

# Non-compact Calabi–Yau Manifolds and Localized Gravity

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## Abstract

We study localization of gravity in flat space in superstring theory. We find that an induced Einstein-Hilbert term can be generated only in four dimensions, when the bulk is a non-compact Calabi-Yau threefold with non-vanishing Euler number. The origin of this term is traced to  $R^4$  couplings in ten dimensions. Moreover, its size can be made much larger than the ten-dimensional gravitational Planck scale by tuning the string coupling to be very small or the Euler number to be very large. We also study the width of the localization and discuss the problems for constructing realistic string models with no compact extra dimensions.

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## 1. Introduction

String models with large extra dimensions have led to many interesting phenomenological developments in the recent years [1]. In such models, gauge degrees of freedom are usually localized on D-branes while gravity, described by closed strings, propagates in the bulk. Moreover, by tuning the size of the latter, one can achieve a large hierarchy between the string and four-dimensional (4d) Planck scales. In fact, the hierarchy problem is traded in this framework for the existence of large compact dimensions, with size much larger than the string length.

On the other hand, models with localized gravity have not yet a clear realization in the context of string theory [2,3,4,5,6,7,8]. Indeed, while largely inspired by stringy developments and having used many string-theoretic techniques, these models generally have not been seen as arising from string theory. Moreover from stringy point of view they often appear to be rather ad hoc. Thus, string derivation of models with localized gravity not only remains as one of the important problems, but may be helpful in establishing the scales and other data that are required otherwise as an input. A particularly attractive possibility is when the bulk is non compact which avoids the problem of fixing the moduli associated to the size of the compactification manifold.

Since curved space is always difficult to handle in string theory, here we concentrate on flat space with gravity localized on a subspace of the bulk (for reasons that will become clear later, we intentionally avoid using the term “brane”), and we shall demonstrate how it can be realized in string theory. The model was introduced in [4,5] and is based on considering simultaneously Einstein-Hilbert (EH) actions in  $D = 4 + n$  and  $d = 4$  dimensions:

$$M^{2+n} \int d^{4+n}x \sqrt{g} \mathcal{R}_{(4+n)} + M_P^2 \int d^4x \sqrt{g} \mathcal{R}_{(4)} \quad ; \quad M_P^2 = M^{2+n} r_c^n, \quad (1.1)$$

with  $M$  and  $M_P$  the respective Planck scales. Depending on the (possibly independent) values of the two gravitational scales, a crossover parameter controls the regime when the effective gravity is the lower- or higher-dimensional one.

In the case of co-dimension one bulk ( $n = 1$ ) and  $\delta$ -function localization, it is easy to see that the crossover scale is  $r_c$ . Indeed, for distances smaller than  $r_c$ , the graviton propagator on the “brane” exhibits four-dimensional behavior with Planck constant  $M_P$ , while at large distances it acquires a five-dimensional fall-off with Planck constant  $M$ . On the other hand, in the presence of non-zero brane thickness  $w$ , a new crossover length-scale,  $R_c = (wr_c)^{1/2}$ , seems to appear. Below this scale the graviton propagator again becomes five-dimensional with an effective larger Planck constant  $M_* = M(r_c/w)^{1/6}$  [7].

The situation changes drastically for more than one non-compact bulk dimensions,  $n > 1$ , due to the ultraviolet properties of the higher-dimensional theories. Thus, in the

limit of zero thickness, the Newton's law is always four-dimensional on the “brane”, while in the presence of a non-zero  $w$  there is only one crossover length-scale,  $R_c$ :

$$R_c = w \left( \frac{r_c}{w} \right)^{\frac{n}{2}}, \quad (1.2)$$

above which one obtains a higher dimensional behavior [7].

In this work, we study the possible stringy origin of two, one higher and one lower dimensional, EH terms of the type (1.1), realizing the idea of localized gravity. Clearly, string theory has a ten-dimensional EH term, and thus, the question is how the lower-dimensional part arises, in how many dimensions, and what are the parameters involved. A particularly important point, which is necessary for the phenomenological viability of this scenario, is that the strength of the lower (four) dimensional term, identified with the Planck mass, must be much stronger than the higher dimensional gravitation scale  $M$ .

It is well-known that the multi-graviton scattering in string theory can generate higher-derivative couplings in curvature [9]. These generally go under the name of  $R^4$  couplings and are well-studied in ten-dimensional flat space  $M_{10}$  [10,11,12,13,14]. So one may wonder if similarly, in certain backgrounds of string theory, localized EH terms can be also generated. As we will show, this indeed happens in type II superstring on  $M_4 \times X_6$ , where  $M_4$  is the four-dimensional Minkowski space and  $X_6$  is a non-compact Ricci flat six-dimensional manifold with non-vanishing Euler number (the compact case has been discussed in [15]). We will first derive the main features using a simple reasoning, based on the analysis of the structure of the  $R^4$  couplings, and then we will confirm the results by performing explicit string computations. An interesting aspect of our analysis is that localization of the EH term is possible only in four dimensions,<sup>1</sup> while in the non-compact case we argue that it comes entirely from the type II closed string sector.

Our paper is organized as follows. In section 2, we discuss the  $R^4$  couplings and show how the localization arises in the context of the effective field theory. We will also present the outline of our results. Sections 3 and 4 (and corresponding appendices A and B) are devoted to string calculations. In section 3, we compute the corrections to the Planck mass from open strings and show that they vanish in the decompactification limit (at least for supersymmetric vacua). In section 4, we compute the corrections from closed strings and we find a universal contribution localized in four dimensions and proportional to the Euler number of the internal space. In section 5, we analyze the width of the localized terms. Finally, in section 6, we discuss the problems and the conditions for constructing realistic string models with no compact extra dimensions and localized gravity.

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<sup>1</sup> Actually, in the strong coupling limit, localization first occurs in five dimensions, as we will argue below by lifting the effective action to M theory.

## 2. Low-energy effective action and outline of the results

In string theory, corrections to the two-derivative EH action are in general very restrictive. For instance, in the heterotic string, they vanish to all orders in perturbation theory [16]. On the other hand, in type II theories, they are constant (moduli independent) and receive contributions only from tree and one loop level (at least for supersymmetric backgrounds) [15]. In this work, we will show that they actually describe localized terms in four dimensions, which therefore survive in the non-compact (decompactification) limit. Finally, in type I theory, there are moduli dependent corrections generated by open strings [17,18], but as we will show in section 3, they vanish in the decompactification limit.

Below, we describe the corrections in type II theories from the effective field theory point of view. In ten dimensions, the type II effective action including the tree-level and one-loop terms is given by [10,12,14,15]:

$$\frac{1}{(2\pi)^7 l_s^8} \int_{M_{10}} \left( e^{-2\phi} \mathcal{R}_{(10)} + \frac{2\zeta(3)}{3 \cdot 2^7} l_s^6 e^{-2\phi} (t_8 t_8 - \frac{1}{8} \epsilon_8 \epsilon_8) R^4 + \frac{2\pi^2 l_s^6}{3^2 \cdot 2^7} (t_8 t_8 \pm \frac{1}{8} \epsilon_8 \epsilon_8) R^4 + \dots \right),$$

where  $l_s = M_s^{-1}$  is the string length scale,  $\phi$  is the dilaton field determining the string coupling  $g_s = e^{\langle \phi \rangle}$ , and  $\mathcal{R}_{(10)}$  is the ten-dimensional curvature scalar. The rank 8 tensor  $t_8$  is defined as  $t_8 M^4 \equiv -6(\text{tr} M^2)^2 + 24 \text{tr} M^4$ ,  $\epsilon_8$  is the rank 8 totally antisymmetric tensor, and the  $+/-$  sign in the last term corresponds to the type IIA/B theory. Here, we have dropped CP-odd terms proportional to  $t_8 \epsilon_8$ . Using differential form notations, we can rewrite the above action as:

$$\begin{aligned} & \frac{1}{(2\pi)^7 l_s^8} \int_{M_{10}} e^{-2\phi} \mathcal{R}_{(10)} + \frac{1}{3(4\pi)^7 l_s^2} \int_{M_{10}} \left( 2\zeta(3) e^{-2\phi} + \frac{2\pi^2}{3} \right) t_8 t_8 R^4 \\ & - \frac{1}{3(4\pi)^7 l_s^2} \int_{M_{10}} \left( 2\zeta(3) e^{-2\phi} \mp \frac{2\pi^2}{3} \right) R \wedge R \wedge R \wedge R \wedge e \wedge e + \dots \end{aligned} \quad (2.1)$$

As already mentioned, we are interested in considering these couplings on a six-dimensional non-compact Calabi-Yau (CY) manifold  $X_6$ , and thus take a background  $M_4 \times X_6$ . The discussion in this section is heuristic and is concerned mostly with analyzing the structure of the various terms in (2.1). In particular, we ignore here issues connected with  $X_6$  being non-compact and more importantly having boundaries. We will see in section 4 how these points get settled by string calculations.

Due to the fact that spacetime is a product of two manifolds, most of the terms in the  $t_8 t_8 R^4$  part drop out and at any rate do not contribute to the Einstein-Hilbert action in four dimensions but to  $R^2$  terms [15]. In principle, on a six-manifold a cosmological constant  $\int_{X_6} t_8 t_8 R^4$  could be induced, but for a CY threefold this contribution vanishes. Indeed, we recall that  $\int_{X_6} (t_8 t_8 - \frac{1}{8} \epsilon_8 \epsilon_8) R^4 = 0$  due to the existence of a covariantly constant

spinor on Ricci-flat Kähler background with  $SU(3)$  holonomy [19]. Since on six dimensional manifolds  $\int_{X_6} \epsilon_8 \epsilon_8 R^4 = 0$ , there is no contribution to the four dimensional cosmological constant from  $t_8 t_8 R^4$ .

