

# Directed flow at RHIC from Lee–Yang zeroes

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Directed flow in ultrarelativistic nucleus-nucleus collisions is analyzed using the reaction plane from elliptic flow, which reduces the bias from nonflow effects. We combine this method with the determination of elliptic flow from Lee–Yang zeroes. The resulting method is more consistent and somewhat easier to implement than the previously used method based on three-particle cumulants, and is also less biased by nonflow correlations. Error terms from residual nonflow correlations are carefully estimated, as well as statistical errors. We discuss the application of the method at RHIC and LHC.

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## I. INTRODUCTION

The standard method of analyzing directed flow ( $v_1$ ) in nucleus-nucleus collisions using an estimate of the reaction plane [1, 2] was shown to be inadequate at ultrarelativistic energies [3], due to the smallness of directed flow and the relatively large magnitude of “nonflow” effects [2, 4]. Trivial nonflow effects such as momentum conservation can be taken into account in the analysis [3], and the standard analysis can be modified for this purpose [5]. A more systematic, model-independent way of eliminating nonflow effects was introduced [6], based on the observation that the three-particle average  $\langle \cos(\phi_1 + \phi_2 - 2\phi_3) \rangle$  (where  $\phi_1$ ,  $\phi_2$  and  $\phi_3$  denote the azimuthal angles of three particles emitted in a collision, and  $\langle \cdot \cdot \cdot \rangle$  denotes an average over triplets of particles and events) is much less sensitive to nonflow effects than the two-particle correlation  $\langle \cos(\phi_1 - \phi_2) \rangle$  used in the standard analysis. Since  $\langle \cos(\phi_1 + \phi_2 - 2\phi_3) \rangle \propto v_1^2 v_2$ , where the elliptic flow  $v_2$  is large at ultrarelativistic energies, this provides an alternative way of analyzing directed flow. This method was first implemented at the CERN SPS by the NA49 Collaboration [7, 8], and recently led to the discovery of directed flow at RHIC [9]. Directed flow at ultrarelativistic energies is interesting in itself [10]; in addition, it is the only way of measuring the sign of elliptic flow, and to check experimentally that it is positive [11].

The practical implementation of the three-particle method presented in Ref. [6] is rather cumbersome and requires to estimate elliptic flow  $v_2$  independently, using cumulants [12] to avoid nonflow effects. In this paper, we suggest to analyze simultaneously elliptic and directed flows using the recently introduced method of Lee–Yang zeroes [13, 14] to analyze elliptic flow. This minimizes the bias from nonflow effects for both  $v_1$  and  $v_2$ . The practical recipe is presented in Sec. II. The theoretical background is briefly discussed in Sec. III. In Sec. IV, we derive the general order of magnitude of the systematic error due to nonflow correlations, which was underestimated in Ref. [6]. We also give analytical expressions of statistical errors.

## II. IMPLEMENTATION

Let us first define useful quantities and notations. For a given event, we define the following complex-valued function

$$g_\epsilon^{\theta_1, \theta_2}(z) = \prod_{j=1}^M [1 + z\epsilon w_1(j) \cos(\phi_j - \theta_1) + zw_2(j) \cos(2(\phi_j - \theta_2))] \quad (1)$$

where  $\theta_1$  and  $\theta_2$  are angles,  $\epsilon$  is a real parameter,  $z$  is a complex variable,  $\phi_j$  are the azimuthal angles of the particles (measured using a fixed reference in the laboratory), and the product runs over all detected particles. As in other methods of analysis,  $w_1$  and  $w_2$  are weights appropriate to directed and elliptic flows, respectively, and can be any functions of particle type, transverse momentum  $p_T$  and rapidity  $y$ . In Eq. (1),  $w_n(j)$  is a shorthand for  $w_n(p_{Tj}, y_j)$ . The best weight is the flow itself [12, 15],  $w_n(p_T, y) = v_n(p_T, y)$ , where  $v_n(p_T, y)$  denotes the value of the flow in a small

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$(p_T, y)$  bin. In practice, one can choose as a first guess the center-of-mass rapidity for directed flow,  $w_1 = y - y_{\text{CM}}$ , and the transverse momentum for elliptic flow  $w_2 = p_T$ , in regions of phase space covered by the detector acceptance. The average of  $g_\epsilon^{\theta_1, \theta_2}(z)$  over events in a centrality bin will be denoted by  $\langle g_\epsilon^{\theta_1, \theta_2}(z) \rangle_{\text{evts}}$ . The same notation holds for any observable associated with the event.

The implementation of the method consists of three logical steps, as in the 3-particle correlation method [6]. The first one is to analyze *integrated* elliptic flow  $V_2$ . By “integrated,” we mean that it is summed over all detected particles with appropriate weights, and averaged over many events:

$$V_2 \equiv \left\langle \sum_{j=1}^M w_2(j) \cos(2(\phi_j - \Phi_R)) \right\rangle_{\text{evts}}, \quad (2)$$

where the sum runs over all particles detected in an event and  $\Phi_R$  is the azimuthal angle of the reaction plane of the event.<sup>1</sup> The second step is the analysis of integrated directed flow  $V_1$ , which is defined similarly:

$$V_1 \equiv \left\langle \sum_{j=1}^M w_1(j) \cos(\phi_j - \Phi_R) \right\rangle_{\text{evts}}. \quad (3)$$

Finally, in a third step, integrated values are used as a reference to analyze directed flow differentially, as a function of transverse momentum and rapidity. In practice, the method requires only two passes through the data: the first pass corresponds to Sec. II A, while the computations in Sec. II B and Sec. II C can be done in a single second pass.

### A. Integrated elliptic flow

The analysis of integrated elliptic flow is identical to that presented in Refs. [13, 14], to which we refer the reader for further practical details. Note, however, that it uses the generating function introduced in Ref. [14], not that of Ref. [13]. Let us just give a short reminder of the recipe.

