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A note on topological amplitudes in hybrid string theory

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Abstract

We study four-dimensional compactifications of type II superstrings on Calabi-Yau spaces using the formalism of hybrid string theory. Chiral and twisted-chiral interactions are rederived, which involve the coupling of the compactification moduli to two powers of the Weyl-tensor and of the derivative of the universal tensor field-strength.

1. Introduction

In this note we study compactifications of type II superstrings on Calabi-Yau manifolds to four spacetime dimensions using the hybrid formulation of string theory [1,2,3]. The low-energy effective actions of these theories are described by $\mathcal{N}=2$ supergravity theories coupled to matter. The massless matter fields are organized in $\mathcal{N}=2$ real chiral or twisted-chiral superfields, describing vector and tensor multiplets, respectively. Hypermultiplets are related to the latter by dualization. Most of these massless superfields contain the compactification moduli and are related to complex structure and Kähler structure deformations of the compactification manifold. Various types of couplings of these fields have been studied: corrections to the moduli space metric of the hypermultiplet or tensor multiplet sector have been discussed in [4,5,6]. The couplings to the universal sectors including supergravity that involve higher-derivative interactions are known for situations where these can be written as chiral or twisted-chiral superspace integrals. Such terms describe couplings of compactification moduli to two powers of the Weyl tensor or of the derivative of the universal tensor field-strength. The description of the full, nonlinear interactions in supergravity relies on the existence of a completely off-shell multiplet calculus for $\mathcal{N}=2$ supergravity [7,8,9]. In string theory these couplings have been calculated for vector multiplets by scattering amplitudes at the linearized level, using the RNS formulation in [10] and using hybrid approach in [2]. The moduli dependence of these amplitudes is thereby given by the partition functions of topological strings on the Calabi-Yau space. The hybrid formulation of string theory is particularly well-suited for studying the spacetime properties of string compactifications since it exhibits manifest four-dimensional $\mathcal{N}=2$ supersymmetry covariance.

Hybrid string theory can be obtained by a field redefinition from the gauge-fixed RNS string or by covariantizing the GS string in light-cone gauge. In this sense, worldsheet reparametrizations are gauge-fixed in the hybrid formulation. Nevertheless, there is no need for ghost-like fields in the formalism since the theory can be formulated as a $\mathcal{N}=4$ topological theory and amplitudes can computed directly by the methods of topological string theory [2]. In this setting, the theory consists of two completely decoupled twisted worldsheet SCFT, one describing the spacetime part, one the internal part. Despite being twisted, hybrid string theory describes the full theory, i.e., it computes also non-topological amplitudes. Hybrid type IIA and IIB string theories are distinguished by the relative twisting of the left- and right-moving sector of the internal SCFT. When working with either type one is therefore committed to a given fixed twisting.

The main application of hybrid strings in this note is to extend the analysis of higher order derivative interactions to the twisted-chiral sector. The procedure is analogous to the computation in the chiral sector given in [2]. Even though one is working with a fixed relative twisting, giving rise to either type IIA or type IIB, it is shown that the chiral and twisted-chiral couplings of each type II theory depend on both the A-model and B-model topological partition functions.

The outline is as follows: in section 2 we briefly review some of the key elements of hybrid string theory that are relevant for the computation. We discuss the worldsheet algebra and vertex operators. In section 3 the definition of amplitudes is reviewed and several important ingredients for calculating amplitudes are discussed. In section 4 we turn to the calculation of the chiral and twisted-chiral couplings of the moduli to the universal supergravity sectors.

2. Hybrid string theory

2.1. The set-up

Hybrid string theory [1,2,3] is a formulation of type II and heterotic string theory compactified on Calabi-Yau three-folds with manifest four-dimensional supersymmetry covariance. The four-dimensional spacetime part is described by four bosons x^m and two pairs of left-moving, canonically conjugate Weyl fermions (p_α, θ^β) and $(\bar{p}^{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}})$. Both have conformal weight $(1,0)$ and are related by hermitian conjugation. In addition there is a chiral boson ρ with period $\rho \sim \rho + 2\pi i$ and

$$\rho(z)\rho(w) \sim -\ln(z-w)\,, \tag{2.1}$$

The background charge for the field ρ is $Q_\rho = -1$. This implies the conformal weight $\mathrm{wt}(e^{q\rho}) = -\frac{1}{2}q(q-1)$. In type II theories these fields are supplemented by two pairs of right-moving fermions, their hermitian conjugates, and a periodic right-moving chiral boson. From the, say, left-moving sector of the spacetime fields one constructs the generators $(T, G^\pm, J)$ of a twisted $\mathcal{N}=2$ superconformal algebra with central charge $c=-3$. As in the RNS formulation of the compactified superstring, the internal part is an untwisted $c=9$, $\mathcal{N}=2$ superconformal field theory with generators (T_C, G_C^+, G_C^-, J_C) with $G_C^- = (G_C^+)^\dagger$. They have trivial OPEs with all spacetime fields. The generators of the combined system are obtained by forming the twisted generators $(T_C + \frac{1}{2}\partial J_C, G_C^+, G_C^-, J_C)$ and adding them to the spacetime generators,

$$T + T_C + \frac{1}{2}\partial J_C\,, \quad G^\pm + G_C^\pm\,, \quad J + J_C\,. \tag{2.2}$$

These fields generate a twisted $c = 6$, $\mathcal{N} = 2$ superconformal algebra. The $U(1)$ R-symmetry current J_C of the internal CFT can be written in terms of a free chiral boson

$$J_C = i\sqrt{3}\,\partial H\,, \quad H(z)H(w) = -\ln(z-w)\,. \tag{2.3}$$

The H-system has a background charge $Q_H = -\sqrt{3}$, i.e. $\mathrm{wt}(e^{i\frac{q}{\sqrt{3}}H}) = \frac{q}{6}(q-3)$. The conformal weights of the fermionic generators are $\mathrm{wt}(G^+) = 1$ and $\mathrm{wt}(G^-) = 2$. Any field $\mathcal{O}$ of the internal sector with definite U(1) charge can be decomposed as $\mathcal{O}^{(q)} = \exp(\frac{iq}{\sqrt{3}}H)\mathcal{O}'$ where $\mathcal{O}'$ is uncharged with respect to J_C. For the generators $G_C^\pm$ this part is in fact independent of H.

