

**Analysis of directed flow from elliptic flow**N. Borghini,<sup>1,\*</sup> P. M. Dinh,<sup>2,†</sup> and J.-Y. Ollitrault<sup>2,‡</sup><sup>1</sup>*Service de Physique Théorique, CP 225, Université Libre de Bruxelles, B-1050 Brussels, Belgium*<sup>2</sup>*Service de Physique Théorique, CEA-Saclay, F-91191 Gif-sur-Yvette cedex, France*

(Received 10 April 2002; published 17 July 2002)

The directed flow of particles produced in ultrarelativistic heavy ion collisions at the CERN Super Proton Synchrotron and the Brookhaven Relativistic Heavy Ion Collider is so small that currently available methods of analysis are at the border of applicability. Standard two-particle and flow-vector methods are biased by large nonflow correlations. On the other hand, cumulants of four-particle correlations, which are free from this bias, are plagued by large statistical errors. Here, we present a new method based on three-particle correlations, which uses the property that elliptic flow is large at these energies. This method may also be useful at intermediate energies, near the balance energy where directed flow vanishes.

DOI: 10.1103/PhysRevC.66.014905

PACS number(s): 25.75.Ld, 25.75.Gz

**I. INTRODUCTION**

The azimuthal angles of particles emitted in a heavy ion collision are correlated with the direction of the impact parameter (reaction plane) of the two incoming nuclei. This correlation is characterized by the Fourier coefficients of the particle azimuthal distribution,  $v_n$  [1],

$$v_n \equiv \langle \cos[n(\phi - \Phi_R)] \rangle, \quad (1)$$

where  $n$  is an integer,  $\phi$  denotes the azimuthal angle of an outgoing particle,  $\Phi_R$  is the azimuth of the reaction plane, and angular brackets denote an average over events and over all particles in a given transverse momentum and rapidity window.

The first Fourier coefficient  $v_1$ , referred to as *directed flow*, is usually positive in the forward hemisphere for protons. This is because particles are deflected away from the target. This effect was first observed at Bevalac in 1984 [2], and reaches its maximum value at about 400 MeV per nucleon [3]. At much lower energies, below 100 MeV per nucleon,  $v_1$  becomes negative due to the attractive nuclear mean field [4,5]. At ultrarelativistic energies, it decreases: at the CERN Super Proton Synchrotron (SPS), it is of the order of 2% [6]. At the Brookhaven Relativistic Heavy Ion Collider (RHIC), it has not been measured yet. Accurate measurements of  $v_1$  are important since this quantity is a sensitive probe of nuclear matter properties [7]. There is an interesting prediction that this quantity may even become negative above SPS energies [8].

Standard methods for analyzing directed flow [9–11] are based on the assumption that all azimuthal correlations between outgoing particles are due to flow. When flow becomes too small, however, other sources of correlation (“nonflow” correlations) can no longer be neglected. Corre-

lations due to global momentum conservation are well known [12], and a systematic method to correct for this effect has recently been proposed [13]. However, many other effects are expected to produce nonflow correlations of the same order of magnitude, such as quantum Hanbury Brown–Twiss correlations [14] and resonance decays [15]. We have proposed new methods, based on higher-order correlations [16,17], which allow one to get rid of nonflow correlations. However, statistical errors on four-particle (or higher-order) correlations are larger, so that these methods may fail if flow is too small.

In this paper, we propose a new method that is free from these limitations. It allows one to obtain an estimate of  $v_1$  which is insensitive to nonflow correlations, with statistical errors usually not much larger than with standard techniques. This method is based on the observation that elliptic flow ( $v_2$ , second Fourier harmonic of the azimuthal distribution) is reasonably large in the situations considered where directed flow is hard to measure:  $v_2$  is roughly 3% at SPS [6], and even larger at RHIC [18]. Our method relies on the measurement of three-particle azimuthal correlations, which involve both  $v_1$  and  $v_2$ ,

$$\langle e^{i(\phi_1 + \phi_2 - 2\phi_3)} \rangle \simeq (v_1)^2 v_2, \quad (2)$$

where  $\phi_1$ ,  $\phi_2$ , and  $\phi_3$  denote azimuthal angles of three particles belonging to the same event, and the average runs over triplets of particles emitted in the collision, and over events. This equation yields  $(v_1)^2 v_2$ , and one then deduces  $v_1$  once  $v_2$  has been obtained from a separate analysis.

The idea of mixing two Fourier harmonics is not new. Similar three-particle correlations were already considered in Ref. [19]. They also underly measurements of elliptic flow relative to the event plane determined from directed flow [20]: in these analyses, one first obtains  $v_1$ , and then  $v_2$  from an equation similar to Eq. (2). In particular, this is the only method to obtain the sign of  $v_2$ , and it was used for this purpose at SPS [6,21]. Here we propose to do the other way round: first measure  $v_2$ , then obtain  $v_1$  from the three-particle correlation.

In Sec. II, we present in more detail the principle of the method, and we explain why it is better than other methods when directed flow is small. Then, in Sec. III, we describe

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the practical implementation of the method. This implementation makes use of two techniques already introduced in Refs. [17,22]: first, a cumulant expansion allows us to eliminate the effects of azimuthal asymmetries in the detector acceptance; second, the formalism of generating functions provides an elegant way of constructing three-particle correlations (and beyond), with a computer time which grows only linearly with the number of particles involved, while taking into account all possible particle triplets. A generalization to higher-order cumulants is presented in Sec. IV. Our results are discussed in Sec. V.

## II. PRINCIPLE AND ORDERS OF MAGNITUDE

In this section, we compare three methods of analyzing directed flow: the standard two-particle technique, the four-particle cumulant, and the three-particle mixed correlation, Eq. (2), which is the focus of this paper. In Sec. II A, we recall how  $v_1$  can be obtained from azimuthal correlations in various ways. In Secs. II B and II C, we estimate for each method the order of magnitude of errors due to nonflow correlations and finite statistics. Numerical estimates are given in Sec. II D, where we show that three-particle correlations provide the most (if not only) reliable way of analyzing directed flow at ultrarelativistic energies. In Sec. II E, we finally explain how to perform detailed measurement of  $v_1$  as a function of transverse momentum and rapidity with this method.

### A. Directed flow from azimuthal correlations

In a given collision, the azimuth of the reaction plane  $\Phi_R$  is unknown, so that the Fourier coefficients  $v_n$  of the azimuthal distribution, defined in Eq. (1), can only be obtained from azimuthal correlations between the outgoing particles. The simplest way is to use correlations between two particles [10], labeled 1 and 2, belonging to the same event,

$$\begin{aligned} \langle e^{i(\phi_1 - \phi_2)} \rangle &= \langle e^{i(\phi_1 - \Phi_R)} e^{i(\Phi_R - \phi_2)} \rangle \\ &= \langle e^{i(\phi_1 - \Phi_R)} \rangle \langle e^{i(\Phi_R - \phi_2)} \rangle \\ &= (v_1)^2, \end{aligned} \quad (3)$$

where, as in the previous equations, averages are taken over pairs of particles and over events. In going from the first to the second line, we have assumed that all azimuthal correlations are due to flow, i.e., that azimuthal angles relative to the reaction plane  $\phi_1 - \Phi_R$  and  $\phi_2 - \Phi_R$  are independent. The ‘‘standard’’ flow analysis [9] proceeds differently: one correlates the azimuth of an individual particle with a ‘‘flow vector’’ obtained by summing over many particles. This correlation is larger than the correlation between two individual particles, typically by a factor of  $\sqrt{M}$ , where  $M$  is the number of particles involved in the flow vector. However, the flow-vector method and the two-particle method are essentially equivalent, inasmuch as both suffer from the same limitations, due to nonflow correlations: the enhancement factor  $\sqrt{M}$  applies to the flow correlation as well as to nonflow

correlations, as was shown explicitly in the case of the correlation due to momentum conservation in Ref. [13].

