Preprint SPhT-T04/020

Genus one contribution to free energy in hermitian two-matrix model

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Abstract. We compute an the genus 1 correction to free energy of Hermitian two-matrix model in terms of theta-functions associated to spectral curve arising in large N limit. We discuss the relationship of this expression to isomonodromic tau-function, Bergmann tau-function on Hurwitz spaces, G-function of Frobenius manifolds and determinant of Laplacian in a singular metric over spectral curve.

1 Two-matrix models: introduction

In this paper we study the partition function of multi-cut two-matrix model:

$$Z_N \equiv e^{-N^2 F} := \int dM_1 dM_2 e^{-N\mathrm{tr}\{V_1(M_1)+V_2(M_2)-M_1M_2\}} \tag{1.1}$$

where the integral is taken over all independent entries of two hermitian matrices M_1 and M_2 such that the eigenvalues of M_1 are concentrated over a finite set of intervals (cuts) with given filling fractions. This integral is to be understood as a formal asymptotic series in N and in the coefficients of the two potentials V_1 and V_2. As a formal series, the questions of convergence of the matrix integral is irrelevant, and the model can be extended to matrices with eigenvalues constrained on contours in the complex plane.

Such asymptotic series play an important role in physics, as generating functions of statistical physics on random discretized polygonal surfaces, i.e. a simplified model of euclidean 2D quantum gravity [3, 6]. The large N expansion $F = \sum_{G=0}^{\infty} N^{-2G} F^G$ (N is the matrix size), called topological expansion, is one of the cornerstones of the theory, since F^G has the meaning of generating function for random discretized polygonal surfaces of genus G. Double scaling limits of these models correspond to statistical physics models on continuous surfaces, with conformal invariance properties. Matrix models thus provide realizations of minimal (p, q) conformal models. The 1-matrix model was shown to correspond to pure gravity (i.e. $q = 2$), and the 2-matrix model was introduced as it produces all (p, q) minimal models.

Recently, the interest in large N matrix models was renewed as it was understood [27], that the large N free energy of matrix models is the low energy effective action for some string theories. The computation of $1/N^2$ expansion for both one-matrix and two-matrix models is based on the loop equations, which was first derived for 1-matrix 1-cut in [8], then for 1-matrix 2-cuts in [9, 10], and recently derived in [12, 13, 11] for 1-matrix model multicut, and in [14, 15] for two-matrix model 1 and 2-cuts. Here, we will extend the results of [14, 15] for an arbitrary number of cuts i.e. for an arbitrary (up to maximal) genus of the spectral curve.

Writing down polynomials V_1 and V_2 in the form

$$V_1(x) = \sum_{k=1}^{d_1+1} \frac{u_k}{k} x^k \ , \qquad V_2(y) = \sum_{k=1}^{d_2+1} \frac{v_k}{k} y^k \ , \tag{1.2}$$

we shall use the following standard notations for operators of differentiation with respect to their coefficients:

$$\frac{\delta}{\delta V_1(x)}\Big|_x := \sum_{k=1}^{d_1+1} x^{-k-1} k \partial_{u_k} \ , \qquad \frac{\delta}{\delta V_2(y)}\Big|_y := \sum_{k=1}^{d_2+1} y^{-k-1} k \partial_{v_k} \ . \tag{1.3}$$

These notations will be used below to shorten some of the formulas; by definition the equality

$$\frac{\delta F}{\delta V_1(x)}\Big|_x = H(x) \quad \text{means that} \quad \frac{\partial F}{\partial u_k} = \frac{1}{2\pi i k} \oint_{x=\infty} x^k H(x) dx \ , \quad k = 1, \ldots, d_1+1 \ ; \tag{1.4}$$

a detailed discussion of this notation is contained in [17]. In fact, formally it is much more convenient not to cut the functions V_1 and V_2 to polynomials, but instead consider the Laurent series

$$V_1(x) = \sum_{k=1}^{\infty} \frac{u_k}{k} x^k \ , \qquad V_2(y) = \sum_{k=1}^{\infty} \frac{v_k}{k} y^k \ . \tag{1.5}$$

In this case we have the formal relations

$$\frac{\delta V_1(x)}{\delta V_1(\tilde{x})} = \frac{1}{\tilde{x} - x} \ , \qquad \frac{\delta V_1{}'(x)}{\delta V_1(\tilde{x})} = \frac{1}{(\tilde{x} - x)^2} \ , \tag{1.6}$$

which are implicitly used in the derivation of loop equation. However, the convergency problem with considering all coefficients in the infinite sums (1.5) to be independent variables forces us to understand all relations involving the operators $\delta/\delta V_1(x)$ and $\delta/\delta V_2(y)$ in the sense of (1.4).

Consider the resolvents (also understood as formal power series)

$$\mathcal{W}(x) = \frac{1}{N}\Big\langle \mathrm{tr}\frac{1}{x - M_1}\Big\rangle \qquad \text{and} \qquad \tilde{\mathcal{W}}(y) = \frac{1}{N}\Big\langle \mathrm{tr}\frac{1}{y - M_2}\Big\rangle \ . \tag{1.7}$$

As a corollary of (1.6), the free energy of two-matrix model (1.1) satisfies the following equations with respect to coefficients of polynomial V_1:

$$\frac{\delta F}{\delta V_1(x)} = \mathcal{W}(x) \ , \qquad \frac{\delta F}{\delta V_2(y)} = \tilde{\mathcal{W}}(y) \ , \tag{1.8}$$

valid in the sense of (1.4).

Assuming existence of $1/N^2$ expansion, the equations (1.8) were solved in [16] in the zeroth order in terms of holomorphic objects associated to the "spectral curve" which arises in $N \to \infty$ limit. The next coefficient F^1 was computed in [14] if the genus of spectral curve equals zero, and in [15] if the genus equals one. The main result of this paper is an expression for F^1 for an arbitrary genus of "spectral curve", which we find using loop equations. We compute F^1 using the algebro-geometric framework (the spectral curve and corresponding machinery) which arises already in zeroth order approximation.

The spectral curve is defined by the following equation:

$$\mathcal{E}^0(x,y) := (V_1{}'(x) - y)(V_2{}'(y) - x) - \mathcal{P}^0(x,y) + 1 = 0 \;, \tag{1.9}$$

where polynomial of two variables $\mathcal{P}^0(x,y)$ is the zeroth order term in $1/N^2$ expansion of polynomial

$$\mathcal{P}(x,y) := \frac{1}{N}\Big\langle \mathrm{tr}\frac{V_1{}'(x) - V_1{}'(M_1)}{x - M_1}\frac{V_2{}'(y) - V_2{}'(M_2)}{y - M_2}\Big\rangle \;; \tag{1.10}$$

the point P of this curve is the pair of complex numbers (x,y) satisfying (1.9).

The spectral curve (1.9) comes together with two meromorphic functions $f(P) = x$ and $g(P) = y$, which project it down to x and y-planes, respectively. These functions have poles only at two points of $\mathcal{L}$, called ∞_f and ∞_g: at ∞_f function $f(P)$ has simple pole, and function $g(P)$ - pole of order d_1 with singular part equal to $V_1{}'(f(P))$. At ∞_g the function $g(P)$ has simple pole, and function $f(P)$ - pole of order d_2 with singular part equal to $V_2{}'(g(P))$. Therefore, one gets the moduli space $\mathcal{M}$ of triples $(\mathcal{L}, f, g)$, where functions f and g have this pole structure. The natural coordinates on this moduli space are coefficients of polynomials V_1 and V_2 and g numbers, called "filling fractions" $\epsilon_\alpha = \frac{1}{2\pi i}\oint_{a_\alpha} g df$, where a_α are (chosen in some way) canonical cycles on $\mathcal{L}$.

Denote the zeros of differential df by $P_1, \dots, P_{m_1}$ ($m_1 = d_2 + 2g + 1$) (these points play the role of ramification points if we realize $\mathcal{L}$ as branched covering by function $f(P)$); their projections on f-plane are the branch points, which we denote we denote by $\lambda_j := f(P_j)$. The zeros of the differential dg (the ramification points if we consider $\mathcal{L}$ as covering defined by function $g(P)$) we denote by $Q_1, \dots, Q_{m_2}$ ($m_2 = d_1 + 2g + 1$); their projections on g-plane (the branch points) we denote by $\mu_j := g(Q_j)$. We shall assume that our potentials V_1 and V_2 are generic i.e. all zeros of differentials df and dg are simple and distinct.

If is well-known [16] how to express all standard algebro-geometrical objects on $\mathcal{L}$ in terms of the previous data. In particular, the Bergmann bidifferential $B(P,Q) = d_P d_Q \ln E(P,Q)$ ($E(P,Q)$ is the prime-form), can be represented as follows:

$$B(P,Q) = \frac{\delta g(P)}{\delta V_1(f(Q))}\Big|_{f(Q)} df(P) df(Q) \tag{1.11}$$

(see [16] for the proof). The Bergmann bidifferential has the following behaviour near diagonal $P \to Q$:

$$B(P,Q) = \left\{\frac{1}{(\tau(P) - \tau(Q))^2} + \frac{1}{6}S_B(P) + o(1)\right\} d\tau(P) d\tau(Q) \;, \tag{1.12}$$

where $\tau(P)$ is some local coordinate; $S_B(P)$ is the Bergmann projective connection ($S_B(P)$ transforms as quadratic differential under Möbius transformations; under an arbitrary coordinate transformation an appropriate Schwarzian derivative is added to it).

Consider also the four-differential $D(P,Q) = d_P d_Q^3 \ln E(P,Q)$, which has on the diagonal the pole of 4th degree: $D(P,Q) = \{6(\tau(P) - \tau(Q))^{-4} + O(1)\} d\tau(P)(d\tau(Q))^3$. From $B(P,Q)$ and $D(P,Q)$ it is easy to construct meromorphic normalized (all a-periods vanish) 1-forms on $\mathcal{L}$ with single pole; in particular, if the pole coincides with ramification point P_k, the natural local parameter near P_k is $x_k(P) = \sqrt{f(P) - \lambda_k}$, and the following objects:

$$B(P,P_k) := \frac{B(P,Q)}{dx_k(Q)}\Big|_{Q=P_k} \;, \qquad D(P,P_k) := \frac{D(P,Q)}{(dx_k(Q))^3}\Big|_{Q=P_k} \tag{1.13}$$

are meromorphic normalized 1-forms on $\mathcal{L}$ with single pole at P_k and the following singular parts:

$$B(P,P_k) = \left\{\frac{1}{x_k(P)^2} + \frac{1}{6}S_B(P_k) + o(1)\right\} dx_k(P)\ ; \qquad D(P,P_k) = \left\{\frac{6}{x_k(P)^4} + O(1)\right\} dx_k(P) \quad (1.14)$$

as $P \to P_k$, where $S_B(P_k)$ is the Bergmann projective connection computed at the branch point P_k with respect to the local parameter $x_k(P)$.

