131, and references therein; see also B. Eynard, *Random Matrices*, Saclay Lecture Notes (2000), available at *http://www-spht.cea.fr/lectures_notes.shtml*

[5] G. Schaeffer, *Conjugaison d'arbres et cartes combinatoires aléatoires*, PhD Thesis, Université Bordeaux I (1998); available at http://www.lix.polytechnique.fr/~schaeffe/Biblio.

[6] G. Schaeffer, *Bijective census and random generation of Eulerian planar maps*, Electronic Journal of Combinatorics, vol. **4(1)** (1997) R20.

[7] J. Bouttier, P. Di Francesco and E. Guitter, *Census of planar maps: from the one-matrix model solution to a combinatorial proof*, Nucl. Phys. **B645**[PM] (2002) 477-499, arXiv:cond-mat/0207682.

[8] J. Bouttier, P. Di Francesco and E. Guitter, *Geodesic distance in planar graphs*, Nucl. Phys. **B663**[FS] (2003) 535-567, arXiv:cond-mat/0303272.

[9] X. Viennot, *Heaps of pieces 1: basic definitions and combinatorial lemmas*, in G. Labelle and P. Leroux Eds. *Combinatoire énumérative*, Lect. Notes in Math. vol **1234** (1986) 321-350.

[10] M. Bousquet-Mélou and A. Rechnitzer, *Lattice animals and heaps of dimers*, Discrete Math. **258** (2002) 235-274.

[11] P. Di Francesco and E. Guitter, *Critical and multicritical semi-random* $(1+d)$*-dimensional lattices and hard objects in* d *dimensions*, J. Phys. A Math. Gen, **35** (2002) 897-927.

[12] J. Bouttier, P. Di Francesco and E. Guitter, *Planar Maps as Labeled Mobiles*, Electronic Journal of Combinatorics, vol. **11(1)** (2004) R69.

[13] R. Cori and B. Vauquelin, *Planar maps are well labeled trees*, Canad. J. Math. **33 (5)** (1981) 1023-1042; D. Arquès, *Les hypercartes planaires sont des arbres très bien*

étiquetés, Discr. Math. **58**(1) (1986) 11-24; M. Marcus and G. Schaeffer, *Une bijection simple pour les cartes orientables*, http://www.lix.polytechnique.fr/~schaeffe/Biblio/.

[14] J. Bouttier, P. Di Francesco and E. Guitter, *Statistics of planar maps viewed from a vertex: a study via labeled trees*, Nucl. Phys. **B675**[FS] (2003) 631-660, arXiv:cond-mat/0307606.

[15] M. Bousquet-Mélou and G. Schaeffer, *Enumeration of planar constellations*, Adv. in Applied Math., **24** (2000) 337-368.

[16] P. Di Francesco, *Geodesic distance in planar graphs: An integrable approach*, preprint (2003), to appear in the Ramanujan Journal

[17] M. Jimbo and T. Miwa, *Solitons and infinite dimensional Lie algebras*, Publ. RIMS, Kyoto Univ. **19** No. 3 (1983) 943-1001, eq.(2.12).

[18] B. Grammaticos, F. Nijhoff and A. Ramani, *Discrete Painlevé equations*, in *The Painlevé property, one century later*, R. Conte Ed., CRM series in Math. Phys. (1999) 413-516.

[19] M. Bousquet-Mélou, *Limit laws for embedded trees. Applications to the integrated superBrownian excursion* preprint arXiv:math.CO/0501266.