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Large N expansion of the 2-matrix model, multicut case

B. Eynard[1]

Service de Physique Théorique de Saclay,
F-91191 Gif-sur-Yvette Cedex, France.

Abstract

We present a method, based on loop equations, to compute recursively, all the terms in the large N topological expansion of the free energy for the 2-hermitian matrix model, in the case where the support of the density of eigenvalues is not connected. We illustrate the method by computing the free energy of a statistical physics model on a discretized torus.

[1] E-mail: eynard@spht.saclay.cea.fr

1 Introduction

Random matrix models [35, 21, 14, 27, 7, 36] have a wide range of applications in mathematics and physics where they constitute a major field of activity. They are involved in condensed matter physics (quantum chaos [27, 33], localization, crystal growths [38],...etc), statistical physics [14,12,9,26] (on a 2d fluctuating surface, also called 2d euclidean quantum gravity, linked to conformal field theory), high energy physics (string theory [15,22], quantum gravity [14, 26, 25], QCD [41],...), and they are very important in mathematics too: (they seem to be linked to the Riemann conjecture [35, 37]), they are important in combinatorics, and provide a wide class of integrable systems [42,7,29,16].

In the 80's, random matrix models were introduced as a toy model for zero–dimensional string theory and quantum gravity [12,14,9].

The free energy of matrix model is conjectured[2] to have a $1/N^2$ expansion [40,14,1,2] called topological expansion (N is the size of the matrix):

$$F = \sum_{h=0}^{\infty} N^{-2h} F^{(h)} \tag{1.1}$$

That expansion is the main motivation for applications to 2-dimensional quantum gravity [14, 21], because each $F^{(h)}$ is the partition function of a statistical physics model on a genus h surface.

The authors of [1] invented an efficient method to compute recursively all the $F^{(h)}$'s for the 1-matrix model, and they improved it in [2].

Their method was generalized in [18] for the 2-matrix model in the so-called 1-cut case.

Here, we extend the result of [18], to multicut cases.

Assuming that the $1/N^2$ expansion exists, the aim of the present work is to give a method to compute recursively the terms of the expansion, similar to that of [18].

The 2-matrix model [10, 11] was first introduced as a model for two-dimensional gravity, with matter, and in particular with an Ising field [31,8]. The diagrammatic expansion of the 2-matrix model's partition function is known to generate 2-dimensional statistical physics models on a random discrete surface [14,12,31]:

$$N^2 F = -\ln Z = \sum_{\text{surfaces}} \sum_{\text{matter}} \mathrm{e}^{-\text{Action}} \tag{1.2}$$

[2]There is at the present time no rigorous proof of the existence of the topological expansion; The Riemann-Hilbert approach seems to be the best way to prove it as in [17]. The Riemann-Hilbert problem for the 2-matrix model has been formulated [6,5,4,30], and seems to be on the verge of being solved [4].

where the Action is the matter action (like Ising's nearest neighboor spin coupling) plus the gravity action (total curvature and cosmological constant) [14]. The cosmological constant couples to the area of the surface, and N (the size of the matrix) couples to the total curvature, i.e. the genus of the surface. The large N expansion thus generates a genus expansion:

$$F = \sum_{h=0}^{\infty} N^{-2h} F^{(h)} \tag{1.3}$$

where $F^{(h)}$ is the partitrion function of the statistical physics model on a random surface of fixed genus h.

$$F^{(h)} = \sum_{\text{genus}\, h\, \text{surfaces}} \sum_{\text{matter}} \mathrm{e}^{-\text{Action}} \tag{1.4}$$

The leading term $F^{(0)}$ computed by [3] (along the method invented by [34] and rigorously established by [28]) is the planar contribution. Our goal in this article is to compute $F^{(1)}$ and present an algorithmic method for computing $F^{(h)}$ for $h \geq 1$. We generalize the method of [18].

1.1 Outline of the article

- In section 2 we introduce the definitions and notations, in particular we define the 1-loop functions and 2-loop functions, loop-insertion operators, and we write the "Master loop equation".
- In section 3, we observe that, to leading order, the master loop equation is an algebraic equation of genus zero, and we study the geometry and the sheet-structure of the underlying algebraic curve.
- In section 5 we include the previously neglected $1/N^2$ term in the loop equation, and we compute the 1-loop function $Y(x)$ to next to leading order. Then we derive the next to leading order free energy $F^{(1)}$ by integrating $Y^{(1)}(x)$. We also discuss how to compute higher order terms.
- In section 6 we complete the calculation for the case where the algebraic curve has genus one.
- section 8 is the conclusion.

2 The 2-matrix model

Let N be an integer, V_1 and V_2 two polynomials of degrees d_1+1 and d_2+1:

$$V_1(x) = g_0 + \sum_{k=1}^{d_1+1} \frac{g_k}{k} x^k \qquad , \qquad V_2(y) = \tilde{g}_0 + \sum_{k=1}^{d_2+1} \frac{\tilde{g}_k}{k} y^k \tag{2.1}$$

Then let g be an integer (called the genus) choosen between $0 \leq g \leq d_1 d_2 - 1$, and let $\vec{\epsilon}$ be a g dimensional vector:

$$\vec{\epsilon} = (\epsilon_1, \epsilon_2, \dots, \epsilon_g)^t \tag{2.2}$$

We define

$$\epsilon_{g+1} = 1 - \sum_{i=1}^{g} \epsilon_i \tag{2.3}$$

All those numbers given, we define the partition function Z and the free energy F by the following matrix integral:

$$Z = \mathrm{e}^{-N^2 F} = \int dM_1 dM_2 \; \mathrm{e}^{-N \operatorname{tr} [V_1(M_1) + V_2(M_2) - M_1 M_2]} \tag{2.4}$$

where the integral is over pairs of hermitian matrices M_1 and M_2 restricted by the following condition:

- the large N limit of the density of eigenvalues of M_1 has a support made of $g+1$ disconnected intervals.

- the filling fraction (integral of the density) in each interval is ϵ_i.

We assume that the free energy F admits a topological $1/N^2$ expansion:

$$F = F^{(0)} + \frac{1}{N^2} F^{(1)} + \dots + \frac{1}{N^{2h}} F^{(h)} + \dots \tag{2.5}$$

That asumption plays a key role in many areas of physics, in particular quantum gravity or string theory [15, 14], and is believe to hold for a wide class of potentials. However, the existence of the $1/N^2$ expansion for the 2-matrix model has never been proven rigorously (for the one matrix model, it has been established by [17]).

The goal of this article is to develop a method to compute $F^{(h)}$ by recursion on h. In particular, we will explicitely compute $F^{(1)}$. Notice that this was already done in [18] in the case $g=0$. Notice that $F^{(0)}$ was computed by [3].

Remark: The model can be extended to normal matrices with support on complex paths, i.e. the eigenvalues are located along some line in the complex plane, not necessarily the real axis. In that case, the potentiasl need not have even degrees and positive leading coefficient, the potentials can be arbitrary complex polynomials, and the complex paths have to be chosen so that the partition function eq. (2.4) makes sense.

2.1 Definition: resolvents

We define:

$$T_{k,l} := \frac{1}{N} \left\langle \operatorname{tr} M_1^k M_2^l \right\rangle \tag{2.6}$$

The resolvents are formaly[3] defined by:

$$W_1(x) := \sum_{k=0}^{\infty} \frac{T_{k,0}}{x^{k+1}} \quad , \quad W_2(y) := \sum_{l=0}^{\infty} \frac{T_{0,l}}{y^{l+1}} \tag{2.7}$$

in other words:

$$W_1(x) = \frac{1}{N} \left\langle \operatorname{tr} \frac{1}{x - M_1} \right\rangle \quad , \quad W_2(y) = \frac{1}{N} \left\langle \operatorname{tr} \frac{1}{y - M_2} \right\rangle \tag{2.8}$$

We also define:

$$Y(x) := V_1'(x) - W_1(x) \quad , \quad X(y) := V_2'(y) - W_2(y) \tag{2.9}$$

We assume that all the $T_{k,l}$ have a $1/N^2$ expansion, and we can write (formaly):

$$Y(x) = Y^{(0)}(x) + \frac{1}{N^2} Y^{(1)}(x) + \ldots + \frac{1}{N^{2h}} Y^{(h)}(x) + \ldots \tag{2.10}$$

$$X(y) = X^{(0)}(y) + \frac{1}{N^2} X^{(1)}(y) + \ldots + \frac{1}{N^{2h}} X^{(h)}(y) + \ldots \tag{2.11}$$

We will recall below that the leading terms $Y^{(0)}(x)$ and $X^{(0)}(y)$ are solutions of algebraic equations. Then we will explain how to compute the first subleading term $Y^{(1)}(x)$. We will show that we can compute all $Y^{(h)}$ by recursion on h.

2.2 Other 1-loop functions

We define the following formal functions:

$$W(x,y) := \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{T_{k,l}}{x^{k+1} y^{l+1}} = \frac{1}{N} \left\langle \operatorname{tr} \frac{1}{x - M_1} \frac{1}{y - M_2} \right\rangle \tag{2.12}$$

$$U(x,y) := \sum_{k=0}^{\infty} \sum_{j=0}^{d_2} \sum_{l=0}^{j-1} g_{j+1}^* \frac{y^{j-1-l}}{x^{k+1}} T_{k,l} = \frac{1}{N} \left\langle \operatorname{tr} \frac{1}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \right\rangle \tag{2.13}$$

$$\begin{aligned} P(x,y) &:= \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} \sum_{k=0}^{i-1} \sum_{l=0}^{j-1} g_{i+1} g_{j+1}^* x^{i-1-k} y^{j-1-l} T_{k,l} \\ &= \frac{1}{N} \left\langle \operatorname{tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \right\rangle \end{aligned} \tag{2.14}$$

Notice that $P(x,y)$ is a polynomial in x and y of degree $d_1 - 1, d_2 - 1$.

All those functions have a $1/N^2$ expansion.

[3] Formaly means the following: the sums in the RHS are not necessarily convergent. $W_1(x)$ is merely a convenient notation to deal with all $T_{k,0}$ at once.

