Hermitian matrix model free energy: Feynman graph technique for all genera

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Hermitian matrix model free energy: Feynman graph technique for all genera

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Abstract: We present the diagrammatic technique for calculating the free energy of the Hermitian one-matrix model to all orders of $1/N$ expansion in the case where the limiting eigenvalue distribution spans arbitrary (but fixed) number of disjoint intervals (curves).

Keywords: Matrix Models, Conformal and W Symmetry, 1/N Expansion.

JHEP03(2006)014

Contents

1. Introduction **1**

2. Matrix models in $1/N$-expansion **2**

2.1 Loop equation and resolvents 2

2.2 Solution in genus zero 4

3. Calculating resolvents. Diagram technique **7**

3.1 Bergmann bidifferential and two-point resolvent 7

3.2 The iterative procedure 9

4. Inverting the loop insertion operator. Free energy **12**

4.1 The H-operator 12

4.2 Scaling relation 15

1. Introduction

Matrix models and their so-called multisupport (multicut) solutions became again important recently due to studies in $\mathcal{N} = 1^1$ SUSY gauge theories by Cachazo, Intrilligator and Vafa [1], [2] followed by the proposal of Dijkgraaf and Vafa [3] to calculate the low energy superpotentials using the partition function of multicut solutions. These solutions, being known already for a long time [4, 5], got a new insight in [6].

The leading order of the $1/N$-expansion in the matrix size of the matrix model is described by the semiclassical tau-function of the so-called universal Whitham hierarchy [7] (see the details about one-matrix and two-matrix cases in [8] and [9]).

One may also consider more general solutions to matrix models, identifying them with generic solutions to the loop (Schwinger–Dyson, or Virasoro) equations [10], to be the Ward identities satisfied by matrix integrals [11].

We consider solutions to the loop equations admitting multi-matrix integral representations [6, 12, 13]. These solutions are associated with families of Riemann surfaces and form a sort of a basis in the space of all solutions to the loop equations [13] (like the

[1] We reserve N to denote the size of matrix in matrix models.

finite-gap solutions form a similar basis in the space of all solutions to an integrable hierarchy). They can be distinguished by their "isomonodromic" properties — switching on higher matrix model couplings and $1/N$-corrections does not change the family of Riemann surfaces, but just reparameterizes the moduli as functions of these couplings. On the side of integrable hierarchies, these solutions must satisfy some equations of multicomponent hierarchies [14], not just the Toda chain hierarchy equations, which are satisfied only by a one-cut solution.

Much progress has been made on the way to constructing explicit solutions of matrix models in the large-N expansion since [15]. It was shown (primarily for the one-cut case [16]) that the variables convenient for describing solutions in the genus expansion are branching points of the corresponding Riemann surface and the so-called moments of the potential. These variables, being expressed highly nonlinearly in times and occupation numbers, allow reducing the problem to solving some algebraic relations. Using this technique, the two-matrix model was solved in the large-N limit in the one-cut case [17], this technique was extended to the multicut case [9, 18] , and the subleading term of $1/N$-expansion was found in [19].

In [20], the consistent diagrammatic technique for finding the loop means (multiresolvents) in the $1/N$-expansion of the Hermitian one-matrix model (1MM) was proposed. The main result of the present paper is the generalization of this technique to describe the free energy contributions of all orders in $1/N$ for multicut solutions of the 1MM.

In section 2, we describe the general properties of multi-cut solutions of 1MM and the solution of the loop equation in the large-N limit.

In section 3, we first present necessary notions of the Riemann geometry describing the Bergmann bidifferentials and the loop insertion operator (which acts from the space of s-differentials to the space of $(s+1)$-differentials) and then describe the diagrammatic technique of [20] in a more convenient terms than in the original presentation.

In section 4, we introduce the operator H, which reduces the degree of forms by one and is in a sense inverse to the loop insertion operator. This eventually makes possible to present the genus k contribution to the free energy through the action of H on the one-point resolvent (for all k except $k=1$; for the latter case, the answer has been already obtained, see [21, 19]).

2. Matrix models in $1/N$-expansion

2.1 Loop equation and resolvents

Let us consider the formal integral of the 1MM [22]

$$\int_{N\times N} DX\, e^{-\frac{N}{t_0}\mathrm{tr}\, V(X)} = e^{-\mathcal{F}}, \tag{2.1}$$

where $V(X) = \sum_{n\geq 1} t_n X^n$ and $\hbar = t_0/N$ is a formal expansion parameter. The integration in (2.1) goes over $N\times N$ Hermitian matrices, and for generic potentials, the integration may

go over curves in the complex plane of each of N proper variables, which are eigenvalues of X. The topological expansion of the Feynman diagrams series is then equivalent to the expansion in even powers of $\hbar$ for

$$\mathcal{F} \equiv \mathcal{F}(\hbar, t_0, t_1, t_2, \ldots) = \sum_{g=0}^{\infty} \hbar^{2g-2} \mathcal{F}_g. \tag{2.2}$$

Customarily, $t_0 = \hbar N$ is the scaled number of eigenvalues. It is convenient, but not compulsory, to assume the potential $V(p)$ to be a polynomial of the fixed degree $m+1$. In general, to have an algebraic curve we need $V'(p)$ to be a rational function [23]

The averages corresponding to partition function (2.1) are defined as usual:

$$\langle F(X) \rangle = \frac{1}{Z} \int_{N \times N} DX\, F(X)\, \exp\left(-\frac{1}{\hbar} \mathrm{tr}\, V(X) \right), \tag{2.3}$$

and it is convenient to use their formal generating functionals: the one-point resolvent

$$W(p) = \hbar \sum_{k=0}^{\infty} \frac{\langle \mathrm{tr}\, X^k \rangle}{p^{k+1}} \tag{2.4}$$

as well as the s-point resolvents ($s \geq 2$)

$$W(p_1, \ldots, p_s) = \hbar^{2-s} \sum_{k_1, \ldots, k_s = 1}^{\infty} \frac{\langle \mathrm{tr}\, X^{k_1} \cdots \mathrm{tr}\, X^{k_s} \rangle_{\mathrm{conn}}}{p_1^{k_1+1} \cdots p_s^{k_s+1}} = \hbar^{2-s} \left\langle \mathrm{tr}\, \frac{1}{p_1 - X} \cdots \mathrm{tr}\, \frac{1}{p_s - X} \right\rangle_{\mathrm{conn}} \tag{2.5}$$

where the subscript "conn" pertains to the connected part.

