The PDEs of biorthogonal polynomials
arising in the two–matrix model [1]

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Abstract:

The two-matrix model can be solved by introducing bi-orthogonal polynomials. In the case the potentials in the measure are polynomials, finite sequences of bi-orthogonal polynomials (called *windows*) satisfy polynomial ODEs as well as deformation equations (PDEs) and finite difference equations (ΔE) which are all Frobenius compatible and define discrete and continuous isomonodromic deformations for the irregular ODE, as shown in previous works of ours.

In the one matrix model an explicit and concise expression for the coefficients of these systems is known and it allows to relate the partition function with the isomonodromic tau-function of the overdetermined system. Here, we provide the generalization of those expressions to the case of bi-orthogonal polynomials, which enables us to compute the determinant of the fundamental solution of the overdetermined system of ODE+PDEs+ΔE.

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1   Introduction

The two-matrix model and biorthogonal polynomials have recently witnessed a renewed interest due to the hope that the asymptotics could be found from a Riemann–Hilbert approach similar to that used in one-matrix models ([**?**] [**?, ?**], [**?**], [**?**] [**?**], [**?**]). Our recent papers have shown some unexpectedly rich algebraic structure associated to the biorthogonal polynomials for two polynomial potentials [**?, ?, ?**]. We recall that the two matrix-model is defined as the ensemble of pairs of hermitian matrices of size N, with the measure:

$$\frac{1}{\mathcal{Z}_N}e^{-\frac{1}{\hbar}[\operatorname{tr} V_1(M_1)+V_2(M_2)-M_1 M_2]}\mathrm{d}M_1\mathrm{d}M_2 \tag{1.1}$$

where $\mathrm{d}M_1\mathrm{d}M_2$ is the standard Lebesgue measure (product of Lebesgue measures of all real components of M_1 and M_2), V_1 and V_2 are two polynomials (called the potentials) of degree $d_1 + 1$ and $d_2 + 1$:

$$V_1(x) = \sum_{k=1}^{d_1+1} \frac{u_k}{k}x^k , \quad V_2(y) = \sum_{k=1}^{d_2+1} \frac{v_k}{k}y^k \tag{1.2}$$

and $\mathcal{Z}_N$ is the partition function:

$$\mathcal{Z}_N = \int e^{-\frac{1}{\hbar}[\operatorname{tr} V_1(M_1)+V_2(M_2)-M_1 M_2]}\mathrm{d}M_1\mathrm{d}M_2 \tag{1.3}$$

The two matrix model has a wide range of applications in physics, in particular Euclidean 2d gravity i.e. the statistical physics of a random surface [**?, ?, ?**].

The 2-matrix model can be solved with the help of two families of bi-orthogonal polynomials

$$\pi_n(x) = x^n + \dots , \quad \sigma_n(y) = y^n + \dots \tag{1.4}$$

which are such that:

$$\int\int \mathrm{d}x\mathrm{d}y \ \pi_n(x)\sigma_m(y) \ e^{-\frac{1}{\hbar}[V_1(x)+V_2(y)-xy]} = h_n\delta_{nm} . \tag{1.5}$$

`biorth`

It was shown in [**?**], that a sequence of $d_2 + 1$ consecutive π_n's (or $d_1 + 1$ consecutive σ_n's) obeys a closed ODE with respect to the "spectral parameter" x (or y)

$$-\hbar\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\psi_{N-d_2}(x)\\ \vdots \\ \psi_N(x)\end{pmatrix} = \underset{N}{D_1}(x)\begin{pmatrix}\psi_{N-d_2}(x)\\ \vdots \\ \psi_N(x)\end{pmatrix} , \tag{1.6}$$

as well as deformation equation (PDEs) with respect to infinitesimal variations of the coefficients of the potentials V_1, V_2 and difference equations with respect to the choice of the consecutive elements (ΔE). The aim of this article is to provide for an explicit expression of said ODE and PDEs, which is similar in spirit to the work done in [**?, ?**] (the ΔE are already very simple and can be found in [**?**]).

Here we follow the same logical lines that were followed in our previous [**?**].

We should remark that an explicit expressions for the ODE is given in [**?**] but is not convenient for practical purposes. For instance it is not at all obvious how to compute the trace of the matrix $\underset{N}{D_1}(x)$ from the expression written in [**?**]. We expect that all the spectral invariants of $D_1(x)$ will play an essential rôle in the future developments on the basis that in the one-matrix model [**?, ?**] the corresponding invariants allow to put in direct relation the partition function of the model with the isomonodromic tau-function in the sense of [**?, ?, ?**]. Indeed it is well known that the one-matrix model is a subcase of the two-matrix model where one of the potential is Gaussian.

The expressions contained in this paper allow us to easily compute the first invariants (traces) of the ODE's and PDE's, thus allowing us to evaluate the determinant of a fundamental system of solutions of the Frobenius compatible system. Hopefully they will prove useful for the higher invariants as well.

Remark 1.1 (Notation) *In the following we will often consider semi-infinite matrices. If M is a semi-infinite matrix we denote:*

- M_+ *the strictly upper triangular part of M,*

- M_- *the strictly lower triangular part of M,*

- M_k *the kth diagonal above the main diagonal if $k \geq 0$, and the $|k|^{\text{th}}$ diagonal below the main diagonal if $k \leq 0$,*

- $M_{+0} = M_+ + \frac{1}{2} M_0$,

- $M_{-0} = M_- + \frac{1}{2} M_0$,

- $M_\geq$ *the upper triangular part (including the main diagonal),*

- $M_\leq$ *the lower triangular part (including the main diagonal).*

We introduce the following semi–infinite matrices:
Λ is the shift matrix:

$$\Lambda_{ij} = \delta_{i+1,j} \ . \tag{1.7}$$

$\underset{N}{\Pi}$ *is the projector on the subspace spanned by the first $N + 1$ basis vectors:*

$$\underset{N}{\Pi} = \operatorname{diag}(\ \overbrace{1, 1, \ldots, 1}^{N+1\text{-}times} , 0, \ldots) \ . \tag{1.8}$$

We have the simple properties (Λ has only a right inverse):

$$\Lambda \underset{N}{\Pi} = \underset{N-1}{\Pi} \Lambda \ , \quad \underset{N}{\Pi} \Lambda^t = \Lambda^t \underset{N-1}{\Pi} \tag{1.9}$$

$$\Lambda^t \Lambda = \mathbf{1} - \underset{0}{\Pi} \ , \quad \Lambda \Lambda^t = \mathbf{1} \ , \quad \Lambda \underset{0}{\Pi} = 0 \ , \quad \underset{0}{\Pi} \Lambda^t = 0 \ . \tag{1.10}$$

Finally the notation for the folded matrices is slightly changed from [?] in order to make it more consistent (in particular the location of certain over/under-scripts).

2 Biorthogonal polynomials: definitions and properties

It is convenient to introduce the wave functions and their Fourier–Laplace transforms, related to the bi-orthogonal polynomials (**??**) by the relations:

$$\psi_n(x) = \frac{1}{\sqrt{h_n}} \pi_n(x) \mathrm{e}^{-\frac{1}{\hbar} V_1(x)} \ , \quad \phi_n(y) = \frac{1}{\sqrt{h_n}} \sigma_n(y) \mathrm{e}^{-\frac{1}{\hbar} V_2(y)} \tag{2.11}$$

$$\underline{\psi}_n(y) = \int \mathrm{d}x \mathrm{e}^{\frac{1}{\hbar} xy} \psi_n(x) \ , \quad \underline{\phi}_n(x) = \int \mathrm{d}y \mathrm{e}^{\frac{1}{\hbar} xy} \phi_n(y) \ . \tag{2.12}$$

The unspecified contour of integration can be chosen in d_1 homologically distinct ways for the functions $\underline{\psi}_n(y)$ (d_2 ways for $\underline{\phi}_n(x)$) as explained in [**?**, **?**]. We introduce also the semi-infinite wave-vectors

$$\underset{\infty}{\Psi} = (\psi_0, \psi_1, \ldots)^t \ , \quad \underset{\infty}{\Phi} = (\phi_0, \phi_1, \ldots)^t \ , \quad \underset{\infty}{\underline{\Psi}} = (\underline{\psi}_0, \underline{\psi}_1, \ldots) \ , \quad \underset{\infty}{\underline{\Phi}} = (\underline{\phi}_0, \underline{\phi}_1, \ldots) \ . \tag{2.13}$$

In order to express the ODEs that these wave-functions satisfy we define the following "windows" of consecutive wave-functions of dimension $d_2 + 1$ (resp. $d_1 + 1$):

$$\underset{N}{\Psi} = (\psi_{N-d_2}, \ldots, \psi_{N-1}, \psi_N)^t \ , \quad \underset{N}{\Phi} = (\phi_{N-d_1}, \ldots, \phi_{N-1}, \phi_N)^t \tag{2.14}$$

$$\underset{N}{\overset{N}{\underline{\Psi}}} = (\underline{\psi}_{N-1}, \underline{\psi}_N, \ldots, \underline{\psi}_{N+d_1-1}) , \quad \underset{N}{\overset{N}{\underline{\Phi}}} = (\underline{\phi}_{N-1}, \underline{\phi}_N, \ldots, \underline{\phi}_{N+d_2-1}) . \tag{2.15}$$

The two windows in each of the pairs $\left(\underset{N}{\Psi}, \underline{\overset{N}{\Phi}} \right)$ and $\left(\underset{N}{\Phi}, \underline{\overset{N}{\Psi}} \right)$ (each constituted of windows of the same size) are called "dual windows" as explained in [?]. It is shown in [?, ?, ?] that –similarly to the ordinary orthogonal polynomials– the semi-infinite wave-vectors obey:

$$x \underset{\infty}{\Psi} = Q \underset{\infty}{\Psi} , \quad y \underset{\infty}{\Phi} = P^t \underset{\infty}{\Phi} , \quad y \underline{\underset{\infty}{\Psi}} = \underline{\underset{\infty}{\Psi}} P^t , \quad x \underline{\underset{\infty}{\Phi}}(x) = \underline{\underset{\infty}{\Phi}} Q \tag{2.16}$$

$$-\hbar \partial_x \underset{\infty}{\Psi} = P \underset{\infty}{\Psi} , \quad -\hbar \partial_y \underset{\infty}{\Phi} = Q^t \underset{\infty}{\Phi} , \quad \hbar \partial_y \underline{\underset{\infty}{\Psi}} = \underline{\underset{\infty}{\Psi}} Q , \quad \hbar \partial_y \underline{\underset{\infty}{\Phi}} = \underline{\underset{\infty}{\Phi}} P^t \tag{2.17}$$

where the semi-infinite matrices P and Q are of finite band size. Componentwise the above equations translate into

$$x\psi_n(x) = \gamma(n)\psi_{n+1} + \sum_{k=0}^{d_2} \alpha_k(n)\psi_{n-k}(x) , \quad y\phi_n(y) = \gamma(n)\phi_{n+1} + \sum_{k=0}^{d_1} \beta_k(n)\phi_{n-k}(y) \tag{2.18}$$

$$x\underline{\phi}_n(x) = \gamma(n-1)\underline{\phi}_{n-1} + \sum_{k=0}^{d_2} \alpha_k(n+k)\underline{\phi}_{n+k}(x) , \quad y\underline{\psi}_n(y) = \gamma(n-1)\underline{\psi}_{n-1} + \sum_{k=0}^{d_1} \beta_k(n+k)\underline{\psi}_{n+k}(y) \tag{2.19}$$

where $\gamma(n) = \sqrt{\frac{h_{n+1}}{h_n}}$.