One must evaluate  $g_\epsilon^{\theta_1, \theta_2}(z)$  for  $\epsilon = \theta_1 = 0$ , for several (typically 4 or 5) equally spaced values of  $\theta_2$  between 0 and  $\pi/2$  (i.e.,  $\theta_2 = 0, \pi/(2n), \dots, (n-1)\pi/(2n)$  with  $n = 4$  or  $5$ ) and many values of  $z$  on the imaginary axis,  $z = ir$ . One then plots the modulus  $|\langle g_0^{0, \theta_2}(ir) \rangle_{\text{evts}}|$  for real, positive  $r$  as a function of  $r$ , for a fixed value of  $\theta_2$ . One determines numerically the value of  $r$  corresponding to the first minimum of this function,  $r = r_0^{\theta_2}$ . The value of  $r_0^{\theta_2}$  will be used in the analysis of directed flow below. It is closely related to elliptic flow, which can be estimated through the following formula [13, 14]:

$$|V_2| = \left\langle \frac{j_{01}}{r_0^{\theta_2}} \right\rangle_{\theta_2}, \quad (4)$$

where  $j_{01} \simeq 2.40483$  is the first root of the Bessel function of the first kind  $J_0(x)$ . The notation  $\langle \dots \rangle_{\theta_2}$  means an average over  $\theta_2$ . The analysis could in principle be performed with a single value of  $\theta_2$ . The averaging is only of practical importance: it reduces the statistical error on  $V_2$  by a factor of  $\sim 2$ .

Note that Eq. (4) yields an estimate of the absolute value of  $V_2$ , not of  $V_2$  itself. The sign of  $V_2$  is determined simultaneously with directed flow, as we shall see below.

### B. Integrated directed flow

The analysis of integrated directed flow involves the generating function  $g_\epsilon^{\theta_1, \theta_2}(z)$ . The angle  $\theta_2$  takes the same values as in Sec. II B. For a given  $\theta_2$ ,  $z$  takes only one value,  $z = ir_0^{\theta_2}$ .

For each event, one first evaluates the generating function for  $\epsilon = \theta_1 = 0$ , as well as its derivative with respect to  $z$ :

$$\frac{\partial g_0^{0, \theta_2}}{\partial z}(ir_0^{\theta_2}) = g_0^{0, \theta_2}(ir_0^{\theta_2}) \sum_{j=1}^M \frac{w_2(j) \cos(2(\phi_j - \theta_2))}{1 + ir_0^{\theta_2} w_2(j) \cos(2(\phi_j - \theta_2))}. \quad (5)$$

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<sup>1</sup>  $\Phi_R$  is the *exact* reaction plane, not an estimated plane;  $\Phi_R$  is unknown on an event-by-event basis.

One then evaluates  $g_\epsilon^{\theta_1, \theta_2}(ir_0^{\theta_2})$  for a single non-zero value of  $\epsilon$ , which is arbitrary but must satisfy the following condition

$$\epsilon \ll \frac{V_2}{V_1}. \quad (6)$$

Since  $V_1$  is a priori unknown, one must guess a reasonable value and check afterwards that the condition is satisfied. If  $\epsilon$  is too small, numerical errors may arise as the procedure amounts to expanding numerically  $g_\epsilon^{\theta_1, \theta_2}(z)$  to order  $\epsilon^2$ . Thus we recommend to perform tests with several values of  $\epsilon$ , and to check the stability of the results, before doing the full analysis. The angle  $\theta_1$  takes 5 (or more) equally spaced values of  $\theta_1$  between 0 and  $2\pi$ , (i.e.  $\theta_1 = 0, 2\pi/n, \dots, 2(n-1)\pi/n$  with  $n \geq 5$ ). Note that the range differs from that of  $\theta_2$  values.

Our estimate of  $V_1$  is defined by

$$(V_1)^2 \text{sgn}(V_2) = \left\langle -\frac{8j_{01}}{\epsilon^2} (r_0^{\theta_2})^{-3} \text{Re} \left( \frac{\left\langle \cos(2(\theta_1 - \theta_2)) g_\epsilon^{\theta_1, \theta_2}(ir_0^{\theta_2}) \right\rangle_{\theta_1, \text{evts}}}{\left\langle \frac{\partial g_0^{0, \theta_2}}{\partial z}(ir_0^{\theta_2}) \right\rangle_{\text{evts}}} \right) \right\rangle_{\theta_2}, \quad (7)$$

where  $\epsilon$  is an arbitrary small number satisfying condition (6),  $\langle \dots \rangle_{\theta_1, \text{evts}}$  denotes an average over  $\theta_1$  and events, and  $\text{Re}$  is the real part of the ratio. Finally,  $\text{sgn}(V_2)$  denotes the sign of  $V_2$ , which is determined by this analysis of  $V_1$ . The sign of  $V_1$ , on the other hand, is not measured and must be postulated, as in any other method of analysis.

As in Sec. II A, the averaging over  $\theta_2$  in Eqs. (7) and (9) is only for practical purposes: it reduces statistical errors by a factor of  $\sim 2$ . In contrast, the averaging over  $\theta_1$  cannot be avoided, for reasons we shall explain below in Sec. III.

### C. Differential directed flow

One can then turn to the analysis of differential flow, i.e., the flow of particles of a given type in a definite phase-space window, which we shall call ‘‘protons’’ for the sake of brevity. A ‘‘proton’’ azimuth will be denoted by  $\psi$ , and the corresponding differential directed flow by  $v'_1$ .