These resolvents are obtained from the free energy $\mathcal{F}$ through the action

$$\begin{aligned} W(p_1, \ldots, p_s) &= -\hbar^2 \frac{\partial}{\partial V(p_s)} \frac{\partial}{\partial V(p_{s-1})} \cdots \frac{\partial \mathcal{F}}{\partial V(p_1)} = \\ &= \frac{\partial}{\partial V(p_s)} \frac{\partial}{\partial V(p_{s-1})} \cdots \frac{\partial}{\partial V(p_2)} W(p_1), \end{aligned} \tag{2.6}$$

of the loop insertion operator[2]

$$\frac{\partial}{\partial V(p)} \equiv -\sum_{j=1}^{\infty} \frac{1}{p^{j+1}} \frac{\partial}{\partial t_j}. \tag{2.7}$$

Therefore, if one knows exactly the one-point resolvent for arbitrary potential, all multi-point resolvents can be calculated by induction. In the above normalization, the genus expansion has the form

$$W(p_1, \ldots, p_s) = \sum_{g=0}^{\infty} \hbar^{2g} W_g(p_1, \ldots, p_s), \quad s \geq 1, \tag{2.8}$$

which is analogous to genus expansion (2.2).

[2]Although this operator contains all partial derivatives w.r.t. the variables t_k's, below we introduce additional variables S_i and, therefore, we use the partial derivative notation here.

The first in the chain of the loop equations of the 1MM is [10]

$$\oint_{\mathcal{C}_\mathcal{D}} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p-\omega} W(\omega) = W(p)^2 + \hbar^2 W(p,p). \tag{2.9}$$

Here and hereafter, $\mathcal{C}_\mathcal{D}$ is a contour encircling clockwise all singular points of $W(\omega)$, but not the point $\omega = p$; this contour integration acts as the projection operator extracting the coefficient of the term p^{-1}. Using Eq. (2.6), one can express the second term in the r.h.s. of loop equation (2.9) through $W(p)$, and Eq. (2.9) becomes an equation on one-point resolvent (2.4).

Substituting genus expansion (2.8) in Eq. (2.9), we find that $W_g(p)$ for $g \geq 1$ satisfy the equation

$$\left(\widehat{K} - 2W_0(p)\right) W_g(p) = \sum_{g'=1}^{g-1} W_{g'}(p) W_{g-g'}(p) + \frac{\partial}{\partial V(p)} W_{g-1}(p), \tag{2.10}$$

where $\widehat{K}$ is the linear integral operator

$$\widehat{K} f(p) \equiv -\oint_{\mathcal{C}_\mathcal{D}} \frac{d\xi}{2\pi i} \frac{V'(\xi)}{p-\xi} f(\xi). \tag{2.11}$$

In Eq. (2.10), $W_g(p)$ is expressed through only the $W_{g_i}(p)$ for which $g_i < g$. This fact permits developing the iterative procedure.

The solution $W_1(p)$ to the loop equation in the multicut case was first found by Akemann [24].[3] He also managed to integrate it in order to obtain the free energy $\mathcal{F}_1$ in the two-cut case. The genus-one partition function in the generic multi-cut case was proposed in [25, 26], where it was observed that the Akemann formula coincides with the correlator of twist fields (that produce cuts on the complex plane and give rise to the hyperelliptic Riemann surface as a cover of complex plane) computed by Al.Zamolodchikov [27], with some corrections due to including star operators (introduced by G.Moore in [28]) into consideration. In [21, 19], the genus-one correction was derived by solving the loop equation, which generalize the Akemann result for partition function to arbitrary number of cuts.

2.2 Solution in genus zero

The form of loop equation (2.9) is based exclusively on the reparameterization invariance of the matrix integral, which retains independently on the details of eigenvalue density distribution. In the 1MM case at $N \to \infty$, the eigenvalues fill in some segments in complex plane, dependently on the shape of potential $V(X)$. For polynomial potentials, the number of segments is finite and the contour $\mathcal{C}_\mathcal{D}$ of integration in (2.10) encircles a finite number n of disjoint intervals

$$\mathcal{D} \equiv \bigcup_{i=1}^{n} [\mu_{2i-1}, \mu_{2i}], \quad \mu_1 < \mu_2 < \ldots < \mu_{2n}. \tag{2.12}$$

[3]The universal critical behavior of the corresponding correlation functions was discussed in [5].

Recall that as

$$W_g(p)|_{p\to\infty} = \frac{t_0}{p}\delta_{g,0} + O(1/p^2), \tag{2.13}$$

all $W_g(p)$ must be total derivatives,

$$W_g(p) = \frac{\partial}{\partial V(p)}\mathcal{F}_g, \quad g \geq 0. \tag{2.14}$$

Inserting genus expansion (2.8) into loop equation (2.9), we obtain

$$\oint_{\mathcal{C}_\mathcal{D}} \frac{d\lambda}{2\pi i}\frac{V'(\lambda)}{p-\lambda}W_0(\lambda) = (W_0(p))^2 \tag{2.15}$$

for genus zero and (2.10) for higher genera. Given $W_0(p)$, one can then determine $W_g(p)$ for $g \geq 1$ iteratively genus by genus provided the operator $(\widehat{K} - 2W_0(p))$ (see (2.11)) can be inverted uniquely.

