

Genuine collective flow from Lee-Yang zeroes

R. S. Bhalerao,¹ N. Borghini,² and J.-Y. Ollitrault²

¹*Department of Theoretical Physics, Tata Institute of Fundamental Research,
Homi Bhabha Road, Colaba, Mumbai 400 005, India*

²*Service de Physique Théorique, CEA-Saclay, F-91191 Gif-sur-Yvette cedex, France*
(Dated: July 10, 2003)

We propose to use the theory of phase transitions of Lee and Yang as a practical tool to analyze long-range correlations in a finite-size system. We apply it to the analysis of anisotropic flow in nucleus-nucleus collisions, and show that this method is more reliable than any other used so far.

PACS numbers: 25.75.Ld, 25.75.Gz, 05.70.Fh

Elliptic flow [1] is widely recognized as a sensitive probe of the dense hadronic matter produced in a nucleus-nucleus collision at RHIC [2]. Elliptic flow, denoted by v_2 , is the second Fourier harmonics of the single-particle azimuthal distribution [3]:

$$v_n \equiv \langle \cos n(\phi - \Phi_R) \rangle, \quad (1)$$

where ϕ denotes the azimuthal angle of an outgoing particle, Φ_R is the azimuthal angle of the impact parameter, both measured in the laboratory frame (see Fig. 1; Φ_R is also called the orientation of the reaction plane), and n is a positive integer. Angular brackets denote an average over many particles belonging to some phase-space region, and over many collisions having approximately the same impact parameter.

While the so-called anisotropic flow v_n , defined by Eq. (1), is a trivial one-particle observable which can easily be computed in a model or an event generator, the experimental situation is quite different. Indeed, the reference direction Φ_R is unknown experimentally, and v_n can only be measured indirectly, from the azimuthal correlations between the detected particles. Furthermore, Φ_R varies randomly from one event to the other. This has a remarkable consequence: anisotropic flow appears as a truly collective motion, in the sense that all outgoing particles in a given event seem to be attracted towards some arbitrary direction.

In this Letter, we introduce a new method to measure v_n from the genuine correlation between a large number of particles. It is both more natural, and more reliable than all methods which have been used so far. Since anisotropic flow appears as a collective effect, involving all particles produced in an event, it is indeed natural to characterize it by means of a global (i.e., multiparticle) observable. While such observables [1, 4] provided early evidence [5] for anisotropic flow, they were unable to yield quantitative, accurate results. As a consequence, the standard method for analyzing flow soon became to correlate particles with an estimate of Φ_R [6]. However, this estimate is itself obtained from the outgoing particles, and one essentially measures a two-particle correlation [7]. Intuitively, such two-body correlations are not

the appropriate tool to probe collective behaviour. Indeed, these two-particle methods were shown to be inadequate at ultrarelativistic energies due to the magnitude of “nonflow” correlations from quantum statistics [8], minijet production [9], and other effects which are to a large extent unknown. At intermediate energies, nonflow correlations from resonance decays may also produce a sizeable bias. All these effects are neglected in the standard methods. Recently, new methods were developed, based on higher-order (typically, four-particle) correlations, together with a cumulant expansion which eliminates low-order nonflow correlations [10]. However, it was argued that experimental results [11] could still be explained by nonflow effects alone [12] at this order. Hence the need for a new method that really probes collective behaviour and isolates it from other effects.

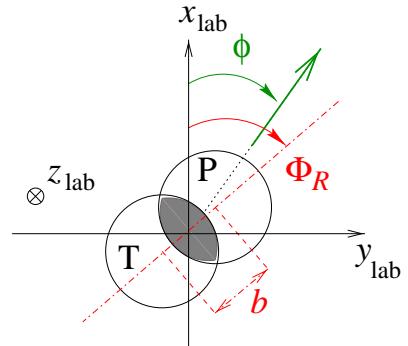


FIG. 1: Schematic picture of a nucleus-nucleus collision viewed in the transverse plane. b is the impact parameter, Φ_R its azimuthal angle. ϕ is the azimuthal angle of an outgoing particle.

