DOUBLE SCALING LIMIT IN RANDOM MATRIX MODELS AND A NONLINEAR HIERARCHY OF DIFFERENTIAL EQUATIONS

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ABSTRACT. We derive the double scaling limit of eigenvalue correlations in the random matrix model at critical points and we relate the limiting correlation functions to a nonlinear hierarchy of ordinary differential equations.

1. INTRODUCTION

We consider the unitary ensemble of random matrices,

$$d\mu_N(M) = Z_N^{-1} \exp\left(-N \operatorname{Tr} V(M)\right) dM, \quad Z_N = \int_{\mathcal{H}_N} \exp\left(-N \operatorname{Tr} V(M)\right) dM, \tag{1.1}$$

on the space $\mathcal{H}_N$ of Hermitian $N \times N$ matrices $M = (M_{ij})_{1 \le i,j \le N}$, where $V(x)$ is a polynomial, $V(x) = v_p x^p + v_{p-1} x^{p-1} + \ldots$, of an even degree p with $v_p > 0$. The ensemble of eigenvalues $\lambda = \{\lambda_j, \; j = 1, \ldots, N\}$ of M is given then by the formula (see e.g. [Meh] or [TW])

$$d\mu_N(\lambda) = \widetilde{Z}_N^{-1} \exp\left(-N H_N(\lambda)\right) d\lambda, \quad \widetilde{Z}_N = \int_{\Lambda_N} \exp\left(-N H_N(\lambda)\right) d\lambda, \tag{1.2}$$

where Λ_N is the symmetrized $\mathbb{R}^N$, $\Lambda_N = \mathbb{R}^N/S(N)$, and

$$H_N(\lambda) = -\frac{2}{N} \sum_{1 \le j < k \le N} \log|\lambda_j - \lambda_k| + \sum_{j=1}^{N} V(\lambda_j). \tag{1.3}$$

Let $d\nu_N(x) = \rho_N(x)dx$ be the distribution of the eigenvalues on the line, so that for any test function $\varphi(x) \in C_0^\infty$,

$$\int_{\Lambda_N} \left[\frac{1}{N} \sum_{j=1}^{N} \varphi(\lambda_j)\right] d\mu_N(\lambda) = \int_{-\infty}^{\infty} \varphi(x) \, d\nu_N(x). \tag{1.4}$$

As $N \to \infty$ there exists a weak limit of $d\nu_N(x)$,

$$d\nu_\infty(x) = \lim_{N \to \infty} d\nu_N(x). \tag{1.5}$$

To determine the limit (cf. [BIPZ], [DGZ], and others), consider the energy functional on the space of probability measures on the line,

$$I(d\nu(x)) = -\iint_{\mathbb{R}^2} \log|x - y|d\nu(x)\,d\nu(y) + \int_{\mathbb{R}} V(x)d\nu(y). \tag{1.6}$$

Then $H_N(\lambda)$ in (1.3) can be written as

$$H_N(\lambda) = N I(d\nu(x; \lambda)), \tag{1.7}$$

where $d\nu(x; \lambda)$ is a discrete probability measure with atoms at λ_j,

$$d\nu(x; \lambda) = \frac{1}{N} \sum_{j=1}^{N} \delta(x - \lambda_j) \, dx. \tag{1.8}$$

Hence,

$$d\mu_N(\lambda) = \widetilde{Z}_N^{-1} \exp\left(-N^2 I(d\nu(x;\lambda))\right) d\lambda. \tag{1.9}$$

Because of the factor N^2 in the exponent, one can expect that as $N \to \infty$, the measures $d\mu_N(\lambda)$ are localized in a shrinking vicinity of an *equilibrium measure* $d\nu_{\text{eq}}(x)$, which minimizes the functional $I(d\nu(x))$, and therefore, one expects the limit (1.5) to exist with $d\nu_\infty(x) = d\nu_{\text{eq}}(x)$. A rigorous proof of the existence and uniqueness of the equilibrium measure, its properties, and the existence of limit (1.5) with $d\nu_\infty(x) = d\nu_{\text{eq}}(x)$, was given in [BPS] and [Joh].

The equilibrium measure $d\nu_{\text{eq}}(x)$ is supported by a finite number of segments $[a_j, b_j]$, $j = 1, \ldots, q$, and it is absolutely continuous with respect to the Lebesgue measure, $d\nu_{\text{eq}}(x) = \rho(x)dx$, with a density function $\rho(x)$ of the form

$$\rho(x) = \frac{1}{2\pi i} h(x) R_+^{1/2}(x), \quad R(x) = \prod_{j=1}^{q} (x - a_j)(x - b_j), \tag{1.10}$$

where $h(x)$ is a polynomial of the degree, $\deg h = p - q - 1$, and $R_+^{1/2}(x)$ means the value on the upper cut of the principal sheet of the function $R^{1/2}(z)$ with cuts on J. The equilibrium measure is uniquely determined by the Euler-Lagrange conditions (see [DKMVZ]): for some real constant l,

$$2 \int_{\mathbb{R}} \log|x - s|\, d\nu_{\text{eq}}(s) - V(x) = l, \quad \text{for} \quad x \in \cup_{j=1}^{q}[a_j, b_j], \tag{1.11}$$

$$2 \int_{\mathbb{R}} \log|x - s|\, d\nu_{\text{eq}}(s) - V(x) \leq l, \quad \text{for} \quad x \in \mathbb{R} \setminus \cup_{j=1}^{q}[a_j, b_j]. \tag{1.12}$$

Equations (1.10), (1.11) imply that

$$\omega(z) = \frac{V'(z)}{2} - \frac{h(z)R^{1/2}(z)}{2}, \tag{1.13}$$

where

$$\omega(z) \equiv \int_J \frac{\rho(x)\, dx}{z - x} = z^{-1} + O(z^{-2}), \quad z \to \infty. \tag{1.14}$$

In addition, (1.11) implies that

$$\int_{b_j}^{a_{j+1}} \frac{h(x)R^{1/2}(x)}{2}\, dx = 0, \quad j = 1, \ldots, q - 1, \tag{1.15}$$

which shows that $h(x)$ has at least one zero on each interval $b_j < x < a_{j+1}$; $j = 1, \ldots, q - 1$. From (1.13) we obtain that

$$V'(z) = \text{Pol}\left[h(z)R^{1/2}(z)\right], \quad \text{Res}_{z=\infty}\left[h(z)R^{1/2}(z)\right] = -2, \tag{1.16}$$

and

$$h(z) = \text{Pol}\left[\frac{V'(z)}{R^{1/2}(z)}\right], \tag{1.17}$$

where $\text{Pol}\,[f(z)]$ is the polynomial part of $f(z)$ at $z = \infty$. The latter equation expresses $h(z)$ in terms of $V(z)$ and the end-points, $a_1, b_1, \ldots, a_q, b_q$. The end-points can be further found from (1.16), which gives $q + 1$ equation on $a_1, \ldots, b_q$, and from (1.15), which gives the remaining $q - 1$ equation.

The equilibrium measure $d\nu_{\text{eq}}(x)$ is called *regular* (otherwise *singular*), see [DKMVZ], if

$$h(x) \neq 0 \quad \text{for} \quad x \in \cup_{j=1}^{q}[a_j, b_j] \tag{1.18}$$

and

$$2 \int \log|x - s|\, d\nu_{\text{eq}}(s) - V(x) < l, \quad \text{for} \quad x \in \mathbb{R} \setminus \cup_{j=1}^{q}[a_j, b_j]. \tag{1.19}$$

The polynomial $V(x)$ is called *critical* if the corresponding equilibrium mesaure $d\nu_{\text{eq}}(x)$ is singular. To study the critical behavior in a vicinity of $V(x)$, one embeds $V(x)$ into a parametric family $V(x;t)$, $t = (t_1, \ldots, t_r)$, so that for some t^c, $V(x;t^c) = V(x)$, and the problem is then to evaluate the asymptotics of eigenvalue

correlation functions as $t \to t^c$. The number of parameters r depends, in general, on the degree of degeneracy of the equilibrium measure $d\nu_{\mathrm{eq}}(x)$.

In this paper we concern with the critical behavior for the polynomial $V(x)$ such that the corresponding equilibrium measure is supported by the segment $[-2, 2]$, with a density function of the form

$$\rho(x) = Z^{-1} (x - c)^{2m} \sqrt{4 - x^2}, \quad Z = \int_{\mathbb{R}} (x - c)^{2m} \sqrt{4 - x^2} \, dx, \tag{1.20}$$

where $-2 < c < 2$ and $m = 1, 2, \ldots$. The choice of the support segment is obviously not important, because by a shift and a dilation one can reduce any segment to $[-2, 2]$. The parameter m determines the degree of degeneracy of the equilibrium measure at $x = c$. Observe that when $c \neq 0$, density (1.20) is not symmetric.