One can solve Eq. (2.15) for the planar one-point resolvent $W_0(p)$ as follows. Deforming the contour in Eq. (2.15) to infinity, we obtain

$$(W_0(p))^2 = V'(p)W_0(p) + \oint_{\mathcal{C}_\infty} \frac{d\lambda}{2\pi i}\frac{V'(\lambda)}{p-\lambda}W_0(\lambda). \tag{2.16}$$

The last term in the r.h.s. is

$$P_{m-1}(p) = \oint_{\mathcal{C}_\infty} \frac{d\lambda}{2\pi i}\frac{V'(\lambda)}{p-\lambda}W_0(\lambda), \tag{2.17}$$

a degree-$m-1$ polynomial to be fixed later, and the solution to (2.16) is then

$$W_0(p) = \frac{1}{2}V'(p) - \frac{1}{2}\sqrt{V'(p)^2 + 4P_{m-1}(p)} \equiv \frac{1}{2}V'(p) - y(p), \tag{2.18}$$

where the minus sign is chosen in order to fulfill asymptotic condition (2.13) and the function $y(p)$ is defined as follows. For the polynomial potential of power $m+1$, the resolvent $W_0(p)$ is a function on complex plane with m cuts, or on a hyperelliptic curve $y^2 = V'(p)^2 + 4P_{m-1}(p)$ of genus $g = m-1$. For generic potential $V(X)$ with $m \to \infty$, this curve may have an infinite genus, but we can still consider solutions with a finite, fixed number n of cuts filled by eigenvalues. For this, we separate the smooth part of the curve introducing

$$y(p) \equiv M(p)\tilde{y}(p), \quad \text{and} \quad \tilde{y}^2(p) \equiv \prod\nolimits_{\alpha=1}^{2n}(p-\mu_\alpha) \tag{2.19}$$

with all branching points μ_α distinct. The variable $\tilde{y}$ defines therefore the new, reduced Riemann surface, which plays a fundamental role in our construction. In what follows, we still assume $M(p)$ to be a polynomial of degree $m-n$, keeping in mind that n is always finite and fixed, while $m \geq n$ can be chosen arbitrarily large.

JHEP03(2006)014

From now on, we distinguish between images of the infinity at two sheets—physical and unphysical — of hyperelliptic Riemann surface (2.19) respectively denoting them ∞_+ and ∞_-. By convention, we set $\tilde{y}|_{p\to\infty_+} \sim p^n$, and $M(p)$ is then[4]

$$M(p) = -\frac{1}{2}\mathrm{res}_{\infty_+} dw \frac{V'(w)}{(w-p)\tilde{y}(w)}. \tag{2.20}$$

Inserting this solution in Eq. (2.18) and deforming the contour back, we obtain the planar one-point resolvent with an n-cut structure,

$$W_0(p) = \frac{1}{2}\oint_{\mathcal{C}_\mathcal{D}} \frac{d\lambda}{2\pi i}\frac{V'(\lambda)}{p-\lambda}\frac{\tilde{y}(p)}{\tilde{y}(\lambda)}, \quad p \notin \mathcal{D}. \tag{2.21}$$

Let us now discuss the parameter counting. Rewriting (2.16) as quadratic equation for $W_0(p)$ we had to imply that polynomial (2.17) depends on some unknown parameters to be fixed later. Indeed, the dependence on coefficients of (2.17) can be retranslated into the dependence on *filling fractions*

$$S_i = \oint_{A_i} \frac{d\lambda}{2\pi i}\, y(\lambda) = \oint_{A_i} \frac{d\lambda}{2\pi i} M(\lambda)\tilde{y}(\lambda), \tag{2.22}$$

where A_i, $i = 1, \ldots, n-1$ is the basis of A-cycles on the hyperelliptic Riemann surface (2.19) (we may conveniently choose them to be the first $n-1$ cuts). Adding the (normalized) total number of eigenvalues

$$t_0 = \oint_{\mathcal{C}_\mathcal{D}} y(\lambda)\frac{d\lambda}{2\pi i} = \mathrm{res}_{\infty_+}\, y(\lambda)d\lambda \tag{2.23}$$

to the set of S_i, we obtain n parameters, which exactly matches the number of coefficients of (2.17) in the nondegenerate case where $m = n$. We assume the occupation number for the last, nth cut to be $t_0 - \sum_{i=1}^{n-1} S_i \equiv S_n$. [5] If $n < m$, we consider variables (2.22) as conditions to which we add, following (2.13), the asymptotic conditions

$$-t_0\delta_{k,n} = \frac{1}{2}\oint_{\mathcal{C}_\mathcal{D}} \frac{d\lambda}{2\pi i}\frac{\lambda^k V'(\lambda)}{\tilde{y}(\lambda)}, \quad k = 0, \ldots, n, \tag{2.24}$$

which provide $n+1$ equations for $2n$ constants μ_α.

In the planar limit of matrix models, filling fractions (2.22) can be considered as *independent* parameters of the theory, since the jumps between different cuts are suppressed as non-perturbative corrections in $\hbar$. (Obviously, no parameters S_i arise in the one-cut case.) In particular, this imposes restrictions

$$\frac{\partial}{\partial V(p)} S_i = 0, \quad i = 1, \ldots, n-1, \qquad \frac{\partial}{\partial V(p)} t_0 = 0. \tag{2.25}$$

[4] By a standard convention, $\mathrm{res}_\infty dx/x = -1$, and the direction of the integration contour at the infinity point therefore coincides with the direction of contour for integrals over $\mathcal{C}_\mathcal{D}$ and over the set of A-cycles, see below.