The estimate of  $v'_1$  involves the derivative of the generating function with respect to the proton weight, evaluated at  $z = ir_0^{\theta_2}$ :

$$\frac{\partial g_\epsilon^{\theta_1, \theta_2}(ir_0^{\theta_2})}{\partial w_1(\psi)} = g_\epsilon^{\theta_1, \theta_2}(ir_0^{\theta_2}) \frac{ir_0^{\theta_2} \epsilon \cos(\psi - \theta_1)}{1 + ir_0^{\theta_2} \epsilon w_1(\psi) \cos(\psi - \theta_1) + ir_0^{\theta_2} w_2(\psi) \cos(2(\psi - \theta_2))}, \quad (8)$$

and is defined by

$$v'_1 = \left\langle -\frac{4j_{01} \text{sgn}(V_2)}{V_1 \epsilon^2} (r_0^{\theta_2})^{-3} \text{Re} \left( \frac{\left\langle \cos(2(\theta_1 - \theta_2)) \frac{\partial g_\epsilon^{\theta_1, \theta_2}(ir_0^{\theta_2})}{\partial w_1(\psi)} \right\rangle_{\theta_1, \psi}}{\left\langle \frac{\partial g_0^{0, \theta_2}}{\partial z}(ir_0^{\theta_2}) \right\rangle_{\text{evts}}} \right) \right\rangle_{\theta_2}, \quad (9)$$

where  $\langle \dots \rangle_{\theta_1, \psi}$  denotes an average over protons and over values of  $\theta_1$ . Note that the value of integrated directed flow  $V_1$  is only required after averaging over protons or events, which means that the ratio between brackets can be computed at the same time as that in Eq. (7). Moreover, the denominator of this ratio is the same as in Eq. (7), so that in the second pass through data, one only need compute three quantities for each event: this common denominator, the numerator of Eq. (7) (actually, a value for each angle  $\theta_1$ ), and the numerator of Eq. (9), taking into account only the ‘‘protons.’’

### D. Relation with the three-particle method of Ref. [6]

The three-particle method we proposed in Ref. [6] is based on averages of the type  $\langle \cos(\phi_1 + \phi_2 - 2\phi_3) \rangle$ , where  $\phi_1$ ,  $\phi_2$  and  $\phi_3$  are the azimuthal angles of three particles belonging to the same event. Let us explain how such three-particle averages appear in the present method: expanding  $g_\epsilon^{\theta_1, \theta_2}(z)$  in Eq. (1) to order  $\epsilon^2 z^3$  produces terms of

the type  $\cos(\phi_1 - \theta_1) \cos(\phi_2 - \theta_1) \cos(2(\phi_3 - \theta_2))$ . Multiplying by  $\cos(2(\theta_1 - \theta_2))$  and averaging over  $\theta_1$  and  $\theta_2$ , as in Eqs. (7) and (9), one obtains  $\frac{1}{8} \cos(\phi_1 + \phi_2 - 2\phi_3)$ , thus recovering the three-particle averages. The present method uses an expansion to order  $\epsilon^2$  (see Sec. III), but for a fixed value of  $z$ , namely  $ir_0^{\theta_2}$ : it therefore also includes the information from higher-order correlations, generated by higher powers of  $z$ , not only three-particle correlations.

### III. THEORY

The general philosophy is that zeroes of generating functions are direct probes of collective effects [13], while they are little sensitive to nonflow effects. The key property of the generating function in Eq. (1) is its factorization property: if a nucleus-nucleus collision can be viewed as the superposition of independent subsystems containing a few particles (for instance, independent nucleon-nucleon collisions), i.e., if there are no collective effects, the average over events  $\langle g_\epsilon^{\theta_1, \theta_2}(z) \rangle_{\text{evts}}$  is the product of the contributions of individual subsystems. The zeroes of the generating function are then simply the zeroes of the contributions of the subsystems. Since the value of the generating function is 1 for  $z = 0$  by construction, the value of  $z$  for which it vanishes verifies  $|z| \sim 1$  for dimensional reasons (in this discussion, we assume unit weights  $w_1 = w_2 = 1$  for simplicity).

If, on the other hand, there is collective flow, nothing prevents the zeroes from moving much closer to the origin,  $|z| \ll 1$ , and this is indeed what happens: the positions of the zeroes scale with the event multiplicity like  $1/M$  (while they would be independent of  $M$  for independent subsystems). This relates our method to the Lee–Yang theory of phase transitions [16], where the zeroes of the grand partition function move closer and closer to the real axis as the size of the system increases when, and only when, there is a phase transition.

Let us now be more quantitative. We first evaluate the average value of  $g_\epsilon^{\theta_1, \theta_2}(z)$  for events having exactly the same reaction-plane orientation  $\Phi_R$  (denoting such an average by  $\langle \dots | \Phi_R \rangle$ ). Taking the logarithm of Eq. (1) and expanding to order  $z$ , we obtain

$$\begin{aligned} \ln \langle g_\epsilon^{\theta_1, \theta_2}(z) | \Phi_R \rangle &\simeq z\epsilon \left\langle \sum_{j=1}^M w_1(j) \cos(\phi_j - \theta_1) | \Phi_R \right\rangle + z \left\langle \sum_{j=1}^M w_2(j) \cos(2(\phi_j - \theta_2)) | \Phi_R \right\rangle \\ &\simeq z\epsilon V_1 \cos(\Phi_R - \theta_1) + zV_2 \cos(2(\Phi_R - \theta_2)), \end{aligned} \quad (10)$$

where we have used Eqs. (2) and (3) and assumed symmetry with respect to the reaction plane, which implies  $\langle \sum_j w_2(j) \sin(2(\phi_j - \Phi_R)) \rangle_{\text{evts}} = \langle \sum_j w_1(j) \sin(\phi_j - \Phi_R) \rangle_{\text{evts}} = 0$ . Terms of order  $z^2$  and higher in the expansion are responsible for systematic errors that will be estimated in Sec. IV A. The average over events is eventually obtained by averaging over  $\Phi_R$ , which is randomly distributed:

$$\langle g_\epsilon^{\theta_1, \theta_2}(z) \rangle_{\text{evts}} = \int_0^{2\pi} \langle g_\epsilon^{\theta_1, \theta_2}(z) | \Phi_R \rangle \frac{d\Phi_R}{2\pi}. \quad (11)$$

