

## Loop equations for the semiclassical 2-matrix model with hard edges

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# Loop equations for the semiclassical 2-matrix model with hard edges

**B Eynard**

Service de Physique Théorique de Saclay, F-91191 Gif-sur-Yvette Cedex, France  
E-mail: [eynard@cea.fr](mailto:eynard@cea.fr) and [eynard@spht.saclay.cea.fr](mailto:eynard@spht.saclay.cea.fr)

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**Abstract.** The 2-matrix model can be defined in a setting more general than polynomial potentials, namely, the semiclassical matrix model. In this case, the potentials are such that their derivatives are rational functions, and the integration paths for eigenvalues are arbitrary homology classes of paths for which the integral is convergent. This choice includes in particular the case where the integration path has fixed endpoints, called hard edges. The hard edges induce boundary contributions in the loop equations. The purpose of this paper is to give the loop equations in that semi-classical setting.

**Keywords:** matrix models, topology and combinatorics

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## 1. Introduction

Orthogonal polynomials and biorthogonal polynomials, in the context of random matrices, have been mostly studied for polynomial potentials, on the real axis, or sometimes on homology classes of contours going from  $\infty$  to  $\infty$  [19]. However, it is possible to define matrix models corresponding to a more general context, in particular, the ‘semi-classical’ (so called, because it contains all the classical polynomials). It is defined as follows (see Bertola [1]).

Consider two potentials (i.e. functions of a complex variable),  $V_1(x)$  and  $V_2(y)$ , whose derivatives  $V_1'(x)$  and  $V_2'(y)$  are rational functions (notice that  $\infty$  may be a pole of  $V_1'$  (resp.  $V_2'$ ); this is the case if  $V_1'$  (resp.  $V_2'$ ) is a polynomial).

Then consider a generalized integration path  $\Gamma = \sum_{i,j} \kappa_{i,j} \gamma_i \times \tilde{\gamma}_j$ , such that the following integral is absolutely convergent:

$$\int_{\Gamma} \exp\left(-\frac{N}{t}[V_1(x) + V_2(y) - xy]\right) dx dy \quad (1.1)$$

where  $t$  is a positive real number, and the  $\kappa_{i,j}$  are arbitrary complex numbers (not all vanishing).

The possible paths  $\gamma$  (resp.  $\tilde{\gamma}$ ) are described in [1]; they can be closed or open.

- If  $\gamma$  (resp.  $\tilde{\gamma}$ ) is a closed contour, the result is non-zero only if it encloses a singularity of  $\mathbf{e}^{-(N/t)V_1}$  (resp.  $\mathbf{e}^{-(N/t)V_2}$ ).
- If  $\gamma$  (resp.  $\tilde{\gamma}$ ) is an open contour, its extremities can be:
  - \* any point in the complex plane, except at the poles of  $V_1'$  (resp.  $V_2'$ ),
  - \* a simple pole of  $(N/t)V_1'$  (resp.  $(N/t)V_2'$ ), with residue  $\in \mathbb{R}_-$ ,
  - \* a degree  $\geq 2$  pole of  $V_1'$  (resp.  $V_2'$ ), if  $\gamma$  (resp.  $\tilde{\gamma}$ ) approaches the pole in a sector where  $\text{Re}(N/t)V_1 > 0$  (resp.  $\text{Re}(N/t)V_2 > 0$ ). This case includes poles at  $\infty$ .

Some of the extremities are such that  $\mathbf{e}^{-(N/t)[V_1(x)+V_2(y)-xy]}$  vanishes: they are the poles of  $V_1'$  (resp.  $V_2'$ ) of degree at least 2, as well as the simple poles of  $(N/t)V_1'$  (resp.  $(N/t)V_2'$ ), with residue  $\in \mathbb{R}_-$ ; we will call them ‘poles’. And some extremities are such that  $\mathbf{e}^{-(N/t)[V_1(x)+V_2(y)-xy]}$  does not vanish: they are arbitrary points in the complex plane; we call them ‘hard edges’, and write them

$$(X_1, X_2, \dots, X_{K_1}) \quad (\text{resp. } (Y_1, Y_2, \dots, Y_{K_2})). \quad (1.2)$$

Then, we define a generalized random matrix integral corresponding to a generalized path  $\Gamma$ , such that it coincides with the Hermitean 2-matrix model for  $\Gamma = \mathbb{R} \times \mathbb{R}$  (see [16] for instance).

**Definition 1.1** Let  $\Gamma = \sum_{i,j} \kappa_{i,j} \gamma_i \times \tilde{\gamma}_j$  be a linear combination of products of paths. We define

$$\begin{aligned} H_N^2(\Gamma) := \{ & M_1, M_2 \in GL_N(\mathbb{C})^2 \setminus \exists U_1, U_2 \in U(N)^2, \\ & \exists (x_1, \dots, x_N) \in \mathbb{C}^N, \exists (y_1, \dots, y_N) \in \mathbb{C}^N, M_1 = U_1 \text{diag}(x_1, \dots, x_N) U_1^\dagger, \\ & M_2 = U_2 \text{diag}(y_1, \dots, y_N) U_2^\dagger, \\ & \forall k = 1, \dots, N, \exists i_k x_k \in \gamma_{i_k}, \forall k = 1, \dots, N, \exists j_k y_k \in \tilde{\gamma}_{j_k} \} \end{aligned} \quad (1.3)$$

with a (not necessarily positive) measure  $d\mu(M_1, M_2)$  such that if  $(x_1, \dots, x_N)$  are the eigenvalues of  $M_1$  and  $(y_1, \dots, y_N)$  are the eigenvalues of  $M_2$ , and  $x_k \in \gamma_{i_k}$  and  $y_k \in \tilde{\gamma}_{j_k}$ :

$$d\mu(M_1 dM_2) := \det_{k,l} (\kappa_{i_k, j_l} \mathbf{e}^{-N[V_1(x_{i_k}) + V_2(y_{j_l}) - x_{i_k} y_{j_l}]} \Delta(x) \Delta(y) dU_1 dU_2 \prod_{i=1}^N dx_i \prod_{i=1}^N dy_i \quad (1.4)$$

where  $\Delta(x) = \prod_{i>j} (x_i - x_j)$  (resp.  $\Delta(y) = \prod_{i>j} (y_i - y_j)$ ) is the Vandermonde determinant,  $dU_1$  (resp.  $dU_2$ ) is the Haar measure on  $U(N)$ , and  $dx_i$  (resp.  $dy_i$ ) are the tangent vectors to the paths  $\gamma$  (resp.  $\tilde{\gamma}$ ). For given  $M_1, M_2 \in H_N^2(\Gamma)$ , the unitary matrix  $U_1$  (resp.  $U_2$ ) and the eigenvalues  $x_i$  (resp.  $y_i$ ) are not uniquely defined; they are defined up to a permutation,

and a conjugation by diagonal unitary matrix. However  $dM_1 dM_2$  is invariant under these operations, and thus is well defined.