Since they implement the multiplication and differentiation by the spectral parameters x or y, the matrices P and Q obey the Heisenberg relation:

$$[P, Q] = \hbar \mathbf{1} \tag{2.20}$$

and they have the properties [?, ?, ?]:

$$P_+ = V_1'(Q)_+ , \quad Q_- = V_2'(P)_- \tag{2.21}$$

$$P_0 = V_1'(Q)_0 , \quad Q_0 = V_2'(P)_0 \tag{2.22}$$

$$P_{n,n-1} = \gamma(n-1) = V_1'(Q)_{n,n-1} - \hbar \frac{n}{\gamma(n-1)} \tag{2.23}$$

$$Q_{n,n+1} = \gamma(n) = V_2'(P)_{n,n+1} - \hbar \frac{n+1}{\gamma(n)} \tag{2.24}$$

The relations (??) can be "folded" [?] onto the finite windows defined in (??,??) using the recursion relations (??) so as to give finite dimensional ODEs:

$$-\hbar \partial_x \underset{N}{\Psi} = \underset{N}{D_1}(x) \underset{N}{\Psi} , \quad -\hbar \partial_y \underset{N}{\Phi} = \underset{N}{D_2}(y) \underset{N}{\Phi} , \quad \hbar \partial_y \underline{\overset{N}{\Psi}} = \underline{\overset{N}{\Psi}} \underline{\overset{N}{D}}_2(y) , \quad \hbar \partial_x \underline{\overset{N}{\Phi}} = \underline{\overset{N}{\Phi}} \underline{\overset{N}{D}}_1(x) \tag{2.25}$$

where $\underset{N}{D_1}(x)$ and $\underline{\overset{N}{D}}_1(x)$ (resp. $\underset{N}{D_2}(y)$ and $\underline{\overset{N}{D}}_2(y)$) are matrices of dimension $d_2 + 1$ (resp. $d_1 + 1$) whose entries are polynomials in x (resp. y) of degree at most d_1 (resp. d_2). They enjoy the duality property ([?], Thm 4.2):

$$\underline{\overset{N}{D}}_1(x) \overset{N}{\mathbb{A}} = \overset{N}{\mathbb{A}} \underset{N}{D_1}(x) , \quad \underline{\overset{N}{D}}_2(y) \overset{N}{\mathbb{B}} = \overset{N}{\mathbb{B}} \underset{N}{D_2}(y) \tag{2.26}$$

where $\overset{N}{\mathbb{A}}$ and $\overset{N}{\mathbb{B}}$ are the Christoffel-Darboux pairing matrices:

$$\overset{N}{\mathbb{A}} = \begin{pmatrix} 0 \ldots & 0 & -\gamma(N-1) \\ \alpha_{d_2}(N) \ldots & \alpha_1(N) & 0 \\ \ddots & \vdots & \vdots \\ 0 & \alpha_{d_2}(N+d_2-1) & 0 \end{pmatrix} \tag{2.27}$$

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$$\underset{N}{\mathbb{B}} = \begin{pmatrix} 0\dots & 0 & -\gamma(N-1) \\ \beta_{d_1}(N)\dots & \beta_1(N) & 0 \\ \ddots & \vdots & \vdots \\ 0 & \beta_{d_1}(N+d_1-1) & 0 \end{pmatrix} . \tag{2.28}$$

The two C-D pairing matrices $\underset{N}{\mathbb{A}}$ and $\underset{N}{\mathbb{B}}$ are the only non-vanishing blocks in the semi-infinite matrix commutators:

$$[Q, \underset{N-1}{\Pi}] \qquad (\text{resp. } [P^t, \underset{N-1}{\Pi}]) \tag{2.29}$$

where $\underset{N-1}{\Pi}$ is the projector defined in (??). These matrices play an essential rôle in the computation of the relevant spectral statistics of the model inasmuch as they enter directly in the kernels used in the Dyson–like formulas [?]

$$
\begin{aligned}
(x'-x)\, K_{11}(x,x') &= \underset{N}{\underline{\Phi}}(x')\, \underset{N}{\mathbb{A}}\, \underset{N}{\Psi}(x) = \underset{\infty}{\underline{\Phi}}(x') \left[Q, \underset{N-1}{\Pi}\right]_{\infty} \Psi(x) \\
(y'-y)\, K_{22}(y,y') &= \underset{N}{\underline{\Psi}}(y')\, \underset{N}{\mathbb{B}}\, \underset{N}{\Phi}(y) = \underset{\infty}{\underline{\Psi}}(y') \left[Q, \underset{N-1}{\Pi}\right]_{\infty} \Phi(y) .
\end{aligned}
\tag{2.30}
$$

2.1 Deformation equations (PDE's)

The deformations of the biorthogonal polynomials (derivations w.r.t the coefficients of the two potentials), are represented by semi-infinite finite-band matrices:

$$\hbar\partial_{u_K} \underset{\infty}{\Psi} = U_K \underset{\infty}{\Psi} \ , \quad \hbar\partial_{v_J} \underset{\infty}{\Psi} = -V_J \underset{\infty}{\Psi} \ , \quad \hbar\partial_{u_K} \underset{\infty}{\underline{\Phi}} = -\underset{\infty}{\underline{\Phi}} U_K \ , \quad \hbar\partial_{v_J} \underset{\infty}{\underline{\Phi}} = \underset{\infty}{\underline{\Phi}} V_J \tag{2.31}$$

$$\hbar\partial_{u_K} \underset{\infty}{\underline{\Psi}} = \underset{\infty}{\underline{\Psi}} U_K^t \ , \quad \hbar\partial_{v_J} \underset{\infty}{\underline{\Psi}} = -\underset{\infty}{\underline{\Psi}} V_J^t \ , \quad \hbar\partial_{u_K} \underset{\infty}{\Phi} = -U_K^t \underset{\infty}{\Phi} \ , \quad \hbar\partial_{v_J} \underset{\infty}{\Phi} = V_J^t \underset{\infty}{\Phi} \tag{2.32}$$

where the semi-infinite matrices U_K, V_J are related to the matrices P, Q by the equations

$$U_K := -\frac{1}{K} Q_{+0}^K \ , \qquad V_J := -\frac{1}{J} P_{-0}^J \ . \tag{2.33}$$

which can be folded onto the windows using eq.(??) as explained in [?]

$$\hbar\partial_{u_K} \underset{N}{\Psi} = \underset{N}{\mathcal{U}_K}(x) \underset{N}{\Psi} \ , \quad \hbar\partial_{v_J} \underset{N}{\Psi} = -\underset{N}{\mathcal{V}_J}(x) \underset{N}{\Psi} \ , \quad \hbar\partial_{u_K} \underset{N}{\underline{\Phi}} = -\underset{N}{\underline{\Phi}}\underset{N}{\underline{\mathcal{U}}_K}(x) \ , \quad \hbar\partial_{v_J} \underset{N}{\underline{\Phi}} = \underset{N}{\underline{\Phi}}\underset{N}{\underline{\mathcal{V}}_J}(x) \tag{2.34}$$

$$\hbar\partial_{u_K} \underset{N}{\underline{\Psi}} = \underset{N}{\underline{\Psi}}\underset{N}{\widetilde{\mathcal{U}}_K}(y) \ , \quad \hbar\partial_{v_J} \underset{N}{\underline{\Psi}} = -\underset{N}{\underline{\Psi}}\underset{N}{\widetilde{\mathcal{V}}_J}(y) \ , \quad \hbar\partial_{u_K} \underset{N}{\Phi} = -\widetilde{\mathcal{U}}_K(y) \underset{N}{\Phi} \ , \quad \hbar\partial_{v_J} \underset{N}{\Phi} = \underset{N}{\widetilde{\mathcal{V}}_J}(y) \underset{N}{\Phi} \tag{2.35}$$

where $\underset{N}{\mathcal{U}_K}(x)$ and $\underset{N}{\underline{\mathcal{U}}_K}(x)$ are matrices of dimension d_2+1 whose entries are polynomials in x of degree at most k, and $\underset{N}{\mathcal{V}_J}(x)$ and $\underset{N}{\underline{\mathcal{V}}_J}(x)$ are matrices of dimension d_2+1 whose entries are polynomials in x of degree at most Jd_1 (and similar statement for the tilde-matrices with d_1 replacing d_2 and vice-versa). They satisfy [?]:

$$\underset{N}{\underline{\mathcal{U}}_K}(x) \underset{N}{\mathbb{A}} + \underset{N}{\mathbb{A}}\underset{N}{\mathcal{U}_K}(x) + \hbar\partial_{u_K} \underset{N}{\mathbb{A}} = 0 \tag{2.36}$$

$$\underset{N}{\underline{\mathcal{V}}_J}(x) \underset{N}{\mathbb{A}} + \underset{N}{\mathbb{A}}\underset{N}{\mathcal{V}_J}(x) - \hbar\partial_{v_J} \underset{N}{\mathbb{A}} = 0 \ . \tag{2.37}$$

The expressions of the matrices U_J, V_K in terms of P and Q can be found with the same notation in [?] but are essentially well known since [?] in a slightly different context and with different notation.

2.2 Deformations with respect to $\hbar$

The $\hbar$-deformations of the biorthogonal polynomials (derivations w.r.t $\hbar$) are also represented by semi-infinite matrices of finite band size which we can write as follows:

$$\hbar^2 \partial_\hbar \underset{\infty}{\Psi} = H \underset{\infty}{\Psi} \ , \quad \hbar^2 \partial_\hbar \underset{\infty}{\Phi} = \widetilde{H} \underset{\infty}{\Phi} \ . \tag{2.38}$$

`deform`

The relationship with the matrices P and Q is easy to derive but since we could not find it in the literature (except in special cases [?]) we give it hereafter. From $\int e^{\frac{xy}{\hbar}} \Psi_\infty(x) \Phi_\infty{}^t(y) = \mathbf{1}$ we promptly obtain

$$H - QP + \widetilde{H}^t = 0 \ . \tag{2.39}$$

Moreover we have

$$(H - V_1(Q))_+ = 0 = \left(\widetilde{H} - V_2(P^t) \right)_+ \tag{2.40}$$

$$\overbrace{H - V_1(Q)}^{\text{lower}} + (V_1(Q) - QP + V_2(P)) + \overbrace{\widetilde{H}^t - V_2(P)}^{\text{upper}} = 0 \tag{2.41}$$

$$(H - V_1(Q))_0 = \left(\widetilde{H} - V_2(P^t) \right)_0 = -\frac{1}{2} \left(V_1(Q) - QP + V_2(P) \right)_0 \tag{2.42}$$

$$(H - V_1(Q))_- = (QP - V_1(Q) - V_2(P))_- \tag{2.43}$$

$$\left(\widetilde{H} - V_2(P^t) \right)_- = \left(P^t Q^t - V_1(Q^t) - V_2(P^t) \right)_- \tag{2.44}$$

so that we have

$$H = V_1(Q)_+ + \frac{1}{2}(QP + V_1(Q) - V_2(P))_0 + (QP - V_2(P))_- = V_1(Q)_{+0} + (QP)_{-0} - V_2(P)_{-0} \tag{2.45}$$

`HinPQ`

$$\widetilde{H} = V_2(P^t)_{+0} + (P^t Q^t)_{-0} - V_1(Q^t)_{-0} \ . \tag{2.46}$$