Equations (1.8) in order $1/N^2$ look as follows (we write only equations with respect to V_1):

$$\frac{\delta F^1}{\delta V_1(f(P))} = -Y^1(P)\ , \tag{1.15}$$

where the Y^1 is the $1/N^2$ contribution to the resolvent $\mathcal{W}$. The function Y^1 can be computed using the loop equations [14], which leads to the following expression:

$$Y^{(1)}(P)df(P) = \sum_{k=1}^{m_1}\left\{-\frac{1}{96g'(P_k)}D(P,P_k) + \left[\frac{g'''(P_k)}{96g'^2(P_k)} - \frac{S_B(P_k)}{24g'(P_k)}\right]B(P,P_k)\right\}\ . \tag{1.16}$$

The solution of (1.15), (1.16) invariant with respect to the projection change (i.e. which satisfies also the required equations with respect to V_2), looks as follows:

$$F^1 = \frac{1}{24}\ln\left\{\tau_f^{12}(v_{d_2+1})^{1-\frac{1}{d_2}}\prod_{k=1}^{m_1} dg(P_k)\right\} + \frac{d_2+3}{24}\ln d_2\ , \tag{1.17}$$

where τ_f is the so-called Bergmann tau-function on Hurwitz space, which satisfies the following system of equations with respect to the branch points λ_k:

$$\frac{\partial}{\partial\lambda_k}\ln\tau_f = -\frac{1}{12}S_B(P_k)\ . \tag{1.18}$$

The Bergmann tau-function (1.18) appears in many important problems: it coincides with isomonodromic tau-function of Hurwitz Frobenius manifolds [18], and gives the main contribution to G-function (solution of Getzler equation) of these Frobenius manifolds; it gives the most non-trivial term in isomonodromic tau-function of Riemann-Hilbert problem with quasi-permutation monodromies. Finally, its modulus square essentially coincides with determinants of Laplace operator in metrics with conical singularities over Riemann surfaces [19]. The solution of the system (1.18) was found in [20] and can be described as follows.

Define the divisor $(df) = -2\infty_f - (d_2+1)\infty_g + \sum_{k=1}^{m_1} P_k := \sum_{k=1}^{m_1+2} r_k D_k$. Choose some initial point $P \in \hat{\mathcal{L}}$ and introduce corresponding vector of Riemann constants K^P and Abel map $\mathcal{A}_\alpha(Q) = \int_P^Q w_\alpha$ (w_α form the basis of normalized holomorphic 1-forms on $\mathcal{L}$). Since all zeros of differential df have multiplicity 1, we can always choose the fundamental cell $\hat{\mathcal{L}}$ of the universal covering of $\mathcal{L}$ in such a way that $\mathcal{A}((df)) = -2K^P$ (for an arbitrary choice of fundamental domain these two vectors coincide only up to an integer combination of periods of holomorphic differentials), where the Abel map is computed along the path which does not intersect the boundary of $\hat{\mathcal{L}}$.

The main ingredient of the Bergmann tau-function is the following holomorphic multivalued $(1-g)g/2$-differential $\mathcal{C}(P)$ on $\mathcal{L}$:

$$\mathcal{C}(P) := \frac{1}{W(P)}\sum_{\alpha_1,\dots,\alpha_g=1}^{g}\frac{\partial^g\Theta(K^P)}{\partial z_{\alpha_1}\dots\partial z_{\alpha_g}}w_{\alpha_1}(P)\dots w_{\alpha_g}(P)\ . \tag{1.19}$$

where

$$W(P) := \det_{1\le\alpha,\beta\le g}||w_\beta^{(\alpha-1)}(P)|| \tag{1.20}$$

denotes the Wronskian determinant of holomorphic differentials at point P. Introduce also the quantity $\mathcal{Q}$ defined by

$$\mathcal{Q} = [df(P)]^{\frac{g-1}{2}}\mathcal{C}(P)\prod_{k=1}^{m+2}[E(P,D_k)]^{\frac{(1-g)r_k}{2}}\ , \tag{1.21}$$

which is independent of the point $P \in \mathcal{L}$. Then the Bergmann tau-function (1.18) of Hurwitz space is given by the following expression:

$$\tau_f = \mathcal{Q}^{2/3}\prod_{k,l=1\ k<l}^{m+n}[E(D_k,D_l)]^{\frac{r_k r_l}{6}}\ ; \tag{1.22}$$

together with (1.17) this gives the answer for $1/N^2$ correction in two-matrix model.

If potential V_2 is quadratic, integration with respect to M_2 in (1.1) can be taken explicitly, and the free energy (1.17) gives rise to the free energy of one-matrix model. The spectral curve $\mathcal{L}$ in this case becomes hyperelliptic, and the formula (1.17) gives, using the expression for τ_f obtained in [23]:

$$F^1 = \frac{1}{24}\ln\left\{\Delta^3\,(\det\mathbf{A})^{12}\prod_{k=1}^{2g+2}g'(\lambda_k)\right\}\ , \tag{1.23}$$

where λ_k, $k = 1, \ldots, 2g+2$ are branch points of $\mathcal{L}$; Δ is their Wronskian determinant; $\mathbf{A}$ is the matrix of a-periods of non-normalized holomorphic differentials on $\mathcal{L}$.

The paper is organised as follows. In section 2, following [14], we write down the loop equations for two-matrix model, and discuss the spectral curve and associated objects which arise in the zeroth order in $1/N^2$ expansion. Here we derive also some new variational formulas, which will be used later in computation of $1/N^2$ correction to free energy. In section 3 we solve the loop equations in $1/N^2$ approximation. Here we also express F^1 in terms of Bergmann tau-function on Hurwitz spaces introduced in [18, 26]. In section 4 we recall the explicit expression for Bergmann tau-function [20], and find its transformation law under the change of projection of the spectral curve to $\mathbb{C}P^1$. This allows to get the formula for F^1 which satisfies the full set of variational equations with respect to polynomials V_1 and V_2. In section 5 we derive variational equation of F^1 with respect to filling fractions. In section 6 we discuss the links between F^1 and other related objects: determinant of Laplace operator, G-function of Frobenius manifolds and isomonodromic tau-function of fuchsian system with quasi-permutation monodromies.

2 Loop equations: leading term

Introduce the function

$$Y(x) = V_1'(x) - \mathcal{W}(x) \tag{2.1}$$

In terms of function Y equations (1.8) for free energy can be written as follows:

$$\frac{\delta F}{\delta V_1(x)} = {V_1}'(x) - Y(x)\ , \tag{2.2}$$

as well as (1.8), valid in the sense of (1.4).

To make use of variational formula (2.2) we need to get some information about the function $Y(x)$. This information is in principle contained in the loop equations, which follow from reparametrization invariance of the partition function (1.1) (see [14] for details). To write them down, apart from resolvent $\mathcal{W}(x)$ (1.7), we need to introduce the following objects:

- Polynomial $\mathcal{P}(x,y)$:

$$\mathcal{P}(x,y) := \frac{1}{N}\Big\langle \mathrm{tr}\frac{V_1(x)-V_1(M_1)}{x-M_1}\frac{V_2(y)-V_2(M_2)}{y-M_2}\Big\rangle \tag{2.3}$$

- Polynomial $\mathcal{E}(x,y)$

$$\mathcal{E}(x,y) := (V_1(x)-y)(V_2(y)-x)-\mathcal{P}(x,y)+1 \tag{2.4}$$

- Function $\mathcal{U}(x,y)$, which is a polynomial in y:

$$\mathcal{U}(x,y) := \frac{1}{N}\Big\langle \mathrm{tr}\frac{1}{x-M_1}\frac{V_2{}'(y)-V_2{}'(M_2)}{y-M_2}\Big\rangle \tag{2.5}$$

- Function $\mathcal{U}(x,y,z)$, which is also a polynomial in y:

$$\mathcal{U}(x,y,z) := \frac{\delta\mathcal{U}(x,y)}{\delta V_1(z)} = \Big\langle \mathrm{tr}\frac{1}{x-M_1}\frac{V_2{}'(y)-V_2{}'(M_2)}{y-M_2}\mathrm{tr}\frac{1}{z-M_1}\Big\rangle - N^2\mathcal{U}(x,y)\mathcal{W}(z) \tag{2.6}$$

Now we are in position to write down the loop equation

$$\mathcal{U}(x,y) = x - V_2{}'(y) + \frac{\mathcal{E}(x,y)}{y-Y(x)} - \frac{1}{N^2}\frac{\mathcal{U}(x,y,x)}{y-Y(x)} \tag{2.7}$$

which arises as a corollary of reparametrization invariance of the matrix integral (1.1) [14].

The residue at $y = Y(x)$ of (2.7) leads to the following loop equation (for polynomials of degree 3 this equation was first derived in [5]) for function $Y(x) := V_1{}'(x) - \mathcal{W}(x)$:

$$\mathcal{E}^0(x, Y(x)) = \frac{1}{N^2}\mathcal{U}(x, Y(x), x)\ . \tag{2.8}$$

To use the loop equation effectively we need to consider the $1/N^2$ expansion of all of their ingredients.