2.3 Loop insertion operators

We define formaly the loop insertion operators:

$$\frac{\partial}{\partial V_1(x)} := \frac{1}{x}\frac{\partial}{\partial g_0} + \sum_{k=1}^{\infty} \frac{k}{x^{k+1}}\frac{\partial}{\partial g_k} \qquad , \qquad \frac{\partial}{\partial V_2(y)} := \frac{1}{y}\frac{\partial}{\partial \tilde{g}_0} + \sum_{k=1}^{\infty} \frac{k}{y^{k+1}}\frac{\partial}{\partial g_k^*} \tag{2.15}$$

These formal definitions actualy mean that for any observable f:

$$\frac{\partial f}{\partial g_k} = \text{Res}\, x^k \frac{\partial f}{\partial V_1(x)} dx \tag{2.16}$$

In particular with the free energy, we read from the partition function:

$$W_1(x) = \frac{\partial}{\partial V_1(x)} F \qquad , \qquad W_2(y) = \frac{\partial}{\partial V_2(y)} F \tag{2.17}$$

2.4 2-loop functions

We define the following functions:

$$\Omega(x;x') := \frac{\partial}{\partial V_1(x')} W_1(x) = -\left\langle \text{tr}\,\frac{1}{x-M_1}\,\text{tr}\,\frac{1}{x'-M_1}\right\rangle_{\text{conn}} \tag{2.18}$$

$$\tilde{\Omega}(y;x') := \frac{\partial}{\partial V_1(x')} W_2(y) = -\left\langle \text{tr}\,\frac{1}{x'-M_1}\,\text{tr}\,\frac{1}{y-M_2}\right\rangle_{\text{conn}} \tag{2.19}$$

$$U(x,y;x') := -\frac{\partial}{\partial V_1(x')} U(x,y) = \left\langle \text{tr}\,\frac{1}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\,\text{tr}\,\frac{1}{x'-M_1}\right\rangle_{\text{conn}} \tag{2.20}$$

2.5 Master loop equation

It is shown in [18, 19, 20] that we have the following system of equations, called the "master loop equations" [39]:

$$\boxed{E(x,Y(x)) = \frac{1}{N^2} U(x,Y(x);x)} \tag{2.21}$$

$$U(x,y) = x - V_2'(y) + \frac{E(x,y)}{y-Y(x)} - \frac{1}{N^2}\frac{U(x,y;x)}{y-Y(x)} \tag{2.22}$$

$$U(x,y;x') = -\frac{\partial}{\partial V_1(x')} U(x,y) \tag{2.23}$$

where $E(x,y)$ is a polynomial in x (degree d_1+1) and y (degree d_2+1):

$$\boxed{E(x,y) := (V_1'(x)-y)(V_2'(y)-x) - P(x,y) + 1} \tag{2.24}$$

and where $P(x,y)$ was defined in eq. (2.14).

We will solve that system of equations, order by order in $1/N^2$:

$$Y(x) = Y^{(0)}(x) + \frac{1}{N^2} Y^{(1)}(x) + \ldots + \frac{1}{N^{2h}} Y^{(h)}(x) + \ldots \tag{2.25}$$

Then, the large N expansion of the free energy is obtained from eq. (2.17):

$$\frac{\partial}{\partial V_1(x)} F^{(h)} = \delta_{h,0} V_1'(x) - Y^{(h)}(x) \tag{2.26}$$

3 leading order: algebraic geometry

To leading order in $1/N^2$, the master loop equation is an algebraic equation for $Y^{(0)}(x)$:

$$E^{(0)}(x, Y^{(0)}(x)) = 0 \tag{3.1}$$

where

$$E^{(0)}(x,y) = (V_1'(x) - y)(V_2'(y) - x) - P^{(0)}(x,y) + 1 \tag{3.2}$$

and $P^{(0)}(x,y)$ is a polynomial of degree $(d_1 - 1, d_2 - 1)$ whose coefficient of $x^{d_1-1}y^{d_2-1}$ is known and is equal to $g_{d_1+1}g^*_{d_2+1}$ (from eq. (2.14)).

Before going to order $1/N^2$ and higher, we need to introduce some concepts of algebraic geometry, and study the geometry of the above algebraic equation.

3.1 Determination of $P^{(0)}(x,y)$

So far, we don't know the other $d_1d_2 - 1$ unknown coefficients of $P^{(0)}(x,y)$. They are determined by the following requirements:

- $E^{(0)}(x,y) = 0$ is a genus g algebraic curve. If g is less than the maximal genus[4] we get $d_1d_2 - 1 - g$ constraints on the coefficients of $P^{(0)}(x,y)$.

- The contour integrals of $Y^{(0)}(x)dx$ along A-cycles are:

$$\frac{1}{2i\pi} \oint_{\mathcal{A}_i} Y^{(0)}(x)dx = \epsilon_i \quad , i = 1, \ldots, g+1 \tag{3.3}$$

where $\mathcal{A}_i$,$\mathcal{B}_i$, $i = 1, \ldots, g$ is a canonical basis of irreducible cycles on the algebraic curve. We get g independent equations for the coefficients of $P^{(0)}(x,y)$.

Therefore we can determine the polynomial $P^{(0)}(x,y)$ (and thus the polynomial $E^{(0)}(x,y)$) completely.

[4]The maximal genus is $d_1d_2 - 1$. It can be computed by various methods. In particular, see [32] for the Newton's polygons method

A physical picture is that the support of the large N average density of eigenvalues of the matrix M_1 is made of $g+1$ intervals $[a_i, b_i]$, $i = 1, \dots, g+1$, and for each i, $\mathcal{A}_i$ is a contour which encloses $[a_i, b_i]$ in the trigonometric direction, and which does not enclose the other a_j or b_j with $j \neq i$. eq. (3.3) means that the interval $[a_i, b_i]$ contains a proportion ϵ_i of the total number of eigenvalues, this why ϵ_i is called a filling fraction.

In practice, we don't have a closed expression for $P^{(0)}(x, y)$ as a function of the coefficients of V_1 and V_2 and the filling fractions ϵ_i. The converse is easier: given a genus g algebraic curve, we can determine V_1, V_2 and the ϵ_i'S.

3.2 Algebraic geometry

Before proceeding, we need to study the geometry of our algebraic curve.

Let us call $\mathcal{E}$ the algebraic curve, and we consider that an abstract point $p \in \mathcal{E}$ is a pair of complex numbers $p = (x, y)$ such that $E^{(0)}(x, y) = 0$. Thus, $x = \mathcal{X}(p)$ and $y = \mathcal{Y}(p)$ are complex-valued functions on the curve.

Here, we sumarize some well known properties of Riemann surfaces and theta-functions. We refer the interested reader to textbooks [23, 24] for proofs and complements.

3.3 Sheet structure

The function $Y^{(0)}(x)$ (resp. $X^{(0)}(y)$) is multivalued, it takes d_2+1 (resp. d_1+1) values, which we note:

$$(Y_0(x), Y_1(x), \dots, Y_{d_2}(x)) \qquad (\text{resp. } (X_0(y), X_1(y), \dots, X_{d_1}(y))\,) \tag{3.4}$$

The zero$^{\text{th}}$ sheet is called the physical sheet, it is the one such that (from eq. (2.9)):

$$Y_0(x) \underset{x\to\infty}{\sim} V_1'(x) - \frac{1}{x} + O(1/x^2) \qquad (\text{resp. } X_0(y) \underset{y\to\infty}{\sim} V_2'(y) - \frac{1}{y} + O(1/y^2)\,) \tag{3.5}$$

An x-sheet (resp. y-sheet) is a domain of $\mathcal{E}$ on which the function $\mathcal{X}$ (resp. $\mathcal{Y}$) is a bijection with $\mathbf{C} \cup \{+\infty\}$. The curve $\mathcal{E}$ is thus decomposed into d_2+1 x-sheets (resp. d_1+1 y-sheets). The decomposition is not unique, and a cannonical possible decomposition will be given below.

For each x (resp. y), there are exactly d_2+1 (resp. d_1+1) points on $\mathcal{E}$, one in each sheet, such that:

$$\begin{aligned} &\mathcal{X}(p) = x \quad \leftrightarrow p = p_k(x) \quad k = 0, \dots, d_2 \\ (\text{resp.} \quad &\mathcal{Y}(p) = y \quad \leftrightarrow p = \tilde{p}_k(y) \quad k = 0, \dots, d_1\,) \end{aligned} \tag{3.6}$$

And thus:

$$Y_k(x) = \mathcal{Y}(p_k(x)) \qquad (\text{resp. } X_k(y) = \mathcal{X}(\tilde{p}_k(y))\,) \tag{3.7}$$

3.4 Points at ∞, poles of $\mathcal{X}$ and $\mathcal{Y}$

In particular, in each x-sheet, there is a point p such that $\mathcal{X}(p) = \infty$. It can be seen from eq. (3.5), that there are exactly two such points on $\mathcal{E}$. We define $p_\pm$ such that:

$$p \to p_+ \leftrightarrow \begin{cases} \mathcal{X}(p) \to \infty \\ \mathcal{Y}(p) \to \infty \\ \mathcal{Y}(p) \sim V_1'(\mathcal{X}(p)) \end{cases} \qquad \text{and} \qquad p \to p_- \leftrightarrow \begin{cases} \mathcal{X}(p) \to \infty \\ \mathcal{Y}(p) \to \infty \\ \mathcal{X}(p) \sim V_2'(\mathcal{Y}(p)) \end{cases} \tag{3.8}$$

p_+ (resp. p_-) is in the x-physical sheet (resp. y-physical sheet), while p_- (resp. p_+) is at the intersection of the other d_2 x-sheets (resp. d_1 y-sheets).

3.5 Endpoints and cuts

The x-endpoints (resp. y-endpoints) correspond to singularities of $Y(x)$ (resp. $X(y)$), i.e. they are such that $dY/dx = \infty$ (resp. $dX/dy = \infty$), i.e. they are the zeroes of $d\mathcal{X}(p)$ (resp. $d\mathcal{Y}(p)$). There are $d_2 + 1 + 2g$ (resp. $d_1 + 1 + 2g$) such endpoints:

$$d\mathcal{X}(p) = 0 \ \leftrightarrow \ p \in \{e_1, e_2, \dots, e_{d_2+1+2g}\} \ , \ d\mathcal{Y}(p) = 0 \ \leftrightarrow \ p \in \{\tilde{e}_1, \dots, \tilde{e}_{d_1+1+2g}\} \tag{3.9}$$

the endpoints are such that $\exists k \neq l$, $p_k(x) = p_l(x)$ (resp. $\tilde{p}_k(y) = \tilde{p}_l(y)$), i.e. they are at the intersection of two sheets.

3.5.1 Critical points

In a generic situation, all the endpoints are distinct. If V_1, V_2 and ϵ_i are chosen so that some endpoints coincide, we say that we are at a critical point. Imagine that e is such a multiple endpoint, near which $d\mathcal{X}$ has a zero of degree $q-1$ and $d\mathcal{Y}$ has a zero of degree $p-1$, then:

$$Y(x) - \mathcal{Y}(e) \sim O\left((x - \mathcal{X}(e))^{p/q}\right) \tag{3.10}$$

which is a typical critical behaviour of a (p,q) conformal minimal model.

From now on, we assume that we are in a generic situation, i.e. all the endpoints are distinct.

3.5.2 Cuts

The cuts are the contours which border the sheets. Like the sheets, they are not uniquely defined, there is some arbitrariness.

