$N = 4$ Supersymmetric Gauge Theory, Twistor Space, and Dualities

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Saclay Lectures, IV

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Course Overview

- Present advanced techniques for calculating amplitudes in gauge theories
- Motivation for hard calculations
- Review gauge theories and supersymmetry
- Color decomposition; spinor-helicity basis; recurrence relations; supersymmetry Ward identities; factorization properties of gauge-theory amplitudes
- Twistor space; Cachazo-Svrcek-Witten rules for amplitudes
- Unitarity-based method for loop calculations; loop integral reductions
- Computation of anomalous dimensions

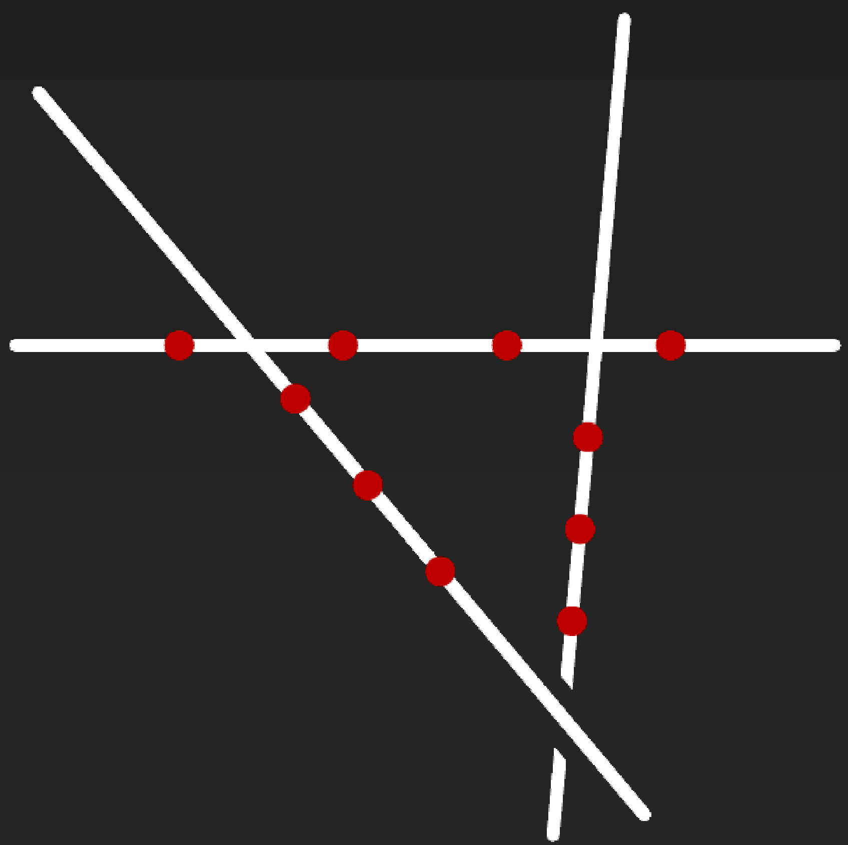
Review of Lecture III

Cachazo-Svrček-Witten construction

MHV vertices

Recursive formulation

Supersymmetry Algebras



Simpler than expected: what does this mean in field theory?

Cachazo–Svrcek–Witten Construction

hep-th/0403047

Amplitudes can be built up out of vertices $\leftrightarrow$ line segments

Intersections $\leftrightarrow$ propagators

Vertices are off-shell continuations of MHV amplitudes: every vertex has two '$-$' helicities, and one or more '$+$' helicities

Includes a three-point vertex

Propagators are scalar ones: i/K^2; helicity projector is in the vertices

Draw all tree diagrams with these vertices and propagator

Different sets of diagrams for different helicity configurations

Off-Shell Continuation

Can decompose a general four-vector into two massless four-vectors, one of which we choose to be the light-cone vector