2.1 Leading order term: algebro-geometric framework

Assume that the function Y admits an expansion into a power series in $1/N^2$:

$$Y(x) = Y^0 + \frac{1}{N^2}Y^1 + \dots\ . \tag{2.9}$$

Then in the leading order the master loop equation (2.8) turns into algebraic equation in two variables: x and $Y^{(0)}(x)$:

$$\mathcal{E}(x, Y^0(x)) = 0\ ,$$

where

$$\mathcal{E}^0(x,y) = (V_1{}'(x)-y)(V_2{}'(y)-x) - \mathcal{P}^0(x,y) + 1\ . \tag{2.10}$$

The polynomial equation

$$\mathcal{E}^0(x,y) = 0 \tag{2.11}$$

defines an algebraic curve $\mathcal{L}$ of some genus g, which we call "spectral curve" (if the spectral curve is non-singular, it has "maximal genus" equal to $d_1 d_2 - 1$); the point P of this curve is a pair of complex numbers (x, y) satisfying the polynomial equation (2.11). Therefore, Y^0 can be considered as multi-valued function of x. The curve $\mathcal{L}$ comes together with two meromorphic functions on it: function $f(P) = x$ and function $g(P) = y$ ($\equiv Y^0(x)$). Since polynomial $\mathcal{P}$ (2.3) and function $\mathcal{E}$ (2.4) are symmetric with respect to substitution $x \leftrightarrow y$, $V_1 \leftrightarrow V_2$, the same algebraic curve appears if we write down the loop equations for $X(y) := V_2{}'(y) - \frac{\delta F}{\delta V_2(y)}$.

Analytical properties of functions $f(P)$ and $g(P)$ on $\mathcal{L}$ are well-known (see [16, 17] and references therein). Namely, $f(P)$ and $g(P)$ are meromorphic functions on $\mathcal{L}$ having poles only at two marked points ∞_f and ∞_g with the following pole structure: function $f(P)$ has simple pole at ∞_f and pole of order d_1 at ∞_g; function $g(P)$ has simple pole at ∞_g and pole of order d_2 at ∞_f. Therefore, near ∞_f we can write the singular part of $g(P)$ as polynomial of $f(P)$; near ∞_g we can represent the singular part of $f(P)$ as polynomial of $g(P)$; coefficients of these polynomials are given by $V_1{}'$ and $V_2{}'$, respectively:

$$g(P) = V_1{}'(f(P)) - \frac{1}{f(P)} + O(f^{-2}(P)) \quad \text{as} \quad P \to \infty_f \;, \tag{2.12}$$

$$f(P) = V_2{}'(g(P)) - \frac{1}{g(P)} + O(g^{-2}(P)) \quad \text{as} \quad P \to \infty_g \;. \tag{2.13}$$

The dimension of the moduli space of triples $(\mathcal{L}, f, g)$ satisfying these conditions equals d_1+d_2+g+2. Let us choose on $\mathcal{L}$ a canonical basis of cycles (a_α, b_α). Then coordinates on the space $\mathcal{M}$ can be chosen as follows:

- $d_1 + 1$ coefficients $u_1, \ldots, u_{d_1+1}$ of polynomial $V_1{}'$.
- $d_2 + 1$ coefficients $v_1, \ldots, v_{d_2+1}$ of polynomial $V_2{}'$.
- The "filling fractions"

$$\epsilon_\alpha := \frac{1}{2\pi i} \oint_{a_\alpha} g df \;. \tag{2.14}$$

In strictly physical situation potentials V_1 and V_2 should be such that, considering $\mathcal{L}$ as a covering defined by function f, one can single out the "physical" sheet (which includes point ∞_f) such that all a-cycles lie on this sheet and each a-cycle encircles exactly one branch cut (all corresponding branch points must be real if potentials V_1 and V_2 are real). Similar requirement comes from g-projection of $\mathcal{L}$. However, here we don't impose these "physical" requirements i.e. consider the "analytical continuation" of physical sector, in the spirit of [27].

Nevertheless, the sheet of the curve $\mathcal{L}$ (realized as $d_2 + 1$-sheeted branched covering by function f), which contains the point ∞_f, is called the "physical" sheet; the physical sheet is well-defined at least in some neighbourhood of ∞_f. Fixing some splitting of $\mathcal{L}$ into $d_2 + 1$ sheets, we denote by $x^{(k)}$ ($k = 1, \ldots, d_2 + 1$) the point of $\mathcal{L}$ belonging to kth sheet such that $f(x^{(k)}) = x$; we assume that point $x^{(1)}$ belongs to the physical sheet of $\mathcal{L}$ i.e. $x^{(1)} \to \infty_f$ as $x \to \infty$.

The polynomial $\mathcal{E}^0(x, y)$ defining the spectral curve $\mathcal{L}$ (2.11) can also be rewritten as follows:

$$\mathcal{E}^0(x,y) = -v_{d_2+1} \prod_{k=1}^{d_2+1} (y - g(x^{(k)})) \tag{2.15}$$

The proof of (2.15) is simple: function $\mathcal{E}^0$ is given by (2.10); since $\mathcal{P}^0$ is a polynomial of degree $d_2 - 1$ with respect to y, function $\mathcal{E}^0$ is a polynomial of degree $d_2 + 1$ in y; its zeros are $Y^0(x^{(k)})$ by definition of points $x^{(k)}$. Comparison of coefficient in front of y^{d_2+1} leads to (2.15).

2.2 Some variational formulas

If a Riemann surface is realized as a branched covering of Riemann sphere, the branch points can be used as natural parameters on the moduli space, and it is easy to differentiate all objects introduced above with respect to the branch points. The answer is given by Rauch variational formulas ([7], for a simple proof see [21]). However, on our moduli space the set of natural coordinates is given by coefficients of polynomials V_1 and V_2 and filling fractions. To differentiate all interesting objects with respect to these coordinated we need to know the matrix of derivatives of branch points (we shall consider only $\{\lambda_k\}$) with respect to coefficients of V_1, V_2 and filling fractions. This matrix was computed in [16]; below we rederive some of these formulas, and prove new variational formulas, required in our context.

In [16] equations (2.2), together with their counterpart with respect to $V_2(y)$, were solved in the leading term i.e. it was found the solution of the system

$$\frac{\delta F^0}{\delta V_1(f(P))}\Big|_{f(P)} = {V_1}'(f(P)) - g(P)$$

$$\frac{\delta F^0}{\delta V_2(g(P))}\Big|_{g(P)} = {V_2}'(g(P)) - f(P)$$

which *a posteriori* turns out to satisfy also the following equations with respect to filling fractions:

$$\frac{\partial F^0}{\partial \epsilon_\alpha} = \Gamma_\alpha := \oint_{b_\alpha} g(P) df(P) \; .$$

To find solution of the equations (2.2) in order $1/N^2$, together with their counterpart with respect to $V_2(y)$) we shall need

Lemma 1 *The following variational formulas take place:*

$$-\frac{\delta \lambda_k}{\delta V_1(f(P))} g'(P_k) df(P) = B(P, P_k) \; , \tag{2.16}$$

$$\frac{\delta\{g'(P_k)\}}{\delta V_1(f(P))}\Big|_{f(P)} df(P) = \frac{1}{4}\left\{ D(P, P_k) - \frac{g'''(P_k)}{g'(P_k)} B(P, P_k) \right\} \tag{2.17}$$

Proof. We start from formula (1.11) for the Bergmann bidifferential:

$$B(P, Q) = \frac{\delta g(P)}{\delta V_1(f(Q))}\Big|_{f(Q)} df(P) df(Q) \; . \tag{2.18}$$

We want to rewrite this formula in the limit $Q \to P_k$ using the local parameter $x_k(Q) = \sqrt{f(Q) - \lambda_k}$. As the first step we notice that for any coordinate t on our moduli space we have the following identity:

$$g_t(Q)|_f(Q) df(Q) = g_t(Q)|_{x_k(Q)} df(Q) - f_t(Q)|_{x_k(Q)} dg(Q) \; , \tag{2.19}$$

which follows from differentiation of composite function $g(t, f(x_k, t))$ with respect to t using the chain rule. In particular,

$$\frac{\delta g(Q)}{\delta V_1(f(P))}\Big|_f (Q) df(Q) = \frac{\delta g(Q)}{\delta V_1(f(P))}\Big|_{x_k(Q)} df(Q) - \frac{\delta g(Q)}{\delta V_1(f(P))}\Big|_{x_k(Q)} dg(Q) \; . \tag{2.20}$$

Consider now first several terms of local expansion of $g(Q)$, $dg(Q)$ and $B(P,Q)$ as $Q \to P_k$ (prime denotes derivative with respect to $x_k := x_k(Q)$):

$$g(Q) = g(P_k) + g'(P_k)x_k + \dots \; , \tag{2.21}$$

$$dg(Q) = \{g'(P_k) + g''(P_k)x_k + \frac{1}{2}g'''(P_k)x_k^2 + \dots\}dx_k \; , \tag{2.22}$$

$$B(P,Q) = \{B(P,P_k) + B'(P,P_k)x_k + \frac{1}{2}B''(P,P_k)x_k^2 + \dots\}dx_k \; . \tag{2.23}$$

Taking into account that $f(Q) = x_k^2 + \lambda_k$, and substituting these relations into (2.20), we get in the order zero the formula (2.16).

The first order terms give relation which defines the dependence of $g(P_k)$ on $\{u_k\}$:

$$\left\{2\frac{\delta g(P_k)}{\delta V_1(f(P))} - \frac{\delta \lambda_k}{\delta V_1(f(P))} g''(P_k)\right\} df(P) = B'(P,P_k) \; ; \tag{2.24}$$

we present this relation only for completeness, since it will not be used below.

Finally, collecting the coefficients in front of x_k^2, we get

$$2\frac{\delta g'(P_k)}{\delta V_1(f(P))} - \frac{1}{2}\frac{\delta \lambda_k}{\delta V_1(f(P))} g'''(P_k) = \frac{1}{2}\frac{B''(P,P_k)}{df(P)} \; ,$$

which leads to (2.17) after using (2.16).

3 Solution of loop equation in $1/N^2$ approximation

The main goal of this paper is to find function F^1 on our moduli space which satisfies the equation

$$\frac{\delta F^1}{\delta V_1(x)} = -Y^1(x) \; , \tag{3.1}$$

where $Y^1(x)$ should be determined from $1/N^2$ expansion of the loop equation (2.8). The equation (3.1) is valid in a neighbourhood of the point ∞_f i.e in a neighbourhood of the point $x = \infty$ on the "physical" (with respect to variable x) sheet of the spectral curve $\mathcal{L}$. The same function $F^{(1)}$ should satisfy the equation

$$\frac{\delta F^1}{\delta V_2(y)} = -X^1(y) \; , \tag{3.2}$$

where function $X^1(y)$ should be found from writing down the loop equation with respect to matrix M_2 in a neighbourhood of point ∞_g. We shall first solve (3.1), and then check the symmetry of the expression with respect to the change of projection $f \leftrightarrow g$.