We will note that

$$d\mu(M_1 dM_2) = e^{-N \text{Tr}[V_1(M_1) + V_2(M_2) - M_1 M_2]} dM_1 dM_2. \quad (1.5)$$

Our goal is to compute the Schwinger–Dyson equations, also called loop equations in the context of random matrices, of the following matrix integral:

$$Z := \int_{H_N^2(\Gamma)} d\mu(M_1, M_2) = \int_{H_N^2(\Gamma)} e^{-N \text{Tr}[V_1(M_1) + V_2(M_2) - M_1 M_2]} dM_1 dM_2. \quad (1.6)$$

We leave as an exercise for the reader to check that equation (1.6) depends only upon  $\Gamma$ , and not on the choice of basis  $\gamma_i, \tilde{\gamma}_j$ .

### 1.1. Notations

Define the following polynomials, which vanish at all the hard edges:

$$s(x) := \prod_{j=1}^{K_1} (x - X_j) = \sum_{r=0}^{K_1} s_r x^r \quad (1.7)$$

$$\tilde{s}(y) := \prod_{j=1}^{K_2} (y - Y_j) = \sum_{r=0}^{K_2} \tilde{s}_r y^r. \quad (1.8)$$

The resolvent (which is a formal series in its large  $x$  expansion) is defined by

$$W(x) := \frac{t}{N} \left\langle \text{Tr} \frac{1}{x - M_1} \right\rangle. \quad (1.9)$$

And, up to a shift by the potential, it is more convenient to use  $Y(x)$ :

$$Y(x) := V_1'(x) - W(x). \quad (1.10)$$

Then, for technical intermediate calculations, define the following expectation values, which are formal series in their large  $x$  expansion:

$$P(x, y) := \frac{t}{N} \left\langle \text{Tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \right\rangle \quad (1.11)$$

$$A(x) := \frac{t}{N} \left\langle \text{Tr} \frac{s(M_1)}{x - M_1} \tilde{s}(M_2) V_2'(M_2) \right\rangle \quad (1.12)$$

$$B(x) := \frac{t}{N} \left\langle \text{Tr} \frac{s(M_1)}{x - M_1} \tilde{s}(M_2) \right\rangle \quad (1.13)$$

$$B_k(x) := \frac{t}{N} \left\langle \text{Tr} \frac{s(M_1)}{x - M_1} M_2^k \right\rangle \quad (1.14)$$

$$D(x) := xB(x) + \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} \tilde{s}_s \frac{t}{N} \langle \text{Tr} M_2^{s-1-j} \rangle B_j(x). \quad (1.15)$$

Notice that  $P(x, y)$  is a rational function of both  $x$  and  $y$ , whose poles are the same as the poles of  $V_1'$  and  $V_2'$ .

**Definition 1.2** For any  $C_1$  and  $C_2$  functions of  $M_1$  and  $M_2$ , we define the connected part of an expectation value as

$$\langle \text{Tr } C_1(M, 1, M_2) \text{Tr } C_2(M, 1, M_2) \rangle_c := \langle \text{Tr } C_1(M, 1, M_2) \text{Tr } C_2(M, 1, M_2) \rangle - \langle \text{Tr } C_1(M, 1, M_2) \rangle \langle \text{Tr } C_2(M, 1, M_2) \rangle. \quad (1.16)$$

For applications of random matrices to combinatorics of discrete surfaces (i.e. also applications to conformal field theory or quantum gravity), the topological expansion implies the factorization theorem:

$$\frac{\langle \text{Tr } C_1 \text{Tr } C_2 \rangle_c}{\langle \text{Tr } C_1 \text{Tr } C_2 \rangle} \sim O(1/N^2). \quad (1.17)$$

This factorization theorem is true in a formal sense for combinatoric generating functions counting discrete surfaces, or under the 1-cut assumption for the convergent matrix model. However, the factorization theorem is not used in the derivation below.

## 2. The loop equations

Loop equations, i.e. Schwinger–Dyson equations, merely express the fact that an integral is invariant under a change of variable. Schwinger equations have been extensively studied in the context of field theory, and have been of great importance for the study of matrix models [20, 6]. For the polynomial 2-matrix model, loop equations were first announced to give algebraic equations by Staudacher [21], and computed precisely for general potentials, including  $1/N^2$  corrections by [13]–[15], [11]. They have been solved recently by [9] following the method of [8]. Here, we follow the method of [15].

### 2.1. Generalities

*2.1.1. Changes of variables.* Consider a change of variable in  $H_n^2(\Gamma)$ , of the form

$$M_1 \rightarrow M'_1 = M_1 + \epsilon f(M_1, M_2) + \bar{\epsilon} f^\dagger(M_1, M_2) + O(|\epsilon|^2). \quad (2.1)$$

**Remark 2.1** Under such a change of variable, the eigenvalues of  $M'_1$  are no longer on paths  $\gamma_{i_k}$ , but on some paths  $\gamma'_{i_k}$ , which, for  $\epsilon$  small enough, are small deformations of  $\gamma_{i_k}$ , and thus the integral is indeed unchanged. This is true only if we can deform the contours, i.e. not at the extremities. At poles, the integrand vanishes, so the integral is still invariant. At hard edges, the integral is invariant only if  $M'_1 = M_1$ , i.e.  $f(M_1, M_2)$  vanishes at hard edges. Thus, in the following, we will choose  $f$  proportional to  $s(M_1)$ .

To order 1 in  $\epsilon$ , one has

$$\begin{aligned} \exp \left[ -\frac{N}{t} \text{Tr}(V_1(M'_1) + V_2(M_2) - M'_1 M_2) \right] \\ = (1 - \epsilon \mathcal{S}(f) - \bar{\epsilon} \overline{\mathcal{S}}(f) + O(|\epsilon|^2, \epsilon^2, \bar{\epsilon}^2)) \\ \times \exp \left[ -\frac{N}{t} \text{Tr}(V_1(M_1) + V_2(M_2) - M_1 M_2) \right] \end{aligned} \quad (2.2)$$

with

$$\mathcal{S}(f) = \frac{N}{t} \text{Tr}(V_1'(M_1) - M_2) f(M_1, M_2). \quad (2.3)$$

The measure  $dM_1$  is multiplied by a Jacobian  $J$ , which we expand to order 1 in  $|\epsilon|$ :

$$dM_1' = J(f) dM_1 = (1 + \epsilon \mathcal{J}(f) + \bar{\epsilon} \bar{\mathcal{J}}(f) + O(|\epsilon|^2, \epsilon^2, \bar{\epsilon}^2)) dM_1. \quad (2.4)$$

Loop equations are obtained by writing that the integral is unchanged, i.e.,

$$\begin{aligned} Z &:= \int dM_1 dM_2 \exp \left[ -\frac{N}{t} \text{Tr}(V_1(M_1) + V_2(M_2) - M_1 M_2) \right] \\ &= \int dM_1' dM_2 \exp \left[ -\frac{N}{t} \text{Tr}(V_1(M_1') + V_2(M_2) - M_1' M_2) \right] \\ &= \int dM_1 dM_2 (1 + \epsilon (\mathcal{J}(f) - \mathcal{S}(f)) + \bar{\epsilon} (\bar{\mathcal{J}}(f) - \bar{\mathcal{S}}(f)) + O(|\epsilon|^2, \epsilon^2, \bar{\epsilon}^2)) \\ &\quad \times \exp \left[ -\frac{N}{t} \text{Tr}(V_1(M_1) + V_2(M_2) - M_1 M_2) \right]. \end{aligned} \quad (2.5)$$

Since this must be true for all  $\epsilon$  complex, i.e.  $\epsilon$  and  $\bar{\epsilon}$  independent, one must have

$$0 = \int dM_1 dM_2 (\mathcal{J}(f) - \mathcal{S}(f)) \exp \left[ -\frac{N}{t} \text{Tr}(V_1(M_1) + V_2(M_2) - M_1 M_2) \right] \quad (2.6)$$

i.e.,

$$\langle \mathcal{J}(f) \rangle = \langle \mathcal{S}(f) \rangle. \quad (2.7)$$

This equation is called a loop equation.