Essentially we have to worry only about contributions from the “geometric” part,  $\int R^4 \wedge e^2$ . Here the discussion closely follows [15], and due to the “internal” part being a total derivative the result is practically identical to the case where  $X_6$  is compact:

$$\frac{1}{(2\pi)^7 l_s^8} \int_{M_4 \times X_6} e^{-2\phi} \mathcal{R}_{(10)} + \frac{\chi}{3(4\pi)^7 l_s^2} \int_{M_4} (-2\zeta(3)e^{-2\phi} \pm 4\zeta(2)) \mathcal{R}_{(4)}. \quad (2.2)$$

A number of conclusions (confirmed by string calculations) can be reached by looking closely at (2.2):

- ▷ Localization requires  $X_6$  to have a non-zero Euler characteristic  $\chi \neq 0$ . Actually,  $\chi$  counts the difference between the numbers of  $\mathcal{N} = 2$  vector multiplets and hypermultiplets:  $\chi = \pm 4(n_V - n_H)$  in type IIA/B (where the graviton multiplet counts as one vector).<sup>2</sup>
- ▷ The structure of the localized  $\mathcal{R}_{(4)}$  term coming from the closed string sector is universal, independent of the background geometry and dependent only on the internal topology.
- ▷ It is a matter of simple inspection to see that if one wants to have a localized EH term in less than ten dimensions, namely something linear in curvature, with non-compact internal space in all directions, the only dimension where this is possible is four.

A brief comment on the last item. We are not attempting to discuss here from first principles why a localized EH action should be preferable to say  $R^2$  gravity. Phenomenologically speaking, this is clear enough. Here we simply pursue the goal of obtaining localized gravity of EH-type, without a priori fixing the number of dimensions, and we see that it is possible only in four.

The next question is to study the conditions for which the localized term becomes much more important than the bulk (ten-dimensional) EH action. It is interesting that this is indeed the case in the weak coupling limit, where the one-loop contribution in (2.2) can be ignored and the relevant scale of the localized four-dimensional (4d) term  $M_s/g_s$  is much larger than the corresponding 10d scale  $M_s/g_s^{1/4}$ . Moreover, the 4d contribution can be further enhanced by a large Euler number  $\chi$ . Note that  $\chi$  should be negative in order to obtain the correct sign for the gravity kinetic terms.<sup>3</sup>

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<sup>2</sup> Note that in the non-compact limit, the Euler number can in general split in different singular points of the internal space, giving rise to different localized terms. This is clear from the orbifold examples that we discuss in more detail in section 4. To simplify our discussion, in the following we consider the simplest case where  $\chi$  is concentrated on one singular point.

<sup>3</sup> The Planck mass receives opposite contributions from vector multiplets and hypermultiplets.

What about the strong coupling limit? Since type IIB theory remains invariant under S-duality, we do not expect to find anything new in this limit. In type IIA on the other hand, the strong coupling limit is taking us to M theory. By lifting the action (2.1) in eleven dimensions and converting everything to the M theory frame, we get

$$\begin{aligned} & \frac{1}{2(2\pi l_M)^9} \int_{M_{11}} \mathcal{R}_{(11)} + \frac{1}{(4\pi)^8 \cdot 3} \int_{M_{10} \times S^1} \left( \frac{2\zeta(3)}{R_{11}^3} + \frac{4\zeta(2)}{l_M^3} \right) t_8 t_8 R^4 \\ & - \frac{1}{(4\pi)^8 \cdot 3} \int_{M_{10} \times S^1} \left( \frac{2\zeta(3)}{R_{11}^3} - \frac{4\zeta(2)}{l_M^3} \right) R \wedge R \wedge R \wedge R \wedge e \wedge e \wedge e + \dots \end{aligned} \quad (2.3)$$

where  $l_M$  is the 11d Planck scale and  $R_{11}$  is the radius of the eleventh dimension with  $M_{11} = M_{10} \times S^1$ . Considering now a background  $M_5 \times X_6$ , with  $M_5 = M_4 \times S^1$  and  $X_6$  a non-compact Calabi-Yau, and taking the large radius limit  $R_{11} \rightarrow \infty$  (string strong coupling), we find the action:

$$\frac{1}{2(2\pi l_M)^9} \int_{M_5 \times X_6} \mathcal{R}_{(11)} + \frac{\chi}{36(4\pi)^5 l_M^3} \int_{M_4 \times S^1} \mathcal{R}_{(5)}. \quad (2.4)$$

As before, essentially we had to worry only for contributions of the “geometric” part,  $\int R^4 \wedge e^3$  (the other geometric piece from eleven dimensions  $\int C_3 \wedge X_8(R)$  [20,21] is not important for our purposes and has been ignored). Thus, as a consequence of the power-dependence on  $R_{11}$  in (2.3), a simultaneous localization of gravity in four and five dimensions is prevented. Weak coupling localizes gravity in four dimensions, while we find a five-dimensional localization in type IIA strong coupling limit. However, in this case, the strength of the 5d localized term is given by the same scale as the eleven-dimensional one  $l_M^{-1}$  and can be enhanced only by considering large  $\chi$ . Therefore, in the following we will concentrate on the ten-dimensional string theory case (2.2), in the weak coupling limit, and we come back in M theory only in the last section.

In open superstring models one can think of diagrams that may in principle induce an Einstein-Hilbert term in six dimensions with  $\mathcal{N} = 1$  supersymmetry. But in the next section we will show that the coefficient of such a term vanishes by the tadpole cancellation condition, and thus no such term is generated. This is a bit surprising since only massless six-dimensional states contribute to the amplitudes [17].

### 3. Planck mass corrections in open string models

As mentioned above, a simple inspection of worldsheets with boundaries suggests that in type I theory one-loop corrections to the Planck mass could be generated already in six dimensions. In fact, the moduli dependence of such corrections in four dimensions were computed in [17,18] for  $\mathcal{N} = 2$  supersymmetric compactifications on  $T^2 \times K3$  and were found to depend only on the  $T^2$  moduli and to be proportional to an index given by the

difference of  $\mathcal{N} = 2$  vector multiplets and hypermultiplets. In this section, we compute the three-graviton amplitude in order to determine the complete form of the corrections and find whether the result decompactifies to a finite non zero contribution in six dimensions. The rules for the normalizations and contractions are the same as in [17].

Only the even-spin structure can contribute to the Planck mass corrections. At the two derivative level, we have to consider the contractions involving four fermions from the following insertions of three graviton vertex operators with corresponding spacetime momenta  $k_i$ :

$$\begin{aligned} \text{at position } z_1 &: (\partial_z x^\mu + \frac{i}{2} k_1 \cdot \psi \psi^\mu) (\bar{\partial}_{\bar{z}} x^\nu - \frac{i}{2} k_1 \cdot \bar{\psi} \bar{\psi}^\nu) e^{i k_1 \cdot x} : (z_1, \bar{z}_1) \\ \text{at position } z_2 &: (\partial_z x^\mu + \frac{i}{2} k_2 \cdot \psi \psi^\mu) (\bar{\partial}_{\bar{z}} x^\nu - \frac{i}{2} k_2 \cdot \bar{\psi} \bar{\psi}^\nu) e^{i k_2 \cdot x} : (z_2, \bar{z}_2) \\ \text{at position } z_3 &: (\partial_z x^\mu + \frac{i}{2} k_3 \cdot \psi \psi^\mu) (\bar{\partial}_{\bar{z}} x^\nu - \frac{i}{2} k_3 \cdot \bar{\psi} \bar{\psi}^\nu) e^{i k_3 \cdot x} : (z_3, \bar{z}_3) \end{aligned}$$

where  $x$  are the spacetime coordinates while  $\psi$  and  $\bar{\psi}$  their left and right 2d fermionic superpartners, depending on the worldsheet positions  $z_i$  and  $\bar{z}_i$ . The possible contractions are:

$$\begin{aligned} A &= \langle \psi(z_1) \psi(z_2) \rangle_s^2 \langle \partial x(z_3) \bar{\partial} x(\bar{z}_2) \rangle \langle \bar{\partial} x(\bar{z}_1) \bar{\partial} x(\bar{z}_3) \rangle \\ &\quad - \langle \psi(z_1) \tilde{\psi}(\bar{z}_2) \rangle_s^2 \langle \partial x(z_3) \partial x(z_2) \rangle \langle \bar{\partial} x(\bar{z}_3) \bar{\partial} x(\bar{z}_1) \rangle + c.c. \\ B &= \langle \psi(z_1) \psi(z_2) \rangle_s^2 \langle \partial x(z_3) \bar{\partial} x(\bar{z}_1) \rangle \langle \bar{\partial} x(\bar{z}_2) \bar{\partial} x(\bar{z}_3) \rangle \\ &\quad - \langle \psi(z_1) \tilde{\psi}(\bar{z}_2) \rangle_s^2 \langle \partial x(z_3) \bar{\partial} x(\bar{z}_1) \rangle \langle \partial x(z_2) \bar{\partial} x(\bar{z}_3) \rangle + c.c. \end{aligned} \tag{3.1}$$

with a corresponding tensorial structure  $T_{A/B}$  in matrix notation. After summation over all permutations of the external states, one finds:

$$T_A = T_B = k_1 \cdot k_2 \text{tr}(\zeta_1 \zeta_3 \zeta_2) - k_1 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_1 \cdot k_2 + \text{perm}, \tag{3.2}$$

with  $\zeta_i^{\mu\nu}$  being the graviton polarization tensors. This is the linearization of the kinetic term of the graviton  $\mathcal{R}_{(4)}$ .

We remind the reader that in order to compute the amplitude correctly, the zero mode (lattice) parts need to be expressed in terms of the open string proper-time  $t$ , while the quantum fluctuations are functions of the closed string proper-time  $\tau$ . This is a consequence of computing the correlators on the double covering of the surface. Skipping the tensorial structure (see appendix A for notations and conventions), we can present any of the three amplitudes of interest ( $\sigma = \mathcal{A}, \mathcal{M}, \mathcal{K}$  for the annulus, Möbius strip and Klein bottle, respectively) in the form

$$\begin{aligned} A_\sigma &= -\frac{2^9}{(4\pi)^2} \int d^2 z_1 d^2 z_2 d^2 z_3 \int_0^\infty \frac{dt}{t^3} \frac{P^{(2)}(t)}{\eta^2(t)} \times \sum_{s=2,3,4} (-1)^{s-1} \frac{\theta_s^2(\tau)}{\eta^4(\tau)} Z_{s,\sigma}^{int}(\tau) \\ &\quad \times (\langle \psi(z_1) \psi(z_2) \rangle_s^2 \langle \partial x(z_3) \bar{\partial} x(\bar{z}_2) \rangle \langle \bar{\partial} x(\bar{z}_1) \bar{\partial} x(\bar{z}_3) \rangle \\ &\quad - \langle \psi(z_1) \tilde{\psi}(\bar{z}_2) \rangle_s^2 \langle \partial x(z_3) \partial x(z_2) \rangle \langle \bar{\partial} x(\bar{z}_3) \bar{\partial} x(\bar{z}_1) \rangle + c.c.), \end{aligned}$$

where  $P^{(2)}(t)/\eta^2(t)$  is the  $T^2$  momentum partition function,  $Z_{s,\sigma}^{int}$  is the contribution of the internal  $\mathcal{N} = 4$  superconformal theory describing  $K3$ , and  $\theta_s^2/\eta^4$  is the oscillator contribution of the bosonic and fermionic coordinates of  $\mathcal{M}^4 \times T^2$  dependent on the even spin structures  $s$ .