One can embed the twisted $c = 6$, $\mathcal{N} = 2$ SCFT into a small twisted $\mathcal{N} = 4$ algebra. This leads to the topological prescription of [2,11] for computing the spectrum and correlation functions of the hybrid string theory.

2.2. Twisted $\mathcal{N} = 4$ algebra

The embedding into a twisted small $\mathcal{N} = 4$ superconformal algebra[1] proceeds as follows: the generators (2.2), which we will denote by T, $G^\pm$, and J in the sequel, generate a $c = 6$, $\mathcal{N} = 2$ algebra. The current $J = \partial\rho + J_C$ is augmented to a triplet of currents (J^{++}, J, J^{--}) defined by

$$J^{\pm\pm}(z) = e^{\pm(\rho(z) + \int^z J_C)}\,. \tag{2.4}$$

The superscripts $\pm$ indicate the total $U(1)$-charges. The conformal weights of the raising and lowering operators are $\mathrm{wt}(J^{++}) = 0$ and $\mathrm{wt}(J^{--}) = 2$. They satisfy the SU(2) relation

$$J^{++}(z)J^{--}(w) \sim \frac{1}{(z-w)^2} + \frac{J(w)}{(z-w)}\,. \tag{2.5}$$

There are two SU(2) doublets of fermionic generators: $(G^+, \widetilde{G}^-)$ and $(G^-, \widetilde{G}^+)$ that transforming in the **2** and **2*** of SU(2), respectively. The $\widetilde{G}^\pm$ are defined via the operator products

$$J^{\pm\pm}(z)G^\mp(w) \sim \mp\frac{\widetilde{G}^\pm(w)}{z-w}\,, \quad J^{\pm\pm}(z)\widetilde{G}^\mp(w) \sim \pm\frac{G^\pm(w)}{z-w}\,. \tag{2.6}$$

[1] Our conventions are based on the $\mathcal{N} = 4$ superconformal algebra presented in [12].

and have $\mathrm{wt}(\tilde{G}^+) = 1$ and $\mathrm{wt}(\tilde{G}^-) = 2$. The other OPEs of $J^{\pm\pm}$ with the fermionic generators are finite. Furthermore, one has

$$G^+(z)\widetilde{G}^+(w) \sim \frac{2J^{++}(w)}{(z-w)^2} + \frac{\partial J^{++}(w)}{z-w} , \quad \widetilde{G}^-(z)G^-(w) \sim \frac{2J^{--}(w)}{(z-w)^2} + \frac{\partial J^{--}(w)}{z-w} . \tag{2.7}$$

The nontrivial OPEs of the supercurrents are

$$G^+(z)G^-(w) \sim \frac{2}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{T(w)}{z-w} , \tag{2.8}$$

and the very same OPE for $\widetilde{G}^+(z)$ and $\widetilde{G}^-(w)$. The explicit form of the fermionic generators is[2]

$$\begin{aligned} G^- &= e^{\rho} d^2 + G_C^- , \\ \widetilde{G}^- &= e^{-2\rho - \int J_C} \bar{d}^2 + e^{-\rho} \tilde{G}_C^{--} , \\ G^+ &= e^{-\rho} \bar{d}^2 + G_C^+ , \\ \widetilde{G}^+ &= e^{2\rho + \int J_C} d^2 + e^{\rho} \tilde{G}_C^{++} . \end{aligned} \tag{2.9}$$

Here $\widetilde{G}_C^{\pm\pm}$ are defined as[3] $\widetilde{G}_C^{\pm\pm} = e^{\pm \int^z J_C}(G_C^{\mp})$. The currents d_α and $\bar{d}_{\dot\alpha}$ are the supersymmetrized versions of the currents p_α and $\bar{p}_{\dot\alpha}$. They commute with the spacetime supersymmetry charges.

It is convenient to label the fermionic generators by indices $i, j = 1, 2$ according to

$$G_i^+ = (G^+, \widetilde{G}^+) , \quad G_i^- = (G^-, \widetilde{G}^-) . \tag{2.10}$$

They satisfy the hermiticity property $(G_i^+)^\dagger = G_i^-$. Consider general linear combinations

$$\widehat{G}_i^- = u_{ij} G_j^- , \quad \widehat{G}_i^+ = u_{ij}^* G_j^+ , \tag{2.11}$$

where the second equation follows from the first by hermitian conjugation. Requiring that $\widehat{G}_i^\pm$ satisfy the same $\mathcal{N} = 4$ relations as $G_i^\pm$ implies that u_{ij} are SU(2) parameters: $u_{11} = u_{22}^* \equiv u_1$ and $u_{21}^* = -u_{12} \equiv u_2$ with $|u_1|^2 + |u_2|^2 = 1$. This shows that the $\mathcal{N} = 4$ algebra has an SU(2) automorphism group that rotates the fermionic generators among each other. More explicitly, we have

$$\begin{aligned} \widehat{\widetilde{G}}{}^+ = \widehat{G}_2^+ &= u_1 \left(e^{2\rho + \int J_C} d^2 + e^{\rho} \widetilde{G}_C^{++} \right) + u_2 \left(e^{-\rho} \bar{d}^2 + G_C^+ \right) , \\ \widehat{G}^- = \widehat{G}_1^- &= u_1 \left(e^{\rho} d^2 + G_C^- \right) - u_2 \left(e^{-2\rho - \int J_C} \bar{d}^2 + e^{-\rho} \widetilde{G}_C^{--} \right) , \end{aligned} \tag{2.12}$$

[2] We are suppressing numerical factors and cocycle factors.

