$1/N^2$ correction to free energy in hermitian two-matrix model

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Abstract. We find an explicit expression for genus 1 correction in Hermitian two-matrix model in terms of holomorphic objects associated to spectral curve arising in large N limit. Our result generalizes known expression for F^1 in hermitian one-matrix model. We discuss the relationship between F^1, isomonodromic tau-function, Bergmann tau-functiopn on Hurwitz spaces, G-function of Frobenius manifolds and determinant of Laplacian over spectral curve.

In this letter we derive an explicit formula for the $1/N^2$ correction to free energy F of hemitian two-matrix model:

$$e^{-N^2F} := \int dM_1 dM_2 e^{-N\mathrm{tr}\{V_1(M_1)+V_2(M_2)-M_1M_2\}} \tag{1}$$

when the eigenvalues of M_1 and M_2 are concentrated over finite sets of intervals (cuts).

It is hard to overestimate the interest to random matrix models in modern prhysics and mathematics; we just mention their appearance in statistical physics condensed matter 2d quantum gravity and number theory (see e.g. books [1, 2] and reviews [3, 4]) . The expansion $F = \sum_{G=0}^{\infty} N^{-2G}F^G$ (N is the matrix size) in hermitian matrix models is one of the cornerstones of the theory, due to its clear physical interpretation as topological expansion of the functional integral, which appears in $N \to \infty$ limit; in statistical physics the term F^G plays the role of free energy fo statistical physics model on genus G Riemann surface.

From the whole zoo if the random matrices one of the simplest is the hermitian one-matrix model with partition function $e^{-N^2F} = \int dM e^{-N\mathrm{tr}V(M)}$ (V is a polynomial), which can be used as testing ground for the methods applied in more general situations of two- and multi- matrix models. In particular, the one-matrix model seems to be the only case for which existence of $1/N^2$ expansion was proved rigorously [5] using the method of Riemann-Hilbert problem proposed in [6]. In two-matrix models, although the existence of this expansion is suggested by many physical reasons, the rigorous proof is still lacking, thus we consider it as an assumption.

The most rigorous way to compute the $1/N^2$ expansion for both one-matrix and two-matrix models is based on the loop equations. The loop equations follow from the reparametrization invariance of matrix integrals; for one-matrix case the loop equations were derived in [7]; in [8] the loop equations were used to compute F^1 for one-cut case of 1-matrix model; in [9] F^1 was computed for the two-cut case. Recently an expression for F^1 of one-matrix model in two-cut case was also derived in [10]; the formula for multi-cut case was derived in [11] following Kostov's ideas [12]. Recently it was confirmed in [13] in loop equations approach.

In [14] the loop equations were written down for the case of two-matrix model, and F^1 was found for the case when the spectral curve has genus zero; in [15] the answer for genus 1 spectral curve was found. For arbitrary genus of spectral curve of two-matrix model only the leading term F^0 is known; it was recently computed in [16].

Let us write down polynomials V_1 and V_2 in the form $V_1(x) = \sum_{k=1}^{d_1+1} \frac{u_k}{k} x^k$ and $V_2(y) = \sum_{k=1}^{d_2+1} \frac{v_k}{k} y^k$. It is sometimes convenient to think of V_1 and V_2 as infinite power series: $V_1(x) = \sum_{k=1}^{\infty} \frac{u_k}{k} x^k$, $V_2(y) =$

$\sum_{k=1}^{\infty} \frac{v_k}{k} y^k$, where coefficients u_k vanish for $k \geq d_1 + 2$, and v_k vanish for $k \geq d_2 + 2$. According to this point of view the operators of differentiation with respect to coefficients of V_1 and V_2 have the following meaning:

$$\frac{\delta}{\delta V_1(x)}\Big|_x := \left\{ \sum_{k=1}^{\infty} x^{-k-1} k \partial_{u_k} \right\} \Big|_{u_k=0\ ,k\geq d_1+2} , \qquad \frac{\delta}{\delta V_2(y)}\Big|_y := \left\{ \sum_{k=1}^{\infty} y^{-k-1} k \partial_{v_k} \right\} \Big|_{v_k=0\ ,k\geq d_1+2} . \tag{2}$$