To express Y^1 in terms of the objects associated to the spectral curve $\mathcal{L}$ we consider the $1/N^2$ term of the master loop equation (2.8); we have:

$$\mathcal{E}(x, Y(x)) = \mathcal{E}^0(f(P), g(P) + \frac{1}{N^2} Y^1(P) + \dots) + \frac{1}{N^2} \mathcal{E}^1(f(P), g(P)) + \dots \,, \tag{3.3}$$

as $P \to \infty_f$, where, as before, in a neighbourhood of ∞_f, $f(P) = x$; $g(P) = Y^0(x)$. The $1/N^2$ expansion of $\mathcal{E}(x, y)$ looks as follows:

$$\mathcal{E}(x, y) = \mathcal{E}^0(x, y) + \frac{1}{N^2} \mathcal{E}^1(x, y) + \dots \,; \tag{3.4}$$

since $\mathcal{E}^1(x, y) = -\mathcal{P}^1(x, y)$, we can further rewrite this expression in a neighbourhood of point ∞_f as follows:

$$\mathcal{E}(x, Y(x)) = \mathcal{E}^0(f(P), g(P) + \frac{1}{N^2} \{\mathcal{E}^1(f(P), g(P)) + Y^1(P) \mathcal{E}^0_y(f(P), g(P))\} + \dots \,. \tag{3.5}$$

Therefore, the $1/N^2$ term of master loop equation (2.8) gives

$$\mathcal{U}^0(f(P), g(P), f(P)) = \mathcal{E}^1(f(P), g(P)) + Y^1(P) \mathcal{E}^0_y(f(P), g(P))$$

as $P \to \infty_f$, or

$$Y^1(P) = \frac{\mathcal{U}^0(f(P), g(P), f(P)) + \mathcal{P}^1(f(P), g(P))}{\mathcal{E}^0_y(f(P), g(P))} \,. \tag{3.6}$$

To make this formula more explicit we need to express $\mathcal{U}^0(f(P), g(P), f(P))$ in terms of known objects using the loop equation (2.7). According to definition of $\mathcal{U}^0(x, y, z)$ we have:

$$\mathcal{U}^0(x, y, z) = -\frac{\delta \mathcal{U}^0(x, y)}{\delta V_1(z)} \tag{3.7}$$

On the other hand, the zeroth order term of (2.7) gives:

$$\mathcal{U}^0(x, y) = x - V_2'(y) + \frac{\mathcal{E}^0(x, y)}{y - g(x^{(1)})} \tag{3.8}$$

(as before, $x^{(1)}$ denotes a point on the physical sheet of $\mathcal{L}$). Therefore,

$$\mathcal{U}^0(x, y, z) = -\frac{\delta \mathcal{E}^0(x, y)/\delta V_1(z)}{y - g(x^{(1)})} - \frac{\mathcal{E}^0(x, y)}{(y - g(x^{(1)}))^2} \frac{\delta g(x^{(1)})}{\delta V_1(z)} \,. \tag{3.9}$$

Using the form (2.15) of the polynomial $\mathcal{E}^0(x, y)$, we can further rewrite this expression as follows:

$$\frac{\delta \mathcal{E}^0(x, y)}{\delta V_1(z)} = -\mathcal{E}^0(x, y) \sum_{k=1}^{d_2+1} \frac{\delta g(x^{(k)})}{\delta V_1(z)} \frac{1}{y - g(x^{(k)})} \,. \tag{3.10}$$

Substituting this expression into (2.15), we get

$$\mathcal{U}^0(x, y, z) = \frac{\mathcal{E}^0(x, y)}{y - g(x^{(1)})} \sum_{k=2}^{d_2+1} \frac{\delta g(x^{(k)})}{\delta V_1(z)} \frac{1}{y - g(x^{(k)})} \,. \tag{3.11}$$

Substituting $z = x = f(P)$ and taking the limit $y \to g(x^{(1)})$, we have:

$$\mathcal{U}^0(f(P), g(P), f(P)) = \mathcal{E}_y^0(f(P), g(P)) \sum_{k=2}^{d_2+1} \frac{\delta g(x^{(k)})}{\delta V_1(f(P))} \frac{1}{g(P) - g(x^{(k)})} \tag{3.12}$$

as $P \equiv x^{(1)} \to \infty_f$. Now (3.6) can be rewritten as follows:

$$Y^1(P) = \frac{\mathcal{P}^1(f(P), g(P))}{\mathcal{E}_y^0(f(P), g(P))} + \sum_{Q \neq P\,:\, f(Q)=f(P)} \frac{\delta g(Q)}{\delta V_1(f(P))} \frac{1}{g(P) - g(Q)} \tag{3.13}$$

as $P \to \infty_f$, which can be further transformed, using the formula (1.11) for the Bergmann bidifferential:

$$Y^1(P) df(P) = \frac{\mathcal{P}^1(f(P), g(P))}{\mathcal{E}_y^0(f(P), g(P))} df(P) + \sum_{Q \neq P\,:\, f(Q)=f(P)} \frac{B(P,Q)}{df(Q)} \frac{1}{g(P) - g(Q)}\ ; \tag{3.14}$$

now we see that the 1-form $Y^1(P)df(P)$ can be analytically continued from a neighbourhood of ∞_f to the whole $\mathcal{L}$.

Lemma 2 *Let the spectral curve $\mathcal{L}$ (2.11) be non-singular. Then the 1-form $Y^1(P)df(P)$(3.14) is a meromorphic 1-form on the spectral curve $\mathcal{L}$ with poles up to fourth order only at the branch points P_k i.e. at the zeros of differential $df(P)$.*

Proof. Let us verify the non-singularity of the first term,

$$\frac{\mathcal{P}^1(f(P), g(P))}{\mathcal{E}_y^0(f(P), g(P))} df(P)\ , \tag{3.15}$$

of (3.14) everywhere on $\mathcal{L}$. For finite $f(P)$ the 1-form (3.15) can be singular only at the zeros of $\mathcal{E}_y^0(f(P), g(P))$, which, if the curve $\mathcal{L}$ is non-singular, are by definition the branch points P_k; these zeros are of the 1st order and are canceled by the zeros of $df(P)$ at the branch points.

To study behaviour of (3.15) at ∞_f and ∞_g we mention that the polynomial $\mathcal{P}(x,y)$ (2.3) (and, therefore, also its first correction $\mathcal{P}^1(x,y)$) is of degree $d_1 - 1$ with respect to x and $d_2 - 1$ with respect to y. However, we can say a bit more about $\mathcal{P}^1(x,y)$. Namely, the coefficient of $\mathcal{P}(x,y)$ in front of $x^{d_1-1}y^{d_2-1}$ equals $u_{d_1+1}v_{d_2+1}$, which does not have any higher corrections. Therefore, the coefficient of $\mathcal{P}^1(x,y)$ in front of $x^{d_1-1}y^{d_2-1}$ vanishes.

Now consider the behaviour of the 1-form (3.15) near ∞_f. We have

$$\mathcal{E}_y^0(f(P), g(P)) = -(V_2{}'(g(P)) - f(P)) - (V_1{}'(f(P)) - g(P))V_2{}''(g(P)) - \mathcal{P}_y^0(f(P), g(P))\ ,$$

which has pole of order $d_1 d_2$ near ∞_f as corollary of asymptotics (2.12) of function $g(P)$ near ∞_f. The 1-form $df(P)$ has at ∞_f the pole of second order. The most singular contribution by $\mathcal{P}^1(f(P), g(P))$ at ∞_f comes from the term $f^{d_1-2}(P)g^{d_2-1}(P)$; it has the pole of order $d_1 - 2 + d_1(d_2 - 1) = d_1 d_2 - 2$. Summing up, we see that (3.15) is non-singular near ∞_f.

Consider the 1-form (3.15) near ∞_g. At this point $df(P)$ has pole of order $d_2 + 1$; the main contribution to $\mathcal{E}_y^0(f(P), g(P))$ is given by the term $(V_1{}'(f(P)) - g(P))V_2{}''(g(P))$, which has pole of order $d_1 d_2 + d_2 - 1$. Finally, the main contribution to $\mathcal{P}^1(f(P), g(P))$ comes from the term $g^{d_2-2}(P)f^{d_1-1}(P)$, which has pole of order $d_1 d_2 - 2$. In total (3.15) is non-singular at ∞_g, too.

Consider now the second term of (3.14):

$$\sum_{Q\neq P\,:\,f(Q)=f(P)} \frac{B(P,Q)}{df(Q)}\cdot\frac{1}{g(P)-g(Q)} \tag{3.16}$$

The Bergmann bidifferential is singular (has second order poles) only at coinciding arguments, i.e. when P coincides with one of the branch points P_k. The denominator $g(P)-g(Q)$ vanishes only if P coincides with Q, (i.e. both of them coincide with one of the branch points P_k). It is slightly more complicated to see that zeros of $df(Q)$ don't give any poles outside of P_k. Obviously, $df(Q)$ is singular if $P\to P_k$ and $Q=P^*$, where P^* is another point such that $f(P^*)=f(P)$ and $P^*\to P_k$ as $P\to P_k$. However, $df(Q)$ is also singular if Q coincides with one of the branch points P_k, while P remains on some other sheet, and does not tend to P_k as $Q\to P_k$. In this case in the sum (3.16) we have two singular terms (with poles of first order), which correspond to Q and Q^*; however, the residues of these terms just differ by sign, and, therefore, the total sum (3.16) remains finite outside the branch points P_k and infinities ∞_f and ∞_g.

As $P\to\infty_f$, all points Q in (3.16) tend to ∞_g; thus all $df(Q)$ have pole of order d_2+2; all other terms remain non-singular and non-vanishing. Therefore, (3.16) has zero of order d_2+1 at ∞_f.