Our results are summarized as follows. We are interested in two problems:

1. The singularity of the infinite volume free energy at the critical point.
2. The double scaling limit, i.e. the limit of rescaled correlation functions as simultaneously the volume goes to infinity and the parameter t goes to t^c, with an appropriate relation between $t - t^c$ and the volume.

Case $m = 1$. Free energy. We evaluate the derivatives in T of the (infinite volume) free energy $F(T)$, where $T > 0$ is the temperature, and we show that $F(T)$ can be analytically continued through the critical value T_c both from below and from above of T_c. In addition, we show that $F(T)$ and its first two derivatives are continuous at $T = T_c$, while the third derivative has a jump. This proves that at $T = T_c$ the phase transition is of the third order. It gives an extension of the result of [GW] where the third order phase transition was shown for the case of a symmetric critical $V(x)$ in the circular ensemble of random matrices.

Case $m = 1$. Double scaling limit. The key problem here is to derive a *uniform* asymptotic formula for the recurrence coefficients of the corresponding orthogonal polynomials. The double scaling limit descibes a transition from a fixed point behavior of the recurrence coefficients to a quasiperiodic behavior (cf. [DKMVZ] and [BDE]), and the problem is to derive a uniform asymptotic formula for the recurrence coefficients in the transition region. We show that under a proper substitution, the recurrence coefficients are expressed, with a uniform error term, in terms of the Hastings-McLeod solution to the Painlevé II differential equation. In the symmetric case ($c = 0$) our solution reduces to the one obtained in [DSS], [PeS]. For a rigorous proof of the double scaling asymptotics in the symmetric case see [BI2], [BDJ]. The both latter papers are based on the Riemann-Hilbert approach, developed in [FIK], [BI1], [DKMVZ]. It is worth mentioning here earlier physical works [BKa], [GM], [DS] which concern with the double scaling limit of the Painlevé I type.

General case, $m \geq 1$. Double scaling limit. In the second half of the paper we consider the double scaling limit for a general $m \geq 1$. Our main result in this part is the derivation of a nonlinear hierarchy of ordinary differential equations which give, under a proper substitution, a uniform asymptotics of the recurrence coefficients in the transition region. This hierarchy is known as the Painlevé II hierarchy [Kit] and it is related to selfsimilar solutions of the mKdV equation [PeS] (see also [Moo]).

2. Critical Behavior for a Nonsymmetric Quartic Polynomial

Let us consider the *critical* quartic polynomial $V_c(x)$ such that

$$V_c'(x) = \frac{1}{T_c} (x^3 - 4c_1 x^2 + 2c_2 x + 8c_1), \quad T_c = 1 + 4c_1^2; \qquad V_c(0) = 0, \tag{2.1}$$

where we denote

$$c_k = \cos k\pi\epsilon, \quad s_k = \sin k\pi\epsilon. \tag{2.2}$$

This corresponds to the critical density

$$\rho_c(x) = \frac{1}{2\pi T_c} (x - 2c_1)^2 \sqrt{4 - x^2}. \tag{2.3}$$

Observe that $0 < \epsilon < 1$ is a parameter of the problem which determines the location of the critical point,

$$-2 < 2c_1 = 2\cos\pi\epsilon < 2. \tag{2.4}$$

Equation (1.13) reads in this case as

$$\omega(z) = \frac{V_c'(z)}{2} - \frac{(z - 2c_1)^2 \sqrt{z^2 - 4}}{2T_c}. \tag{2.5}$$

The correlations between eigenvalues in the matrix model are expressed in terms of orthogonal polynomials $P_n(x) = x^n + \ldots$ on the line with respect to the weight $e^{-NV_c(x)}$. Let

$$\psi_n(x) = \frac{1}{\sqrt{h_n}} P_n(x) e^{-NV_c(x)/2}, \qquad n = 0, 1, \ldots, \tag{2.6}$$

be the corresponding psi-functions, which form an orthonormal basis in L^2. They satisfy the basic recurrence relation (see e.g. [Sze]),

$$x\psi_n(x) = \gamma_{n+1}\psi_{n+1} + \beta_n\psi_n + \gamma_n\psi_{n-1} \qquad , \qquad \gamma_n = \sqrt{\frac{h_n}{h_{n-1}}}, \tag{2.7}$$

and the differential equation,

$$\frac{1}{N}\psi_n'(x) + \frac{V_c'(x)}{2}\psi_n(x) = \frac{n}{N}\frac{1}{\gamma_n}\psi_{n-1} + \frac{1}{T_c}\gamma_n\gamma_{n-1}\left(\beta_n + \beta_{n-1} + \beta_{n-2} - 4c_1\right)\psi_{n-2}$$

$$+ \frac{1}{T_c}\gamma_n\gamma_{n-1}\gamma_{n-2}\psi_{n-3}. \tag{2.8}$$

The compatibility condition of equations (2.7), (2.8) leads to the string equations,

$$T_c\frac{n}{N}\frac{1}{\gamma_n^2} = \gamma_n^2 + \gamma_{n-1}^2 + \gamma_{n+1}^2 + \beta_n^2 + \beta_{n-1}^2 + \beta_n\beta_{n-1} - 4c_1(\beta_n + \beta_{n-1}) + 2c_2, \tag{2.9}$$

$$0 = V_c'(\beta_n) + \gamma_n^2(2\beta_n + \beta_{n-1}) + \gamma_{n+1}^2(2\beta_n + \beta_{n+1}) - 4c_1(\gamma_n^2 + \gamma_{n+1}^2). \tag{2.10}$$

To study the critical asymptotics we embed $V_c(x)$ into a parametric family of polynomials. To that end for any $T > 0$ we define the polynomial

$$V(x; T) = \frac{1}{T}V(x), \tag{2.11}$$

where $V(x)$ is such that

$$V'(x) = x^3 - 4c_1x^2 + 2c_2x + 8c_1, \quad V(0) = 0. \tag{2.12}$$

Then $V_c'(x) = V'(x; T_c)$. We call T *temperature* and T_c *critical temperature*. Denote $\Delta T = T - T_c$. Let $\rho(x; T)$ be the equilibrium density for the polynomial $V(x; T)$. Equation (1.13) reads in this case,

$$\omega(z; T) \equiv \int_{J(T)} \frac{\rho(x; T)\,dx}{z - x} = \frac{V'(z)}{2T} - \frac{h(z; T)R^{1/2}(z; T)}{2T}, \tag{2.13}$$

where $h(z; T)$ is a monic polynomial in z.

Free energy near the critical point. The (infinite volume) free energy is defined as

$$F(T) = -T\lim_{N\to\infty}\frac{1}{N^2}\ln Z_N(T), \quad Z_N(T) = \int_{\mathcal{H}_N}\exp\left(-\frac{N}{T}\operatorname{Tr}V(M)\right)dM. \tag{2.14}$$

We will show that at $T = T_c$, $F(T)$ is not analytic. To evaluate the type of nonanalyticity at $T = T_c$, consider the function

$$F_1(T) = T^2\frac{d}{dT}\left(\frac{F(T)}{T}\right) = \lim_{N\to\infty}\frac{1}{Z_N(T)}\int_{\mathcal{H}_N}\frac{1}{N}\operatorname{Tr}V(M)\exp\left(-\frac{N}{T}\operatorname{Tr}V(M)\right)dM. \tag{2.15}$$

It can be evaluated as

$$F_1(T) = \int_{J(T)} V(x)\rho(x; T)dx = -\frac{1}{4\pi iT}\oint_C V(z)h(z; T)\sqrt{R(z; T)}\,dz$$

$$= \frac{1}{2\pi i}\oint_C V(z)\omega(z; T)\,dz, \tag{2.16}$$

where C is any contour with positive orientation around $J(T)$, the support of equilibrium measure. Observe that limits in (2.14), (2.15) exist for any polynomial $V(x)$, due to the weak convergence of eigenvalue correlation functions (cf. [Joh]). On the contrary, for the second derivative of $F(T; N) \equiv -(T/N^2) \ln Z_N(T)$, the convergence $F''(T; N) \to F''(T)$ does *not* hold if the equilibrium measure has two cuts or more, because of quasiperiodic oscillations of $F''(T; N)$ as a function of N (see [BDE]). From (2.16) it follows that since $\omega(z; T)$ is continuous on C in T at $T = T_c$, $F_1(T)$ is continuous as well. Therefore, $F'(T)$ is *continuous* at $T = T_c$. Consider $F''(T)$.

