

Duality, Biorthogonal Polynomials and Multi–Matrix Models¹

M. Bertola^{†‡2}, B. Eynard^{†*3} and J. Harnad^{†‡4}

[†] *Centre de recherches mathématiques, Université de Montréal
C. P. 6128, succ. centre ville, Montréal, Québec, Canada H3C 3J7*

[‡] *Department of Mathematics and Statistics, Concordia University
7141 Sherbrooke W., Montréal, Québec, Canada H4B 1R6*

^{*} *Service de Physique Théorique, CEA/Saclay
Orme des Merisiers F-91191 Gif-sur-Yvette Cedex, FRANCE*

Abstract

The statistical distribution of eigenvalues of pairs of coupled random matrices can be expressed in terms of integral kernels having a generalized Christoffel–Darboux form constructed from sequences of biorthogonal polynomials. For measures involving exponentials of a pair of polynomials V_1, V_2 in two different variables, these kernels may be expressed in terms of finite dimensional “windows” spanned by finite subsequences having length equal to the degree of one or the other of the polynomials V_1, V_2 . The vectors formed by such subsequences satisfy “dual pairs” of first order systems of linear differential equations with polynomial coefficients, having rank equal to one of the degrees of V_1 or V_2 and degree equal to the other. They also satisfy recursion relations connecting the consecutive windows, and deformation equations, determining how they change under variations in the coefficients of the polynomials V_1 and V_2 . Viewed as overdetermined systems of linear difference-differential-deformation equations, these are shown to be compatible, and hence to admit simultaneous fundamental systems of solutions. The main result is the demonstration of a spectral duality property; namely, that the spectral curves defined by the characteristic equations of the pair of matrices defining the dual differential systems are equal upon interchange of eigenvalue and polynomial parameters.

¹Work supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC) and the Fonds FCAR du Québec.

²bertola@crm.umontreal.ca

³eynard@spht.saclay.cea.fr

⁴harnad@crm.umontreal.ca

1 Introduction

1.1 Random matrices

1.1.1 Background and motivation

Random matrices [?, ?] play an important rôle in many areas of physics. They were first introduced by Wigner [?] in the context of the spectra of large nuclei, and the theory was greatly developed in pioneering work of Mehta, Gaudin [?, ?] and Dyson [?, ?]. It has found subsequent applications in solid state physics [?] (e.g., conduction in mesoscopic devices, quantum chaos and, lately, crystal growth[?]), in particle physics [?], $2d$ -quantum gravity and string theory [?, ?, ?]. The reason for the success and large range of applications of random matrices is, due, in part, to their universality property; when the size of the matrices N becomes large, the statistics of the eigenvalues tend to be independent of the model, and determined only by its symmetries and the spectral region considered, relative to critical points and edges in the spectral density. Matrix integrals are also known to give special realizations of KP , Toda and isomonodromic τ -functions, and thus have a close relationship to integrable systems [?, ?, ?, ?, ?, ?, ?, ?].

Random matrices also have important applications in pure mathematics, for example, in the statistical distribution of the zeros of ζ -functions [?, ?, ?]. They are also related to other statistical problems such as random word growth and the lengths of nondecreasing subsequences of random sequences [?, ?].

The model we shall consider here is called the “2-matrix model” [?, ?, ?, ?, ?, ?, ?]. This involves an ensemble consisting of pairs of $N \times N$ hermitian matrices M_1 and M_2 , with a $U(N)$ invariant probability measure of the form:

$$\frac{1}{\tau_N} d\mu(M_1, M_2) := \frac{1}{\tau_N} \exp \operatorname{tr} (-V_1(M_1) - V_2(M_2) + M_1 M_2) dM_1 dM_2, \quad (1-1)$$

where $dM_1 dM_2$ is the standard Lebesgue measure for pairs of Hermitian matrices, V_1 and V_2 are polynomials of degrees $d_1 + 1$, $d_2 + 1$ respectively, called the potentials, with coefficients viewed as deformation parameters, and the normalization factor (partition function) is

$$\tau_N = \int_{M_1} \int_{M_2} d\mu, \quad (1-2)$$

which is known to be a KP τ -function in each set of deformation parameters, as well as providing solutions to the two-Toda equations [?, ?, ?].