As $P\to\infty_g$, the situation is slightly more complicated. If we enumerate the sheets of $\mathcal{L}$ such that, as $x\to\infty$, $x^{(1)}\to\infty_f$, and $x^{(2)},\dots,x^{(d_2+1)}\to\infty_g$, and assume that $P=x^{(d_2+1)}$, then (3.16) can be split as follows:

$$\frac{B(x^{(1)},x^{(d_2+1)})}{df(x^{(1)})}\frac{1}{g(x^{(d_2+1)})-g(x^{(1)})}+\sum_{j=2}^{d_2}\frac{B(x^{(j)},x^{(d_2+1)})}{df(x^{(j)})}\frac{1}{g(x^{(d_2+1)})-g(x^{(j)})}\;. \tag{3.17}$$

As $x\to\infty$, the first term in (3.17) has zero of order two ($df(x^{(1)})$ has pole of order two, other multipliers remain non-singular and non-vanishing). In each term of the sum in (3.17) the Bergmann bidifferential has pole of second order as $x\to\infty$; however, $df(x^{(j)})$ has pole of order d_2+1, and $g(x^{(d_2+1)})-g(x^{(j)})$ has simple pole as $x\to\infty$; therefore, the whole expression (3.17) is non-singular (and even vanishing) as $x\to\infty$.

Remark 1 The condition of non-singularity of the spectral curve (2.11) made in lemma 2 means in physical language that the spectral curve has maximal possible genus equal to d_1d_2-1 for given degrees of polynomials V_1 and V_2. If the genus of the spectral curve is less than the maximal genus, it must be singular; then the non-singularity of 1-form $Y^1(P)df(P)$ at the double points can not be verified rigorously. However, this non-singularity is suggested by physical consideration: since we assume that at the double points one does not have any eigenvalues of M_1 or M_2 in large N limit (i.e. corresponding filling fractions are equal to zero), there is no physical reason for corresponding resolvents to be singular at these points in large N limit. Therefore, in the sequel we shall assume that $Y^1(P)df(P)$ is non-singular outside of branch points of $\mathcal{L}$ both for maximal and non-maximal genus. We should mention that this assumption was also made (explicitly or implicitly) in the previous papers [8, 9, 13, 15].

The singular parts of $Y^1(P)df(P)$ at the branch points P_k can be found from (3.14). If, say, $P\to P_k$, the only term in (3.14) which contributes to singular part at P_k corresponds to $Q=P^*$. Thus

$$Y^1(P)df(P)=\frac{B(P,P^*)}{df(P^*)}\frac{1}{g(P)-g(P^*)}+O(1)\;;\qquad \text{as}\qquad P\to P_k\;. \tag{3.18}$$

Consider the local expansion of all ingredients of this expression as $P \to P_k$ in terms of the local parameter $x_k(P) = \sqrt{f(P) - \lambda_k}$:

$$g(P) = g(P_k) + x_k(P)g'(P_k) + \frac{1}{2}x_k^2(P)g''(P_k) + \frac{1}{6}x_k^3(P)g'''(P_k) + \dots \ ,$$

$$g(P^*) = g(P_k) - x_k(P)g'(P_k) + \frac{1}{2}x_k^2(P)g''(P_k) - \frac{1}{6}x_k^3(P)g'''(P_k) + \dots \ ,$$

$$df(P^*) = 2x_k(P)dx_k(P) \ ,$$

$$B(P,P^*) = \left(\frac{1}{(2x_k(P))^2} + \frac{1}{6}S_B(P_k) + \dots\right) dx_k(P)(-dx_k(P)) \ .$$

Therefore,

$$\frac{1}{g(P) - g(P^*)} = \frac{1}{2x_k(P)g'(P_k)}\left(1 - \frac{x_k(P)^2}{6}\frac{g'''(P_k)}{g'(P_k)}\right) + \dots$$

and, as $P \to P_k$,

$$\frac{B(P,P^*)}{df(P^*)}\frac{1}{g(P)-g(P^*)} = \left\{-\frac{1}{16x_k^4(P)g'(P_k)} + \left(\frac{1}{96}\frac{g'''(P_k)}{g'^2(P_k)} - \frac{S_B}{24g'(P_k)}\right)\frac{1}{x_k^2(P)} + O(1)\right\} dx_k(P) \ . \tag{3.19}$$

Since, according to our assumption, the 1-form $Y^1(P)df(P)$ is non-singular on $\mathcal{L}$ outside of the branch points, we can express this 1-form in terms of differentials $B(P,P_k)$ and $D(P,P_k)$ (1.13) using their behaviour near P_k;

$$Y^{(1)}(P)df(P) = \sum_{k=1}^{m_1}\left\{-\frac{1}{96g'(P_k)}D(P,P_k) + \left[\frac{g'''(P_k)}{96g'^2(P_k)} - \frac{S_B(P_k)}{24g'(P_k)}\right]B(P,P_k)\right\} \ ; \tag{3.20}$$

as a result we rewrite the equation (3.1) for F^1 as follows:

$$\frac{\delta F^1}{\delta V_1(f(P))}df(P) = \sum_{k=1}^{m_1}\left\{\frac{1}{96g'(P_k)}D(P,P_k) + \left[-\frac{g'''(P_k)}{96g'^2(P_k)} + \frac{S_B(P_k)}{24g'(P_k)}\right]B(P,P_k)\right\} \ . \tag{3.21}$$

Proposition 1 *A general solutions F^1 of the system (3.21) can be written as follows:*

$$F^1 = \frac{1}{2}\ln\tau_f + \frac{1}{24}\ln\left\{\prod_{k=1}^{m_1} g'(P_k)\right\} + C(\{v_k\},\{\epsilon_\alpha\}) \tag{3.22}$$

where $C(\{v_k\},\{\epsilon_\alpha\})$ is a function on our moduli space depending only on coefficients of polynomial V_2 and filling fractions $\{\epsilon_\alpha\}$; function τ_f (the Bergmann tau-function on Hurwitz space) is defined by the system of equations with respect to the branch points $\{\lambda_k\}$:

$$\frac{\partial}{\partial\lambda_k}\ln\tau_f = -\frac{1}{12}S_B(P_k) \ ; \tag{3.23}$$

function τ_f depends on coordinates $\{u_k,\ v_k,\ \epsilon_\alpha\}$ as a composite function.

Proof. The derivative of τ_f with respect to $V_1(f(P))$ is computed by chain rule using variational formula (2.16); derivatives of $g'(P_k)$ with respect to $V_1(f(P))$ are given by (2.17). Collecting all these terms together we see that derivative of (3.22) coincides with (3.21).

Therefore, to compute F^1 it remains to find the Bergmann tau-function τ_f and to make sure that "constant" $C(\{v_k\},\{\epsilon_\alpha\})$ is chosen such that the final expression is symmetric with respect to the change of "projection" i.e that F^1 satisfies also equations (3.2).

4 $F^{(1)}$ and Bergmann tau-function on Hurwitz spaces

4.1 Bergmann tau-function on Hurwitz spaces

Here, following [20], we discuss the Bergmann tau-function on Hurwitz spaces for the stratum of Hurwitz space arising in the application to two-matrix model.

The Hurwitz space $H_{g,N}$ is the space of equivalence classes of pairs $(\mathcal{L},\, f)$, where $\mathcal{L}$ is a compact Riemann surface of genus g and f is a meromorphic functions of degree N. The Hurwitz space is stratified according to multiplicities of poles of function f. By $H_{g,N}(k_1,\dots,k_n)$, where $k_1+\dots+k_n = N$, we denote the stratum of $H_{g,N}$ consisting of meromorphic functions which have n poles on $\mathcal{L}$ with multiplicities $k_1,\dots,k_n$. (In applications to two-matrix model we need to study the tau-function on the stratum $H_{g,N}(1,N-1)$, on which the function f has only two poles: one simple pole and one pole of order $N-1$.)

Suppose that all critical points of the function f are simple; denote them by $P_1,\dots P_M$ ($m = 2N+2g-2$ according to Riemann-Hurwitz formula); the critical values $\lambda_k = \pi(P_k)$ can be used as (local) coordinates on $H_{g,N}(k_1,\dots,k_n)$. Function f defines the realization of the Riemann surface $\mathcal{L}$ as an N-sheeted branched covering of $\mathbb{C}P^1$ with ramification points $P_1,\dots,P_m$ and branch points $\lambda_k = f(P_k)$; the points at infinity we denote by $\infty_1,\dots,\infty_n$. In a neighbourhood of the ramification point P_k the local coordinate is chosen to be $x_k := \sqrt{\lambda-\lambda_k}$, $k=1,\dots,m$; in a neighbourhood of the point ∞_j the local parameter is $x_{m+j} := \lambda^{-1/k_j}$

The Bergmann bidifferential $B(P,Q)$ has the second order pole as $Q\to P$ with the local behaviour (1.12): $B(P,Q)/\{dx(P)dx(Q)\} = (x(P)-x(Q))^{-2} + \frac{1}{6}S_B(x(P)) + o(1)$, where $x(P)$ is a local coordinate; $S_B(x(P))$ is the Bergmann projective connection.

We define the Bergmann τ-function $\tau_f(\lambda_1,\dots,\lambda_m)$ locally by the system of equations (3.23):

$$\frac{\partial}{\partial\lambda_k}\ln\tau_f = -\frac{1}{12}S_B(x_k)|_{x_k=0}\;, \qquad k=1,\dots,m\;. \tag{4.1}$$

compatibility of this system is a simple corollary of Rauch variational formulas [21].

Consider the divisor of the differential df: $(df) = \sum_{k=1}^{m+n} r_k D_k$ where $D_k := P_k$, $r_k := 1$ for $k=1,\dots,m$ and $D_{m+j} = \infty_j$, $r_{m+j} = -(k_j+1)$ for $j=1,\dots,n$; the corresponding local parameters x_k, $k=1,\dots,m+n$ were introduced above.

Here and below, if an argument of a differential coincides with a point D_j of divisor (df), we evaluate this differential at this point with respect to local parameter x_j. In particular, for the prime form we shall use the following conventions:

$$E(D_k,D_l) := E(P,Q)\sqrt{dx_k(P)}\sqrt{dx_l(Q)}|_{P=D_k,\;Q=D_l}\;, \tag{4.2}$$

for $k,l=1,\dots,m+N$. The next notation correspond to prime-forms, evaluated at points of divisor (df) with respect to only one argument:

$$E(P,D_l) := E(P,Q)\sqrt{dx_l(Q)}|_{Q=D_l}\;, \tag{4.3}$$

$l=1,\dots,m+n$; in contrast to $E(D_k,D_l)$, which are just scalars, $E(P,D_l)$ are $-1/2$-forms with respect to P.