It is straightforward to generalize Eq. (3) to higher-order correlations, such as the four-particle correlation,

$$\langle e^{i(\phi_1 + \phi_2 - \phi_3 - \phi_4)} \rangle = (v_1)^4, \quad (4)$$

where the average now involves all possible quadruplets of particles belonging to the same event. This result will be used below.

In this paper, we shall be concerned with yet another type of correlation, namely, a three-particle correlation which mixes the first two Fourier harmonics,

$$\begin{aligned} \langle e^{i(\phi_1 + \phi_2 - 2\phi_3)} \rangle &= \langle e^{i(\phi_1 - \Phi_R)} e^{i(\phi_2 - \Phi_R)} e^{2i(\Phi_R - \phi_3)} \rangle \\ &= (v_1)^2 v_2. \end{aligned} \quad (5)$$

If  $v_2$  is already known from a previous analysis, using the three-particle correlation Eq. (5) allows one to extract an estimate of  $v_1$ .

### B. Nonflow correlations

The above estimates were derived under the assumption that all azimuthal correlations are due to flow. However, there are also other, ‘‘nonflow,’’ contributions to azimuthal correlations due to various effects [11,14,15]. Let us recall the order of magnitude of these correlations by taking a simple example: if a resonance decays into  $k$  particles, these  $k$  particles will be strongly correlated by the decay kinematics. Now, the probability that  $k$  arbitrary particles seen in a detector originate from the same resonance scales like the total multiplicity  $M$  of the event like  $1/M^{k-1}$ . This is generally the order of magnitude of the genuine  $k$ -particle correlation due to nonflow effects. In particular, the two-particle nonflow correlation is of the order of  $1/M$ , so that Eq. (3) reads in fact

$$\langle e^{i(\phi_1 - \phi_2)} \rangle = (v_1)^2 + \mathcal{O}\left(\frac{1}{M}\right). \quad (6)$$

When  $v_1$  becomes smaller than  $1/\sqrt{M}$ , the resulting error on  $v_1$  due to the unknown nonflow term becomes as large as  $v_1$  itself. This is probably the case at SPS, as shown in Ref. [15], and *a fortiori* at RHIC energies.

A similar reasoning applies to higher-order correlations. In the case of the four-particle correlation, Eq. (4), it may for instance happen that two pions labeled 1 and 3 originate from the same  $\rho$  meson, while 2 and 4 come from another  $\rho$  meson. This gives a four-particle correlation of order  $\mathcal{O}(1/M)^2$ . However, these pairwise correlations can be subtracted from the measured four-particle correlation so as to isolate the genuine four-particle correlation. This is the principle of the cumulant expansion that was proposed in Refs. [16,17] to get rid of nonflow correlations in the flow analysis. The cumulant of the four-particle correlation is defined as

$$\begin{aligned} \langle\langle e^{i(\phi_1 + \phi_2 - \phi_3 - \phi_4)} \rangle\rangle & \\ \equiv \langle e^{i(\phi_1 + \phi_2 - \phi_3 - \phi_4)} \rangle &- \langle e^{i(\phi_1 - \phi_3)} \rangle \langle e^{i(\phi_2 - \phi_4)} \rangle \\ &- \langle e^{i(\phi_1 - \phi_4)} \rangle \langle e^{i(\phi_2 - \phi_3)} \rangle. \end{aligned} \quad (7)$$

According to the above discussion, the contribution of nonflow effects to this genuine four-particle correlation is of order  $1/M^3$ , much smaller than  $1/M^2$ . The contribution of flow follows in a straightforward way from Eqs. (3) and (4), so that one may finally write

$$\langle\langle e^{i(\phi_1+\phi_2-\phi_3-\phi_4)} \rangle\rangle = -(v_1)^4 + \mathcal{O}\left(\frac{1}{M^3}\right). \quad (8)$$

Finally, let us estimate nonflow contributions to the mixed three-particle correlation, Eq. (5). Unlike the four-particle correlation Eq. (4), this quantity does not receive any contribution from two-particle correlations, since quantities such as  $\langle e^{i(\phi_1+\phi_2)} \rangle$  or  $\langle e^{i(\phi_1-2\phi_3)} \rangle$  vanish by symmetry. The only nonflow correlation is the genuine three-particle correlation, of order  $1/M^2$ , and Eq. (5) becomes

$$\langle e^{i(\phi_1+\phi_2-2\phi_3)} \rangle = (v_1)^2 v_2 + \mathcal{O}\left(\frac{1}{M^2}\right). \quad (9)$$

In the following, we shall denote by  $v_1\{2\}$ ,  $v_1\{4\}$ , and  $v_1\{3\}$  the estimates of  $v_1$  obtained from Eqs. (6), (8), and (9), respectively, ignoring the nonflow term. Using these equations, one finds that the differences due to nonflow correlations between these estimates and the exact value  $v_1$  are of order

$$\begin{aligned} v_1\{2\} - v_1 &= \mathcal{O}\left(\frac{1}{Mv_1}\right), \\ v_1\{4\} - v_1 &= \mathcal{O}\left(\frac{1}{(Mv_1)^3}\right), \\ v_1\{3\} - v_1 &= \mathcal{O}\left(\frac{1}{(Mv_1)(Mv_2)}\right). \end{aligned} \quad (10)$$

As explained in Ref. [16], it is possible to measure  $v_n$  only if  $Mv_n \gg 1$ , and we assume throughout this paper that this condition holds both for  $v_1$  and  $v_2$ . Then, Eq. (10) shows that estimates of  $v_1$  from three- or four-particle correlations are much less biased by nonflow correlations than standard estimates from two-particle correlations.

### C. Statistical errors

In practice, the use of higher-order correlations is limited by statistical errors due to limited statistics. The various correlations encountered are quantities of the type  $(1/N)\sum_{j=1}^N \cos(\Delta\phi_j)$ , where the  $\Delta\phi_j$  are various combinations of the particle azimuths. If the  $\Delta\phi_j$  are independent and randomly distributed, the standard error on such a quantity is  $1/\sqrt{2N}$ .

In the case of two-particle correlations, Eq. (3), one can construct  $M(M-1)/2$  different pairs of particles in an event with multiplicity  $M$ , so that the total number of combinations is  $N \approx N_{\text{evts}} M^2/2$  for  $N_{\text{evts}}$  events. In the case of four-particle correlations, Eq. (4), one can construct  $M(M-1)(M-2) \times (M-3)/8$  independent quadruplets, so that  $N \approx N_{\text{evts}} M^4/8$ .

Finally, in the case of three-particle correlations, there are  $M(M-1)(M-2)/2$  independent triplets, so that  $N \approx N_{\text{evts}} M^3/2$ .