This model was introduced in [?, ?] as a toy model for quantum gravity and string theory. The main interest was in a special “double scaling” limit, where $N \rightarrow \infty$ and the potentials V_1 and V_2 are fine-tuned to critical potentials. The asymptotic behaviour in such limits is related to finite dimensional irreducible representations of the 2D-conformal group [?, ?, ?]. The best known example is when V_1 and V_2 are cubic polynomials, tuned to their critical values, which reproduces the critical behaviour of the Ising model on a random surface [?, ?]. It is important to note that the 2-matrix model contains more critical points than a 1-matrix model; for instance, the 1-matrix-model cannot have an Ising transition [?].

String theorists have also introduced a generalization, known as the “multi-matrix model” [?, ?, ?, ?], where one has a set of $m \geq 2$ matrices ($N \times N$ hermitian) coupled together in a chain, with a measure of the form

$$\frac{1}{\tau_N} d\mu(M_1, \dots, M_m) = \frac{1}{\tau_N} \exp \operatorname{tr} \left(- \sum_{j=1}^m V_j(M_j) + \sum_{j=1}^{m-1} M_j M_{j+1} \right) \prod_{i=1}^m dM_i, \quad (1-3)$$

and the V_j 's are again polynomials in their arguments. This model has the same universal behaviour as the 2-matrix model and, in some sense, does not seem to contain any more information. Throughout

the main body of this work, we will concentrate on the 2-matrix-model, for which the statistics of the eigenvalues can be calculated using biorthogonal polynomials [?, ?, ?, ?, ?, ?]. In the appendix, it will be explained how to extend all the results in the present work from the 2-matrix model to the differential systems associated with this multi-matrix model.

1.1.2 Relation to biorthogonal polynomials

By biorthogonal polynomials, we mean two sequences of monic polynomials

$$\pi_n(x) = x^n + \dots, \quad \sigma_n(y) = y^n + \dots, \quad n = 0, 1, \dots \quad (1-4)$$

which are orthogonal with respect to a coupled measure on the product space:

$$\int \int dx dy \pi_n(x) \sigma_m(y) e^{-V_1(x) - V_2(y) + xy} = h_n \delta_{mn}, \quad (1-5)$$

where $V_1(x)$ and $V_2(y)$ are polynomials chosen to be the same as those appearing in the 2-matrix model measure (??), and a suitable contour is chosen to make the integrals convergent. The orthogonality relations determine the two families. Once the biorthogonal polynomials are known, they may be used to compute four different kernels:

$$\overset{N}{K}_{12}(x, y) = \sum_{n=0}^{N-1} \frac{1}{h_n} \pi_n(x) \sigma_n(y) e^{-V_1(x) - V_2(y)}, \quad \overset{N}{K}_{11}(x, x') = \int dy \overset{N}{K}_{12}(x, y) e^{x'y}, \quad (1-6)$$

$$\overset{N}{K}_{22}(y', y) = \int dx \overset{N}{K}_{12}(x, y) e^{xy'}, \quad \overset{N}{K}_{21}(y', x') = \int \int dx dy \overset{N}{K}_{12}(x, y) e^{xy'} e^{x'y}. \quad (1-7)$$

All the statistical properties of the spectra of the 2-matrix ensemble may then be expressed in terms of these kernels [?] and the corresponding Fredholm integral operators $\overset{N}{\mathbf{K}}_{ij}$, $i, j = 1, 2$. For instance the density of eigenvalues of the first matrix is:

$$\overset{N}{\rho}_1(x) = \frac{1}{N} \overset{N}{K}_{11}(x, x), \quad (1-8)$$

the correlation function of two eigenvalues of the first matrix is:

$$\overset{N}{\rho}_{11}(x, x') = \frac{1}{N^2} \left(\overset{N}{K}_{11}(x, x) \overset{N}{K}_{11}(x', x') - \overset{N}{K}_{11}(x, x') \overset{N}{K}_{11}(x', x) \right), \quad (1-9)$$

and the correlation function of two eigenvalues, one of the first matrix and one of the second is:

$$\overset{N}{\rho}_{12}(x, y) = \frac{1}{N^2} \left(\overset{N}{K}_{11}(x, x) \overset{N}{K}_{22}(y, y) - \overset{N}{K}_{12}(x, y) \overset{N}{K}_{21}(y, x) - e^{xy} \right). \quad (1-10)$$

Any other correlation function of m eigenvalues can similarly be written as a determinant involving these four kernels only.