As it was discussed in detail in [17], to think of V_1 and V_2 as about infinite power series is not really necessary: one can build the whole formalizm without appealing to infinite series; however, here we use the notations (2), since this significantly shortens the presentation. In particular, according to these notations,

$$\frac{\delta V_1(x)}{\delta V_1(\tilde{x})} = \frac{1}{\tilde{x} - x} , \qquad \frac{\delta {V_1}'(x)}{\delta V_1(\tilde{x})} = \frac{1}{(\tilde{x} - x)^2} \tag{3}$$

Consider the resolvents

$$\mathcal{W}(x) = \frac{1}{N} \Big\langle \mathrm{tr} \frac{1}{x - M_1} \Big\rangle \qquad \text{and} \qquad \tilde{\mathcal{W}}(y) = \frac{1}{N} \Big\langle \mathrm{tr} \frac{1}{y - M_2} \Big\rangle \tag{4}$$

As a corollary of (3), the free energy of two-matrix model (1) satisfies the following equations with respect to coefficients of polynomials V_1 and V_2:

$$\frac{\delta F}{\delta V_1(x)} = \mathcal{W}(x) , \qquad \frac{\delta F}{\delta V_2(y)} = \tilde{\mathcal{W}}(y) . \tag{5}$$

Assuming existence of $1/N^2$ expansion, the equations (5) were solved in [16] in the zeroth order assuming the finite-gap structure of distribution of eigenvalues of M_1 (and, *a posteriori*, also of M_2) as $N \to \infty$.

Here we find the next coefficient F^1, using the loop equations and a natural additional assumption. The spectral curve $\mathcal{L}$ is defined by the following equation, which arises in the zeroth order approximation:

$$\mathcal{E}^0(x, y) := ({V_1}'(x) - y)({V_2}'(y) - x) - \mathcal{P}^0(x, y) + 1 = 0 \tag{6}$$

where the polynomial of two variables $\mathcal{P}^0(x, y)$ is the zeroth order term in $1/N^2$ expansion of the polynomial

$$\mathcal{P}(x, y) := \frac{1}{N} \Big\langle \mathrm{tr} \frac{{V_1}'(x) - {V_1}'(M_1)}{x - M_1} \frac{{V_2}'(y) - {V_2}'(M_2)}{y - M_2} \Big\rangle ; \tag{7}$$

the point $P \in \mathcal{L}$ of the curve is the pair of complex numbers (x, y) satisfying (6).

The spectral curve (6) comes together with two mermorphic functions $f(P) = x$ and $g(P) = y$, which project it down to x and y-planes, respectively. These functions have poles only at two points of $\mathcal{L}$, called ∞_f and ∞_g: at ∞_f function $f(P)$ has simple pole, and function $g(P)$ - pole of order d_1 with singular part equal to ${V_1}'(f(P))$. At ∞_g the function $g(P)$ has simple pole, and function $f(P)$ - pole of order d_2 with singular part equal to ${V_2}'(g(P))$. Therefore, one gets the moduli space $\mathcal{M}$ of triples $(\mathcal{L}, f, g)$, where functions f and g have this pole structure. The natural coordinates on this moduli space are coefficients of polynomials V_1 and V_2 and g numbers, called "filling fractions" $\epsilon_\alpha = \frac{1}{2\pi i} \oint_{a_\alpha} g df$, where (a_α, b_α) is some basis of canonical cycles on $\mathcal{L}$. The additional constraints which should be imposed *a posteriori* to make the "filling fractions" dependent on coefficients of polynomials V_1 and V_2 are (according to one-matrix model experience, these conditions correspond to non-tunneling between different intervals of eigenvalues support):

$$\oint_{b_a} g df = 0 \; .$$

Denote the zeros of differential df by $P_1, \dots, P_{m_1}$ ($m_1 = d_2 + 2g + 1$) (these points play the role of ramification points if we realize $\mathcal{L}$ as branched covering by function $f(P)$); their projections on x-plane are the branch points, which we denote we denote by $\lambda_j := f(P_j)$. The zeros of the differential dg (the ramification points if we consider $\mathcal{L}$ as covering defined by function $g(P)$) we denote by $Q_1, \dots, Q_{m_2}$ ($m_2 = d_1 + 2g + 1$); there projections on y-plane (the branch points) we denote by $\mu_j := g(Q_j)$. We shall assume hat our potentials V_1 and V_2 are generic i.e. all zeros of differentials df and dg are simple.