Our new method is based on the following global observable, which is defined for each event:

$$Q^\theta = \sum_{j=1}^M \cos n(\phi_j - \theta), \quad (2)$$

where n is the Fourier harmonic under study ($n = 1$ for directed flow v_1 , $n = 2$ for elliptic flow), the sum runs

over all M detected particles, ϕ_j are their azimuthal angles, and θ is an arbitrary reference direction. This quantity is nothing but a projection of the “event flow-vector” which is used in other methods to estimate the orientation of the reaction plane, Φ_R [6]. In practice, the sum in Eq. (2) is often weighted: weights depending on the particle mass, transverse momentum and rapidity are used in order to enhance the contribution of particles with a larger v_n . They are omitted here for sake of simplicity, but should be included in the actual analysis.

The probability distribution of Q^θ is fully characterized by the moment generating function [13]

$$G(z) \equiv \langle e^{zQ^\theta} \rangle, \quad (3)$$

where z is a complex variable, and angular brackets now denote an average over a large number of events with the same impact parameter. The cumulants c_k of the probability distribution are then defined by [13]

$$\ln G(z) \equiv \sum_{k=1}^{+\infty} \frac{c_k}{k!} z^k. \quad (4)$$

The first two terms in this power-series expansion correspond to the average value of Q^θ , and the square of its standard deviation, respectively:

$$\begin{aligned} c_1 &= \langle Q^\theta \rangle \\ c_2 &= \langle (Q^\theta)^2 \rangle - \langle Q^\theta \rangle^2. \end{aligned} \quad (5)$$

Note that c_1 vanishes by symmetry if the detector has uniform azimuthal coverage. For particles emitted with uncorrelated, randomly distributed azimuthal angles, $c_2 = M/2$.

Cumulants have the following additivity property: if Q^θ is a sum of independent variables, then the cumulants of the distribution of Q^θ are the sum of the cumulants for each individual variable. This is because $G(z)$ factorizes into the product of the contributions of each variable, which is converted into a sum by the logarithm. Hence, cumulants scale *linearly* with the size of the system, and they are much smaller than the moments $\langle (Q^\theta)^k \rangle$. The physical reason is that cumulants essentially isolate the contribution of genuine k -particle correlations, by subtracting out most contributions from lower-order correlations. Taking the logarithm in Eq. (4) effectively performs this subtraction. The scaling still holds if particles are correlated within small clusters, but does not when collective effects are present.

Collective effects are by definition correlations among a large number of particles. They increase the value of the cumulant c_k for arbitrarily large k . The asymptotic behaviour of c_k is dominated by the singularities of $\ln G(z)$ which lie closest to the origin in the complex plane. The closer they are to $z = 0$, the larger the cumulants. Since $G(z)$ has no singularity, the only possible singularities of

$\ln G(z)$ are the zeroes of $G(z)$. This shows that collective effects, if any, are thus uniquely determined by the zeroes of $G(z)$ closest to $z = 0$.

So far, our analysis is general: Q^θ could be replaced by any extensive variable associated with the system. Our approach is in fact deeply related to the well-known theory of Yang and Lee, which characterizes phase transitions by the zeroes of the grand partition function [14], defined as

$$\mathcal{G}(\mu) = \sum_{N=0}^{+\infty} Z_N e^{\mu N/kT}, \quad (6)$$

where Z_N is the canonical partition function for N particles at temperature T in a volume V (both T and V are fixed). Let μ_c denote a reference value of the chemical potential. The probability P_N to have N particles in the system at $\mu = \mu_c$ is

$$P_N \equiv \frac{Z_N e^{\mu_c N/kT}}{\mathcal{G}(\mu_c)}. \quad (7)$$

The moment generating function of this probability distribution can be simply expressed in terms of the grand partition function, Eq. (6)

$$G(z) \equiv \sum_{N=0}^{\infty} P_N e^{zN} = \frac{\mathcal{G}(\mu_c + kTz)}{\mathcal{G}(\mu_c)}. \quad (8)$$

Lee and Yang studied the repartition of the zeroes of $G(z)$ in the complex plane (they actually chose the variable $y = e^z$ instead of z). There is no zero for z on the real axis, since Eq. (8) is a sum of positive terms. If a phase transition occurs at $\mu = \mu_c$, however, the zeroes of $G(z)$ come closer and closer to the origin $z = 0$ as the volume of the system, V , increases [14]. If no phase transition occurs at $\mu = \mu_c$, on the other hand, the zeroes remain at a finite distance from the origin.