Second derivative of the free energy. From (2.16),

$$\frac{d[TF_1(T)]}{dT} = \frac{1}{2\pi i} \oint_C V(z) \frac{d}{dT}[T\omega(z; T)]\, dz. \tag{2.17}$$

For $T > T_c$, the equilibrium measure corresponding to $V(x : T)$ is supported by one cut $[a, b]$ and the equilibrium density is written as

$$\rho(x; T) = \frac{1}{2\pi T}[(x - c)^2 + d^2]\sqrt{(b - x)(x - a)}, \tag{2.18}$$

where $a = a(T)$, $b = b(T)$, $c = c(T)$, $d = d(T)$. In the one-cut case we have the equation,

$$\frac{d}{dT}[T\omega(z; T)] = \frac{1}{\sqrt{(z - a)(z - b)}}, \tag{2.19}$$

see Appendix below, hence

$$\frac{d[TF_1(T)]}{dT} = \frac{1}{2\pi i} \oint_C \frac{V(z)}{\sqrt{(z - a)(z - b)}}\, dz. \tag{2.20}$$

This implies that

$$\frac{d[TF_1(T)]}{dT}\bigg|_{T=T_c^+} = \frac{1}{2\pi i} \oint_C \frac{V(z)}{\sqrt{z^2 - 4}}\, dz. \tag{2.21}$$

We should mention here another useful formula valid for $T \geq T_c$ (see [DGZ], [Eyn]):

$$\frac{d^2[TF(T)]}{dT^2} = 2\ln\frac{b - a}{4}. \tag{2.22}$$

For $T < T_c$, the equilibrium measure corresponding to $V(x)$ is supported by two cuts $[a_1, b_1]$ and $[a_2, b_2]$. The equilibrium density is written in this case as

$$\rho(x; T) = \frac{1}{2\pi T}(x - c)\sqrt{(b_1 - x)(x - a_1)(b_2 - x)(x - a_2)}, \tag{2.23}$$

where where a_1, b_1, a_2, b_2, c depend on T and $b_1 < c < a_2$. In the two-cut case we have the equation,

$$\frac{d}{dT}[T\omega(z; T)] = \frac{z - x_0}{\sqrt{(z - a_1)(z - b_1)(z - a_2)(z - b_2)}}, \tag{2.24}$$

where $x_0 = x_0(T)$, $b_1 < x_0 < a_2$, is determined from the condition that

$$\int_{b_1}^{a_2} \frac{x - x_0}{\sqrt{(x - a_1)(x - b_1)(x - a_2)(x - b_2)}}\, dx = 0, \tag{2.25}$$

see Appendix below, hence

$$\frac{d[TF_1(T)]}{dT} = \frac{1}{2\pi i} \oint_C \frac{V(z)(z - x_0)}{\sqrt{(z - a_1)(z - b_1)(z - a_2)(z - b_2)}}\, dz. \tag{2.26}$$

We have that $a_2 = b_1 = x_0 = 2c_1$ at $T = T_c^-$, hence

$$\frac{d[TF_1(T)]}{dT}\bigg|_{T=T_c^-} = \frac{1}{2\pi i} \oint_C \frac{V(z)}{\sqrt{z^2 - 4}}\, dz. \tag{2.27}$$

Thus,

$$
\left. \frac{d[TF_1(T)]}{dT} \right|_{T=T_c^-} = \left. \frac{d[TF_1(T)]}{dT} \right|_{T=T_c^+} , \tag{2.28}
$$

so that $F''(T)$ is *continuous* at $T = T_c$. Consider now $F'''(T)$.

Third derivative of the free energy. In the one-cut case we have that

$$
\frac{d}{dT}[(x-c)^2 + d^2]\sqrt{(b-x)(x-a)} = -\frac{2}{\sqrt{(b-x)(x-a)}} \tag{2.29}
$$

and

$$
\frac{da}{dT} = \frac{4}{h(a)(a-b)} , \qquad \frac{db}{dT} = \frac{4}{h(b)(b-a)} ; \qquad h(x) = (x-c)^2 + d^2 , \tag{2.30}
$$

see Appendix below. From (1.17) we find that

$$
c = 2c_1 - \frac{a+b}{4} , \qquad d^2 = \frac{5}{16}(a+b)^2 - c_1(a+b) - \frac{1}{2}ab - 2 , \tag{2.31}
$$

and then that

$$
h(a)|_{a=-2,\, b=2} = 4(c_1 + 1)^2 , \qquad h(b)|_{a=-2,\, b=2} = 4(c_1 - 1)^2 . \tag{2.32}
$$

Therefore, $a(T)$ and $b(T)$ are analytic at $T = T_c^+$, as a solution of system (2.30) with analytic coefficients. Equation (2.20) implies that $F_1(T)$, and hence $F(T)$, are *analytic* at $T = T_c^+$. From (2.30) and (2.31) we obtain

$$
\left. \frac{da}{dT} \right|_{T=T_c^+} = -\frac{1}{4(1+c_1)^2} , \qquad \left. \frac{db}{dT} \right|_{T=T_c^+} = \frac{1}{4(1-c_1)^2} . \tag{2.33}
$$

The analyticity at $T = T_c^-$ is more difficult. In the two-cut case we have that

$$
\frac{d}{dT}(z-c)\sqrt{(z-a_1)(z-b_1)(z-a_2)(z-b_2)} = -\frac{2(z-x_0)}{\sqrt{(z-a_1)(z-b_1)(z-a_2)(z-b_2)}} . \tag{2.34}
$$

where $b_1 < x_0 < a_2$ solves equation (2.25), and

$$
\frac{da_1}{dT} = \frac{4(a_1 - x_0)}{(a_1 - c)(a_1 - b_1)(a_1 - a_2)(a_1 - b_2)} , \qquad \frac{db_2}{dT} = \frac{4(b_2 - x_0)}{(b_2 - c)(b_2 - a_1)(b_2 - b_1)(b_2 - a_2)} , \tag{2.35}
$$

see Appendix below. At $T = T_c^-$ this gives that

$$
\left. \frac{da_1}{dT} \right|_{T=T_c^-} = -\frac{1}{4(1+c_1)^2} , \qquad \left. \frac{db_2}{dT} \right|_{T=T_c^-} = \frac{1}{4(1-c_1)^2} . \tag{2.36}
$$

Define d and δ such that

$$
b_1 = c - d + \delta, \quad a_2 = c + d + \delta . \tag{2.37}
$$

Then $d, \delta \to 0$ as $T \to T_c^-$ and from (1.15),

$$
\begin{aligned}
0 &= \int_{c-d+\delta}^{c+d+\delta} (x-c)\sqrt{(c+d+\delta-x)(x-c+d-\delta)}\,\sqrt{(x-a_1)(b_2-x)}\,dx \\
&= \int_{-d}^{d} (u+\delta)\sqrt{d^2 - u^2}\,\sqrt{(u+c-a_1+\delta)(b_2-c-\delta-u)}\,du \\
&= d^3 \int_{-1}^{1} \left(v + \frac{\delta}{d}\right)\sqrt{1-v^2}\,\sqrt{(vd+c-a_1+\delta)(b_2-c-\delta-vd)}\,dv \\
&= d^3 \sqrt{(c-a_1)(b_2-c)}\left[\frac{d}{2}\frac{(a_1+b_2-2c)}{(c-a_1)(b_2-c)}\int_{-1}^{1} v^2\sqrt{1-v^2}\,dv + \frac{\delta}{d}\int_{-1}^{1}\sqrt{1-v^2}\,dv \right. \\
&\quad \left. +O\left(d^3 + \frac{\delta^2}{d}\right)\right] = d^3\sqrt{(c-a_1)(b_2-c)}\left[\frac{\pi}{16}\frac{(a_1+b_2-2c)}{(c-a_1)(b_2-c)}d + \frac{\pi}{2}\frac{\delta}{d} + O\left(d^3 + \frac{\delta^2}{d}\right)\right] ,
\end{aligned} \tag{2.38}
$$

hence

$$\delta = -\frac{1}{8} \frac{(a_1 + b_2 - 2c)}{(c - a_1)(b_2 - c)} d^2 + O(d^4) . \tag{2.39}$$

By the implicit function theorem, δ is an analytic function of d^2 at $d = 0$. Equation (1.17) gives that

$$c = 2c_1 - \frac{a_1 + b_2}{4} - \frac{1}{2} \delta ,$$

$$d^2 = 2(1 + c_1)da_1 - 2(1 - c_1)db_2 - \frac{1}{2} \delta^2 - \frac{5}{8} (da_1^2 + db_2^2) - \frac{1}{4} da_1 db_2 ;$$

$$da_1 \equiv a_1 + 2, \quad db_2 \equiv -2 + b_2 . \tag{2.40}$$

Therefore, from (2.36) and (2.39) we obtain that as $T \to T_c^-$,

$$d^2 = -\frac{1}{s_1^2}\Delta T + O(\Delta T^2), \qquad \delta = -\frac{c_1}{8 s_1^4} \Delta T + O(\Delta T^2);$$

$$c = 2c_1 - \frac{3c_1}{16 s_1^4} \Delta T + O(\Delta T^2) , \qquad \frac{b_1 + a_2}{2} = 2c_1 - \frac{5c_1}{16 s_1^4} \Delta T + O(\Delta T^2) . \tag{2.41}$$