$$K = k^\flat + f(K)q.$$

We can solve for f using the condition $(k^\flat)^2 = 0$

$$f(K) = \frac{K^2}{2q \cdot K}$$

The rule for continuing k_j off-shell in an MHV vertices is then just

$$\langle j\, j' \rangle \to \langle j^\flat\, j' \rangle\,,$$

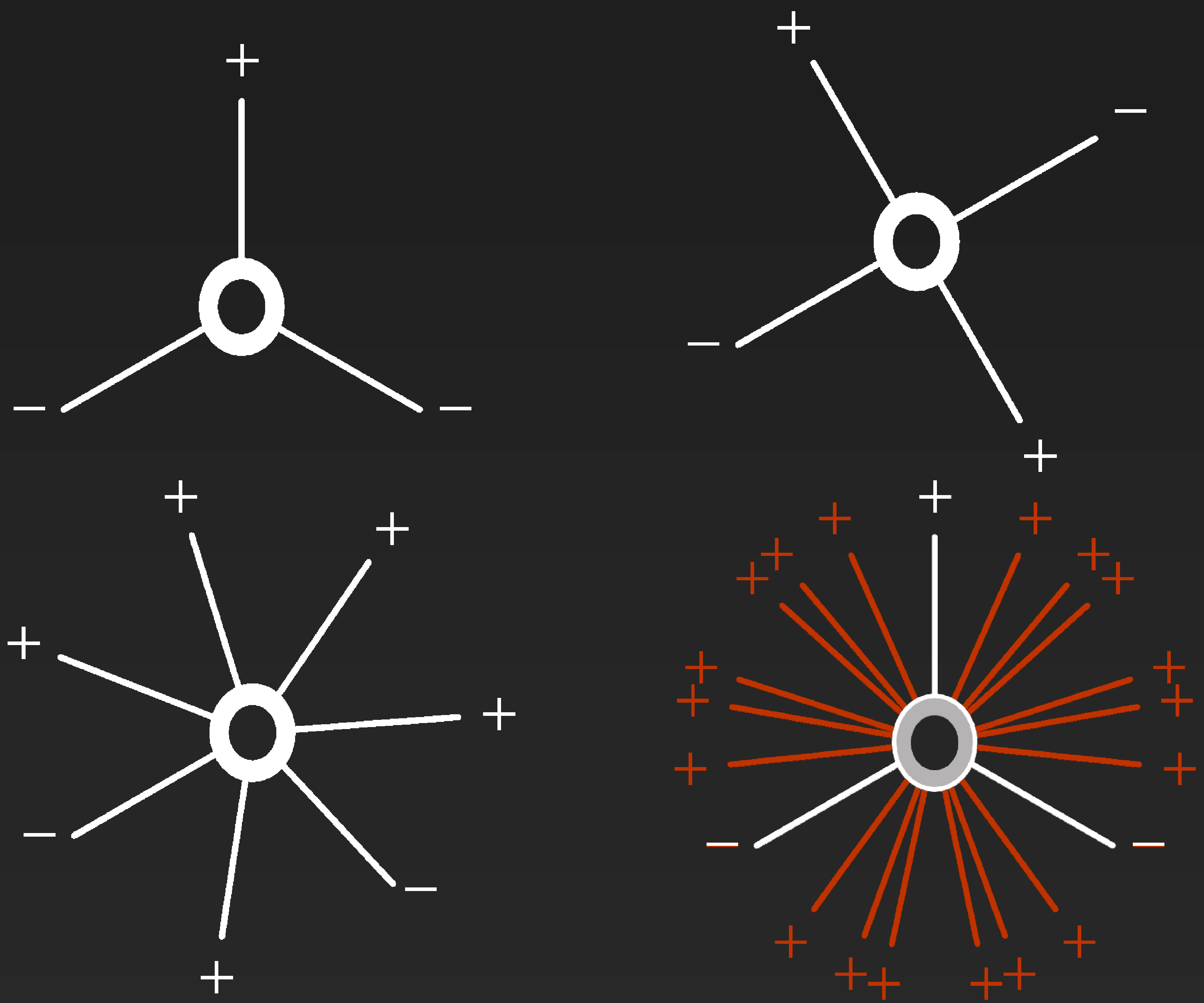
Off-Shell Vertices

General MHV vertex

$$\frac{\langle m_1\, m_2\rangle^4}{\langle 1\, 2\rangle\, \langle 2\, 3\rangle \cdots \langle (j-1)\, j^\flat\rangle\, \langle j^\flat\, (j+1)\rangle \cdots \langle (n-1)\, n\rangle\, \langle n\, 1\rangle}$$

Three-point vertex is just n = 3 case

$$\frac{\langle m_1\, m_2\rangle^4}{\langle 1\, 2\rangle\, \langle 2\, 3^\flat\rangle\, \langle 3^\flat\, 1\rangle}$$



Supersymmetry Ward Identities

Color-ordered amplitudes don't distinguish between quarks and gluinos $\Rightarrow$ same for QCD and N=1 SUSY

Supersymmetry should relate amplitudes for different particles in a supermultiplet, such as gluons and gluinos

Supercharge annihilates vacuum

$$\langle 0|[Q, \Phi_1 \Phi_2 \cdots \Phi_n]|0\rangle = 0 = \sum_{i=1}^{n} \langle 0|\Phi_1 \cdots [Q, \Phi_i] \cdots \Phi_n|0\rangle$$

Grisaru, Pendleton & van Nieuwenhuizen (1977)

Use a practical representation of the action of supersymmetry on the fields. Multiply by a spinor wavefunction & Grassman parameter θ

$$[Q, G^{\pm}(k)] = \pm\Gamma^{\pm}(k,q)\Lambda^{\pm}(k)$$
$$[Q, \Lambda^{\pm}(k)] = \mp\Gamma^{\mp}(k,q)G^{\pm}(k)$$

where $\Gamma^{-} = \theta\,[q\,k]\,,\ \Gamma^{+} = \theta\,\langle q\,k\rangle$

With explicit helicity choices, we can use this to obtain equations relating different amplitudes

Typically start with Q acting on an 'amplitude' with an odd number of fermion lines (overall a bosonic object)

Supersymmetry WI in Action

All helicities positive:

$$
\begin{aligned}
0 &= \langle 0 | [Q_q, \Lambda_1^+ G_2^+ \cdots G_n^+] | 0 \rangle \\
&= -\Gamma^-(k_1, q) A_n^{\rm tree}(1^+, \ldots, n^+) \\
&\qquad + \Gamma^+(k_2, q) A_n^{\rm tree}(1_\Lambda^+, 2_\Lambda^+, 3^+, \ldots, n^+) \\
&\qquad + \cdots + \Gamma^+(k_n, q) A_n^{\rm tree}(1_\Lambda^+, 2^+, \ldots, (n-1)^+, n_\Lambda^+)
\end{aligned}
$$

Helicity conservation implies that the fermionic amplitudes vanish

$$
0 = -\Gamma^-(k_1, q) A_n^{\rm tree}(1^+, \ldots, n^+)
$$

so that we obtain the first Parke–Taylor equation

With two negative helicity legs, we get a non-vanishing relation

$$
\begin{aligned}
0 &= \langle 0|[Q_q, \Lambda_1^+ G_2^- G_3^+ \cdots G_j^- G_{j+1}^+ \cdots G_n^+]|0\rangle \\
&= -\Gamma^-(k_1, q) A_n^{\rm tree}(1^+, 2^-, \ldots, n^+) \\
&\quad - \Gamma^+(k_2, q) A_n^{\rm tree}(1_\Lambda^+, 2_\Lambda^-, 3^+, \ldots, j^-, \ldots, n^+) \\
&\quad - \Gamma^+(k_j, q) A_n^{\rm tree}(1_\Lambda^+, 2^-, 3^+, \ldots, j_\Lambda^-, \ldots, n^+)
\end{aligned}
$$

Choosing $q = k_2$

$$
\begin{aligned}
&A_n^{\rm tree}(1_\Lambda^+, 2^-, 3^+, \ldots, j_\Lambda^-, \ldots, n^+) = \\
&\qquad - \frac{\langle 1\, 2\rangle}{\langle 2\, j\rangle} A_n^{\rm tree}(1^+, 2^-, 3^+, \ldots j^-, \ldots, n^+)
\end{aligned}
$$

Additional Relations in Extended SUSY

$$A_n^{\text{tree}}(1^+, \ldots, j^-, \ldots, l^-, \ldots, n^+) =$$
$$\frac{\langle j\, l\rangle^4}{\langle 1\, 2\rangle^4} A_n^{\text{tree}}(1^-, 2^-, 3^+, \ldots, n^+)$$

Tree-level mplitudes with external gluons or one external fermion pair are given by supersymmetry even in QCD.