The property that the zeroes of $G(z)$ come closer to the origin expresses that the cumulants of the distribution of N increase faster than linearly with the volume V . At a first-order transition point, a liquid-gas transition, say, the system can either be in a pure low-density gas phase or in a pure high-density liquid phase. The probability distribution P_N in Eq. (8), instead of being sharply peaked around its average value, is widely spread between two values N_{\min} (gas) and N_{\max} (liquid) which both scale like the volume V . Then, the partition function $G(z)$ depends on the volume V essentially through the combination zV , and consequently its zeroes scale with the volume like $1/V$.

We can now come back to our generating function, Eq. (3), and discuss what happens when there is anisotropic flow. Let us repeat the discussion of Eqs. (3–5), but with all average values taken for a fixed orientation of the reaction plane Φ_R . Using the definition of v_n , Eq. (1), and symmetry with respect to the reaction

plane (which implies $\langle \sin n(\phi - \Phi_R) \rangle = 0$), and assuming for simplicity that the multiplicity M is the same for all events, one obtains from Eq. (2):

$$c_1 = \langle Q^\theta | \Phi_R \rangle = M v_n \cos(n(\Phi_R - \theta)). \quad (9)$$

We neglect terms c_3 and higher in Eq. (4). This amounts to assuming that the probability distribution of Q^θ is gaussian for a fixed Φ_R : this is the central limit theorem, which holds if M is large enough, and if there is no other collective effect in the system. We further neglect the Φ_R -dependence of c_2 . Finally, averaging over Φ_R , one obtains the following theoretical expression of $G(z)$, which we denote by $G_{\text{c.l.}}(z)$ since it corresponds to the central limit approximation:

$$G_{\text{c.l.}}(z) = e^{c_2 z^2/2} I_0(M v_n z), \quad (10)$$

where I_0 is a modified Bessel function.

The first zeroes of $G_{\text{c.l.}}(z)$ lie on the imaginary axis at

$$z_0 = i r_0 = \frac{i j_{01}}{M v_n}, \quad (11)$$

and at $-z_0$, where $j_{01} \simeq 2.405$ is the first positive root of the Bessel function $J_0(x)$. As expected from the general discussion above, anisotropic flow v_n , being a collective effect, is completely determined by z_0 . The situation is analogous to a first-order phase transition, in the sense that the position of the zero scales like $1/M$, and the multiplicity M is the analogue of the volume V in Lee-Yang's theory. The important difference with statistical physics is that the system size is much smaller. As a consequence, zeroes never come very close to the origin, but the physics involved is essentially the same.

In a second paper [15], Lee and Yang further showed that all zeroes lie on the imaginary axis of the variable z (or, equivalently, on the unit circle for $y = e^z$) for a general class of models. It is interesting to note that our theoretical estimate, Eq. (10), has the same property.

Figure 2 displays the variation of $|G(ir)|$ as a function of r for simulated data. The data set contained $N_{\text{evts}} = 20000$ events with a detected multiplicity $M = 300$, and a simulated elliptic flow $v_2 = 6\%$. These are typical values for mid-central Au+Au collision at $\sqrt{s_{\text{NN}}} = 130$ GeV, as analyzed by the STAR Collaboration [16]. The global observable Q^θ in Eq. (2) was constructed for each event with $n = 2$ and $\theta = 0$. The numerical results are compared with the theoretical estimate, Eq. (10), where we have taken $c_2 = M/2$ (the expected value for independent particles). The excellent agreement justifies the approximations made in deriving Eq. (10). However, a closer look at the numerical results (inlay in Fig. 2) shows that unlike the theoretical estimate, $|G(ir)|$ does not strictly vanish: due to statistical fluctuations, the zeroes of $G(z)$