[3] The expression $A(B(w))$ denotes the residue in the OPE of $A(z)$ with $B(w)$ and equals the commutator $[\oint A, B(w)\}$, where $\oint A = \frac{1}{2\pi i} \oint dz A(z)$.

and analogous expressions for $\widehat{G}^+ = \widehat{G}_1^+$ and $\widehat{\widehat{G}}{}^- = \widehat{G}_2^-$, which involve the complex conjugate parameters u_i^*. The u_i's parameterize the different embeddings of the $\mathcal{N}=2$ subalgebras into the $\mathcal{N}=4$ algebra. These different embeddings play a role in the definition of scattering amplitudes, cf. section 3.

For notational simplicity we discuss mostly type IIB string theory. In this case one has a set of identically defined left- and right-moving generators. We will use the subscripts "L" and "R" in order to distinguish left-moving from right-moving fields and adopt the notation $|A|^2 = A_L A_R$. As mentioned before, for type IIA theories the right-moving part of the algebra is obtained by the opposite twisting as compared to (2.2). Operationally, the expressions for IIA can be obtained from those of IIB by replacing $(J_C)_R \to -(J_C)_R$ (thereby reversing the background charge) in above definitions of the currents and by reversing, e.g., $(G_C^{\pm})_R \to (G_C^{\mp})_R$. The spacetime part remains unaffected.

2.3. Vertex operators

Vertex operators are defined in terms of the cohomologies of the operators $\oint \widehat{G}^+$ and $\oint \widehat{\widehat{G}}{}^+$. Integrated vertex operators have zero total U(1)-charge and can be written in the form

$$U = \int d^2 z \, |\widehat{G}^- \widehat{G}^+|^2 V \,. \tag{2.13}$$

We have $\int d^2 z \, |\widehat{G}^- \widehat{G}^+|^2 V = \int d^2 z \, |\widehat{G}^+ \widehat{G}^-|^2 V$ where one drops a total derivative under the integral. Further, if V is an SU(2)-singlet one has $\int d^2 z \, |\widehat{G}^- \widehat{G}^+|^2 V = \int d^2 z \, |\widehat{\widehat{G}}{}^- \widehat{\widehat{G}}{}^+|^2 V$. Therefore, as will be used later, $\widehat{G}^+ U = \widehat{\widehat{G}}{}^+ U = 0$. Of particular interest are the universal, compactification independent vertex operators contained in the real superfield $V = V(x, \theta_L, \bar\theta_L, \theta_R, \bar\theta_R)$ which was discussed in [3]. It contains the degrees of freedom of $\mathcal{N}=2$ supergravity and those of the universal tensor multiplet. It satisfies the $\mathcal{N}=2$ primarity constraints which imply transversality constraints and linearized equations of motion for the component fields. In the amplitude computations of the next section, we will pick a certain fixed term in the u_i expansion of the integrated vertex operators (2.13), namely $U = \int |G^+ G^-|^2 V = \int |\tilde{G}^+ \tilde{G}^-|^2 V$. These operators satisfy the same properties listed below (2.13) as the full u_i-dependent operators (2.13). For this choice, the corresponding integrated vertex operator U contains (among other parts) the field strengths of the supergravity and universal tensor multiplets:

$$\int d^2 z |\bar{d}^{\dot\alpha} D^2 \bar{D}_{\dot\alpha} + d^\alpha \bar{D}^2 D_\alpha|^2 V = \int d^2 z (d_L^\alpha d_R^\beta P_{\alpha\beta} + d_L^\alpha \bar{d}_R^{\dot\beta} Q_{\alpha\dot\beta}) + \text{h.c.} \,, \tag{2.14}$$

where $P_{\alpha\beta} = (\bar{D}^2 D_\alpha)_L (\bar{D}^2 D_\beta)_R V$ and $Q_{\alpha\dot{\beta}} = (\bar{D}^2 D_\alpha)_L (D^2 \bar{D}_{\dot{\alpha}})_R V$ are chiral and twisted-chiral superfields[4]. As discussed below, on-shell, these superfields describe the linearized Weyl multiplet and the derivative of the linearized field-strength multiplet of the universal tensor. For later purposes we also introduce U' and U'' defined by $U = |G^+|^2 U'$ and $U = |\tilde{G}^+|^2 U''$, i.e., $U' = \int d^2z |e^\rho d^\alpha D_\alpha|^2 V$ and $U'' = \int d^2z |e^{-2\rho - \int J_C} \bar{d}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}}|^2 V$.

The complex structure moduli are in one-to-one correspondence to elements of $H^{2,1}(CY)$ and related to primary fields Ω_c of the chiral (c,c) ring [13]. The corresponding type IIB hybrid vertex operators are[5]

$$V_{cc} = |e^\rho \bar{\theta}^2|^2 M_c \Omega_c \,, \quad V_{aa} = (V_{cc})^\dagger = |e^{-\rho} \theta^2|^2 \bar{M}_c \bar{\Omega}_c \,, \tag{2.15}$$

where M_c is a real chiral superfield (vector multiplet). Note that in the (twisted) type IIB theory Ω_c has conformal weight $h_L = h_R = 0$, while $\bar{\Omega}_c$ has conformal weight $h_L = h_R = 1$. The complexified Kähler moduli are in one-to-one correspondence to elements of $H^{1,1}(CY)$ and related to primary fields Ω_{tc} of the twisted-chiral ring (c,a):

$$V_{ca} = e^{\rho_L - \rho_R} \, \bar{\theta}_L^2 \, \theta_R^2 \, M_{tc} \Omega_{tc} \,, \quad V_{ac} = (V_{ca})^\dagger = e^{-\rho_L + \rho_R} \, \theta_L^2 \, \bar{\theta}_R^2 \, \bar{M}_{tc} \bar{\Omega}_{tc} \,, \tag{2.16}$$

where M_{tc} are real twisted-chiral superfields (tensor multiplets). The conformal weight of Ω_{tc} is $h_L = 0$ and $h_R = 1$. The integrated vertex operators are

$$U_{cc} = \int d^2z \, M_c |G_C^-|^2 \Omega_c + \dots \,, \quad U_{ca} = \int d^2z \, M_{tc} (G_C^-)_L (G_C^+)_R \Omega_{tc} + \dots \,, \tag{2.17}$$

where we have suppressed terms involving derivatives acting on M_c and M_{tc}. These terms carry nonzero ρ-charge and will not play a role in the discussion of the amplitudes in section 4.