For  $\epsilon = 0$ , substituting the estimate (10) into Eq. (11), one obtains

$$\langle g_0^{0, \theta_2}(z) \rangle_{\text{evts}} = I_0(|V_2|z), \quad (12)$$

where  $I_0$  is the modified Bessel function of order 0. Its zeroes lie on the imaginary axis, the first one being located at  $ij_{01}$ . This explains how  $|V_2|$  is obtained in Sec. II A, Eq. (4). Since zeroes are expected to be on the imaginary axis except for irrelevant statistical fluctuations and/or detector effects [13], we suggest to look for the position of the first minimum of  $|\langle g_0^{0, \theta_2}(ir) \rangle_{\text{evts}}|$  on the upper imaginary axis, denoted by  $z = ir_0^{\theta_2}$  in Sec. II A, rather than that of the first zero of  $\langle g_0^{0, \theta_2}(z) \rangle_{\text{evts}}$  in the complex plane.

In order to obtain an estimate of  $V_1$ , we shall now study how the first zero of  $\langle g_\epsilon^{\theta_1, \theta_2}(z) \rangle_{\text{evts}}$ , which we denote by  $z_\epsilon$ , varies for small  $\epsilon$ . For small enough  $\epsilon$ , we may write

$$z_\epsilon - z_0 = - \frac{\langle g_\epsilon^{\theta_1, \theta_2}(z_0) \rangle_{\text{evts}}}{\left\langle \frac{\partial g_0^{0, \theta_2}}{\partial z}(z_0) \right\rangle_{\text{evts}}}, \quad (13)$$

where  $z_0$  is the value of  $z_\epsilon$  for  $\epsilon = 0$ . According to our general philosophy, the value of  $z_\epsilon$  is directly related to collective flow, and little sensitive to nonflow effects. Then, this is also true of the right-hand side (r.h.s.) of Eq. (13), which is experimentally measurable. Following the discussion above, we replace the exact zero  $z_0$  by  $ir_0^{\theta_2}$ . To relate

the quantity in Eq. (13) to anisotropic flow, we use our theoretical estimate of the generating function, defined by Eqs. (10) and (11). The denominator is simply obtained by differentiating Eq. (12):

$$\left\langle \frac{\partial g_0^{0,\theta_2}}{\partial z}(z_0) \right\rangle_{\text{evts}} = |V_2| I_1(|V_2|z_0). \quad (14)$$

The numerator is obtained by substituting Eq. (10) into (11), and expanding to order  $\epsilon^2$ :

$$\langle g_\epsilon^{\theta_1,\theta_2}(z_0) \rangle_{\text{evts}} = \frac{\epsilon^2}{4} z_0^2 I_1(V_2 z_0) V_1^2 \cos(2(\theta_1 - \theta_2)). \quad (15)$$

Since  $I_1$  is an odd function, we may write  $I_1(V_2 z_0) = I_1(|V_2|z_0) \text{sgn}(V_2)$ . Putting together the last two equations, replacing  $z_0$  with  $ir_0^{\theta_2}$  and  $|V_2|$  with its estimate  $j_{01}/r_0^{\theta_2}$  (see Eq. (4)), we obtain

$$V_1^2 \text{sgn}(V_2) \cos(2(\theta_1 - \theta_2)) = -\frac{4j_{01}}{\epsilon^2} \left(r_0^{\theta_2}\right)^{-3} \frac{\langle g_\epsilon^{\theta_1,\theta_2}(ir_0^{\theta_2}) \rangle_{\text{evts}}}{\left\langle \frac{\partial g_0^{0,\theta_2}}{\partial z}(ir_0^{\theta_2}) \right\rangle_{\text{evts}}}. \quad (16)$$

This yields in principle an estimate of  $V_1$  for each value of  $\theta_1$  and  $\theta_2$ . However, the expansion of  $\langle g_\epsilon^{\theta_1,\theta_2}(ir_0^{\theta_2}) \rangle_{\text{evts}}$  in powers of  $\epsilon$  generally yields a non-vanishing term of order  $\epsilon$  due to statistical fluctuations. Multiplying the previous equation by  $\cos(2(\theta_1 - \theta_2))$ , and averaging over 5 or more equally spaced values of  $\theta_1$ , one eliminates this term. Averaging over  $\theta_2$ , one obtains our final estimate, Eq. (7).

We can now turn to differential directed flow. For this purpose, let us study how the zero of  $g_\epsilon^{\theta_1,\theta_2}(z)$  moves when the weights  $w_1$  of all protons are shifted by some small quantity  $\delta w_1$ :

$$z_{\delta w} - z_0 = -\frac{\left\langle \frac{\partial g_\epsilon^{\theta_1,\theta_2}(z_0)}{\partial w_1(\psi)} \right\rangle_\psi}{\left\langle \frac{\partial g_0^{0,\theta_2}}{\partial z}(z_0) \right\rangle_{\text{evts}}}. \quad (17)$$