One thus obtains the following expressions for the relative statistical errors on the various estimates of  $v_1$ :

$$\begin{aligned} \frac{\delta v_1\{2\}}{v_1} &\approx \frac{1}{2\sqrt{N_{\text{evts}}}} \frac{1}{(v_1\sqrt{M})^2}, \\ \frac{\delta v_1\{4\}}{v_1} &\approx \frac{1}{2\sqrt{N_{\text{evts}}}} \frac{1}{(v_1\sqrt{M})^4}, \\ \frac{\delta v_1\{3\}}{v_1} &\approx \frac{1}{2\sqrt{N_{\text{evts}}}} \frac{1}{(v_1\sqrt{M})^2 (v_2\sqrt{M})}. \end{aligned} \quad (11)$$

These estimates are correct as long as  $v_1\sqrt{M}$  and  $v_2\sqrt{M}$  are not larger than unity, otherwise correlations due to flow must be taken into account in the calculation of statistical errors [17]. A more accurate formula for  $\delta v_1\{3\}$  will be given below in Sec. III D.

In this paper, we are interested in the situation where directed flow is very small,  $v_1\sqrt{M} \ll 1$ , which is the case when the standard flow analysis fails due to large nonflow correlations (see Sec. II B). In that case, one sees that the statistical uncertainty on  $v_1\{4\}$ , i.e., the estimate from the fourth-order cumulant, is much larger than the error on the standard estimate  $v_1\{2\}$ . On the other hand, if elliptic flow is significantly large,  $v_2\sqrt{M}$  is not much lower than unity: in such a case, the mixed-correlation technique provides an estimate of  $v_1$  with a statistical error of the same order as the standard analysis, but which is much less biased by nonflow correlations.

### D. Orders of magnitude at SPS and RHIC

Let us estimate numerically the various errors discussed above in a realistic situation. In practice, evaluating the magnitude of nonflow correlations requires a detailed modeling. Equations (10), strictly speaking, represent scaling laws rather than orders of magnitude. Detailed studies of various effects at SPS energies [13–15] show that the contribution of nonflow correlations to  $v_1$  obtained from the standard analysis (which is equivalent to  $v_1\{2\}$  above) is of the same order as  $v_1$  itself. For midcentral collisions,  $v_2$  is about 3% [6] and the number of detected particles  $M \sim 300$ . According to Eq. (10), one expects that using three-particle correlations (i.e.,  $v_1\{3\}$ ) will reduce systematic errors due to nonflow correlations by a factor of at least  $Mv_2$ , that is, a factor of 10. Thus one expects estimates using higher-order correlations  $v_1\{3\}$  and  $v_1\{4\}$  to be little biased by nonflow correlations, contrarily to  $v_1\{2\}$ .

Statistical errors can be estimated quantitatively using Eqs. (11). Realistic values at SPS are  $N_{\text{evts}} = 50\text{k}$  events,  $M = 300$  particles,  $v_1 \approx 2\%$  and  $v_2 \approx 3\%$ . One then obtains

$$\begin{aligned}
\frac{\delta v_1\{2\}}{v_1} &\simeq 2\%, \\
\frac{\delta v_1\{4\}}{v_1} &\simeq 16\%, \\
\frac{\delta v_1\{3\}}{v_1} &\simeq 4\%.
\end{aligned} \tag{12}$$

The relative statistical error on  $v_1\{4\}$  is too large to allow detailed measurements of  $v_1$  as a function of  $p_T$  or  $y$ . On the other hand, the uncertainty on  $v_1\{3\}$  is only twice larger than the error on  $v_1\{2\}$ , while we have seen above that the gain on the systematic error due to nonflow correlations is typically a factor of 10.

At RHIC,  $v_2$  is larger, typically 5% for midcentral collisions [18], so that measuring directed flow from elliptic flow ( $v_1\{3\}$ ) is even more appropriate than at SPS. This is reflected in both systematic errors due to nonflow correlations, and in statistical errors. While  $v_2$  is larger than at SPS,  $v_1$  is expected to be smaller if one extrapolates the decrease observed at SPS compared to the Brookhaven Alternating Gradient Synchrotron (AGS). The number of detected particles  $M$  can be estimated using the values of  $v_2$  and the event plane resolution given in Ref. [18], and is similar to that used above for SPS,  $M \simeq 300$ .

Since  $v_1$  is smaller than at SPS, one expects from Eq. (10) that the bias on  $v_1\{2\}$  from nonflow correlations will be even worse. However, the decrease in  $v_1$  may be partially compensated by the increase in  $v_2$ , so that the error on  $v_1\{3\}$  remains of the same order. With  $N_{\text{evts}} = 50\text{k}$  events,  $v_1 = 1\%$ ,  $v_2 = 5\%$ , one obtains the following statistical errors:

$$\begin{aligned}
\frac{\delta v_1\{2\}}{v_1} &\simeq 7\%, \\
\frac{\delta v_1\{4\}}{v_1} &\simeq 250\%, \\
\frac{\delta v_1\{3\}}{v_1} &\simeq 11\%,
\end{aligned} \tag{13}$$

where we have used the formula derived in Sec. III D for  $\delta v_1\{3\}$  [here  $v_2\sqrt{M}$  is of order unity, so that the third of Eqs. (11) no longer applies]. The statistical uncertainty is only 50% larger on  $v_1\{3\}$  than on  $v_1\{2\}$ , while once again the gain on the systematic error more than compensates for that loss.

These numerical estimates show clearly that three-particle mixed correlations offer the best compromise to measure  $v_1$  at ultrarelativistic energies. The corresponding estimate  $v_1\{3\}$  is much less affected by nonflow correlations than standard two-particle methods. In this respect, it shares the advantages of higher-order estimates such as  $v_1\{4\}$ . In addition,  $v_1\{3\}$  is much less limited by statistics than the latter, which would require millions of events at SPS and RHIC energies.

### E. Integrated flow, differential flow

As with other methods, the analysis proceeds in two steps. One first estimates the average value of  $v_1$  over phase space. This is done by averaging  $e^{i(\phi_1 + \phi_2 - 2\phi_3)}$  over all possible triplets, as in Eq. (2). However,  $v_1$  and  $v_2$  depend strongly on rapidity and transverse momentum (for instance,  $v_1$  has opposite signs in the backward and forward hemispheres). In practice, one therefore performs a *weighted* average, and Eq. (2) becomes

$$\langle w_1(1)w_1(2)w_2(3)e^{i(\phi_1 + \phi_2 - 2\phi_3)} \rangle = \langle w_1v_1 \rangle^2 \langle w_2v_2 \rangle, \tag{14}$$

where  $w_1$  and  $w_2$  are weights appropriate to directed flow and elliptic flow, respectively, which may be any function of the particle type, its transverse momentum  $p_T$  and rapidity  $y$ . In this equation,  $w_n(k)$  is a shorthand for  $w_n(p_{Tk}, y_k)$ . The right-hand side (rhs) of Eq. (14) naturally involves *weighted* averages  $\langle w_nv_n \rangle$ , rather than  $v_n$ .

The best choice for the weights is that which leads to the smallest statistical errors. Repeating the discussion in Sec. II C, one easily shows that this is done by maximizing  $\langle w_nv_n \rangle / \sqrt{\langle w_n^2 \rangle}$ . Therefore the best weight is the flow itself [16,23],  $w_n(p_T, y) = v_n(p_T, y)$ , where  $v_n(p_T, y)$  denotes the value of the flow in a small  $(p_T, y)$  bin. In practice, one can choose as a first guess the center-of-mass rapidity for directed flow,  $w_1 = y - y_{\text{CM}}$ , and the transverse momentum for elliptic flow  $w_2 = p_T$ , in regions of phase space covered by the detector acceptance.