The spacing distributions (the probability that two neighbouring eigenvalues are at some given distance) can be computed as Fredholm determinants. For example, the probability that some subset J of the real axis contains no eigenvalue of the first matrix is the Fredholm determinant:

$$p_J^{N,1} = \det \left(\mathbf{1} - \overset{N}{\mathbf{K}}_{11} \circ \chi_J \right), \quad (1-11)$$

where χ_J is the characteristic function of the set J .

An important feature in the study of the $N \rightarrow \infty$ limit is that the kernels K_{ij} may be expressed [?] in terms of sums involving only a fixed number of terms (either $d_1 + 1$ or $d_2 + 1$), independently of N , as a consequence of a “generalized Christoffel–Darboux” formula [?, ?] following from the recursion relations satisfied by the biorthogonal polynomials. This allows one, in the $N \rightarrow \infty$ limit, with suitable scaling in the spectral variables, depending on the region considered, to treat N as just a parameter.

1.2 Duality

1.2.1 Dual isomonodromic deformations

The notion of *duality* arises in a number of contexts, both in relation to isospectral flows [?] and isomonodromic [?, ?, ?] deformations. What is meant here by “duality” in the case of isomonodromic deformations is the existence of a pair of parametric families of meromorphic covariant derivative operators on the Riemann sphere

$$\mathcal{D}_1 := \frac{\partial}{\partial x} + \mathbf{L}(x, \mathbf{u}) , \quad \mathcal{D}_2 := \frac{\partial}{\partial y} + \mathbf{M}(y, \mathbf{u}) , \quad (1-12)$$

where $\mathbf{L}(x, \mathbf{u})$ and $\mathbf{M}(y, \mathbf{u})$ are, respectively, $l \times l$ and $m \times m$ matrices that are rational functions of the complex variables $x \in \mathbf{P}^1$ and $y \in \mathbf{P}^1$, with pole divisors of fixed degrees, depending smoothly on a set of deformation parameters $\mathbf{u} = (u_1, u_2, \dots)$ in such a way that:

- 1) The matrices $\mathbf{L}(x, \mathbf{u})$ and $\mathbf{M}(y, \mathbf{u})$ are obtained from the integral curves of a set of commuting (in general, nonautonomous) vector fields defined on a phase space \mathcal{M} by composition with a prescribed pair of maps (possibly depending explicitly on the deformation parameters) from \mathcal{M} to the spaces of rational, $l \times l$ or $m \times m$ matrix valued rational functions of the spectral parameter x or y , respectively, with pole divisors of fixed degree.
- 2) The generalized monodromy data of both the operators \mathcal{D}_1 and \mathcal{D}_2 are invariant under the \mathbf{u} -deformations. (This includes the monodromy representation of the fundamental group of the punctured Riemann sphere obtained by removing the locus of poles and, in the case of non-Fuchsian systems, the Stokes matrices and connection matrices [?, ?].)
- 3) The spectral curves determined by the characteristic equations:

$$\det(\mathbf{L}(x, \mathbf{u}) - y\mathbf{1}) = 0 , \quad \det(\mathbf{M}(y, \mathbf{u}) - x\mathbf{1}) = 0 \quad (1-13)$$

are biholomorphically equivalent.

(A similar definition can be given for the case of dual isospectral flows of matrices $\mathbf{L}(x, \mathbf{u})$ and $\mathbf{M}(y, \mathbf{u})$.)

Such “dual pairs” of isomonodromic families occur in many applications. They are related to the solution of “dual” matrix Riemann-Hilbert (RH) problems [?, ?, ?] which, in certain cases, are equivalent to determining the resolvents of a special class of “integrable” Fredholm integral operators \mathbf{K} , $\tilde{\mathbf{K}}$, with kernels of the form:

$$K(x, x') = \sum_{i=1}^l \frac{f_i(x)g_i(x')}{x - x'} , \quad (1-14)$$

$$\tilde{K}(y, y') = \sum_{a=1}^m \frac{\tilde{f}_a(y)\tilde{g}_a(y')}{y - y'} . \quad (1-15)$$