Let us emphasize again that equation (2.5), and thus (2.7), holds only if  $f$  vanishes at the hard edges.

**2.1.2. Split and merge rules.** The rules to compute  $\mathcal{J}(f)$  are called split and merge rules and can be found in the literature, for instance in [15]. Notice that  $\mathcal{J}$  is linear, and if  $f$  is a product,  $\mathcal{J}(f)$  can be computed by the chain rule. Thus, it is useful to determine  $\mathcal{J}(f)$  for some particular  $f$  s. For any two matrices  $A$  and  $B$ , one has the following.

- *Split rule:*

$$\mathcal{J} \left( A \frac{1}{x - M_1} B \right) = \text{Tr} A \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} B \quad (2.8)$$

or equivalently

$$\mathcal{J} (A M_1^k B) = \sum_{j=0}^{k-1} \text{Tr} A M_1^j \text{Tr} M_1^{k-1-j} B. \quad (2.9)$$

- *Merge rule:*

$$\mathcal{J} \left( A \text{Tr} \left( \frac{1}{x - M_1} B \right) \right) = \text{Tr} \frac{1}{x - M_1} A \frac{1}{x - M_1} B \quad (2.10)$$

or equivalently

$$\mathcal{J} (A \text{Tr} (M_1^k B)) = \sum_{j=0}^{k-1} \text{Tr} M_1^j A M_1^{k-j-1} B. \quad (2.11)$$

## 2.2. Equations

The purpose of this section is to derive an equation for  $Y(x)$  as a function of  $x$ , in a form where the LHS is a rational function of  $x$  and  $Y(x)$ , and the RHS is proportional to  $1/N^2$ :

$$s(x)\tilde{s}(Y(x)) E(x, Y(x)) = \frac{t^2}{N^2} L(x). \quad (2.12)$$

To this purpose, we first compute the auxiliary functions  $B(x)$ ,  $B_k(x)$ ,  $D(x)$  and  $A(x)$ , in terms of rational functions of  $x$  and  $Y(x)$  and  $1/N^2$  terms. Then, a last equation is used to close the system.

Throughout all this section, we write the change of variable, i.e. function  $f$ , and the corresponding loop equation of type equation (2.7). We always write the contribution of the Jacobian  $\mathcal{J}$  in the LHS and the contribution of the action  $\mathcal{S}$  in the RHS.

- Computation of  $B(x)$ : from

$$\begin{aligned} f(M_1, M_2) &= s(M_1) \frac{1}{x - M_1} \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \\ &= s(x) \frac{1}{x - M_1} \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} - \frac{s(x) - s(M_1)}{x - M_1} \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \end{aligned} \quad (2.13)$$

(which indeed vanishes at all hard edges) we get

$$\begin{aligned} s(x)W(x) \frac{t}{N} \left\langle \text{Tr} \frac{1}{x - M_1} \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle \\ + s(x) \frac{t^2}{N^2} \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle_c \\ - \frac{t^2}{N^2} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} s_r \left\langle \text{Tr} M_1^{r-1-i} \text{Tr} \frac{x^i - M_1^i}{x - M_1} \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle \\ = \frac{t}{N} \left\langle \text{Tr} \frac{s(M_1)}{x - M_1} (V_1'(M_1) - M_2) \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle \end{aligned} \quad (2.14)$$

i.e.,

$$\begin{aligned} B(x) &= s(x)\tilde{s}(Y(x))W(x) \\ &- s(x) \frac{t}{N} \left\langle \text{Tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle \\ &- \frac{t}{N} \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} \tilde{s}(M_2) \right\rangle \\ &- \frac{t}{N} \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} (V_1'(M_1) - M_2) \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle \\ &+ \frac{t^2}{N^2} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} s_r \left\langle \text{Tr} M_1^{r-1-i} \text{Tr} \frac{x^i - M_1^i}{x - M_1} \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle \\ &- s(x) \frac{t^2}{N^2} \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle_c. \end{aligned} \quad (2.15)$$



Since  $W(x) = V_1'(x) - Y(x)$ , the RHS is of the desired form, i.e. involves only rational functions of  $x$  and  $Y(x)$  or  $1/N^2$  terms.

- Computation of  $B_k(x)$ : similarly, from

$$\begin{aligned} f(M_1, M_2) &= s(M_1) \frac{1}{x - M_1} \frac{Y(x)^k - M_2^k}{Y(x) - M_2} \\ &= s(x) \frac{1}{x - M_1} \frac{Y(x)^k - M_2^k}{Y(x) - M_2} - \frac{s(x) - s(M_1)}{x - M_1} \frac{Y(x)^k - M_2^k}{Y(x) - M_2} \end{aligned} \quad (2.16)$$

we get

$$\begin{aligned} s(x)W(x) &\frac{t}{N} \left\langle \text{Tr} \frac{1}{x - M_1} \frac{Y(x)^k - M_2^k}{Y(x) - M_2} \right\rangle \\ &+ s(x) \frac{t^2}{N^2} \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} \frac{Y(x)^k - M_2^k}{Y(x) - M_2} \right\rangle_c \\ &- \frac{t^2}{N^2} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} s_r \left\langle \text{Tr} M_1^{r-1-i} \text{Tr} \frac{x^i - M_1^i}{x - M_1} \frac{Y(x)^k - M_2^k}{Y(x) - M_2} \right\rangle \\ &= \frac{t}{N} \left\langle \text{Tr} \frac{s(M_1)}{x - M_1} (V_1'(M_1) - M_2) \frac{Y(x)^k - M_2^k}{Y(x) - M_2} \right\rangle \end{aligned} \quad (2.17)$$

i.e.,

$$\begin{aligned} B_k(x) &= Y(x)^k s(x)W(x) \\ &- s(x) \frac{t}{N} \left\langle \text{Tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{Y(x)^k - M_2^k}{Y(x) - M_2} \right\rangle \\ &- \frac{t}{N} \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} (V_1'(M_1) - M_2) \frac{Y(x)^k - M_2^k}{Y(x) - M_2} \right\rangle \\ &- \frac{t}{N} \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} M_2^k \right\rangle \\ &+ \frac{t^2}{N^2} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} s_r \left\langle \text{Tr} M_1^{r-1-i} \text{Tr} \frac{x^i - M_1^i}{x - M_1} \frac{Y(x)^k - M_2^k}{Y(x) - M_2} \right\rangle \\ &- s(x) \frac{t^2}{N^2} \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} \frac{Y(x)^k - M_2^k}{Y(x) - M_2} \right\rangle_c. \end{aligned} \quad (2.18)$$