Using the value of the integrated (and weighted) elliptic flow  $\langle w_2v_2 \rangle$  obtained from a separate analysis, one finally obtains the integrated directed flow  $\langle w_1v_1 \rangle$  from Eq. (14). Naturally, one must use the same weight  $w_2$  in the reference analysis that gives  $\langle w_2v_2 \rangle$  and in the mixed-correlation analysis.

The second step is to analyze differential flow, i.e., to obtain values of  $v_1$  as a function of transverse momentum  $p_T$  and/or rapidity  $y$ . For that purpose, one averages  $e^{i(\phi_1 + \phi_2 - 2\phi_3)}$  over all  $\phi_2$  and  $\phi_3$ , but restricts  $\phi_1$  to a given particle type in a particular  $(p_T, y)$  bin,

$$\langle w_1(2)w_2(3)e^{i(\phi_1 + \phi_2 - 2\phi_3)} \rangle = \langle w_1v_1 \rangle \langle w_2v_2 \rangle v_1(p_T, y). \tag{15}$$

Note that there is no weight for the ‘‘differential’’ particle labeled 1, for which we do not perform any phase space average. With the previously derived values of  $\langle w_2v_2 \rangle$  and  $\langle w_1v_1 \rangle$ , one obtains the differential flow  $v_1(p_T, y)$ . This differential flow  $v_1(p_T, y)$  will be denoted by  $v_1'$  in this paper, while we keep the notation  $v_1$  for the integrated value only.

### III. IMPLEMENTATION

In this section, we show how to analyze directed flow in practice using the three-particle correlation method. The method proposed here is a straightforward generalization of the one introduced in Ref. [17], which involves the formalism of cumulants and generating functions. One may believe at first sight that this formalism is a useless complication

here. But in fact, it provides an efficient and elegant solution to the following problems.

(1) Taking into account azimuthal asymmetries in the detector acceptance, which always exist, even if the detector has full azimuthal coverage.

(2) Eliminating correlations due to detector effects.

(3) Dealing with the combinatorics, i.e., averaging over all possible triplets.

We define the cumulants in Sec. III A. The generating functions used in analyzing integrated and differential flow are introduced in Sec. III B. Then we give interpolation formulas that can be used to extract the relevant cumulants from these generating functions (Sec. III C). Finally, in Sec. III D, we derive the standard statistical errors on both integrated and differential flow.

### A. Cumulants

Even if the detector has full azimuthal coverage, its acceptance is not perfectly isotropic, so that averages like  $\langle e^{i\phi} \rangle$  do not strictly vanish. A general way to take into account such effects consists in using cumulants. For instance, the cumulant associated with the two-particle correlation Eq. (3) is defined as

$$\langle\langle e^{i(\phi_1 - \phi_2)} \rangle\rangle \equiv \langle e^{i(\phi_1 - \phi_2)} \rangle - \langle e^{i\phi_1} \rangle \langle e^{-i\phi_2} \rangle. \quad (16)$$

If the detector is perfectly isotropic, the last term vanishes and the cumulant reduces to the two-particle correlation.

With a realistic detector, however, the cumulant isolates the physical correlation by subtracting the contribution of detector effects. Similarly, the cumulant of the three-particle correlation is defined as

$$\begin{aligned} \langle\langle e^{i(\phi_1 + \phi_2 - 2\phi_3)} \rangle\rangle &\equiv \langle e^{i(\phi_1 + \phi_2 - 2\phi_3)} \rangle - \langle e^{i(\phi_1 + \phi_2)} \rangle \langle e^{-2i\phi_3} \rangle \\ &\quad - \langle e^{i\phi_1} \rangle \langle e^{i(\phi_2 - 2\phi_3)} \rangle - \langle e^{i\phi_2} \rangle \langle e^{i(\phi_1 - 2\phi_3)} \rangle \\ &\quad + 2 \langle e^{i\phi_1} \rangle \langle e^{i\phi_2} \rangle \langle e^{-2i\phi_3} \rangle. \end{aligned} \quad (17)$$

As in the previous case, this isolates the genuine three-particle correlation from effects of detector inefficiencies, and from spurious correlations induced by detector effects.

If the acceptance is almost azimuthally symmetric, then all acceptance corrections are taken into account automatically by the cumulant expansion: in order to obtain the flow, one simply need replace the left-hand side (lhs) of Eq. (5) by the cumulant Eq. (17). If detector asymmetries are stronger, a multiplicative factor appears in the rhs of Eq. (5) relating the correlation to the flow. This correction is derived in the Appendix.

### B. Generating functions

Generating functions provide an elegant way of summing over all possible  $n$ -uplets in a given event. For a given event with  $M$  particles seen in the detector, we define the following real-valued function of two complex variables  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$ :

$$\begin{aligned} G(z_1, z_2) &= \prod_{j=1}^M \left[ 1 + \frac{w_1(j)}{M} (z_1^* e^{i\phi_j} + z_1 e^{-i\phi_j}) + \frac{w_2(j)}{M} (z_2^* e^{2i\phi_j} + z_2 e^{-2i\phi_j}) \right] \\ &= \prod_{j=1}^M \left[ 1 + \frac{w_1(j)}{M} [2x_1 \cos(\phi_j) + 2y_1 \sin(\phi_j)] + \frac{w_2(j)}{M} [2x_2 \cos(2\phi_j) + 2y_2 \sin(2\phi_j)] \right], \end{aligned} \quad (18)$$

where  $z_1^* \equiv x_1 - iy_1$  and  $z_2^* \equiv x_2 - iy_2$  are the complex conjugates of  $z_1$  and  $z_2$ , respectively, and  $w_1(j)$  and  $w_2(j)$  are the weights mentioned in Sec. II E. For sake of simplicity, we drop these weights from now on, unless otherwise stated.

The generating function  $G(z_1, z_2)$  generalizes the generating function  $G_n(z)$  introduced in Ref. [17]: the latter involved only one Fourier harmonic at a time, while we are now mixing two Fourier harmonics. More specifically, we recover the results of our earlier work in the limiting cases when either  $z_1$  or  $z_2$  vanishes:  $G(z, 0) = G_1(z)$ ,  $G(0, z) = G_2(z)$ .

Neither the generating function  $G(z_1, z_2)$ , nor the complex numbers  $z_1$ ,  $z_2$  have a physical meaning. They are a formal trick which allows us to extract azimuthal correlations to all orders, if necessary. This is done by averaging  $G(z_1, z_2)$  over events [we denote this average by  $\langle G(z_1, z_2) \rangle$ ], and then expanding in power series of  $z_1$ ,  $z_1^*$ ,  $z_2$ , and  $z_2^*$ . For instance, the coefficient of  $z_1^* z_2$  in the expansion is

$$\langle G(z_1, z_2) \rangle = \dots + \frac{z_1^* z_2}{M^3} \left\langle \sum_{j,k,l} e^{i(\phi_j + \phi_k - 2\phi_l)} \right\rangle + \dots \quad (19)$$

The sum runs over nonequivalent triplets, i.e.,  $j < k$ . The values of  $(j, k, l)$  are all different, so that autocorrelations are automatically avoided. Since there are  $M(M-1)(M-2)/2$  nonequivalent triplets, one obtains for large  $M$ ,

$$\langle G(z_1, z_2) \rangle = \dots + \frac{z_1^* z_2}{2} \langle e^{i(\phi_1 + \phi_2 - 2\phi_3)} \rangle + \dots \quad (20)$$

One recognizes here the three-particle correlation Eq. (2), averaged over triplets of particles and over events.