Denote by $w_1,\dots,w_g$ normalized ($\oint_{a_\alpha} w_\beta = \delta_{\alpha\beta}$) holomorphic differentials on $\mathcal{L}$; $\mathbf{B}_{\alpha\beta} = \oint_{b_\alpha} w_\beta$ is the corresponding matrix of b-periods; $\Theta(z|\mathbf{B})$ is the theta-function.

Choose some initial point $P\in\hat{\mathcal{L}}$ and introduce corresponding vector of Riemann constants K^P and Abel map $\mathcal{A}_\alpha(Q) = \int_P^Q w_\alpha$. Since zeros of differential df have multiplicity 1, we can always choose the fundamental cell $\hat{\mathcal{L}}$ of the universal covering of $\mathcal{L}$ in such a way that $\mathcal{A}((df)) = -2K^P$ (for an arbitrary choice of fundamental domain these two vectors coincide only up to an integer combination of periods of holomorphic differentials), where the Abel map is computed along the path which does not intersect the boundary of $\hat{\mathcal{L}}$.

The key entry of the Bergmann tau-function is the following holomorphic multivalued $(1-g)g/2$-differential $\mathcal{C}(P)$ on $\mathcal{L}$:

$$\mathcal{C}(P) := \frac{1}{W(P)} \sum_{\alpha_1,\dots,\alpha_g=1}^{g} \frac{\partial^g \Theta(K^P)}{\partial z_{\alpha_1} \dots \partial z_{\alpha_g}} w_{\alpha_1}(P) \dots w_{\alpha_g}(P) \; . \tag{4.4}$$

where

$$W(P) := \det_{1\leq\alpha,\beta\leq g} ||w_\beta^{(\alpha-1)}(P)|| \tag{4.5}$$

denotes the Wronskian determinant of holomorphic differentials at point P.

The following theorem is a slight modification of the theorem proved in [20].

Theorem 1 *The Bergmann tau-function (4.1) of Hurwitz space $H_{g,N}(k_1,\dots,k_n)$ is given by the following expression:*

$$\tau_f = \mathcal{Q}^{2/3} \prod_{k,l=1\ \ k<l}^{m+n} [E(D_k, D_l)]^{\frac{r_k r_l}{6}} \tag{4.6}$$

where the quantity $\mathcal{Q}$ defined by

$$\mathcal{Q} = [df(P)]^{\frac{g-1}{2}} \mathcal{C}(P) \prod_{k=1}^{m+N} [E(P, D_k)]^{\frac{(1-g)r_k}{2}} \; ; \tag{4.7}$$

is independent of the point $P \in \mathcal{L}$.

The proof of the theorem is very similar to [20]. The only technical difference is the appearance of higher order poles of function f.

4.2 Dependence of Bergmann tau-function on the choice of the projection

Theorem 2 *Let τ_f and τ_g be Bergmann tau-functions (4.6) corresponding to divisors (df) and (dg), respectively. Then*

$$\left(\frac{\tau_f}{\tau_g}\right)^{12} = C \frac{(u_{d_1+1})^{1-\frac{1}{d_1}}}{(v_{d_2+1})^{1-\frac{1}{d_2}}} \frac{\prod_k df(Q_k)}{\prod_k dg(P_k)} \tag{4.8}$$

where

$$C = \frac{d_1^{d_1+3}}{d_2^{d_2+3}} \tag{4.9}$$

is a constant independent of moduli parameters.

Proof. As above, we assume that the fundamental cell $\hat{\mathcal{L}}$ is chosen in such a way that $\mathcal{A}((df)) = \mathcal{A}((dg)) = -2K^P$. Denote divisors (df) and (dg) as follows:

$$(df) = \sum_{k=1}^{m_1} P_k - 2\infty_f - (d_2+1)\infty_g := \sum_{k=1}^{m_1+2} r_k D_k \; , \tag{4.10}$$

$$(dg) = \sum_{k=1}^{m_2} Q_k - 2\infty_g - (d_1+1)\infty_f := \sum_{k=1}^{m_2+2} s_k G_k \ . \tag{4.11}$$

Since $\deg(df) = \deg(dg) = 2g-2$, we have $\sum_{k=1}^{m_1+2} r_k = \sum_{k=1}^{m_2+2} s_k = 2g-2$. Then, according to the expression (4.7) for the Bergmann tau-function, we have

$$(\tau_f)^{12} = \mathcal{C}^8(P)[df(P)]^{4g-4} \prod_{k,j=1}^{m_1+2} \{E(D_k,D_j)\}^{2r_k r_j} \prod_{k=1}^{m_1+2} \{E(P,D_k)\}^{r_k(4-4g)} \ , \tag{4.12}$$

where the values of prime-forms at the points of divisor (df) are evaluated in the system of local parameters defined by function f i.e. near P_k the local parameter is $x_k = \sqrt{f(P)-\lambda_k}$; near ∞_f the local parameter is $x_{m_1+1} = 1/f(P)$, and near ∞_g it is $x_{m_1+2} = [f(P)]^{-1/d_2}$.

Similarly, we have

$$(\tau_g)^{12} = \mathcal{C}^8(P)[dg(P)]^{4g-4} \prod_{k,j=1}^{m_2+2} \{E(G_k,G_j)\}^{2s_k s_j} \prod_{k=1}^{m_2+2} \{E(P,G_k)\}^{s_k(4-4g)} \ , \tag{4.13}$$

where the values of prime-forms at the points of divisor (dg) should be evaluated in the system of local parameters defined by function g i.e. near Q_k the local parameter is $y_k = \sqrt{g(P)-\mu_k}$; near ∞_f the local parameter is $y_{m_2+1} = 1/g(P)$, and near ∞_g it is $y_{m_2+2} = [g(P)]^{-1/d_2}$.

Therefore,

$$\left(\frac{\tau_f}{\tau_g}\right)^{12} = \frac{\prod_{k,j=1}^{m_1+2}\{E(D_k,D_j)\}^{2r_k r_j}}{\prod_{k,j=1}^{m_2+2}\{E(G_k,G_j)\}^{2s_k s_j}} \left\{ \frac{df(P)}{dg(P)} \frac{\prod_{k=1}^{m_2+2}\{E(P,G_k)\}^{s_k}}{\prod_{k=1}^{m_1+2}\{E(P,D_k)\}^{r_k}} \right\}^{4g-4} \ . \tag{4.14}$$

Using independence of this expression of the choice of point P we can split the $(4g-4)$th power in this formula into the product over points of divisor $(df)+(dg)$ (whose degree equals exactly $4g-4$!). The subtlety which arises is that, evaluating the prime-forms and differentials df and dg at the points D_k and G_k we fix the local parameters (these local parameters at the points of (df) are defined via function f, and at the points of (dg) via function g as explained above). Since divisors (df) and (dg) have common points (∞_f and ∞_g), in a neighbourhood of each of these points we introduce two essentially different local parameters, and it is important to remember in each case in which local parameter the prime-forms are computed.

Another subtlety is that, being considered as functions of P, different multipliers in (4.14) either vanish or become singular if $P \in (df)+(dg)$; cancellation of these singularities should be accurately traced down.

Consider the first "half" of this expression, namely, the product over $P \in (df)$:

$$\left\{ \frac{df(P)}{dg(P)} \frac{\prod_{k=1}^{m_2+2}\{E(P,G_k)\}^{s_k}}{\prod_{k=1}^{m_1+2}\{E(P,D_k)\}^{r_k}} \right\}^{2g-2} = \prod_{l=1}^{m_1+2} \lim_{P\to D_l} \left\{ \frac{df(P)}{dg(P)} \frac{\prod_{k=1}^{m_2+2}\{E(P,G_k)\}^{s_k}}{\prod_{k=1}^{m_1+2}\{E(P,D_k)\}^{r_k}} \right\}^{r_l} \tag{4.15}$$

$$= \prod_{k,l=1\,,\,k<l}^{m_1+2} \{E(D_l,D_k)\}^{-2r_k r_l} \prod_{k=1}^{m_1+2} \left\{ \lim_{P\to D_k} \frac{df(P)}{\{E(P,D_k)\}^{r_k}} \right\}^{r_k} \prod_{l=1}^{m_1+2} \left\{ \lim_{P\to D_l} \frac{\prod_{k=1}^{m_2+2}\{E(P,G_k)\}^{s_k}}{dg(P)} \right\}^{r_l} \ . \tag{4.16}$$

The first product looks nice since it cancels out against the first product in the numerator of (4.14). Let us evaluate other ingredients of this expression. We have $D_k = P_k$, $r_k = 1$ for $k = 1, \ldots, m_1$, $D_{m_1+1} = \infty_f$, $k_{m_1+1} = -2$, $D_{m_1+2} = \infty_g$, $k_{m_1+2} = -(d_2+1)$. Therefore,

$$\prod_{k=1}^{m_1+2} \left\{ \lim_{P \to D_k} \frac{df(P)}{\{E(P, D_k)\}^{r_k}} \right\}^{r_k}$$

$$= \left\{ \lim_{P \to D_{m_1+1}} \{df(P) E^2(P, D_{m_1+1})\} \right\}^{-2} \left\{ \lim_{P \to D_{m_1+2}} \{df(P) E^{d_2+1}(P, D_{m_1+2})\} \right\}^{-d_2-1} \prod_{k=1}^{m_1} \lim_{P \to P_k} \frac{df(P)}{\{E(P, P_k)\}} , \tag{4.17}$$

where we don't write ∞_f and ∞_g instead of D_{m_1+1} and D_{m_1+2}, respectively, to remember that we need to use the system of local parameters related to $f(P)$. The last term in (4.17) product is the easiest one:

$$\lim_{P \to P_k} \frac{df(P)}{\{E(P, P_k)\}} = \lim_{x_k(P) \to 0} \frac{2x_k}{x_k} = 2 . \tag{4.18}$$