If is well-known [16] how to express all standard algebro-geometrical objects on $\mathcal{L}$ in terms of the previous data. In particular, the Bergmann bidifferential $B(P,Q) = d_P d_Q \ln E(P,Q)$ ($E(P,Q)$ is the prime-form) can be represented as follows:

$$B(P,Q) = \frac{\delta g(P)}{\delta V_1(f(Q))}\Big|_{f(Q)} df(P) df(Q) \tag{8}$$

The Bergmann bidifferential has the following behaviour near diagonal $P \to Q$: $B(P,Q) = \Big\{ \frac{1}{(z(P)-z(Q))^2} + \frac{1}{6} S_B(P)$ where $z(P)$ is some local coordinate; $S_B(P)$ is the Bergmann projective connection . Consider also the four-differential $D(P,Q) = d_P d_Q^3 \ln E(P,Q)$, which has on the diagonal the pole of 4th degree: $D(P,Q) = \{6(z(P)-z(Q))^{-4} + O(1)\} dz(P)(dz(Q))^3$. From $B(P,Q)$ and $D(P,Q)$ it is easy to construct meromorphic normalized (all a-periods vanish) 1-forms on $\mathcal{L}$ with single pole; in particular, if the pole coincides with ramification point P_k, the natural local parameter near P_k is $x_k(P) = \sqrt{f(P) - \lambda_k}$; then $B(P,P_k) := \frac{B(P,Q)}{dx_k(Q)}\Big|_{Q=P_k}$ and $D(P,P_k) := \frac{D(P,Q)}{(dx_k(Q))^3}\Big|_{Q=P_k}$ are meromorphic normalized 1-forms on $\mathcal{L}$ with single pole at P_k and the following singular parts:

$$B(P,P_k) = \left\{ \frac{1}{[x_k(P)]^2} + \frac{1}{6} S_B(P_k) + o(1) \right\} dx_k(P) \; ; \qquad D(P,P_k) = \left\{ \frac{6}{[x_k(P)]^4} + O(1) \right\} dx_k(P) \tag{9}$$

as $P \to P_k$, where $S_B(P_k)$ is the Bergmann projective connection computed at the branch point P_k with respect to the local parameter $x_k(P)$.

Equations (5) in order $1/N^2$ look as follows (we write only equations with respect to V_1):

$$\frac{\delta F^1}{\delta V_1(f(P))} = -Y^1(P) \tag{10}$$

where the Y^1 is the (taken with minus sign) $1/N^2$ contribution to the resolvent $\mathcal{W}$. The function Y^1 can be computed using the loop equations [14] and an additional assumption that

$$\oint_{a_\alpha} Y^1(P) df(P) = 0 \tag{11}$$

over all basic a-cycles (which means that the "filling fractions" do not have the $1/N^2$ correction; the arguments in favor of this assumption come from comparison of the answer with known expression of F^1 in one-matrix model and symmetry of the final answer with respect to change of x and y).

To write down the loop equation we introduce also the polynomial

$$\mathcal{E}(x,y) := (V_1(x)-y)(V_2(y)-x) - \mathcal{P}(x,y) + 1 \; , \tag{12}$$

the function $\mathcal{U}(x,y)$, which is a polynomial in y and rational function in x:

$$\mathcal{U}(x,y) := \frac{1}{N}\left\langle \mathrm{tr} \frac{1}{x-M_1} \frac{V_2{}'(y) - V_2{}'(M_2)}{y-M_2} \right\rangle \; , \tag{13}$$

and rational function $\mathcal{U}(x,y,z)$:

$$\mathcal{U}(x,y,z) := \frac{\delta \mathcal{U}(x,y)}{\delta V_1(z)} = \left\langle \mathrm{tr} \frac{1}{x-M_1} \frac{V_2{}'(y) - V_2{}'(M_2)}{y-M_2} \mathrm{tr} \frac{1}{z-M_1} \right\rangle - N^2 \mathcal{U}(x,y)\mathcal{W}(z) \; . \tag{14}$$