In our averaging over events, we have assumed that the number of particles  $M$  is the same for all events. Although the method can accommodate small fluctuations of the multiplicity  $M$ , this is a possible source of error [17]. Our rec-

ommendation is the following: in a given centrality bin, where the total number of detected particles  $M_{\text{tot}}$  fluctuates from one event to the other, choose a *fixed* number of particles  $M \leq M_{\text{tot}}$  to construct the generating function Eq. (18). Although using only a fraction of the total number of particles results in a loss in statistics, this avoids the uncontrolled effects due to fluctuations in the multiplicity.

Note that the generating function previously introduced in Refs. [17,22] was a function of only one complex variable  $z$ . Here, we need two independent variables  $z_1$  and  $z_2$  because we are mixing two different Fourier harmonics of the azimuthal distribution.

Generating functions are not only a convenient way of summing over all  $n$ -uplets of particles. They also allow one to construct easily cumulants of arbitrary order. The generating function of cumulants for *integrated* flow,  $\mathcal{C}(z_1, z_2)$ , is defined by [17,22]

$$\mathcal{C}(z_1, z_2) \equiv M(\langle G(z_1, z_2) \rangle^{1/M} - 1). \quad (21)$$

Expanding in power series of  $z_1$ ,  $z_1^*$ ,  $z_2$ , and  $z_2^*$ , one obtains cumulants of arbitrary order. In particular, the coefficient of  $z_1^{*2} z_2$  in the expansion is

$$\langle \mathcal{C}(z_1, z_2) \rangle = \dots + \frac{z_1^{*2} z_2}{2} \langle \langle e^{i(\phi_1 + \phi_2 - 2\phi_3)} \rangle \rangle + \dots \quad (22)$$

An explicit calculation using Eqs. (18) and (21) shows that the expression of  $\langle \langle e^{i(\phi_1 + \phi_2 - 2\phi_3)} \rangle \rangle$  thus defined coincides with Eq. (17) in the limit of large  $M$ .

While the generating function  $\mathcal{C}(z_1, z_2)$  is real valued, the cumulant  $\langle \langle e^{i(\phi_1 + \phi_2 - 2\phi_3)} \rangle \rangle$  defined by Eq. (22) is in general a complex number. However, the imaginary part results from detector effects and statistical fluctuations, and only the real part is relevant. For sake of brevity, we denote this cumulant of three-particle correlations by  $c\{3\}$  in the following:

$$c\{3\} \equiv \text{Re}(\langle \langle e^{i(\phi_1 + \phi_2 - 2\phi_3)} \rangle \rangle), \quad (23)$$

where  $\text{Re}$  denotes the real part. Restoring the weights, this cumulant gives an estimate of the weighted integrated directed flow, which we denote by  $\langle w_1 v_1 \{3\} \rangle$  [see Eq. (14)],

$$c\{3\} = \langle w_1 v_1 \{3\} \rangle^2 \langle w_2 v_2 \rangle, \quad (24)$$

where the integrated elliptic flow  $\langle w_2 v_2 \rangle$  comes from an independent analysis.

Let us now turn to differential flow. We shall denote by  $\psi$  the azimuth of the differential particle under study, and by  $v_1'$  its flow,  $v_1' \equiv \langle e^{i(\psi - \Phi_R)} \rangle$ , and call it a ‘‘proton’’ (although it can be any type of particle). In opposition, we call ‘‘pions’’ the particles used to estimate integrated flow.

The overall procedure in the analysis is quite similar to the analysis of integrated flow. We first introduce a generating function of the azimuthal correlations between the proton and the pions. It is given by the average value over *protons* of  $e^{i\psi} G(z_1, z_2)$ , where  $G(z_1, z_2)$  is evaluated for the event where the protons belong,

$$\langle e^{i\psi} G(z_1, z_2) \rangle = \langle e^{i\psi} \rangle + z_1 \langle e^{i(\psi - \phi_1)} \rangle + \dots \quad (25)$$

Note that in the averaging procedure, an event with two protons is counted twice, while an event with no proton does not contribute.

Then we define a generating function of the cumulants for *differential* flow,  $\mathcal{D}(z_1, z_2)$ , by [17,22]

$$\mathcal{D}(z_1, z_2) \equiv \frac{\langle e^{i\psi} G(z_1, z_2) \rangle}{\langle G(z_1, z_2) \rangle}, \quad (26)$$

where, in the denominator,  $\langle G(z_1, z_2) \rangle$  denotes an average over *all* events. The cumulants are the coefficients in the expansion in power series of this generating function. As in the case of integrated flow, they are in general complex numbers, but only the real part is relevant from the physical point of view. For instance, the coefficient of  $z_1^* z_2$  defines the cumulant of order 3 which we shall use to extract differential flow,

$$d\{3\} \equiv \text{Re}(\langle \langle e^{i(\psi + \phi_1 - 2\phi_2)} \rangle \rangle). \quad (27)$$

Note the similarity between this expression and Eq. (23). In fact, the cumulant  $d\{3\}$  shares the same features as  $c\{3\}$ : it is free from detector effects, and reflects physical three-particle correlations, due either to direct three-body correlations, or to flow. Restoring the weights, the relation between the cumulant and flow reads [see Eq. (15)]

$$d\{3\} = \langle w_1 v_1 \rangle \langle w_2 v_2 \rangle v_1' \{3\}. \quad (28)$$

We denote by  $v_1' \{3\}$  this estimate of  $v_1'$  obtained from three-particle correlations.

### C. Interpolating the cumulants

In this section, we show how to extract the cumulants for integrated and differential flows, Eqs. (23) and (27), numerically, from the computation of the generating function Eq. (18) for various values of  $z_1$  and  $z_2$ . We introduce the interpolation points  $(z_{1,p}, z_{2,q}) = (x_{1,p} + iy_{1,p}, x_{2,q} + iy_{2,q})$  with

$$\begin{aligned} x_{1,p} &= r_0 \cos\left(\frac{p\pi}{8}\right), & y_{1,p} &= r_0 \sin\left(\frac{p\pi}{8}\right), \\ x_{2,q} &= r_0 \cos\left(\frac{q\pi}{4}\right), & y_{2,q} &= r_0 \sin\left(\frac{q\pi}{4}\right), \end{aligned} \quad (29)$$

for  $p=0, \dots, 7$  and  $q=0, \dots, 3$ , and where  $r_0$  is a real number, which must be neither too large, otherwise the error due to higher-order terms in the power-series expansion of  $G(z_1, z_2)$  is large, nor too small, to avoid numerical errors.