In a similar way we evaluate the first term:

$$\lim_{P \to D_{m_1+1}} \{df(P) E^2(P, D_{m_1+1})\} = -1 , \tag{4.19}$$

and the second one:

$$\lim_{P \to D_{m_1+2}} \{df(P) E^{d_2+1}(P, D_{m_1+2})\} = -d_2 . \tag{4.20}$$

It remains to evaluate the third product in (4.16):

$$\prod_{l=1}^{m_1+2} \left\{ \lim_{P \to D_l} \frac{\prod_{k=1}^{m_2+2} \{E(P, G_k)\}^{s_k}}{dg(P)} \right\}^{r_l} = \left(\prod_{l=1}^{m_1} \{dg(P_l)\}^{-1} \right) \left(\prod_{\text{all } k,l \text{ such that } D_l \neq G_k} \{E(D_l, G_k)\}^{r_l s_k} \right)$$

$$\times \left(\lim_{P \to D_{m_1+1}} \{E(P, G_{m_2+2})\}^{d_1+1} dg(P) \right)^2 \left(\lim_{P \to D_{m_1+2}} \{E(P, G_{m_2+1})\}^2 dg(P) \right)^{d_2+1} . \tag{4.21}$$

Consider the first limit in (4.21):

Lemma 3

$$\lim_{P \to D_{m_1+1}} \left(\{E(P, G_{m_2+2})\}^{d_1+1} dg(P) \right)^2 = (d_1^2)(u_{d_1+1})^{1-\frac{1}{d_1}} \tag{4.22}$$

Proof. Two different local parameters at the point $\infty_f \equiv D_{m_1+1} \equiv G_{m_2+2}$ which we need to use are $x_{m_1+1}(P) = f^{-1}(P)$ and $y_{m_2+2}(P) = g^{-1/d_1}(P)$. We have

$$E(P, G_{m_2+2}) = \frac{(y_{m_2+2}(P) + \ldots)}{d\sqrt{y_{m_2+2}(P)}} = \sqrt{\frac{dx_{m_1+1}}{dy_{m_2+2}}(\infty_f)} \frac{(y_{m_2+2}(P) + \ldots)}{\sqrt{dx_{m_1+1}(P)}} . \tag{4.23}$$

We have also $g(P) = y_{m_2+2}^{-d_1}$; thus

$$dg(P) = -(d_1)(y_{m_2+2})^{-d_1-1} \left(\frac{dy_{m_2+2}}{dx_{m_1+1}}(\infty_f) \right) dx_{m_1+1}(P) . \tag{4.24}$$

Taking in (4.22) the limit $P \to D_{m_1+1}$ we indicate that all differentials in the bracket should be evaluated with respect to the local parameter x_{m_1+1}. Therefore, in (4.22) we ignore all factors $dx_{m_1+1}(P)$ and (4.22) turns out to be equal to

$$(d_1^2)\left(\frac{dy_{m_2+2}}{dx_{m_1+1}}(\infty_f)\right)^{1-d_1} = (d_1^2)(u_{d_1+1})^{1-\frac{1}{d_1}}\ , \tag{4.25}$$

where we take into account that, as $P \to \infty_f$, $g = u_{d_1+1}x^{d_1} + \dots$; thus $(dy_{m_2+2}/dx_{m_1+1})(\infty_f) = (u_{d_1+1})^{-1/d_1}$.

Consider now the second limit in (4.21):

Lemma 4

$$\lim_{P\to D_{m_1+2}} \{E(P, G_{m_2+1})\}^2 dg(P) = -1 \tag{4.26}$$

Proof. In analogy to (4.22) we have to evaluate the prime-form and the differential dg with respect to the local parameter related to function f i.e. with respect to $x_{m_1+2}(P) = (f(P))^{-1/d_2}$, while the local parameter arising from function g is $y_{m_2+1}(P) = (g(P))^{-1}$. We have near D_{m_1+2}:

$$E(P, G_{m_2+1}) = \frac{y_{m_2+1}(P) + \dots}{\sqrt{y_{m_2+1}(P)}} = \sqrt{\frac{x_{m_1+2}}{dy_{m_2+1}}(\infty_g)}\frac{y_{m_2+1}(P) + \dots}{\sqrt{dx_{m_1+2}(P)}}\ . \tag{4.27}$$

and

$$dg(P) = d\left(\frac{1}{y_{m_2+1}(P)}\right) = -\frac{dy_{m_2+1}}{x_{m_1+2}}(\infty_g)\frac{dx_{m_1+2}(P)}{y^2_{m_2+1}(P)}\ . \tag{4.28}$$

As before, substituting these expressions to (4.26) and ignoring the arising power of $dx_{m_1+2}(P)$, we see that this limit equals -1.

Substituting this -1, together with the answer (4.25) for the limit (4.22), into (4.21), and collecting all terms in (4.16), we get

$$\left\{\frac{df(P)}{dg(P)}\frac{\prod_{k=1}^{m_2+2}\{E(P,G_k)\}^{s_k}}{\prod_{k=1}^{m_1+2}\{E(P,D_k)\}^{r_k}}\right\}^{2g-2}$$
$$= \left\{2d_1^2 d_2^{-(d_2+1)}\right\}(u_{d_1+1})^{1-\frac{1}{d_1}}\left(\prod_{l=1}^{m_1}\{dg(P_l)\}^{-1}\right)\frac{\prod_{D_l\neq G_k}\{E(D_l,G_k)\}^{r_l s_k}}{\prod_{k,l=1\,,\,k<l}^{m_1+2}\{E(D_l,D_k)\}^{2r_k r_l}}\ . \tag{4.29}$$

Now, computing the second "half" of (4.14) i.e. taking the product analogous to (4.16) over points of divisor (dg), and taking the product with (4.29), we get the statement of theorem 2.

4.3 Bergmann tau-function and $F^{(1)}$

Theorem 3 *The F^1 solution of equations (3.1), (3.2) (3.20) is given by any of the following two equivalent formulas:*

$$F^1 = \frac{1}{24}\ln\left\{\tau_f^{12}(v_{d_2+1})^{1-\frac{1}{d_2}}\prod_{k=1}^{m_1} dg(P_k)\right\} + \frac{d_2+3}{24}\ln d_2 + C \tag{4.30}$$

or

$$F^{(1)} = \frac{1}{24}\ln\left\{\tau_g^{12}(u_{d_1+1})^{1-\frac{1}{d_1}}\prod_{k=1}^{m_2} df(Q_k)\right\} + \frac{d_1+3}{24}\ln d_1 + C\ . \tag{4.31}$$

Here τ_f and τ_g are Bergmann tau-function (4.6) built from divisors (df) and (dg), respectively; C is a constant.

Proof. From the formulas (4.8), (4.9) it follows that expressions (4.30) and (4.31) define the same function. According to Proposition 1, expression (4.30) satisfies equations (3.1), (3.20) with respect to coefficients of V_1. Similarly, expression (4.31) satisfies analogous system (3.2) with respect to coefficients of V_2.

Remark 2 (higher order branch points) If potentials V_1 and V_2 are non-generic i.e. some (or all) of the branch points have multiplicity higher than 1, formula (4.31) should be only slightly modified. Namely, the expression for Bergmann tau-function (4.6) formally remains the same in terms of divisor of differential df (the zeros of df can now have arbitrary multiplicities). The expression for F^1 then looks as follows:

$$F^1 = \frac{1}{48}\ln\left\{\tau_f^{24}(v_{d_2+1})^{2-\frac{2}{d_2}}\prod_{k=1}^{m_1}\mathrm{res}|_{P_m}\frac{(dg)^2}{df}\right\} + \frac{d_2+3}{24}\ln d_2 + C\ . \tag{4.32}$$

The proof of (4.32) is slightly more involved technically than the generic case and will be published separately.

5 Equations with respect to filling fractions

it is well-known (see for example [17]) that the normalized ($\oint_{a_\alpha} w_\beta = \delta_{ab}$) holomorphic differentials can be expressed as follows in terms of f and g:

$$2\pi i w_\alpha(P) = \frac{\partial g(P)}{\partial\epsilon_\alpha}\Big|_{f(P)} df(P) \tag{5.1}$$

(Sketch of the proof: differentiating (2.14) with respect to ϵ_β, we verify the normalization conditions for differentials (5.1). The 1-form gdf is singular at ∞_f and ∞_g; at ∞_f we have $g = V_1'(f) - 1/f + \ldots$; this singularity disappear since coefficients of V_1 and V_2 are independent of filling fractions. Singularities at branch points P_k of derivative of g with respect to ϵ_α get cancelled by zeros of df at these points. At ∞_g we have: $x = V_2'(g) - 1/g + \ldots$; due to thermodynamic identity

$$\frac{\partial g}{\partial\epsilon_\alpha}\Big|_f df = -\frac{\partial f}{\partial\epsilon_\alpha}\Big|_g dg\ .$$

Since coefficients of V_2 are independent of ϵ_α, singularity of gdf at ∞_g also disappears after differentiation.)

To obtain equations for derivatives of $F^{(1)}$ with respect to the filling fractions we shall prove the following analog of lemma 1:

Lemma 5 *The following deformation equations with respect to filling fractions take place:*

$$\partial_{\epsilon_\alpha}\lambda_k = -2\pi i\frac{w_\alpha(P_k)}{g'(P_k)} \tag{5.2}$$

$$\frac{\partial\{g'(P_k)\}}{\partial\epsilon_\alpha} = \frac{\pi i}{2}\left\{w''_a(P_k) - \frac{g'''(P_k)}{g'(P_k)}w_\alpha(P_k)\right\} \; . \tag{5.3}$$

Proof is parallel to the proof of (2.16) and (2.17): from (5.1) we have

$$\frac{\partial g(P)}{\partial\epsilon_\alpha}\Big|_{x_k(P)} df(P) - \frac{\partial f(P)}{\partial\epsilon_\alpha}\Big|_{x_k(P)} dg(P) = 2\pi i w_\alpha(P) \; . \tag{5.4}$$

Substituting in (5.4) the local expansions (2.21) of $g(P)$ and (2.22) of $dg(P)$, and expansion of $w_\alpha(P)$

$$w_\alpha(P) = (w_\alpha(P_k) + w'_\alpha(P_k)x_k + \frac{w''_\alpha(P_k)}{2}x_k^2 + \dots)dx_k \; , \tag{5.5}$$

we get, since $f(P) = x_k^2(P) + \lambda_k$ and $df(P) = 2x_k(P)dx_k(P)$:

$$(\partial_{\epsilon_\alpha} g(P_k) + x_k\partial_{\epsilon_\alpha} g'(P_k) + \frac{1}{2}\partial_{\epsilon_\alpha} g''(P_k) + \dots)2x_k dx_k - \partial_{\epsilon_\alpha} f_k(g'(P_k) + g''(P_k)x_k + \frac{1}{2}g'''(P_k)x_k^2 + \dots)dx_k$$

$$= 2\pi i(w_\alpha(P_k) + w'_\alpha(P_k)x_k + \frac{1}{2}w''_\alpha(P_k)x_k^2)dx_k \; .$$

The zeroth order term gives (5.2). Collecting the coefficients in front of x_k^2, and using(5.2), we get (5.3).