It is also interesting to consider another derivation of (??), from scaling properties. Consider the change of variables $x \mapsto \frac{x}{\lambda}$ and $y \mapsto \frac{y}{\mu}$, it gives

$$(\lambda\mu)^{n+1} h_n(\{u_K\}, \{v_J\}, \hbar) \delta_{nm} = \int \mathrm{d}x \mathrm{d}y \lambda^n \pi_n \left(\frac{x}{\lambda}, \{u_K\}, \{v_J\}, \hbar \right) \mu^m \sigma_m \left(\frac{y}{\mu}, \{u_K\}, \{v_J\}, \hbar \right) e^{-\frac{\widetilde{V}_1(x) + \widetilde{V}_2(y) - xy}{\hbar}} \tag{2.47}$$

where

$$\widetilde{V}_1(x) := \lambda\mu V_1 \left(\frac{x}{\lambda} \right) \ , \qquad \widetilde{V}_2(y) := \lambda\mu V_2 \left(\frac{y}{\mu} \right) \ , \qquad \widetilde{\hbar} = \lambda\mu\hbar \ . \tag{2.48}$$

This immediately implies the identities

$$\pi_n(x; \{u_K \lambda^{1-K}\mu\}, \{v_J \mu^{1-J}\lambda\}, \lambda\mu\hbar) = \lambda^n \pi_n \left(\frac{x}{\lambda}, \{u_K\}, \{v_J\}, \hbar \right) \tag{2.49}$$

$$\sigma_n(y; \{u_K \lambda^{1-K}\mu\}, \{v_J \mu^{1-J}\lambda\}, \lambda\mu\hbar) = \mu^n \sigma_n \left(\frac{y}{\mu}, \{u_K\}, \{v_J\}, \hbar \right) \tag{2.50}$$

$$h_n \left(\{u_K \lambda^{1-K}\mu\}, \{v_J \mu^{1-J}\lambda\}, \lambda\mu\hbar \right) = (\lambda\mu)^{n+1} h_n \left(\{u_K\}, \{v_J\}, \hbar \right) \tag{2.51}$$

$$\psi_n \left(x, \{u_K \lambda^{1-K}\mu\}, \{v_J \mu^{1-J}\lambda\}, \lambda\mu\hbar \right) = \lambda^{\frac{n-1}{2}} \mu^{-\frac{n+1}{2}} \psi_n \left(\frac{x}{\lambda}; \{u_K\}, \{v_J\}, \hbar \right) \tag{2.52}$$

$$\phi_n \left(y, \{u_K \lambda^{1-K}\mu\}, \{v_J \mu^{1-J}\lambda\}, \lambda\mu\hbar \right) = \mu^{\frac{n-1}{2}} \lambda^{-\frac{n+1}{2}} \phi_n \left(\frac{y}{\mu}; \{u_K\}, \{v_J\}, \hbar \right) \tag{2.53}$$

In infinitesimal form, from $\lambda \partial_\lambda|_{\lambda=\mu=1}$ and $\mu \partial_\mu|_{\lambda=\mu=1}$ we obtain

$$\left(\frac{\widehat{n}-1}{2} + x\partial_x - \sum_K (K-1)u_K \partial_{u_K} + \sum_J v_J \partial_{v_J} + \hbar\partial_\hbar \right) \underset{\infty}{\Psi}(x) = 0 \tag{2.54}$$

$$\left(-\frac{1+\widehat{n}}{2} + \sum_K u_K \partial_{u_K} - \sum_J (J-1)v_J \partial_{v_J} + \hbar \partial_\hbar \right) \underset{\infty}{\Psi}(x) = 0 \tag{2.55}$$

$$\left(-\frac{1+\widehat{n}}{2} - \sum_K (K-1)u_K \partial_{u_K} + \sum_J v_J \partial_{v_J} + \hbar \partial_\hbar \right) \underset{\infty}{\Phi}(y) = 0 \tag{2.56}$$

$$\left(\frac{\widehat{n}-1}{2} + y\partial_y + \sum_K u_K \partial_{u_K} - \sum_J (J-1)v_J \partial_{v_J} + \hbar \partial_\hbar \right) \underset{\infty}{\Phi}(y) = 0 \ , \tag{2.57}$$

where $\widehat{n} := \mathrm{diag}(0,1,2,3,4,\ldots)$. Using eq. (**??**) we immediately obtain that

$$-H = \frac{\hbar(\widehat{n}-1)}{2} - PQ - \sum_K (K-1)u_K U_K - \sum_J v_J V_J = -\frac{\hbar(\widehat{n}+1)}{2} + \sum_K u_K U_K + \sum_J (J-1)v_J V_J \tag{2.58}$$

$$-\widetilde{H}^t = -\frac{\hbar(\widehat{n}+1)}{2} + \sum (K-1)u_K U_K + \sum v_J V_J = \frac{\hbar(\widehat{n}-1)}{2} - PQ - \sum u_K U_K - \sum (J-1)v_J V_J \ , \tag{2.59}$$

which, using (**??**, **??**), is identical to (**??**).

We also get the identity

$$-PQ = \sum_J J v_J V_J + \sum_K K u_K U_K - \hbar \widehat{n} \ . \tag{2.60} \qquad \boxed{\texttt{PQrel}}$$

This last relation is nothing but the expression of the scaling properties of the biorthogonal polynomials and will prove itself very useful later on.

2.3   Ladder recursion relations (ΔE's)

We give here only the main statements without proofs and refer the reader to ref. [**?**] for further details. Let us introduce the sequence of companion–like matrices $\underset{N}{\mathbf{a}}(x)$ and $\underset{N}{\underline{\mathbf{a}}}(x)$ of size $d_2 + 1$

$$\underset{N}{\mathbf{a}}(x) \quad := \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\alpha_{d_2}(N)}{\gamma(N)} & \cdots & \frac{-\alpha_1(N)}{\gamma(N)} & \frac{(x-\alpha_0(N))}{\gamma(N)} \end{bmatrix} \ , \quad N \geq d_2 \ , \tag{2.61} \qquad \boxed{\texttt{aNdef}}$$

$$\underset{}{\overset{N}{\underline{\mathbf{a}}}}(x) \quad := \quad \begin{bmatrix} \frac{x-\alpha_0(N)}{\gamma(N-1)} & 1 & 0 & 0 \\ \frac{-\alpha_1(N+1)}{\gamma(N-1)} & 0 & \ddots & 0 \\ \vdots & & 0 & 0 & 1 \\ \frac{-\alpha_{d_2}(N+d_2)}{\gamma(N-1)} & 0 & 0 & 0 \end{bmatrix} \ , \quad N \geq 1, \tag{2.62} \qquad \boxed{\texttt{abarN}}$$

and also the analogous sequence of matrices $\underset{N}{\mathbf{b}}(y)$ and $\overset{N}{\underline{\mathbf{b}}}(y)$ of size $d_1 + 1$

$$\underset{N}{\mathbf{b}}(y) \quad := \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\beta_{d_1}(N)}{\gamma(N)} & \cdots & \frac{-\beta_1(N)}{\gamma(N)} & \frac{(y-\beta_0(N))}{\gamma(N)} \end{bmatrix} \ , \quad N \geq d_1 \ , \tag{2.63} \qquad \boxed{\texttt{bNdef}}$$

$$\overset{N}{\underline{\mathbf{b}}}(y) \quad := \quad \begin{bmatrix} \frac{y-\beta_0(N)}{\gamma(N-1)} & 1 & 0 & 0 \\ \frac{-\beta_1(N+1)}{\gamma(N-1)} & 0 & \ddots & 0 \\ \vdots & & 0 & 0 & 1 \\ \frac{-\beta_{d_1}(N+d_1)}{\gamma(N-1)} & 0 & 0 & 0 \end{bmatrix} \ , \quad N \geq 1. \tag{2.64} \qquad \boxed{\texttt{bbarN}}$$

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The equations in (**??**) imply the following

Lemma 2.1 *The sequences of matrices* $\underset{N}{\mathbf{a}}$, $\underset{N}{\mathbf{b}}$ *and* $\underset{N}{\underline{\mathbf{a}}}$, $\underset{N}{\underline{\mathbf{b}}}$ *implement the shift* $N \mapsto N+1$ *and* $N \mapsto N-1$ *in the windows of quasi-polynomials and Fourier–Laplace transforms in the sense that*

$$\underset{N}{\mathbf{a}}\,\underset{N}{\Psi}(x) = \underset{N+1}{\Psi}(x) \ , \qquad \underset{N}{\underline{\Phi}}(x) = \underset{N+1}{\underline{\Phi}}(x)\,\underset{N}{\underline{\mathbf{a}}} \ , \tag{2.65}$$

$$\underset{N}{\mathbf{b}}\,\underset{N}{\Phi}(y) = \underset{N+1}{\Phi}(y) \ , \qquad \underset{N}{\underline{\Psi}}(y) = \underset{N+1}{\underline{\Psi}}(y)\,\underset{N}{\underline{\mathbf{b}}} \ . \tag{2.66}$$

and in general

$$\underset{N+j}{\Psi} = \underset{N+j-1}{\mathbf{a}}\cdots\underset{N}{\mathbf{a}}\,\underset{N}{\Psi} \ , \qquad \underset{N}{\underline{\Phi}} = \underset{N+j}{\underline{\Phi}}\,\underset{N+j-1}{\underline{\mathbf{a}}}\cdots\underset{N}{\underline{\mathbf{a}}} \ , \tag{2.67}$$

$$\underset{N+j}{\Phi} = \underset{N+j-1}{\mathbf{b}}\cdots\underset{N}{\mathbf{b}}\,\underset{N}{\Phi} \ , \qquad \underset{N}{\underline{\Psi}} = \underset{N+j}{\underline{\Psi}}\,\underset{N+j-1}{\underline{\mathbf{b}}}\cdots\underset{N}{\underline{\mathbf{b}}} \ , \tag{2.68}$$

where $\underset{N}{\underline{\Psi}}$, $\underset{N}{\underline{\Phi}}$ *here denotes a window in any of the Fourier–Laplace transforms defined in eqs. (**??**).*

It was proven in [**?**] that the differential-deformation-recursion relations (**??, ??, ??**) are compatible and admit a common fundamental solution.

2.4 Folding

Important remark. From this point on we focus on the wave-functions $\psi_n(x)$ and their dual $\underline{\phi}_n(x)$, but everything being said can be immediately extended to the wave-functions $\phi_n(y), \underline{\psi}_n(y)$ by interchanging the rôles of the spectral parameters, the potentials and the matrices P and Q.

We recall that the notion of "folding" is the following: to express any quasipolynomial $\psi_n(x)$ as a linear combination of $d_2 + 1$ *fixed* consecutive quasipolynomials ψ_{N-j} , $j = 0, \ldots d_2$ *with polynomial coefficients*. We now provide a way of computing the folding which is different from (but equivalent to) the one used in [**?**] where the ladder matrices were used instead.