Here, the vector valued functions

$$\mathbf{f}(x, \mathbf{u}) = (f_1(x, \mathbf{u}), \dots, f_l(x, \mathbf{u})) ; \quad \mathbf{g}(x, \mathbf{u}) = (g_1(x, \mathbf{u}), \dots, g_l(x, \mathbf{u})) \quad (1-16)$$

$$\tilde{\mathbf{f}}(y, \mathbf{u}) = (\tilde{f}_1(y, \mathbf{u}), \dots, \tilde{f}_m(y, \mathbf{u})) ; \quad \tilde{\mathbf{g}}(y, \mathbf{u}) = (\tilde{g}_1(y, \mathbf{u}), \dots, \tilde{g}_m(y, \mathbf{u})) . \quad (1-17)$$

depend on the spectral variables x and y as well as on some, but not necessarily all the deformation parameters (u_1, u_2, \dots) . They satisfy overdetermined, compatible differential systems in these variables which imply the invariance of the monodromy of associated “vacuum” isomonodromic families of covariant derivative operators

$$\mathcal{D}_{0,1} := \frac{\partial}{\partial x} + \mathbf{L}_0(x, \mathbf{u}) , \quad \mathcal{D}_{0,2} := \frac{\partial}{\partial y} + \mathbf{M}_0(y, \mathbf{u}) . \quad (1-18)$$

The dual kernels $K(x, x')$ and $\tilde{K}(y, y')$ are related to each other by applying partial integral transforms with respect to one of the two spectral variables (x, y) (e.g., Fourier-Laplace transforms) to an integral operator on the product space $\mathbf{P}^1 \times \mathbf{P}^1$. Application of the Riemann-Hilbert dressing method for suitably chosen “dual” sets of discontinuity data then gives rise to the “dressed” families (??), which have a similar relation to the resolvent kernels of the two operators. (Additional deformation parameter dependence may enter, besides that contained in the vacuum equations, characterizing the support of these operators.) The Fredholm determinants $\det(\mathbf{1} - \mathbf{K})$ and $\det(\mathbf{1} - \tilde{\mathbf{K}})$ may then be shown, through deformation formulae, to coincide with the corresponding isomonodromic tau functions, and with each other (see e.g. [?, ?]).

Such systems arise naturally, as discussed above, in the study of the spectral statistics of random matrix ensembles, both in the finite case, and in suitable infinite limits. An example of such dual pairs is given by the class of exponential kernels in which:

$$f_i(x) = (-1)^i e^{u_i x} , \quad g_i(x) = e^{-u_i x} , \quad (1-19)$$

$$\tilde{f}_a(x) = (-1)^a e^{v_a x} , \quad \tilde{g}_a(x) = e^{-v_a x} , \quad (1-20)$$

For the case $l = m = 2$, these include the sine kernel

$$K_S(x, x') = \frac{\sin(u(x - x'))}{(x - x')} , \quad \tilde{K}_S(y, y') = \frac{\sin(v(y - y'))}{(y - y')} \quad (1-21)$$

governing the spectral statistics in the scaling limit of the GUE in the bulk region [?].

Other examples include the various Painlevé equations $P_{II}, P_{IV}, P_V, P_{VI}$ [?, ?, ?] which each possess “dual” isomonodromic representations. A last class of examples, unrelated to random matrices, but including a special case of P_{VI} , consists of the isomonodromic representations of the WDVV equations of topological 2D gravity entering in the theory of Frobenius manifolds [?]. These possess both Fuchsian representations with n finite poles with residues of rank 1, and non-Fuchsian $n \times n$ representations, having a single irregular singular point of Poincaré index 1 at ∞ .

1.2.2 Duality in the large N Limit

It is proved in [?], under a suitable large N assumption, and conjectured for other cases [?, ?] that in a particular large N limit (where the coefficients $\{u_K, v_J\}$ scale as N , and the supports of $\rho_1(x) := \lim_{N \rightarrow \infty} \rho_1^N(x)$ and $\rho_2(y) := \lim_{N \rightarrow \infty} \rho_2^N(y)$ are connected intervals $[a_1, b_1], [a_2, b_2]$ respectively), the following functions:

$$f(x) := \frac{1}{N} V_1'(x) - \int_{a_1}^{b_1} \frac{\rho_1(x')}{x - x'} dx' , \quad g(y) := \frac{1}{N} V_2'(y) - \int_{a_2}^{b_2} \frac{\rho_2(y')}{y - y'} dy' , \quad (1-22)$$

are inverses of each other. In other words, if $y = f(x)$ then $g(y) = x$. One can see that the functions $f(x)$ and $g(y)$ are related to the eigenvalues of the operators which implement the derivative with respect to x and y for the biorthogonal polynomials. The spectral duality theorems ?? and ?? which we present in this work are a more precise statement related to this conjecture, with a rigorous proof valid for all N .