Beyond tree level, there are additional contributions, but the Ward identities are still useful.

For supersymmetric theories, they hold to all orders in perturbation theory

Factorization Properties of Amplitudes

As sums of external momenta approach poles,

$$p^2 = (k_1 + k_2)^2 \to m_X^2$$

amplitudes factorize

$$A(1{+}2 \to \ldots) \to A_L(1{+}2 \to X)\frac{i}{p^2 - m^2}A_R(X \to \ldots)$$

More generally as $p^2 = (k_1 + \cdots + k_n)^2 \to m_X^2$

$$A(1{+}\cdots{+}n \to \ldots) \to A_L(1{+}\cdots{+}n \to X)\frac{i}{p^2 - m^2}A_R(X \to \ldots)$$

In massless theories, the situation is more complicated but at tree level it can be understood as an expression of unitarity

In gauge theories, it holds (at tree level) for $n \geq 3$ but breaks down for $n = 2$: $A_3 = 0$ so we get $0/0$

However A_3 only vanishes linearly, so the amplitude is not finite in this limit, but should $\sim 1/k$, that is $1/\sqrt{s_{12}}$

This is a collinear limit $(k_1 + k_2)^2 = 2k_1 \cdot k_2 \to 0$

$$\Longrightarrow k_1 \propto k_2, \;\; i.e., \; k_1 \parallel k_2$$

Universal Factorization

Amplitudes have a universal behavior in this limit

$$A_n^{\text{tree}}(\ldots, a^{h_a}, b^{h_b}, \ldots) \xrightarrow{k_a \| k_b}$$

$$\sum_{h=\pm} \text{Split}_{-h}^{\text{tree}}(a^{h_a}, b^{h_b}) A_{n-1}^{\text{tree}}(\ldots, (k_a + k_b)^h, \ldots)$$

$$+ \ \text{non-singular}$$

Depend on a collinear momentum fraction z

$$k_a = z(k_a + k_b), \qquad k_b = (1 - z)(k_a + k_b)$$

Splitting Amplitudes

Compute it from the three-point vertex

$$
\begin{aligned}
\mathrm{Split}^{\mathrm{tree}}_{-}(a^+, b^+) &= -\frac{\sqrt{2}}{s_{ab}} \left[k_b \cdot \varepsilon_a \, \varepsilon_b \cdot \varepsilon_{a+b} - k_a \cdot \varepsilon_b \, \varepsilon_a \cdot \varepsilon_{a+b} \right] \\
&= -\frac{1}{s_{ab}} \left[\frac{\langle q\, b\rangle \, [b\, a]}{\langle q\, a\rangle} \frac{\langle q\, (a{+}b)\rangle \, [q\, b]}{\langle q\, b\rangle \, [(a{+}b)\, q]} \right. \\
&\qquad \left. - \frac{\langle q\, a\rangle \, [a\, b]}{\langle q\, b\rangle} \frac{\langle q\, (a{+}b)\rangle \, [q\, a]}{\langle q\, a\rangle \, [(a{+}b)\, q]} \right] \\
&= \frac{1}{\langle a\, b\rangle} \left[\sqrt{\frac{1-z}{z}} + \sqrt{\frac{z}{1-z}} \right] \\
&= \frac{1}{\sqrt{z(1-z)} \; \langle a\, b\rangle} \, .
\end{aligned}
$$

Explicit Values

$$\mathrm{Split}_{-}^{\mathrm{tree}}(a^{-}, b^{-}) = 0$$

$$\mathrm{Split}_{-}^{\mathrm{tree}}(a^{+}, b^{+}) = \frac{1}{\sqrt{z(1-z)}\ \langle a\, b\rangle}$$

$$\mathrm{Split}_{-}^{\mathrm{tree}}(a^{+}, b^{-}) = -\frac{z^2}{\sqrt{z(1-z)}\ [a\, b]}$$

$$\mathrm{Split}_{-}^{\mathrm{tree}}(a^{-}, b^{+}) = -\frac{(1-z)^2}{\sqrt{z(1-z)}\ [a\, b]}$$

Collinear Factorization at One Loop

$$A_n^{\text{1-loop; LC}}(\ldots, a^{h_a}, b^{h_b}, \ldots) \xrightarrow{k_a \| k_b}$$

$$\sum_{h=\pm} \Bigg(\text{Split}^{\text{tree}}_{-h}(a^{h_a}, b^{h_b}) A_{n-1}^{\text{1-loop;LC}}(\ldots, (k_a+k_b)^h, \ldots)$$

$$+ \text{Split}^{\text{1-loop}}_{-h}(a^{h_a}, b^{h_b}) A_{n-1}^{\text{tree}}(\ldots, (k_a+k_b)^h, \ldots) \Bigg)$$

$$+ \ \text{non-singular}$$

Anomalous Dimensions & Amplitudes

In QCD, one-loop anomalous dimensions of twist-2 operators in the OPE are related to the tree-level Altarelli-Parisi function,

Twist-2 Anomalous Dimension	Mellin Transform ⇔	Altarelli-Parisi function	=	Helicity-summed splitting amplitude

Relation understood between two-loop anomalous dimensions & one-loop splitting amplitudes

DAK & Uwer (2003)

Loop Calculations: Textbook Approach

Sew together vertices and propagators into loop diagrams

Obtain a sum over [2,n]-point [0,n]-tensor integrals, multiplied by coefficients which are functions of k and ε

Reduce tensor integrals using Brown-Feynman & Passarino-Veltman brute-force reduction, or perhaps Vermaseren-van Neerven method

Reduce higher-point integrals to bubbles, triangles, and boxes

Can apply this to color-ordered amplitudes, using color-ordered Feynman rules

Can use spinor-helicity method at the end to obtain helicity amplitudes

BUT

This fails to take advantage of gauge cancellations early in the calculation, so a lot of calculational effort is just wasted.