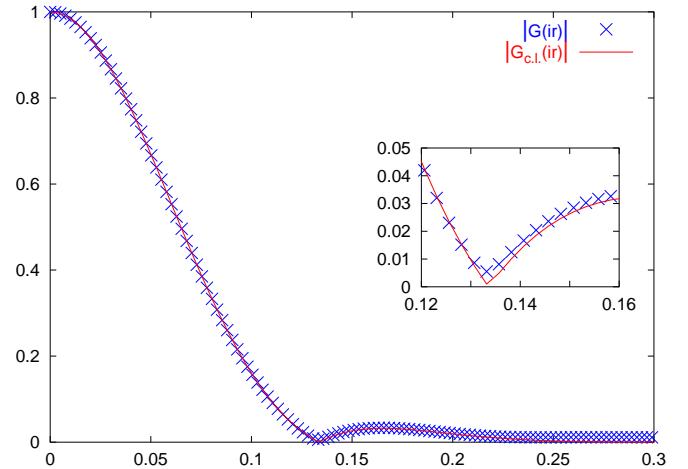


FIG. 2: Variation of $|G(ir)|$ with r . The crosses are the values of $|G(ir)|$. The solid line displays the expected value $|G_{\text{c.l.}}(ir)|$ [defined by Eq. (10)].

are slightly off the imaginary axis. This small deviation is physically irrelevant, and we choose to investigate the minima of $|G(z)|$, rather than the zeroes of $G(z)$. We denote by r_0^θ the first minimum of $|G(ir)|$, where the superscript θ recalls that it may depend on the reference angle θ in Eq. (2). Identifying $G(z)$ with the theoretical estimate $G_{\text{c.l.}}(z)$, and using Eq. (11), we obtain the following estimate of v_n , which may also depend on θ :

$$v_n^\theta \equiv \frac{j_{01}}{M r_0^\theta}. \quad (12)$$

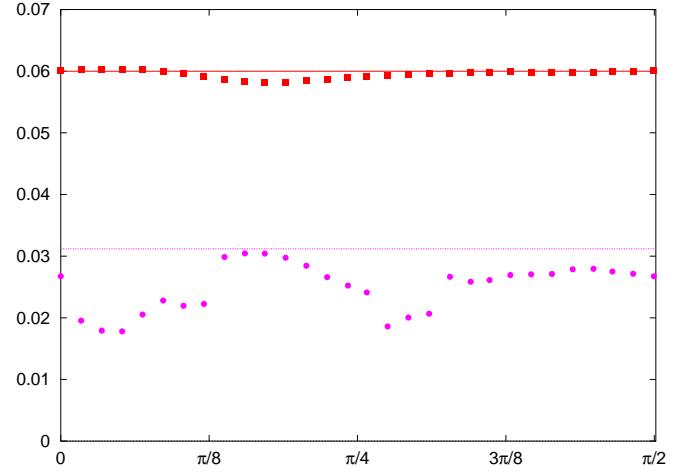


FIG. 3: The reconstructed value v_2^θ as a function of θ . Squares: simulated data with input $v_2 = 6\%$ [same data as in Fig. 2]. Circles: simulated data with input $v_2 = 0$.

This procedure was applied to the simulated data. The result is shown in Fig. 3. v_2^θ coincides with the input

value $v_2 = 6\%$, up to statistical fluctuations. Performing the analysis for several values of θ , and averaging v_2^θ over θ , reduces this statistical error. We finally obtain 5.95%: v_2 is reconstructed with great accuracy.

For the sake of illustration, we also applied the same procedure to simulated data with no flow. The procedure yields a spurious flow, due to statistical fluctuations, which is also shown in Fig. 3. The magnitude of this spurious flow can easily be understood. The average in Eq. (3) is evaluated over a finite number of events, N_{evts} . As a consequence, $G(ir)$ has statistical fluctuations, whose typical magnitude is $1/\sqrt{N_{\text{evts}}}$. For large enough r , they become as large as the expectation value given by Eq. (10), in which we set $v_n = 0$ and $c_2 = M/2$. This occurs when

$$e^{-Mr^2/4} \sim \frac{1}{\sqrt{N_{\text{evts}}}}. \quad (13)$$

As soon as r is larger than this value, fluctuations can produce a minimum of $|G(ir)|$. The corresponding “spurious flow” given by the analysis, Eq. (12), satisfies

$$v_n^\theta \lesssim \frac{j_{01}}{\sqrt{2M \ln N_{\text{evts}}}}. \quad (14)$$

For our simulated data, the right-hand side (rhs) is about 3.1%, which is depicted as the dashed line in Fig. 3. As expected, the values of v_2^θ are below this value, but only slightly. This is the main limitation of our method: v_n can be safely reconstructed only if it is larger than the rhs of Eq. (14).