To obtain the cumulants, one should for each event choose randomly  $M$  particles among the  $M_{\text{tot}}$  detected, and, with the particle azimuths  $\phi_j$  (and possibly with their transverse momenta and rapidities, if  $p_T$  and/or  $y$ -dependent weights are used) compute the generating function Eq. (18)

at the points, Eq. (29). Then one must average the values  $G(z_{1,p}, z_{2,q})$  over events, and calculate the generating function of cumulants, Eq. (21). We denote by  $C_{p,q}$  the values of the generating function  $\mathcal{C}(z_1, z_2)$  evaluated at the interpolation points (29),

$$C_{p,q} \equiv \mathcal{C}(z_{1,p}, z_{2,q}). \quad (30)$$

From these quantities, we then build

$$\begin{aligned} (C_p)_x &\equiv \frac{1}{4r_0} (C_{p,0} - C_{p,2}), \\ (C_p)_y &\equiv \frac{1}{4r_0} (C_{p,3} - C_{p,1}), \end{aligned} \quad (31)$$

which correspond to the real and imaginary parts of the derivative of  $\mathcal{C}(z_1, z_2)$  with respect to  $z_2$ . Finally, the third-order cumulant we are interested in,  $c\{3\}$ , is given by

$$\begin{aligned} c\{3\} &= \frac{1}{4r_0^2} [(C_0)_x - (C_1)_y - (C_2)_x + (C_3)_y \\ &\quad + (C_4)_x - (C_5)_y - (C_6)_x + (C_7)_y]. \end{aligned} \quad (32)$$

Consider now differential flow. We denote by  $D_{p,q}$  the values of the generating function  $\mathcal{D}(z_1, z_2)$  evaluated at the interpolation points (29),

$$D_{p,q} \equiv (D_{p,q})_x + i(D_{p,q})_y \equiv \mathcal{D}(z_{1,p}, z_{2,q}), \quad (33)$$

where in fact one only needs even values of  $p$ : the interpolation of the ‘‘differential’’ cumulant, which is only a second-order derivative, requires less points than the ‘‘integrated’’ cumulant, which is a derivative of third order. With the quantities Eq. (33), we then build

$$\begin{aligned} (D_p)_x &\equiv \frac{1}{4r_0} [(D_{p,0})_x - (D_{p,2})_x + (D_{p,1})_y - (D_{p,3})_y], \\ (D_p)_y &\equiv \frac{1}{4r_0} [(D_{p,0})_y - (D_{p,2})_y + (D_{p,3})_x - (D_{p,1})_x], \end{aligned}$$

and the cumulant  $d\{3\}$  is finally given by

$$d\{3\} = \frac{1}{4r_0} [(D_0)_x - (D_2)_y - (D_4)_x + (D_6)_y]. \quad (34)$$

Naturally, one may prefer using a different interpolation scheme. In any case, one should check that the final values,  $c\{3\}$  and  $d\{3\}$ , do not depend on the parameters introduced in the interpolation: here, one should try different values of  $r_0$ , and check the stability of the result.

#### D. Statistical errors

The standard deviation of the cumulant  $c\{3\}$ , Eq. (23), is given by

$$(\delta c\{3\})^2 = \frac{\langle w_1^2 \rangle^2 \langle w_2^2 \rangle}{2M^3 N_{\text{evts}}} (2 + 4\chi_1^2 + 2\chi_2^2 + 4\chi_1^2 \chi_2^2 + \chi_1^4). \quad (35)$$

In this equation,  $\chi_1$  and  $\chi_2$  are the resolution parameters appropriate for directed flow and elliptic flow, respectively,

$$\chi_n \equiv \frac{\langle w_n v_n \rangle}{\sqrt{\langle w_n^2 \rangle}} \sqrt{M}. \quad (36)$$

From Eq. (35), and the relation between  $c\{3\}$  and the flow Eq. (24) one can calculate the statistical error on  $\langle w_1 v_1 \rangle$ . If one neglects the statistical error on  $v_2$ , one recovers the simple estimate (11) with unit weights, in the limit where  $\chi_1 \ll 1$  and  $\chi_2 \ll 1$ . A more careful estimate must take into account the error on  $\langle w_2 v_2 \rangle$  in deriving the error on  $\langle w_1 v_1 \rangle$ . Statistical errors on the cumulant are Gaussian. Since the relation between the cumulant and the integrated directed flow, Eq. (24), is quadratic rather than linear, the resulting error bars on  $\langle w_1 v_1 \rangle$  are asymmetric when the error is large, as discussed in detail in Appendix D of Ref. [17].

Similarly, the standard deviation of the differential cumulant  $d\{3\}$ , Eq. (27), is given by

$$(\delta d\{3\})^2 = \frac{\langle w_1^2 \rangle \langle w_2^2 \rangle}{2M^2 N'} (1 + \chi_1^2)(1 + \chi_2^2), \quad (37)$$

where  $N'$  denotes the number of protons used in the differential flow analysis. Using the relation (28) between this cumulant and the differential flow,  $v_1'\{3\}$ , we easily obtain the statistical error on  $v_1'\{3\}$ . Since the analysis is done in a narrow phase space bin, one may reasonably assume here that the error on  $d\{3\}$  is dominated by the error on  $v_1'\{3\}$ , which yields

$$\delta v_1'\{3\} = \frac{1}{\sqrt{2N'}} \frac{\sqrt{1 + \chi_1^2}}{\chi_1} \frac{\sqrt{1 + \chi_2^2}}{\chi_2}. \quad (38)$$

This result can be understood simply in two limiting cases. When both  $\chi_1$  and  $\chi_2$  are large compared to unity, the reaction plane  $\Phi_R$  can be reconstructed accurately. The error on  $\langle \cos(\psi - \Phi_R) \rangle$  estimated with  $N'$  values of  $\psi$  is then  $1/\sqrt{2N'}$ , to which the error reduces for  $\chi_1 \gg 1$ ,  $\chi_2 \gg 1$ . In the opposite case  $\chi_1 \ll 1$ ,  $\chi_2 \ll 1$ , comparing Eqs. (35) and (37), one finds for unit weights  $\delta v_1'\{3\}/\delta v_1\{3\} = \sqrt{MN_{\text{evts}}/N'}$ : errors scale like the inverse square root of the number of particles involved, and the determination of the integrated flow  $v_1$  involves a total number of  $MN_{\text{evts}}$  particles while the differential flow  $v_1'$  involves  $N'$  particles.

#### IV. HIGHER ORDERS

In the preceding section, we have studied the three-particle correlation, which is the lowest-order nontrivial result obtained with the generalized generating function. Higher orders can also be derived in a straightforward way. Expanding the generating function of cumulants  $\mathcal{C}(z_1, z_2)$ , defined in Eq. (21), in power series of  $z_1$ ,  $z_1^*$ ,  $z_2$ , and  $z_2^*$ , yields cumulants of arbitrary order,

$$\mathcal{C}(z_1, z_2) \equiv \sum_{j,k,l,m} \frac{z_1^{*j} z_1^k z_2^{*l} z_2^m}{j!k!l!m!} \langle\langle e^{i[\phi_1 + \dots + \phi_j - \phi_{j+1} - \dots - \phi_{j+k} + 2(\phi_{j+k+1} + \dots + \phi_{j+k+l} - \phi_{j+k+l+1} - \dots - \phi_{j+k+l+m})]} \rangle\rangle. \quad (39)$$

The only relevant cumulants are those which are invariant under a simultaneous shift of all azimuthal angles  $\phi_j \rightarrow \phi_j + \alpha$ . Other cumulants vanish except for statistical fluctuations and detector effects. According to Eq. (19), this shift is equivalent to the change of variables  $z_1 \rightarrow z_1 e^{-i\alpha}$ ,  $z_2 \rightarrow z_2 e^{-2i\alpha}$ . The only terms in Eq. (39) which are invariant under this transformation are those with  $j+2l=k+2m$ . Any of these cumulants can be used to extract the flow. Until now, we have explored only a few possibilities: the case considered in Sec. III is  $(j,k,l,m) = (2,0,0,1)$ ; the cumulants used in Ref. [17] to extract  $v_1$  and  $v_2$  are those with  $j=k$ ,  $l=m$

$=0$  and those with  $j=k=0$ ,  $l=m$  (denoted by  $c_1\{2j\}$  and  $c_2\{2l\}$ , respectively).