[5] It is sometimes convenient to consider S_n instead of t_0 as a canonical variable. However, in all instants we use S_n, we specially indicate it for not confusing S_n with the "genuine" filling fraction variables S_i, $i = 1, \ldots, n-1$.

JHEP03(2006)014

In accordance with the original matrix model formulation (see e.g. [4], [24]) one must look for the genuine minimum of the matrix model effective action with respect to *all* variables: S_i together with the times t_j. This implies vanishing of partial derivatives of $\mathcal{F}_0$ with respect to S_i, and one can find (see [4], [3], and [29]) that these derivatives are differences of chemical potential on disjoint cuts, equal to the integrals over *dual* B-cycles on (2.19):

$$\frac{\partial \mathcal{F}_0}{\partial S_i} = \oint_{B_i} y(\lambda) d\lambda \equiv \Pi_i \tag{2.26}$$

Note that the geometric definition of $\mathcal{F}_0$ is *not* modular invariant, i.e., it depends on the choice of A- and B-cycle basis on (2.19). Under the change of homology basis, $\mathcal{F}_0$ transforms in accordance with the duality transformations [30]. The higher-genus corrections become also scheme dependent: choosing S_i or Π_i as independent variables, we obtain different expressions for the genus-one free energy.

3. Calculating resolvents. Diagram technique

In this section, we rederive the diagrammatic technique of [20] in a more concise form of differentials on Riemann surface (2.19). We choose S_i to be independent variables. Our main goal is to invert loop equation (2.10) to obtain the expression for $W_k(p)$ for any $k \geq 1$.

3.1 Bergmann bidifferential and two-point resolvent

Let us consider the *Bergmann kernel* (canonically normalized bidifferential in Fay's terminology) which is the bi-differential on a Riemann surface Σ_g that is symmetrical in its arguments $P, Q \in \Sigma_g$ and has the only singularity at the coinciding arguments where it has the behavior (see [31], [32])

$$B(P,Q) = \left(\frac{1}{(\tau(P) - \tau(Q))^2} + \frac{1}{6} S_B(P) + o(1) \right) d\tau(P) d\tau(Q), \tag{3.1}$$

in some local coordinate $\tau(P)$; $S_B(P)$ is the Bergmann projective connection and we fix the normalization (the possibility to add symmetrical bilinear forms composed from holomorphic 1-differentials in variables Q and P) claiming vanishing all the integrals over A-cycles of $B(P,Q)$:

$$\oint_{A_i} B(P,Q) = 0, \text{ for } i = 1, \ldots, g. \tag{3.2}$$

We then have the following standard Rauch variational formulas relating $B(P,Q)$ with other objects on a (general, not necessarily hyperelliptic) Riemann surface:

$$\frac{\partial}{\partial \mu_\alpha} B(P,Q) = \frac{1}{2} B(P,[\mu_\alpha]) B([\mu_\alpha],Q), \tag{3.3}$$

and

$$\oint_{B_i} B(P,Q) = 2\pi i dw_i(P), \tag{3.4}$$

where μ_α is any simple branching point of the complex structure, then, by definition, in the vicinity of μ_α,

$$B(P,Q)|_{Q\to\mu_\alpha} = B(P,[\mu_\alpha])\left(\frac{dq}{\sqrt{q-\mu_\alpha}} + O(\sqrt{q-\mu_\alpha})dq\right), \tag{3.5}$$

and $dw_i(P)$ are canonically normalized holomorphic differentials:

$$\oint_{A_j} dw_i(P) = \delta_{ij}. \tag{3.6}$$

We also introduce the 1-form $dE_{Q,Q_0}(P)$, which is the primitive of $B(P,Q)$:

$$dE_{Q,Q_0}(P) = \int_{Q_0}^{Q} B(P,\xi), \qquad dE_{Q,Q_0}(P)|_{P\to Q} = \frac{d\tau(P)}{\tau(P)-\tau(Q)} + \text{finite}\,. \tag{3.7}$$

Then, obviously,

$$\oint_{A_i} dE_{Q,Q_0}(P) = 0. \tag{3.8}$$

The form $dE_{Q,Q_0}(P)$ is single-valued w.r.t. P on the Riemann surface and multiple-valued w.r.t. the variable Q: from (3.4),

$$\begin{aligned} dE(P)_{Q+\oint_{B_i},Q_0}(P) &= 2\pi i dw_i(P) + dE_{Q,Q_0}(P),\\ dE(P)_{Q+\oint_{A_i},Q_0}(P) &= dE_{Q,Q_0}(P). \end{aligned}$$

The reference point Q_0 does not play any role in the construction; we keep it only for the consistency of notation.