Can We Take Advantage?

Of tree-level techniques for reducing computational effort?

Of the CSW construction?

Unitarity

Basic property of any quantum field theory: conservation of probability. In terms of the scattering matrix,

$$S^\dagger S = 1$$

In terms of the transfer matrix $iT = S - 1$ we get,

$$-i(T - T^\dagger) = T^\dagger T$$

or

$$2 \text{ “Im” } T_{fi} = (T^\dagger T)_{fi}$$

with the Feynman iε

$$\mathrm{Disc}\, T = T^\dagger T$$

This has a direct translation into Feynman diagrams, using the Cutkosky rules. If we have a Feynman integral,

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2 + i\delta} \cdots \frac{1}{(\ell - K)^2 + i\delta}$$

and we want the discontinuity in the K^2 channel, we should replace

$$\frac{1}{\ell^2 + i\delta} \longrightarrow -2\pi i\delta^{(+)}(\ell^2)$$

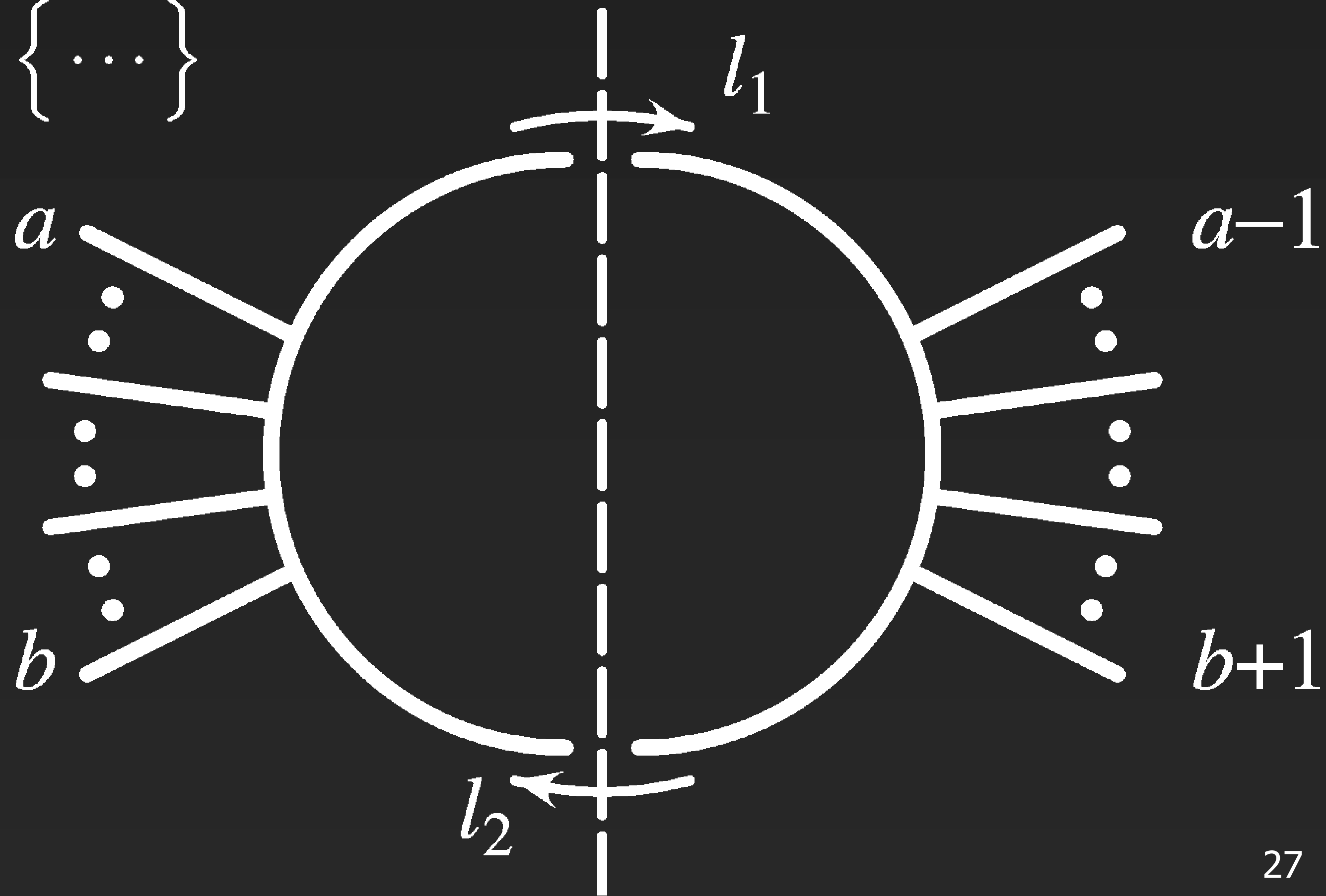
$$\frac{1}{(\ell - K)^2 + i\delta} \longrightarrow -2\pi i\delta^{(+)}((\ell - K)^2)$$

$$\delta^{(+)}(k^2) = \Theta(k^0)\delta(k^2)$$

When we do this, we obtain a phase-space integral

$$\int \frac{d^D\ell}{(2\pi)^{D-1}}\,\delta^{(+)}(\ell^2)\delta^{(+)}((\ell-K)^2)\Big\{\cdots\Big\} =$$

$$\int d^D\mathrm{LIPS}\,\Big\{\cdots\Big\}$$



In the Bad Old Days of Dispersion Relations

To recover the full integral, we could perform a dispersion integral

$$\mathrm{Re}\, f(s) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dw \, \frac{\mathrm{Im}\, f(w)}{w-s} + \mathrm{Re}\, C_\infty$$

in which $C_\infty = 0$ so long as $f(w) \to 0$ when $w \to \infty$

If this condition isn't satisfied, there are 'subtraction' ambiguities corresponding to terms in the full amplitude which have no discontinuities

But it’s better to obtain the full integral by identifying which Feynman integral(s) the cut came from.