The vertex operators of IIA associated to elements of the (c,c) (complex structure) and (c,a) ring (Kähler) are

$$V_{cc} = e^{\rho_L - \rho_R} \, \bar{\theta}_L^2 \, \theta_R^2 \, M_{tc} \Omega_c \,, \quad V_{ca} = |e^\rho \bar{\theta}^2|^2 M_c \Omega_{tc} \,. \tag{2.18}$$

[4] Chiral superfields M_c satisfy $\bar{D}_{\dot{\alpha}L} M_c = 0 = \bar{D}_{\dot{\alpha}R} M_c$ and twisted-chiral superfields M_{tc} satisfy $\bar{D}_{\dot{\alpha}L} M_{tc} = 0 = D_{\alpha R} M_{tc}$. D_α and $\bar{D}_{\dot{\alpha}}$ are the covariant fermionic derivatives that commute with the supersymmetry charges. Real chiral superfield satisfy in addition $D_L^2 M_c = \bar{D}_R^2 \bar{M}_c$ and real twisted-chiral superfields $D_L^2 M_{tc} = D_R^2 \bar{M}_{tc}$, cf. [3].

[5] We are suppressing the indices distinguishing between the different elements of the ring.

For type IIA the conformal weights of Ω_c are $h_L = 0$ and $h_R = 1$ while Ω_{tc} has weight $h_L = h_R = 0$. The integrated vertex operators involve

$$U_{cc} = \int d^2z\, M_{tc}(G^-_C)_L(G^-_C)_R\Omega_c + \dots\,, \quad U_{ca} = \int d^2z\, M_c(G^-_C)_L(G^+_C)_R\Omega_{tc} + \dots\,. \tag{2.19}$$

3. Amplitudes and correlation functions

In this section we review the definition of scattering amplitudes on Riemann surfaces Σ_g with genus $g \geq 2$ as given in [2]. We also collect correlation functions for chiral bosons.

3.1. Amplitudes

Scattering amplitudes of hybrid string theory are defined in [2] for $g \geq 2$ as[6]

$$F_g(u_L, u_R) = \int_{\mathcal{M}} \frac{[dm_g]}{\det(\mathrm{Im}\tau)} \prod_{i=1}^{g} \left\langle \int d^2v_i \prod_{j=1}^{g-1} |\,\widehat{\widetilde{G}}^+(v_j)|^2 |J(v_g)|^2 \prod_{k=1}^{3g-3} |(\mu_k, \widehat{G}^-)|^2 \prod_{l=1}^{N} U_l \right\rangle . \tag{3.1}$$

Since $F_g(u_L, u_R)$ is a homogeneous polynomial in both u_{iL} and u_{iR} of degree $4g-4$ (we are taking U to carry no $u_{iL,R}$ dependence as is explained above (2.14)) this definitions provides a whole set of amplitudes $F_g^{n,m}$ given by the coefficients in the $u_{i\,L,R}$-expansion:

$$F_g(u_L, u_R) = \sum_{n,m} \begin{pmatrix} 4g-4 \\ 2g-2-n \end{pmatrix} \begin{pmatrix} 4g-4 \\ 2g-2-m \end{pmatrix} F_g^{n,m} u_{1L}^{2g-2+n} u_{2L}^{2g-2-n} u_{1R}^{2g-2+m} u_{2\,R}^{2g-2-m} , \tag{3.2}$$

where $2-2g \leq m, n \leq 2g-2$. We focus on either the left- or right-moving sector in the following. In view of (2.12) it is clear that F_g^n involves $2g-2+n$ insertions of $\widetilde{G}^+$ and G^- and $2g-2-n$ insertions of G^+ and $\widetilde{G}^-$. It is shown in [2] that up to contact terms all distributions of $\widetilde{G}^+$'s, G^-'s, G^+'s, and $\widetilde{G}^-$'s satisfying these constraints are equivalent. We can therefore determine $F_g^{n,m}$ (3.2) by evaluating a single amplitude with an admissible distribution of insertions.

In addition there is a selection rule that relies on the cancellation of the R-parity anomaly [2]. The R-charge is

$$R = \oint \left(\partial\rho - \frac{1}{2}\theta d + \frac{1}{2}\bar{\theta}\bar{d} \right) , \tag{3.3}$$

[6] This differs by the factor $(\det(\mathrm{Im}\tau))^{-1}$ from the expression given in [2] and [11]. We will comment on this below (3.13).

with background charge $1-g$. In the RNS formulation R precisely coincides with the superconformal ghost-number (picture) operator. $\widetilde{G}^{\pm}$ carry R-charges ∓ 1 while those of $G^{\pm}$ are zero. The contribution to the R-charge of the insertions is $g-1-n$. The anomaly is therefore canceled only if the vertex operators insertions have total R-charge n. Put differently: given vertex operators $\prod_{i=1}^{N} U_i$ with total R-charge n, the only non-vanishing contribution to (3.2) is F_g^n. This selection rule is completely analogous to the one that relies on picture charge in the RNS formulation.