Then the loop equation looks as follows:

$$\mathcal{U}(x,y) = x - V_2{}'(y) + \frac{\mathcal{E}(x,y)}{y-Y(x)} - \frac{1}{N^2}\frac{\mathcal{U}(x,y,x)}{y-Y(x)} \; ; \tag{15}$$

it arises as a corollary of reparametrization invariance of the matrix integral (1) [14]. The residue at $y = Y(x)$ of (15) leads to "master" loop equation for function $Y(x) := V_1{}'(x) - \mathcal{W}(x)$:

$$\mathcal{E}(x, Y(x)) = \frac{1}{N^2}\mathcal{U}(x, Y(x), x) \; . \tag{16}$$

To use the loop equations effectively we need to consider the $1/N^2$ expansion of all of their ingredients. In this way we get the following expression for Y^1:

$$Y^1(P)df(P) = \frac{\mathcal{P}^1(f(P),g(P))df(P)}{\mathcal{E}^0_y(f(P),g(P))} + \sum_{Q\neq P\,:\,f(Q)=f(P)} \frac{B(P,Q)}{df(Q)} \frac{1}{g(P)-g(Q)} \; ; \tag{17}$$

all ingredients of this expression arise already in the leading term, except $\mathcal{P}^1$. However, from (7) we see that $\mathcal{P}(x,y)$ is a polynomial of degree $d_1 - 1$ with respect to x and $d_2 - 1$ with respect to y; moreover, the coefficient in front of $x^{d_1-1}y^{d_2-1}$ does not have $1/N^2$ correction. Thus we can conclude that the one-form $Y^1(P)df(P)$ is non-singular on the spectral curve outside of the branch points P_m (where it has poles of order 4); moreover, the first term in (17) is non-singular on $\mathcal{L}$ (the first order zeros of $\mathcal{E}^0_y$ at the branch points are cancelled by first order zeros of $df(P)$ at these points). The form of singular parts at P_m allows to determine $Y^1(P)df(P)$ completely in terms of differentials $B(P,P_k)$ and $D(P,P_k)$ if we make the assumption of vanishing of all a-periods of the 1-form $Y^1(P)df(P)$ (11). This assumption is justified *a posteriori* by comparison of final answer for F^1 with known expression for F^1 in the partial case of 1-matrix model, and by the symmetry of final answer with respect to the change of projection $x \leftrightarrow y$.

Taking into account (11), we can write down the meromorphic 1-form $Y^1(P)df(P)$ in terms of differentials $B(P,P_k)$ and $D(P,P_k)$:

$$Y^{(1)}(P)df(P) = \sum_{k=1}^{m_1}\left\{ -\frac{1}{96 g'(P_k)} D(P,P_k) + \left[\frac{g'''(P_k)}{96 g'^2(P_k)} - \frac{S_B(P_k)}{24 g'(P_k)}\right] B(P,P_k) \right\} \tag{18}$$

Then solution of (10), (18), which is symmetric with respect to the projection change (and, therefore, satisfies also equations (5) with respect to V_2), looks as follows:

$$F^1 = \frac{1}{24} \ln \left\{ \tau_f^{12} (v_{d_2+1})^{1-\frac{1}{d_2}} \prod_{k=1}^{m_1} dg(P_k) \right\} + \frac{d_2+3}{24} \ln d_2 \tag{19}$$

where τ_f is the so-called Bergmann tau-function on Hurwitz space [21], which satisfies the following system of equations with respect to the branch points λ_k:

$$\frac{\partial}{\partial \lambda_k} \ln \tau_f = -\frac{1}{12} S_B(P_k) \; ; \tag{20}$$