- Computation of  $D(x)$ : from the definition (equation (1.15)) of  $D(x)$ , we find

$$\begin{aligned} D(x) &= xB(x) + \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} \tilde{s}_s \frac{t}{N} \langle \text{Tr} M_2^{s-1-j} \rangle B_j(x) \\ &= xs(x)\tilde{s}(Y(x))W(x) \\ &+ s(x)W(x) \frac{t}{N} \left\langle \text{Tr} \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle \\ &- xs(x) \frac{t}{N} \left\langle \text{Tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle \end{aligned}$$

$$\begin{aligned}
 & -x \frac{t}{N} \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} \tilde{s}(M_2) \right\rangle \\
 & -x \frac{t}{N} \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} (V_1'(M_1) - M_2) \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle \\
 & + x \frac{t^2}{N^2} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} s_r \left\langle \text{Tr} M_1^{r-1-i} \text{Tr} \frac{x^i - M_1^i}{x - M_1} \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle \\
 & - s(x) \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} \tilde{s}_s \frac{t}{N} \left\langle \text{Tr} M_2^{s-1-j} \right\rangle \frac{t}{N} \left\langle \text{Tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{Y(x)^j - M_2^j}{Y(x) - M_2} \right\rangle \\
 & - \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} \tilde{s}_s \frac{t}{N} \left\langle \text{Tr} M_2^{s-1-j} \right\rangle \\
 & \times \frac{t}{N} \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} (V_1'(M_1) - M_2) \frac{Y(x)^j - M_2^j}{Y(x) - M_2} \right\rangle \\
 & - \frac{t}{N} \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} (V_2'(M_2) - M_1) \tilde{s}(M_2) \right\rangle \\
 & + \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} \tilde{s}_s \frac{t}{N} \left\langle \text{Tr} M_2^{s-1-j} \right\rangle \\
 & \times \frac{t^2}{N^2} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} s_r \left\langle \text{Tr} M_1^{r-1-i} \text{Tr} \frac{x^i - M_1^i}{x - M_1} \frac{Y(x)^j - M_2^j}{Y(x) - M_2} \right\rangle \\
 & - xs(x) \frac{t^2}{N^2} \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle_c \\
 & - s(x) \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} \tilde{s}_s \frac{t}{N} \left\langle \text{Tr} M_2^{s-1-j} \right\rangle \frac{t^2}{N^2} \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} \frac{Y(x)^j - M_2^j}{Y(x) - M_2} \right\rangle_c \\
 & + \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} \tilde{s}_s \frac{t^2}{N^2} \left\langle \text{Tr} M_2^{s-1-j} \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} M_2^j \right\rangle_c. \tag{2.19}
 \end{aligned}$$

- Computation of  $A(x)$ : by doing a change of variable on  $M_2$  of the form

$$\tilde{f}(M_1, M_2) = s(M_1) \frac{1}{x - M_1} \tilde{s}(M_2) \tag{2.20}$$

we get the loop equation

$$\begin{aligned}
 & \frac{t^2}{N^2} \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} \tilde{s}_s \left\langle \text{Tr} \frac{s(M_1)}{x - M_1} M_2^j \text{Tr} M_2^{s-1-j} \right\rangle = A(x) - \frac{t}{N} \left\langle \text{Tr} s(M_1) \frac{M_1}{x - M_1} \tilde{s}(M_2) \right\rangle \\
 & = A(x) - xB(x) + \frac{t}{N} \langle \text{Tr} s(M_1) \tilde{s}(M_2) \rangle \tag{2.21}
 \end{aligned}$$

i.e.,

$$A(x) = D(x) - \frac{t}{N} \langle \text{Tr } s(M_1) \tilde{s}(M_2) \rangle + \frac{t^2}{N^2} \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} \tilde{s}_s \left\langle \text{Tr} \frac{s(M_1)}{x - M_1} M_2^j \text{Tr } M_2^{s-1-j} \right\rangle_c. \quad (2.22)$$

• Main computation: from

$$\begin{aligned} f(M_1, M_2) &= s(M_1) \frac{1}{x - M_1} \frac{\tilde{s}(Y(x))V_2'(Y(x)) - \tilde{s}(M_2)V_2'(M_2)}{Y(x) - M_2} \\ &= s(x) \frac{1}{x - M_1} \frac{\tilde{s}(Y(x))V_2'(Y(x)) - \tilde{s}(M_2)V_2'(M_2)}{Y(x) - M_2} \\ &\quad - \frac{s(x) - s(M_1)}{x - M_1} \frac{\tilde{s}(Y(x))V_2'(Y(x)) - \tilde{s}(M_2)V_2'(M_2)}{Y(x) - M_2} \end{aligned} \quad (2.23)$$

we get

$$\begin{aligned} s(x) \frac{t}{N} W(x) &\left\langle \text{Tr} \frac{1}{x - M_1} \frac{\tilde{s}(Y(x))V_2'(Y(x)) - \tilde{s}(M_2)V_2'(M_2)}{Y(x) - M_2} \right\rangle \\ &+ s(x) \frac{t^2}{N^2} \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} \frac{\tilde{s}(Y(x))V_2'(Y(x)) - \tilde{s}(M_2)V_2'(M_2)}{Y(x) - M_2} \right\rangle_c \\ &- \frac{t^2}{N^2} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} s_r \left\langle \text{Tr } M_1^{r-1-i} \text{Tr} \frac{x^i - M_1^i}{x - M_1} \frac{\tilde{s}(Y(x))V_2'(Y(x)) - \tilde{s}(M_2)V_2'(M_2)}{Y(x) - M_2} \right\rangle \\ &= \frac{t}{N} \left\langle \text{Tr } s(M_1) \frac{1}{x - M_1} (V_1'(M_1) - M_2) \frac{\tilde{s}(Y(x))V_2'(Y(x)) - \tilde{s}(M_2)V_2'(M_2)}{Y(x) - M_2} \right\rangle \end{aligned} \quad (2.24)$$

i.e.,

$$\begin{aligned} s(x) \frac{t^2}{N^2} &\left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} \frac{\tilde{s}(Y(x))V_2'(Y(x)) - \tilde{s}(M_2)V_2'(M_2)}{Y(x) - M_2} \right\rangle_c \\ &= s(x) \tilde{s}(Y(x))V_2'(Y(x))W(x) \\ &\quad - s(x) \frac{t}{N} \left\langle \text{Tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{\tilde{s}(Y(x))V_2'(Y(x)) - \tilde{s}(M_2)V_2'(M_2)}{Y(x) - M_2} \right\rangle \\ &\quad - \frac{t}{N} \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} (V_1'(M_1) - M_2) \frac{\tilde{s}(Y(x))V_2'(Y(x)) - \tilde{s}(M_2)V_2'(M_2)}{Y(x) - M_2} \right\rangle \\ &\quad + \frac{t^2}{N^2} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} s_r \left\langle \text{Tr } M_1^{r-1-i} \text{Tr} \frac{x^i - M_1^i}{x - M_1} \frac{\tilde{s}(Y(x))V_2'(Y(x)) - \tilde{s}(M_2)V_2'(M_2)}{Y(x) - M_2} \right\rangle \\ &\quad - \frac{t}{N} \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} \tilde{s}(M_2)V_2'(M_2) \right\rangle - A(x) \end{aligned} \quad (2.25)$$