These quantities are related to the Prime form $E(P,Q)$:

$$B(P,Q) = d_P d_Q \log E(P,Q), \qquad dE_{Q,Q_0}(P) = d_P \log\frac{E(P,Q)}{E(P,Q_0)}, \tag{3.9}$$

where the Prime form is defined in the standard way. Let us consider the Jacobian J, which is a g-dimensional torus related to the curve Σ_g. Recall that the Abel map $\Sigma_g \mapsto J : Q \to \vec{x} \equiv \left\{\int_{Q_0}^{Q} d\omega_i\right\}$, where Q_0 is a reference point, set into the correspondence to each point Q of the complex curve the vector in the Jacobian, and we also introduce the theta function $\Theta_{[\alpha]}(\vec{x})$ of an odd characteristic $[\alpha]$ that becomes zero at $\vec{x}=0$. We introduce the normalizing functions (1/2-differentials) $h_\alpha(x)$ determined for the subset of $\vec{x}\in J$ that are image points of $Q\in\Sigma_g$

$$h_\alpha^2(\vec{x}) = \sum_{i=1}^{g} \frac{\partial\Theta_{[\alpha]}}{\partial z_i}(0) d\omega_i(Q).$$

The explicit expression for the Prime form $E(x,y)$ that has a single zero on the Riemann surface Σ_g then reads

$$E(P,Q) = \frac{\Theta_{[\alpha]}(\vec{x}-\vec{y})}{h_\alpha(\vec{x})h_\alpha(\vec{y})}. \tag{3.10}$$

We can now express the 2-point resolvent $W_0(p,q)$ in terms of $B(P,Q)$. Let us denote p and $\overline{p}$ the points on the respective physical and unphysical sheets. Then,

$$\frac{\partial V'(p)}{\partial V(q)} = -B(p,q) - B(p,\overline{q}) = -\frac{dp\,dq}{(p-q)^2}$$

since it has double poles with unit quadratic residues at $p = q$ and $p = \overline{q}$. The 2-point resolvent (2.18) is nonsingular at coinciding points; therefore,

$$\frac{\partial y(p)}{\partial V(q)} = -\frac{1}{2}(B(p,q) - B(p,\overline{q})), \tag{3.11}$$

and

$$W_0(p,q) = -B(p,\overline{q}). \tag{3.12}$$

3.2 The iterative procedure

We can determine higher genus contributions iteratively by inverting genus expanded loop equation (2.10). All multi-point resolvents of the same genus can be obtained from $W_g(p)$ merely by applying the loop insertion operator $\frac{\partial}{\partial V(p)}$.

Claiming all the higher free energy terms F_g to depend only on μ_α and a *finite* number of the moments $M_\alpha^{(k)}$, which are derivatives of $(k-1)$th order of the polynomial $M(p)$ at branching points, allows no freedom of adding the terms depending only on t_0 and S_i to F_g.

We first prove that the operator $\widehat{K} - 2W_0(p)$ acting on

$$\oint_{\mathcal{C}_\mathcal{D}} dE_{q,q_0}(p)\frac{dq}{2\pi i}\frac{1}{2y(q)}f(q), \tag{3.13}$$

where $f(q)$ is the combination of resolvents in the r.h.s. of (2.10) whose only singularities are poles of finite orders at μ_α and the point p lies outside $\mathcal{C}_\mathcal{D}$, just gives $f(p)$.

For this, we first note that as $V'(\xi)$ has no square root singularities at μ_α, instead of evaluating the integral over $\mathcal{C}_\mathcal{D}$, we can just evaluate *residues* on Riemann surface (2.19) w.r.t. the variables ξ and q. (This effectively corresponds to doubling the integral over $\mathcal{C}_\mathcal{D}$ and it gives additional factor 1/2.) We therefore denote $\mathcal{C}^{(\xi)}$ the contour that encircles all branching points of $\tilde{y}(\xi)$ and assume that the contour $\mathcal{C}^{(q)}$ of integration w.r.t. the variable q lies inside $\mathcal{C}^{(\xi)}$.

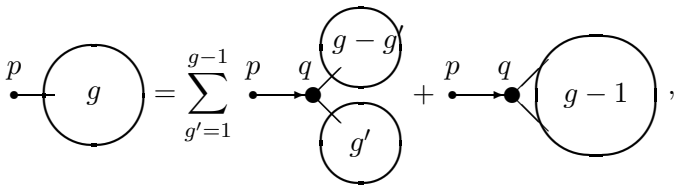
We now implement the Riemann bilinear identities. The contributions of A- and B-cycle integrations vanish due to (2.25) and (3.4). Following (2.18), we substitute $2(W_0(\xi) + y(\xi))$ for $V'(\xi)$ in (2.11) and calculate the part with $W_0(\xi)$ taking the contour of integration to the infinity; the integrand with $dE_{q,q_0}(\xi)$ is regular, and the only contribution comes from poles at $\xi = p$ and $\xi = \overline{p}$; this contribution exactly cancels the term $-2W_0(p)$.

The remaining integral is

$$-\oint_{p>\mathcal{C}^{(\xi)}>\mathcal{C}^{(q)}} \frac{d\xi}{2\pi i}\frac{dq}{2\pi i}\frac{y(\xi)}{p-\xi}\frac{\tilde{y}(p)}{\tilde{y}(\xi)}dE_{q,q_0}(\xi)\frac{f(q)}{2y(q)}.$$

Since $y(\xi)/\tilde{y}(\xi) = M(\xi)$ is regular at $\xi = \mu_\alpha$, pushing the integration contour $\mathcal{C}^{(\xi)}$ through $\mathcal{C}^{(q)}$ we obtain zero upon the integration over ξ, and the only nonzero contribution comes from the simple pole of $dE_{q,q_0}(\xi) \sim 1/(\xi - q)$ at $\xi = q$. Terms with $y(q)$ in the numerator and denominator then cancel each other and the remaining integral w.r.t. q can be taken pushing the contour again to infinity evaluating the residues at $q = p$ and $q = \overline{p}$, which gives $\frac{1}{2}(f(p) + f(\overline{p})) = f(p)$ for a function analytic in p outside the branching points.