Using the following identity valid for  $K3$  models [17]

$$\frac{1}{2} \sum_{s=2,3,4} (-1)^{s-1} \theta_s^2(\tau|u) Z_{s,\sigma}^{int} = \theta_1^2(\tau|u) \times Z_{s=1,\sigma}^{int}, \quad (3.3)$$

the even spin structure sum and the fermionic contractions yield:

$$A_\sigma = -\frac{2^6}{\pi^2} Z_{s=1,\sigma}^{int} \int_0^\infty \frac{dt}{t^3} P^{(2)}(t) \times \int d^2 z_1 d^2 z_2 d^2 z_3 \times (\langle \partial x(z_3) \bar{\partial} x(\bar{z}_2) \rangle \langle \bar{\partial} x(\bar{z}_1) \bar{\partial} x(\bar{z}_3) \rangle - \langle \partial x(z_3) \partial x(z_2) \rangle \langle \bar{\partial} x(\bar{z}_3) \bar{\partial} x(\bar{z}_1) \rangle + c.c.). \quad (3.4)$$

Note that  $Z_{s=1,\sigma}^{int}$  is an index in the odd spin structure  $s = 1$ ,  $t$ -independent, determined from the massless content of the theory in six dimensions. Using now the identity (5.7) of [17], that for a periodic function on the covering torus  $\mathcal{T}$  of the open surface

$$\int_\sigma \partial_w f(w) - \partial_w f(I_\sigma(w)) = \int_{\mathcal{T}} \partial_w f(w) = 0,$$

with  $I_\sigma$  the corresponding  $\mathbb{Z}_2$  involution, the amplitude reduces to

$$\begin{aligned} A_\sigma &= -\frac{2^7}{\pi^2} \times Z_{s=1,\sigma}^{int} \int_0^\infty \frac{dt}{t^3} P^{(2)}(t) \int d^2 z_1 d^2 z_2 d^2 z_3 \times \left( \frac{\pi}{4\tau_2} \right)^2 \\ &= -Z_\sigma^{int} \int_0^\infty \frac{dt}{t^3} \times \tau_2 \eta^4(\tau) \times \frac{P^{(2)}(t)}{\eta^4(\tau)}. \end{aligned} \quad (3.5)$$

Here, we used that in terms of the closed string proper-time  $\tau = \tau_1 + i\tau_2$ , all diagrams  $\sigma$  have the same volume  $\tau_2/2$ .

Next, we convert the amplitude to the transverse closed string channel with the following sequence of changes of variables for each diagram (see [22,23,24] and Appendix A):

$$\begin{aligned} \sigma = \mathcal{K} : \quad \tau &= 2it \rightarrow -\frac{1}{\tau} = i\ell \\ \sigma = \mathcal{A} : \quad \tau &= \frac{it}{2} \rightarrow -\frac{1}{\tau} = i\ell \\ \sigma = \mathcal{M} : \quad \tau &= \frac{1+it}{2} \rightarrow -\frac{1}{\tau} \rightarrow -\frac{1}{\tau} + 2 \rightarrow \left( \frac{1}{\tau} - 2 \right)^{-1} = -\frac{1}{2} + \frac{i}{2t} = i\ell - \frac{1}{2} \end{aligned} \quad (3.6)$$

The one-loop correction to the Planck mass takes the form

$$\delta = -T_2 \int_0^\infty d\ell \left[ \frac{1}{8} W^{(2)} \left( \frac{\ell}{2} \right) Z_{\mathcal{A}}^{int} + 2W^{(2)} \left( \frac{\ell}{2} \right) Z_{\mathcal{M}}^{int} + 8W^{(2)}(2\ell) Z_{\mathcal{K}}^{int} \right] \quad (3.7)$$



where  $T_2$  is the volume of the two-torus,  $W^{(2)}$  is the winding sum of  $T^2$  (obtained by Poisson resumming  $P^{(2)}(t)$ ), and it is understood that the internal partition function is restricted to the odd spin structure  $s = 1$  (which we dropped for simplicity) and thus  $Z_{\mathcal{A},\mathcal{M},\mathcal{K}}^{int}$  are numbers. The divergence for  $\ell \rightarrow \infty$  is given by

$$\delta_{divergent} = -T_2 \left[ \frac{1}{8}Z_{\mathcal{A}}^{int} + 2Z_{\mathcal{M}}^{int} + 8Z_{\mathcal{K}}^{int} \right] \int^{\infty} d\ell \ell \quad (3.8)$$

where we used that in the  $\ell \rightarrow \infty$  limit  $W^{(2)} \rightarrow 1$ . In appendix A we explain how the tadpole cancellation conditions [22,23,24] imply that the prefactor of the divergence vanishes and the amplitude is finite.

On the other hand, the decompactification limit of (3.5) to six dimensions is given by precisely the same expression (3.8). Therefore, there are no corrections to the six-dimensional Planck mass consistently with the interpretation that all contributions come from the localized fixed points.<sup>4</sup> It follows that the total correction to the EH action is given only by the four dimensional contribution, whose  $U$ -dependence was analyzed in [17]:

$$\delta = -\frac{Z_2(U, \bar{U})}{T_2} \left[ \frac{1}{2}Z_{\mathcal{A}}^{int} + \frac{1}{2}Z_{\mathcal{M}}^{int} + 2Z_{\mathcal{K}}^{int} \right], \quad (3.9)$$

where  $U = U_1 + iU_2$  is the complex structure (shape) modulus of  $T^2$ , and the function  $Z_2$  is given by:

$$Z_2(U, \bar{U}) = \sum_{(m,n) \neq (0,0)} \frac{U_2^2}{|m + nU|^4} = 2\zeta(4)U_2^2 + \pi\zeta(3)U_2^{-1} + \mathcal{O}(\exp(-U_2)).$$

Note that in ref. [17] only the derivative with respect to the  $U$ -modulus of the correction to the Planck mass was obtained, namely:

$$4U_2^2 \partial_U \partial_{\bar{U}} \delta = -2 \frac{Z_2(U, \bar{U})}{T_2} \left[ \frac{1}{2}Z_{\mathcal{A}}^{int} + \frac{1}{2}Z_{\mathcal{M}}^{int} + 2Z_{\mathcal{K}}^{int} \right]. \quad (3.10)$$

This relation follows trivially from (3.9) using the property that the non-holomorphic Eisenstein series  $Z_s(U, \bar{U})$  is an eigenfunction of the  $Sl(2, \mathbb{Z})$ -Laplacian with eigenvalue  $s(s-1)$ . Thus, our analysis completely fixed the arbitrariness of possible zero modes of the Laplacian, since we showed that the integration constant ( $U$ -independent piece)

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<sup>4</sup> Strictly speaking, in the non-compact case, tadpole cancellation is not required. However, here we define non-compact spaces as decompactification limits of compact ones. In the orbifold limits of  $K3$  only twisted states contribute to the correction to the Planck mass, for which the tadpole condition has to be imposed even in the non-compact cases.

of obtaining (3.9) from (3.10) is vanishing after imposing the global tadpole cancellation condition.

The square bracket of (3.9) contains the information about the matter content of the  $K3$ -model, that we will denote  $\chi_I$  since this quantity counts the number of  $\mathcal{N} = 2$  vector-multiplets minus the number of hyper-multiplets, in analogy with the Euler number  $\chi$  of the Calabi-Yau manifold in type IIA compactifications.

One may also study the five dimensional limit by considering a square  $T^2$  torus with radii  $R_{1,2}$  and taking the limit  $R_1 \rightarrow \infty$ . In this case  $T_2 = R_1 R_2$  and  $U = iR_1/R_2$ . Expanding the modular function  $Z_2(U, \bar{U})$ , the Planck mass correction  $\delta$  becomes:

$$\delta = -\frac{\pi}{2} \chi_I R_1 \times \left( 2\zeta(3) \frac{1}{R_1^3} + 4\zeta(2) \frac{\pi}{15} \frac{1}{R_2^3} + \mathcal{O}(\exp(-U_2)) \right) \quad (3.11)$$

and leads to a localized term in five dimensions:

$$\sim -\chi_I \times \frac{1}{R_2^3} \int d^5x \sqrt{g^{(5)}} \mathcal{R}_{(5)}.$$

It is interesting to observe that this result reproduces the one found in the M theory context (2.4), upon the identification  $R_{11} = R_1$  and  $l_M = R_2$ . At the same time, the first term proportional to  $\zeta(3)$  in (3.11) reproduces the subleading contribution proportional to  $1/R_{11}^3$  in the second line of eq. (2.3). Notice also the relative positive sign between the two contributions of (3.11), since we are in type I compactifications, corresponding to the type IIB choice of (2.2).<sup>5</sup>

One can also ask the question whether a localized EH term can be generated already at the disk level, in analogy to the gauge kinetic terms. In fact, it is known that gauge couplings in orientifold models are in general given (to lowest order) as linear combinations of the dilaton  $e^{-\phi}$  and the various twisted moduli  $m$  (blowing up modes), which are closed string excitations localized at the orbifold fixed points:  $1/g_a^2 = e^{-\phi} + s_a m$ , with  $s_a$  calculable constants [25]. In the decompactification limit,  $e^{-\phi}$  leads to the usual 10d kinetic terms (actually  $(p+1)$ -dimensional for a  $p$ -brane), while the terms proportional to  $m$  yield additional contributions localized, say, at the origin of the internal space. These terms can be computed by studying the one-loop infrared divergence of the annulus amplitude in the closed string channel, generated by the propagation of the massless twisted states. Following this method for gravity, one should look for one-loop infrared divergences in the closed string channel of an amplitude involving at least two gravitons with two spacetime momenta each. A simple inspection of such amplitudes, analyzed for instance in [18], shows however the absence of power divergences, at least for  $\mathcal{N} = 2$  supersymmetric compactifications. It follows that there are no perturbative open string contributions that give rise to localized graviton kinetic terms in six dimensions.