A canonical choice for the cuts is the following: the cuts are the sets of $p \in \mathcal{E}$ such that

$$\exists q \neq p, \ \ \mathcal{X}(p) = \mathcal{X}(q) \text{ and } \mathrm{Re} \int_p^q \mathcal{Y}(u) d\mathcal{X}(u) = 0 \tag{3.11}$$

3.6 Irreducible cycles

We have already introduced a basis of irreducible cycles, $\mathcal{A}_i$, $\mathcal{B}_i$,$i = 1, \dots, g$, such that:

$$\mathcal{A}_i \bigcap \mathcal{B}_j = \delta_{i,j} \tag{3.12}$$

Moreover, we assume that the A-cycles are cuts, and that the A and B cycles do not intersect a line L which joins p_+ and p_-.

We have:

$$\frac{1}{2i\pi} \oint_{\mathcal{A}_i} \mathcal{Y}(p) d\mathcal{X}(p) = \epsilon_i \tag{3.13}$$

and we define:

$$\Gamma_j := \oint_{\mathcal{B}_j} \mathcal{Y}(p) d\mathcal{X}(p) \tag{3.14}$$

We then define the period-matrix τ by:

$$\tau_{i,j} := \frac{1}{2i\pi} \frac{\partial \Gamma_i}{\partial \epsilon_j} \tag{3.15}$$

Remark:

It is proven in the appendix 8 that:

$$\Gamma_j = \frac{\partial F^{(0)}}{\partial \epsilon_j} \quad , \quad \tau_{i,j} = \frac{1}{2i\pi} \frac{\partial^2 F^{(0)}}{\partial \epsilon_i \epsilon_j} \tag{3.16}$$

where $F^{(0)}$ is computed from eq. (2.17). Notice that τ is symmetric.

3.7 Holomorphic differentials

We define the following differential one-forms:

$$du_i(p) := \frac{1}{2i\pi} \frac{\partial}{\partial \epsilon_i} \left(\mathcal{Y}(p) d\mathcal{X}(p) \right) \tag{3.17}$$

they are holomorphic. Indeed, the pole of $\mathcal{Y}(p)d\mathcal{X}(p) = dV_1(\mathcal{X}(p)) - W_1(\mathcal{X}(p))d\mathcal{X}(p)$ at $p = p_+$ is independent of ϵ_i (because V_1 is independent of ϵ_i and $\mathrm{Res} W_1(x)dx = 1$ is independent of ϵ_i), therefore $du_i(p)$ has no pole at p_+. By the same argument, $du_i(p)$ has no pole at p_-, and $du_i(p)$ is holomorphic. Moreover we have (from eq. (3.13) and eq. (3.14)):

$$\oint_{\mathcal{A}_i} du_j(p) = \delta_{i,j} \quad , \quad \oint_{\mathcal{B}_i} du_j(p) = \tau_{i,j} \tag{3.18}$$

Remark:

Anticipating a little bit, it follows from eq. (3.14), eq. (4.27) and eq. (4.30), that:

$$du_i(p) = -\frac{1}{2i\pi} \frac{\partial \Gamma_i}{\partial V_1(\mathcal{X}(p))} d\mathcal{X}(p) \tag{3.19}$$

3.8 Abelian differential of the third kind

On the Riemann surface $\mathcal{E}$, there exists a unique abelian differential of the third kind dS, with two simple poles at $p = p_\pm$, such that:

$$\mathop{\rm Res}_{p_+} dS = -1 = -\mathop{\rm Res}_{p_-} dS \quad \text{and} \quad \forall i \ \oint_{\mathcal{A}_i} dS = 0 \tag{3.20}$$

We choose an arbitrary point $p_0 \in \mathcal{E}$, which does not belong to any cut or any irreducible cycle, and we choose a line L joining p_+ to p_-, which does not intersect any cycle and does not contain p_0. Then we define the following functions on $\mathcal{E}\backslash(\cup_i \mathcal{A}_i \cup_i \mathcal{B}_i \cup L)$:

$$S(p) := \int_{p_0}^{p} dS \qquad , \qquad \Lambda(p) := \exp S(p) \tag{3.21}$$

where the line of integration does not intersect any cycle neither the line L (notice that the integral around L vanishes because it encloses p_+ and p_- which have opposite residues). We have:

$$\Lambda(p) = \frac{E(p,p_-)}{E(p,p_+)} \tag{3.22}$$

$S(p)$ has logarithmic singularities near p_+ and p_-, and is discontinuous along L, the discontinuity is:

$$\delta S(p) = 2i\pi \qquad p \in L \tag{3.23}$$

$S(p)$ is continuous along the B-cycles, and discontinuous along the A-cycles, the discontinuity is:

$$\delta S(p) = \eta_i \qquad p \in \mathcal{A}_i \qquad , \qquad \eta_i := \oint_{\mathcal{B}_i} dS = u_i(p_+) - u_i(p_-) \tag{3.24}$$

$\Lambda(p)$ has no discontinuity along L, it has a simple pole at p_+, and a simple zero at p_-, therefore the following quantities are well defined:

$$\gamma := \lim_{p \to p_+} \mathcal{X}(p)/\Lambda(p) \qquad , \qquad \tilde\gamma := \lim_{p \to p_-} \mathcal{Y}(p)\Lambda(p) \tag{3.25}$$

Remark: By an appropriate choice of p_0, it should be possible to have $\gamma = \tilde\gamma$, however, we will not make that asumption.

4 2-loop functions and the Bargmann kernel

Consider the 2-loop function defined in eq. (2.18):

$$\Omega(x;x') = \frac{\partial W_1(x)}{\partial V_1(x')} = \frac{1}{(x-x')^2} - \frac{\partial Y(x)}{\partial V_1(x')} = -\left\langle \mathrm{tr}\, \frac{1}{x-M_1}\, \mathrm{tr}\, \frac{1}{x'-M_1} \right\rangle_{\rm conn} \tag{4.26}$$

and define the bilinear differential (where $x = \mathcal{X}(p)$ and $x' = \mathcal{X}(p')$):

$$B(p,p') := \frac{\partial Y(x)}{\partial V_1(x')}\, dx\, dx' \qquad (4.27)$$

$\Omega(x;x')$ has the following properties:

- $\Omega(x;x') = \Omega(x';x)$ is symmetric.
- since $Y(x)$ has square root singularities near the endpoints e_k, $\Omega(x;x')$ has inverse square root singularities near the endpoints (i.e. simple poles in $p \to e_k$). Therefore $B(p,p')$ has no pole in $p = e_k$.
- since $W_1(x)$ behaves like $\frac{1}{x} + O(1/x^2)$ in the physical sheet, i.e. when $x \to p_+$, we must have: $\Omega(x;x') \sim O(1/x^2)$. In particular $B(p,p')$ is finite when $p \to p_+$.
- since $Y(x)$ behaves like $V_2'(Y(x)) - \frac{1}{Y(x)} + O(1/Y^2(x)) \sim x$ when $x \to p_-$, we must have that $B(p,p')$ is finite when $p \to p_-$.
- $\Omega(x;x')$ has no pole at $x = x'$ in the same sheet (i.e. when $p = p'$). This implies that $B(p,p') \sim (\mathcal{X}(p) - \mathcal{X}(p'))^{-2} dx(p) dx(p')$ when $p \to p'$.
- since $Y(x)$ satisfies eq. (3.3) and ϵ is independent on V_1, we must have:

$$\forall i \quad \int_{\mathcal{A}_i} \Omega(x;x') dx = 0 \qquad (4.28)$$

This allows to determine $\Omega(x;x')$. Indeed, $B(p,p')$ is a meromorphic bilinear differential on $\mathcal{E}$, with only one normalized double pole at $p = p'$, and normalized A-cycles, therefore $B(p,p')$ is the Bargmann kernel, i.e. the unique meromorphic bilinear differential with such properties.

It can be written (see appendix 8):

$$B(p,p') = \partial_i \partial_j \ln \theta(\vec{u}(p) - \vec{u}(p') - \vec{z}) \;\; du_i(p) du_j(p') \qquad (4.29)$$

It has the property that:

$$\oint_{p \in \mathcal{B}_i} B(p,p') = 2i\pi du_i(p') \qquad (4.30)$$

Notice that in eq. (4.27), the derivative is taken at fixed $x = \mathcal{X}(p)$.

5 $1/N^2$ Expansion

We are now interested in the $1/N^2$ expansion of the free energy and loop functions:

$$F = F^{(0)} + \frac{1}{N^2} F^{(1)} + \dots \quad , \quad Y(x) = Y^{(0)}(x) + \frac{1}{N^2} Y^{(1)}(x) + \dots \tag{5.31}$$

where

$$\frac{\partial}{\partial V_1(x)} F^{(h)} = \delta_{h,0} V_1'(x) - Y^{(h)}(x) \tag{5.32}$$

So far, we have explained how to compute $Y^{(0)}(x)$. Once $Y^{(0)}(x)$ is known, $F^{(0)}$ can in principle be computed from eq. (5.32), this has been done in [3].

Our goal is to compute $Y^{(1)}(x)$, $F^{(1)}$, and then define a recursive procedure to compute $Y^{(h)}(x)$ and $F^{(h)}$ for all $h > 1$.

5.1 $1/N^2$ term

First we expand the polynomial $E(x, y)$:

$$E(x, y) = E^{(0)}(x, y) + \frac{1}{N^2} E^{(1)}(x, y) + \dots \tag{5.33}$$

where $E^{(1)}(x, y) = -P^{(1)}(x, y)$ is a polynomial of degree $(d_1 - 1, d_2 - 1)$ whose coefficient of $x^{d_1-1} y^{d_2-1}$ vanishes. And we write similar expansions for all other loop functions, in particular $U(x, y)$ and $U(x, y; x')$.

$$U(x, y) = U^{(0)}(x, y) + \frac{1}{N^2} U^{(1)}(x, y) + \dots \tag{5.34}$$

$$U(x, y; x') = U^{(0)}(x, y; x') + \frac{1}{N^2} U^{(1)}(x, y; x') + \dots \tag{5.35}$$

Then we expand eq. (2.21) to order $1/N^2$:

$$E^{(1)}(x, Y^{(0)}(x)) + Y^{(1)}(x) E_y^{(0)}(x, Y^{(0)}(x)) = U^{(0)}(x, Y^{(0)}(x); x) \tag{5.36}$$

i.e.:

$$Y^{(1)}(x) = \frac{P^{(1)}(x, Y^{(0)}(x)) + U^{(0)}(x, Y^{(0)}(x); x)}{E_y^{(0)}(x, Y^{(0)}(x))} \tag{5.37}$$

So far, the polynomial $P^{(1)}(x, y)$ is unknown, i.e. we have $d_1 d_2 - 1$ unknown coefficients.

We expect that order by order in the $1/N^2$ expansion, the resolvent $W_1(x) = V_1'(x) - Y(x)$ has no singularities appart from the endpoints, so we require that $Y^{(1)}$ has singularities only at the endpoints.