It is convenient to rewrite (3.1) in the form

$$
\begin{aligned}
&F_g(u_L,u_R) = \\
&= \int_{\mathcal{M}} \frac{[dm_g]}{\det(\mathrm{Im}\tau)} \prod_{i=1}^{g} \Big\langle \int d^2 v_i \prod_{j=1}^{g} |\, \widehat{\widehat{G}}^{+}(v_j)|^2 \prod_{k=1}^{3g-4} |(\mu_k, \widehat{G}^{-})|^2 |(\mu_{3g-3}, J^{--})|^2 \prod_{l=1}^{N} U_l \Big\rangle . \qquad (3.4)
\end{aligned}
$$

This is obtained from (3.1) by contour deformation using $\widehat{G}^{-} = \oint \widehat{\widehat{G}}^{+} J^{--}$ and $\widehat{\widehat{G}}^{+} = -\oint \widehat{\widehat{G}}^{+} J$ and the fact that $\oint \widehat{\widehat{G}}^{+}$ has no non-trivial OPE with any of the other insertions except a simple pole with J. Consider the integrand of (3.4). As a function of, say, v_1, it has a pole only at the insertion point of J^{--}. But the residue $\langle \prod_{i=2}^{g} \widehat{\widehat{G}}^{+}(v_i) \prod_{j=1}^{3g-3} (\mu_j, \widehat{G}^{-}) \prod_l U_l \rangle$ vanishes: each of the remaining $\widehat{\widehat{G}}^{+}(v_i)$ can be written as $-\oint \widehat{\widehat{G}}^{+} J(v_i)$ and $\widehat{\widehat{G}}^{+}$ has no singular OPE with any of the other insertions. Analyticity and the fact that $\widehat{\widehat{G}}^{+}$ are Grassmann odd and of weight one, fixes the v-dependence of the integrand as $\det(\omega_i(v_j))$. The ω_i are the g holomorphic one-forms on Σ_g. In (3.4) we can thus replace

$$
\prod \widehat{\widehat{G}}^{+}(v_i) = \det(\omega_i(v_j))\, \frac{\prod \widehat{\widehat{G}}^{+}(\tilde{v}_l)}{\det(\omega_k(\tilde{v}_l))} \,, \qquad (3.5)
$$

where $\tilde{v}_k$ are g arbitrary points on Σ_g that can be chosen for convenience. Combining left- and right-movers the v-integrations can be performed with the result

$$
\prod_{i=1}^{g} \int d^2 v_i |\det(\omega_k(v_l))|^2 \propto \det(\mathrm{Im}\,\tau) \,. \qquad (3.6)
$$

τ is the period matrix of Σ_g. Using similar arguments one can rewrite

$$
\frac{1}{\det(\mathrm{Im}\tau)} \Big(\int_{\Sigma_g} |\widehat{\widehat{G}}^{+}|^2 \Big)^g \propto \Big| \prod_{i=1}^{g} \oint_{a_i} \widehat{\widehat{G}}^{+} \Big|^2 . \qquad (3.7)
$$

The reason for the insertion $\oint \widehat{\widehat{G}}^{+}$ on every a-cycle of Σ_g was presented in [2,11]: it projects to the reduced Hilbert space formed by the physical fields of an $\mathcal{N}=2$ twisted theory. Amplitudes for these states can be calculated using the rules of $\mathcal{N}=2$ topological strings.

3.2. Correlators of chiral bosons

In this section we provide the correlation functions which are necessary to compute the amplitudes, cf. [14,15,16]. In the hybrid formulation there is no sum over spin structures and no need for a GSO projection. The correlation functions are with periodic boundary conditions around all homology cycles of the Riemann surface Σ_g.

We start with the correlators of the chiral boson H:

$$\Big\langle \prod_k e^{i\frac{q_k}{\sqrt{3}}H(z_k)} \Big\rangle = Z_1^{-1/2} F(\tfrac{1}{\sqrt{3}}\sum q_k z_k - Q_H\Delta) \prod_{i<j} E(z_i,z_j)^{\frac{1}{3}q_i q_j} \prod_l \sigma(z_l)^{\frac{1}{\sqrt{3}}Q_H q_l} \,, \quad (3.8)$$

where Z_1 is the chiral determinant of [14]. The prime forms $E(z,w)$ express the pole and zero structure of the correlation function while the σ's express the coupling to the background charge. Of the remaining part F, which is due to the zero-modes of H, only the combination in which the insertion points enter will be relevant. It is, in fact, an appropriately defined theta-function [10]. Also $F(-z) = F(z)$. In the above expression (and below), z either means a point on Σ_g or its image under the Jacobi map, i.e., $\vec{I}(z) = \int_{p_0}^{z} \vec{\omega}$, depending on the context.

The ρ-correlation functions are subtle. The field ρ is very much like the chiral boson ϕ which appears in the 'bosonization' of the superconformal (β,γ) ghost system in the RNS formulation, the only difference being the value of its background charge. In the RNS superconformal ghost system ϕ is accompanied by a fermionic spin 1 (η,ξ) system. Expressions for correlation functions of products of $e^{q_i\phi(z_i)}$ which are used in RNS amplitude calculations are always done in the context of the complete (β,γ) ghost system. Following [2] our strategy will be to combine an auxiliary fermionic spin 1 (η,ξ) system with the ρ-scalar to build a bona-fide spin 1 (β,γ) system. We then compute correlation functions as in the RNS formulation, which we divide by the contribution of the auxiliary (η,ξ)-system. Following [16], we obtain

$$\Big\langle \prod_k e^{q_k\rho(z_k)} \Big\rangle_{(\beta,\gamma)} = \frac{Z_1^{1/2}}{\theta(\sum q_k z_k - Q_\rho\Delta)} \prod_{k<l} E(z_k,z_l)^{-q_k q_l} \prod_r \sigma(z_r)^{-Q_\rho q_r} \quad (3.9)$$

with $Q_\rho = -1$. As in [16], the correlation function had to be regularized due to the fact that the zero-mode contribution of the ρ-field diverges. The regularization involved a projection of the ρ-momentum plus the momentum of the regulating (η,ξ) system in the loops to arbitrary but fixed values. These projections were accompanied by factors $\oint_{a_i}\eta$ for each a-cycle on Σ and one factor of ξ to absorb its (constant) zero mode. The contribution of (η,ξ) has to be divided out in order to obtain the regulated correlators of the ρ-system. This means that (3.9) must be divided by

$$\Big\langle \prod_{i=1}^{g} \oint_{a_i} \frac{dz_i}{2\pi i} \eta(z_i)\,\xi(w) \Big\rangle = Z_1\,. \tag{3.10}$$

Altogether we thus find

$$\Big\langle \prod_k e^{q_k \rho(z_k)} \Big\rangle_{\text{reg.}} = \frac{Z_1^{-1/2}}{\theta(\sum q_k z_k + \Delta)} \prod_{k<l} E(z_k,z_l)^{-q_k q_l} \prod_r \sigma(z_r)^{q_r}\,. \tag{3.11}$$