Define now δ_0 such that

$$b_1 = x_0 - d + \delta_0, \quad a_2 = x_0 + d + \delta_0 . \tag{2.42}$$

Then from (2.25),

$$0 = \int_{x_0 - d + \delta_0}^{x_0 + d + \delta_0} \frac{x - x_0}{\sqrt{(x_0 + d + \delta_0 - x)(x - x_0 + d - \delta_0)} \sqrt{(x - a_1)(b_2 - x)}} dx$$

$$= \int_{-d}^{d} \frac{u + \delta_0}{\sqrt{d^2 - u^2} \sqrt{(u + x_0 - a_1 + \delta_0)(b_2 - x_0 - \delta_0 - u)}} du$$

$$= d \int_{-1}^{1} \frac{v + \frac{\delta_0}{d}}{\sqrt{1 - v^2} \sqrt{(vd + c - a_1 + \delta_0)(b_2 - c - \delta_0 - vd)}} dv$$

$$= \frac{d}{\sqrt{(c - a_1)(b_2 - c)}} \left[-\frac{d}{2} \frac{(a_1 + b_2 - 2c)}{(c - a_1)(b_2 - c)} \int_{-1}^{1} \frac{v^2}{\sqrt{1 - v^2}} dv + \frac{\delta_0}{d} \int_{-1}^{1} \frac{1}{\sqrt{1 - v^2}} dv \right.$$

$$\left. + O\left(d^3 + \frac{\delta_0^2}{d} \right) \right] = \frac{d}{\sqrt{(c - a_1)(b_2 - c)}} \left[-\frac{\pi}{2} \frac{(a_1 + b_2 - 2c)}{(c - a_1)(b_2 - c)} d + \pi \frac{\delta_0}{d} + O\left(d^3 + \frac{\delta_0^2}{d} \right) \right] , \tag{2.43}$$

hence

$$\delta_0 = \frac{1}{2} \frac{(a_1 + b_2 - 2c)}{(c - a_1)(b_2 - c)} d^2 + O\left(d^4 \right) , \tag{2.44}$$

so that as $T \to T_c^-$,

$$\delta_0 = \frac{c_1}{2 s_1^4} \Delta T + O\left(\Delta T^2 \right) . \tag{2.45}$$

By the implicit function theorem δ_0 is an analytic function of d^2 at $d = 0$. Using (2.41) we obtain that

$$x_0 = 2c_1 - \frac{13c_1}{16 s_1^4} \Delta T + O\left(\Delta T^2 \right) . \tag{2.46}$$

By (2.37) and (2.42),

$$x_0 = c + \delta - \delta_0, \tag{2.47}$$

hence equations (2.35) can be written as

$$\frac{da_1}{dT} = \frac{4(a_1 - c - \delta + \delta_0)}{(a_1 - c)(d^2 - (a_1 - c - \delta)^2)(a_1 - b_2)} ,$$

$$\frac{db_2}{dT} = \frac{4(b_2 - c - \delta + \delta_0)}{(b_2 - c)(d^2 - (b_2 - c - \delta)^2)(b_2 - a_2)} . \tag{2.48}$$

Observe that d^2 is an analytic function of a_1, b_2, and δ, δ_0 are analytic functions of d^2. This gives the right hand side in (2.48) as analytic functions of a_1, b_2 and hence a_1, b_2 are analytic as functions of T at $T = T_c^-$. Set

$$m = \frac{b_1 + a_2}{2}. \tag{2.49}$$

Then $b_1 = m - d$, $a_2 = m + d$, hence, by (2.26)

$$\frac{d[TF_1(T)]}{dT} = \frac{1}{2\pi i} \oint_C \frac{V(z)(z - x_0)}{\sqrt{(z - a_1)(z^2 - 2mz + m^2 - d^2)(z - b_2)}} \, dz. \tag{2.50}$$

Since x_0, m and d^2 are analytic in T at $T = T_c^-$, we obtain that $F(T)$ is *analytic* at $T = T_c^-$.

By (2.20),

$$\left. \frac{d^2[TF_1(T)]}{dT^2} \right|_{T=T_c^+} = \frac{1}{2\pi i} \oint_C V(z) \left. \frac{d}{dT} \left(\frac{1}{\sqrt{(z-a)(z-b)}} \right) \right|_{T=T_c^+} dz, \tag{2.51}$$

and by (2.50),

$$\left. \frac{d^2[TF_1(T)]}{dT^2} \right|_{T=T_c^-} = \frac{1}{2\pi i} \oint_C V(z) \left[\frac{d}{dT} \left(\frac{1}{\sqrt{(z-a_1)(z-b_2)}} \right) \frac{(z - x_0)}{\sqrt{z^2 - 2mz + m^2 - d^2}} \right.$$
$$\left. + \frac{1}{\sqrt{(z-a_1)(z-b_2)}} \frac{d}{dT} \left(\frac{(z - x_0)}{\sqrt{z^2 - 2mz + m^2 - d^2}} \right) \right]_{T=T_c^-} dz. \tag{2.52}$$

Observe that by (2.33), (2.36),

$$\left. \frac{da}{dT} \right|_{T=T_c^+} = \left. \frac{da_1}{dT} \right|_{T=T_c^-} = -\frac{1}{4(1 + c_1)^2} \, ; \qquad \left. \frac{db}{dT} \right|_{T=T_c^+} = \left. \frac{db_2}{dT} \right|_{T=T_c^-} = \frac{1}{4(1 - c_1)^2} \, , \tag{2.53}$$

hence

$$\left. \frac{d^2[TF_1(T)]}{dT^2} \right|_{T=T_c^+} - \left. \frac{d^2[TF_1(T)]}{dT^2} \right|_{T=T_c^-}$$
$$= -\frac{1}{2\pi i} \oint_C V(z) \frac{1}{\sqrt{z^2 - 4}} \left. \frac{d}{dT} \left(\frac{(z - x_0)}{\sqrt{z^2 - 2mz + m^2 - d^2}} \right) \right|_{T=T_c^-} dz. \tag{2.54}$$

From (2.41) and (2.46) we find that

$$\left. \frac{d}{dT} \left(\frac{(z - x_0)}{\sqrt{z^2 - 2mz + m^2 - d^2}} \right) \right|_{T=T_c^-} = -\frac{1 - c_1 z + c_1^2}{s_1^4 (z - 2c_1)^2} \, , \tag{2.55}$$

hence

$$\left. \frac{d^2[TF_1(T)]}{dT^2} \right|_{T=T_c^+} - \left. \frac{d^2[TF_1(T)]}{dT^2} \right|_{T=T_c^-} = \frac{1}{2\pi i} \oint_C V(z) \frac{1}{\sqrt{z^2 - 4}} \frac{1 - c_1 z + c_1^2}{s_1^4 (z - 2c_1)^2} \, dz$$
$$= -\frac{3 + 25c_1^2 + 2c_1^4}{s_1^4} < 0. \tag{2.56}$$

Thus, $F(T)$ is *analytic* both at $T = T_c^+$ and $T = T_c^-$, and $F'''(T)$ has a *jump* at $T = T_c$. Therefore, $T = T_c$ is a critical point of the *third order* phase transition.

Recurrence coefficients near the critical point. The recurrence coefficients γ_n, β_n approach fixed values for $T > T_c$, see e.g. [DKMVZ]. Namely,

$$\lim_{n,N\to\infty; \, \frac{n}{N} \to \frac{T}{T_c}} \gamma_n = \gamma(T), \qquad \lim_{n,N\to\infty; \, \frac{n}{N} \to \frac{T}{T_c}} \beta_n = \beta(T), \tag{2.57}$$

where $\gamma = \gamma(T)$, $\beta = \beta(T)$ are fixed points of (2.9), (2.10), so that

$$T \frac{1}{\gamma^2} = 3\gamma^2 + 3\beta^2 - 8c_1\beta + 2c_2, \tag{2.58}$$

$$0 = V'(\beta) + 6\gamma^2\beta - 8c_1\gamma^2. \tag{2.59}$$

The values $\gamma = \gamma(T)$, $\beta = \beta(T)$ can be expressed in terms of the end-points of the cut as

$$\gamma = \frac{b-a}{4}, \quad \beta = \frac{b+a}{2}, \tag{2.60}$$

see e.g. [DGZ]. Therefore, by (2.33),

$$\gamma = 1 + \frac{1+c_1^2}{8s_1^4}\Delta T + O(\Delta T^2), \quad \beta = \frac{c_1}{2s_1^4}\Delta T + O(\Delta T^2), \quad \Delta T \to 0^+. \tag{2.61}$$

For $T < T_c$, the recurrence coefficients are asymptotically quasi-periodic, see [DKMVZ]. More precisely,

$$\lim_{n,N\to\infty;\,\frac{n}{N}\to\frac{T}{T_c}}[\gamma_n - \gamma(\omega\,n + \varphi)] = 0, \qquad \lim_{n,N\to\infty;\,\frac{n}{N}\to\frac{T}{T_c}}[\beta_n - \beta(\omega\,n + \varphi)] = 0, \qquad (T < T_c), \tag{2.62}$$

where

$$\omega = \omega(T) = 1 - \frac{1}{K}\int_{b_2}^{\infty}\frac{dz}{\sqrt{R(z)}}, \qquad K = \int_{b_1}^{a_2}\frac{dz}{\sqrt{R(z)}}; \tag{2.63}$$
$$R(z) = (z - a_1)(z - b_1)(z - a_2)(z - b_2),$$