Let N (i.e. a window) be fixed; we seek to describe the folding of the infinite wave-vector $\underset{\infty}{\Psi}(x)$ onto the window at N by means of a single matrix $\underset{N}{\mathcal{F}}(x)$ of size $\infty \times (d_2 + 1)$ and with polynomial entries such that:

$$\forall n, \qquad \psi_n(x) = \sum_{k=N-d_2}^{N} \underset{N}{\mathcal{F}}_{n,k}(x)\psi_k(x) \ . \tag{2.69}$$

In fact it is more convenient to think of $\underset{N}{\mathcal{F}}$ as a $\infty \times \infty$ matrix with only a vertical band of width $d_2 + 1$ of nonzero entries (with column index in the range $N - d_2, \ldots, N$). In order to describe as explicitly as possible the matrix $\underset{N}{\mathcal{F}}$ we first introduce a convenient notation to express the band-matrices P, Q:

$$Q = \gamma\Lambda + \sum_{j=0}^{d_2} \alpha_j \Lambda^{tj} \ , \quad P = \Lambda^t\gamma + \sum_{j=0}^{d_1} \Lambda^j \beta_j \tag{2.70}$$

where Λ is the shift matrix defined in (**??**), and $\alpha_j, \beta_j, \gamma$ here above denote the diagonal matrices

$$\alpha_j := \mathrm{diag}(\alpha_j(0), \ldots, \alpha_j(n), \ldots) \ , \qquad \beta_j := \mathrm{diag}(\beta_j(0), \ldots, \beta_j(n), \ldots) \ , \qquad \gamma := \mathrm{diag}(\gamma(0), \ldots, \gamma(n), \ldots) \ . \tag{2.71}$$

It is known that the existence of biorthogonal polynomials satisfying (**??**) is equivalent to saying that $\gamma(n) \neq 0$, $\forall n$. Therefore we assume that γ is invertible.

Eqs. (**??**) imply that

$$\alpha_{d_2}\Lambda^{t\,d_2} = v_{d_2+1}(\Lambda^t\gamma)^{d_2}\ , \tag{2.72}$$

or, more transparently,

$$\alpha_{d_2}(n) = v_{d_2+1}\gamma(n-1)\cdots\gamma(n-d_2)\ . \tag{2.73}$$

Let us now define the following semi-infinite matrices:

$$A := \mathbf{1} - \frac{1}{v_{d_2+1}}(\gamma^{-1}\Lambda)^{d_2}\,(Q-x)\ , \quad B := \mathbf{1} - \underset{0}{\Pi} - (\Lambda^t\gamma^{-1})\,(Q-x)\ . \tag{2.74}$$ ⬚ Afol

We note that A is strictly upper triangular while B is strictly lower triangular.

When acting on the semi–infinite wave-vector $\underset{\infty}{\Psi}(x)$ by definition we have $(Q-x)\underset{\infty}{\Psi} = 0$ so that:

$$\underset{\infty}{\Psi} = A\underset{\infty}{\Psi}\ , \quad \underset{\infty}{\Psi} = \left(B + \underset{0}{\Pi}\right)\underset{\infty}{\Psi} \tag{2.75}$$

Notice that $\mathbf{1} - A$ and $\mathbf{1} - B$ are invertible because they are upper (resp. lower) triangular with ones on the main diagonal. Notice also that the matrix $Q - x$ has a left inverse[4]:

$$\frac{1}{v_{d_2+1}}\,(\mathbf{1}-A)^{-1}\,(\gamma^{-1}\Lambda)^{d_2}(Q-x) = \mathbf{1} \tag{2.76}$$

and a right inverse (write $\mathbf{1} - B = \underset{0}{\Pi} + \Lambda^t\gamma^{-1}(Q-x)$, and multiply by $\gamma\Lambda$ on the left and by $(\mathbf{1}-B)^{-1}\Lambda^t\gamma^{-1}$ on the right):

$$(Q-x)(\mathbf{1}-B)^{-1}\,(\Lambda^t\gamma^{-1}) = \mathbf{1} \tag{2.77}$$

but they do not coincide (one is upper and the other is lower triangular).

2.4.1 Upper folding

We start our construction of the folding matrix $\underset{N}{\mathcal{F}}$ by first looking at the "upper" folding, i.e. the part of $\underset{N}{\mathcal{F}}$ with first index greater than N.

We introduce the following notations

$$\overset{N}{\Pi} := \mathbf{1} - \underset{N}{\Pi}\ , \qquad \overset{M}{\underset{N}{\Pi}} := \underset{N}{\Pi} - \underset{M}{\Pi} = \overset{M}{\Pi} - \overset{N}{\Pi} = \underset{N}{\Pi}\overset{M}{\Pi}\ , \tag{2.78}$$

and we remark the following formulas which hold since B is strictly lower triangular

$$\underset{N}{\Pi}B\overset{N}{\Pi} = 0\ ; \qquad \overset{N}{\Pi}B\overset{N}{\Pi} = B\overset{N}{\Pi}. \tag{2.79}$$

With these formulas in mind we compute

$$\overset{N}{\Pi}\underset{\infty}{\Psi} = \overset{N}{\Pi}B\underset{\infty}{\Psi} = \overset{N}{\Pi}B\left(\underset{N}{\Pi}+\overset{N}{\Pi}\right)\underset{\infty}{\Psi} = \overset{N}{\Pi}B\underset{N}{\Pi}\underset{\infty}{\Psi} + \overset{N}{\Pi}B\overset{N}{\Pi}\underset{\infty}{\Psi} = \overset{N}{\Pi}B\underset{N}{\Pi}\underset{\infty}{\Psi} + \overset{N}{\Pi}B\left(\overset{N}{\Pi}B\underset{N}{\Pi}\underset{\infty}{\Psi} + \overset{N}{\Pi}B\overset{N}{\Pi}\underset{\infty}{\Psi}\right) =$$

$$= (\mathbf{1}+B)\overset{N}{\Pi}B\underset{N}{\Pi}\underset{\infty}{\Psi} + \overset{N}{\Pi}B\overset{N}{\Pi}B\overset{N}{\Pi}\underset{\infty}{\Psi} = (\mathbf{1}+B)\overset{N}{\Pi}B\underset{N}{\Pi}\underset{\infty}{\Psi} + B^2\overset{N}{\Pi}\underset{\infty}{\Psi} = [\text{iterating } r\text{-times}] =$$

$$= \sum_{j=0}^{r}B^j\overset{N}{\Pi}B\underset{N}{\Pi}\underset{\infty}{\Psi} + B^{r+1}\overset{N}{\Pi}\underset{\infty}{\Psi}\ . \tag{2.80}$$

---

[4]The fact that $Q - x$ has a left inverse $(Q-x)_{\mathrm{L}}^{-1}$ is not in contradiction with $(Q-x)\Psi_\infty = 0$.

Indeed, $\left((Q-x)_{\mathrm{L}}^{-1}(Q-x)\right)\Psi_\infty \neq (Q-x)_{\mathrm{L}}^{-1}\left((Q-x)\Psi_\infty\right)$ because both LHS and RHS are infinite sums, and the order of summation cannot be exchanged.

Given that B is lower triangular (in a sloppy sense it is nilpotent) the remainder in the formula above is a vector whose nonzero components are only those with index greater than $N+r+1$. Since in our problems we are only folding finite band matrices and we will never need arbitrarily high indices, we can disregard the remainder and send $r \to \infty$ (i.e. we take an inductive limit). The result is then

$$\prod_{\infty}^{N} \Psi = (1-B)^{-1} \prod^{N} B \prod_{N}^{N} \Psi_{\infty} \ . \tag{2.81}$$

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Finally note that, since the matrix B has only d_2 bands below the main diagonal, we have

$$\prod^{N} B \prod_{N} \prod_{N-d_2}^{N} = 0 \ . \tag{2.82}$$

In other words the expression in eq.(??) contains only the quasipolynomials $\psi_{N-d_2}, \ldots, \psi_N$. Therefore we have achieved the first part of our computation

$$\prod_{N}^{N} \mathcal{F} = (1-B)^{-1} \prod^{N} B \prod_{N} \ , \tag{2.83}$$

which can be simplified further as follows

$$
\begin{aligned}
\prod_{N}^{N} \mathcal{F} \quad &= \quad (\mathbf{1}-B)^{-1} \prod^{N} B \prod_{N} = \\
&= \quad -(\mathbf{1}-B)^{-1} \prod^{N} (\Lambda^t \gamma^{-1}) (Q-x) \prod_{N} = \\
&= \quad -(\mathbf{1}-B)^{-1} (\Lambda^t \gamma^{-1}) \prod^{N-1} (Q-x) \prod_{N} = \\
&= \quad -(\mathbf{1}-B)^{-1} (\Lambda^t \gamma^{-1}) (Q-x) \prod^{N-1} \prod_{N} -(1-B)^{-1} (\Lambda^t \gamma^{-1}) \left[Q-x, \prod_{N-1} \right] \prod_{N} = \\
&= \quad -(\mathbf{1}-B)^{-1} \left(\mathbf{1} - B - \prod_{0} \right) \prod^{N-1}_{N} -(1-B)^{-1} (\Lambda^t \gamma^{-1}) \left[Q, \prod_{N-1} \right] = \\
&= \quad -\prod^{N-1}_{N} -(\mathbf{1}-B)^{-1} (\Lambda^t \gamma^{-1}) \left[Q, \prod_{N-1} \right]
\end{aligned}
$$
$$\tag{2.84}$$

Concluding this part we have

$$\prod_{N}^{N} \mathcal{F} = -\prod_{N}^{N-1} -(1-B)^{-1} (\Lambda^t \gamma^{-1}) \left[Q, \prod_{N-1} \right] \ . \tag{2.85}$$

upperf

2.4.2   Lower folding

Similarly to the case before we have now

$$\prod^{N-d_2-1} A \prod_{N-d_2-1} = 0 \ ; \quad \prod_{N-d_2-1} A \prod_{N-d_2-1} = A \prod_{N-d_2-1} \tag{2.86}$$

and hence we promptly obtain along the same lines (we set for convenience $M = N - d_2 - 1$)

$$
\begin{aligned}
\prod_{M}\prod_{\infty} \Psi &= \prod_{M} A \prod_{\infty} \Psi = \prod_{M} A \prod^{M}_{\infty} \Psi + \prod_{M} A \prod_{M}\prod_{\infty} \Psi = \\
&= \prod_{M} A \prod^{M}_{\infty} \Psi + \prod_{M} A \left(\prod_{M} A \prod^{M}_{\infty} \Psi + \prod_{M} A \prod_{M}\prod_{\infty} \Psi \right) = \\
&= (1+A) \prod_{M} A \prod^{M}_{\infty} \Psi + A^2 \prod_{M}\prod_{\infty} \Psi = [\text{iterating } r\text{-times}] = \\
&= \sum_{j=0}^{r} A^j \prod_{M} A \prod^{M}_{\infty} \Psi + A^{r+1} \prod_{M}\prod_{\infty} \Psi \ .
\end{aligned}
$$
$$\tag{2.87}$$

For the same reason used above we disregard the remainder term and thus we obtain

$$\Pi_{M} \Psi_{\infty} = (1-A)^{-1} \Pi_{M} A \overset{M}{\Pi} \Psi_{\infty} \ . \tag{2.88}$$

Once more this expression contains only the quasipolynomials within the window at N. Therefore:

$$\Pi_{M} \mathcal{F}_{N} = (1-A)^{-1} \Pi_{M} A \overset{M}{\Pi} \tag{2.89}$$

Once and again this expression can be simplified further as follows

$$
\begin{aligned}
\Pi_{M} \mathcal{F}_{N} &= (1-A)^{-1} \Pi_{M} A \overset{M}{\Pi} = \\
&= -\frac{1}{v_{d_2+1}} (1-A)^{-1} \Pi_{M} (\gamma^{-1}\Lambda)^{d_2} (Q-x) \overset{M}{\Pi} = \\
&= -\frac{1}{v_{d_2+1}} (1-A)^{-1} (\gamma^{-1}\Lambda)^{d_2} \Pi_{N-1} (Q-x) \overset{M}{\Pi} = \\
&= -\frac{1}{v_{d_2+1}} (1-A)^{-1} (\gamma^{-1}\Lambda)^{d_2} (Q-x) \Pi_{N-1} \overset{M}{\Pi} \\
&\quad +\frac{1}{v_{d_2+1}} (1-A)^{-1} (\gamma^{-1}\Lambda)^{d_2} [Q-x, \Pi_{N-1}] = \\
&= -(1-A)^{-1} (1-A) \Pi_{N-1}^{M} \\
&\quad +\frac{1}{v_{d_2+1}} (1-A)^{-1} (\gamma^{-1}\Lambda)^{d_2} [Q, \Pi_{N-1}] = \\
&= -\Pi_{N-1}^{M} +\frac{1}{v_{d_2+1}} (1-A)^{-1} (\gamma^{-1}\Lambda)^{d_2} [Q, \Pi_{N-1}] \ . \tag{2.90}
\end{aligned}
$$