A useful identity is the 'bosonization formula'[14]:

$$\prod_{i=1}^{g} E(z_i,w)\sigma(w) = \frac{\prod_{i<j} E(z_i,z_j) \prod_{i=1}^{g} \sigma(z_i)}{Z_1^{3/2} \det(\omega_i(z_j))} \theta(\textstyle\sum_{i=1}^{g} z_i - w - \Delta)\,. \tag{3.12}$$

Using this identity one finds

$$\Big\langle \prod_{k=1}^{g} e^{-\rho(z_k)}\, e^{\rho(w)} \Big\rangle_{\text{reg.}} = \frac{1}{Z_1^2 \det(\omega_k(z_l))}\,, \tag{3.13}$$

which differs by a factor of $\det(\mathrm{Im}\tau)$ from the corresponding expression used in [2].

4. Topological Amplitudes

4.1. Generalities

The expressions for F_g^n that one obtains by inserting the generators (2.12) into (3.1) in general are very involved. Certain restrictions are imposed by background charge cancellation. Since the total U(1) charge of the vertex operators is zero the insertions of $\widehat{\widehat{G}}^+$ and $\widehat{G}^-$ in (3.1) are precisely such that they cancel the anomaly of the total U(1) current. It is therefore sufficient to study the constraints imposed by requiring cancellation of the background charge of the ρ-field.[7] A consequence of this constraint is that if the vertex operators are not charged under $\partial\rho$ then $|n| \le g-1$. For $|n| < g-1$ there are several possibilities how the various parts of the operators (2.12) can contribute. For $|n| = g-1$ and uncharged vertex operators there is only a single amplitude that must be considered. These cases are studied in the following. We restrict to the case with $2g$ vertex operator insertions. There are then just enough insertions of θ and p to absorb their zero modes an no nontrivial contractions occur.

[7] Since the J_C current is a linear combination of the $\partial\rho$ and the total U(1)-current, background charge cancellation for H is then automatic.

4.2. R-charge $(g-1,g-1)$

This amplitude was computed in the RNS formalism in [10]. In this section we review the computation in the hybrid formalism of [2]. Imposing ρ and H background charge saturation (3.1) leads to

$$\begin{aligned}\mathcal{A}_g = \int_{\mathcal{M}}[dm_g]\frac{1}{|\det(\omega_i(\tilde{v}_j))|^2}\Big\langle\Big|\prod_{j=1}^{m}e^{\rho}\tilde{G}_C^{++}(\tilde{v}_j)\prod_{j=m+1}^{g}e^{-\rho}\bar{d}^2(\tilde{v}_j)\\ \times\prod_{l=1}^{m}(\mu_l,e^{-2\rho-\int J_C}\bar{d}^2)\prod_{l=m+1}^{3g-3}(\mu_l,G_C^-)\Big|^2 U'U^{2g-1}\Big\rangle\,.\end{aligned}\tag{4.1}$$

We have used the fact that $\oint e^{-\rho}\bar{d}^2$, when pulled off from U', only gets stuck at $J(v_g)$. $0\le m\le g-1$ parametrizes different ways to saturate the background charges.[8] We now use the freedom to choose $\tilde{v}_l = z_l$ for $l=1,\ldots,g$ where z_l are the arguments of the Beltrami differentials μ_l (which are integrated over). This is possible since the OPEs which one encounters are the naive products (no poles or zeros). This gives

$$\mathcal{A}_g = \int_{\mathcal{M}}[dm_g]\int\prod_{l=1}^{g}d^2z_l\frac{1}{|\det(\omega_i(z_l))|^2}\Big\langle\big|(\mu_l,e^{-\rho}G_C^-\bar{d}^2(z_l))\big|^2\prod_{k=1}^{2g-3}\big|(\mu_k,G_C^-)\big|^2U'U^{2g-1}\Big\rangle \tag{4.2}$$

which is independent of m.[9] Its evaluation is straightforward. One easily sees that there are just enough operator insertions to absorb the p and $\bar{p}$ zero modes. θ and $\bar{\theta}$ then also only contribute with their (constant) zero modes. The p zero modes must come from the explicit d-dependence of the vertex operator. The $(p,\theta)_L$ and $(p,\theta)_R$ correlation functions contribute a factor $|Z_1|^4(\det\mathrm{Im}\tau)^2$, where the integrals over the insertion points have already been performed. What is left is the integral over the θ zero-modes which are the Grassmann odd co-ordinates of $\mathcal{N}=2$ chiral superspace. The spinor indices arrange themselves to produce $(P_{\alpha\beta}P^{\alpha\beta})^{g-1}P_{\gamma\delta}D_L^{\gamma}D_R^{\delta}V$. The $(\bar{p},\bar{\theta})$ correlators give a term $|Z_1|^4|\det\omega_i(z_l)|^4$, leaving only the $\bar{\theta}$ zero-mode integrations. They can be performed using $\int(d^2\bar{\theta})_L(d^2\bar{\theta})_R\Psi = \bar{D}_L^2\bar{D}_R^2\Psi|_{\bar{\theta}_L=\bar{\theta}_R=0}$. Since $\bar{D}_{\dot{\alpha}}P_{\beta\gamma}=0$, the only effect of this is to convert $D_L^{\alpha}D_R^{\beta}V$ to $P^{\alpha\beta}$. Finally, the ρ-correlator gives, using (3.11) and (3.12), $(|Z_1|^4|\det\omega_i(z_l)|^2)^{-1}$. The partition function of the x^m contributes a factor

[8] For notational simplicity we have chosen the same m for the left- and for the right-movers.

[9] This shows that for this amplitude all admissible distributions of vertex operators parametrized by m indeed lead to the same result and that the only subtleties that arise from contact terms are the ones analyzed in [17,18]. We are not aware of an argument that this is generally the case.