The value of  $z_{\delta w}$  is directly related to collective effects. Following the same procedure as for integrated directed flow, we estimate the value of the r.h.s. when flow is present. To evaluate the numerator, we first compute the average for a fixed  $\Phi_R$ . Using Eq. (8), assuming that the proton and the other particles are uncorrelated for fixed  $\Phi_R$  (i.e., that the correlation between the proton and the other particles is only due to flow), and neglecting the contribution of the proton to the generating function, we obtain

$$\begin{aligned} \left\langle \frac{\partial g_\epsilon^{\theta_1,\theta_2}(z_0)}{\partial w_1(\psi)} \Big| \Phi_R \right\rangle &\simeq \epsilon z_0 \langle \cos(\psi - \theta_1) | \Phi_R \rangle \langle g_\epsilon^{\theta_1,\theta_2}(z) | \Phi_R \rangle \\ &= \epsilon z_0 v'_1 \cos(\Phi_R - \theta_1) \exp[z\epsilon V_1 \cos(\Phi_R - \theta_1) + zV_2 \cos(2(\Phi_R - \theta_2))], \end{aligned} \quad (18)$$

where we have used Eq. (10), the definition of  $v'_1 = \langle \cos(\psi - \Phi_R) \rangle$ , and assumed symmetry with respect to the reaction plane,  $\langle \sin(\psi - \Phi_R) \rangle = 0$ . We then expand the exponential to order  $\epsilon$  and integrate over  $\Phi_R$ :

$$\left\langle \frac{\partial g_\epsilon^{\theta_1,\theta_2}(z_0)}{\partial w_1(\psi)} \right\rangle = \frac{\epsilon^2}{2} z_0^2 v'_1 V_1 I_1(V_2 z_0) \cos(2(\theta_1 - \theta_2)). \quad (19)$$

Using Eq. (14), and replacing  $z_0$  with  $ir_0^{\theta_2}$ , we obtain

$$v'_1 V_1 \text{sgn}(V_2) \cos(2(\theta_1 - \theta_2)) = -\frac{2j_{01}}{\epsilon^2} \left(r_0^{\theta_2}\right)^{-3} \frac{\left\langle \frac{\partial g_\epsilon^{\theta_1,\theta_2}(ir_0^{\theta_2})}{\partial w_1(\psi)} \right\rangle_\psi}{\left\langle \frac{\partial g_0^{0,\theta_2}}{\partial z}(ir_0^{\theta_2}) \right\rangle_{\text{evts}}}. \quad (20)$$

Multiplying both sides of this equation by  $\cos(2(\theta_1 - \theta_2))$  and averaging over  $\theta_1$ , one eventually obtains Eq. (9).

For the sake of consistency, one must recover the integrated flow  $V_1$  by integrating the differential flow  $v'_1$  over phase space. Our estimates (7) and (9) satisfy this sum rule, provided  $\epsilon$  is small enough. In order to prove this, we weight Eq. (9) with  $w_1(\psi)$  and integrate over phase space. The following quantity appears

$$\sum_{j=1}^M w_1(j) \frac{\partial g_\epsilon^{\theta_1,\theta_2}(ir_0^{\theta_2})}{\partial w_1(j)} = \epsilon \frac{\partial g_\epsilon^{\theta_1,\theta_2}(ir_0^{\theta_2})}{\partial \epsilon}. \quad (21)$$

Next, we use the fact that  $g_\epsilon^{\theta_1, \theta_2}(ir_0^{\theta_2})$  scales like  $\epsilon^2$  for small  $\epsilon$ , hence

$$\epsilon \frac{\partial g_\epsilon^{\theta_1, \theta_2}(ir_0^{\theta_2})}{\partial \epsilon} = 2g_\epsilon^{\theta_1, \theta_2}(ir_0^{\theta_2}). \quad (22)$$

This completes the proof.

#### IV. ERRORS

The main motivation for analyzing collective flow with multiparticle correlations is that one thereby reduces spurious, “nonflow” effects. The magnitude of errors due to residual nonflow effects is estimated in Sec. IV A. The price to pay for this greater reliability is an increase in statistical errors, which are evaluated in Sec. IV B. Finally, azimuthal asymmetries in the detector acceptance are a potential source of bias in the analysis. They are carefully studied in Sec. IV C.

##### A. Systematic errors from nonflow effects

There are several tricks to check experimentally whether or not the analysis is biased by nonflow effects. A first one is to perform the analysis twice, using different weights in Eq. (1), e.g., one analysis with  $w_2 = p_t$  and another with  $w_2 = 1$ , or redoing the analysis with zero weights for the particles in one hemisphere. The final differential flow results should be independent of the weights, while no such independence is expected for nonflow effects. However, using non-optimal weights increases statistical errors, sometimes by large amounts, so that this is doable in practice only if statistical errors are not a limitation. A second trick is to check that the final results look “reasonable:” this means in particular that they must be consistent with the known symmetries of the system, i.e.,  $v_1(-y) = -v_1(y)$ ,  $v_2(-y) = v_2(y)$  for a symmetric collision. Nonflow effects can destroy these symmetries: for instance, effects of total momentum conservation may contaminate the measurement of  $v_1$ , yielding an estimate that does not vanish near midrapidity [5]. This may however occur only when the weights in Eq. (1) are not symmetric themselves (i.e.,  $w_1(-y) \neq -w_1(y)$ ,  $w_2(-y) \neq w_2(y)$ ). With symmetric weights, the fact that  $v_1$  vanishes at midrapidity is not an indication that it is not biased by nonflow effects.

In this paper, we are interested in directed flow at ultrarelativistic energies, which is most often at the border of observability, so that it is not always possible to use the above tricks. It is therefore important to derive estimates of the magnitude of nonflow effects using purely theoretical arguments. There is of course no way to derive quantitative estimates, due to the variety of the physical effects involved [3]. In most cases, one can at best rely on orders of magnitude and simple scaling rules [12], which are derived below. More quantitative statements can be made concerning momentum conservation, a well-known bias in analyses of directed flow [5, 17, 18].

With standard methods of flow analysis [1, 2] and also, to a lesser degree, with cumulants [12], the analysis may yield a non-zero value of  $v_1$  and  $v_2$  even when they are in fact zero, i.e., even if only “nonflow” effects are present [19]. This is not the case with Lee–Yang zeroes, where a non-vanishing result beyond statistical errors can be considered a clear signal of anisotropic flow.