$\gamma(x) = \gamma(x;T)$, $\beta(x) = \beta(x;T)$ are explicit analytic even periodic functions of period 1 in x, and φ is an explicit phase, see [BDE]. The extrema of $\gamma(x)$ and $\beta(x)$ are expressed in terms of the end-points of the cuts,

$$\min_x \gamma(x) = \frac{b_2 - a_1 - (a_2 - b_1)}{4}, \qquad \max_x \gamma(x) = \frac{b_2 - a_1 + (a_2 - b_1)}{4},$$
$$\min_x \beta(x) = \frac{b_2 + a_1 - (a_2 - b_1)}{2}, \qquad \max_x \beta(x) = \frac{b_2 + a_1 + (a_2 - b_1)}{2}. \tag{2.64}$$

Using (2.41), we obtain that as $\Delta T \to 0^-$,

$$K = \frac{\pi}{2s_1} + O(\Delta T), \qquad \int_{b_2}^{\infty}\frac{dz}{\sqrt{R(z)}} = \frac{\pi(1-\varepsilon)}{2s_1} + O(\Delta T), \tag{2.65}$$

hence

$$\omega = \varepsilon + O(\Delta T), \quad \Delta T \to 0^-. \tag{2.66}$$

As concerns the extrema of $\gamma(x)$ and $\beta(x)$, they behave as

$$\min_x, \max_x \gamma(x) = 1 \pm \frac{1}{2s_1}\left(\frac{|\Delta T|}{2}\right)^{1/2} + \frac{1+c_1^2}{8s_1^4}\Delta T + O(|\Delta T|^{3/2}); \tag{2.67}$$

$$\min_x, \max_x \beta(x) = \pm\frac{1}{s_1}\left(\frac{|\Delta T|}{2}\right)^{1/2} + \frac{c_1}{2s_1^4}\Delta T + O(|\Delta T|^{3/2}), \quad \Delta T \to 0^-. \tag{2.68}$$

2.1. **Double Scaling Limit for Recurrence Coefficients.** We considered above the case when T was fixed: we took first the limit n, $N \to \infty$, $\frac{n}{N} \to \frac{T}{T_c}$ and then the limit $T \to T_c$. Here we will consider the double scaling limit, when n, $N \to \infty$, $\frac{n}{N} \to 1$, with an appropriate scaling of $n - N$. We start with the following ansatz, which reproduces the quasiperiodic behavior of the recurrence coefficients:

$$\frac{n}{N} = 1 + N^{-2/3}t, \tag{2.69}$$

$$\gamma_n^2 = 1 + N^{-1/3}u(t)\cos 2n\pi\epsilon$$
$$+ N^{-2/3}\left(v_0(t) + v_1(t)\cos 2n\pi\epsilon + v_2(t)\cos 4n\pi\epsilon\right)$$
$$+ N^{-1}\left(w_0(t) + w_1(t)\cos 2n\pi\epsilon + w_2(t)\cos 4n\pi\epsilon + w_3(t)\cos 6n\pi\epsilon + w_4(t)\sin 4n\pi\epsilon\right), \tag{2.70}$$

$$\beta_n = 0 + N^{-1/3}u(t)\cos(2n+1)\pi\epsilon$$
$$+ N^{-2/3}\left(\widetilde{v}_0(t) + \widetilde{v}_1(t)\cos(2n+1)\pi\epsilon + \widetilde{v}_2(t)\cos(4n+2)\pi\epsilon\right)$$
$$+ N^{-1}\left(\widetilde{w}_0(t) + \widetilde{w}_1(t)\cos(2n+1)\pi\epsilon + \widetilde{w}_2(t)\cos(4n+2)\pi\epsilon + \widetilde{w}_3(t)\cos(6n+3)\pi\epsilon\right.$$
$$\left. + \widetilde{w}_4(t)\sin(4n+2)\pi\epsilon\right), \tag{2.71}$$

where $u(t)$, $v_0(t), \ldots, \widetilde{w}_4(t)$ are unknown functions. We substitute the ansatz into string equations (2.9), (2.10) and equate terms of the same order.

Order $N^{-1/3}$. Our ansatz is automatically satisfied at this order.

Order $N^{-2/3}$. We obtain from (2.9), (2.10) that

$$\begin{pmatrix} c_1^2 + 1 & -2c_1 \\ -2c_1 & c_1^2 + 1 \end{pmatrix} \begin{pmatrix} v_0 \\ \widetilde{v}_0 \end{pmatrix} = \frac{u^2}{4} \begin{pmatrix} c_1^2 \\ -c_1 \end{pmatrix} + \frac{T_c t}{4} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.72}$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ \widetilde{v}_1 \end{pmatrix} = \frac{u'}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \tag{2.73}$$

$$\begin{pmatrix} c_1^2 + c_2^2 & -2c_1 c_2 \\ -2c_1 c_2 & c_1^2 + c_2^2 \end{pmatrix} \begin{pmatrix} v_2 \\ \widetilde{v}_2 \end{pmatrix} = \frac{u^2}{4} \begin{pmatrix} 1 \\ -c_3 \end{pmatrix}. \tag{2.74}$$

By solving these equations we obtain that

$$v_0 = -\frac{c_1^2}{4s_1^2} u^2 + \frac{1 + c_1^2}{4s_1^4} t T_c, \qquad \widetilde{v}_0 = -\frac{c_1}{4s_1^2} u^2 + t T_c \frac{c_1}{2s_1^4}, \tag{2.75}$$

and

$$\widetilde{v}_1 - v_1 = \frac{1}{2} u', \qquad v_2 = \frac{u^2}{4s_1^2}, \qquad \widetilde{v}_2 = \frac{c_1 u^2}{4s_1^2}. \tag{2.76}$$

Order N^{-1}. We obtain from (2.9), (2.10) and (2.75), (2.76) that

$$\begin{pmatrix} c_1^2 + 1 & -2c_1 \\ -2c_1 & c_1^2 + 1 \end{pmatrix} \begin{pmatrix} w_0 \\ \widetilde{w}_0 \end{pmatrix} = \frac{uu'}{4} \begin{pmatrix} 0 \\ -c_1 \end{pmatrix} + \frac{uv_1}{2} \begin{pmatrix} c_1^2 \\ -c_1 \end{pmatrix} + c_1 \begin{pmatrix} -\widetilde{v}_0' \\ v_0' \end{pmatrix}, \tag{2.77}$$

$$8c_1^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ \widetilde{w}_1 \end{pmatrix} = \frac{u^3}{2s_1^2} \begin{pmatrix} -1 \\ c_2 \end{pmatrix} + \frac{T_c t u}{2s_1^4} \begin{pmatrix} 2c_1^4 + 3c_1^2 - 1 \\ -2c_1^4 - c_1^2 - 1 \end{pmatrix} + u'' \begin{pmatrix} -c_2 \\ 1 \end{pmatrix} + v_1' \begin{pmatrix} -4c_1^2 \\ 4c_1^2 \end{pmatrix}, \tag{2.78}$$

$$\begin{pmatrix} c_1^2 + c_2^2 & -2c_1 c_2 \\ -2c_1 c_2 & c_1^2 + c_2^2 \end{pmatrix} \begin{pmatrix} w_2 \\ \widetilde{w}_2 \end{pmatrix} = \frac{uv_1}{2} \begin{pmatrix} 1 \\ -c_3 \end{pmatrix} + \frac{uu'}{4s_1^2} \begin{pmatrix} -2c_1^2 c_2 \\ 4c_1^5 - 3c_1^3 + c_1 \end{pmatrix}, \tag{2.79}$$

$$\begin{pmatrix} c_1^2 + c_3^2 & -2c_1 c_3 \\ -2c_1 c_3 & c_1^2 + c_3^2 \end{pmatrix} \begin{pmatrix} w_3 \\ \widetilde{w}_3 \end{pmatrix} = \frac{u^3 c_1^2}{8s_1^2} \begin{pmatrix} -c_2 + 2 \\ -2c_1 c_3 + 1 \end{pmatrix}, \tag{2.80}$$

$$\begin{pmatrix} c_1^2 + c_2^2 & -2c_1 c_2 \\ -2c_1 c_2 & c_1^2 + c_2^2 \end{pmatrix} \begin{pmatrix} w_4 \\ \widetilde{w}_4 \end{pmatrix} = \frac{uu'}{8} \begin{pmatrix} -2s_2 \\ (4c_1^4 - 3c_1^2 - 1)/s_1 \end{pmatrix} \tag{2.81}$$

(we did symbolic calculations with MAPLE). Consider equation (2.78). The matrix on the left in this equation is degenerate, hence we have the compatibility condition,

$$\boxed{2s_1^2 u'' = u^3 + \frac{T_c}{s_1^2} t u,} \tag{2.82}$$

which is the Painlevé II equation. When $\epsilon = 1/2$ it reduces to $2u'' = u^3 + tu$. The function $u(t)$ behaves as

$$u(t) \underset{t \to -\infty}{\sim} \frac{1}{s_1} \sqrt{-T_c t}, \qquad u(t) \underset{t \to +\infty}{\sim} \mathrm{Ai}\,(\kappa t), \qquad \kappa = \left(\frac{T_c}{2s_1^4} \right)^{1/3}. \tag{2.83}$$

Here and in what follows we use the following notations: $f(x) \sim g(x)$ as $x \to a$ means that $\lim_{x \to a} \frac{f(x)}{g(x)} = 1$, and $f(x) \approx g(x)$ as $x \to a$ means that $\lim_{x \to a} [f(x) - g(x)] = 0$.