Allows us to take advantage of sophisticated techniques for evaluating Feynman integrals: identities, modern reduction techniques, differential equations, reduction to master integrals, etc.

Computing Amplitudes Not Diagrams

The cutting relation can also be applied to sums of diagrams, in addition to single diagrams

Looking at the cut in a given channel s of the sum of all diagrams for a given process throws away diagrams with no cut — that is diagrams with one or both of the required propagators missing — and yields the sum of all diagrams on each side of the cut.

Each of those sums is an on-shell tree amplitude, so we can take advantage of all the advanced techniques we've seen for computing them.

Unitarity-Based Method at One Loop

Compute cuts in a set of channels

Compute required tree amplitudes

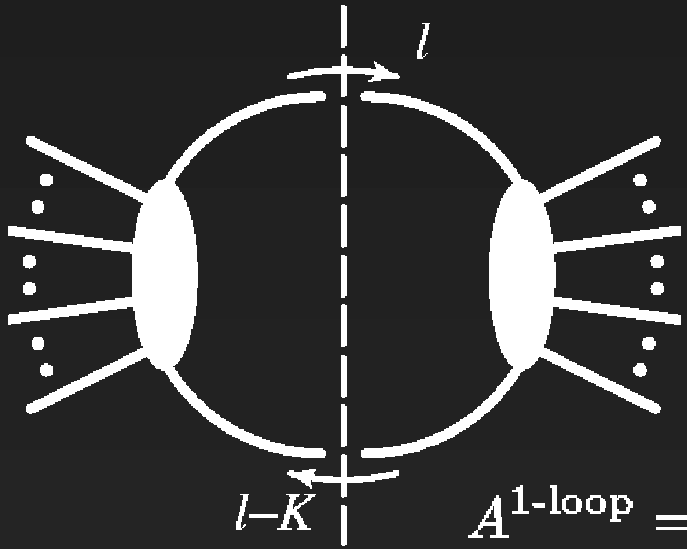
Form the phase-space integrals

Reconstruct corresponding Feynman integrals

Perform integral reductions to a set of master integrals

Assemble the answer

Unitarity-Based Calculations



Bern, Dixon, Dunbar, & DAK (1994)

$$A^{\text{1-loop}} = \sum_{\text{cuts } K^2} \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \frac{i}{\ell^2} A^{\text{tree}}_{\text{left}} \frac{i}{(\ell-K)^2} A^{\text{tree}}_{\text{right}}$$

In general, work in D=4-2ϵ $\Rightarrow$ full answer

van Neerven (1986): dispersion relations converge

At one loop in D=4 for SUSY $\Rightarrow$ full answer

Merge channels rather than blindly summing: find function w/given cuts in all channels

The Three Roles of Dimensional Regularization

Ultraviolet regulator;

Infrared regulator;

Handle on rational terms.

Dimensional regularization effectively removes the ultraviolet divergence, rendering integrals convergent, and so removing the need for a subtraction in the dispersion relation

Pedestrian viewpoint: dimensionally, there is always a factor of $(-s)^{-\varepsilon}$, so at higher order in ε, even rational terms will have a factor of $\ln(-s)$, which has a discontinuity

Integral Reductions

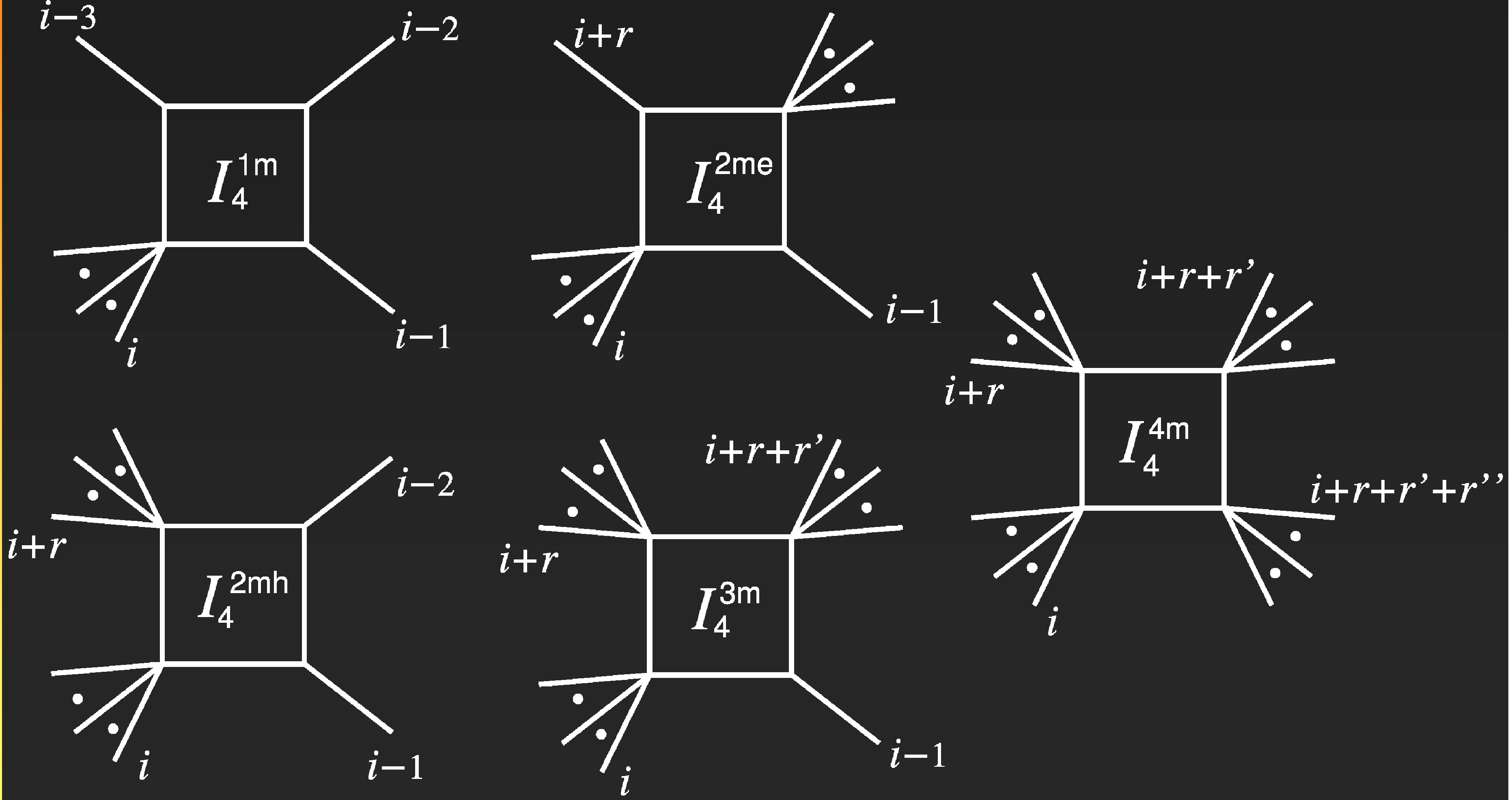
At one loop, all n≥5-point amplitudes in a massless theory can be written in terms of nine different types of scalar integrals:

boxes (one-mass, 'easy' two-mass, 'hard' two-mass, three-mass, and four-mass);

triangles (one-mass, two-mass, and three-mass);

bubbles

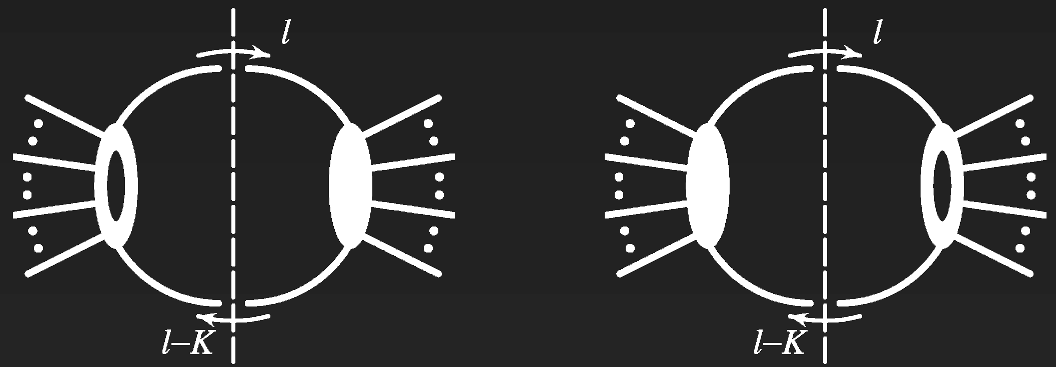
In an N=4 supersymmetric theory, only boxes are needed.



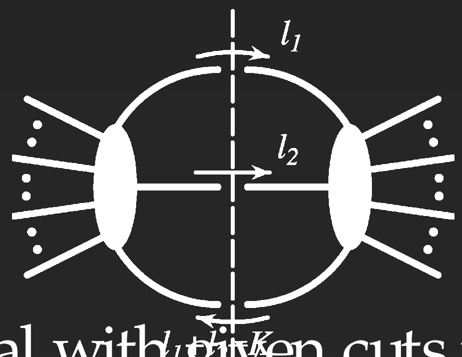
i−3
i−2
I_4^1m
i
i−1
i+r
I_4^2me
i
i−1
i+r+r’
i+r
I_4^4m
i+r+r’+r’’
i
i−2
i+r
I_4^2mh
i
i−1
i+r+r’
i+r
I_4^3m
i
i−1

Unitarity-Based Method at Higher Loops

Loop amplitudes on either side of the cut



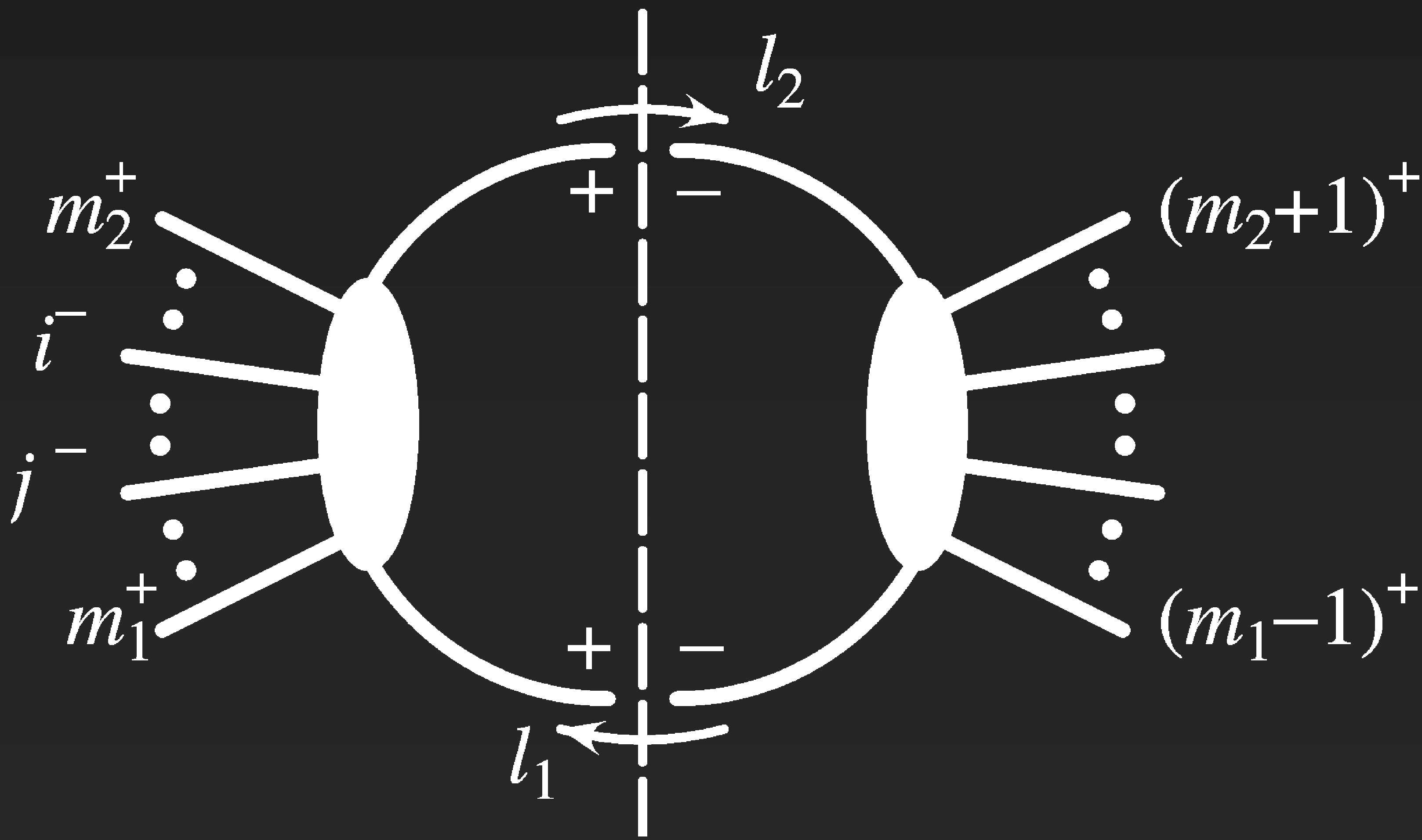
Multi-particle cuts in addition to two-particle cuts