i.e.,

$$\begin{aligned}
 & s(x) \frac{t^2}{N^2} \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} \frac{\tilde{s}(Y(x))V_2'(Y(x)) - \tilde{s}(M_2)V_2'(M_2)}{Y(x) - M_2} \right\rangle_c \\
 & + \frac{t^2}{N^2} \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} \tilde{s}_s \left\langle \text{Tr} \frac{s(M_1)}{x - M_1} M_2^j \text{Tr} M_2^{s-1-j} \right\rangle_c \\
 & = s(x) \tilde{s}(Y(x)) V_2'(Y(x)) W(x) \\
 & - s(x) \frac{t}{N} \left\langle \text{Tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{\tilde{s}(Y(x))V_2'(Y(x)) - \tilde{s}(M_2)V_2'(M_2)}{Y(x) - M_2} \right\rangle_c \\
 & - \frac{t}{N} \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} (V_1'(M_1) - M_2) \frac{\tilde{s}(Y(x))V_2'(Y(x)) - \tilde{s}(M_2)V_2'(M_2)}{Y(x) - M_2} \right\rangle_c \\
 & + \frac{t^2}{N^2} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} s_r \left\langle \text{Tr} M_1^{r-1-i} \text{Tr} \frac{x^i - M_1^i}{x - M_1} \frac{\tilde{s}(Y(x))V_2'(Y(x)) - \tilde{s}(M_2)V_2'(M_2)}{Y(x) - M_2} \right\rangle_c \\
 & - \frac{t}{N} \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} \tilde{s}(M_2) V_2'(M_2) \right\rangle_c \\
 & + \frac{t}{N} \langle \text{Tr} s(M_1) \tilde{s}(M_2) \rangle - D(x). \tag{2.26}
 \end{aligned}$$

Now, on inserting the value of  $D(x)$ , and after tedious but straightforward computations, we get

$$\begin{aligned}
 & s(x) \tilde{s}(Y(x)) \frac{t^2}{N^2} \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} \frac{V_2'(Y(x)) - V_2'(M_2)}{Y(x) - M_2} \right\rangle_c \\
 & + s(x) \frac{t^2}{N^2} \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} (V_2'(M_2) - M_1) \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle_c \\
 & - s(x) \frac{t^2}{N^2} \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle_c \\
 & + s(x) \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} \tilde{s}_s \frac{t^2}{N^2} \left\langle \text{Tr} M_2^{s-1-j} \text{Tr} \frac{1}{x - M_1} M_2^j \right\rangle_c \\
 & - s(x) \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} \tilde{s}_s \frac{t}{N} \langle \text{Tr} M_2^{s-1-j} \rangle \frac{t^2}{N^2} \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} \frac{Y(x)^j - M_2^j}{Y(x) - M_2} \right\rangle_c \\
 & + \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} \tilde{s}_s \frac{t}{N} \langle \text{Tr} M_2^{s-1-j} \rangle \\
 & \times \frac{t^2}{N^2} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} s_r \left\langle \text{Tr} M_1^{r-1-i} \text{Tr} \frac{x^i - M_1^i}{x - M_1} \frac{Y(x)^j - M_2^j}{Y(x) - M_2} \right\rangle_c \\
 & - \frac{t^2}{N^2} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} s_r \left\langle \text{Tr} M_1^{r-1-i} \text{Tr} \frac{x^i - M_1^i}{x - M_1} (V_2'(M_2) - M_1) \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle_c \\
 & = s(x) \tilde{s}(Y(x)) ((V_2'(Y(x)) - x)(V_1'(x) - Y(x)) - P(x, y) + t)
 \end{aligned}$$

$$\begin{aligned}
 & -s(x) \frac{t}{N} \left\langle \text{Tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} (V_2'(M_2) - M_1) \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle \\
 & + s(x) \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} \tilde{s}_s \frac{t}{N} \left\langle \text{Tr} M_2^{s-1-j} \right\rangle \frac{t}{N} \left\langle \text{Tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{Y(x)^j - M_2^j}{Y(x) - M_2} \right\rangle \\
 & - \tilde{s}(Y(x)) \frac{t}{N} \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} (V_1'(M_1) - M_2) \frac{V_2'(Y(x)) - V_2'(M_2)}{Y(x) - M_2} \right\rangle \\
 & + \tilde{s}(Y(x)) \frac{t^2}{N^2} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} s_r \left\langle \text{Tr} M_1^{r-1-i} \text{Tr} \frac{x^i - M_1^i}{x - M_1} \frac{V_2'(Y(x)) - V_2'(M_2)}{Y(x) - M_2} \right\rangle \\
 & - \frac{t}{N} \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} \right. \\
 & \times (V_1'(M_1)V_2'(M_2) - M_2V_2'(M_2) - M_1V_1'(M_1) + M_1M_2) \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \left. \right\rangle \\
 & - \frac{t^2}{N^2} \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} \text{Tr} \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle \\
 & + \frac{t^2}{N^2} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} s_r \left\langle \text{Tr} M_1^{r-1-i} \right\rangle \left\langle \text{Tr} \frac{x^i - M_1^i}{x - M_1} (V_2'(M_2) - M_1) \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle \\
 & + \frac{t^2}{N^2} \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} \tilde{s}_s \left\langle \text{Tr} M_2^{s-1-j} \right\rangle \\
 & \times \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} (V_1'(M_1) - M_2) \frac{Y(x)^j - M_2^j}{Y(x) - M_2} \right\rangle \\
 & - \frac{t^3}{N^3} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} s_r \tilde{s}_s \left\langle \text{Tr} M_1^{r-1-i} \right\rangle \\
 & \times \left\langle \text{Tr} M_2^{s-1-j} \right\rangle \left\langle \text{Tr} \frac{x^i - M_1^i}{x - M_1} \frac{Y(x)^j - M_2^j}{Y(x) - M_2} \right\rangle
 \end{aligned} \tag{2.27}$$

which we write as follows:

$$\boxed{s(x) \tilde{s}(Y(x)) E(x, Y(x)) = \frac{t^2}{N^2} L(x)} \tag{2.28}$$

where  $E(x, y)$  is a rational fraction of both  $x$  and  $y$  with poles at the poles of  $V_1'$  and  $V_2'$ , and at the hard edges,