$|Z_1|^{-4}(\det \mathrm{Im}\tau)^{-2}$. To the given order of spacetime derivatives, the x^m-dependence of the vertex operators is only through its zero mode. Combining arguments we obtain

$$\mathcal{A}_g = \int (d^2\theta)_L (d^2\theta)_R (P_{\alpha\beta}P^{\alpha\beta})^g \int_{\mathcal{M}} [dm_g] \Big\langle \prod_{i=1}^{3g-3} |(\mu_i, G_C^-)|^2 \Big\rangle . \tag{4.3}$$

The last part of this expression is the string partition function of the topological B-model:

$$F_g^B = \int_{\mathcal{M}} [dm_g] \Big\langle \prod_{i=1}^{3g-3} |(\mu_i, G_C^-)|^2 \Big\rangle . \tag{4.4}$$

To determine the dependence of F_g^B on the chiral or twisted-chiral moduli one inserts the appropriate expressions (2.17) into these correlation functions. It can be shown, using the arguments of [17], that F_g^B does not depend on perturbations induced by either (c,a) or (a,c) operators. It therefore depends only on the complex structure moduli and the amplitudes calculated are therefore vector multiplet couplings (type IIB).

4.3. R-charge $(1-g, 1-g)$

Starting from (3.1)and imposing ρ and H-background charge saturation, one obtains, in close analogy to (4.1),

$$\begin{aligned} \mathcal{A}'_g = \int_{\mathcal{M}} [dm_g] \frac{1}{|\det(\omega_i(\tilde{v}_j))|^2} \Big\langle \Big| \prod_{j=1}^{m} G_C^+(\tilde{v}_j) \prod_{j=m+1}^{g} e^{2\rho + \int J_C} d^2(\tilde{v}_j) \prod_{l=1}^{m} (\mu_l, e^{\rho} d^2) \\ \times \prod_{l=m+1}^{3g-3} (\mu_l, e^{-\rho} \tilde{G}_C^{--}) \Big|^2 U'' U^{2g-1} \Big\rangle . \end{aligned} \tag{4.5}$$

$0 \leq m \leq g-1$ parametrizes the different ways of saturating the background charges. By appropriate choices of the $\tilde{v}_j$ this amplitude can be brought to the form

$$\mathcal{A}'_g = \int_{\mathcal{M}} [dm_g] \int \prod_{j=1}^{g} d^2 z_j \frac{1}{|\det \omega_i(z_j)|^2} \langle |(\mu(z_j), e^{\rho} d^2 G_C^+(z_j)) \prod_{k=1}^{2g-3} (\mu_k, e^{-\rho} \tilde{G}_C^{--})|^2 U'' U^{2g-1} \Big\rangle , \tag{4.6}$$

which shows that also this amplitude is independent of m. However, its evaluation is most easily done for a different choice of the insertion points $\tilde{v}_j$. To fix them, we start from (4.5) with the choice $m = 0$ and compute the ρ and the H correlators. Their product is, using (3.8) and (3.11),

$$\frac{1}{Z_1} \cdot \frac{F(\sqrt{3}\sum \tilde{v}_j - \frac{2}{\sqrt{3}} \sum z_k - \sqrt{3} w + \sqrt{3}\Delta)}{\theta(2\sum \tilde{v}_j - \sum z_k - 2w + \Delta)} \cdot \frac{\prod_{k<l} E(z_k, z_l)^{\frac{1}{3}} \prod_j E(\tilde{v}_j, w) \prod_k \sigma(z_k)\sigma(w)}{\prod_{i<j} E(\tilde{v}_i, \tilde{v}_j) \prod_j \sigma(\tilde{v}_j)} , \tag{4.7}$$

where we have only displayed the holomorphic part. With the help of the identity (3.12) this is equal to

$$\frac{1}{(Z_1)^{\frac{5}{2}}}\frac{F(\sqrt{3}\sum\tilde{v}_j-\frac{2}{\sqrt{3}}\sum z_k-\sqrt{3}w+\sqrt{3}\Delta)}{\theta(2\sum\tilde{v}_j-\sum z_k-2w+\Delta)}\cdot\frac{\theta(\sum\tilde{v}_j-w-\Delta)}{\det w_i(\tilde{v}_j)}\cdot\prod_{k<l}E(z_k,z_l)^{\frac{1}{3}}\prod_k\sigma(z_k)\,. \tag{4.8}$$

We now choose the g positions $\tilde{v}_j$ such that $\vec{I}(\sum\tilde{v}_j-w-\Delta)=\vec{I}(2\sum\tilde{v}_j-\sum z_k-2w+\Delta)$. Then the theta functions cancel and the remaining terms are

$$\frac{1}{(Z_1)^{\frac{5}{2}}\det\omega_i(\tilde{v}_j)}\cdot F(\tfrac{1}{\sqrt{3}}\sum z_k-\sqrt{3}\Delta)\cdot\prod_{k<l}E(z_k,z_l)^{\frac{1}{3}}\prod_k\sigma(z_k)\,. \tag{4.9}$$

This can be written as

$$\frac{1}{Z_1^2\det\omega_i(\tilde{v}_j)}\Big\langle\prod_{k=1}^{3g-3}e^{-\frac{i}{\sqrt{3}}H(z_k)}\Big\rangle\,. \tag{4.10}$$

The $p,\theta,\bar{p}$ and $\bar{\theta}$ correlators are as in the previous amplitude (with the roles on barred and unbarred variables interchanged) and one finally obtains

$$\mathcal{A}'_g=\int(d^2\bar{\theta})_L(d^2\bar{\theta})_R(\bar{P}_{\dot{\alpha}\dot{\beta}}\bar{P}^{\dot{\alpha}\dot{\beta}})^g\int_{\mathcal{M}}[dm_g]\Big\langle\prod_{i=1}^{3g-3}|(\mu_i,\check{G}^-_C)|^2\Big\rangle\,. \tag{4.11}$$