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<sup>5</sup> A more detailed analysis is however needed to understand the precise numerical factors entering in this identification.

## 4. Planck mass corrections in type II orbifolds models

We return now to the contributions of closed type II strings propagating in non-compact Calabi-Yau (CY) threefolds, that were outlined in section 2, in the context of the effective field theory. As we pointed out, the localized term becomes dominant at the weak coupling limit; it is determined at the string tree-level from the four-loop beta-function of the two-dimensional sigma model [26], and is proportional to the Euler number of the CY manifold.

The aim of this and the following section is to study in detail the localization properties of the EH term in the context of string theory, and derive in particular the relevant width which determines the crossover scale (1.2). Since it is difficult to work in a generic Calabi-Yau space, even at the string tree-level and in the non-compact limit, we would like to treat a simple example, such as the orbifold case. However, in the orbifold limit, the tree-level contribution to the localized EH action vanishes. This is easy to see by inspection, for instance, of the Kähler metric of the untwisted  $\mathcal{N} = 2$  vector multiplets that should receive a perturbative 2d  $\sigma$ -model correction proportional to  $\zeta(3)\chi$ . Such a correction is though absent for orbifolds, since the tree-level metric of untwisted fields can be exactly determined by truncation of the dimensionally reduced action from ten dimensions, and can also be verified by a direct string computation of the 4-point amplitude on the sphere.

For the above reason, in the following, we will restrict our analysis to the one-loop correction and we will show that in the limit where the localized gravity kinetic terms become dominant, their width is fixed by their strength, which is in fact the only natural scale in this limit, as it has been also argued in the literature before [5,7]. We will thus work in the context of type IIB theory compactified on the orbifold Calabi-Yau space  $CY^{(n_V, n_H)} = T^6/\mathbb{Z}_N$  with  $n_V$  vector multiplets and  $n_H$  hyper-multiplets. We will be particularly interested in the decompactification limit when all the internal radii are sent to infinity.

The partition function of the model decomposes into a sum over three sectors preserving different amount of supersymmetry: the  $\mathcal{N} = (4, 4)$  sector where none of the coordinates is twisted, the  $\mathcal{N} = (2, 2)$  sector with two untwisted internal coordinates, and the  $\mathcal{N} = (1, 1)$  sector where all the Calabi-Yau coordinates are twisted:

$$\mathcal{Z} = \mathcal{Z}^{(4,4)} + \mathcal{Z}^{(2,2)} + \mathcal{Z}^{(1,1)}.$$

Following the computation of ref. [15], only the odd-odd spin structure contributes to the two derivative graviton kinetic terms. Moreover, only the  $\mathcal{N} = (1, 1)$  sector without internal fermionic zero modes gives non-vanishing result. Since in the odd spin structure,

the partition functions of twisted fermions and bosons cancel among each other, we are left with a constant proportional to the multiplicity of twisted states:

$$\mathcal{Z}_{odd}^{int} = \mathcal{Z}_{odd}^{(1,1)} = \sum_{f=0, \dots, n_f} \chi_f = \chi, \quad (4.1)$$

where  $f = 0, \dots, n_f$  labels the fixed points and  $\chi_f$  is the corresponding contribution to the Euler number.

We consider now the one-loop amplitude involving two graviton zero modes with a possible emission of a Kaluza-Klein (KK) excitation along the internal directions. The vertex operator for the emission of an untwisted state, with compact momentum  $p$  and winding  $\omega$ , is given in the  $(0,0)$ -ghost picture by the usual vertex operator expressed in terms of the corresponding twisted coordinates  $x_L$  and  $x_R$  [27]:

$$\begin{aligned} V_{(p,\omega)} &=: e^{ip_L \cdot x_L(z) + ip_R \cdot x_R(\bar{z})} ; \\ V_{(p,\omega)}^{inv} &= \frac{1}{N} \sum_{k=0}^{N-1} V_{\gamma^k \cdot (p,\omega)}, \end{aligned} \quad (4.2)$$

where  $p_L = (p + \omega)/\sqrt{2}$  and  $p_R = (p - \omega)/\sqrt{2}$  are the internal left and right momenta of the Narain lattice along the orbifold directions [27]. The invariant vertex operator is obtained after summation over its images under the representation  $\gamma$  of the action of the orbifold group (normalized to the identity operator for  $V_{(0,0)}^{inv}$ ).<sup>6</sup>

In the odd-odd spin structure, we will need to take one graviton vertex in the  $(-1, -1)$ -ghost picture:

$$V_{(-1,-1)} = \zeta_{MN} : \psi^M \bar{\psi}^N e^{-\varphi - \bar{\varphi}} V_{(p,\omega)}^{inv} e^{ik \cdot x} : ,$$

where  $\varphi$  and  $\bar{\varphi}$  are the 2d superghosts. The other vertices are in the  $(0,0)$ -ghost picture:

$$V_{(0,0)} = \zeta_{MN} : (\partial x^M + \frac{i}{2} k \cdot \psi \psi^M) (\bar{\partial} x^N - \frac{i}{2} k \cdot \bar{\psi} \bar{\psi}^N) V_{(p,\omega)}^{inv} e^{ik \cdot x} : ,$$

where the indices  $M = (\mu, I)$  can lie along the (non-compact) 4d directions  $\mu$  and the (compact) 6d internal directions  $I$ . Besides the graviton vertices, one has to consider insertions of the holomorphic and anti-holomorphic world-sheet supercurrents,  $T_F = \partial x_\mu \psi^\mu + G_{IJ} \partial x^I \psi^J$  and  $\bar{T}_F = \bar{\partial} x_\mu \bar{\psi}^\mu + G_{IJ} \bar{\partial} x^I \bar{\psi}^J$ , respectively.

As mentioned above, the order  $\mathcal{O}(k^2)$  gets contributions only from the odd-odd spin-structure of the  $\mathcal{N} = (1,1)$ -twisted sector of the three-graviton amplitude [15]. The even-spin-structure and the other sectors with more supersymmetry start contributing

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<sup>6</sup> See appendix B for more details on the construction of this vertex operator for  $\mathbb{Z}_N$ -orbifolds and computation of the twist correlator.

from the order  $\mathcal{O}(k^4)$ . At the order  $\mathcal{O}(k^2)$ , after soaking the four spacetime fermionic zero-modes  $\psi^{\mu=0,\dots,3}$ , and taking into account that the contribution from the CY part reduces to the twisted partition function  $\mathcal{Z}_{odd}^{(1,1)}$  in the odd-odd spin structure, which is the index (4.1), one finds:

$$\langle (V_{(0,0)})^3 \rangle = \mathcal{R} \chi \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} + \mathcal{O}(k^4), \quad (4.3)$$

where the integration over  $\tau$  is restricted in the fundamental domain  $\mathcal{F}$  for  $Sl(2, \mathbb{Z})$ , and the linearized tensorial structure  $\mathcal{R}$  is given by:

$$\mathcal{R} = \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \varepsilon^{\nu_1 \nu_2 \nu_3 \nu_4} k_{\mu_1}^1 k_{\nu_1}^2 \zeta_{\mu_2 \nu_2}^1 \zeta_{\mu_3 \nu_3}^2 \zeta_{\mu_4 \nu_4}^3 + \text{permutations}.$$

Note however that in the decompactification limit of the orbifold, the resulting localized term at a given fixed point  $f$  is obtained by replacing  $\chi$  in (4.3) with the corresponding contribution  $\chi_f$  defined in (4.1).

We turn now to the amplitude involving one Kaluza-Klein (KK) excitation of the graviton with KK momentum  $q$  and zero winding. Picking up the zero modes of the fermions along the non-compact directions from  $V_{(0,0)}(z_1)$  and  $V_{(-1,-1)}(z_2)$  and the zero mode parts from the contractions  $\langle \partial x^{\mu_2}(w) \bar{\partial} x^{\nu_3}(\bar{z}_3) \rangle$  and  $\langle \bar{\partial} x^{\nu_2}(\bar{w}) \partial x^{\mu_3}(z_3) \rangle$  between the supercurrents and the vertex operator  $V_{(0,0)}(z_3)$ , we get using momentum conservation and the mass-shell conditions  $\sum_i k_i = 0$ ,  $(k_1)^2 = (k_2)^2 = 0$  and  $(k_3)^2 = -q^2$ :

$$\begin{aligned} \langle (V_{(0,0)})^2 V_{(-1,-1)} \rangle^{(1)} &= \mathcal{R} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \int \prod_{1 \leq i \leq 3} \frac{d^2 z_i}{\tau_2} \left\langle \prod_{1 \leq l \leq 3} : e^{ik_l \cdot x} : \right\rangle \frac{1}{N} \sum'_{(h,g)} \langle V_{(q,0)}^{inv} \rangle^{(h,g)} \\ &= \mathcal{R} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \int \prod_{1 \leq i \leq 3} \frac{d^2 z_i}{\tau_2} \prod_{1 \leq i < j \leq 3} \chi_{ij}^{4\alpha' k_i \cdot k_j} \frac{1}{N} \sum'_{(h,g)} \langle V_{(q,0)}^{inv} \rangle^{(h,g)} \\ &= \mathcal{R} \frac{1}{N^2} \sum_{\substack{f=0 \dots n_f \\ k=0 \dots N-1}} e^{i\gamma^k q \cdot x_f} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \int \prod_{1 \leq i \leq 3} \frac{d^2 z_i}{\tau_2} \sum'_{(h,g)} e^{\alpha' q^2 F_{(h,g)}(\tau, z_i)} \end{aligned} \quad (4.4)$$

where  $(h, g) = (l, m) \times v/N$  label the twisted boundary conditions, with  $l, m = 0, \dots, N-1$ ;  $v/N$  is one of the three angles of the orbifold action with  $v$  being an integer between 1 and  $N-1$ , usually restricted by the crystallographic action and the supersymmetry restriction. The prime in the sum excludes the untwisted sector  $(h, g) = (0, 0)$ , which does not contribute to the amplitude because of the zero modes. In the second line of (4.4), we performed the bosonic contractions on the torus defining as usual [9]:

$$\chi_{ij} = 2\pi e^{-\pi \frac{(\Im m(z_i - z_j))^2}{\tau_2}} \left| \frac{\theta_1(z_i - z_j | \tau)}{\theta_1'(0 | \tau)} \right|. \quad (4.5)$$