1.3 Outline of the article

1.3.1 Biorthogonal polynomials and differential systems

In Section 2, we consider the normalized quasi-polynomials

$$\psi_n(x) = \frac{1}{\sqrt{h_n}} \pi_n(x) e^{-V_1(x)}, \quad \phi_n(y) = \frac{1}{\sqrt{h_n}} \sigma_n(y) e^{-V_2(y)}, \quad n = 0, \dots, \infty. \quad (1-23)$$

Viewing these as the components of a pair of column vectors

$$\Psi_\infty = (\psi_0, \psi_1, \dots, \psi_n, \dots)^t \quad \text{and} \quad \Phi_\infty = (\phi_0, \phi_1, \dots, \phi_n, \dots)^t, \quad (1-24)$$

we obtain a pair of semi-infinite matrices Q and P that implement multiplication of Ψ_∞ by x and derivation with respect to x , respectively. Equivalently, we obtain the transposes Q^t and P^t by applying $-\frac{d}{dy}$ or multiplication by $-y$ to Φ_∞ . By construction, these satisfy the Heisenberg commutation relations

$$[P, Q] = \mathbf{1}, \quad (1-25)$$

and, as shown in Section 2, they are finite band matrices; Q has nonvanishing elements only along diagonals that range from 1 above the principal diagonal to d_2 below it, and P has nonvanishing elements only along the diagonals from 1 below the principal to d_1 above it, where $d_1 + 1$ and $d_2 + 1$ are the degrees of the polynomials $V_1(x)$ and $V_2(y)$, respectively.

The first result (Prop.??) following from the finite recursion relations satisfied by the quasi-polynomials $\{\psi_n(x)\}$ and $\{\phi_n(y)\}$ is a set of “generalized Christoffel–Darboux relations [?, ?], which imply that the kernels $\overset{N}{K}_{11}(x, x')$ and $\overset{N}{K}_{22}(y', y)$ may be expressed as:

$$\overset{N}{K}_{11}(x, x') = -\frac{\left(\overset{N-1}{\underline{\Phi}}(x'), \overset{N}{\mathbb{A}} \overset{N}{\Psi}(x) \right)}{x - x'}, \quad \overset{N}{K}_{22}(y', y) = \frac{\left(\overset{N-1}{\underline{\Psi}}(y'), \overset{N}{\mathbb{B}} \overset{N}{\Phi}(y) \right)}{y' - y}, \quad (1-26)$$

where $\overset{N}{\Psi}(x)$ and $\overset{N}{\Phi}(y)$ are the $d_2 + 1$ and $d_1 + 1$ dimensional column vectors with components $[\psi_{N-d_2}, \dots, \psi_N]$

and $[\phi_{N-d_2}, \dots, \phi_N]$, respectively, and $\overset{N-1}{\underline{\Psi}}(y)$ and $\overset{N-1}{\underline{\Phi}}(x)$ are the $d_2 + 1$ and $d_1 + 1$ component row vectors with components $[\underline{\psi}_{N-1}, \dots, \underline{\psi}_{N+d_2-1}]$ and $[\underline{\phi}_{N-1}, \dots, \underline{\phi}_{N+d_1-1}]$, respectively, where the underbarred quantities $\{\underline{\psi}_n(y)\}$ and $\{\underline{\phi}_n(y)\}$ designate the Fourier-Laplace transforms of the quasi-polynomials $\{\psi_n(y)\}$

and $\{\phi_n(y)\}$. The matrices $\overset{N}{\mathbb{A}}$ and $\overset{N}{\mathbb{B}}$ are, essentially, the nonvanishing parts of the matrices obtained by commuting Q and P , respectively, with the projectors to the appropriate finite-dimensional subspace. A similar “differential” form of the generalized Christoffel–Darboux relations holds, following from applying the derivations $\partial_x + \partial_{x'}$ and $\partial_y + \partial_{y'}$ to the kernels $\overset{N}{K}_{11}(x, x')$ and $\overset{N}{K}_{22}(y', y)$.

The recursion relations satisfied by the quasi-polynomials $\{\psi_n(x)\}_{n=0\dots\infty}$ and $\{\phi_n(y)\}_{n=0\dots\infty}$ may be conveniently expressed (Lemma ??) as:

$$\overset{\mathbf{a}}{N} \overset{N}{\Psi}(x) = \overset{N}{N+1} \overset{N}{\Psi}(x), \quad \overset{\mathbf{b}}{N} \overset{N}{\Phi}(y) = \overset{N}{N+1} \overset{N}{\Phi}(y),$$

where the “ladder” matrices \mathbf{a}_N and \mathbf{b}_N are linear in x and in y , respectively, and are formed from the rows of Q and the columns of P (see eqs. (??, ??) for their exact definitions).