In derivation of (18) we have used the following variational formulas, which can be easily proved in analogy to rauch variational formulas [21]:

$$-\frac{\delta \lambda_k}{\delta V_1(f(P))} g'(P_k) df(P) = B(P, P_k) \; , \tag{21}$$

$$\frac{\delta\{g'(P_k)\}}{\delta V_1(f(P))}\Big|_{f(P)} df(P) = \frac{1}{4} \left\{ D(P, P_k) - \frac{g'''(P_k)}{g'(P_k)} B(P, P_k) \right\} \tag{22}$$

The Bergmann tau-function (20) appears in many important problems: it coincides with isomonodromic tau-function of Hurwitz Frobenius manifolds [18], and gives the main contribution to G-function (solution of Getzler equation) of these Frobenius manifolds; it gives the most non-trivial term in isomonodromic tau-function of Riemann-Hilbert problem with quasi-permutation monodromies. Finally, its modulus square essentially coincides with determinants of Laplace operator in metrics with conical singularities over Riemann surfaces [19]. The solution of the system (20) looks as follows [20]. Define the divisor $(df) = -2\infty_f - (d_2+1)\infty_g + \sum_{k=1}^{m_2} P_k := \sum_{k=1}^{m_2+2} r_k D_k$. Choose some initial point $P \in \hat{\mathcal{L}}$ and introduce corresponding vector of Riemann constants K^P and Abel map $\mathcal{A}_\alpha(Q) = \int_P^Q w_\alpha$ (w_α form the basis of normalized holomorphic 1-froms on $\mathcal{L}$). Since some points of divisor (df) have multiplicity 1, we can always choose the fundamental cell $\hat{\mathcal{L}}$ of the universal covering of $\mathcal{L}$ in such a way that $\mathcal{A}((df)) = -2K^P$ (for an arbitrary choice of fundamental domain these two vectors coincide only up to an integer combination of periods of holomorphic differentials), where the Abel map is computed along the path which does not intersect the boundary of $\hat{\mathcal{L}}$.

The main ingredient of the Bergmann tau-function is the following holomorphic maltivalued $(1-g)g/2$-differential $\mathcal{C}(P)$ on $\mathcal{L}$:

$$\mathcal{C}(P) := \frac{1}{W(P)} \sum_{\alpha_1,\dots,\alpha_g=1}^{g} \frac{\partial^g \Theta(K^P)}{\partial z_{\alpha_1} \dots \partial z_{\alpha_g}} w_{\alpha_1}(P) \dots w_{\alpha_g}(P) \; . \tag{23}$$

where

$$W(P) := \det_{1 \le \alpha,\beta \le g} || w_\beta^{(\alpha-1)}(P) || \tag{24}$$

denotes the Wronskian determinant of holomorphic differentials at point P; K^P is the vector of Riemann constants with basepoint P. Introduce the quantity $\mathcal{Q}$ defined by

$$\mathcal{Q} = [df(P)]^{\frac{g-1}{2}} \mathcal{C}(P) \prod_{k=1}^{m+N} [E(P, D_k)]^{\frac{(1-g)r_k}{2}} \; ; \tag{25}$$

which is independent of the point $P \in \mathcal{L}$. Then the Bergmann tau-fuction (20) of Hurwitz space is given by the following expression [20]:

$$\tau_f = \mathcal{Q}^{2/3} \prod_{k,l=1 \;\; k<l}^{m+n} [E(D_k, D_l)]^{\frac{r_k r_l}{6}} \; ; \tag{26}$$

together with (19) this gives the answer for $1/N^2$ correction in two-matrix model under the assumptions discussed above.

If τ_f and τ_g are Bergmann tau-functions (20) corresponding to divisors (df) and (dg), respectively, then

$$\left(\frac{\tau_f}{\tau_g}\right)^{12} = C \frac{(u_{d_1+1})^{1-\frac{1}{d_1}}}{(v_{d_2+1})^{1-\frac{1}{d_2}}} \frac{\prod_k df(Q_k)}{\prod_k dg(P_k)} \tag{27}$$

where $C = d_1^{d_1+3}/d_2^{d_2+3}$ is a constant independent of moduli parameters. Using the transformation (27) of the Bergmann tau-function under projection change, we find that the solution expression (19) for F^1 satisfies also the necessary equations with respect to V_2. This could be considered as a verification of consistency of our computation, which indirectly justifies our main assumption of non-singularity of resolvents outside of the branch points.