2.2. **Scaled Differential Equations at the Critical Point.** Equations (2.7), (2.8) can be used to derive a closed system of differential equations,

$$\frac{T_c}{N} \frac{d}{dx} \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix} = \begin{pmatrix} -\frac{TV'(x)}{2} - \gamma_n^2 A_n(x) & \gamma_n B_n(x) \\ -\gamma_n B_{n-1}(x) & \frac{TV'(x)}{2} + \gamma_n^2 A_n(x) \end{pmatrix} \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}, \tag{2.84}$$

where

$$A_n(x) = x - 4c_1 + \beta_n + \beta_{n-1}, \tag{2.85}$$

$$B_n(x) = x^2 + x(\beta_n - 4c_1) + \beta_n^2 - 4c_1\beta_n + 2c_2 + \gamma_n^2 + \gamma_{n+1}^2. \tag{2.86}$$

To derive a scaled system at the critical point $x = 2c_1$ we set

$$x = 2c_1 + yN^{-1/3}. \tag{2.87}$$

Then

$$\frac{T_c V'(x)}{2} + \gamma_n^2 A_n(x) = 2c_1(1 - \gamma_n^2) + \gamma_n^2(\beta_n + \beta_{n-1}) + (\gamma_n^2 - 1)yN^{-1/3}$$

$$+ c_1 y^2 N^{-2/3} + \frac{y^3 N^{-1}}{2}, \tag{2.88}$$

$$B_n(x) = y^2 + \beta_n y + \gamma_n^2 + \gamma_{n+1}^2 - 2 - 2c_1\beta_n + \beta_n^2. \tag{2.89}$$

Substituting ansatz (2.69)-(2.71) we obtain that

$$\frac{T_c V'(x)}{2} + \gamma_n^2 A_n(x) = N^{-2/3}\left(c_1 y^2 + \frac{c_1 u^2}{2} + \frac{c_1 T_c t}{2s_1^2} + yu\cos 2n\pi\epsilon - s_1 u'\sin 2n\pi\epsilon\right)$$

$$+ O(N^{-1}), \tag{2.90}$$

$$\gamma_n B_n(x) = N^{-2/3}\left(y^2 + \frac{u^2}{2} + \frac{T_c t}{2s_1^2} + yu\cos(2n+1)\pi\epsilon + s_1 u'\sin(2n+1)\pi\epsilon\right)$$

$$+ O(N^{-1}), \tag{2.91}$$

$$\gamma_n B_{n-1} = N^{-2/3}\left(y^2 + \frac{u^2}{2} + \frac{T_c t}{2s_1^2} + yu\cos(2n-1)\pi\epsilon + s_1 u'\sin(2n-1)\pi\epsilon\right)$$

$$+ O(N^{-1}). \tag{2.92}$$

Thus, system (2.84) reduces to

$$T_c \frac{d}{dy}\begin{pmatrix}\psi_n \\ \psi_{n-1}\end{pmatrix} = \begin{pmatrix}a_{11}(y) & a_{12}(y) \\ a_{21}(y) & a_{22}(y)\end{pmatrix}\begin{pmatrix}\psi_n \\ \psi_{n-1}\end{pmatrix}, \tag{2.93}$$

where up to $O(N^{-1/3})$,

$$a_{11}(y) = -c_1\left(y^2 + \frac{u^2}{2} + \frac{T_c t}{2s_1^2}\right) - yu\cos 2n\pi\epsilon + s_1 u'\sin 2n\pi\epsilon, \tag{2.94}$$

$$a_{12}(y) = y^2 + \frac{u^2}{2} + \frac{T_c t}{2s_1^2} + yu\cos(2n+1)\pi\epsilon + s_1 u'\sin(2n+1)\pi\epsilon, \tag{2.95}$$

$$a_{21}(y) = -\left(y^2 + \frac{u^2}{2} + \frac{T_c t}{2s_1^2}\right) - yu\cos(2n-1)\pi\epsilon - s_1 u'\sin(2n-1)\pi\epsilon, \tag{2.96}$$

$$a_{22}(y) = c_1\left(y^2 + \frac{u^2}{2} + \frac{T_c t}{2s_1^2}\right) + yu\cos 2n\pi\epsilon - s_1 u'\sin 2n\pi\epsilon. \tag{2.97}$$

When $\epsilon = 1/2$, this simplifies to

$$a_{11}(y) = -a_{22}(y) = -(-1)^n yu, \tag{2.98}$$

$$a_{12}(y) = y^2 + \frac{u^2 + t}{2} + (-1)^n u', \tag{2.99}$$

$$a_{21}(y) = -\left(y^2 + \frac{u^2 + t}{2}\right) + (-1)^n u'. \tag{2.100}$$

Under the substitution

$$\psi_n(y) = \cos\left(n + \frac{1}{2}\right)\pi\epsilon\, f(y) - \sin\left(n + \frac{1}{2}\right)\pi\epsilon\, g(y), \tag{2.101}$$

$$\psi_{n-1}(y) = \cos\left(n - \frac{1}{2}\right)\pi\epsilon\, f(y) - \sin\left(n - \frac{1}{2}\right)\pi\epsilon\, g(y), \tag{2.102}$$

system (2.93) reduces, up to $O(N^{-1/3})$, to

$$\boxed{\frac{T_c}{s_1}\frac{d}{dy}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} s_1 u' & \left(y^2 + \frac{u^2}{2} + \frac{Tt}{2s_1^2}\right) + yu \\ -\left(y^2 + \frac{u^2}{2} + \frac{Tt}{2s_1^2}\right) + yu & -s_1 u' \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix},} \tag{2.103}$$

the differential ψ-equation for Painlevé II equation (2.82).

2.3. **Universal Kernel.** To eliminate dependence on ϵ consider new variables $\tilde{t}$, $\tilde{u}$ and $\tilde{y}$ such that

$$t = \left(\frac{2s_1^4}{T_c}\right)^{1/3}\tilde{t}, \quad u = \left(\frac{4T_c}{s_1}\right)^{1/3}\tilde{u}, \quad y = \left(\frac{4T_c}{s_1}\right)^{1/3}\tilde{y}. \tag{2.104}$$

Then equations (2.82) and (2.103) reduce to

$$\tilde{u}'' = \tilde{t}\,\tilde{u} + 2\tilde{u}^3, \quad (') = \frac{d}{d\tilde{t}}, \tag{2.105}$$

and

$$\frac{d}{d\tilde{y}}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 2\tilde{u}' & \left(4\tilde{y}^2 + 2\tilde{u}^2 + \tilde{t}\right) + 4\tilde{y}\tilde{u} \\ -\left(4\tilde{y}^2 + 2\tilde{u}^2 + \tilde{t}\right) + 4\tilde{y}\tilde{u} & -2\tilde{u}' \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}. \tag{2.106}$$

Equations (2.104) give the scaling as

$$\frac{n}{N} = 1 + N^{-2/3}\left(\frac{2s_1^4}{T_c}\right)^{1/3}\tilde{t}, \tag{2.107}$$

$$\gamma_n^2 = 1 + N^{-1/3}\left(\frac{4T_c}{s_1}\right)^{1/3}\tilde{u}\,\cos 2n\pi\epsilon + O(N^{-2/3}), \tag{2.108}$$

$$\beta_n = 0 + N^{-1/3}\left(\frac{4T_c}{s_1}\right)^{1/3}\tilde{u}\,\cos(2n+1)\pi\epsilon + O(N^{-2/3}), \tag{2.109}$$

$$x = 2c_1 + N^{-1/3}\left(\frac{4T_c}{s_1}\right)^{1/3}\tilde{y}. \tag{2.110}$$

The Dyson integral kernel for the double scaling limit correlation functions is then:

$$\boxed{K(\tilde{y}_1, \tilde{y}_2) = \frac{f(\tilde{y}_1)g(\tilde{y}_2) - g(\tilde{y}_1)f(\tilde{y}_2)}{\tilde{y}_1 - \tilde{y}_2}.} \tag{2.111}$$