The condition that $Y^{(1)}$ has singularities only at the endpoints, implies that in eq. (5.37), the poles at the zeroes of $E_y^{(0)}(x, Y^{(0)}(x))$ which are not endpoints should cancel. Since there are $d_1 d_2 - 1$ such points, we can determine $P^{(1)}(x, y)$, and thus we can determine $Y^{(1)}(x)$. In other words, $P^{(1)}$ is determined by the condition that $Y^{(1)}$ has singularities only at the endpoints.

5.2 The function $U(x, y; x')$ to leading order

Consider x in the physical sheet, so that $Y^{(0)}(x) = Y_0(x)$. From eq. (2.22) we have:

$$U^{(0)}(x,y;x') = -\frac{\partial U^{(0)}(x,y)}{\partial V_1(x')} = -\frac{\frac{\partial E^{(0)}(x,y)}{\partial V_1(x')}}{y - Y^{(0)}(x)} - \frac{\partial Y^{(0)}(x)}{\partial V_1(x')} \frac{E^{(0)}(x,y)}{(y - Y^{(0)}(x))^2} \tag{5.38}$$

Notice that

$$E^{(0)}(x,y) = -g^*_{d_2+1} \prod_{k=0}^{d_2} (y - Y_k(x)) \tag{5.39}$$

(indeed, both sides are polynomials in y with the same degree, the same zeroes and the same leading term). Therefore:

$$\frac{\partial E^{(0)}(x,y)}{\partial V_1(x')} = -E^{(0)}(x,y) \sum_{k=0}^{d_2} \frac{\partial Y_k(x)}{\partial V_1(x')} \frac{1}{y - Y_k(x)} \tag{5.40}$$

and thus:

$$U^{(0)}(x,y;x') = \frac{E^{(0)}(x,y)}{y - Y_0(x)} \sum_{k=1}^{d_2} \frac{\partial Y_k(x)}{\partial V_1(x')} \frac{1}{y - Y_k(x)} \tag{5.41}$$

Notice that we have considerably simplified the derivation given in [18]

In particular, when $y = Y_0(x)$ and $x' = x$, we have:

$$U^{(0)}(x, Y_0(x); x) = E^{(0)}_y(x, Y_0(x)) \sum_{k=1}^{d_2} \frac{\partial Y_k(x)}{\partial V_1(x)} \frac{1}{Y_0(x) - Y_k(x)} \tag{5.42}$$

5.3 $Y^{(1)}$

Using eq. (5.37) and eq. (5.42) we have:

$$\boxed{Y^{(1)}(x) = \frac{P^{(1)}(x, Y^{(0)}(x))}{E^{(0)}_y(x, Y^{(0)}(x))} + \sum_{k=1}^{d_2} \frac{\partial Y_k(x)}{\partial V_1(x)} \frac{1}{Y_0(x) - Y_k(x)}} \tag{5.43}$$

and $Y^{(1)}$ has poles (of degree up to 5) only at the endpoints. In other words, $Y^{(1)}(x)dx$ is a one form, with poles only at the endpoints (no pole near p_+ and p_-).

5.3.1 Behaviour near the endpoints

Recall that the endpoints are the zeroes of $d\mathcal{X}(e) = 0$. If p is near an endpoint e_k, there exists a unique (because we have assumed that the potentials are generic) p' such that $\mathcal{X}(p') = \mathcal{X}(p)$ and p' is near e_k.

We have

$$Y^{(1)}(x(p))d\mathcal{X}(p) = \frac{B(p,p')}{d\mathcal{X}(p')}\,\frac{1}{\mathcal{Y}(p)-\mathcal{Y}(p')} + O(1) \quad \text{when } p\to e_k \tag{5.44}$$

where $B(p,p')$ is the Bargmann kernel.

This can also be written:

$$Y^{(1)}(x(p))d\mathcal{X}(p) = \mathop{\rm Res}_{p''\to p'} \frac{B(p,p'')}{(\mathcal{X}(p)-\mathcal{X}(p''))(\mathcal{Y}(p)-\mathcal{Y}(p''))} + O(1) \quad \text{when } p\to e_k \tag{5.45}$$

By adding only $O(1)$ quantities, we arrive at:

$$Y^{(1)}(x(p))d\mathcal{X}(p) = \mathop{\rm Res}_{p''\to p} \frac{B(p,p'')}{(\mathcal{X}(p)-\mathcal{X}(p''))(\mathcal{Y}(p)-\mathcal{Y}(p''))} + O(1) \quad \text{when } p\to e_k \tag{5.46}$$

Since that quantity is symmetric in x and y, we have:

$$\boxed{\begin{aligned} Y^{(1)}(\mathcal{X}(p))d\mathcal{X}(p) + X^{(1)}(\mathcal{Y}(p))d\mathcal{Y}(p) = & \mathop{\rm Res}_{p'\to p} \frac{B(p,p')}{(\mathcal{X}(p)-\mathcal{X}(p'))(\mathcal{Y}(p)-\mathcal{Y}(p'))} \\ & + \sum_{i=1}^{g} C_i du_i(p) \end{aligned}} \tag{5.47}$$

where C_i are some constants. Indeed, the difference between the LHS and RHS has no pole, it is a holomorphic one-form.

5.3.2 local coordinate near an endpoint

Consider that $z(p)$ is a local coordinate near an endpoint e_k, we have:

$$\mathcal{X}(p) = \mathcal{X}(e_k) + \frac{z^2}{2}\mathcal{X}''(e_k) + \frac{z^3}{6}\mathcal{X}'''(e_k) + \frac{z^4}{24}\mathcal{X}^{IV}(e_k) + \ldots \tag{5.48}$$

$$\mathcal{Y}(p) = \mathcal{Y}(e_k) + z\mathcal{Y}'(e_k) + \frac{z^2}{2}\mathcal{Y}''(e_k) + \frac{z^3}{6}\mathcal{Y}'''(e_k) + \ldots \tag{5.49}$$

$$B(p,p') = \left(\frac{1}{(z-z')^2} + \frac{1}{6}S(e_k) + \ldots\right) dz\, dz' \tag{5.50}$$

where $S(p)$ is the projective connection.

$\mathcal{X}(p') = \mathcal{X}(p)$ implies:

$$z' = -z(1 + r_k z + r_k^2 z^2 + (2r_k^3 + t_k)z^3 + \ldots) \tag{5.51}$$

where

$$r_k = \frac{1}{3}\frac{\mathcal{X}'''(e_k)}{\mathcal{X}''(e_k)} \quad , \quad s_k = \frac{1}{6}\frac{\mathcal{X}^{IV}(e_k)}{\mathcal{X}''(e_k)} \quad , \quad t_k = \frac{1}{60}\frac{\mathcal{X}^{V}(e_k)}{\mathcal{X}''(e_k)} - r_k s_k \tag{5.52}$$

That gives:

$$\begin{aligned} Y^{(1)}(\mathcal{X}(p))\frac{d\mathcal{X}(p)}{dz} &= \frac{1}{8\mathcal{X}''(e_k)\mathcal{Y}'(e_k)}z^{-4} - \frac{\mathcal{X}'''(e_k)}{24\mathcal{X}''^2(e_k)\mathcal{Y}'(e_k)}z^{-3} \\ &+ \frac{\frac{\mathcal{X}'''(e_k)^2}{\mathcal{X}''(e_k)^2} - \frac{\mathcal{X}^{IV}(e_k)}{\mathcal{X}''(e_k)} + \frac{\mathcal{X}'''(e_k)}{\mathcal{X}''(e_k)}\frac{\mathcal{Y}''(e_k)}{\mathcal{Y}'(e_k)} - \frac{\mathcal{Y}'''(e_k)}{\mathcal{Y}'(e_k)}}{48\mathcal{X}''(e_k)\mathcal{Y}'(e_k)}z^{-2} \\ &- \frac{S(e_k)}{12\mathcal{X}''(e_k)\mathcal{Y}'(e_k)}z^{-2} + O(1) \end{aligned} \tag{5.53}$$

This is in principle sufficient to determine $Y^{(1)}$.

5.4 Free energy

Then, we want to find the free energy $F^{(1)}$ such that:

$$Y^{(1)}(x) = -\frac{\partial F^{(1)}}{\partial V_1(x)} \tag{5.54}$$

In this purpose, we have to compute the derivatives of various quantities with respect to $V_1(x)$, and in particular, how the theta-function parametrization changes with the potential V_1.

We conjecture:

$$F^{(1)} = -\frac{1}{24}\ln\prod_i Y'(e_i) \tag{5.55}$$

where

$$\ln Y'(e_i) := \int_{p=p_+}^{e_i}\int_{p=p_+}^{e_i}\left(B(p,p') - \frac{dy(p)dy(p')}{(p-p')^2}\right) \tag{5.56}$$

5.5 Higher orders

Imagine we already know all quantities up to order $h-1$, and write eq. (2.21) to order h:

$$\begin{aligned} &\sum_{j=0}^{h} N^{-2h+2j}E^{(h-j)}\left(x, \sum_{k=0}^{j} N^{-2k}Y^{(k)}(x)\right) \\ &= \sum_{j=0}^{h-1} N^{-2h+2j}U^{(h-j)}\left(x, \sum_{k=0}^{j} N^{-2k}Y^{(k)}(x); x\right) + O(N^{-2h}) \end{aligned} \tag{5.57}$$

The only unknown quantities in that equation are: $Y^{(h)}(x)$ and $E^{(h)}(x, Y^{(0)}(x))$. The polynomial $E^{(h)}(x,y)$ must be chosen such that $Y^{(h)}(x)$ has no other singularities than the endpoints, and is completely determined by this consition. That allows to find $Y^{(h)}(x)$ as well as $E^{(h)}(x,y)$ and $U^{(h)}(x,y)$ to order h.

The procedure can be repeated recursively to find $Y^{(h)}(x)$ to any order.

6 Genus 1 case

Let us recall that the case $g = 0$ was done in [18]. The case $g = 1$ is treated in this section.

We require that $E(x,y) = 0$ be a **genus one** algebraic curve. Therefore, there must exist an elliptic uniformization. We choose it of the following form:

$$\begin{aligned} x = \mathcal{X}(u) &= \gamma \frac{\prod_{i=0}^{d_2} \theta(u - \sigma_i(0))}{\theta(u-u_\infty)\theta(u+u_\infty)^{d_2}} \frac{\theta(2u_\infty)^{d_2+1}}{\prod_i \theta(u_\infty - \sigma_i(0))} \\ y = \mathcal{Y}(u) &= \tilde{\gamma} \frac{\prod_{i=0}^{d_1} \theta(u - \tilde{\sigma}_i(0))}{\theta(u-u_\infty)^{d_1}\theta(u+u_\infty)} \frac{\theta(2u_\infty)^{d_1+1}}{\prod_i \theta(u_\infty + \tilde{\sigma}_i(0))} \end{aligned} \tag{6.1}$$

and we denote τ the modulus. Here, θ denotes θ_1, i.e. the prime form for genus 1. A definition of the θ-function and its properties can be found in appendix 8.