$$\begin{aligned}
 E(x, y) &= (V_2'(y) - x)(V_1'(x) - y) - P(x, y) + t \\
 & - \frac{1}{\tilde{s}(y)} \frac{t}{N} \left\langle \text{Tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} (V_2'(M_2) - M_1) \frac{\tilde{s}(y) - \tilde{s}(M_2)}{y - M_2} \right\rangle \\
 & + \frac{1}{\tilde{s}(y)} \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} \tilde{s}_s \frac{t}{N} \left\langle \text{Tr} M_2^{s-1-j} \right\rangle \frac{t}{N} \left\langle \text{Tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{y^j - M_2^j}{y - M_2} \right\rangle
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{1}{s(x)} \frac{t}{N} \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} (V_1'(M_1) - M_2) \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \right\rangle \\
 & + \frac{1}{s(x)} \frac{t^2}{N^2} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} s_r \left\langle \text{Tr} M_1^{r-1-i} \text{Tr} \frac{x^i - M_1^i}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \right\rangle \\
 & - \frac{1}{s(x)\tilde{s}(y)} \frac{t}{N} \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} \right. \\
 & \times (V_1'(M_1)V_2'(M_2) - M_2V_2'(M_2) - M_1V_1'(M_1) + M_1M_2) \frac{\tilde{s}(y) - \tilde{s}(M_2)}{y - M_2} \left. \right\rangle \\
 & - \frac{1}{s(x)\tilde{s}(y)} \frac{t^2}{N^2} \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} \text{Tr} \frac{\tilde{s}(y) - \tilde{s}(M_2)}{y - M_2} \right\rangle \\
 & + \frac{1}{s(x)\tilde{s}(y)} \frac{t^2}{N^2} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} s_r \langle \text{Tr} M_1^{r-1-i} \rangle \\
 & \times \left\langle \text{Tr} \frac{x^i - M_1^i}{x - M_1} (V_2'(M_2) - M_1) \frac{\tilde{s}(y) - \tilde{s}(M_2)}{y - M_2} \right\rangle \\
 & + \frac{1}{s(x)\tilde{s}(y)} \frac{t^2}{N^2} \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} \tilde{s}_s \langle \text{Tr} M_2^{s-1-j} \rangle \\
 & \times \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} (V_1'(M_1) - M_2) \frac{y^j - M_2^j}{y - M_2} \right\rangle \\
 & - \frac{1}{s(x)\tilde{s}(y)} \frac{t^3}{N^3} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} s_r \tilde{s}_s \langle \text{Tr} M_1^{r-1-i} \rangle \\
 & \times \langle \text{Tr} M_2^{s-1-j} \rangle \left\langle \text{Tr} \frac{x^i - M_1^i}{x - M_1} \frac{y^j - M_2^j}{y - M_2} \right\rangle \tag{2.29}
 \end{aligned}$$

and in the RHS of equation (2.28), we have

$$\begin{aligned}
 L(x) = & \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} \frac{V_2'(Y(x)) - V_2'(M_2)}{Y(x) - M_2} \right\rangle_c \\
 & + \frac{1}{\tilde{s}(Y(x))} \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} (V_2'(M_2) - M_1) \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle_c \\
 & - \frac{1}{\tilde{s}(Y(x))} \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle_c \\
 & + \frac{1}{\tilde{s}(Y(x))} \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} \tilde{s}_s \left\langle \text{Tr} M_2^{s-1-j} \text{Tr} \frac{1}{x - M_1} M_2^j \right\rangle_c \\
 & - \frac{1}{\tilde{s}(Y(x))} \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} \tilde{s}_s \frac{t}{N} \langle \text{Tr} M_2^{s-1-j} \rangle \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} \frac{Y(x)^j - M_2^j}{Y(x) - M_2} \right\rangle_c \\
 & + \frac{1}{s(x)\tilde{s}(Y(x))} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} \sum_{s=0}^{K_2} \sum_{j=0}^{s-1} s_r \tilde{s}_s
 \end{aligned}$$

$$\begin{aligned}
 & \times \frac{t}{N} \left\langle \text{Tr } M_2^{s-1-j} \right\rangle \left\langle \text{Tr } M_1^{r-1-i} \text{Tr } \frac{x^i - M_1^i}{x - M_1} \frac{Y(x)^j - M_2^j}{Y(x) - M_2} \right\rangle_c \\
 & - \frac{1}{s(x)\tilde{s}(Y(x))} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} s_r \left\langle \text{Tr } M_1^{r-1-i} \text{Tr } \frac{x^i - M_1^i}{x - M_1} (V_2'(M_2) - M_1) \right. \\
 & \times \left. \frac{\tilde{s}(Y(x)) - \tilde{s}(M_2)}{Y(x) - M_2} \right\rangle_c. \tag{2.30}
 \end{aligned}$$

### 2.3. Examples

Equation (2.28) looks rather terrible, but it is actually very simple to use. Let us illustrate it on simple examples.

*2.3.1. No hard edges.* If there are no hard edges, we have  $s(x) = 1$  and  $\tilde{s}(y) = 1$ , and the loop equation becomes

$$\begin{aligned}
 & (V_2'(Y(x)) - x)(V_1'(x) - Y(x)) - P(x, Y(x)) + t \\
 & = \frac{t^2}{N^2} \left\langle \text{Tr } \frac{1}{x - M_1} \text{Tr } \frac{1}{x - M_1} \frac{V_2'(Y(x)) - V_2'(M_2)}{Y(x) - M_2} \right\rangle_c \tag{2.31}
 \end{aligned}$$

which is the well-known loop equation of the 2-matrix model with polynomial potentials [15].

*2.3.2. 1-matrix model.* Consider the 1-matrix model with weight  $\mathbf{e}^{-(N/t) \text{Tr } V(M)}$ . It is equivalent to a 2-matrix model where  $M_2$  is Gaussian, i.e.  $V_2'(y) = y$ ,  $\tilde{s}(y) = 1$ ,  $V'(x) = V_1'(x) - x$ . Equation (2.28) becomes

$$\begin{aligned}
 E(x, y) &= (y - x)(V'(x) + x - y) - \frac{t}{N} \left\langle \text{Tr } \frac{V'(x) - V'(M)}{x - M} \right\rangle \\
 & - \frac{1}{s(x)} \frac{t}{N} \left\langle \text{Tr } \frac{s(x) - s(M)}{x - M} V'(M) \right\rangle \\
 & + \frac{1}{s(x)} \frac{t^2}{N^2} \sum_{r=0}^{K_1} \sum_{i=0}^{r-1} s_r \left\langle \text{Tr } M^{r-1-i} \text{Tr } \frac{x^i - M^i}{x - M} \right\rangle \tag{2.32}
 \end{aligned}$$

and in the RHS of equation (2.28), we have

$$L(x) = \left\langle \text{Tr } \frac{1}{x - M} \text{Tr } \frac{1}{x - M} \right\rangle_c. \tag{2.33}$$

Since  $E(x, y)$  is quadratic in  $y$ , this equation defines a hyperelliptical curve.

*2.3.3. 1-matrix model with no hard edges.* If we choose  $s(x) = 1$ , we recover the well-known loop equation for the 1-matrix model in polynomial potential [20, 6]:

$$(V'(x) - W(x))W(x) - \frac{t}{N} \left\langle \text{Tr } \frac{V'(x) - V'(M)}{x - M} \right\rangle = \frac{t^2}{N^2} \left\langle \text{Tr } \frac{1}{x - M} \text{Tr } \frac{1}{x - M} \right\rangle_c \tag{2.34}$$

where we have used that  $Y(x) = V_1'(x) - W(x)$ .