Find integrand/integral with given cuts in all channels

In practice, replace loop amplitudes by their cuts too

Example: MHV at One Loop



Start with the cut

$$\int d^4\mathrm{LIPS}(\ell_1,-\ell_2)\; A^{\mathrm{tree}}(-\ell_2, m_2+1,\ldots,m_1-1,\ell_1) \\ \times A^{\mathrm{tree}}(-\ell_1, m_1,\ldots,m_2,\ell_2)$$

Use the known expressions for the MHV amplitudes

$$-\int d^4\mathrm{LIPS}(\ell_1,-\ell_2)\; \frac{\langle (-\ell_1)\,\ell_2\rangle^3}{\langle (-\ell_1)\,m_1\rangle\,\langle\langle m_1\cdots m_2\rangle\rangle\,\langle m_2\,\ell_2\rangle} \\ \times \frac{\langle i\,j\rangle^4}{\langle (-\ell_2)\,(m_2+1)\rangle\,\langle\langle (m_2+1)\cdots(m_1-1)\rangle\rangle\,\langle (m_1-1)\,\ell_1\rangle\,\langle \ell_1\,(-\ell_2)\rangle}$$

Most factors are independent of the integration momentum

$$iA^{\rm tree}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)$$

$$\times \int d^4{\rm LIPS}(\ell_1,-\ell_2)\; \frac{\langle (m_1-1)\, m_1\rangle\, \langle m_2\,(m_2+1)\rangle\, \langle \ell_1\,\ell_2\rangle^2}{\langle \ell_1\, m_1\rangle\, \langle m_2\,\ell_2\rangle\, \langle (m_1-1)\,\ell_1\rangle\, \langle \ell_2\,(m_2+1)\rangle}$$

$$= iA^{\rm tree}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)$$

$$\times \int d^4{\rm LIPS}(\ell_1,-\ell_2)\;\; \langle (m_1-1)\, m_1\rangle\, \langle \ell_1\,\ell_2\rangle^2\, \langle m_2\,(m_2+1)\rangle$$

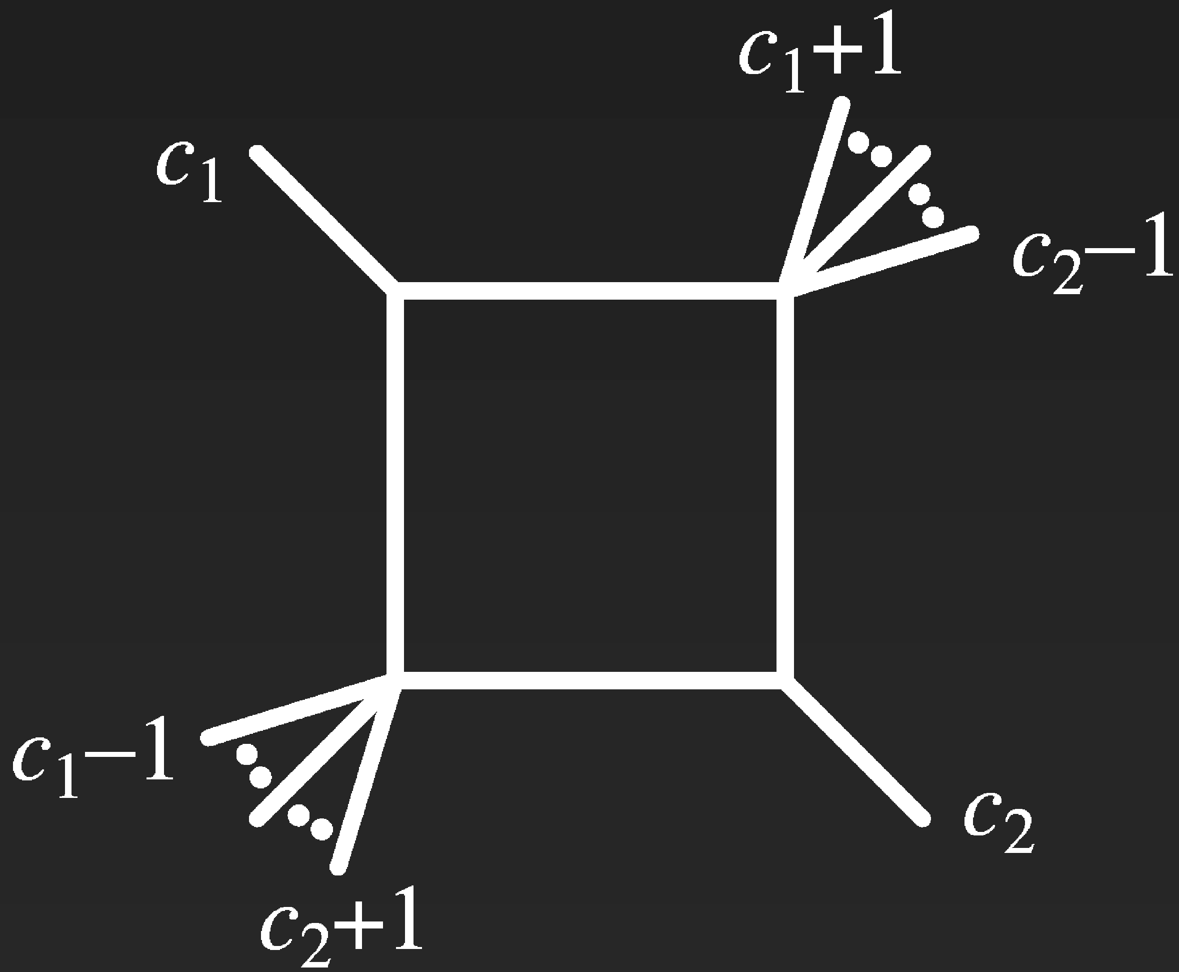
$$\times \frac{[m_1\,\ell_1]\,[\ell_2\, m_2]\,[(m_1-1)\,\ell_1]\,[\ell_2\,(m_2+1)]}{(\ell_1-k_{m_1})^2(\ell_2+k_{m_2})^2(\ell_1+k_{m_1-1})^2(\ell_2-k_{m_2+1})^2}$$