Theorem 4 *Derivatives of function F^1 (4.30), (4.31) with respect to the filling fractions look as follows:*

$$\frac{\partial F^1}{\partial\epsilon_\alpha} = -\oint_{b_\alpha} Y^1(P)df(P) \; , \tag{5.6}$$

where $Y^1 df$ is defined by (3.20) .

Proof. The vectors of b-periods of these 1-forms $B(P, P_k)$ and $D(P, P_k)$ can be expressed in terms of holomorphic differentials via the following standard formulas:

$$\oint_{b_a} B(P, P_k) = 2\pi i w_a(P_k) \; , \qquad \oint_{b_\alpha} D(P, P_k) = 2\pi i w''_\alpha(P_k) \; . \tag{5.7}$$

Therefore, the b-periods of the 1-form $-Y^{(1)}(P)df(P)$ (3.20) are given by the following expression:

$$-\oint_{b_\alpha} Y^{(1)}(P)df(P) = 2\pi i\sum_{k=1}^{m_1}\left\{-\frac{w''_a(P_k)}{96g'(P_k)} + \frac{g'''(P_k)w_a(P_k)}{96g'^2(P_k)} + \frac{S_B(P_k)w_a(P_k)}{24g'(P_k)}\right\} \; . \tag{5.8}$$

On the other hand, derivatives of F^1 (4.30) with respect to ϵ_α can be computed using (5.2), (5.3) and equations for Bergmann tau-function (3.23), which also leads to (5.8).

6 F^1 of two-matrix model, isomonodromic tau-function, G-function of Frobenius manifolds, and determinant of Laplace operator

Here we outline some links between the expression (4.30), (4.31) for F^1 and other well-known objects.

6.1 F^1, isomonodromic tau-function and G-function of Frobenius manifolds

We recall that the genus 1 correction to free energy in topological field theories is given by so-called G-function of associated Frobenius manifolds. The G-function is a solution of Getzler equation [30]; for Frobenius manifolds related to quantum cohomologies, the G-function was intensively studied as generating function of elliptic Gromov-Witten invariants (see [24, 31] for references). In [24] it was found the following formula for G-function of an arbitrary m-dimensional Frobenius manifold:

$$G = \ln \frac{\tau_I}{\prod_{k=1}^m \eta_{kk}^{1/48}} \tag{6.1}$$

where τ_I is the Jimbo-Miwa tau-function of Riemann-Hilbert problem associated to a given Frobenius manifold [18]; η_{kk} are elements of Egoroff-Darboux (pseudo) metric (written in canonical coordinates) corresponding to the Frobenius manifold.

One of the well-studied classes of Frobenius manifolds arises from Hurwitz spaces [18]. For these Frobenius manifolds the isomonodromic tau-function τ_I [18] is related to Bergmann tau-function τ_f (3.23) as follows [25]:

$$\tau_I = \tau_f^{-1/2}\,, \tag{6.2}$$

where f stands for meromorphic function on Riemann surface $\mathcal{L}$. Therefore, the tau-function terms, which are the main ingredients of the formulas (4.30) for F^1 and (6.1) for the G-function coincide (up to a sign, which is related to the choice of the sign in the exponent in the definition (1.1) of the free energy). The solution of Fuchsian system corresponding to tau-function τ_I is not known explicitly. However, the same function τ_I, being multiplied with certain theta-functional factor, gives tau-function of an arbitrary Riemann-Hilbert problem with quasi-permutation monodromy matrices which was solved in [26].

The metric coefficients of Darboux-Egoroff metric corresponding to Hurwitz Frobenius manifold are defined in terms of an "admissible" 1-form φ, defining the Frobenius manifold:

$$\eta_{kk} = \text{res}|_{P_k} \frac{\varphi^2}{df}\,. \tag{6.3}$$

If, trying to develop an analogy with our formula (4.30) for F^1, we formally choose $\varphi(P) = dg(P)$, we get $\eta_{kk} = g'^2(P_k)/2$ and the formula (6.1) coincides with (4.30) up to small details like sign, additive constant, and the highest coefficient of polynomial V_2 arising from requirement of symmetry $f \leftrightarrow g$.

Therefore, we got complete formal analogy between our expression (4.30) for F^1 and Dubrovin-Zhang formula (6.1) for G-function. Unfortunately, for the moment this analogy remains only formal, since, from the point of view of Dubrovin's theory [18], the differential dg is not admissible; therefore, the metric $\eta_{kk} = g'^2(P_k)/2$ built from this differential is not flat, and, strictly speaking, it does not correspond to any Frobenius manifold. Therefore, the true origin of the analogy between the G-function of Frobenius manifolds and F^1 still has to be explored.

6.2 F^1 and determinant of Laplace operator

Existence of close relationship between F^1 and determinant of certain Laplace operator was suggested by several authors (see e.g. [27] for hermitial one-matrix model, [15] for hermitian two-matrix model and, finally, [28] for normal two-matrix model with simply-connected support of eigenvalues, where F^1 is claimed to coincide with determinant of Laplace operator in the domain with Dirichlet boundary conditions).

However, in the context of hermitial two-matrix model (as well as in the case of hermitian one matrix model [27]) this relationship is more subtle.

First, if we don't impose any reality conditions on coefficients of polynomials V_1 and V_2, function F^1 is holomorphic function of our moduli parameters (i.e. coefficients of V_1, V_2 and filling fractions), while detΔ is always a real function. The Laplace operator Δ^f which should be playing a role here corresponds to the singular metric of infinite volume $|df|^2$.

This problem disappears if we start from more physical situation, when all these moduli parameters are real, as well as the branch points of the Riemann surface $\mathcal{L}$ with respect to both projections. In this case F^1 is real itself, as well as determinant of Laplace operator. However, little is known about rigorous definition for determinants of such Laplace operators, although such objects were actively used by string theorists without rigorous mathematical justification [32, 34, 33]. According to empirical results of [34], the regularised determinant of Laplace operator Δ^f is given by the formula

$$\frac{\det\Delta^f}{\mathcal{A}\det\Im\mathbf{B}} = C|\tau_f|^2 \,, \tag{6.4}$$

where $\mathcal{A}$ is a regularised area of $\mathcal{L}$, Δ^f is Laplace operator defined in singular metric $|df(P)|^2$, $\mathbf{B}$ is the matrix of b-periods of $\mathcal{L}$, C is a constant.

In the "physical" case of real moduli parameters the empirical expression (6.4) for $\ln\{\det\Delta^f\}$ coincides with F^1 (4.30) up to a simple power and additional multipliers.

Therefore, the relationship between hermitial and normal two-matrix models [28] on the level of F^1 is not as straightforward as on the level of functions F^0 (F^0 for hermitian two-matrix mode can be obtained from F^1 for normal two-matrix model by a simple analytical continuation [16, 17, 35]).

From the point of view of determinants of Laplace operators the theorem 2 which tells how the Bergmann tau-function depends on the projection choice is nothing but a version of Alvarez-Polyakov formula [36], which describes variation of detΔ if the metric changes within given conformal class.

7 From two-matrix to one-matrix model: hyperelliptic curves

Suppose that $d_2 = 1$, i.e. polynomial V_2 is quadratic. Then integration with respect to M_2 in (1.1) can be carried out explicitly, and we get

$$Z_N \equiv e^{-N^2F} = C\int dMe^{-N\mathrm{tr}V(M)} \tag{7.1}$$

where $M := M_1$, $V := V_1$ and C is a constant. Hence, in this case (1.1) gives rise to the partition function of one-matrix model.

For $d_2 = 1$ the function $f(P)$ has two poles of order 1 at ∞_f and ∞_g; thus, the spectral curve $\mathcal{L}$ is hyperelliptic and function $f(P)$ defines two-sheeted covering of $CP1$. The number of branch points in this case equals $m_1 \equiv 2g+2$; as before, we call them $\lambda_1, \dots, \lambda_{2g+2}$. The Bergmann tau-function (3.23) for hyperelliptic curves was computed in [23]; in this case it admits the following, alternative to (4.6), (4.7) expression:

$$\tau_f = \Delta^{1/4}\det\mathbf{A} \tag{7.2}$$

where

$$\Delta := \prod_{j<k\,,\; j,k=1}^{2g+2}(\lambda_j - \lambda_k)\,, \tag{7.3}$$

$\mathbf{A}$ is the matrix of a-periods of non-normalized holomorphic differentials on $\mathcal{L}$:

$$\mathbf{A}_{\alpha\beta} = \oint_{a_\alpha} \frac{x^{\beta-1}dx}{\nu} \; ; \tag{7.4}$$

where

$$\nu^2 = \prod_{k=1}^{2g=2} (x - \lambda_k)$$

is the equation of spectral curve $\mathcal{L}$.

Substituting formula (7.2) into (4.30), and ignoring coefficient v_{d_2+1} (it becomes part of constant C), we get the expression

$$F^1 = \frac{1}{24} \ln \left\{ \Delta^3 \left(\det\mathbf{A}\right)^{12} \prod_{k=1}^{2g+2} g'(\lambda_k) \right\} \tag{7.5}$$

which agrees with previously known results [9, 10, 13, 11].

Acknowledgements We thank M.Bertola, L.Chekhov, B.Dubrovin, T.Grava, V.Kazakov, I.Kostov, M.Staudacher and S.Theisen for important discussions. The work of BE was partially supported by the EC ITH Network HPRN-CT-1999-000161. The work of DK was partially supported by NSERC, NATEQ and Humboldt foundation. AK and DK thank Max-Planck Institute for Mathematics in Bonn for support and nice working conditions. DK thanks also SISSA and CEA for support and hospitality. BE thanks CRM for support and hospitality.

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