However, the interference of collective flow and nonflow effects may produce a small relative error, which is the sum of two terms:

$$\frac{(\delta v_1)_{\text{nonflow}}}{v_1} = \mathcal{O}\left(\frac{v_2}{M v_1^2}\right) + \mathcal{O}\left(\frac{1}{M v_2}\right). \quad (23)$$

Let us explain how such terms arise. Our determination of directed flow with the present method involves three-particle averages such as  $\langle \cos(\phi_1 + \phi_2 - 2\phi_3) \rangle$  (see Sec. II D).<sup>2</sup> The contribution of flow to this average is  $v_1^2 v_2$ . Let us now evaluate the contribution of nonflow effects. As a simple model, assume that the  $M$  particles detected in an event are emitted in  $M/2$  collinear pairs (“jet-like correlations”). There is a probability  $1/(M-1) \simeq 1/M$  that  $\phi_2 = \phi_1$ , in which case the three-particle average becomes  $\langle \cos 2(\phi_1 - \phi_3) \rangle = v_2^2$ . This gives the first term in Eq. (23). Similarly, there is a probability of order  $1/M$  that  $\phi_1 = \phi_3$ , in which case the three-particle average reads  $\langle \cos(\phi_1 - \phi_2) \rangle = v_1^2$ ; this yields the second term in Eq. (23). Since the present method is expected to be useful essentially when  $v_1 < v_2$ ,

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<sup>2</sup> One can check that the systematic errors due to higher-order correlations are at most of the same order as those due to three-particle correlations.

this term is subleading. Finally, one can check explicitly that momentum conservation does not contribute to the error terms in Eq. (23).

These orders of magnitude can be recovered by using toy models of nonflow effects, as e.g. the “jet-like correlations” above, pushing the expansion of the logarithm of the generating function, Eq. (10), to order  $z^2$ , and studying the resulting flow estimates.

With the standard, event-plane method [1, 2], the error due to nonflow effects is much larger:

$$\frac{(\delta v_1)_{\text{nonflow}}}{v_1} = \mathcal{O}\left(\frac{1}{M v_1^2}\right). \quad (24)$$

Comparing with Eq. (23), our method reduces the nonflow error by a factor of  $1/v_2$ , i.e., typically 15 to 20 at RHIC.

The errors with the present method, Eq. (23), also apply to the three-particle cumulant method, but they were not mentioned in Ref. [6]. The cumulant method has an additional error term, coming from the “pure nonflow” contribution to the three-particle average  $\langle \cos(\phi_1 + \phi_2 - 2\phi_3) \rangle$ , of order  $1/M^2$ . This gives an additional error term in Eq. (23)

$$\frac{(\delta v_1)_{\text{nonflow}}}{v_1} = \mathcal{O}\left(\frac{1}{M^2 v_2 v_1^2}\right). \quad (25)$$

This contribution, which was the only error term mentioned in Ref. [6], disappears with Lee–Yang zeroes. In practice at ultrarelativistic energies  $M v_2^2 \gtrsim 1$ , so that the additional term is typically of the same order of magnitude as the first term of Eq. (23). In addition, momentum conservation produces an error of this order, so that the Lee–Yang zeroes method does a better job in eliminating effects of momentum conservation. Finally, the three-particle method requires to estimate  $v_2$  independently. If  $v_2$  is estimated from two-particle cumulants or by the event-plane method, the resulting error on  $v_1$  is  $(\delta v_1)_{\text{nonflow}}/v_1 = \mathcal{O}(1/M v_2^2)$ , which may dominate over both terms in Eq. (23).

## B. Statistical errors

Statistical uncertainties on  $V_1$  and  $v_1'$  naturally involve both directed and elliptic flows. Both errors turn out to scale like the statistical error on the differential elliptic flow  $v_2'$  determined from Lee–Yang zeroes [13], which is natural since we use elliptic flow as a reference. The statistical error on elliptic flow involves the associated *resolution parameter*

$$\chi_2 \equiv \frac{V_2}{\sigma_2}, \quad (26)$$

where  $\sigma_2$  is a measure of event-by-event fluctuations. Provided nonflow effects are not too large,  $\sigma_2$  can be obtained experimentally by the following formula:

$$\sigma_2^2 \simeq \left\langle \sum_{j=1}^M w_2(j)^2 \right\rangle_{\text{evts}}. \quad (27)$$

More accurate determinations are discussed in Ref. [13]. Once  $\chi_2$  has been determined, the statistical error on  $v_2'$  is [13]:

$$(\delta v_2')^2 = \frac{1}{4N' J_1(j_{01})^2} \left\langle \exp\left(\frac{j_{01}^2}{2\chi_2^2} \cos 2\theta_2\right) J_0(2j_{01} \sin \theta_2) - \exp\left(-\frac{j_{01}^2}{2\chi_2^2} \cos 2\theta_2\right) J_0(2j_{01} \cos \theta_2) \right\rangle_{\theta_2}, \quad (28)$$

where  $N'$  is the number of “protons” in the phase-space region under study. For large  $\chi_2$ , this yields  $\delta v_2' = 1/\sqrt{2N'}$ , which is the expected statistical error when the reaction plane is exactly known. When  $\chi_2$  becomes significantly smaller than unity, on the other hand, the statistical error on  $v_2'$  increases exponentially as  $\delta v_2' \propto \exp(2.9/\chi_2^2)$ , and the method cannot be applied [13].

Let us now discuss statistical errors on the directed flow estimates. These errors involve two parameters  $\sigma_1$  and  $\chi_1$  defined as in Eqs. (26) and (27), namely,  $\sigma_1^2 \simeq \langle \sum_j w_j^2 \rangle_{\text{evts}}$  and  $\chi_1 = V_1/\sigma_1$ . The latter is the resolution parameter associated with directed flow.<sup>3</sup> In order to obtain the square of the statistical error on  $V_1^2 \text{sgn}(V_2)$ , one then does the

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<sup>3</sup> Please note that the resolution parameter  $\chi$  as defined in Ref. [2] is larger by a factor of  $\sqrt{2}$

following substitution in Eq. (28):

$$\frac{1}{N'} \rightarrow \frac{2\sigma_1^4(1+\chi_1^2)}{N_{\text{evts}}}. \quad (29)$$

For large  $\chi_1$  and  $\chi_2$ , this yields  $\delta V_1 = \sigma_1/\sqrt{2N_{\text{evts}}}$ , which is the expected value of the statistical error if the reaction plane is exactly known.