Derivatives of function F^1 (19) with respect to the filling fractions look as follows:

$$\frac{\partial F^1}{\partial \epsilon_\alpha} = -\oint_{b_\alpha} Y^1(P) df(P) \; ; \tag{28}$$

these equations are $1/N^2$ counterparts of Seiberg-Witten type equations for F^0 found in [22, 16].

If potential V_2 is quadratic, integration with respect to M_2 in (1) can be taken explicitly, and the free energy (19) gives rise to the free energy of one-matrix model. The spectral curve $\mathcal{L}$ in this case becomes hyperelliptic, and the formula (19) gives, using the expression for τ_f obtained in [23]:

$$F^1 = \frac{1}{24} \ln \left\{ \Delta^3 \, (\det \mathbf{A})^{12} \prod_{k=1}^{2g+2} g'(\lambda_k) \right\} \tag{29}$$

where λ_k, $k = 1, \dots, 2g+2$ are branch points of $\mathcal{L}$; Δ is their Wronskian determinant; $\mathbf{A}$ is the matrix of a-periods of non-normalized holomorphic differentials on $\mathcal{L}$. Comparing this answer with previous results, we see that in genus 1 it agrees with formulas of [10] (there is a slight deviation with formulas of Akemann [9] due to a different choice of normalization). In higher genus formula (29) coincides with formula (4.5) of [11] (whose derivation, based on the use of so-called star operator, follows unpublished notes by I.Kostov); in the framework of loop equations expression (29) was verified in [13].

In strictly physical situation potentials V_1 and V_2 should be such that, considering $\mathcal{L}$ as a covering defined by function f, one can single out the "physical" sheet (which includes point ∞_f) such that all a-cycles lie on this sheet and each a-cycle encircles exactly one branch cut (all corresponding branch points must be real if potentials V_1 and V_2 are real). Similar requirement comes from g-projection of $\mathcal{L}$.

F^1, isomonodromic tau-function and G-function of Frobenius manifolds. The genus 1 correction to free energy in topological field theories is given by so-called G-function (solution of Getzler equation of associated Frobenius manifolds. In [24] it was found the following formula for G-function of an arbitrary m-dimensional Frobenius manifold:

$$G = \ln \frac{\tau_I}{\prod_{k=1}^m \eta_{kk}^{1/48}} \tag{30}$$

where τ_I is the Jimbo-Miwa tau-function of Riemann-Hilbert problem associated to a given Frobenius manifold [18]; η_{kk} are metric elements of Egoroff-Darboux (i.e. flat diagonal potential) metric corresponding to the Frobenius manifold, written in canonical coordinates.

One of the well-studied classes of Frobenius manifolds (corresponding to topological field theories of type B) arises from Hurwitz spaces [18]. For these Frobenius manifolds the isomonodromic tau-function τ_I [18] is related to Bergmann tau-function τ_f (26) as follows [25]: $\tau_I = \tau_f^{-1/2}$, where f stands for meromorphic function on Riemann surface $\mathcal{L}$. Threfore, the tau-function terms, which are the main ingredients of the formulas (19) for F^1 and (30) for the G-function coincide (up to a sign, which is related to the choice of the sign in the exponent in the definition (1) of the free energy). The monodromy group of Fuchsian system corresponding to tau-function τ_I is not known explicitly [18]; presumably, this monodromy group is generated by matrices of reflection. However, the same function τ_I, being multiplied with certain theta-functional factor, gives tau-function of an arbitrary Riemann-Hilbert problem with quasi-permutation monodromy matrices [26].

The metric coefficients of Darboux-Egoroff metric corresponding to Hurwitz Frobenius manifold are defined in terms of an "admissible" 1-form φ, defining the Frobenius manifold:

$$\eta_{kk} = \mathrm{res}|_{P_k} \frac{\varphi^2}{df} \; . \tag{31}$$

If, trying to build an analogy with our formula (19) for F^1, we formally choose $\varphi(P) = dg(P)$, we get $\eta_{kk} = g'^2(P_k)/2$ and the formula (30) coincides with (19) up to small details like sign, additive constant, and the highest coefficient of polynomial V_2 arising from requirement of symmetry $f \leftrightarrow g$.