We now derive the relations between cumulants of arbitrary orders and the flow coefficients  $v_1$  and  $v_2$ . For this purpose, we compute the average value of  $G(z_1, z_2)$  in the presence of flow. We first average for a given orientation of the reaction plane  $\Phi_R$ . For an arbitrary particle, by definition of  $v_n$ , we may write  $\langle e^{in\phi} | \Phi_R \rangle = v_n e^{in\Phi_R}$ . Replacing in Eq. (19), dropping the weights for simplicity, and neglecting all nonflow correlations, we obtain

$$\begin{aligned} \langle G(z_1, z_2) | \Phi_R \rangle &= \left( 1 + \frac{z_1^* v_1 e^{i\Phi_R} + z_1 v_1 e^{-i\Phi_R} + z_2^* v_2 e^{2i\Phi_R} + z_2 v_2 e^{-2i\Phi_R}}{M} \right)^M \\ &\simeq \exp(z_1^* v_1 e^{i\Phi_R} + z_1 v_1 e^{-i\Phi_R}) \exp(z_2^* v_2 e^{2i\Phi_R} + z_2 v_2 e^{-2i\Phi_R}). \end{aligned} \quad (40)$$

The next step is to average over  $\Phi_R$ . We make use of the following formula:

$$\exp(z^* e^{i\phi} + z e^{-i\phi}) = \sum_{q=-\infty}^{+\infty} e^{-iq\phi} \left( \frac{z}{|z|} \right)^q I_q(2|z|), \quad (41)$$

where  $I_q$  is the modified Bessel function of order  $q$ . Applying this identity to each term of Eq. (40) and integrating over  $\Phi_R$ , one obtains

$$\begin{aligned} \langle G(z_1, z_2) \rangle &= \int_0^{2\pi} \langle G(z_1, z_2) | \Phi_R \rangle \frac{d\Phi_R}{2\pi} \\ &= \sum_{q=-\infty}^{+\infty} \left( \frac{z_1^{*2} z_2}{|z_1|^2 |z_2|} \right)^q I_{2q}(2|z_1|v_1) I_q(2|z_2|v_2). \end{aligned} \quad (42)$$

In the limiting cases  $z_1=0$  or  $z_2=0$ , only the term  $q=0$  contributes to the sum in the rhs, and we recover  $\langle G(z_1, 0) \rangle = I_0(2|z_1|v_1)$ ,  $\langle G(0, z_2) \rangle = I_0(2|z_2|v_2)$ , already derived in Ref. [17].

Expanding the generating function of cumulants,  $\mathcal{C}(z_1, z_2) \simeq \ln \langle G(z_1, z_2) \rangle$ , in powers of  $z_1$ ,  $z_1^*$ ,  $z_2$ ,  $z_2^*$ , and identifying with Eq. (39), one obtains the relations between the various cumulants and flow. As expected, the only non-vanishing terms are those that satisfy the condition  $j+2l=k+2m$  derived above, and the corresponding cumulants are proportional to  $(v_1)^{j+k} (v_2)^{l+m}$ , with an integer multiplicative constant depending on the values of  $j, k, l, m$ . To order  $z_1^{*2} z_2^* z_2^2$ , for instance, one obtains

$$\langle\langle e^{i(\phi_1 + \phi_2 - 2\phi_3 + 2\phi_4 - 2\phi_5)} \rangle\rangle = -(v_1)^2 (v_2)^3. \quad (43)$$

If  $v_2$  is measured independently, this equation yields an estimate of  $v_1$  from five-particle correlations, which we denote by  $v_1\{5\}$ . The statistical error on  $v_1\{5\}$  is not much larger than the error on the earlier three-particle estimate  $v_1\{3\}$  if  $v_2$  is large enough (more precisely, if the resolution parameter  $\chi_2 \sim v_2 \sqrt{M}$  is larger than unity). These higher orders may be useful at RHIC, not at SPS where  $v_2$  is too small. Cumulants with  $j+k>2$  in Eq. (39) involve higher powers of  $v_1$  and are useless in the situation we are interested in, namely small values of  $v_1$ .

## V. DISCUSSION

In the preceding sections, we have presented a new method for analyzing directed flow ( $v_1$ ), through the help of an independent measurement of elliptic flow ( $v_2$ ). It allows one to measure both integrated and differential flow, the value of integrated flow being used in the differential analysis.

Our method relies on a study of *three*-particle correlations. Unlike standard methods, based on two-particle correlations, it is not biased by two-particle nonflow correlations, which are an important bias at ultrarelativistic energies. This can be checked experimentally by studying  $v_1$  near midrapidity: while standard two-particle estimates  $v_1\{2\}$  generally do not cross zero, especially if they are not corrected for momentum conservation [13], our estimate  $v_1\{3\}$  should naturally vanish at midrapidity, as expected.

In experiments where analyses of directed flow are al-



ready available, it would be interesting to compare these standard estimates from two-particle correlations,  $v_1\{2\}$ , with our new estimate from three-particle correlations  $v_1\{3\}$ . If they are in agreement (within statistical error bars), it is a good hint that they indeed coincide with the true directed flow. If they differ, it is instructive to study the centrality dependence of the product  $M(v_1\{3\}^2 - v_1\{2\}^2)$ , where  $M$  is the event multiplicity. If the difference between  $v_1\{2\}$  and  $v_1\{3\}$  is due to two-particle nonflow correlations, this product should be approximately constant [25] (remember that two-particle nonflow correlations scale as  $1/M$ .) If the product differs significantly from a constant, then another explanation must be looked for: the difference between  $v_1\{2\}$  and  $v_1\{3\}$  may be due to fluctuations, either fluctuations of the impact parameter within a given centrality class of events [24], or, more interestingly, physical fluctuations of the flow event by event.

The price to pay for eliminating nonflow effects is an increase in statistical errors, compared to standard two-particle methods. However, this increase is moderate, a factor of 2 or less at SPS and RHIC. This new method is thus much less statistics demanding than those based on correlations between four (or more) particles. All in all, three-particle correlations seem to be the most appropriate way to measure  $v_1$  when it is small, and especially if  $v_2$  is strong: near the balance energy, at SPS, RHIC, and the forthcoming experiments at LHC.

### ACKNOWLEDGMENTS

We thank Marek Gaździcki, Art Poskanzer, Herbert Ströbele, Sergei Voloshin, and Alexander Wetzler for discussions. N.B. acknowledges support from the “Actions de Recherche Concertées” of “Communauté Française de Belgique” and IISN-Belgium.

### APPENDIX: ACCEPTANCE CORRECTIONS

In the case of a realistic detector, with a nonisotropic acceptance, the relations between the cumulants and flow, Eqs. (24) and (28), no longer hold. In this appendix, we derive the relevant modifications taking into account the detector acceptance.