We must have:

$$\sum_i \sigma_i(0) = (d_2 - 1)u_\infty \quad , \quad \sum_i \tilde{\sigma}_i(0) = (d_1 - 1)u_\infty \tag{6.2}$$

All this means that for every (x, y) which satisfy $E(x, y) = 0$, there exists at least one u (in the fundamental paralellogram of sides $1, \tau$) such that $x = \mathcal{X}(u)$ and $y = \mathcal{Y}(u)$.

An alternative parametrization is:

$$\mathcal{X}(u) = \gamma \frac{\theta(2u_\infty)}{\theta'(0)} (Z(u - u_\infty) - Z(u + u_\infty)) + A_0 + \sum_{k=2}^{d_2} \frac{A_k}{k-1!} \phi^{(k-2)}(u + u_\infty) \tag{6.3}$$

$$\mathcal{Y}(u) = -\tilde{\gamma} \frac{\theta(2u_\infty)}{\theta'(0)} (Z(u + u_\infty) - Z(u - u_\infty)) + \tilde{A}_0 + \sum_{k=2}^{d_1} \frac{\tilde{A}_k}{k-1!} \phi^{(k-2)}(u - u_\infty) \tag{6.4}$$

where Z is the Zeta-function, i.e. the log-derivative of θ_1, and ϕ is the Weierstrass function, i.e. $\phi = -Z'$.

We note the inverse functions:

$$x = \mathcal{X}(s) \leftrightarrow s = \sigma(x) \quad , \quad y = \mathcal{Y}(s) \leftrightarrow s = \tilde{\sigma}(y) \tag{6.5}$$

The functions $\sigma(x)$ and $\tilde{\sigma}(y)$ are multivalued, we will discuss their sheet structure below. The functions $Y(x)$ and $X(y)$ are:

$$Y(x) = \mathcal{Y}(\sigma(x)) \quad , \quad X(y) = \mathcal{X}(\tilde{\sigma}(y)) \tag{6.6}$$

They are multivalued too, and their sheet structure will be discussed below.

6.1 The parameters

Our parametrization depends on d_1+d_2+6 parameters which are: $\sigma_k(0)$ ($k=1,\dots,d_2$), $\tilde{\sigma}_j(0)$ ($j=1,\dots,d_1$), γ, $\tilde{\gamma}$, u_∞ and τ. The condition eq. (6.2) means that only d_1+d_2+4 of them are independent.

Equations eq. (3.5) read:

$$g_k = \frac{1}{2i\pi}\oint_{u_\infty} ds \frac{\mathcal{Y}(s)\mathcal{X}'(s)}{\mathcal{X}(s)^k} \qquad k=1,\dots,d_1+1 \tag{6.7}$$

$$\tilde{g}_k = \frac{1}{2i\pi}\oint_{-u_\infty} ds \frac{\mathcal{X}(s)\mathcal{Y}'(s)}{\mathcal{Y}(s)^k} \qquad k=1,\dots,d_2+1 \tag{6.8}$$

and

$$1 = \frac{1}{2i\pi}\oint_{u_\infty} ds \mathcal{Y}(s)\mathcal{X}'(s) = \frac{1}{2i\pi}\oint_{-u_\infty} ds \mathcal{X}(s)\mathcal{Y}'(s) \tag{6.9}$$

Where the contour of integrations are small cylces around $\pm u_\infty$.

And eq. (3.3) reads:

$$2i\pi\epsilon = \int_0^\tau ds\ \mathcal{Y}(s)\mathcal{X}'(s) \tag{6.10}$$

We thus have d_1+d_2+4 equations, therefore we can, in principle, determine all the parameters.

In principle, it should be possible to revert these formula, and compute the $\sigma(0)$'s and $\tilde{\sigma}(0)$'s as functions of the coupling constants g and $\tilde{g}$. This can be done at least numerically.

Remark: Another point of view is interesting too: once eq. (3.3) and eq. (3.5) are taken into account, we have d_1+d_2+2 independent parameters. This is precisely the number of coefficients of the potentials V_1 and V_2. We can consider that **the parameters are merely a reparametrization of the coefficients of the potentials**, according to eq. (6.7) and eq. (6.8).

6.2 endpoints and cuts

The endpoints in the x-plane (resp. y-plane), i.e. the singularities of $Y(x)$ (resp. $X(y)$) are such that:

$$\mathcal{X}'(s) = 0 \qquad (\text{resp.}\, \mathcal{Y}'(s) = 0) \tag{6.11}$$

There are d_2+3 (resp. d_1+3) such endpoints:

$$\mathcal{X}'(s)=0 \ \leftrightarrow\ s\in\{e_1,e_2,\dots,e_{d_2+3}\} \qquad , \qquad \mathcal{Y}'(s)=0 \ \leftrightarrow\ s\in\{\tilde{e}_1,\dots,\tilde{e}_{d_1+3}\} \tag{6.12}$$

We can write:

$$\mathcal{X}'(s) = -\gamma \frac{\theta'(0)\theta(2u_\infty)^{d_2+2}}{\prod_i \theta(u_\infty - e_i)} \frac{\prod_{i=1}^{d_2+3} \theta(s-e_i)}{\theta(s-u_\infty)^2 \theta(s+u_\infty)^{d_2+1}} \tag{6.13}$$

$$\mathcal{Y}'(s) = \tilde{\gamma} \frac{\theta'(0)\theta(2u_\infty)^{d_1+2}}{\prod_i \theta(u_\infty + \tilde{e}_i)} \frac{\prod_{i=1}^{d_1+3} \theta(s-\tilde{e}_i)}{\theta(s-u_\infty)^{d_1+1} \theta(s+u_\infty)^{2}} \tag{6.14}$$

Since $\mathcal{X}'(s)$ and $\mathcal{Y}'(s)$ are elliptical functions, we must have:

$$\sum_i e_i = -(d_2-1)u_\infty \quad , \quad \sum_i \tilde{e}_i = (d_1-1)u_\infty \tag{6.15}$$

and since $\mathcal{X}'(s)$ and $\mathcal{Y}'(s)$ are the derivatives of elliptical functions, we must have:

$$\sum_i Z(u_\infty - e_i) = (d_2+1)Z(2u_\infty) \quad , \quad \sum_i Z(u_\infty + \tilde{e}_i) = (d_1+1)Z(2u_\infty) \tag{6.16}$$

Notice that eq. (6.7) for $k = d_1+1$ and eq. (6.8) for $k = d_2+1$ imply:

$$\tilde{\gamma} = -d_1 g_{d_1+1} \gamma^{d_1} \frac{\prod_i \theta(u_\infty + \tilde{e}_i)}{\prod_i \theta(u_\infty - \tilde{e}_i)} \quad , \quad \gamma = -d_2 \tilde{g}_{d_2+1} \tilde{\gamma}^{d_2} \frac{\prod_i \theta(u_\infty - e_i)}{\prod_i \theta(u_\infty + e_i)} \tag{6.17}$$

6.3 2-loop functions and the Bargmann kernel

Consider the 2-loop function defined in eq. (2.18):

$$\Omega(x;x') = \frac{\partial W_1(x)}{\partial V_1(x')} = \frac{1}{(x-x')^2} - \frac{\partial Y(x)}{\partial V_1(x')} = -\left\langle \mathrm{tr}\, \frac{1}{x-M_1}\, \mathrm{tr}\, \frac{1}{x'-M_1} \right\rangle_{\mathrm{conn}} \tag{6.18}$$

$\Omega(x;x')$ has the following properties:

- $\Omega(x;x') = \Omega(x';x)$ is symmetric.
- since $Y(x)$ has square root singularities near the endpoints $\mathcal{X}(e_k)$, $\Omega(x;x')$ has inverse square root singularities near the endpoints (i.e. simple poles in $\sigma(x)-e_k$). Therefore $\mathcal{X}'(s)\Omega(\mathcal{X}(s);x')$ is finite when $s \to e_k$.
- since $W_1(x)$ behaves like $\frac{1}{x} + O(1/x^2)$ when $x \to \infty$ (i.e. $\sigma(x) \to +u_\infty$), we must have: $\Omega(x;x') \sim O(1/x^2)$. In particular $\mathcal{X}'(s)\Omega(\mathcal{X}(s);x')$ is finite when $s \to +u_\infty$.
- since $Y(x)$ behaves like $V_2'(Y(x)) - \frac{1}{Y(x)} + O(1/Y^2(x)) \sim x$ when $\sigma(x) \to -u_\infty$, we must have that $\mathcal{X}'(s)\Omega(\mathcal{X}(s);x')$ is finite when $s \to -u_\infty$.
- $\Omega(x;x')$ has no pole at $x = x'$ when x and x' are in the same sheet, i.e. when $\sigma(x) = \sigma(x')$. This implies that $\frac{\partial Y(x)}{\partial V_1(x')} \sim \frac{1}{(x-x')^2}$ when $\sigma(x) \to \sigma(x')$.

- since $Y(x)$ satisfies eq. (3.3) and ϵ is independent on V_1, we must have:

$$\int_0^1 \Omega(\mathcal{X}(s);x')\partial s = 0 \tag{6.19}$$

This allows to determine $\Omega(x;x')$. Indeed, the function $\mathcal{X}'(s)\mathcal{X}'(u)\frac{\partial Y(\mathcal{X}(s))}{\partial V_1(\mathcal{X}(u))}$ is an elliptical function of s, with only one double pole at $s=u$. Therefore (see appendix 8):

$$\mathcal{X}'(s)\mathcal{X}'(u)\frac{\partial Y(\mathcal{X}(s))}{\partial V_1(\mathcal{X}(u))} = \phi(s-u)+C \tag{6.20}$$

where ϕ is the Weierstrass function, and where the constant C must be equal to zero in order to satisfy eq. (4.28).

We recognize the Bargmann kernel:

$$\boxed{\frac{\partial Y(x)}{\partial V_1(x')} = \sigma'(x)\sigma'(x')\phi(\sigma(x)-\sigma(x')) = \partial_x\partial_{x'}\ln\theta(\sigma(x)-\sigma(x'))} \tag{6.21}$$

In a similar fashion, we find that the function:

$$\tilde\Omega(y;x') = \frac{\partial W_2(y)}{\partial V_1(x')} = -\left\langle \mathrm{tr}\,\frac{1}{y-M_2}\,\mathrm{tr}\,\frac{1}{x'-M_1}\right\rangle_{\mathrm{conn}} \tag{6.22}$$

is:

$$\boxed{\tilde\Omega(y;x') = -\frac{\partial X(y)}{\partial V_1(x')} = \tilde\sigma'(y)\sigma'(x')\phi(\tilde\sigma(y)-\sigma(x')) = \partial_y\partial_{x'}\ln\theta(\tilde\sigma(y)-\sigma(x'))} \tag{6.23}$$

We are now equipped to compute the next to leading order functions...