We can use the Schouten identity to rewrite the remaining parts of the integrand,

$$
\begin{aligned}
&(\ell_1 + k_{m_1-1})^2(\ell_2 - k_{m_2+1})^2 \frac{1}{2}\,\mathrm{Tr}((1+\gamma_5)\not{\ell}_1 \not{k}_{m_2} \not{\ell}_2 \not{k}_{m_1}) \\
&\quad - \{k_{m_1-1} \leftrightarrow -k_{m_1}\} - \{k_{m_2+1} \leftrightarrow -k_{m_2}\} \\
&\quad + \{k_{m_1-1} \leftrightarrow -k_{m_1}, k_{m_2+1} \leftrightarrow -k_{m_2}\}\,.
\end{aligned}
$$

Two propagators cancel, so we're left with a box — the γ_5 leads to a Levi-Civita tensor which vanishes

What's left over is the same function which appears in the denominator of the box: $-st + m_2^2 m_4^2$

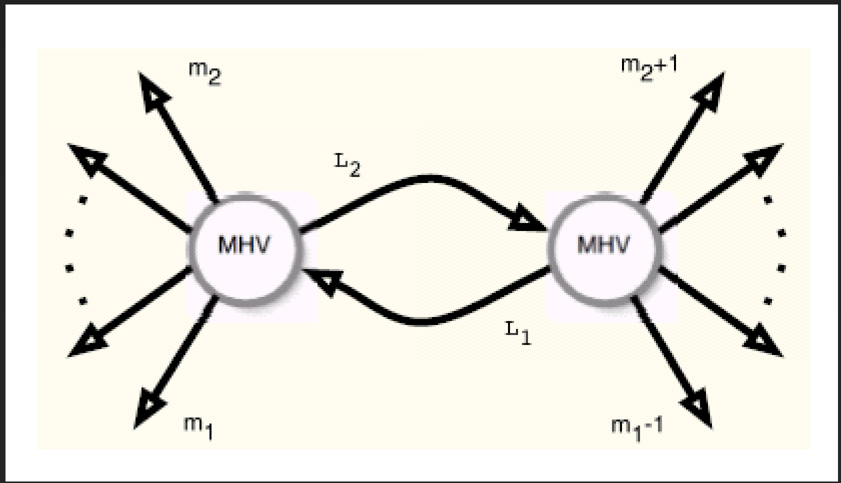

c_1+1
c_1
c_2-1
c_1-1
c_2
c_2+1

We obtain the result,

$$A^{\mathrm{tree}}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)$$
$$\times \sum_{\mathrm{easy2\ mass}} \mathrm{Box}\cdot(\mathrm{its\ denominator})$$

From Trees To Loops Using CSW

Sew together two MHV vertices



Brandhuber, Spence, & Travaglini (2004)

Can't integrate the CSW form directly; but we can change variables

$$\frac{d^D L}{(2\pi)^4} \frac{1}{L^2} \longrightarrow \frac{dz}{z} \frac{d^D \ell}{(2\pi)^D} \delta^{(+)}(\ell^2)$$

Integration over the cut

Holomorphic 'Anomaly'

Applying differential operators to MHV loop amplitudes appeared to yield a complicated picture

Appeared to be in contradiction to BST calculation

Resolution: need to account for poles in a holomorphic function

$$\partial_{\bar{z}}\,(1/z) \neq 0$$

Cachazo, Svrček, & Witten (9/2004)

With explicit computation, one can define a modified line operator, $F' = F - \Delta$

$$F' A = 0$$

Bena, Bern, Roiban, DAK (10/2004)

Direct Algebraic Equations?

Cachazo (10/2004)

Basic structure of differential equations

$$D\,A = 0$$

To solve for A, need a non-trivial right-hand side; 'anomaly' provides one!

$$D\,A = a$$

Operators annihilate coefficients of boxes with an anomaly in a given cut channel;

Evaluate only discontinuity, using known basis set of integrals
$\Rightarrow$ system of algebraic equations

Challenges Ahead

Many problems remain to be worked on

Incorporation of massive particles into the picture $\Rightarrow$ $D = 4-2\varepsilon$ helicities

Interface with non-QCD parts of amplitudes

Complete machinery for one-loop calculations: reduce one-loop calculations to purely algebraic ones, polynomial-time in an automated fashion; do so in an analytic context, in avoiding intermediate-expression swell

Twistor string at one loop (conformal supergravitons); connected-curve picture?