<code>lowerf</code>

Finally we remark that $\mathcal{F}_{N \, i,j}$ for $N-d_2 \le i,j \le N$ is the d_2+1 identity matrix on the selected window so that, putting eqs. (**??, ??**) together we have:

$$
\begin{aligned}
\mathcal{F}_{N} &= \Pi_{M} \mathcal{F}_{N} + \overset{N}{\Pi} \mathcal{F}_{N} + \overset{M}{\Pi}_{N} \\
&= -(1-B)^{-1} (\Lambda^t \gamma^{-1}) \left[Q, \Pi_{N-1} \right] + \frac{1}{v_{d_2+1}} (1-A)^{-1} (\gamma^{-1}\Lambda)^{d_2} \left[Q, \Pi_{N-1} \right] \\
&\qquad (2.91)
\end{aligned}
$$

We can summarize the above computations into the single formula

$$\boxed{\mathcal{F}_{N} = \left(\frac{1}{v_{d_2+1}} (1-A)^{-1} (\gamma^{-1}\Lambda)^{d_2} - (1-B)^{-1} (\Lambda^t \gamma^{-1}) \right) \left[Q, \Pi_{N-1} \right]} \tag{2.92}$$

Using this folding operator we can immediately find the folded version of any other operator, viz, if L is any semi-infinite matrix with finite band size, its folded counterpart is:

$$L \ \longrightarrow \ \mathcal{L}(x) = \overset{M}{\underset{N}{\Pi}} L \, \mathcal{F}_{N} \ . \tag{2.93}$$

<code>opfold</code>

Note that –strictly speaking– formula (**??**) defines a semiinfinite matrix with only a nonzero square block of size d_2+1 located on the main diagonal

$$\mathcal{L}(x)_{ij} \equiv 0 \ , i,j \notin N-d_2, \ldots, N \ . \tag{2.94}$$

By abuse of notation we will denote by the same symbol the $(d_2+1) \times (d_2+1)$ matrix corresponding to the nonzero block. In this fashion we will not distinguish between tracing over the finite or the semiinfinite matrix (with some advantages which will appear later).

2.5 Dual Folding

For completeness we report the formulas for the folding of the dual wave vector $\underset{\infty}{\underline{\Phi}}$ onto the window $N - 1 \leq n \leq N + d_2 - 1$. The computation is completely parallel to the above using now the matrices (remember that the dual wave vector is a row-vector and hence we must act on the right)

$$\underline{B} = \mathbf{1} - (Q - x)\Lambda^t\gamma^{-1} \ , \qquad \underline{A} = \mathbf{1} - \underset{d_2-1}{\Pi} - (Q - x)\frac{(\gamma^{-1}\Lambda)^{d_2}}{v_{d_2+1}} \ . \tag{2.95}$$

With this in mind it is straightforward to obtain

$$\underset{\infty}{\underline{\Phi}}\underset{N-2}{\Pi} = \underset{\infty}{\underline{\Phi}}\overset{N-2}{\Pi}\ \underline{B}\ \underset{N-2}{\Pi}(\mathbf{1} - \underline{B})^{-1} = -\overset{N-2}{\underset{N-1}{\Pi}} - \left[Q, \underset{N-1}{\Pi}\right]\Lambda^t\gamma^{-1}(\mathbf{1} - \underline{B})^{-1}$$

$$\underset{\infty}{\underline{\Phi}}\overset{N+d_2-1}{\Pi} = \underset{\infty}{\underline{\Phi}}\underset{N+d_2-1}{\Pi}\ \underline{A}\ \overset{N+d_2-1}{\Pi}(\mathbf{1} - \underline{A})^{-1} = -\underset{N-d_2-1}{\overset{N-1}{\Pi}} + \frac{1}{v_{d_2+1}}\left[Q, \underset{N-1}{\Pi}\right](\gamma^{-1}\Lambda)^{d_2}(\mathbf{1} - \underline{A})^{-1} \tag{2.96}$$

$$\underset{}{\overset{N}{\underline{\mathcal{F}}}} = \underset{\infty}{\underline{\Phi}}\overset{N-2}{\Pi}\ \underline{B}\ \underset{N-2}{\Pi}(\mathbf{1} - \underline{B})^{-1} = \left[Q, \underset{N-1}{\Pi}\right]\left(-\Lambda^t\gamma^{-1}(\mathbf{1} - \underline{B})^{-1} + \frac{1}{v_{d_2+1}}(\gamma^{-1}\Lambda)^{d_2}(\mathbf{1} - \underline{A})^{-1}\right) \tag{2.97}$$

3 Folded deformation matrices $\underset{N}{\mathcal{U}_K}(x)$

From eq. (??) and using the folding relation (??) we obtain the desired formula for the deformation matrices $\underset{N}{\mathcal{U}_K}$ (the folded version of $\frac{1}{K}Q_{+0}^K$), indeed (recall $M = N - d_2 - 1$):

$$
\begin{aligned}
-K\underset{N}{\mathcal{U}_K} &= \underset{N}{\overset{M}{\Pi}}\ Q_{+0}^K\mathcal{F}_N = \underset{N}{\overset{M}{\Pi}}\ Q_{+0}^K\ \overset{M}{\Pi}\ \mathcal{F}_{=} \\
&= \underset{N}{\overset{M}{\Pi}}\ Q_{+0}^K\ \underset{N}{\overset{M}{\Pi}}\ \mathcal{F}_+ \underset{N}{\overset{M}{\Pi}}\ Q^K\ \underset{N}{\overset{N}{\Pi}}\ \mathcal{F}_N = \\
&= \underset{N}{\overset{M}{\Pi}}\ Q_{+0}^K\ \underset{N}{\overset{M}{\Pi}} - \underset{N}{\overset{M}{\Pi}}\ Q^K\ \underset{N}{\overset{N-1}{\Pi}} - \underset{N}{\overset{M}{\Pi}}\ Q^K(\mathbf{1} - B)^{-1}\left(\Lambda^t\gamma^{-1}\right)\left[Q, \underset{N-1}{\Pi}\right] = \\
&= \underset{N}{\overset{M}{\Pi}}\ Q_{+0}^K\ \underset{N-1}{\overset{M}{\Pi}} - \underset{N}{\overset{M}{\Pi}}\ Q_{-0}^K\ \underset{N}{\overset{N-1}{\Pi}} \\
&\quad - \underset{N}{\overset{M}{\Pi}}(Q^K - x^K)(\mathbf{1} - B)^{-1}\left(\Lambda^t\gamma^{-1}\right)\left[Q, \underset{N-1}{\Pi}\right] \\
&\quad - x^K \underset{N}{\overset{M}{\Pi}}(\mathbf{1} - B)^{-1}\left(\Lambda^t\gamma^{-1}\right)\left[Q, \underset{N-1}{\Pi}\right] = \\
&= \underset{N}{\overset{M}{\Pi}}\ Q_{+0}^K\ \underset{N-1}{\overset{M}{\Pi}} - \underset{N}{\overset{M}{\Pi}}\ Q_{-0}^K\ \underset{N}{\overset{N-1}{\Pi}} - \underset{N}{\overset{M}{\Pi}}\frac{Q^K - x^K}{Q - x}\left[Q, \underset{N-1}{\Pi}\right] - x^K \underset{N}{\overset{N-1}{\Pi}} = \\
&= \underset{N}{\overset{M}{\Pi}}\ Q_{+0}^K\ \underset{N-1}{\overset{M}{\Pi}} - \underset{N}{\overset{M}{\Pi}}\ Q_{-0}^K\ \underset{N}{\overset{N-1}{\Pi}} - \underset{N}{\overset{M}{\Pi}}W_K(x)\left[Q, \underset{N-1}{\Pi}\right] - x^K \underset{N}{\overset{N-1}{\Pi}} = \\
&= \underset{N}{\overset{M}{\Pi}}\ Q_{+0}^K\ \underset{N-1}{\overset{M}{\Pi}} - \underset{N}{\overset{M}{\Pi}}W_K(x)\left[Q, \underset{N-1}{\Pi}\right] + \left(x^K - \frac{1}{2}Q_{N,N}^K\right)\underset{N}{\overset{N-1}{\Pi}} \ , \tag{3.98}
\end{aligned}
$$

`defU`

where we have defined

$$W_K(x) := \frac{Q^K - x^K}{Q - x} \ . \tag{3.99}$$

This formula generalizes similar formulas for the one-matrix model appeared in [?] which allowed us to perform explicit computations linking the isomonodromic tau function and the partition function of the model (see Conclusion). **Trace Formulas**

We compute the trace of the deformation matrices by noticing that the last term in (??) has zero trace because $W_K(x)$

commutes with Q and because of the cyclicity of the trace

$$\text{Tr} \prod_{N}^{M} W_K(x) \left[Q, \prod_{n-1}^{0} \right] = \text{Tr} \, W_K(x) \left[Q, \prod_{n-1}^{0} \right] \prod_{N}^{M} = \text{Tr} \, W_K(x) \left[Q, \prod_{n-1}^{0} \right] = 0 \tag{3.100}$$

where Tr means the trace of the (finite rank) semi-infinite matrix. Therefore

$$-K \operatorname{tr} \underset{N}{\mathcal{U}}_K(x) = x^K - \frac{1}{2} Q_{N,N}^K + \sum_{m=1}^{d_2} \frac{1}{2} Q_{N-m,N-m}^K . \tag{3.101}$$

Now, the deformation equations imply the deformation equation for the normalization coefficients of the biorthogonal polynomials,

$$\frac{1}{K} Q_{N,N}^K = -\hbar \partial_{u_K} \ln h_N \tag{3.102}$$

which implies promptly

$$\operatorname{tr} \underset{N}{\mathcal{U}}_K(x) = -\frac{1}{K} x^k + \frac{\hbar}{2} \partial_{u_k} \ln \left(\frac{\prod_{m=1}^{d_2} h_{N-m}}{h_N} \right) = -\frac{1}{K} x^k + \frac{\hbar}{2} \partial_{u_k} \ln \left(\frac{\mathcal{Z}_N^{\,2}}{\mathcal{Z}_{N-d_2} \mathcal{Z}_{N+1}} \right) , \tag{3.103}$$

$\boxed{\text{traceU}}$

where we have used the fact that [?]

$$\mathcal{Z}_N = C_N \prod_{j=0}^{N-1} h_j , \tag{3.104}$$

with C_n dependent only on N but independent on the V_i's.