6.4 Computation of $Y^{(1)}$

We have (see eq. (5.43)):

$$Y^{(1)}(x) = \frac{P^{(1)}(x,Y_0(x))}{E_y(x,Y_0(x))} + \sum_{k=1}^{d_2}\frac{\phi(\sigma_k(x)-\sigma_0(x))}{\mathcal{X}'(\sigma_k(x))\mathcal{X}'(\sigma_0(x))(Y_0(x)-Y_k(x))} \tag{6.24}$$

and we require that $Y^{(1)}(x)$'s only poles (of degree up to 5) are the endpoints $\mathcal{X}(e_k)$. Note that $\mathcal{X}'(s)Y^{(1)}(\mathcal{X}(s))$ has no pole at $s=\pm u_\infty$, therefore we may write:

$$Y^{(1)}(\mathcal{X}(s)) = \frac{1}{\mathcal{X}'(s)}\sum_{k=1}^{d_2+3} A_k\phi''(s-e_k)+B_k\phi'(s-e_k)+C_k\phi(s-e_k)+D_kZ(s-e_k) \tag{6.25}$$

The coefficients A_k, B_k, C_k, D_k are determined by matching the poles in eq. (6.24) ($P^{(1)}$ doesnot contribute to them because $E_y(x, Y_0(x))$ has only single poles at the endpoints). Below, we will find that $D_k = 0$.

Let $k \in [1, d_2 + 3]$, and choose s close to e_k:

$$s = e_k + \epsilon \tag{6.26}$$

there must exist $\tilde{s}$ (unique because the potentials are non-critical) such that $\mathcal{X}(\tilde{s}) = \mathcal{X}(s)$ and $\tilde{s}$ is close to e_k ($\tilde{s}$ is the $\sigma_k(x)$ in eq. (6.24)):

$$\tilde{s} = e_k - \eta \quad , \quad \eta = O(\epsilon) \tag{6.27}$$

By solving $\mathcal{X}(s) = \mathcal{X}(\tilde{s})$ order by order in ϵ we get:

$$\eta = \lambda\epsilon \quad , \quad \lambda = 1 + r_k\epsilon + r_k^2\epsilon^2 + (2r_k^3 + t_k)\epsilon^3 + O(\epsilon^4) \tag{6.28}$$

where

$$r_k = \frac{1}{3}\frac{\mathcal{X}'''(e_k)}{\mathcal{X}''(e_k)} \quad , \quad s_k = \frac{1}{6}\frac{\mathcal{X}^{IV}(e_k)}{\mathcal{X}''(e_k)} \quad , \quad t_k = \frac{1}{60}\frac{\mathcal{X}^{V}(e_k)}{\mathcal{X}''(e_k)} - r_k s_k \tag{6.29}$$

From eq. (6.24) we must have:

$$6A_k - 2B_k\epsilon + C_k\epsilon^2 + D_k\epsilon^3 = -\frac{\epsilon^4\phi(\epsilon+\eta)}{\mathcal{X}'(e_k-\eta)(\mathcal{Y}(e_k+\epsilon) - \mathcal{Y}(e_k-\eta))} + O(\epsilon^4) \tag{6.30}$$

We note the 3rd degree polynomial:

$$P_k(\epsilon) = 6A_k - 2B_k\epsilon + C_k\epsilon^2 + D_k\epsilon^3 \tag{6.31}$$

i.e.

$$\begin{aligned}
P_k(\epsilon) &= ((1+\lambda)^{-2} - \zeta_1\epsilon^2) \\
&\quad (-\lambda\mathcal{X}''(e_k) + \epsilon\frac{\lambda^2}{2}\mathcal{X}'''(e_k) - \epsilon^2\frac{\lambda^3}{6}\mathcal{X}^{IV}(e_k) + \epsilon^3\frac{\lambda^4}{24}\mathcal{X}^{V}(e_k))^{-1} \\
&\quad ((1+\lambda)\mathcal{Y}'(e_k) + \frac{\epsilon}{2}(1-\lambda^2)\mathcal{Y}''(e_k) + \frac{\epsilon^2}{6}(1+\lambda^3)\mathcal{Y}'''(e_k) \\
&\quad + \frac{\epsilon^3}{24}(1-\lambda^4)\mathcal{Y}^{IV}(e_k))^{-1} + O(\epsilon^4) \\
&= -(\mathcal{X}''(e_k)\mathcal{Y}'(e_k))^{-1}\ (1+\lambda)^{-3}\lambda^{-1} \\
&\quad (1 - \frac{3}{2}\epsilon\lambda r_k + \epsilon^2\lambda^2 s_k - \frac{5}{2}\epsilon^3(t_k + r_k s_k))^{-1} \\
&\quad (1 + \frac{\epsilon}{2}(1-\lambda)\frac{\mathcal{Y}''(e_k)}{\mathcal{Y}'(e_k)} + \frac{\epsilon^2}{6}(1-\lambda+\lambda^2)\frac{\mathcal{Y}'''(e_k)}{\mathcal{Y}'(e_k)})^{-1} \\
&\quad + \frac{\zeta_1}{2\mathcal{X}''(e_k)\mathcal{Y}'(e_k)}\epsilon^2 + O(\epsilon^4)
\end{aligned} \tag{6.32}$$

It is easy to see that t_k as well as $\mathcal{Y}^{IV}(e_k)$ disappear, and that $D_k = 0$.

After a straightforward calculation, one finds:

$$\begin{aligned} -8\mathcal{X}''(e_k)\mathcal{Y}'(e_k)(6A_k - 2B_k\epsilon + C_k\epsilon^2) \quad = \quad & 1 - \frac{1}{3}\epsilon\frac{\mathcal{X}'''(e_k)}{\mathcal{X}''(e_k)} \\ & + \frac{1}{6}\epsilon^2\left(\frac{\mathcal{X}'''(e_k)^2}{\mathcal{X}''(e_k)^2} - \frac{\mathcal{X}^{IV}(e_k)}{\mathcal{X}''(e_k)}\right. \\ & \left. + \frac{\mathcal{X}'''(e_k)\mathcal{Y}''(e_k)}{\mathcal{X}''(e_k)\mathcal{Y}'(e_k)} - \frac{\mathcal{Y}'''(e_k)}{\mathcal{Y}'(e_k)} - 24\zeta_1\right) \end{aligned} \tag{6.33}$$

After substitution into eq. (6.25), we find the genus one correction to the resolvent:

$$\boxed{\begin{aligned} Y^{(1)}(\mathcal{X}(s)) \quad = \quad & \sum_{k=1}^{d_2+3} \frac{1}{48\mathcal{X}'(s)\mathcal{X}''(e_k)\mathcal{Y}'(e_k)}\phi''(s-e_k) \\ & + \sum_{k=1}^{d_2+3} \frac{\mathcal{X}'''(e_k)}{48\mathcal{X}'(s)\mathcal{X}''^2(e_k)\mathcal{Y}'(e_k)}\phi'(s-e_k) \\ & + \sum_{k=1}^{d_2+3} \frac{\frac{\mathcal{X}'''(e_k)^2}{\mathcal{X}''(e_k)^2} - \frac{\mathcal{X}^{IV}(e_k)}{\mathcal{X}''(e_k)} + \frac{\mathcal{X}'''(e_k)}{\mathcal{X}''(e_k)}\frac{\mathcal{Y}''(e_k)}{\mathcal{Y}'(e_k)} - \frac{\mathcal{Y}'''(e_k)}{\mathcal{Y}'(e_k)}}{48\mathcal{X}'(s)\mathcal{X}''(e_k)\mathcal{Y}'(e_k)}\phi(s-e_k) \\ & - \sum_{k=1}^{d_2+3} \frac{\zeta_1}{2\mathcal{X}'(s)\mathcal{X}''(e_k)\mathcal{Y}'(e_k)}\phi(s-e_k) \end{aligned}} \tag{6.34}$$

That function represents the partition function of a statistical physics model on a genus one surface with one + boundary.

Notice that

$$\frac{\partial F^{(1)}}{\partial \epsilon} = \Gamma^{(1)} = \sum_{k=1}^{d_2+3} \frac{\frac{\mathcal{X}'''(e_k)^2}{\mathcal{X}''(e_k)^2} - \frac{\mathcal{X}^{IV}(e_k)}{\mathcal{X}''(e_k)} + \frac{\mathcal{X}'''(e_k)}{\mathcal{X}''(e_k)}\frac{\mathcal{Y}''(e_k)}{\mathcal{Y}'(e_k)} - \frac{\mathcal{Y}'''(e_k)}{\mathcal{Y}'(e_k)} - 24\zeta_1}{48\mathcal{X}''(e_k)\mathcal{Y}'(e_k)} \tag{6.35}$$

6.5 The free energy

We are now going to find the free energy $F^{(1)}$ such that:

$$Y^{(1)}(x) = -\frac{\partial F^{(1)}}{\partial V_1(x)} \tag{6.36}$$

conjecture:

$$F^{(1)} = -\frac{1}{24}\ln\left(\gamma^4\tilde{\gamma}^4 \prod_{i=1}^{d_2+3}\prod_{j=1}^{d_1+3}\left(\frac{\theta(e_i-\tilde{e}_j)\theta(2u_\infty)}{\theta(e_i-u_\infty)\theta(\tilde{e}_j+u_\infty)}\right)\right) \tag{6.37}$$

question: how to extend that conjecture to genus $g > 1$???

7 Variations with respect to the potentials

7.0.1 Variations with u fixed

Now, we would like to compute:

$$\dot{\mathcal{X}}(u) := \mathcal{X}'(s)\frac{\partial \mathcal{X}(u)}{\partial V_1(\mathcal{X}(s))} \qquad \text{and} \qquad \dot{\mathcal{Y}}(u) := \mathcal{X}'(s)\frac{\partial \mathcal{Y}(u)}{\partial V_1(\mathcal{X}(s))} \tag{7.38}$$

with u fixed (we will not write the s dependence for lisibility in this section).