This provides a basis for the diagrammatic representation for resolvents in 1MM [20]. Let us represent the form $dE_{q,q_0}(p)$ as the vector directed from p to q, the three-point vertex as the dot in which we assume the integration over q, $\bullet \equiv \oint \frac{dq}{2\pi i}\frac{1}{2y(q)}$, and the Bergmann 2-form $B(p,q)$ as a nonarrowed edge connecting points p and q. The graphic representation for a solution of (2.10) then looks as follows. Representing the multiresolvent $W_{g'}(p_1,\dots,p_s)$ as the block with s external legs and with the index g', we obtain [20]



(3.14)

which provides the diagrammatic representation for $W_k(p_1,\dots,p_s)$. The multiresolvent $W_k(p_1,\dots,p_s)$ can be presented as a finite sum of all possible connected graphs with k loops and s external legs and with only three-valent internal vertices (the total number of edges is then $2s+3k-3$, and we assume $s \geq 1$ for $k \geq 1$ and $s \geq 3$ for $k = 0$). We also set arrows on some (exactly $2k+s-2$) of edges for the arrowed edges to constitute the rooted tree subgraph with all arrows directed from the root. That means that we choose one of the external legs, say, p_1 (the choice is arbitrary due to the symmetry of $W_k(p_1,\dots,p_s)$), to be the root vertex the tree starts with; for each three-valent vertex there must exist exactly one incoming edge of the tree subgraph, all external edges (except the root edge) are nonarrowed, and all internal nonarrowed edges either start and terminate at the same vertex (we then associate with such an edge the Bergmann differential $B(p,\overline{p})$) or connect two different vertices such that there exists the directed path composed by arrowed edges connecting these two vertices. At each internal vertex we have the integration $\oint_{\mathcal{C}^{(q)}} \frac{dq}{2\pi i}\frac{1}{2y(q)}$, while the arrangement of the integration contours at different vertices is prescribed by the arrowed subtree: the closer is a vertex to the root, the more outer is the integration contour.

We now demonstrate the consistency of this diagram technique by calculating the action of loop insertion operator (2.7) on its elements.

We first calculate the action of $\partial/\partial V(r)$ on $B(P,Q)$. Using (3.3), we represent this action through the action of partial derivatives in μ_α subsequently calculating the latter from relation (3.11). Let

$$y(x)dx|_{x\to\mu_\alpha} = y([\mu_\alpha])\sqrt{x-\mu_\alpha}dx + O(\sqrt{x-\mu_\alpha})^3 dx.$$

Then, since

$$\left.\frac{\partial y(p)dp}{\partial V(r)}\right|_{p\to\mu_\alpha} \simeq -\frac{1}{2}y([\mu_\alpha])\frac{dp}{\sqrt{p-\mu_\alpha}}\frac{\partial\mu_\alpha}{\partial V(r)},$$

we have

$$\frac{\partial\mu_\alpha}{\partial V(r)} = \frac{2B([\mu_\alpha],q)}{y([\mu_\alpha]),} \tag{3.15}$$

and, therefore

$$\frac{\partial}{\partial V(r)}B(P,Q) = \sum_{\alpha=1}^{2n}\frac{B(P,[\mu_\alpha])B([\mu_\alpha],Q)B([\mu_\alpha],r)}{y([\mu_\alpha])}. \tag{3.16}$$

For expressing this in terms of the differentials, we integrate one of the Bergmann bidifferentials in order to obtain the 1-differential $dE_{Q,Q_0}(\xi)$.[6] Note that the local variable in the vicinity of μ_α is $\xi(x) = \sqrt{x-\mu_\alpha}$, this gives the additional factor 1/2, and we eventually obtain

$$\frac{\partial}{\partial V(r)}B(P,Q) = \sum_{\alpha=1}^{2n}\mathrm{res}_{\mu_\alpha}\frac{B(P,\xi(x))dE_{\xi(x),Q_0}(Q)B(\xi(x),r)}{2y(xi(x))d\xi(x)}. \tag{3.17}$$

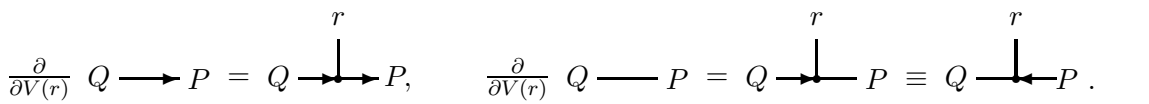
From this relation it obviously follows that

$$\frac{\partial}{\partial V(r)}dE_{P,Q_0}(Q) = \sum_{\alpha=1}^{2n}\mathrm{res}_{\mu_\alpha}\frac{dE_{P,Q_0}(\xi(x))dE_{\xi(x),Q_0}(Q)B(\xi(x),r)}{2y(xi(x))d\xi(x)}, \tag{3.18}$$

and the last quantity to evaluate is

$$\frac{\partial}{\partial V(r)}\frac{1}{2y(p)} = -\frac{B(p,r)}{2y^2(p)}. \tag{3.19}$$

Note that the point P in (3.18) is outside the integration contour. (This is irrelevant in formula (3.17) as the Bergmann bidifferential has zero residue at the coinciding arguments, in contrast to $dE_{P,Q_0}(Q)$. Multiplying by $1/(2y(P))$ both sides of (3.18), using (3.19), and pushing the integration contour through the point P, we observe that the contribution of the simple pole of $dE_{P,Q_0}(\xi(x))$ at $\xi(x) = P$ cancels exactly the variation of $1/(2y(P))$. We therefore attain the prescribed contour ordering and can graphically present the action of $\partial/\partial V(r)$ as



(3.20)

In the second case, it is our choice on which of edges to set the arrow. Recall, however, that the points P and Q were already ordered as prescribed by the diagram technique. That is, if "$P \to Q$", we must choose the first variant and if "$Q \to P$", we must choose the second variant of arrows arrangement.

[6] This differs from the commonly accepted viewpoint that one should instead take the primitive of $y(\xi)$. But the 1-differential ydx, being the Whitham differential dS in other terminology [7], [8], plays the fundamental role, not its primitive.