3. Nonlinear Hierarchy

3.1. **Basic Ansatz.** For $m = 1, 2, \ldots$, we consider the model critical density

$$\rho(x) = \frac{1}{2\pi T_c}(x - 2c_1)^{2m}\sqrt{4 - x^2}, \tag{3.1}$$

where

$$T_c = \frac{1}{2\pi}\int_{-2}^{2}(x - 2c_1)^{2m}\sqrt{4 - x^2}\, dx. \tag{3.2}$$

The corresponding polynomial $V(x)$ is such that

$$V'(x) = \frac{1}{T_c}\mathrm{Pol}\left[(x - 2c_1)^{2m}\sqrt{4 - x^2}\right], \tag{3.3}$$

where $\text{Pol}[f(x)]$ means a polynomial part of a function $f(x)$ at infinity. In particular,

$$m = 1: \quad V'(x) = \frac{1}{T_c}\left[x^3 - 4c_1 x^2 + 2c_2 x + 8c_1\right], \quad T_c = 1 + 4c_1^2; \tag{3.4}$$

$$m = 2: \quad V'(x) = \frac{1}{T_c}\left[x^5 - 8c_1 x^4 + (-2 + 24c_1^2)x^3 - 16c_1 c_2 x^2 + (-2 + 16c_1^4 - 48c_1^2)x\right.$$
$$\left. + 16(c_1 + 4c_1^3)\right], \quad T_c = 2 + 24c_1^2 + 16c_1^4, \tag{3.5}$$

and so on. In fact, our considerations will be very general and (3.1) is only an example. They hold for any density (1.10) which satisfies regularity conditions (1.18), (1.19) everywhere except one point c lying strictly inside one of the cuts, and such that as $z \to c$, $h(z) \sim C(x - c)^{2m}$, $C \neq 0$.

In the double scaling limit we define variables K, t and y as

$$K = N^{-1/(2m+1)}, \qquad \frac{n}{N} = 1 + K^{2m} s_1 t, \qquad x = 2c_1 + 2Ky. \tag{3.6}$$

Our ansatz for the orthogonal polynomials is the following:

$$\psi(n, x) = \cos{(n + 1/2)\pi\epsilon}\, f(t, y) - \sin{(n + 1/2)\pi\epsilon}\, g(t, y)$$
$$+ K\left[\cos{(n + 1/2)\pi\epsilon}\, f_1(t, y) - \sin{(n + 1/2)\pi\epsilon}\, g_1(t, y)\right.$$
$$\left. + \cos{3(n + 1/2)\pi\epsilon}\, \widetilde{f}(t, y) - \sin{3(n + 1/2)\pi\epsilon}\, \widetilde{g}(t, y)\right] + O(K^2), \tag{3.7}$$

[cf. (2.101)], and for the recurrence coefficients,

$$\gamma_n = 1 + K u(t) \cos 2n\pi\epsilon + O(K^2), \qquad \beta_n = 2K u(t) \cos{(2n + 1)\pi\epsilon} + O(K^2) \tag{3.8}$$

[cf. (2.70), (2.71)]. See also the work [PeS] where an intimately related ansatz for the recurrence coefficients was suggested in the case of a *symmetric* potential $V(x)$ in the circular ensemble. We substitute the ansatz into the 3-terms recursion relation,

$$x\psi(n, x) = \gamma_{n+1}\psi(n + 1, x) + \beta_n \psi(n, x) + \gamma_n \psi(n - 1, x) \tag{3.9}$$

and in the first order in K we obtain two systems of equations,

$$\partial_t \begin{pmatrix} f(t, y) \\ g(t, y) \end{pmatrix} = L \begin{pmatrix} f(t, y) \\ g(t, y) \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & y + u(t) \\ -y + u(t) & 0 \end{pmatrix} \tag{3.10}$$

(at frequency 1) and

$$\begin{pmatrix} \widetilde{f}(t, y) \\ \widetilde{g}(t, y) \end{pmatrix} = \frac{c_1 u(t)}{4s_1^2} \begin{pmatrix} f(t, y) \\ g(t, y) \end{pmatrix} \tag{3.11}$$

(at frequency 3).

3.2. **Differential System.** We would like to derive a differential equation in y,

$$\partial_y \begin{pmatrix} f(t, y) \\ g(t, y) \end{pmatrix} = D(t, y) \begin{pmatrix} f(t, y) \\ g(t, y) \end{pmatrix}. \tag{3.12}$$

We are looking for $D(t, y)$ in the form

$$D(t, y) = \begin{pmatrix} -A(t, y) & yB(t, y) + C(t, y) \\ yB(t, y) - C(t, y) & A(t, y) \end{pmatrix} \tag{3.13}$$

[cf. (2.103)], where A, B and C are even polynomials in y of the following degrees:

$$\deg A = 2m - 2, \quad \deg B = 2m - 2, \quad \deg C = 2m. \tag{3.14}$$

We will assume that C is a monic polynomial, so that $C = y^{2m} + \dots$. The general case can be reduced to this one by the change of variables, $t = \kappa \widetilde{t}$, $y = \frac{\widetilde{y}}{\kappa}$, $u(t) = \frac{\widetilde{u}(\widetilde{t})}{\kappa}$, which preserves the structure of the operator L in (3.10). The consistency condition of equations (3.10) and (3.12),

$$[D, L] = \partial_y L - \partial_t D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \partial_t D, \tag{3.15}$$

implies that

$$\partial_t B = 2A\,, \quad \partial_t C = 1 + 2uA\,, \quad \partial_t A = -2y^2 B + 2uC\,. \tag{3.16}$$

• Example: $m = 1$. According to (3.14), $A = a(t)$, $B = b(t)$, $C = y^2 + c(t)$. From the last equation in (3.16) we obtain that $b = u$ and then that

$$a = \frac{u'}{2}, \quad b = u, \quad c = t + \frac{u^2}{2} + t_0, \tag{3.17}$$

where t_0 is a free constant, and

$$\frac{u''}{2} = u^3 + 2(t + t_0)u, \tag{3.18}$$

the Painlevé II equation. By changing $t + t_0$ to t we can reduce it to $t_0 = 0$.

We would like to construct solutions to (3.16) for $m > 1$. To that end, define recursively functions $A_m(t, y)$, $B_m(t, y)$, $C_m(t, y)$ by the equations

$$C_{m+1} = y^2 C_m + f_m(u)\,, \tag{3.19}$$

$$B_{m+1} = y^2 B_m + R_m(u)\,, \tag{3.20}$$

$$A_{m+1} = y^2 A_m + \frac{1}{2}\,\partial_t R_m(u)\,, \tag{3.21}$$

where $R_m(u)$, $f_m(u)$ solve the recursive equations

$$R_{m+1}(u) = u f_m(u) - \frac{1}{4}\,\partial_{tt} R_m(u), \tag{3.22}$$

$$\partial_t f_m(u) = u \partial_t R_m(u), \qquad f_m(0) = 0, \tag{3.23}$$

with the initial data

$$A_0 = B_0 = 0, \quad C_0 = 1, \quad R_0(u) = u, \quad f_0(u) = \frac{u^2}{2}\,. \tag{3.24}$$

We solve recursively (3.22)-(3.24) as

$$R_1(u) = \frac{1}{2}u^3 - \frac{1}{4}\,u'', \qquad f_1(u) = \frac{3}{8}u^4 - \frac{1}{4}\,uu'' + \frac{1}{8}u'^2\,; \tag{3.25}$$

$$R_2(u) = \frac{3}{8}\,u^5 - \frac{5}{8}u^2 u'' - \frac{5}{8}\,uu'^2 + \frac{1}{16}\,u^{(4)}, \tag{3.26}$$

$$f_2(u) = \frac{5}{16}\,u^6 - \frac{5}{8}\,u^3 u'' - \frac{5}{16}\,u^2 u'^2 + \frac{1}{16}uu^{(4)} - \frac{1}{16}u' u''' + \frac{1}{32}u''^2\,; \tag{3.27}$$

$$R_3(u) = \frac{5}{16}\,u^7 - \frac{35}{32}u^4 u'' - \frac{35}{16}\,u^3 u'^2 + \frac{7}{32}\,u^2 u^{(4)} + \frac{7}{8}\,uu' u''' + \frac{21}{32}uu''^2 + \frac{35}{32}\,u'^2 u'' - \frac{1}{64}u^{(6)}, \tag{3.28}$$

and so on,

$$R_m(u) = \frac{(2m)!}{2^{2m}(m!)^2}\,u^{2m+1} + \ldots + \frac{(-1)^m}{2^{2m}}u^{(2m)}\,. \tag{3.29}$$