We have:

$$\left.\frac{\partial \mathcal{Y}(u)}{\partial V_1(\mathcal{X}(s))}\right|_u = \left.\frac{\partial Y(\mathcal{X}(u))}{\partial V_1(\mathcal{X}(s))}\right|_u = \left.\frac{\partial Y(x)}{\partial V_1(\mathcal{X}(s))}\right|_{x=\mathcal{X}(u)} + Y'(\mathcal{X}(u))\left.\frac{\partial \mathcal{X}(u)}{\partial V_1(\mathcal{X}(s))}\right|_u \tag{7.39}$$

which implies:

$$\boxed{\dot{\mathcal{X}}(u)\mathcal{Y}'(u) - \dot{\mathcal{Y}}(u)\mathcal{X}'(u) = \phi(s-u)} \tag{7.40}$$

In particular at $u = e_i$ we have:

$$\dot{\mathcal{X}}(e_i) = \frac{1}{\mathcal{Y}'(e_i)}\phi(s-e_i) \tag{7.41}$$

We set:

$$\alpha_i := \frac{\phi(s-e_i)}{\mathcal{X}''(e_i)\mathcal{Y}'(e_i)} \tag{7.42}$$

Moreover, we have:

$$\frac{\dot{\mathcal{X}}(u+1)}{\mathcal{X}'(u+1)} = \frac{\dot{\mathcal{X}}(u)}{\mathcal{X}'(u)} \quad , \quad \frac{\dot{\mathcal{X}}(u+\tau)}{\mathcal{X}'(u+\tau)} = \frac{\dot{\mathcal{X}}(u)}{\mathcal{X}'(u)} - \dot{\tau} \tag{7.43}$$

where $\dot{\tau} := \mathcal{X}'(s)\frac{\partial \tau}{\partial V_1(\mathcal{X}(s))}$. That means that the function:

$$f(u) := \frac{\dot{\mathcal{X}}(u)}{\mathcal{X}'(u)} - \sum_{i=1}^{d_2+3} \alpha_i Z(u-e_i) \tag{7.44}$$

has no poles and satisfies:

$$f(u+1) = f(u) \quad , \quad f(u+\tau) = f(u) - \dot{\tau} + 2i\pi \sum_i \alpha_i \tag{7.45}$$

$f'(u)$ is an elliptical function with no pole, therefore it is a constant, and $f(u)$ is a constant too.

That allows to write:

$$\boxed{\dot{\mathcal{X}}(u) = \mathcal{X}'(u)\left[C + \sum_{i=1}^{d_2+3} \alpha_i Z(u-e_i)\right]} \tag{7.46}$$

that implies:

$$\dot{\tau} = 2i\pi \sum_{i=1}^{d_2+3} \alpha_i \tag{7.47}$$

and C is a constant which is determined by the behaviours near $\pm u_\infty$. Indeed, consider $\dot{\mathcal{X}}'(u)/\mathcal{X}'(u)$, you get:

$$\frac{2\dot{u_\infty}}{u-u_\infty} = -\frac{2}{u-u_\infty}\left[C + \sum_i \alpha_i Z(u_\infty - e_i)\right] \tag{7.48}$$

and

$$-\frac{(d_2+1)\dot{u_\infty}}{u-u_\infty} = -\frac{d_2+1}{u-u_\infty}\left[C + \sum_i \alpha_i Z(-u_\infty - e_i)\right] \tag{7.49}$$

where $\dot{u}_\infty := \mathcal{X}'(s)\frac{\partial u_\infty}{\partial V_1(\mathcal{X}(s))}$. That implies

$$C = -\frac{1}{2}\sum_i \alpha_i (Z(u_\infty - e_i) + Z(-u_\infty - e_i)) \tag{7.50}$$

and

$$-2\dot{u_\infty} = \sum_i \alpha_i (Z(u_\infty - e_i) + Z(u_\infty + e_i)) \tag{7.51}$$

i.e.

$$\boxed{\dot{\mathcal{X}}(u) = \frac{1}{2}\mathcal{X}'(u)\sum_i \alpha_i [2Z(u-e_i) + Z(u_\infty + e_i) - Z(u_\infty - e_i)]} \tag{7.52}$$

We also have:

$$\boxed{\dot{\mathcal{Y}}(u) = \frac{1}{2}\mathcal{Y}'(u)\sum_i \alpha_i [2Z(u-e_i) + Z(u_\infty + e_i) - Z(u_\infty - e_i)] - \frac{\phi(s-u)}{\mathcal{X}'(u)}} \tag{7.53}$$

In particular near $u = e_i$, we find:

$$\dot{e_i} = -C - \sum_{j\neq i} \alpha_j Z(e_i - e_j) - \frac{1}{2}\alpha_i \frac{\mathcal{X}'''(e_i)}{\mathcal{X}''(e_i)} \tag{7.54}$$

and near $u = u_\infty$, we write:

$$\mathcal{X}(u) \sim \frac{A}{u-u_\infty} \tag{7.55}$$

$$\frac{\dot{A}}{A} = \sum_i \alpha_i \phi(u - u_\infty) \tag{7.56}$$

and near $u = -u_\infty$, we write:

$$\mathcal{Y}(u) \sim \frac{\tilde{A}}{u + u_\infty} \tag{7.57}$$

$$\frac{\dot{\tilde{A}}}{\tilde{A}} = \sum_i \alpha_i \phi(u + u_\infty) \tag{7.58}$$

note that:

$$A = \gamma \frac{\theta(2u_\infty)}{\theta'(0)} \quad , \quad \tilde{A} = -\tilde{\gamma}\frac{\theta(2u_\infty)}{\theta'(0)} \tag{7.59}$$

Note also that:

$$\mathcal{X}'(s)\frac{\partial \ln \theta'(0)}{\partial V_1(\mathcal{X}(s))} = \frac{3}{4i\pi}\dot{\tau}\zeta_1 = \frac{3}{2}\sum_i \alpha_i \zeta_1 \tag{7.60}$$

7.0.2 variation of $\mathcal{Y}'(e_i)$

$$\begin{aligned} \mathcal{X}'(s)\frac{\partial\left(\mathcal{Y}'(e_i)\right)}{\partial V_1(\mathcal{X}(s))} &= \dot{\mathcal{Y}}'(e_i) + \dot{e}_i \mathcal{Y}''(e_i) \\ &= -\mathcal{Y}'(e_i)\left(\sum_{j\neq i}\alpha_j\phi(e_i - e_j) - \zeta_1\alpha_i\right) + \frac{1}{2}\alpha_i\mathcal{Y}'''(e_i) \\ &\quad -\frac{1}{2}\alpha_i\mathcal{Y}''(e_i)\frac{\mathcal{X}'''(e_i)}{\mathcal{X}''(e_i)} + \frac{\phi(e_i - s)}{\mathcal{X}''(e_i)}\left(\frac{1}{6}\frac{\mathcal{X}^{IV}(e_i)}{\mathcal{X}''(e_i)} - \frac{1}{4}\frac{\mathcal{X}'''^2(e_i)}{\mathcal{X}''^2(e_i)}\right) \\ &\quad +\frac{1}{2}\frac{\phi'(e_i - s)\mathcal{X}'''(e_i)}{\mathcal{X}''^2(e_i)} - \frac{1}{2}\frac{\phi''(e_i - s)}{\mathcal{X}''(e_i)} \end{aligned} \tag{7.61}$$

i.e.

$$\begin{aligned} \mathcal{X}'(s)\frac{\partial\left(\ln\mathcal{Y}'(e_i)\right)}{\partial V_1(\mathcal{X}(s))} &= -\sum_{j\neq i}\alpha_j\phi(e_i - e_j) + \zeta_1\alpha_i \\ &\quad +\frac{\alpha_i}{2}\left(\frac{\mathcal{Y}'''(e_i)}{\mathcal{Y}'(e_i)} + \frac{1}{3}\frac{\mathcal{X}^{IV}(e_i)}{\mathcal{X}''(e_i)} - \frac{1}{2}\frac{\mathcal{X}'''^2(e_i)}{\mathcal{X}''^2(e_i)} - \frac{\mathcal{Y}''(e_i)}{\mathcal{Y}'(e_i)}\frac{\mathcal{X}'''(e_i)}{\mathcal{X}''(e_i)}\right) \\ &\quad +\frac{1}{2}\frac{\phi'(e_i - s)\mathcal{X}'''(e_i)}{\mathcal{X}''^2(e_i)\mathcal{Y}'(e_i)} - \frac{1}{2}\frac{\phi''(e_i - s)}{\mathcal{X}''(e_i)\mathcal{Y}'(e_i)} \end{aligned} \tag{7.62}$$

thus:

$$\begin{aligned} \mathcal{X}'(s)\frac{\partial \ln\left(\prod_i \mathcal{Y}'(e_i)\right)}{\partial V_1(\mathcal{X}(s))} &= -\sum_i\sum_{j\neq i}\alpha_j\phi(e_i - e_j) + \zeta_1\sum_i\alpha_i \\ &\quad +\sum_i\frac{\alpha_i}{2}\left(\frac{\mathcal{Y}'''(e_i)}{\mathcal{Y}'(e_i)} + \frac{1}{3}\frac{\mathcal{X}^{IV}(e_i)}{\mathcal{X}''(e_i)} - \frac{1}{2}\frac{\mathcal{X}'''^2(e_i)}{\mathcal{X}''^2(e_i)}\right. \\ &\quad \left. - \frac{\mathcal{Y}''(e_i)}{\mathcal{Y}'(e_i)}\frac{\mathcal{X}'''(e_i)}{\mathcal{X}''(e_i)}\right) \end{aligned}$$

$$+\frac{1}{2}\sum_i \frac{\phi'(e_i-s)\mathcal{X}'''(e_i)}{\mathcal{X}''^2(e_i)\mathcal{Y}'(e_i)} - \frac{1}{2}\sum_i \frac{\phi''(e_i-s)}{\mathcal{X}''(e_i)\mathcal{Y}'(e_i)} \tag{7.63}$$

Notice that:

$$\frac{\mathcal{X}''(u)}{\mathcal{X}'(u)} = \sum_i Z(u-e_i) - 2Z(u-u_\infty) - (d_2+1)Z(u+u_\infty) \tag{7.64}$$

implies (expand to order 1 in $u-e_j$):

$$\sum_{i\neq j}\phi(e_i-e_j) = \zeta_1 - \frac{1}{3}\frac{\mathcal{X}^{IV}(e_j)}{\mathcal{X}''(e_j)} + \frac{1}{4}\frac{\mathcal{X}'''^2(e_j)}{\mathcal{X}''^2(e_j)} + 2\phi(e_j-u_\infty) + (d_2+1)\phi(e_j+u_\infty) \tag{7.65}$$

therefore

$$\begin{aligned}\mathcal{X}'(s)\frac{\partial\ln\left(\prod_i \mathcal{Y}'(e_i)\right)}{\partial V_1(\mathcal{X}(s))} &= -\sum_j \alpha_j(2\phi(e_j-u_\infty)+(d_2+1)\phi(e_j+u_\infty)) \\ &\quad +\sum_i \frac{\alpha_i}{2}\left(\frac{\mathcal{Y}'''(e_i)}{\mathcal{Y}'(e_i)} + \frac{\mathcal{X}^{IV}(e_i)}{\mathcal{X}''(e_i)} - \frac{\mathcal{X}'''^2(e_i)}{\mathcal{X}''^2(e_i)} \right. \\ &\quad \left. - \frac{\mathcal{Y}''(e_i)}{\mathcal{Y}'(e_i)}\frac{\mathcal{X}'''(e_i)}{\mathcal{X}''(e_i)}\right) \\ &\quad +\frac{1}{2}\sum_i \frac{\phi'(e_i-s)\mathcal{X}'''(e_i)}{\mathcal{X}''^2(e_i)\mathcal{Y}'(e_i)} - \frac{1}{2}\sum_i \frac{\phi''(e_i-s)}{\mathcal{X}''(e_i)\mathcal{Y}'(e_i)}\end{aligned} \tag{7.66}$$

i.e.