In the last line of (4.4), using theta-function identities, we included the twisted sector contribution into the function

$$F_{(h,g)}(\tau, z_i) = -\ln(\chi_{12})^2 + \ln(\chi_{13})^2 + \ln(\chi_{23})^2 - \ln \Delta_h, \quad (4.6)$$

where  $\Delta_h$  is the coupling constant between twisted and untwisted states on the torus:

$$\ln \Delta_h = 2\Psi(1) - \Psi(h) - \Psi(1-h) + \sum_{\substack{n \in \mathbb{Z} \\ m > 0}} \frac{1}{|n+h|} e^{2i\pi m(g+(n+h)\tau_1) - 2\pi m\tau_2|n+h|} + c.c. \quad (4.7)$$

It depends on the modular parameter  $\tau$  and is derived in Appendix B, as well as its modular properties,  $\ln \Delta_h(-1/\tau) = \ln \Delta_g(\tau) - \ln |\tau|^2$ . In the field theory limit,  $\tau_2 \rightarrow \infty$ , the above expression reduces to the standard tree-level coupling  $\delta_h^{-\alpha' q^2}$ , with  $\ln \delta_h = 2\Psi(1) - \Psi(h) - \Psi(1-h)$ , between two twisted and one untwisted state [27,28].

A second contribution to the amplitude comes by picking up the two fermionic zero modes from  $V_{(0,0)}(z_1)$ , one from  $V_{(0,0)}(z_3)$  and another from  $V_{(-1,-1)}(z_2)$ . After contracting two internal fermions on each side (left and right movers) and taking the bosonic zero modes from the  $\langle \partial x \bar{\partial} x \rangle$  contraction between the supercurrents, we find

$$\begin{aligned} \left\langle (V_{(0,0)})^2 V_{(-1,-1)} \right\rangle^{(2)} &= \mathcal{R} \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} \int \prod_{1 \leq i \leq 3} \frac{d^2 z_i}{\tau_2} \left\langle \prod_{1 \leq l \leq 3} : e^{ik_l \cdot x} : \right\rangle \\ &\times \frac{1}{N} \sum_{(h,g)} \tau_2 k_L^3 k_L^1 G_{IJ} G_{I'J'} \left\langle \partial x^I(w) \bar{\partial} x^{I'}(\bar{w}) \right\rangle^{(h,g)} \left\langle \psi^J \psi^L \right\rangle^{(h,g)} \left\langle \bar{\psi}^{J'} \bar{\psi}^{L'} \right\rangle^{(h,g)} \\ &= 0. \end{aligned}$$

The above expression vanishes because in the twisted sector bosonic coordinates do not have zero modes:  $\left\langle \partial x^I(w) \bar{\partial} x^{I'}(\bar{w}) \right\rangle^{(h,g)} = 0$ . Therefore, we are left over only with the contribution (4.4).

## 5. Analysis of the width

In order to extract the information on the effective width of the localized term at the orbifold fixed point, we Fourier transform the amplitude with respect to the KK momentum  $q$  (in the Euclidean region  $q^2 < 0$ ) in all internal directions and take the decompactification limit of all radii  $R \rightarrow \infty$ . Sitting at a fixed point of the orbifold, for instance the origin, all other fixed points go to infinity and we are left over with a Gaussian profile for the Planck mass correction  $\delta$  in the 6d internal position space  $y$ :

$$\delta(y) = \frac{1}{N} \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} \int \prod_{1 \leq i \leq 3} \frac{d^2 z_i}{\tau_2} \sum_{(h,g)} \frac{1}{(F_{(h,g)}(\tau, z_i))^3} e^{-\frac{y^2}{\alpha' F_{(h,g)}(\tau, z_i)}}. \quad (5.1)$$

Here, the sum over  $k$  in (4.4) cancels one factor of  $N$ . From this expression we extract a form factor with a width  $w$  associated with the fixed point, and the corresponding localized induced EH effective action reads:

$$\frac{\chi_0}{l_s^2} \int d^4x d^6y \sqrt{g} \delta_w(y) \mathcal{R}, \quad (5.2)$$

where we have defined  $\delta(y) \equiv \chi_0 M_s^2 \delta_w(y)$ . The width  $w$  of the form factor acts as a UV cutoff for the modes of the 4d graviton propagating in the bulk [5,7].

Indeed, the one-loop correction to the Planck mass  $\delta(q)$  modifies the Laplace equation for the Green's function as [4,5]:

$$M^{2+n}(k^2 + q^2)G(k, q) + M_P^2 k^2 \delta_w \star G(k, q) = 1, \quad (5.3)$$

where the star stands for a convolution integral in a self-explanatory notation. In the limit of vanishing width where  $\delta_w(q) = 1$ , one can partially Fourier transform from  $q$  to  $y$  and sit at the origin to find:

$$G(k, y=0) \sim \frac{D(k, 0)}{1 + r_c^6 k^2 D(k, 0)}, \quad D(k, 0) = \int \frac{d^6q}{k^2 + q^2}, \quad (5.4)$$

where we utilized (1.1) and for notational simplicity we use the same symbol for a function and its Fourier transform. The above expression is of course formal, since the bulk propagator  $D$  has a short distance singularity at  $y = 0$ , or at  $q \rightarrow \infty$ . A finite width  $w$  regulates the singularity and leads to the crossover scale (1.2) for  $n = 6$ , upon replacing  $D(k, 0) \sim w^{-4}$ .

In order to determine in our case the effective width  $w$  of the localized gravitational kinetic term, we have to examine more closely the exponent of the Gaussian profile (5.1). In our example, it is obviously fixed trivially by the string length, which is the only available scale, times a numerical constant. An additional parameter can however be introduced by varying the Euler number  $\chi$ . Thus, the one-loop induced four-dimensional Planck mass in eq. (1.1) becomes  $M_P^2 \sim \chi M_s^2$ , while the ten-dimensional bulk gravitational scale is  $M^8 \sim M_s^8/g_s^2$ .

Using dimensional analysis in the limit  $M_P \rightarrow \infty$ , we expect the effective width to vanish as a power of  $l_P \equiv M_P^{-1}$ :  $w \sim l_P^\nu/l_s^{\nu-1}$  with  $\nu > 0$ . In refs. [5,7], it was argued that  $\nu = 1$  and thus  $w \sim l_P$ , which is the only left-over scale in the decoupling limit of the effective field theory on the ‘‘brane’’. However, we do not see any a-priori reason for this argument to be valid in the context of string theory which contains a finite fundamental length  $l_s$ . We thus allow for a general positive exponent  $\nu > 0$ , that we are going to determine from the dependence of  $w$  on  $\chi$  in the limit  $\chi \rightarrow \infty$ .

To analyze the dependence of the exponent in (5.1) on the Euler number, we should study generic orbifolds. To simplify the discussion, we will consider  $\mathbb{Z}_N$  orbifolds with  $N$  prime and take the limit of  $\chi$  (and therefore  $N$ ) large, in order to enhance the strength of the localized term. We thus have to relax the crystallographic restriction on the action of the orbifold group, which will also break supersymmetry, but this is not important for the purpose of our computation. Note that in this limit the strength of the induced term in eq. (5.2)  $\chi_0 \sim N$ , since the sum over  $g$  and  $h$  in eqs. (4.4) and (5.1) brings a factor of  $N^2$ .

To deduce the width of the localized term on the fixed-point, i.e. at  $y = 0$ , it is necessary to study the large- $q$  limit of the correction  $\delta(q)$ . The problem is in finding the stationary points of  $F_{(h,g)}$  in (4.6). In the general  $\mathbb{Z}_N$  orbifold case and in the large  $N$  limit the dominant contribution arises when  $F$  approaches zero asymptotically. This can be done by considering small values of  $h \sim 1/N$  and sending  $\tau_2$  to infinity in an appropriate way.

In the large- $\tau_2$  limit, the oscillator contributions in the expression (4.5) of  $\chi_{ij}$  disappear and we are left with the zero mode part:

$$\ln \chi(z) \simeq -\frac{\pi}{\tau_2} (\Im m z)^2 + \ln |2 \sin \pi z|. \quad (5.5)$$

The positions  $z_{1,2,3}$  of the graviton vertex operators have to be chosen such that  $F_{(h,g)}$  in (4.6) reaches a minimum. It follows that  $\Im m(z_i - z_j) \sim \tau_2$  go to infinity and one is left with the minimization of

$$-\ln \chi(z_{12}) + \ln \chi(z_{23}) + \ln \chi(z_{31}) \quad ; \quad \ln \chi(z) \simeq -\frac{\pi}{\tau_2} (\Im m z)^2 + \pi |\Im m z|, \quad (5.6)$$

where  $z_{ij} = z_i - z_j$ . Using the constraints  $z_{12} + z_{23} + z_{31} = 0$  and  $\Im m z_i \in [0, \tau_2]$ , one finds:

$$\Im m z_{12} = -\frac{2}{3}\tau_2 \quad , \quad \Im m z_{23} = \Im m z_{31} = \frac{\tau_2}{3}. \quad (5.7)$$

Note the similarity of these values with the Gross and Mende configuration that extremizes the four-point amplitude, in which case the positions of the vertices are separated by half the period of the torus [29].