Finally, the statistical error on the differential directed flow  $v'_1$  is simply related to the error on  $v'_2$ :

$$(\delta v'_1)^2 = \frac{1+\chi_1^2}{\chi_1^2} (\delta v'_2)^2. \quad (30)$$

One notes that the statistical error on directed flow obtained with this method is always larger than the statistical error on elliptic flow in absolute value. When  $\chi_2$  is large enough,  $\delta v'_1$  is the same as with the reaction plane analysis [2].

When using 3-particle cumulants [6, 9], the statistical error is given by a similar formula, where  $\delta v'_2$  is the error on elliptic flow obtained with the reaction plane method (or by 2-particle correlations). Going from the 3-particle method to the present method, the increase in statistical errors will be the same as when going from the standard method to Lee–Yang zeroes in the analysis of elliptic flow. For a semi-central Au–Au collision at RHIC analyzed with the STAR detector, the resolution parameter  $\chi_2$  is about 1, which means a factor of  $\simeq 2$  increase in statistical errors [13]. This is compensated by the smaller error from nonflow effects.

### C. Azimuthal asymmetries in the detector acceptance

A nice feature of all flow analyses based on cumulants or Lee–Yang zeroes (which amount to cumulants of many-particle correlations [13]) is that they automatically eliminate most effects of azimuthal asymmetries in the detector acceptance: more precisely, such asymmetries cannot produce a signal by themselves, even if the detector has a very partial azimuthal coverage. This is not the case with the reaction plane analysis, where several flattening procedures are required [2], and with two-particle correlation methods [20, 21], which use mixed events to correct for acceptance effects.

Acceptance asymmetries generally have two effects with cumulants or Lee–Yang zeroes [12, 13]: 1) the flow given by the analysis differs from the true flow by some factor, which can be computed analytically once the acceptance profile is known; 2) different harmonics interfere, so that a measurement of  $v_1$  is generally biased by  $v_2$  and vice-versa. These interference terms can also be calculated analytically.

Such interference terms between  $v_1$  and  $v_2$  are also present here, and they turn out to be a rather serious problem. Indeed, as an explicit calculation will show, acceptance asymmetries produce an additional term proportional to  $v_2$  in the left-hand side (l.h.s.) of Eq. (7): even if there is only elliptic flow in the system, the analysis then yields a spurious directed flow  $v_1 \propto \sqrt{v_2}$ . Since we are typically interested in a situation where  $v_2$  is large and  $v_1$  is small, such an interference is most unwelcome, and it must be thoroughly studied.

In order to focus on the interference with  $v_2$ , we assume that only elliptic flow is present in the system,  $v_1 = 0$ , and we evaluate the spurious directed flow. The following study is limited to the integrated flow  $V_1$ . For a given reaction plane  $\Phi_R$ , the azimuthal distribution of a particle of type  $j$  is, before detection,

$$\frac{dN}{d\phi_j} \propto 1 + 2v_2(j) \cos(2(\phi_j - \Phi_R)). \quad (31)$$

With this distribution, and with a non-symmetric detector, the first term in Eq. (10) becomes

$$\left\langle \sum_{j=1}^M w_1(j) \cos(\phi_j - \theta_1) | \Phi_R \right\rangle = \text{Re} \left[ V_0^{\text{acc}} e^{-i\theta_1} + V_2^{\text{acc}} e^{i(2\Phi_R - \theta_1)} + V_2'^{\text{acc}} e^{i(-2\Phi_R - \theta_1)} \right], \quad (32)$$

where  $V_0^{\text{acc}}$ ,  $V_2^{\text{acc}}$  and  $V_2'^{\text{acc}}$  are complex coefficients defined by

$$V_0^{\text{acc}} = \left\langle \sum_{j=1}^M w_1(j) e^{i\phi_j} \right\rangle$$

$$V_2^{\text{acc}} = \left\langle \sum_{j=1}^M w_1(j) v_2(j) e^{-i\phi_j} \right\rangle$$



$$V_2'^{\text{acc}} = \left\langle \sum_{j=1}^M w_1(j) v_2(j) e^{3i\phi_j} \right\rangle, \quad (33)$$

with the averages taken over detected particles. For a symmetric acceptance, the three coefficients vanish. With these values, one can compute the generating function  $\langle g_\epsilon^{\theta_1, \theta_2}(z) \rangle$  to order  $\epsilon^2$  following essentially the same steps as in Sec. III. Let us give the final result, skipping tedious intermediate calculations: to leading order in acceptance asymmetries, one must replace  $(V_1)^2$  with  $\text{Re}[2V_0^{\text{acc}}V_2^{\text{acc}}]$  in the l.h.s. of Eq. (7).

For moderate inhomogeneities in the detector acceptance, the interference of elliptic flow on the estimate of directed flow remains under control. Thus, assuming an elliptic flow value  $v_2 = 0.06$ , variations of 20% in the detector efficiency will result in a spurious  $v_1$  of at most 0.01. Using appropriate weights (e.g., weighting with the inverse of the detector efficiency profile) would significantly decrease this spurious directed flow value. However, we fear that working with a detector with incomplete azimuthal coverage might prove prohibitive.

Finally, note that the three-particle method is affected by acceptance asymmetries in essentially the same way, but the corresponding interference term was unfortunately omitted in Ref. [6].