*2.3.4. 1-matrix model with only one hard edge.* In particular, consider  $s(x) = (x - a)$ ; we have

$$E(x, y) = (y - x)(V'(x) + x - y) - \frac{t}{N} \left\langle \text{Tr} \frac{V'(x) - V'(M)}{x - M} \right\rangle - \frac{1}{x - a} \frac{t}{N} \langle \text{Tr} V'(M) \rangle. \quad (2.35)$$

*2.3.5. 1-matrix model with only two hard edges.* In particular, consider  $s(x) = (x - a)(x - b)$ ; we have

$$E(x, y) = (y - x)(V'(x) + x - y) - \frac{t}{N} \left\langle \text{Tr} \frac{V'(x) - V'(M)}{x - M} \right\rangle - \frac{1}{(x - a)(x - b)} \frac{t}{N} \langle \text{Tr}(x + M - a - b) V'(M) \rangle + \frac{t^2}{(x - a)(x - b)}. \quad (2.36)$$

*2.3.6. 2-matrix model with only one hard edge.* Consider  $s(x) = (x - a)$  and  $\tilde{s}(y) = 1$ ; we have

$$E(x, y) = (V'_2(y) - x)(V'_1(x) - y) - P(x, y) + t - \frac{1}{x - a} \frac{t}{N} \left\langle \text{Tr}(V'_1(M_1) - M_2) \frac{V'_2(y) - V'_2(M_2)}{y - M_2} \right\rangle \quad (2.37)$$

and in the RHS of equation (2.28), we have

$$L(x) = \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} \frac{V'_2(Y(x)) - V'_2(M_2)}{Y(x) - M_2} \right\rangle_c. \quad (2.38)$$

*2.3.7. 2-matrix model with only two hard edges.* Consider  $s(x) = (x - a)(x - b)$  and  $\tilde{s}(y) = 1$ ; we have

$$E(x, y) = (V'_2(y) - x)(V'_1(x) - y) - P(x, y) + t - \frac{1}{s(x)} \frac{t}{N} \left\langle \text{Tr}(x + M_1 - a - b) (V'_1(M_1) - M_2) \frac{V'_2(y) - V'_2(M_2)}{y - M_2} \right\rangle + \frac{1}{s(x)} \frac{t^2}{N} \left\langle \text{Tr} \frac{V'_2(y) - V'_2(M_2)}{y - M_2} \right\rangle \quad (2.39)$$

and in the RHS of equation (2.28), we have

$$L(x) = \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} \frac{V'_2(Y(x)) - V'_2(M_2)}{Y(x) - M_2} \right\rangle_c. \quad (2.40)$$



2.3.8. *2-matrix model with only one hard edge in  $x$  and one hard edge in  $y$ .* Consider  $s(x) = (x - a)$  and  $\tilde{s}(y) = (y - b)$ ; we have

$$\begin{aligned} E(x, y) = & (V_2'(y) - x)(V_1'(x) - y) - P(x, y) + t \\ & - \frac{1}{\tilde{s}(y)} \frac{t}{N} \left\langle \text{Tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} (V_2'(M_2) - M_1) \right\rangle \\ & - \frac{1}{s(x)} \frac{t}{N} \left\langle \text{Tr} (V_1'(M_1) - M_2) \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \right\rangle \\ & - \frac{1}{s(x)\tilde{s}(y)} \frac{t}{N} \langle \text{Tr} (V_1'(M_1)V_2'(M_2) - M_2V_2'(M_2) - M_1V_1'(M_1) + M_1M_2) \rangle \\ & - \frac{t^2}{s(x)\tilde{s}(y)} \end{aligned} \quad (2.41)$$

and in the RGS of equation (2.28), we have

$$\begin{aligned} L(x) = & \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} \frac{V_2'(Y(x)) - V_2'(M_2)}{Y(x) - M_2} \right\rangle_c \\ & + \frac{1}{\tilde{s}(Y(x))} \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x - M_1} (V_2'(M_2) - M_1) \right\rangle_c. \end{aligned} \quad (2.42)$$

### 3. Large $N$ limit, algebraic curve

In the large  $N$  limit, equation (2.28) formally reduces to an algebraic equation:

$$E(x, Y(x)) = 0. \quad (3.1)$$

This assumes that we can use the factorization theorem equation (1.17)  $\langle \text{Tr} \text{Tr} \rangle = \langle \text{Tr} \rangle \langle \text{Tr} \rangle$  (see e.g. [6]), which is valid only in the so-called 1-cut case for convergent integrals (i.e. genus zero algebraic curve; see for instance [11] section 3.3), or is valid if we consider formal matrix integrals as combinatoric generating functions for enumerating discrete surfaces. Notice that in equation (3.1),  $x$  and  $y$  play symmetric roles, which is the expression of the duality discovered by Matytsin [18].

We see that  $E(x, y)$  has poles only at the poles of  $V_1'$  and zeros of  $s$  in  $x$  and at the poles of  $V_2'$  and the zeros of  $\tilde{s}$  in  $y$ .

#### 3.1. Behaviour near hard edges

Near a hard edge  $x \rightarrow X_i$ , such that  $s(X_i) = 0$ , we have

$$\begin{aligned} Y^2(x) \underset{x \rightarrow X_i}{\sim} & - \frac{1}{s(x)} \frac{t}{N} \left\langle \text{Tr} \frac{s(x) - s(M_1)}{x - M_1} (V_1'(M_1) - M_2) \right\rangle \\ & + \frac{1}{s(x)} \sum_{r=2}^{K_1} \sum_{l=1}^{r-1} s_r \frac{t}{N} \langle \text{Tr} M_1^{r-1-l} \rangle \frac{t}{N} \left\langle \text{Tr} \frac{x^l - M_1^l}{x - M_1} \right\rangle \\ & + \text{finite}. \end{aligned} \quad (3.2)$$

Thus

$$\begin{aligned} \operatorname{Res}_{X_i} Y^2(x) dx &= \frac{1}{s'(X_i)} \frac{t}{N} \left\langle \operatorname{Tr} \frac{s(M_1)}{X_i - M_1} (V_1'(M_1) - M_2) \right\rangle \\ &+ \frac{1}{s'(X_i)} \sum_{r=2}^{K_1} \sum_{l=1}^{r-1} s_r \frac{t}{N} \left\langle \operatorname{Tr} M_1^{r-1-l} \right\rangle \frac{t}{N} \left\langle \operatorname{Tr} \frac{X_i^l - M_1^l}{X_i - M_1} \right\rangle. \end{aligned} \quad (3.3)$$

Hard edges are at the same time poles of  $Y(x)$ , and zeros of  $dx$ , so  $Y(x) dx$  is finite.