4 Differential system $\underset{N}{D}_1(x)$

From the formulas for the deformation equations we can derive the formula for the differential equation. Recall that P is of finite band size and that we have:

$$P = P_{-1} + P_{\geq} = P_- + V_1'(Q)_{\geq} = \Lambda^t \gamma + \sum_{K=0}^{d_1} u_{K+1} \left(Q_{+0}^K + \frac{1}{2} Q_0^K \right) . \tag{4.105}$$

Using (??) and (??) we get:

$$P = \Lambda^t \gamma + u_1 \mathbf{1} - \sum_{K=1}^{d_1} K u_{K+1} U_K + \frac{1}{2} \sum_{K=1}^{d_1} u_{K+1} Q_0^K . \tag{4.106}$$

$\boxed{\text{Pdeco}}$

Since $\underset{N}{D}_1(x)$ is nothing but the folded version of P, and the folding is linear, we have:

$$\underset{N}{D}_1(x) = \prod_{N}^{M} \Lambda^t \gamma \mathcal{F}_N + u_1 \mathbf{1}_{d_2+1} - \sum_{K=1}^{d_1} u_{K+1} K \underset{N}{\mathcal{U}}_K(x) + \frac{1}{2} \sum_{K=1}^{d_1} u_{K+1} \underset{N0}{q}^K \tag{4.107}$$

where $\underset{N}{\gamma}$ and $\underset{N0}{q}^K$ are the diagonal matrices:

$$\underset{N}{\gamma} = \operatorname{diag}\left(\gamma(N-d_2-1), \ldots, \gamma(n-1) \right) , \quad \underset{N0}{q}^K = \operatorname{diag}\left(Q_{N-d_2,N-d_2}^K, \ldots, Q_{N,N}^K \right) . \tag{4.108}$$

We leave to the reader to check that the first term is

$$\prod_{N}^{M} \Lambda^t \gamma \underset{N}{\mathcal{F}} = \prod_{N}^{M} \Lambda^t \gamma \prod_{N}^{M} + \prod_{N}^{M} \Lambda^t \gamma \prod_{M}^{M} \underset{N}{\mathcal{F}} = \begin{pmatrix} \gamma(M) & & \\ & \ddots & \\ & & \gamma(N-1) \end{pmatrix} \begin{pmatrix} -\frac{\alpha_{d_2-1}(N-1)}{\alpha_{d_2}(N-1)} & \cdots & \frac{x-\alpha_0(N-1)}{\alpha_{d_2}(N-1)} & -\frac{\gamma(N-1)}{\alpha_{d_2}(N-1)} \\ 1 & & & 0 \\ & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} , \tag{4.109}$$

where the second matrix in the product is nothing but the inverse of the ladder matrix $\underset{N-1}{\mathbf{a}}(x)$. Using the expression (??) for the folded matrices $\underset{N}{\mathcal{U}_K}$ and with simple manipulations one obtains ($M = N - d_2 - 1$).

$$\underset{N}{D_1}(x) = \prod_{N}^{M} V_1'(Q)_{\geq} \prod_{N-1}^{M} - \prod_{N}^{M} W(x)\left[Q, \prod_{N-1}\right] + V_1'(x) \prod_{N}^{N-1} + \prod_{N}^{M} \Lambda^t \gamma \mathcal{F}_N \tag{4.110}$$

where $W(x)$ is the semi-infinite matrix (polynomial in x and Q):

$$W(x) = \frac{V_1'(Q) - V_1'(x)}{Q - x} \ . \tag{4.111} \quad \boxed{\texttt{defW}}$$

In a totally explicit fashion we have the expression of the matrix $\underset{N}{D_1}(x)$:

$$
\begin{aligned}
\underset{N}{D_1}(x) \;=\; & \begin{pmatrix} V_1'(Q)_{N-d_2,N-d_2} & \cdots & V_1'(Q)_{N-d_2,N-1} & 0 \\ 0 & \ddots & \vdots & \vdots \\ 0 & 0 & V_1'(Q)_{N-1,N-1} & 0 \\ 0 & \cdots & 0 & V_1'(x) \end{pmatrix} \\
& + \begin{pmatrix} \gamma(M) & & \\ & \ddots & \\ & & \gamma(N-1) \end{pmatrix} \begin{pmatrix} -\frac{\alpha_{d_2-1}(N-1)}{\alpha_{d_2}(N-1)} & \cdots & \frac{x-\alpha_0(N-1)}{\alpha_{d_2}(N-1)} & -\frac{\gamma(N-1)}{\alpha_{d_2}(N-1)} \\ 1 & & & 0 \\ & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \\
& - \begin{pmatrix} W(x)_{M+1,N-1} & \cdots & W(x)_{M+1,N+d_2-1} \\ \vdots & & \vdots \\ W(x)_{N,N-1} & \cdots & W(x)_{N,N+d_2-1} \end{pmatrix} \underset{N}{\mathbb{A}} \ ,
\end{aligned}
\tag{4.112}
$$

which is the exact analogue of the very explicit formula appeared in [?].

Notice that (using (??) and (??)), if $k \geq l$, we have:

$$V_1'(Q)_{N-k,N-l} = -P_{N-k,N-l} = \beta_{k-l}(N-l) \tag{4.113}$$

which allows to rewrite:

$$
\begin{aligned}
\underset{N}{D_1}(x) \;=\; & \begin{pmatrix} \beta_0(N-d_2) & \cdots & \beta_{d_2-1}(N-1) & 0 \\ \gamma(N-d_2) & \ddots & \vdots & \vdots \\ 0 & \ddots & \beta_0(N-1) & 0 \\ 0 & \cdots & \gamma(N-1) & V_1'(x) \end{pmatrix} + \\
& -\frac{\gamma(M)}{\alpha_{d_2}(M-1)} \begin{pmatrix} \alpha_{d_2-1}(N-1) & \cdots & \alpha_0(N-1)-x & \gamma(N-1) \\ 0 & \cdots & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \end{pmatrix} + \\
& - \begin{pmatrix} W(x)_{N-d_2,N-1} & \cdots & W(x)_{N-d_2,N+d_2-1} \\ \vdots & & \vdots \\ W(x)_{N,N-1} & \cdots & W(x)_{N,N+d_2-1} \end{pmatrix} \underset{N}{\mathbb{A}} \ .
\end{aligned}
\tag{4.114}
$$

Trace formula

Using the previous formulas for the deformation matrices we can promptly obtain the trace of the matrix $\underset{N}{D_1}$.

$$\operatorname{tr} \underset{N}{D_1}(x) = V_1'(x) + \sum_{k=1}^{d_2} \beta_0(N-k) - \gamma(M)\frac{\alpha_{d_2-1}(N-1)}{\alpha_{d_2}(N-1)} \tag{4.115}$$

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and using eq. (**??**) we have:

$$\alpha_{d_2}(N-1) = \prod_{k=1}^{d_2} \gamma(N-1-k) \ v_{d_2+1} \tag{4.116}$$

and

$$\alpha_{d_2-1}(N-1) = \prod_{k=1}^{d_2-1} \gamma(N-1-k) \left(v_{d_2} + v_{d_2+1} \sum_{j=1}^{d_2} \beta_0(N-j)) \right) \tag{4.117}$$

therefore:

$$\mathrm{tr}\, \underset{N}{D_1}(x) = V_1'(x) - \frac{v_{d_2}}{v_{d_2+1}} \ . \tag{4.118}$$

`TraceD`

Note the remarkable fact that the trace of $\underset{N}{D_1}$ does not depend on N. We also point out that this result cannot be obtained directly from the formula contained in [**?**].

4.1 Folded deformation matrices $\underset{N}{\mathcal{V}}_J(x)$

We first consider the deformations w.r.t. v_J, $0 \le J \le d_2$ because they have simpler expression and allow to compute the deformation w.r.t. v_{d_2+1} by folding of (**??**). The semiinfinite deformation matrix is the finite-band size matrix

$$V_J = -\frac{1}{J}(P^J)_{-0} \tag{4.119}$$

Its folded counterpart as per eq. (**??**) is (for brevity in the formulas we set $M := N - d_2 - 1$):

$$
\begin{aligned}
-J \underset{N}{\mathcal{V}}_J(x) &= \overset{M}{\underset{N}{\Pi}}(P^J)_{-0}\, \mathcal{F}_N = \overset{M}{\underset{N}{\Pi}}(P^J)_{-0} \underset{N}{\Pi}\, \mathcal{F}_N \\
&= \overset{M}{\underset{N}{\Pi}}(P^J)_{-0} \overset{M}{\underset{N}{\Pi}}\, \mathcal{F}_N + \overset{M}{\underset{N}{\Pi}} P^J \underset{M}{\Pi}\, \mathcal{F}_N \\
&= \overset{M}{\underset{N}{\Pi}}(P^J)_{-0} \overset{M}{\underset{N}{\Pi}} + \overset{M}{\underset{N}{\Pi}} P^J (1-A)^{-1} \underset{M}{\Pi} A \overset{M}{\Pi} \ .
\end{aligned}
\tag{4.120}
$$

`iits`

From the expression (**??**) for A we have the relation

$$
\begin{aligned}
\underset{M}{\Pi} A \overset{M}{\Pi} &= -\frac{1}{v_{d_2+1}} \underset{M}{\Pi}(\gamma^{-1}\Lambda^t)^{d_2}(Q-x) \overset{M}{\Pi} = -\frac{1}{v_{d_2+1}}(\gamma^{-1}\Lambda^t)^{d_2} \underset{N-1}{\Pi}(Q-x) \overset{M}{\Pi} = \\
&= -\frac{1}{v_{d_2+1}}(\gamma^{-1}\Lambda^t)^{d_2}(Q-x) \underset{N-1}{\Pi} \overset{M}{\Pi} + \frac{1}{v_{d_2+1}}(\gamma^{-1}\Lambda^t)^{d_2}\left[Q, \underset{N-1}{\Pi}\right] = \\
&= -(1-A) \underset{N-1}{\Pi} \overset{M}{\Pi} + \frac{1}{v_{d_2+1}}(\gamma^{-1}\Lambda^t)^{d_2}\left[Q, \underset{N-1}{\Pi}\right] \ .
\end{aligned}
\tag{4.121}
$$

`iits1`

Plugging (**??**) into (**??**) we obtain

$$
-J \underset{N}{\mathcal{V}}_J(x) = \overset{M}{\underset{N}{\Pi}} P^J_{-0} \overset{M}{\underset{N}{\Pi}} - \overset{M}{\underset{N}{\Pi}} P^J \overset{M}{\underset{N-1}{\Pi}} + \frac{1}{v_{d_2+1}} \overset{M}{\underset{N}{\Pi}} P^J (1-A)^{-1}(\gamma^{-1}\Lambda)^{d_2}\left[Q, \underset{N-1}{\Pi}\right] \ .
\tag{4.122}
$$

`vd21`

The last deformation matrix is obtained from eq. (**??**) by also recalling that $\Pi^{d_2+1} Q \mathcal{F}_N = x \Pi^{d_2+1} \mathcal{F}_N$ to be

$$
-(d_2+1)v_{d_2+1}\underset{N}{\mathcal{V}}_{d_2+1} = x \underset{N}{D_1}(x) + \sum_{J=1}^{d_2} J\, v_J \underset{N}{\mathcal{V}}_J + \sum_{K=1}^{d_1+1} K\, u_K \underset{N}{\mathcal{U}}_K - \hbar\,\mathrm{diag}(N-d_2,\dots,N)
\tag{4.123}
$$