In what follows, we assume that the various classes of events analyzed, for instance, the different centrality bins, are determined with an *independent* detector (e.g., a zero degree calorimeter), which has a full azimuthal coverage (at least approximately). This is meant to make sure that the centrality assigned to a given event is not strongly correlated to the orientation of its reaction plane, which would bias the sample of events used in the flow analysis.

To describe a detector, we introduce its acceptance/ efficiency function  $A(j, \phi, p_T, y)$ , which is the probability that a particle of type  $j$  with azimuth  $\phi$ , transverse momentum  $p_T$ , and rapidity  $y$  be detected [17,22]. Obviously,  $A(j, \phi, p_T, y)$  will vary from a detector to another, and a “perfect” detector corresponds to  $A(j, \phi, p_T, y) = 1$  for every particle type in the whole phase space. In practice,  $A(j, \phi, p_T, y)$  is proportional to the number of hits in a  $(\phi, p_T, y)$  bin: to obtain its

shape for a given detector and for each  $(p_T, y)$  bin, one only has to count the number of hits in each  $\phi$  bin while reading the data to analyze them, and in the end divide by the maximal number encountered. The Fourier coefficients of the acceptance function are defined by

$$A_n(j, p_T, y) \equiv \int_0^{2\pi} A(j, \phi, p_T, y) e^{-in\phi} \frac{d\phi}{2\pi}. \quad (\text{A1a})$$

These differential coefficients can be integrated, with appropriate weighting and a sum over the various types of particles used for the flow analysis, so as to describe the “integrated” acceptance of the detector,

$$a_n[w] = \frac{\sum_j \int w(j, p_T, y) A_n(j, p_T, y) dp_T dy}{\sum_j \int w(j, p_T, y) A_0(j, p_T, y) dp_T dy}. \quad (\text{A1b})$$

Note our introducing the weights  $w(j, p_T, y)$ , which are of course the same as in Eq. (18), either  $w_1(j)$  or  $w_2(j)$ . In the following, we assume that both weights are equal, so that there is only one set of  $a_n$  coefficients. If two different weights are used—a weight  $w_1$  that maximizes  $v_1$  and the weight  $w_2$  that was used to derive the reference  $v_2$ —one should keep track of the two different sets of coefficients  $a_n[w_1]$ ,  $a_n[w_2]$  in the calculation that we now sketch.

To compute the contribution of flow to the cumulant  $c\{3\}$ , we follow the same procedure as in Ref. [17]. We first average the generating function Eq. (18) over events with the same azimuth of the reaction plane  $\Phi_R$ ; then we average over  $\Phi_R$ . For simplicity, we neglect nonflow correlations in the derivation. Denoting by  $\langle x | \Phi_R \rangle$  the average of a quantity  $x$  for fixed  $\Phi_R$ , the average value of  $e^{in\phi}$  for a fixed  $\Phi_R$  is given by [17]

$$\langle e^{in\phi} | \Phi_R \rangle = a_n^* + \sum_{p \neq 0} (a_{p-n} - a_p a_n^*) v_p e^{ip\Phi_R}, \quad (\text{A2})$$

where  $a_n^* = a_{-n}$  is the complex conjugate of  $a_n$ . Thus, a nonisotropic acceptance mixes the various flow harmonics. In the case of a perfect acceptance, Eqs. (A1) show that all coefficients  $a_n$  vanish, except  $a_0 = 1$ , and the identity (A2) reads  $\langle e^{in\phi} | \Phi_R \rangle = v_n e^{in\Phi_R}$ , which follows in a straightforward way from Eq. (1).

Inserting the average value, Eq. (A2) for  $n=1$  and  $n=2$  in Eq. (18), one obtains the generating function averaged over events with the same orientation of the reaction plane,  $\langle G(z_1, z_2) | \Phi_R \rangle$ . The latter must then be averaged over  $\Phi_R$ , then one computes the cumulants using Eqs. (21) and (39), keeping only the real part. In particular, the third cumulant reads

$$\begin{aligned} c\{3\} = & \text{Re}((1 - |a_2|^2)(1 - |a_1|^2)^2 \\ & + 2[(a_1 - a_1^* a_2)^2 (a_2^* - a_1^{*2}) + (1 - |a_1|^2) a_3 \\ & - a_1 a_2]^2 + (a_4 - a_2^2)(a_2^* - a_1^{*2})^2) v_1^2 v_2, \end{aligned} \quad (\text{A3})$$

instead of Eq. (24), where we have assumed that only  $v_1$  and  $v_2$  are nonvanishing. When the detector is perfect, one recovers Eq. (24). Even if the detector is not perfect, but nevertheless does not have too bad an acceptance, the factor in front of  $v_1^2 v_2$  in Eq. (A3) will remain close to 1, since the correction terms are at least quadratic in the  $a_n$  coefficients. Thus, Eq. (24) remains a good approximation, except for detectors with a very bad azimuthal coverage.

The contribution of flow to the ‘‘differential’’ cumulant  $d\{3\}$ , Eq. (27), can be calculated along the same lines. In that calculation, one may assume that integrated and differential flows are measured using two different detectors: e.g., a large acceptance detector for integrated flow, and a smaller one, but with better particle identification or  $p_T$  determination, for differential flow. We thus denote by  $A'(j, \psi, p_T, y)$  the corresponding acceptance function and by  $A'_k(j, p_T, y)$  its Fourier coefficients defined as in Eq. (A1a). The differential acceptance coefficients  $a'_k$  are then defined as in Eq. (A1b), without the weights and the summation over  $j$  (since one usually measures the differential flow of identified particles) and with the integration over  $p_T$  and  $y$  restricted to the phase-space region under interest (typically, one integrates over  $p_T$  or  $y$ , so as to obtain  $v'_1$  as a function of  $y$  or  $p_T$ , respectively).

The average value over protons in the numerator of Eq. (26) is then computed in two steps, first averaging over protons detected in events with the same orientation of the reaction plane, then averaging over  $\Phi_R$ . The denominator of Eq. (26),  $\langle G(z_1, z_2) | \Phi_R \rangle$ , is the same as in the calculation of  $c\{3\}$ . Finally,  $d\{3\}$  is given by the coefficient of  $z_1^* z_2$  in the expansion in power series of  $\mathcal{D}(z_1, z_2)$ ,

$$\begin{aligned} d\{3\} = & \text{Re}[(1 - |a_1|^2)(1 - |a_2|^2) + (a_1 - a_1^* a_2)^2 a_2'^* \\ & + |a_3 - a_1 a_2|^2 + (a_2^* - a_1'^2)(a_4 - a_2^2) a_2'^*] v'_1 v_2 \\ & + \text{Re}[(1 - |a_1|^2)(a_3 - a_1 a_2) a_3'^* \\ & + (a_2^* - a_1'^2)(a_1 - a_1^* a_2) a_1'] v_2^2 v_1^2. \end{aligned} \quad (\text{A4})$$

A nonisotropic acceptance will cause interference between the various differential flow harmonics: the measurement of directed differential flow  $v'_1$  is perturbed by the elliptic differential flow  $v'_2$ . It is worth noting that as soon as the acceptance of the detector used for integrated flow is perfect, Eq. (A4) reduces to Eq. (28).

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[25] Similarly, the product  $M(v_n\{4\}^2 - v_n\{2\}^2)$  may be used to probe the possible difference between a four-particle flow estimate  $v_n\{4\}$  derived with the method proposed in Ref. [17] and an estimate  $v_n\{2\}$  obtained with some two-particle method.