## V. CONCLUSIONS AND PERSPECTIVES

In the foregoing, we have introduced a new method to analyze directed flow  $v_1$  in cases where it is too small to be reliably obtained with standard two-particle methods without contamination from nonflow effects, and also too small to be determined by more recent methods as cumulants or Lee–Yang zeroes, which would both give large statistical uncertainties. Moreover, the method relies on the implicit assumption that elliptic flow is reasonably large, as when using three-particle cumulants [6]. With respect to the latter, the new method is easier to implement: instead of interpolating successive derivatives of a function, one need just determine the position  $r_0$  of the first minimum of a function, and then compute the values at  $r_0$  of the three quantities that appear in the numerators and denominator of Eqs. (7) and (9). In addition, the method is conceptually more elegant, since it relies on the deep relation there exists between the behaviour of the zeroes of generating functions and the presence of collective effects in the systems these functions describe [13].

We have also carefully estimated both systematic errors due to nonflow effects and statistical uncertainties arising from finite available statistics. We may use these estimates to discuss the applicability of the method to present and future heavy-ion experiments. At RHIC, the three-particle cumulant method was already successfully applied [9]. In mid-central collisions, statistical uncertainties should be about a factor 2 larger with the new method, but with the high statistics of Run 4, this should not be a problem. On the other hand, systematic errors will be reduced: the error term (25), which arises in particular from momentum conservation, disappears with the new method. The method could be useful for new measurements of directed flow at the CERN SPS, although this would require detectors with a larger coverage than what single experiments had in the past runs.

Finally, one can expect that the present method will allow measurements of directed flow at LHC, if any, using muons from decay pions seen in the ALICE spectrometer at forward rapidities ( $2.5 < \eta < 4$ ) or hits in the CMS very forward hadronic calorimeter ( $3 < \eta < 5$ ). Since  $v_2$  at LHC is expected to be at least as large as at RHIC, and the multiplicity will be higher, elliptic flow will be analyzed with an excellent resolution. Then, the statistical errors with our new method will be barely larger than with the standard reaction plane method: statistics will not be a limitation down to values of  $v_1$  of a fraction of a percent. Systematic errors from nonflow effects may be a more severe problem, although our method minimizes their magnitude. Comparing Eq. (30), with  $\chi_1 \sim v_1\sqrt{M}$ , and the first term of Eq. (23), one sees that as  $v_1$  decreases, systematic errors tend to become larger than statistical errors. One may reasonably hope, however, that the huge particle multiplicity  $M$  expected at LHC will compensate for the smaller value of  $v_1$ , and that  $v_1$  will eventually be observed.

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- [1] P. Danielewicz and G. Odyniec, Phys. Lett. B **157**, 146 (1985).
  - [2] A. M. Poskanzer and S. A. Voloshin, Phys. Rev. C **58**, 1671 (1998) [nucl-ex/9805001].
  - [3] N. Borghini, P. M. Dinh and J.-Y. Ollitrault, Phys. Rev. C **62**, 034902 (2000) [nucl-th/0004026].
  - [4] J.-Y. Ollitrault, Nucl. Phys. A **590**, 561c (1995).
  - [5] N. Borghini, P. M. Dinh, J.-Y. Ollitrault, A. M. Poskanzer and S. A. Voloshin, Phys. Rev. C **66**, 014901 (2002) [nucl-th/0202013].
  - [6] N. Borghini, P. M. Dinh and J.-Y. Ollitrault, Phys. Rev. C **66**, 014905 (2002) [nucl-th/0204017].
  - [7] N. Borghini, P. M. Dinh and J.-Y. Ollitrault, Nucl. Phys. A **715**, 629c (2003) [nucl-th/0208014].
  - [8] C. Alt *et al.* [NA49 Collaboration], Phys. Rev. C **68**, 034903 (2003) [nucl-ex/0303001].

- [9] J. Adams *et al.* [STAR Collaboration], Phys. Rev. Lett. **92**, 062301 (2004) [nucl-ex/0310029].
- [10] R. J. M. Snellings, H. Sorge, S. A. Voloshin, F. Q. Wang and N. Xu, Phys. Rev. Lett. **84**, 2803 (2000) [nucl-ex/9908001]; L. P. Csernai and D. Röhrich, Phys. Lett. B **458**, 454 (1999) [nucl-th/9908034]; M. Bleicher and H. Stöcker, Phys. Lett. B **526**, 309 (2002) [hep-ph/0006147].
- [11] J.-Y. Ollitrault, Phys. Rev. D **46**, 229 (1992).
- [12] N. Borghini, P. M. Dinh and J.-Y. Ollitrault, Phys. Rev. C **63**, 054906 (2001) [nucl-th/0007063]; Phys. Rev. C **64**, 054901 (2001) [nucl-th/0105040].
- [13] R. S. Bhalerao, N. Borghini and J.-Y. Ollitrault, Phys. Lett. B **580**, 157 (2004) [nucl-th/0307018]; Nucl. Phys. A **727**, 373 (2003) [nucl-th/0310016].
- [14] N. Borghini, R. S. Bhalerao and J.-Y. Ollitrault, nucl-th/0402053.
- [15] P. Danielewicz, Phys. Rev. **C51**, 716 (1995).
- [16] C. N. Yang and T. D. Lee, Phys. Rev. **87**, 404 (1952).
- [17] P. Danielewicz *et al.*, Phys. Rev. C **38**, 120 (1988).
- [18] C. A. Ogilvie *et al.*, Phys. Rev. C **40**, 2592 (1989).
- [19] Y. V. Kovchegov and K. L. Tuchin, Nucl. Phys. A **708**, 413 (2002) [hep-ph/0203213]; Nucl. Phys. A **717**, 249 (2003) [nucl-th/0207037].
- [20] S. Wang *et al.*, Phys. Rev. C **44**, 1091 (1991).
- [21] K. Adcox *et al.* [PHENIX Collaboration], Phys. Rev. Lett. **89**, 212301 (2002) [nucl-ex/0204005]; N. N. Ajitanand [PHENIX Collaboration], Nucl. Phys. A **715**, 765c (2003) [nucl-ex/0210007].