`vd21`

4.1.1 Trace

We will compute first the traces of the deformation matrices for $J \le d_2$, whereas the case $J = d_2 + 1$ will be obtained by tracing the identity (**??**). Hereafter the notation tr denotes the finite dimensional trace for the folded matrices (of dimension $(d_2+1) \times (d_2+1)$) while Tr denotes the trace of a semiinfinite matrix. Bearing this in mind we compute

$$
-J\,\mathrm{tr}\,\underset{N}{\mathcal{V}}_J(x) = \frac{1}{2}\,\mathrm{Tr}\left(P^J \overset{N-d_2-1}{\underset{N}{\Pi}} \right) - \mathrm{Tr}\left(P^J \overset{N-d_2-1}{\underset{N-1}{\Pi}} \right) + \frac{1}{v_{d_2+1}}\,\mathrm{Tr}\left(P^J (1-A)^{-1} (\gamma^{-1}\Lambda)^{d_2}\left[Q, \underset{N-1}{\Pi}\right] \right) =
$$

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$$= \frac{1}{2}P^J_{N,N} - \frac{1}{2}\sum_{j=M}^{N-1} P^k_{j,j} + \frac{1}{v_{d_2+1}} \operatorname{Tr}\left(P^J \left(1-A\right)^{-1}\left(\gamma^{-1}\Lambda\right)^{d_2} \left[Q, \underset{N-1}{\Pi}\right]\right) \quad (4.124)$$

Let us focus our attention on the last term. First, notice that:

$$\frac{1}{v_{d_2+1}} \operatorname{Tr} P^J \left(1-A\right)^{-1}\left(\gamma^{-1}\Lambda\right)^{d_2}\left[Q, \Pi_{N-1}\right]$$

$$= \frac{1}{v_{d_2+1}} \operatorname{Tr} P^J \left(1-\widetilde{A}\right)^{-1}\left(\gamma^{-1}\Lambda\right)^{d_2}\left[V_2'(P)-x, \Pi_{N-1}\right]$$

$$(4.125)$$

where $\widetilde{A}$ is the following strictly upper triangular matrix

$$\widetilde{A} := 1 - (\gamma^{-1}\Lambda)^{d_2}(V_2'(P)-x) \ . \tag{4.126}$$ `Atilde`

Indeed, for $J \le d_2$, the difference is the trace of a strictly upper triangular matrix. Then we have:

$$\operatorname{Tr} P^J \left(1-\widetilde{A}\right)^{-1}\left(\gamma^{-1}\Lambda\right)^{d_2}\left[V_2'(P)-x, \underset{N-1}{\Pi}\right] = \operatorname{Tr} P^J \left[\left(1-\widetilde{A}\right)^{-1}\left(\gamma^{-1}\Lambda\right)^{d_2}, V_2'(P)-x\right] \underset{N-1}{\Pi} \tag{4.127}$$

Let us denote by $y_j = y_j(x)$, $j = 1, \ldots, d_2$ the solutions of algebraic equation $V_2'(y) = x$ and define the following polynomials in y:

$$S_j(y) := \frac{V_2'(y)-x}{y-y_j} \tag{4.128}$$

We have in particular $S_j(y_j) = V_2''(y_j)$.

We also define the following strictly upper triangular matrices:

$$R_j := 1 - (\gamma^{-1}\Lambda)(P-y_j) \tag{4.129}$$

We are now going to use the following identity, whose proof can be found in appendix **??**:

$$(1-\widetilde{A})^{-1}(\gamma^{-1}\Lambda)^{d_2} = \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)}\left(1-R_j\right)^{-1}\gamma^{-1}\Lambda \tag{4.130}$$ `inappe`

Using (**??**) we have:

$$\frac{1}{v_{d_2+1}} \operatorname{Tr} P^J \left[\left(1-\widetilde{A}\right)^{-1}\left(\gamma^{-1}\Lambda\right)^{d_2}, V_2'(P)-x\right] \underset{N-1}{\Pi}$$

$$= \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \operatorname{Tr} P^J \left[\left(1-R_j\right)^{-1}\gamma^{-1}\Lambda, V_2'(P)-x\right] \underset{N-1}{\Pi}$$

$$= \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \operatorname{Tr} P^J \left(1-R_j\right)^{-1}\gamma^{-1}\Lambda(V_2'(P)-x) \underset{N-1}{\Pi}$$

$$-\sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \operatorname{Tr} P^J \left(V_2'(P)-x\right)\left(1-R_j\right)^{-1}\gamma^{-1}\Lambda \underset{N-1}{\Pi} =$$

$$= \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \operatorname{Tr} P^J \overbrace{\left(1-R_j\right)^{-1}\gamma^{-1}\Lambda(P-y_j)}^{=1} \frac{V_2'(P)-x}{P-y_j} \underset{N-1}{\Pi} +$$

$$-\sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \operatorname{Tr} P^J \frac{V_2'(P)-x}{P-y_j}(\Lambda^t\gamma\gamma^{-1}\Lambda + \underset{0}{\Pi})(P-y_j)\left(1-R_j\right)^{-1}\gamma^{-1}\Lambda \underset{N-1}{\Pi}$$

$$= \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \operatorname{Tr} P^J S_j(P) \underset{N-1}{\Pi} - \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \operatorname{Tr} P^J S_j(P)\Lambda^t\Lambda \underset{N-1}{\Pi}$$

$$- \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \, \text{Tr} \, P^J \, S_j(P) \, \underset{0}{\Pi}(P - y_j)(1 - R_j)^{-1} \, \gamma^{-1} \Lambda \, \underset{N-1}{\Pi}$$

$$= \quad \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \, \text{Tr} \, P^J \, S_j(P) \, \underset{0}{\Pi} - \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \, \text{Tr} \, P^J \, S_j(P) \, \underset{0}{\Pi}(P - y_j)(\mathbf{1} - R_j)^{-1} \, \gamma^{-1} \Lambda \, \underset{N-1}{\Pi}$$

(4.131)

Notice that if $N > 2d_2$ the Π_{N-1} in the last trace is irrelevant and therefore that trace is independent of N. Assuming thus $N > 2d_2$ we obtain can carry on with our computation

$$\frac{1}{v_{d_2+1}} \, \text{Tr} \, P^J \, (1 - A)^{-1} \, (\gamma^{-1}\Lambda)^{d_2} \, [Q, \Pi_{N-1}]$$

$$= \quad \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \, \text{Tr} \, P^J \, S_j(P) \, \underset{0}{\Pi} - \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \, \text{Tr} \, \underset{0}{\Pi}(P - y_j)(1 - R_j)^{-1} \, \gamma^{-1}\Lambda \, P^J \, S_j(P)$$

$$= \quad \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \, \text{Tr} \, P^J \, S_j(P) \, \underset{0}{\Pi} - \sum_{j=1}^{d_2} \frac{y_j^k}{V_2''(y_j)} \, \text{Tr} \, \underset{0}{\Pi}(P - y_j)(1 - R_j)^{-1} \, \gamma^{-1}\Lambda \, S_j(P)$$

$$\quad - \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \, \text{Tr} \, \underset{0}{\Pi}(P - y_j)(1 - R_j)^{-1} \, \gamma^{-1}\Lambda \, (P^J - y_j^J) \, S_j(P)$$

$$= \quad \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \, \text{Tr} \, P^J \, S_j(P) \, \underset{0}{\Pi} - \sum_{j=1}^{d_2} \frac{y_j^J}{V_2''(y_j)} \, \text{Tr} \, \underset{0}{\Pi}(P - y_j)(1 - R_j)^{-1} \, \gamma^{-1}\Lambda \, S_j(P)$$

$$\quad - \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \, \text{Tr} \, \underset{0}{\Pi}(P - y_j)(1 - R_j)^{-1} \, \gamma^{-1}\Lambda \, (P - y_j) \frac{P^J - y_j^J}{P - y_j} \, S_j(P)$$

$$= \quad \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \, \text{Tr} \, P^J \, S_j(P) \, \underset{0}{\Pi} - \sum_{j=1}^{d_2} \frac{y_j^J}{V_2''(y_j)} \, \text{Tr} \, \underset{0}{\Pi}(P - y_j)(1 - R_j)^{-1} \, \gamma^{-1}\Lambda \, (S_j(P) - S_j(y_j))$$

$$\quad - \sum_{j=1}^{d_2} y_j^J \, \text{Tr} \, (P - y_j)(1 - R_j)^{-1} \, \gamma^{-1}\Lambda \, \underset{0}{\Pi} - \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \, \text{Tr} \, \underset{0}{\Pi}(P - y_j) \frac{P^J - y_j^J}{P - y_j} \, S_j(P) \, . \qquad (4.132) \quad \boxed{\text{1212}}$$

Recalling that $\Lambda \underset{0}{\Pi} = 0$ we continue

$$(\mathbf{??}) \quad = \quad \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \, \text{Tr} \, P^J \, S_j(P) \, \underset{0}{\Pi} - \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \, \text{Tr} \, \underset{0}{\Pi}(P^J - y_j^J) \, S_j(P)$$

$$\quad - \sum_{j=1}^{d_2} \frac{y_j^J}{V_2''(y_j)} \, \text{Tr} \, \underset{0}{\Pi}(P - y_j)(1 - R_j)^{-1} \, \gamma^{-1}\Lambda \, (P - y_j) \frac{S_j(P) - S_j(y_j)}{P - y_j}$$

$$= \quad \sum_{j=1}^{d_2} \frac{y_j^J}{V_2''(y_j)} \, \text{Tr} \, S_j(P) \, \underset{0}{\Pi} - \sum_{j=1}^{d_2} \frac{y_j^J}{V_2''(y_j)} \, \text{Tr} \, \underset{0}{\Pi}(P - y_j) \frac{S_j(P) - S_j(y_j)}{P - y_j}$$

$$= \quad \sum_{j=1}^{d_2} \frac{y_j^J}{V_2''(y_j)} \, \text{Tr} \, S_j(P) \, \underset{0}{\Pi} - \sum_{j=1}^{d_2} \frac{y_j^J}{V_2''(y_j)} \, \text{Tr} \, \underset{0}{\Pi}(S_j(P) - S_j(y_j))$$

$$= \quad \sum_{j=1}^{d_2} y_j^J \, \text{Tr} \, \underset{0}{\Pi} = \sum_{j=1}^{d_2} y_j^J \qquad\qquad (4.133) \quad \boxed{\text{final1}}$$

With eq. (**??**) we can finally state

$$\boxed{-J \, \text{tr} \, \underset{N}{\mathcal{Y}}_J(x) = \frac{1}{2} P_{N,N}^J - \frac{1}{2} \sum_{j=M}^{N-1} P_{j,j}^J + \sum_{j=1}^{d_2} y_j^J}$$

(4.134) $\boxed{\text{traces}}$

Using eq. (**??**) and tracing (**??**) one can obtain the following formulas for the traces

$$\operatorname{tr} \underset{N}{\mathcal{Y}}_J(x) = -\frac{\hbar}{2}\partial_{v_J} \ln\left(\frac{\prod_{j=1}^{d_2} h_{N-j}}{h_N}\right) + \oint dy\, \frac{y^J}{J}\frac{V_2''(y)}{V_2'(y)-x}\ , \quad J = 1,\dots,d_2$$

$$\operatorname{tr} \underset{N}{\mathcal{Y}}_{d_2+1}(x) = -\frac{\hbar}{2}\partial_{v_{d_2+1}} \ln\left(\frac{\prod_{j=1}^{d_2} h_{N-j}}{h_N}\right) + \oint dy\, \frac{y^{d_2+1}}{d_2+1}\frac{V_2''(y)}{V_2'(y)-x} - \hbar\left(N-\frac{d_2}{2}\right)\frac{1}{v_{d_2+1}}\ . \tag{4.135} \boxed{\text{guess2}}$$

4.2 Determinant of the fundamental system

It was proven in [**?**] that the following overdetermined set of differential equations for a $(d_2+1)\times(d_2+1)$ matrix $\underset{N}{\boldsymbol{\Psi}}$ is Frobenius compatible:

$$-\hbar\partial_x \underset{N}{\boldsymbol{\Psi}}(x) = \underset{N}{D}_1(x)\,\underset{N}{\boldsymbol{\Psi}}(x) \tag{4.136} \boxed{11}$$

$$\hbar\partial_{u_K} \underset{N}{\boldsymbol{\Psi}}(x) = \underset{N}{\mathcal{U}}_K(x)\,\underset{N}{\boldsymbol{\Psi}}(x) \tag{4.137} \boxed{12}$$

$$-\hbar\partial_{v_J} \underset{N}{\boldsymbol{\Psi}}(x) = \underset{N}{\mathcal{V}}_J(x)\,\underset{N}{\boldsymbol{\Psi}}(x) \tag{4.138} \boxed{13}$$

$$\underset{N}{\mathbf{a}}(x)\,\underset{N}{\boldsymbol{\Psi}}(x) = \underset{N+1}{\boldsymbol{\Psi}}(x) \tag{4.139} \boxed{14}$$

The biorthogonal polynomials constitute one column of $\underset{N}{\boldsymbol{\Psi}}(x)$, while the other d_2 can be also explicitly obtained as in [**?**].