### 3.2. Behaviour near poles of the potential

Near a finite pole  $\xi$  of  $V_1'(x)$ , we have

$$Y(x) \underset{x \rightarrow \xi}{\sim} V_1'(x) - W(\xi) + O(x - \xi) \quad (3.4)$$

thus

$$\operatorname{Res}_{\xi} Y(x) dx = \operatorname{Res}_{\xi} V_1'(x) dx. \quad (3.5)$$

Near a pole at  $\infty$ , we have

$$Y(x) \underset{x \rightarrow \infty}{\sim} V_1'(x) - \frac{t}{x} + O(1/x^2) \quad (3.6)$$

thus

$$\operatorname{Res}_{\xi} Y(x) dx = t. \quad (3.7)$$

### 3.3. Determination of the algebraic equation $E(x, y)$

So far, we know that  $E(x, y)$  is a rational function of  $x$  and  $y$ , we know its form, and its behaviour near poles, but most of the coefficients are not determined by the loop equations.

The remaining coefficients of  $E(x, y)$ , as usual, are determined by extra requirements, which depend on how the matrix model is defined, i.e. on the purpose for which we introduce the matrix model.

The two most frequent definitions of the matrix model follow in sections 3.3.1 and 3.3.2 (we mainly follow the presentation of [16]).

**3.3.1. Case of the convergent matrix model.** In this case, the matrix model is defined by the convergent integral equation (1.6). For generic potentials and hard edges,  $(1/N^2) \ln Z$  has a large  $N$  limit  $F_0$ , but has no  $1/N$  series expansion. The resolvent, and thus the function  $Y(x)$  also, has a large  $N$  limit, but no  $1/N$  expansion. This fact can be understood from the work of [4].

The large  $N$  limit of  $Y(x)$  obeys an algebraic equation  $E(x, Y(x)) = 0$ .

Now, consider an arbitrary  $\tilde{E}(x, y)$  satisfying the correct behaviour near poles. It gives a function  $\tilde{Y}(x)$ , and from it one can compute the free energy  $F_0$  (see the formula in [2]). The  $Y(x)$  which is the large  $N$  limit of the resolvent is the one for which  $\operatorname{Re}(F_0)$

is minimal (see [16, 4]). That implies in particular that for any contour  $\mathcal{C}$  on the algebraic curve  $E(x, y) = 0$ , one has

$$\forall \mathcal{C}, \quad \operatorname{Re} \oint_{\mathcal{C}} Y(x) dx = 0. \quad (3.8)$$

On an algebraic curve of the type  $E(x, y) = 0$ , there are at most  $2 \times$  genus independent irreducible cycles (plus contours around poles), and one can check that the number of constraints of type equation (3.8) exactly matches the number of coefficients of  $E(x, y)$  not determined by the pole behaviour. Thus, condition equation (3.8) determines  $E(x, y)$  completely. In case there are several solutions, one determines a unique one, by choosing the absolute minimum of  $F_0$ .

One should notice that in the so-called 1-cut case, i.e. when the algebraic curve  $E(x, y) = 0$  has genus zero, the  $1/N^2$  expansion of the free energy and all expectation values exist. In that case, since the genus is zero, there are no filling fractions and equation (3.8) is trivially satisfied.

**3.3.2. Case of the formal matrix model.** The formal matrix model can be defined in a combinatoric way, as a formal series, generating discrete surfaces. The formal expansion is obtained by expanding the matrix integral equation (1.6) with the Feynman graph techniques [5, 17, 6].

In that model,  $\ln Z$  has a  $1/N$  expansion by definition, as well as the resolvent, and all expectation values. The resolvent of the formal matrix model is thus obtained by solving the loop equation (2.28), order by order in  $1/N^2$ .

That model, in addition to the potentials and hard edges, depends on a ‘vacuum’ around which the Feynman expansion is performed. This vacuum is characterized by a set of ‘filling fractions’, as follows (considered for instance in the appendix of [4], or as an application to string theory in [7]): the potential  $V_1(x) + V_2(y) - xy$  has a certain number of extrema, which are given by the algebraic equation

$$V_1'(x) = y, \quad V_2'(y) = x \quad (3.9)$$

i.e.,

$$V_2'(V_1'(x)) = x. \quad (3.10)$$

Let us call  $K$  the degree of this algebraic equation, i.e. the number of its solutions:

$$(\bar{x}_1, \bar{y}_1), \dots, (\bar{x}_K, \bar{y}_K). \quad (3.11)$$

The eigenvalues of matrices  $M_1, M_2$  which extremize  $\operatorname{Tr}(V_1(M_1) + V_2(M_2) - M_1 M_2)$  must be among the  $K$  solutions described above, or can be trapped on contours stopping at hard edges. The filling fractions  $(\epsilon_1, \dots, \epsilon_{K+K_1+K_2})$  represent the number of eigenvalues equal to each solutions of equation (3.10), i.e. there are  $N\epsilon_1$  eigenvalues of  $M_1$  equal to  $\bar{x}_1, \dots$ . One must have

$$\sum_{i=1}^{K+K_1+K_2} \epsilon_i = 1. \quad (3.12)$$

The average number of eigenvalues of  $M_1$  in the vicinity of a point  $(\bar{x}, \bar{y})$  is a contour integral of the resolvent, along a contour which surrounds  $(\bar{x}, \bar{y})$ :

$$2i\pi N\epsilon_i = - \oint_{\mathcal{A}_i} W(x) dx = \oint_{\mathcal{A}_i} Y(x) dx \quad (3.13)$$

and it can be non-zero only on irreducible cycles of the algebraic curve.

Thus, in the formal matrix model, a set of irreducible cycle contour integrals is fixed. One can verify that the number of filling fractions matches exactly the number of coefficients of  $E(x, y)$  not determined by the pole behaviour. Thus, equation (3.13) is sufficient to determine completely the rational function  $E(x, y)$ .

#### 4. Conclusion

The purpose of this paper was to write down the loop equations for the so-called semiclassical 2-matrix model. We find that the loop equation becomes an algebraic equation  $E(x, y) = 0$  in the large  $N$  limit, with poles at the poles of the potentials, and at the hard edges. The hard edges are such that the resolvent has a simple pole, and  $dx$  has a zero, so the differential form  $y dx$  is regular.

The loop equation determines the form of the algebraic equation  $E(x, y)$ , but does not determine its coefficients. The coefficients are determined by extra assumptions, related to which definition of the matrix model is considered. In the formal matrix model, the  $\mathcal{A}$  cycle integrals of  $y dx$  are fixed parameters of the model, and that determines  $E(x, y)$  completely. In the convergent matrix model, the real parts of both  $\mathcal{A}$  and  $\mathcal{B}$  cycle integrals of  $y dx$  must vanish, and that determines  $E(x, y)$  completely.

Let us also remark that the function equation (2.28)  $E(x, y)$  is unchanged under the exchange  $x \leftrightarrow y$ ,  $V_1 \leftrightarrow V_2$ ,  $s \leftrightarrow \tilde{s}$ , which is the generalization of Matytsin's duality property [18]:

$$X(Y(x)) = x. \quad (4.1)$$

The consequences of that algebraic equation can then be studied. This is done for instance in [3]. One can also expect to generalize the works of [10], or [8], or [9], i.e. the computation of all correlation functions and their  $1/N^2$  expansion, and further, compute the expansion of the free energy [11, 12].

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