In addition,

$$A_1 = \frac{1}{2}\,u', \quad B_1 = u, \quad C_1 = y^2 + \frac{1}{2}\,u^2\,; \tag{3.30}$$

$$A_2 = \frac{1}{2}\,u'y^2 + \frac{3}{4}\,u^2 u' - \frac{1}{8}\,u''', \quad B_2 = uy^2 + \frac{1}{2}\,u^3 - \frac{1}{4}\,u'', \tag{3.31}$$

$$C_2 = y^4 + \frac{1}{2}\,u^2 y^2 + \frac{3}{8}\,u^4 - \frac{1}{4}\,uu'' + \frac{1}{8}\,(u')^2\,; \tag{3.32}$$

and so on. It is easy to check that the functions $A_m(t, y)$, $B_m(t, y)$, $C_m(t, y)$ defined by (3.19)-(3.24) solve the equations

$$\partial_t B_m = 2A_m, \quad \partial_t C_m = 2uA_m, \quad \partial_t A_m = -2y^2 B_m + 2uC_m - 2R_m(u). \tag{3.33}$$

Indeed, by (3.24) it holds for $m = 0$. Assume that it holds for some m. Then by (3.19)-(3.23) and (3.33),

$$\partial_t C_{m+1} = y^2 \partial_t C_m + \partial_t f_m(u) = 2y^2 u A_m + u \, \partial_t R_m(u) = 2u A_{m+1} \,, \tag{3.34}$$

$$\partial_t B_{m+1} = y^2 \partial_t B_m + \partial_t R_m(u) = 2y^2 A_m + \partial_t R_m(u) = 2 A_{m+1} \,, \tag{3.35}$$

$$\partial_t A_{m+1} = y^2 \partial_t A_m + \frac{1}{2} \, \partial_{tt} R_m(u) = y^2 \left[-2y^2 B_m + 2u C_m - 2 R_m(u) \right] + \frac{1}{2} \, \partial_{tt} R_m(u)$$

$$= -2y^2 \left[y^2 B_m + R_m(u) \right] + 2u \left[y^2 C_m + f_m(u) \right] - 2 \left(u f_m(u) - \frac{1}{4} \, \partial_{tt} R_m(u) \right)$$

$$= -2y^2 B_{m+1} + 2u C_{m+1} - 2 R_{m+1}(u) \,, \tag{3.36}$$

which proves (3.33) for $m+1$ and hence for all $m = 0, 1, 2, \ldots$. Comparing (3.16) with (3.33) we obtain that

$$A = A_m, \quad B = B_m, \quad C = t + C_m \tag{3.37}$$

solve equation (3.16), provided $u(t)$ is a solution of the equation

$$R_m(u) + tu = 0 \,. \tag{3.38}$$

The sequence of equations (3.38) for $m = 1, 2, \ldots$ forms a hierarchy of ordinary differential equations which is known as the Painlevé II hierarchy [Kit] (see also [Moo], [PeS]). We can formulate now the following result.

THEOREM **3.1.** *Define*

$$D_m(t, y) = \begin{pmatrix} -A_m(t, y) & y B_m(t, y) + C_m(t, y) \\ y B_m(t, y) - C(t, y) & +A_m(t, y) \end{pmatrix} \tag{3.39}$$

Then if $u(t)$ is a solution of equation (3.38), then the matrix

$$D(t, y) = \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} + D_m(t, y) \tag{3.40}$$

is a solution to (3.15). More generally, if $t_1, \ldots, t_m$ are arbitrary constants and $u(t)$ is a solution of the equation

$$\sum_{k=1}^{m} t_k R_k(u) + tu = 0, \tag{3.41}$$

then the matrix

$$D(t, y) = \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} + \sum_{k=1}^{m} t_k D_k(t, y) \tag{3.42}$$

is a solution to (3.15).

Remark: It can be shown that (3.42) is a *general* solution to equation (3.15).

The meaning of the constants $t_1, \ldots, t_m$ in (3.42) is the following. Observe that the differential equation in y, (3.12) describes the double scaling limit for a critical polynomial of degeneracy $2m$. In this case the space of transversal fluctuations to the manifold of critical polynomials has dimension m. The variables $t_1, \ldots, t_m$ serve as coordinates in the space of transversal fluctuations, and (3.42) gives the matrix describing the double scaling limit of the recurrence coefficients in the direction $\tau = (t_1, \ldots, t_m)$.

4. CONCLUSION

In this paper we considered critical polynomials which violate the regularity conditions at exactly one point, inside the support of the equilibrium measure. It is characterized by the degree $2m$ of degeneracy of the equilibrium density at the critical point. Our main results are the following:

- When $m = 1$, the infinite volume free energy exhibits the phase transition of the third order. This extends the result of [GW] to nonsymmetric critical polynomials.

- When $m = 1$, the double scaling limit of the recurrence coefficients is described, under a proper substitution, by the Hastings-McLeod solution to the Painlevé II differential equation. Before this result was known only for symmetric critical polynomials [DSS], [PeS] (for rigorous results see [BI2], [BDJ]).
- For $m > 1$, we derive a hierarchy of ordinary differential equations describing the double scaling limit of the recurrence coefficients.

5. APPENDIX. ONE USEFUL IDENTITY

In some cases it is convenient to consider a one-parameter family in (1.1),

$$V(z;T) = \frac{V(z)}{T}, \tag{5.1}$$

where $T > 0$ is the temperature, so that

$$\mu_N(dM;T) = Z_N(T)^{-1} \exp\left(-\frac{N}{T} \operatorname{Tr} V(M)\right) dM, \qquad Z_N(T) = \int_{\mathcal{H}_N} \exp\left(-\frac{N}{T} \operatorname{Tr} V(M)\right) dM. \tag{5.2}$$

Then $\rho(x)$ and $\omega(z)$ depend on T. The following identity is useful in many questions.

Proposition. *Assume that the number of cuts does not change in a neighborhood of a given $T > 0$. Then*

$$\frac{d}{dT}[T\omega(z)] = \frac{\prod_{j=1}^{q-1}(z - x_j)}{R^{1/2}(z)}, \tag{5.3}$$

where the numbers $b_j < x_j < a_{j+1}$, $j = 1, \ldots, q-1$, solve the equations,

$$\int_{b_k}^{a_{k+1}} \frac{\prod_{j=1}^{q-1}(x - x_j)}{R^{1/2}(x)} \, dx = 0, \quad k = 1, \ldots, q-1. \tag{5.4}$$

The neighborhood can be one-sided, then the derivative in T is also one-sided.

Proof. Equation (2.13) gives that

$$T\omega(z) = \frac{V'(z)}{2} - \frac{h(z)R^{1/2}(z)}{2}. \tag{5.5}$$

Since $V(z)$ does not depend on T,

$$\frac{d}{dT}[T\omega(z)] = -\frac{d}{dT}\frac{h(z)R^{1/2}(z)}{2}. \tag{5.6}$$

The function on the right can be written as

$$-\frac{d}{dT}\frac{Th(z)R^{1/2}(z)}{2} = \frac{P(z)}{R^{1/2}(z)}, \tag{5.7}$$

where $P(z)$ is a polynomial with real coefficients. Since

$$\frac{d}{dT}[T\omega(z)] = \frac{1}{z} + O(z^{-2}), \tag{5.8}$$

we obtain that

$$\frac{P(z)}{R^{1/2}(z)} = \frac{1}{z} + O(z^{-2}), \tag{5.9}$$

which shows that $P(z) = z^{q-1} + \ldots$. By (1.15),

$$\int_{b_j}^{a_{j+1}} \frac{h(x)R^{1/2}(x)}{2} \, dx = 0, \quad j = 1, \ldots, q-1. \tag{5.10}$$

By differentiting with respect to T we obtain that

$$\int_{b_j}^{a_{j+1}} \frac{P(x)}{R^{1/2}(x)} \, dx = 0, \quad j = 1, \ldots, q-1. \tag{5.11}$$

This is possible only if $P(x)$ has a zero in each interval $[b_j, a_{j+1}]$. Thus, (5.4) is proven.

As a corollary, from (5.6) we get that

$$\frac{d}{dT}\left[h(z)R^{1/2}(z)\right] = -\frac{2\prod_{j=1}^{q-1}(z-x_j)}{R^{1/2}(z)}\,.\tag{5.12}$$

This implies that

$$\frac{d}{dT}\ln\left[h(z)R^{1/2}(z)\right] = -\frac{2\prod_{j=1}^{q-1}(z-x_j)}{h(z)R(z)}\,.\tag{5.13}$$

Comparing the residue of the both sides at $z = a_k, b_k$ we obtain that

$$\frac{da_k}{dT} = \frac{4\prod_{j=1}^{q-1}(a_k-x_j)}{h(a_k)(a_k-b_k)\prod_{j\,:\,j\neq k}[(a_k-a_j)(a_k-b_j)]}\,,$$

$$\frac{db_k}{dT} = \frac{4\prod_{j=1}^{q-1}(a_k-x_j)}{h(b_k)(b_k-a_k)\prod_{j\,:\,j\neq k}[(b_k-a_j)(b_k-b_j)]}\,,\quad 1\leq k\leq q.\tag{5.14}$$

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