4. Inverting the loop insertion operator. Free energy

4.1 The H-operator

We now introduce the operator that is in a sense inverse to loop insertion operator (2.7). Let[7]

$$H\cdot = \frac{1}{2}\mathrm{res}_{\infty_+}V(x)\cdot\ -\frac{1}{2}\mathrm{res}_{\infty_-}V(x)\cdot\ -t_0\int_{\infty_-}^{\infty_+}\cdot\ -\sum_{i=1}^{n-1}S_i\oint_{B_i}\cdot. \tag{4.1}$$

The arrangement of the integration contours see in Fig. 1. We now calculate the action of H on the Bergmann bidifferential $B(x,q)$ using again the Riemann bilinear identities. We first note that $B(x,q) = \partial_x dE_{x,Q_0}(q)$ and we can evaluate residues at infinities by parts. Then, since $dE_{x,Q_0}(q)$ is regular at infinities, we substitute for $V'(x)$ $2y(x)+2t_0/x$ as $x\to\infty_+$ and $-2y(x)+2t_0/x$ as $x\to\infty_-$ thus obtaining

$$\begin{aligned}-\mathrm{res}_{\infty_+}\left(y(x)+\frac{t_0}{x}\right)dE_{x,Q_0}(q)dx+\mathrm{res}_{\infty_-}\left(-y(x)+\frac{t_0}{x}\right)dE_{x,Q_0}(q)dx\\ -t_0dE_{x,Q_0}(q)\Big|_{x=\infty_-}^{x=\infty_+}-\sum_{i=1}^{n-1}S_i\oint_{B_i}B(q,x),\end{aligned} \tag{4.2}$$

whence the cancelation of terms containing t_0 is obvious, and it remains to take the combination of residues at infinities involving $y(x)$. For this, we cut the surface along A- and B-cycles taking into account the residue at $x=q$. The boundary integrals on two sides of the cut at B_i then differ by $dE_{x,Q_0}(q)-dE_{x+\oint_{A_i},Q_0}(q)=0$, while the integrals on two sides of the cut at A_i differ by $dE_{x,Q_0}(q)-dE_{x+\oint_{B_i},Q_0}(q)=\oint_{B_i}B(q,x)$, and we obtain for the boundary term the expression

$$\sum_{i=1}^{n-1}\oint_{A_i}y(x)dx\oint_{B_i}B(q,\xi),$$

which exactly cancel the last term in (4.2). It remains only the contribution from the pole at $x=q$, which is just $-y(q)$. We have therefore proved that

$$H\cdot B(\cdot,q) = -y(q)dq. \tag{4.3}$$

Let us now consider the action of H on $W_k(\cdot)$ subsequently evaluating the action of loop insertion operator (2.7) on the result. Note first that the only result of action of $\partial/\partial V(p)$ on the operator H itself are derivatives $\partial V(x)/\partial V(p) = -1/(p-x)$ (and recall that by definition $|p|>|x|$, i.e., instead of evaluating residues at infinities one should take residues at $x=p$, and we obtain

$$\frac{\partial}{\partial V(p)}\left(H\cdot W_k(\cdot)\right) = W_k(p)+H\cdot W_k(\cdot,p). \tag{4.4}$$

[7]This definition works well when acting on 1-forms regular at infinities. Otherwise (say, in the case of $W_0(p)$), the integral in the third term must be regularized, e.g., by replacing it by the contour integral around the logarithmic cut stretched between two infinities.

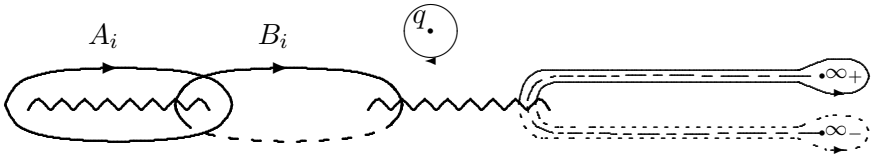
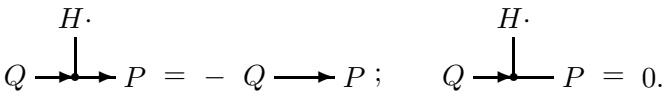


Figure 1: The arrangement of integration contours on the Riemann surface.

For the second term, due to the symmetry of $W_k(p,q)$, we may take the point p as the root of the tree subgraph. Then, the operator H always acts on $B(\cdot,\xi)$ where ξ are integration variables of internal vertices. However, if this vertex is an innermost (i.e., there is no arrowed edges coming out of it), then the 1-form $y(\xi)d\xi$ arising under the action of H (4.3) cancels the corresponding form in the integration expression, and the residue vanishes being nonsingular at the branching point. If there is an outgoing arrowed edge, say $dE_{\rho,Q_0}(\xi)$ (can be just one as one line is external), then, again, we can push the integration contour for ξ through the one for ρ; the only contribution comes again only from the pole at $\xi=\rho$. The value of the residue is doubled, and we come to the following graphical representation for the action of the operator H:



(4.5)