At the minimum (5.7) and in the limit  $\tau_2 \rightarrow \infty$  and  $h \rightarrow 0$ , the function  $F_{(h,g)}$  becomes:

$$\begin{aligned} F_{(h,g)} \Big|_{\min} &\simeq \frac{1}{h} \left[ \frac{4}{9} \pi h \tau_2 + 1 - \frac{1}{1 - \exp(2i\pi(g + h\tau))} - \frac{1}{1 - \exp(-2i\pi(g + h\bar{\tau}))} \right] + \mathcal{O}(h) \\ &= \frac{1}{h} \left[ \frac{4}{9} \pi h \tau_2 - \frac{1}{2} \coth(i\pi(g + h\bar{\tau})) - \frac{1}{2} \coth(-i\pi(g + h\tau)) \right] + \mathcal{O}(h) \end{aligned} \quad (5.8)$$

up to terms exponentially suppressed in  $\tau_2$ . The terms of order  $h$  come from the expansion of the  $\Psi$  functions in the vertex (4.7) and are independent of  $\tau_2$ . It is now clear that  $h\tau$  can be chosen in a way that the  $1/h$  term vanishes, and thus  $F_{(h,g)}$  becomes of order  $h \sim 1/N$ . Indeed, to leading order,  $\pi h \tau_2$  is given as a solution of the equation  $4x/9 = \coth x$ . As a result, the width decreases as

$$w \simeq l_s / N^{\frac{1}{2}} \simeq l_P, \quad (5.9)$$

and is given by the induced Planck length.



## 6. Discussion of the results

In the previous section we argued that in the limit where the strength of the localized gravity kinetic terms is much larger than the higher dimensional gravitational scale,  $M_P \gg M$  in eq. (1.1), their effective width is fixed by their strength, i.e. the 4d Planck length,  $w \sim l_P$ . Although our computation was done in a particular class of orbifold models where the tree level correction was absent, the result confirms the field theory expectations of previous works [5,7] and in the following we will assume that it remains valid in general.

We can now summarize our previous analysis for type II string theories on a 6d non-compact internal space. The relevant gravitational kinetic terms are:

$$\frac{M_s^8}{g_s^2} \int d^{10}x \sqrt{g} \mathcal{R}_{(10)} + \chi M_s^2 f(g_s) \int d^4x \sqrt{g} \mathcal{R}_{(4)}, \quad (6.1)$$

where for simplicity we omitted numerical factors and we introduced the function  $f(g_s)$  in order to treat simultaneously tree level and one-loop contributions in (non-compact) Calabi-Yau manifolds and orbifolds. In the weak coupling limit

$$f(g_s) = -\frac{c_0}{g_s^2} \pm c_1 + \dots, \quad (6.2)$$

where  $c_0$  and  $c_1$  are positive numerical constants given in (2.2), while the dots stand for exponentially suppressed terms. In the case of  $\mathbb{Z}_N$  orbifolds,  $c_0$  vanishes and  $\chi \sim N$  for large  $N$ . It follows that  $M_P^2 = \chi f(g_s) M_s^2$ , implying that  $\chi f(g_s) \simeq 10^{32}$  for  $M_s \simeq 1$  TeV. On the other hand, from eq. (1.1) one has

$$r_c = [\chi g_s^2 f(g_s)]^{\frac{1}{6}} l_s, \quad (6.3)$$

while from eq. (1.2) with  $w \sim l_P$  one obtains the crossover scale

$$R_c = \frac{r_c^3}{w^2} \sim g_s \sqrt{\chi f(g_s)} \frac{l_s^3}{l_P^2} = g_s \frac{l_s^4}{l_P^3} \simeq g_s \times 10^{32} \text{ cm}, \quad (6.4)$$

for  $M_s \simeq 1$  TeV. At 4d distances smaller than  $R_c$ , between points on the “brane” where gravity is localized, one recovers ordinary Newton’s law, while at larger distances gravity becomes ten-dimensional. Imposing  $R_c$  to be larger than the size of the universe,  $R_c \gtrsim 10^{28}$  cm, one obtains that the string coupling can be relatively small,  $g_s \gtrsim 10^{-4}$ , while the Euler number must be huge (and negative if  $c_0 \neq 0$ ):  $|\chi| \sim g_s^2 \times 10^{32} \gtrsim 10^{24}$ .

Thus, the hierarchy is obtained mainly due to the large value of  $\chi$ , which can be lowered only if one imposes a weaker bound on  $R_c$ , depending on our actual knowledge of gravity at very large distance scales. Note that in the case of  $\mathbb{Z}_N$  orbifolds we studied in the previous section,  $f(g_s) \sim \mathcal{O}(1)$  and  $\chi \sim N$  determines completely the hierarchy:  $|\chi| \sim (l_s/l_P)^2 \simeq 10^{32}$ . It is worth noticing that adjusting only one parameter  $\chi$  and keeping

$g_s$  of order unity, one can account for the hierarchy and simultaneously obtain  $R_c$  larger than the size of the universe. Actually, having large Euler number implies only a large number of closed string massless particles with no a-priori constraint on the observable gauge and matter sector which has different origin in type II theories, as we discuss below. As mentioned in section 2, in the case of Calabi-Yau manifolds,  $\chi$  counts the difference between the numbers of vector multiplets and hypermultiplets,  $\chi = \pm 4(n_V - n_H)$  for type IIA/B, and thus must be negative/positive since  $c_0$  of eq. (6.2) is not vanishing. On the other hand, in the case of orbifolds, the contribution comes from  $c_1$  requiring always a surplus of closed string twisted vectors [30]. All these particles are localized at the orbifold fixed points and should have sufficiently suppressed gravitational-type couplings, so that their presence with such a huge multiplicity does not contradict observations.

Note that these results depend crucially on the scaling of the width  $w$  in terms of the Planck length:  $w \sim l_P$ , implying  $R_c \sim 1/l_P^{2\nu+1}$  (in string units) for  $\nu = 1$ . If there are models with  $\nu > 1$ , the required value of  $\chi$  would be much lower, becoming  $\mathcal{O}(1)$  for  $\nu \geq 3/2$ . In this case, the hierarchy would be determined by tuning the string coupling to infinitesimal values,  $g_s \sim 10^{-16}$ , in analogy to the compact string models studied in refs. [31,32]. An alternative way to avoid introducing large  $\chi$  is to compactify the six internal dimensions at a length scale  $l_c \lesssim 10^{16}$  cm. Indeed, for  $\chi$  of order unity, eq. (6.4) with  $f \sim \mathcal{O}(1/g_s^2)$  leads a crossover scale of a size of the solar system:  $R_c \sim l_s^3/l_P^2 \simeq 10^{16}$  cm. Then, at distances larger than  $l_c$  KK modes decouple, while at distances shorter than  $l_c$  localized terms dominate. As a result, Newton's law remains four-dimensional at all scales [7].

A similar situation is obtained in the context of M theory. As discussed in section 2, localization of gravity now arises in five dimensions and, upon compactification in four, the relevant kinetic terms become:

$$M^9 R_{11} \int d^{10}x \sqrt{g} \mathcal{R}_{(10)} + \chi M^3 R_{11} \int d^4x \sqrt{g} \mathcal{R}_{(4)}, \quad (6.5)$$

where  $M = l_M^{-1}$  is the M theory scale and omitted numerical factors can be trivially restored from eq. (2.4). It follows that  $\chi R_{11}/l_M \sim (l_M/l_P)^2 \simeq 10^{32}$  for  $M \simeq 1$  TeV, while  $R_c \sim \sqrt{\chi} l_M^3/l_P^2$  implying again  $\chi \gtrsim 10^{24}$  and  $R_{11}/l_M \lesssim 10^8$ , or equivalently a compactification scale for the eleventh dimension bigger than about 10 keV.

Finally, we would like to discuss the question of matter localization. In a realistic model with six non-compact dimensions and four-dimensional localized gravity, one should also have matter and gauge interactions localized at the same point of the non-compact Calabi-Yau manifold. One possibility is to use D3 branes located at this point. In this case, the string coupling is fixed by the gauge coupling  $g_s \sim g_{YM}^2 \sim \mathcal{O}(1)$  and thus the hierarchy should be accounted by the Euler number. Another possibility is to consider singular points where massless charged states can arise as D-branes wrapped around the collapsing cycles.

However, although chiral matter indeed comes localized in particular singular points of the internal space, gauge fields are generically localized only in co-dimension four surfaces, and thus propagate in two extra dimensions. This can be understood by considering for instance CY threefolds obtained as  $K3$  fibrations on a two-dimensional base. Massless gauge fields then arise by wrapping D-branes around  $K3$  singularities and propagate freely along the base. In this case, one is forced to consider only four non-compact dimensions and compactify the remaining two at the string scale.

On the other hand, models with gauge fields localized at co-dimension six singularities might emerge in more general CY spaces, as suggested by the analysis of ref. [33]. Another way to avoid this problem in models with small string coupling, is to identify the gauge sector with D4 branes stretched between two parallel (Neveu-Schwarz) NS5 branes located at the same point where gravity is localized. The gauge coupling is then independent of  $g_s$  and is given, instead, by the coupling of the little string theory,  $g_{l_{st}} = g_s l_s / L = g_{YM}^2$ , in the limit  $g_s, L \rightarrow 0$ , with  $L$  the separation of the two NS5 branes. The resulting models are very similar to those examined in ref. [32] in the context of TeV little strings.

In conclusion, we have studied gravity localization and non-compact flat dimensions in string theory and proposed possible realizations in a consistent perturbative framework. It would be interesting to study in detail the phenomenological consequences of these models and compare to other realizations of TeV strings with compact dimensions.

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## Appendix A. Tadpole cancellation condition

We rederive the tadpole cancellation condition for the type I models on  $T^2 \times K3$  studied in ref. [17] since it is intimately connected with the calculations in Section 2.

The procedure follows the methods explained in the standard texts [22,23]. Note that the definition of the open string proper-time  $t$  and the closed string proper-time  $\tau$  differ from the conventions of the papers [17,23,24] where similar calculations were performed.