The strength of this method is its stability with respect to nonflow correlations. As a simple model for such correlations, assume that instead of emitting M particles in each event, we emit M clusters, each cluster containing k collinear particles. Then, Q^θ is increased by a factor of k . As a consequence, the position of the first minimum, r_0^θ , is smaller by a factor of k . Since the event multiplicity is now kM , one must replace M with kM in the denominator of Eq. (12), so that the flow estimate v_n^θ is strictly the same, as it should.

The method can be extended to the analysis of differential flow, i.e., the analysis of v_n as a function of transverse momentum and rapidity. This will be explained in a forthcoming publication [17], where we also discuss in detail errors due to nonflow correlations, statistical fluctuations, and show that the method is remarkably insensitive to azimuthal asymmetries in the detector acceptance.

We have shown that Lee-Yang zeroes provide a sensitive probe of collective effects. It would be interesting to extend the present approach to other observables, in order to look for critical fluctuations which may occur in the vicinity of a phase transition. However, one must

beware of trivial fluctuations. For instance, a analysis similar to ours was performed in high-energy physics, with Q^θ replaced by the multiplicity; but it was soon realized that the position of the zeroes merely reflects general, well-known features of the multiplicity distribution [18]. To avoid such effects, one studies fluctuations of ratios [19] (typically, the mean transverse momentum per particle in an event) rather than extensive observables, and generalizing our approach to such quantities is not obvious.

Acknowledgments

R. S. B. acknowledges the hospitality of the SPhT, CEA, Saclay; J.-Y. O. acknowledges the hospitality of the Department of Theoretical Physics, TIFR, Mumbai. Both acknowledge the financial support from CEFIPRA, New Delhi, under its project no. 2104-02.

-
- [1] J.-Y. Ollitrault, Phys. Rev. D **46**, 229 (1992).
 - [2] K. H. Ackermann *et al.* [STAR Collaboration], Phys. Rev. Lett. **86**, 402 (2001).
 - [3] S. Voloshin and Y. Zhang, Z. Phys. C **70**, 665 (1996).
 - [4] P. Danielewicz and M. Gyulassy, Phys. Lett. B **129**, 283 (1983).
 - [5] H. A. Gustafsson *et al.*, Phys. Rev. Lett. **52**, 1590 (1984); J. Barrette *et al.* [E877 Collaboration], Phys. Rev. Lett. **73**, 2532 (1994).
 - [6] P. Danielewicz and G. Odyniec, Phys. Lett. B **157**, 146 (1985).
 - [7] S. Wang *et al.*, Phys. Rev. C **44**, 1091 (1991).
 - [8] P. M. Dinh, N. Borghini and J.-Y. Ollitrault, Phys. Lett. B **477**, 51 (2000); N. Borghini, P. M. Dinh and J.-Y. Ollitrault, Phys. Rev. C **62**, 034902 (2000).
 - [9] Y. V. Kovchegov and K. L. Tuchin, Nucl. Phys. A **708**, 413 (2002).
 - [10] N. Borghini, P. M. Dinh and J.-Y. Ollitrault, Phys. Rev. C **63**, 054906 (2001); Phys. Rev. C **64**, 054901 (2001).
 - [11] C. Adler *et al.* [STAR Collaboration], Phys. Rev. C **66**, 034904 (2002).
 - [12] Y. V. Kovchegov and K. L. Tuchin, Nucl. Phys. A **717**, 249 (2003).
 - [13] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).
 - [14] C. N. Yang and T. D. Lee, Phys. Rev. **87**, 404 (1952).
 - [15] T. D. Lee and C. N. Yang, Phys. Rev. **87**, 410 (1952).
 - [16] K. H. Ackermann *et al.* [STAR Collaboration], Phys. Rev. Lett. **86**, 402 (2001).
 - [17] R. S. Bhalerao, N. Borghini and J.-Y. Ollitrault, in preparation.
 - [18] T. C. Brooks, K. L. Kowalski and C. C. Taylor, Phys. Rev. D **56**, 5857 (1997).
 - [19] S. Jeon and V. Koch, hep-ph/0304012.