For $H_q \cdot W_k(q,p) = H_q \cdot \frac{\partial}{\partial V(q)} W_k(p)$, we obtain that for each arrowed edge on which the action of (2.7) produces the new vertex, the inverse action of $H_q\cdot$ just give the factor -1 and on each nonarrowed edge on which the action of (2.7) produces the new vertex, the inverse action of $H_q\cdot$ just gives zero. As the total number of arrowed edges is $2k-1$ for any graph contributing to the sum of diagrams, we obtain that

$$H_q \cdot W_k(q,p) = -(2k-1)W_k(p)$$

and, combining with (4.4), we just obtain

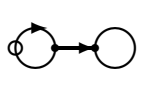
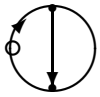
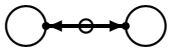
$$\frac{\partial}{\partial V(p)}\left(H_q \cdot W_k(q)\right) = -(2-2k)\frac{\partial}{\partial V(p)}F_k, \tag{4.6}$$

and, since all the dependence on filling fractions and t_0 is fixed by the claim that the answer depends only on μ_α and derivatives of $M(p)$ of finite orders at the branching points, we conclude that

$$F_k = \frac{1}{2k-2} H \cdot W_k. \tag{4.7}$$

This is our final answer for the free energy. It permits us to calculate all F_k except the contribution at $k=1$ (torus approximation). It was however calculated by a direct integration in [21]. All other orders can be consistently calculated. For this, we only introduce one new vertex $\circ$ in which we have $\oint_{C(\xi)} \frac{d\xi}{2\pi i} \frac{\int_{\mu_\alpha}^{\xi} y(s)ds}{y(\xi)}$. Although the integral term

$\int_{Q_0}^{\xi} y(s)ds$ is nonlocal, its constant part $\int_{Q_0}^{\mu_\alpha}$ drops out of the residue in the 1MM case, and we can integrate it in the neighborhoods of every branching point μ_α separately. That is, we need only the expansion of this integral as $\xi \to \mu_\alpha$. Then, say, the genus two contribution is provided by the sum of three diagrams

$2F_2 \;=\; 2$  $+2$  $+$  (4.8)

For completeness, we also present

$$F_0 = \frac{1}{2\pi i}\oint_{\mathcal{C}_\mathcal{D}} V(p)y(p)dp - \frac{1}{(2\pi i)^2}\oint_{\mathcal{C}_\mathcal{D}}\oint_{\mathcal{C}_\mathcal{D}} y(p)y(q)\log|p-q|dp\,dq$$

which can also be written like in (4.7)

$$F_0 = -\frac{1}{2}H_{\rm reg}\cdot W_0. \tag{4.9}$$

where $H_{\rm reg}$ is explained in many articles (see [18]), and

$$F_1 = -\frac{1}{24}\log\left(\prod_{\alpha=1}^{2n} M(\mu_\alpha)\Delta^4(\det\sigma)^{12}\right), \tag{4.10}$$

where Δ is the Vandermonde determinant for μ_α and σ is $(n-1)\times(n-1)$-matrix

$$\sigma_{ij} = \oint_{A_i}\frac{\xi^{j-1}d\xi}{\tilde{y}(\xi)};$$

note that $\det\sigma = \det{}^{-1}\widehat{H}_{ij}$, where $\widehat{H}_{ij}$ are coefficients of the polynomials expressing canonically normalized holomorphic 1-differentials:

$$dw_i = \frac{\widehat{H}_i(\xi)}{\tilde{y}(\xi)}d\xi,$$

and, for instance,

$$dE_{q,q_0}(\xi) - dE_{\bar{q},q_0}(\xi) = \frac{\tilde{y}(q)d\xi}{(\xi-q)\tilde{y}(\xi)} - \sum_{i=1}^{n-1}\frac{\widehat{H}_i(\xi)d\xi}{\tilde{y}(\xi)}\oint_{A_i}\frac{d\rho\tilde{y}(q)}{(\rho-q)\tilde{y}(\rho)}.$$

Meanwhile, it is easy to see that $H\cdot W_1$ is constant. Indeed,

$$H\cdot W_1 = \sum_{\alpha=1}^{2n}{\rm res}_{\mu_\alpha}\frac{\int_{\mu_\alpha}^{\xi} y(s)}{y(\xi)}B(\xi,\bar{\xi})d\xi,$$

the first term has simple zero with residue $1/2$ to be compensated by the double pole of $B(\xi,\bar{\xi}) \simeq \frac{1}{4(\xi-\mu_\alpha)^2}$, and the total answer is then just the constant $n/4$.

4.2 Scaling relation

We now a little demystify relation (4.6). Indeed, recall that for any functional F of a finite number of "local" variables, which are in our case the branching points μ_α and the moments

$$M_\alpha^{(k)} = \frac{1}{(k-1)!}\frac{d^{k-1}}{dp^{k-1}}M(p)\Big|_{p=\mu_\alpha},$$

we have the relations

$$\frac{\partial F}{\partial t_0} = \int_{\infty_-}^{\infty_+}\frac{\partial F}{\partial V(\xi)}d\xi \tag{4.11}$$

and

$$\frac{\partial F}{\partial S_i} = \oint_{B_i}\frac{\partial F}{\partial V(\xi)}d\xi. \tag{4.12}$$

Relation (4.6) is then equivalent to the relation

$$\left[\sum_{k=1}^{\infty} t_k\frac{\partial}{\partial t_k} + t_0\frac{\partial}{\partial t_0} + \sum_{i=1}^{n-1} S_i\frac{\partial}{\partial S_i} + \hbar\frac{\partial}{\partial\hbar}\right]F = 0 \tag{4.13}$$

satisfied by the total 1MM free energy. Indeed, coming back to basics of the matrix model, note that the operator in (4.13) just generates the scaling transformations: the part with derivatives w.r.t. t_k multiplies all vertices by the same scaling factor ρ simultaneously multiplying propagators by ρ^{-1}. The action of derivatives in t_0 and S_i results in multiplying all index loops (faces of the fat graph) by ρ. Therefore, for any graph, the total factor is

$$\rho^{\#\text{ vertices} - \#\text{ edges} + \#\text{ faces}} = \rho^{2-2g},$$

and it is exactly canceled by the scaling of the formal expansion parameter $\hbar \to \rho\hbar$.

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JHEP03(2006)014