From (**??**) it follows that the dependence on the coefficients of V_2 of the determinant of the fundamental system is

$$\hbar d_{\mathbf{v}} \ln\det\left(\underset{N}{\boldsymbol{\Psi}}(x)\right) := -\sum_{J=1}^{d_2+1} dv_J \hbar\partial_{v_J} \ln\det\left(\underset{N}{\boldsymbol{\Psi}}(x)\right) = \sum_{J=1}^{d_2+1} dv_J \operatorname{tr}\left(\underset{N}{\mathcal{V}}_J(x)\right) = \tag{4.140}$$

$$= -\hbar d_{\mathbf{v}} \ln\left(\sqrt{\frac{\prod_{j=1}^{d_2} h_{N-j}}{h_N}}\right) + \sum_{J=1}^{d_2+1} dv_J \oint dy\, \frac{y^J}{J}\frac{V_2''(y)}{V_2'(y)-x} - \hbar\left(N-\frac{d_2}{2}\right)d\ln(v_{d_2+1}) \tag{4.141}$$

Now one can check directly that

$$\oint dy\, \frac{y^J}{J}\frac{V_2''(y)}{V_2'(y)-x} = \partial_{v_J}\left(\oint dy\, (V_2(y)-xy)\frac{V_2''(y)}{V_2'(y)-x}\right) \tag{4.142}$$

Therefore, as a consequence of eqs. (**??**, **??**, **??**) we find that any matrix fundamental solution to the compatible system of equations satisfies the equations

$$\hbar\partial_x \ln\left(\det\underset{N}{\boldsymbol{\Psi}}(x)\right) = -V_1'(x) + \frac{v_{d_2}}{v_{d_2+1}} \tag{4.143}$$

$$\hbar d_{\mathbf{u}} \ln\left(\det\underset{N}{\boldsymbol{\Psi}}(x)\right) = d_{\mathbf{u}}\left(-V_1(x) + \frac{\hbar}{2}\ln\left(\frac{\prod_{m=1}^{d_2} h_{N-m}}{h_N}\right)\right) \tag{4.144}$$

$$\hbar d_{\mathbf{v}} \ln\left(\det\underset{N}{\boldsymbol{\Psi}}(x)\right) = -\hbar d_{\mathbf{v}}\left[\ln\left(v_{d_2+1}^{\frac{d_2}{2}-N}\sqrt{\frac{\prod_{m=1}^{d_2} h_{N-m}}{h_N}}\right) + \oint dy\, (V_2(y)-xy)\frac{V_2''(y)}{V_2'(y)-x}\right] \tag{4.145}$$

$$\det\left(\underset{N+1}{\boldsymbol{\Psi}}(x)\underset{N}{\boldsymbol{\Psi}}(x)^{-1}\right) = \det(\underset{N}{\mathbf{a}}(x)) = v_{d_2+1}(-)^{d_2+1}\sqrt{\frac{h_N{}^2 h_{N-d_2-1}}{h_N+1}}. \tag{4.146}$$

and hence we have finally the complete formula

$$\det\left[\underset{N}{\boldsymbol{\Psi}}(x)\right] = (v_{d_2+1})^{N-\frac{d_2}{2}}(-1)^{N(d_2+1)}\sqrt{\frac{\prod_{m=1}^{d_2} h_{N-m}}{h_N}}\exp\left[-\frac{1}{\hbar}\left(V_1(x) + \oint dy(V_2(y)-xy)\frac{V_2''(y)}{V_2'(y)-x}\right)\right]\ . \tag{4.147}$$

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Note that the contour integral just returns the sum of the critical values of $V_2(y) - xy$.

We know from [?] that a joint solution of the dual system of overdetermined equations for the dual window denoted by $\underline{\overset{N}{\Phi}}(x)$ has the property that

$$\underline{\overset{N}{\Phi}}(x) \overset{N}{\mathbb{A}} \underset{N}{\boldsymbol{\Psi}}(x) = C \ , \tag{4.148}$$

where C is an invertible matrix which does not depend on any of the potentials or n or $\hbar$ or x and can be conveniently be normalized to unity (see [?] for an explicit construction). Therefore

$$\det(\underline{\overset{N}{\Phi}}(x)) \quad = \quad \det(\overset{N}{\mathbb{A}})^{-1} \det(\underset{N}{\boldsymbol{\Psi}}(x))^{-1} \tag{4.149}$$

5 Dual folded system

For completeness we add the formulas for the relevant folded operators for the dual window $\overset{N}{\Phi} = (\phi_{N-1}, \ldots, \phi_{N+d_2-1})$. Since the steps are essentially the same we give only the results (here we set $L = N + d_2 - 1$)

$$K\underline{\overset{N}{U}}_K = \prod_L^{N-1} Q_{+0}^K \prod_L^{N-1} - [Q, \Pi_{N-1}] W_K(x) \prod_L^{N-1} + \left(x^K - \frac{1}{2} Q_{N-1,N-1}^K \right) \prod_N^{N-1} \tag{5.150}$$

$$J\underline{\overset{N}{V}}_J = \prod_L^{N-1} P_{-0}^J \prod_L^{N-1} + \prod_L \underline{A} \prod_L^L (\mathbf{1} - \underline{A})^{-1} P^J \prod_L^{N-1} \tag{5.151}$$

$$\underline{\overset{N}{D}}_1(x) = \begin{pmatrix} V_1'(x) & 0 & \ldots & 0 \\ \gamma(N-1) & \beta_0(N) & \ldots & \beta_{d_2-1}(L) \\ 0 & \ddots & \ddots & \vdots \\ 0 & \ldots & \gamma(L-1) & \beta_0(L) \end{pmatrix} + \tag{5.152}$$

$$-\frac{\gamma(L)}{\alpha_{d_2}(L+1)} \begin{pmatrix} 0 & \ldots & 0 & \gamma(N-1) \\ 0 & & & \alpha_0(N) - x \\ 0 & \ddots & & \vdots \\ 0 & \ldots & 0 & \alpha_{d_2-1}(L) \end{pmatrix} - \overset{N}{\mathbb{A}} \begin{pmatrix} W(x)_{N-d_2,N-1} & \ldots & W(x)_{N-d_2,N+d_2-1} \\ \vdots & & \vdots \\ W(x)_{N,N-1} & \ldots & W(x)_{N,N+d_2-1} \end{pmatrix} \tag{5.153}$$

6 Conclusion

We have given the most explicit construction for the matrices describing the differential-deformation-difference folded system. This has allowed us to determine explicitly the determinant of the fundamental solution of the system in terms of the partition function of the model and the two potentials. The final expression is quite simple in comparison with the complexity of the computation, expecially for the traces of the deformation matrices $\underset{N}{\mathcal{V}_J}$.

It is our hope (in fact it is our plan) that these formula be used to relate explicitly the partition functio of the two–matrix model to a (suitably defined) isomonodromic tau–function. Indeed the system (??,??,??,??) can (and should) be regarded as a monodromy–preserving set of infinitesimal (??, ??) and finite (??) deformation equation for the ODE (??) which has an irregular, degenerate singularity at infinity. As such it is envisionable that one can define the associated notion of tau–function although there are technical difficulties due to the degeneracy at infinity. In the similar case of the ODE+PDE+ΔE for the orthogonal polynomials in the one–matrix model a similar approach has given very satisfactory results [?, ?].

A   Proof of formula (??)

We need to prove formula (**??**)

$$(\mathbf{1} - \widetilde{A})^{-1}(\gamma^{-1}\Lambda)^{d_2} = \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \, (\mathbf{1} - R_j)^{-1}\gamma^{-1}\Lambda \, , \tag{A.154}$$

which can be written more transparently as

$$\left((\gamma^{-1}\Lambda)^{d_2}(V_2'(P) - x)\right)^{-1} (\gamma^{-1}\Lambda)^{d_2} = \sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \left(\gamma^{-1}\Lambda(P - y_j)\right)^{-1} \gamma^{-1}\Lambda \, . \tag{A.155}$$

We remark for the better understanding of the reader that if the matrices Λ, $(V_2'(P) - x)$ were invertible this would amount simply to the partial fraction expansion of $1/(V_2'(P) - x)$. Multiplying both sides by the invertible matrix $(\mathbf{1} - \widetilde{A})$ on the left and recalling the definition (**??**) of $\widetilde{A}$

$$
\begin{aligned}
(\gamma^{-1}\Lambda)^{d_2} &= (\gamma^{-1}\Lambda)^{d_2} \left(\sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} (V_2'(P) - x)(\mathbf{1} - R_j)^{-1}\gamma^{-1}\Lambda \right) \\
&= (\gamma^{-1}\Lambda)^{d_2} \left(\sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \frac{V_2'(P) - x}{P - y_j}(P - y_j)(\mathbf{1} - R_j)^{-1}\gamma^{-1}\Lambda \right) \\
&= (\gamma^{-1}\Lambda)^{d_2} \left(\sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \frac{V_2'(P) - x}{P - y_j}(\underset{0}{\Pi} + \Lambda^t\gamma\gamma^{-1}\Lambda)(P - y_j)(\mathbf{1} - R_j)^{-1}\gamma^{-1}\Lambda \right)
\end{aligned}
$$
$$(A.156)$$

Noticing now that $(\gamma^{-1}\Lambda)^{d_2} \dfrac{V_2'(P) - x}{P - y_j} \underset{0}{\Pi} = 0$ we continue

$$
\begin{aligned}
(\gamma^{-1}\Lambda)^{d_2} &= (\gamma^{-1}\Lambda)^{d_2} \left(\sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \frac{V_2'(P) - x}{P - y_j}\Lambda^t\gamma\gamma^{-1}\Lambda \right) \\
&= (\gamma^{-1}\Lambda)^{d_2} \left(\sum_{j=1}^{d_2} \frac{1}{V_2''(y_j)} \frac{V_2'(P) - x}{P - y_j}\left(\mathbf{1} - \underset{0}{\Pi}\right) \right) \\
&= (\gamma^{-1}\Lambda)^{d_2} \left(\mathbf{1} - \underset{0}{\Pi}\right) = (\gamma^{-1}\Lambda)^{d_2} \, , \tag{A.157}
\end{aligned}
$$

where we have used that $(\gamma^{-1}\Lambda)^{d_2} \underset{0}{\Pi} = 0$. The identity is thus proved.

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