Therefore, we got complete formal analogy between our expression (19) for F^1 and Dubrovin-Zhang formula (30) for G-function. Unfortunately, for the moment this analogy remains only formal, since, from the point of view of Dubrovin's theory [18], the differential dg is not admissible; therefore, the metric $\eta_{kk} = g'^2(P_k)/2$ built from this differential is not flat, and, strictly speaking, it does not correspond to any Frobenius manifold. Therefore, the true origin of the analogy between the G-function of Frobenius manifolds and F^1 still has to be explored.

F^1 and determinant of Laplace operator

Existence of close relationship between F^1 and determinant of certain Laplace operator was suggested by several authors (see e.g. [27] for hermitial one-matrix model, [15] for hermitian two-matrix model and, finally, [28] for normal two-matrix model with simply-connected support of eigenvalues, where F^1 is claimed to coincide with determinant of Laplace operator in the domain with Dirichlet boundary conditions).

Howevere, even if we assume that the conjectures of existence of $1/N^2$ expansion of F and the conjecture (11) are correct (which leads to our final formula (19),(26)) in the context of hermitial two-matrix model (as well as in the case of hermitian one matrix model [27]) this statement is not completely correct.

First, if we don't impose any reality conditions on coefficients of polynomials V_1 and V_2, function F^1 is holomorphic function of our moduli parameters (i.e. coefficients of V_1, V_2 and filling fractions), while $\det\Delta$ is always a real function. The Laplace operator Δ^f which should be playing a role here corresponds to the singular metric $|df|^2$.

This problem disappears if we start from more physical situation, when all these moduli parameters are real, as well as the branch points of the Riemann surface $\mathcal{L}$ with respect to both projections. In this case F^1 is also real. However, there is another problem: namely, little is known about rigorous definition (and, moreover, explicit expression) for determinants of such Laplace operators, although such objects were actively used by string theorists without rigorous mahematical justification [29, 30] (rigorous results concerning Laplace determinants for metrics of this type in genus 0 were obtained recently in [31], but the explicit formulas, and even rigorous definition of this object on Riemann surfaces of higher genus is still missing).

The natural conjecture [19] is that, after an appropriate regularization of the type [31], the determinant of Laplace operator Δ^f is given by the formula

$$\frac{\det\Delta^f}{\mathcal{A}\det\Im\mathbf{B}} = C|\tau_f|^2 \tag{32}$$

where $\mathcal{A}$ is a reqularized area of $\mathcal{L}$, Δ^f is Laplace operator defined in singular metric $|df(P)|^2$, $\mathbf{B}$ is the matrix of b-periods of $\mathcal{L}$, C is a constant. In the "physical" case of real moduli parameters the conjectured expression (32) for $\ln\{\det\Delta^f\}$ coincides with our formula (19) up to a simple power and explicit multipliers.

The relationship between hermitial and normal two-matrix models [28] on the level of F^1 seems to be not as straigtforward as on the level of functions F^0 (F^0 for hermitian two-matrix mode can be obtained from F^1 for normal two-matrix model by a simple analytical continuation [16, 17, 32]).

From the point of view of determinants of Laplace opertors the formula 27 which tells how the Bergmann tau-function depends on the projection choice is nothing but a version of Polyakov-Alvarez formula [33] for singular metrics.

Acknowledgements We thank M.Bertola, L.Chekhov, B.Dubrovin, T.Grava, V.Kazakov, I.Kostov, M.Staudacher and S.Theisen for important discussions. The work of DK was partially supported by NSERC, NATEQ, Humboldt foundation and Research Chair program of Concordia University. AK and DK thank Max-Planck Institute for Mathematics in Bonn, where most of their work on this paper was done, for support, warm hospitality and nice working conditions. DK thanks also SISSA and CEA for support and hospitality.

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