Here $\check{G}^-_C=e^{-\frac{i}{\sqrt{3}}H}G'_C$ where G'_C is defined to be $G^+_C=e^{\frac{i}{\sqrt{3}}H}G'_C$. Note that G^-_C and $\check{G}^-_C$ both have conformal weight two. The internal amplitude multiplying the spacetime part is the complex conjugate of the B-model amplitude (4.4): this follows from the fact that the expression (4.9) can be written as

$$\frac{1}{Z_1^2\det\omega_i(\tilde{v}_j)}\Big\langle\prod_{k=1}^{3g-3}e^{\frac{i}{\sqrt{3}}H(z_k)}\Big\rangle_{Q_H=\sqrt{3}}\,, \tag{4.12}$$

where we used (3.8) but with the reversed background charge as compared to (4.10). This happens if one chooses the opposite twisting in (2.2). Since the operators $\check{G}^-_C$ and G^+_C both contain the same operator G'_C, the internal part of the amplitude (4.11) is equal to

$$\Big\langle\prod_{i=1}^{3g-3}|(\mu_i,\check{G}^-_C)|^2\Big\rangle_{++}=\Big\langle\prod_{i=1}^{3g-3}|(\mu_i,G^+_C)|^2\Big\rangle_{--}\,. \tag{4.13}$$

The subscripts refer to the two possible twistings $T\to T+\frac{1}{2}\partial J$ and $T\to T-\frac{1}{2}\partial J$ for left- and right-movers. Finally, since for unitary theories $(G^-_C)^\dagger=G^+_C$, the right-hand side of (4.13) is the complex conjugate of F^B_g given in (4.4), and therefore $\mathcal{A}'_g$ defined in (4.5) is the complex conjugate of the chiral amplitude $\mathcal{A}_g$ of (4.1).

4.4. R-charges $(g-1,1-g)$ and $(1-g,g-1)$

The 'mixed' amplitudes with R-charges $(g-1,1-g)$ and $(1-g,g-1)$ can now be written down immediately. They are expressed as integrals over twisted chiral superspace and involve the superfields $Q_{\alpha\dot\beta}$ and $\bar Q_{\dot\alpha\beta}$. They are

$$\mathcal{A}''_g = \int (d^2\theta)_L (d^2\bar\theta)_R (Q_{\alpha\dot\beta}Q^{\alpha\dot\beta})^g \int_{\mathcal{M}} [dm_g] \Big\langle \prod_{i=1}^{3g-3} (\mu_i, G^-_C)_L (\bar\mu_i, \check G^-_C)_R \Big\rangle + \text{c.c.} \tag{4.14}$$

By the same arguments as given before, one shows that this type IIB string amplitude only depends on deformations in the (a,c) (and (c,a) for the complex conjugate piece) ring, i.e., on Kähler moduli. In type IIB, these are in tensor multiplets. From the discussion in section 4.3 it also follows that

$$\int_{\mathcal{M}} [dm_g] \Big\langle \prod_{i=1}^{3g-3} (\mu_i, G^-_C)_L (\bar\mu_i, \check G^-_C)_R \Big\rangle_{++} = \int_{\mathcal{M}} [dm_g] \Big\langle \prod_{i=1}^{3g-3} (\mu_i, G^-_C)_L (\bar\mu_i, G^+_C)_R \Big\rangle_{+-} = F^A_g \,, \tag{4.15}$$

which is the topological A-model amplitude.

So far we have computed amplitudes of type IIB string theory. To compute type IIA amplitudes we need to twist the left- and right-moving internal SCFTs oppositely. In the amplitudes this induces the following changes: $(G^-_C)_R \to (G^+_C)_R$ and $(\check G^-_C)_R \to (\check G^+_C)_R$ where $\check G^+_C = e^{\frac{i}{\sqrt3}H}\bar G'_C$. Due to the opposite twist, the conformal weights are preserved under this operation. For instance, the spacetime part of (4.3) gets combined with F^A_g, that of (4.14) with F^B_g. According to (2.19), F^A_g depends on the moduli contained in vector multiplets, F^B_g on those contained in tensor multiplets.

4.5. Summary

We have recomputed certain chiral and twisted-chiral couplings that involve g powers of P^2 or Q^2, respectively, using hybrid string theory. The amplitudes involve the topological string partition functions F^A_g and F^B_g. F^A_g depends on the moduli parametrizing the (c,a) ring, F^B_g on those of the (c,c) ring. In type IIA or type IIB, these are contained in spacetime chiral (vector) or twisted-chiral (tensor) multiplets, as summarized in the table. The dependence on the moduli of the complex conjugate rings is only through the holomorphic anomaly [17].

type IIA		type IIB	
$(P^2)^g F_g^A$	(c,a): vector	$(P^2)^g F_g^B$	(c,c): vector
$(Q^2)^g F_g^B$	(c,c): tensor	$(Q^2)^g F_g^A$	(c,a): tensor

As discussed in [3,19], on-shell, the superfield $P_{\alpha\beta}$ describes the linearization of the Weyl multiplet. Its lowest component is the selfdual part of the graviphoton field strength, $P_{\alpha\beta}| = F_{\alpha\beta}$. The $\theta_L\theta_R$-component is the selfdual part $C_{\alpha\beta\gamma\delta}$ of the Weyl tensor. The bosonic components of $Q_{\alpha\dot\beta}$ are $Q_{\alpha\dot\beta}| = \partial_{\alpha\dot\beta}Z$, where Z is the complex R-R-scalar of the RNS formulation of the type II string; its $\theta_L\bar\theta_R$-component is $\partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta}S$. The real component of S is the dilaton, its imaginary component is dual to the antisymmetric tensor of the NS-NS-sector.These results can be obtained by explicit computation from the θ-expansion of the superfield V. After integrating (4.3) and (4.14) over chiral and twisted-chiral superspace, respectively, $2g-2$ powers of $F_{\alpha\beta}$ are coupled to two powers of $C_{\alpha\beta\gamma\delta}$, while $2g-2$ powers of ∂Z are coupled to two powers of $\partial^2 S$, with the tensorial structure discussed in [10]. In [3,20,21] the question is addressed how these (and other) couplings can be described in an off-shell (projective) superspace description at the non-linearized level.

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