The partition function of the model receives contributions from the torus  $\mathcal{T}$ , the Klein bottle  $\mathcal{K}$ , the annulus  $\mathcal{A}$  and the Möbius strip  $\mathcal{M}$

$$\mathcal{Z} = \mathcal{T} + \mathcal{K} + \mathcal{A} + \mathcal{M} .$$

Since the fundamental domain of the torus has a natural ultraviolet cutoff, the torus amplitude does not contribute to the tadpole cancellation condition. In the other surfaces, the general form of the one-loop amplitude is [22,23]:

$$Z_\sigma = C_\sigma \times \int_0^\infty \frac{dt}{t} \frac{1}{t^2} \frac{1}{\eta^2(\tau)} \frac{P^{(2)}(t)}{\eta^2(\tau)} \times \sum_{s=1,2,3,4} (-1)^{s-1} \frac{\theta_s^2(\tau)}{\eta^2(\tau)} \times Z_s(\tau), \quad (\text{A.1})$$

where  $C_\sigma$  are normalization constants depending on the Riemann surface:  $C_{\mathcal{A}} = 1/4$ ,  $C_{\mathcal{M}} = -1/4$  and  $C_{\mathcal{K}} = 1/2$ .  $Z_s$  is the  $K3$  partition function including the zero-modes, while the sum is performed over all the spin structures  $s$ . In the main text and the following of this appendix, we follow the conventions of [17] where the internal partition function contains the symmetry factor  $C_\sigma$ , so that  $Z_{s,\sigma}(\tau) = C_\sigma Z_s(\tau)$ .  $\theta_s^2/\eta^2$  is the contribution of the fermionic oscillators associated with the  $T^2$ ,  $P^{(2)}/\eta^2$  represents the bosonic contribution from KK states and oscillators of  $T^2$ , and the  $1/t^2\eta^2$  comes from the non-compact coordinates. All zero-mode parts are evaluated in terms of the proper-time  $t$ , while oscillator contributions are computed on the double-cover of the “open” Riemann surfaces and are functions of the closed string proper-time  $\tau$ . The  $d$ -dimensional KK-lattice sum  $P^{(d)}$  and winding-lattice sum  $W^{(d)}$  are given by:

$$P^{(d)}(t) = \sum_{p \in \Gamma_a} e^{-\pi t p^2}, \quad W^{(d)}(t) = \sum_{w \in \Gamma_a} e^{-\pi t w^2},$$

with  $p^2 = |m + nU|^2/T_2 U_2$  and  $w^2 = T_2 |m + nU|^2/U_2$  for  $d = 2$ .

▷ The Klein bottle vacuum amplitude reads:

$$\mathcal{K} = \int_0^\infty \frac{dt}{t^3} \frac{P^{(2)}(t)}{\eta^4(2it)} \sum_{s=1,2,3,4} (-1)^{s-1} \frac{\theta_s^2(2it)}{\eta^2(2it)} \times Z_{s,\mathcal{K}}(2it)$$

In order to extract the tadpole, we have to convert the amplitude to the closed string channel and express the divergence in terms of the closed string proper-time  $\ell$ . The latter is given by  $2it = \tau \rightarrow -1/\tau = i\ell$  (3.6). Now we can extract the tadpole contribution:

$$\tilde{\mathcal{K}} = 2^3 T_2 \int_0^\infty d\ell \frac{W^{(2)}(2\ell)}{\eta^4(i\ell)} \times \sum_{s=1,2,3,4} (-1)^{s-1} \frac{\theta_s^2(i\ell)}{\eta^2(i\ell)} \times \tilde{Z}_{s,\mathcal{K}}(i\ell),$$

which together with the sum over the spin structure (3.3) valid for  $K3$  models gives

$$\tilde{\mathcal{K}} \Big|_{\text{tadpole}} = T_2 (1-1) 2^3 Z_{s=1,\mathcal{K}}^{int} \int_0^\infty d\ell. \quad (\text{A.2})$$

▷ The annulus amplitude is given by

$$\mathcal{A} = \int_0^\infty \frac{dt}{t^3} \frac{P^{(2)}(t)}{\eta^4(it/2)} \sum_{s=1,2,3,4} (-1)^{s-1} \frac{\theta_s^2(it/2)}{\eta^2(it/2)} \times Z_{s,\mathcal{A}}(it/2)$$

After transforming the amplitude into the closed string channel using (3.6):  $it/2 = \tau \rightarrow -1/\tau = i\ell$ , and summing over the spin structure (3.3) for  $K3$  model, one can extract the tadpole contribution:

$$\tilde{\mathcal{A}} \Big|_{\text{tadpole}} = T_2 (1-1) \frac{1}{2^3} Z_{s=1,\mathcal{A}}^{int} \int_0^\infty d\ell. \quad (\text{A.3})$$

▷ The Möbius strip amplitude is given by

$$\mathcal{M} = \int_0^\infty \frac{dt}{t^3} \frac{P^{(2)}(t)}{\eta^4(1/2 + it/2)} \sum_{s=1,2,3,4} (-1)^{s-1} \frac{\theta_s^2(1/2 + it/2)}{\eta^2(1/2 + it/2)} \times Z_{s,\mathcal{M}}(1/2 + it/2)$$

Transforming again the amplitude into the closed string channel using (3.6):  $1/2 + it/2 = \tau \rightarrow -1/\tau \rightarrow -1/\tau + 2 \rightarrow (1/\tau - 2)^{-1} = -1/2 + i/2t = i\ell - 1/2$ , and using the sum over the spin structure (3.3) for  $K3$  model, we extract the corresponding tadpole contribution:

$$\tilde{\mathcal{M}} = T_2 (1-1) \frac{1}{2} Z_{s=1,\mathcal{M}}^{int} \int_0^\infty d\ell. \quad (\text{A.4})$$

Tadpole cancellation implies the vanishing of the sum of the coefficient of the integral:

$$\frac{1}{8} Z_{s=1,\mathcal{A}}^{int} + 2 Z_{s=1,\mathcal{M}}^{int} + 8 Z_{s=1,\mathcal{K}}^{int} = 0 \quad (\text{A.5})$$

which is the coefficient found in (3.8). Note the absence of the  $K3$  volume in the above tadpole condition, following from the fact that we considered only the odd spin structure  $s = 1$  associated with a localized term. In the orbifold limit, it reduces to the twisted tadpole which has to vanish locally at the fixed points even in the non-compact case.

## Appendix B. Untwisted vertex operator emission for orbifolds

In this appendix we describe some properties on the construction of the untwisted closed string vertex operator  $V_{(p,\omega)}$  (4.2) used in the main text. For simplicity, we restrict ourselves to one orbifold coordinate  $S^1/\mathbb{Z}_N$ .

The various sectors under the orbifold action will be labelled by  $h$  and  $g$ .  $h$  corresponds to twisted states running around the loop, *i.e.* taking a trace over the twisted states  $t$ ,  $\sum_t \langle t | \cdots | t \rangle$  and  $g$  to the insertion of the orbifold projector inside the trace, *i.e.*  $\text{Tr}(\gamma^g \cdots)$ .

### B.1. Definitions and notations

This vertex operator is constructed in a similar way as for the heterotic string [9], using the  $\mathbb{Z}_N$  twisted coordinate  $X$  under the orbifold action [27]:

$$V_{(p,\omega)} =: e^{ip_L X_L + ip_R X_R} : ,$$

with the coordinates twisted by the orbifold action  $\exp(2i\pi l v/N)$ .  $X_{L,R}$  have the following mode expansion [34] with  $h = l v/N$

$$\begin{aligned} X_L &= x_L + i\sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \left( \frac{\alpha_{n-h}}{n-h} z^{-(n-h)} + \frac{\alpha_{n+h}^\dagger}{n+h} z^{-(n+h)} \right) \\ X_R &= x_R + i\sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \left( \frac{\tilde{\alpha}_{n+h}}{n-h} \bar{z}^{-(n-h)} + \frac{\tilde{\alpha}_{n-h}^\dagger}{n+h} \bar{z}^{-(n+h)} \right) . \end{aligned}$$

Here,  $x_f = (x_L + x_R)/\sqrt{2}$  is the fixed point  $x_f = f 2\pi R/(1 - \gamma)$ , with  $f = 0, 1$  and  $\gamma = -1$  for a  $\mathbb{Z}_2$ -orbifold. The quantization of the oscillators implies:

$$[\alpha_{n-h}, \alpha_{m+h}^\dagger] = [\tilde{\alpha}_{n-h}, \tilde{\alpha}_{m+h}^\dagger] = (m+h)\delta_{n+m,0} ,$$

while the zero modes satisfy [35]:

$$[x_L, x_R] = i \frac{2\pi}{1 - \gamma} .$$

The standard definition of the normal ordering gives the following split between the zero mode and oscillator parts:

$$V_{(p,w)} = \Delta_h^{-\alpha'(p^2 + \omega^2)} e^{i(p_L X_L^{(-)} + p_R X_R^{(-)})} e^{i(p_L x_L + p_R x_R)} e^{i(p_L X_L^{(+)} + p_R X_R^{(+)})} , \quad (\text{B.1})$$

with  $X_{L,R}^{(-)}$  and  $X_{L,R}^{(+)}$  the negative and positive frequency parts of  $X_{L,R}$ , respectively.

▷ The zero-mode part of (B.1) is given by  $\exp(i(p_L x_L + p_R x_R))$ . Changing variables to  $(x_f, q)$  with  $x_{L/R} = (x_f \pm q)/\sqrt{2}$ , we get the canonical commutation relation

$$[x_f, q] = i \frac{2\pi}{1 - \gamma},$$

and the zero-mode part takes the form [35]:

$$e^{i(p_L x_L + p_R x_R)} = e^{-i\pi p \omega} e^{i p x_f} e^{i \omega q}.$$

Since  $q$  is the canonical conjugate variable to the position of the fixed point, the last term is just the shift operation of the position of the fixed point (this is the operator  $S(\omega)$  introduced in the definition of the vertex operator in [27]):

$$e^{i \omega q} |x_f\rangle = |x_{f+\omega}\rangle.$$

▷ The standard definition of normal ordering [9] does not include the factor  $\exp(-\alpha' p^2 \sum_n 1/n)$  from the commutator  $\exp([X^{(-)}, X^{(+)}]/2)$ . According to this definition we find a normal ordering prefactor, on the cylinder, given by:

$$\begin{aligned} e^{\frac{1}{2}(p_L^2 [X_L^{(-)}, X_L^{(+)}] + p_R^2 [X_R^{(-)}, X_R^{(+)}])} \times e^{\alpha'(p_L^2 + p_R^2) \sum_n' \frac{1}{n}} &= e^{-\alpha'(p_L^2 + p_R^2) \sum_n \frac{1}{|n-h|} - \sum_n' \frac{1}{|n|}} \\ &= \Delta_h^{-\alpha'(p^2 + \omega^2)}. \end{aligned}$$

The factor  $\Delta_h$  defined above is the ratio of the normal ordering factor between twisted and untwisted states [27]. It can be obtained as the difference between the coincident point correlators  $X^2(0)$  in the twisted and untwisted sector:

$$\ln \Delta_h = \langle X(0)X(0) \rangle_{\text{twisted}} - \langle X(0)X(0) \rangle_{\text{untwisted}}. \quad (\text{B.2})$$

This definition will allow us to derive the corresponding  $\ln \Delta_h$  factor on the torus.