$$\begin{aligned}\mathcal{X}'(s)\frac{\partial\ln\left(\prod_i \mathcal{Y}'(e_i)\right)}{\partial V_1(\mathcal{X}(s))} &= -2\frac{\dot{A}}{A} - (d_2+1)\frac{\dot{\tilde{A}}}{\tilde{A}} - \sum_i \frac{\alpha_i}{2}\zeta_1 \\ &\quad +\sum_i \frac{\alpha_i}{2}\left(\frac{\mathcal{Y}'''(e_i)}{\mathcal{Y}'(e_i)} + \frac{\mathcal{X}^{IV}(e_i)}{\mathcal{X}''(e_i)} - \frac{\mathcal{X}'''^2(e_i)}{\mathcal{X}''^2(e_i)} \right. \\ &\quad \left. - \frac{\mathcal{Y}''(e_i)}{\mathcal{Y}'(e_i)}\frac{\mathcal{X}'''(e_i)}{\mathcal{X}''(e_i)}\right) \\ &\quad +\frac{1}{2}\sum_i \frac{\phi'(e_i-s)\mathcal{X}'''(e_i)}{\mathcal{X}''^2(e_i)\mathcal{Y}'(e_i)} - \frac{1}{2}\sum_i \frac{\phi''(e_i-s)}{\mathcal{X}''(e_i)\mathcal{Y}'(e_i)}\end{aligned} \tag{7.67}$$

therefore:

$$F^{(1)} = K(V_2,\epsilon) - \frac{1}{24}\ln\theta'(0)^8\, A^2\, \tilde{A}^{(d_2+1)} \prod_i \mathcal{Y}'(e_i) \tag{7.68}$$

where K may depend on V_2 and ϵ but not on V_1. Using:

$$d_2\tilde{g}_{d_2+1} = \gamma\tilde{\gamma}^{-d_2}\prod_i \frac{\theta(e_i+u_\infty)}{\theta(e_i-u_\infty)} \tag{7.69}$$

we can write it in a more symmetric form:

$$\boxed{F^{(1)} = K(\epsilon) - \frac{1}{24}\ln\gamma^4\tilde{\gamma}^4\theta'(0)^8 \prod_{i,j}\frac{\theta(e_i-\tilde{e}_j)\theta(2u_\infty)}{\theta(e_i-u_\infty)\theta(\tilde{e}_j+u_\infty)}} \tag{7.70}$$

where K depends neither on V_1 nor on V_2. K may still depend on ϵ.

The calculation of derivatives wrt ϵ is exactly the same, with replacing α_i by $1/\mathcal{X}''(e_i)\mathcal{Y}'(e_i)$, and we find that K does not depend on ϵ.

The limit where $\epsilon \to 0$ is the genus zero case, and we have $K = 0$.

8 Conclusion

We have computed the free energy to order $1/N^2$, in the genus one case, and it should not be too difficult to do the calculation for all genus. It is conjectured that $F^{(1)}$ should be the log of the determinant of the Laplacian on the algebraic curve.

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Appendix A Calculation

We shall prove here that:

$$\Gamma_j = \frac{\partial F^{(0)}}{\partial \epsilon_j} \quad , \quad \tau_{i,j} = \frac{1}{2i\pi}\frac{\partial^2 F^{(0)}}{\partial \epsilon_i \epsilon_j} \tag{A.1}$$

Notice that the second of these two equalities follows from the first.

The compatibility condition coming from eq. (2.17):

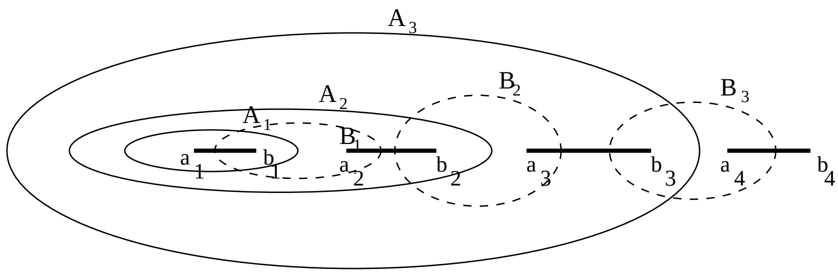
$$\frac{\partial}{\partial V_1(x)}\frac{\partial F^{(0)}}{\partial \epsilon_j} = \frac{\partial}{\partial \epsilon_j}\frac{\partial F^{(0)}}{\partial V_1(x)} = \frac{\partial}{\partial \epsilon_j}W_1(x) = -\frac{\partial}{\partial \epsilon_j}Y(x) \tag{A.2}$$

as well as eq. (3.17) and eq. (3.19) imply that $\Gamma_j - \frac{\partial F^{(0)}}{\partial \epsilon_j}$ is independent of V_1, and by the same argument after an integration by parts, it must be independent of V_2 too.

In order to prove that $\Gamma_j - \frac{\partial F^{(0)}}{\partial \epsilon_j} = 0$, we can choose V_1 and V_2 such that the x-physical sheet contains all the A-cylces. That means that the function $Y_0(x)$ has $g+1$ cuts in the physical sheet, i.e. the large N average density of eigenvalues of matrix M_1 has a support made of $g+1$ connected intervals:

$$\rho(x) = \frac{1}{2i\pi}(Y(x+i0) - Y(x-i0)) \quad , \quad \operatorname{supp}\rho = \bigcup_{i=1}^{g+1}[a_i,b_i] \tag{A.3}$$

We choose the A_i contours as follows:



and we can choose V_1 and V_2 such that the filling fraction of interval $[a_i, b_i]$ is:

$$\frac{n_i}{N} := \int_{a_i}^{b_i} \rho(x) dx = \frac{1}{2i\pi} \oint_{[a_i,b_i]} y dx = \epsilon_i - \epsilon_{i-1} \tag{A.4}$$

n_i is the average number of eigenvalues of M_1 in interval $[a_i, b_i]$.

Now, following the method of [13] for the one-matrix model, compute the energy cost of moving one single eigenvalue x from a to b, where a belongs to cut i and b to cut $i+1$. Basicaly: $n_i \to n_i - 1$ and $n_{i+1} \to n_{i+1} + 1$, i.e. $\epsilon_i \to \epsilon_i - 1/N$. That energy cost can be written:

$$-\frac{1}{N}\frac{\partial N^2 F}{\partial \epsilon_i} = N \int_a^b V'_{\text{eff}}(x) dx \tag{A.5}$$

where the effective mean field potential V_{eff} experienced by one eigenvalue of M_1 in the presence of the potentials and the interaction with all other eigenvalues in their equilibrium position was computed in [34, 43, 4, 32], and it reduces to:

$$\frac{\partial F}{\partial \epsilon_i} = \frac{1}{2i\pi} \oint_{B_i} Y(x) dx = \Gamma_i \tag{A.6}$$

Appendix B Theta functions in genus one

Consider a complex number τ such that $\operatorname{Im}\tau > 0$, called the modulus. We define the theta-function by:

$$\theta(u) = \sum_{n=0}^{\infty} e^{i\pi\tau n^2} \sin(2\pi n u) \tag{B.1}$$

We have:

$$\theta(u+1) = \theta(u) \quad , \quad \theta(u+\tau) = -\theta(u) e^{-i\pi(2u+\tau)} \quad , \quad \theta(-u) = -\theta(u) \tag{B.2}$$

We also introduce:

$$Z(u) = \frac{\partial}{\partial u} \ln \theta(u) \tag{B.3}$$

we have:

$$Z(u+1) = Z(u) \quad , \quad Z(u+\tau) = Z(u) - 2i\pi \quad , \quad Z(-u) = -Z(u) \tag{B.4}$$

$Z(u)$ has a single pole at $u = 0$, with residue 1:

$$Z(u) \sim \frac{1}{u} + \zeta_1 u + O(u^3) \tag{B.5}$$

And we introduce the Weierstrass function:

$$\phi(u) = -\frac{\partial}{\partial u} Z(u) \tag{B.6}$$

we have:

$$\phi(u+1) = \phi(u+\tau) = \phi(-u) = \phi(u) \tag{B.7}$$

$\phi(u)$ is thus an elliptical function (doubly periodic). It has a double pole at $u = 0$:

$$\phi(u) \sim \frac{1}{u^2} - \zeta_1 + O(u^2) \tag{B.8}$$

Consider an elliptical function f such that:

$$f(u+1) = f(u+\tau) = f(u) \tag{B.9}$$

then:

- if f is entire, then f is a constant.
- if f is meromorphic (its singularities are poles), it must have at least two poles (possibly a double pole).
- if f has n poles $e_1, \ldots, e_n$ with multiplicities $m_1, \ldots, m_n$, in the fundamental parallelogram of side $(1, \tau)$, then there exist $m = \sum_{k=1}^{n} m_k$ complex numbers $A, f_1, f_2, \ldots, f_{m-1}$ such that:

  $$f(u) = A \frac{\prod_{k=1}^{m} \theta(u - f_k)}{\prod_{k=1}^{n} \theta(u - e_k)^{m_k}} \tag{B.10}$$

  where we have defined f_m such that:

  $$\sum_{k=1}^{m} f_k = \sum_{k=1}^{n} m_k e_k \tag{B.11}$$

- An alternative representation of f is the following: there exist there exist $m+1$ numbers $A_0, A_{1,1}, \ldots, A_{1,m_1}, \ldots, A_{n,1}, \ldots, A_{n,m_n}$ such that:

  $$f(u) = A_0 + \sum_{k=1}^{n} \sum_{l=1}^{m_k} A_{k,l} Z^{(l-1)}(u - e_k) \tag{B.12}$$

  with the condition that:

  $$\sum_{k=1}^{n} A_{k,1} = 0 \tag{B.13}$$

Appendix C Theta functions arbitrary genus

C.1 Genus $g > 1$

$$\theta(u) = \sum_n \mathrm{e}^{i\pi n^t \tau n} \mathrm{e}^{2i\pi n^t u} \tag{C.1}$$

we have:

$$\theta(u+n) = \theta(u) = \theta(-u) \tag{C.2}$$

and

$$\theta(u+\tau n) = \theta(u)\, \mathrm{e}^{-i\pi[2n^t u + n^t \tau n]} \tag{C.3}$$

References

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