We have to compute the vacuum expectation value  $\langle V_{(p,\omega)}(z_3) \rangle$  of the vertex operator (B.1), associated with the emission of an untwisted state from the twisted sectors. A naive evaluation of this operator using Wick theorem would set this expectation value to zero, but the correct answer is

$$\langle V_{(p,\omega)}(z_3) \rangle \Big|_h = \Delta_h^{-\alpha'(p^2 + \omega^2)} \sum_{\text{fixed points}} e^{i p x_f}. \quad (\text{B.3})$$

In the following, we derive this correlator from a massive deformation of the partition function of a free (twisted) boson.

## B.2. The twisted correlator on the torus

On the torus, the twisted coordinates are expanded as

$$X = x_f + \sum_{m,n \in \mathbb{Z}} \alpha_{n+h,m+g} \phi_{n+h,m+g}(\tau, z),$$

with  $\phi_{n+h,m+g}(\tau, z)$  a basis of eigen-functions of  $-\Delta_z$  with the eigenvalue  $\lambda_{n+h,m+g} = (2\pi)^2 |(n+h)\tau + m+g|^2 / \tau_2^2$  for the boundary conditions specified by  $h$  and  $g$  [36]. The coincident point correlator is extracted from the partition of a twisted boson with mass  $(2\pi\mu)^2$  after differentiating with respect to  $(2\pi\mu)^2$ . The eigenvalues of the massive Laplace operator  $-\Delta_z + (2\pi\mu)^2$  are  $\lambda_{n+h,m+g} = (2\pi)^2 [|(n+h)\tau + m+g|^2 / \tau_2^2 + \mu^2]$ , and the partition function, computed using the standard zeta-function regularization following the references [36,37], is

$$\begin{aligned} \mathcal{Z}(h, g; \mu^2) &= \int \mathcal{D}X e^{-\int_{\mathcal{T}^2} d^2z X(-\Delta_z + (2\pi\mu)^2)X} = \prod_{n,m} \frac{1}{\lambda_{n+h,m+g}} \\ &= e^{2\pi\tau_2\gamma_h} \left| \prod_{n \in \mathbb{Z}} \left( 1 - e^{2i\pi(g+(n+h)\tau_1) - 2\pi\tau_2\sqrt{(n+h)^2 + \mu^2}} \right) \right|^{-2}. \end{aligned} \quad (\text{B.4})$$

The prefactor depends on the mass  $\mu^2$  and the twist  $h$  only, and represents the zero-point energy of the massive boson [38,39]

$$\gamma_h = \frac{1}{2\pi} \sum_{n \neq 0} \int_0^\infty dt e^{-\pi t n^2 - \pi \mu^2 / t + 2i\pi n h}.$$

We will need its expansion with respect to  $\mu^2$ , given in [37]:

$$\begin{aligned} \gamma_{h \neq 0} &= \left( h - \frac{1}{2} \right)^2 - \frac{1}{12} + \frac{\mu^2}{2} (\ln(4\pi)^2 + \Psi(h) + \Psi(1-h)) + \frac{\mu^4}{4} \int_0^1 dl \sum_{n \in \mathbb{Z}} \frac{(1-l)}{((n+h)^2 + l\mu^2)^{\frac{3}{2}}} \\ \gamma_0 &= \frac{1}{6} - \mu + \frac{\mu^2}{2} (\ln(4\pi)^2 + 2\Psi(1)) + \frac{\mu^4}{4} \int_0^1 dl \sum_{n \neq 0} \frac{(1-l)}{(n^2 + l\mu^2)^{\frac{3}{2}}}. \end{aligned}$$

The correlator  $\langle X(0)X(0) \rangle$  in the  $(h, g)$  sector is deduced by differentiating the logarithm of the massive partition function (B.4), and  $\ln \Delta_h$  is given by

$$\ln \Delta_h = -\frac{1}{\pi\tau_2} \partial_{\mu^2} \ln \mathcal{Z}(h, g; \mu^2) \Big|_{\mu^2=0} + \ln(4\pi)^2 + 2\Psi(1). \quad (\text{B.5})$$



▷ The result for the twisted sector  $h \neq 0$  is

$$\ln \Delta_{h \neq 0} = 2\Psi(1) - \Psi(h) - \Psi(1-h) + \sum_{\substack{n \in \mathbb{Z} \\ m > 0}} \frac{1}{|n+h|} e^{2i\pi m(g+(n+h)\tau_1) - 2\pi m\tau_2|n+h|} + c.c. \quad (\text{B.6})$$

In the field theory limit  $\tau_2 \rightarrow \infty$ , only the first term survives, representing the tree-level coupling between twisted and untwisted states obtained in [27,28].

▷ In the untwisted sector  $h = 0$  (but  $g \neq 0$ ) we have

$$\ln \Delta_{h=0} = \frac{\pi\tau_2}{2\sin(\pi g)^2} + \sum_{\substack{n > 0 \\ m > 0}} \frac{1}{n} \left( e^{2i\pi m(g+n\tau)} + e^{2i\pi m(g-n\bar{\tau})} + c.c. \right). \quad (\text{B.7})$$

Because of the term linear in  $\tau_2$ , the amplitude vanishes in the field theory limit  $\tau_2 \rightarrow \infty$ , since the contributions from the untwisted states  $h = 0$  running in the loop vanish due to momentum conservation. For  $g = 0$ , the expression is infinite due to the presence of a zero mode. Actually, it is easy to check that the expression (B.7) is the  $h \rightarrow 0$  limit of (B.6).

▷ In order to study the modular invariance of these expressions we follow [38,39], where the modular properties of  $\mathcal{Z}(h, g; \mu^2|\tau) = \mathcal{Z}(-g, h; \mu^2|\tau|| - 1/\tau)$  were studied. We consider the following integral representation:

$$\begin{aligned} \ln \mathcal{Z}(h, g; \mu^2) &= 2\pi\tau_2\gamma_h + \sum_{\substack{n \in \mathbb{Z} \\ m > 0}} \frac{1}{m} e^{2i\pi m(g+(n+h)\tau_1) - 2\pi m\tau_2\sqrt{(n+h)^2 + \mu^2}} + c.c. \\ &= 2\pi\tau_2\gamma_h + \sum_{\substack{n \in \mathbb{Z} \\ m > 0}} \int_0^\infty \frac{dt}{t^{1/2}} e^{-\pi t m^2 - \frac{\pi\tau_2^2}{t}((n+h)^2 + \mu^2) + 2i\pi m(g+(n+h)\tau_1)} + c.c. \end{aligned} \quad (\text{B.8})$$

and perform a Poisson resummation on  $n$ :

$$\ln \mathcal{Z}(h, g; \mu^2) = 2\pi\tau_2\gamma_h + \frac{1}{\tau_2} \sum_{\substack{\hat{n} \in \mathbb{Z} \\ m > 0}} \int_0^\infty dt e^{-\pi\tau_2^2\mu^2/t - \pi t m^2 - t\pi(\hat{n} + m\tau_1)^2/\tau_2^2 + 2i\pi(mg - \hat{n}h)} + c.c. \quad (\text{B.9})$$

Differentiating with respect to  $\mu^2$ , one finds:

$$\begin{aligned}
\frac{1}{2\pi\tau_2}\partial_{\mu^2}\ln\mathcal{Z}(h,g;\mu^2) &= \partial_{\mu^2}\gamma_h - \frac{1}{2}\sum_{\substack{\hat{n}\in\mathbb{Z} \\ m>0}}\int_0^\infty\frac{dt}{t}e^{-\frac{\pi}{t}\tau_2^2\mu^2-\pi tm^2-t\pi(\hat{n}+m\tau_1)^2/\tau_2^2+2i\pi(mg-\hat{n}h)}+c.c. \\
&= \partial_{\mu^2}\gamma_h - \frac{1}{2}\sum_{\substack{\hat{n}\in\mathbb{Z} \\ m\neq 0}}\int_0^\infty\frac{dt}{t}e^{-\frac{\pi}{t}\tau_2^2\mu^2-\pi tm^2-\frac{t\pi}{\tau_2^2}(\hat{n}+m\tau_1)^2+2i\pi(mg-\hat{n}h)} \\
&= \partial_{\mu^2}\gamma_h - \frac{1}{2}\sum_{\substack{\hat{n}\in\mathbb{Z} \\ m\neq 0}}\int_0^\infty\frac{dt}{t}e^{-\frac{\pi}{t}\tau_2^2\mu^2-\frac{\pi t|\tau|^2}{\tau_2^2}\left(m+\hat{n}\frac{\tau_1}{|\tau|^2}\right)^2-\frac{t\pi}{\tau_2^2}\hat{n}^2+2i\pi(mg-\hat{n}h)} \\
&= \frac{1}{|\tau|^2}\partial_{\mu^2}\gamma_g - \ln|\tau| \\
&\quad - \frac{\tau_2}{2|\tau|}\sum_{\substack{\hat{m}\in\mathbb{Z} \\ \hat{n}\neq 0}}\int_0^\infty\frac{dt}{t^{3/2}}e^{-\frac{\pi}{t}\tau_2^2\mu^2-\frac{\pi t\tau_2^2}{t|\tau|^2}(\hat{m}+g)^2-\frac{t\pi}{|\tau|^2}\hat{n}^2-2i\pi\hat{n}\left(h+(g+\hat{m})\frac{\tau_1}{|\tau|^2}\right)}
\end{aligned} \tag{B.10}$$

The last equality has been obtained after Poisson resummation on  $m$ . In the first line of the last equality, the modular ‘‘anomaly’’ arises from the sector  $\hat{n} = \hat{m} = 0$  [37]. In the second line, we can set  $\mu^2 = 0$ , and by rescaling  $t$  as  $t \rightarrow |\tau|^2 t$  and relabeling the integers as  $(\hat{n}, \hat{m}) \rightarrow (m, n)$ , we bring (B.10) into the same form as the  $\mu^2$ -derivative of the second line of (B.8) with the exchange of  $h$  and  $g$  together with  $\tau \rightarrow -1/\tau$ .

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