The vectors $\underline{\Psi}_N(x)$ and $\underline{\Phi}_N(y)$ also satisfy the following differential equations (Lemma ??)

$$\frac{\partial}{\partial x} \underline{\Psi}_N = -\underline{D}_1(x) \underline{\Psi}_N, \quad \frac{\partial}{\partial y} \underline{\Phi}_N = -\underline{D}_2(y) \underline{\Phi}_N, \quad (1-27)$$

where the matrices $\underline{D}_1(x)$ and $\underline{D}_2(y)$ are, respectively, of size $(d_2 + 1) \times (d_2 + 1)$ and $(d_1 + 1) \times (d_1 + 1)$, with entries that are polynomials in the indicated variables of degrees d_1 and d_2 , respectively (and also polynomials in the matrix entries of Q and P). Furthermore, if $\{u_K\}_{K=1\dots d_1+1}$ and $\{v_J\}_{J=1\dots d_2+1}$ are the coefficients of the polynomials $V_1'(x)$ and $V_2'(y)$, respectively, and these are varied smoothly, the effect of such deformations is given by the following system of PDE's (Lemma ??)

$$\begin{aligned} \frac{\partial}{\partial u_K} \underline{\Psi}_N &= \underline{\mathbf{U}}_K^{N,\Psi} \underline{\Psi}_N, & \frac{\partial}{\partial v_J} \underline{\Psi}_N &= -\underline{\mathbf{V}}_J^{N,\Psi} \underline{\Psi}_N, \\ \frac{\partial}{\partial u_K} \underline{\Phi}_N &= \underline{\mathbf{U}}_K^{N,\Phi} \underline{\Phi}_N, & \frac{\partial}{\partial v_J} \underline{\Phi}_N &= -\underline{\mathbf{V}}_J^{N,\Phi} \underline{\Phi}_N, \end{aligned} \quad (1-28)$$

where the matrices $\underline{\mathbf{U}}_K^{N,\Psi}(x)$, $\underline{\mathbf{V}}_J^{N,\Psi}(x)$, $\underline{\mathbf{U}}_K^{N,\Phi}(y)$ and $\underline{\mathbf{V}}_J^{N,\Phi}(y)$ are again polynomials in the indicated variables and in the matrix entries of Q and P .

1.3.2 Compatibility

So far, these statements are just a re-writing of the infinite series of recursion relations, differential equations and deformation equations satisfied by the functions $\{\psi_n(x)\}_{n=0\dots\infty}$ and $\{\phi_n(y)\}_{n=0\dots\infty}$, projected onto the finite “windows” represented by the vectors $\underline{\Psi}_N$ and $\underline{\Phi}_N$. However, we may now view these equations as defining an overdetermined system of finite difference-differential-deformation equations for vector functions of the variable $\{N, x, y, u_K, v_J\}$, and ask whether, as such, these systems are *compatible*; i.e., whether they admit a *basis* of simultaneous linearly independent solutions. The affirmative answer to this question is provided in Section 3 by Prop. ??, which states that sequences of invertible $(d_2 + 1) \times (d_2 + 1)$ and $(d_1 + 1) \times (d_1 + 1)$ matrices $\underline{\Psi}_N(x)$ and $\underline{\Phi}_N(y)$ exist (fundamental solutions), for which all the column vectors satisfy the above difference-differential-deformation equations simultaneously.

The compatibility of the deformation equations and finite difference equations with the x and y differential equations imply, in particular, that the (generalized) monodromy of the polynomial covariant derivative operators

$$\frac{\partial}{\partial x} + \underline{D}_1(x), \quad \frac{\partial}{\partial y} + \underline{D}_2(y) \quad (1-29)$$

is invariant under both the $\{u_K, v_J\}$ deformations and the shifts in N . A similar statement can be made of the corresponding operators

$$\frac{\partial}{\partial x} - \underline{D}_1(x), \quad \frac{\partial}{\partial y} - \underline{D}_2(y) \quad (1-30)$$

annihilating the Fourier-Laplace transformed vectors $\underline{\Phi}^{N-1}(x)$ and $\underline{\Psi}^{N-1}(y)$.

1.3.3 Spectral duality

A result that is not at all obvious from the above discussion is proved in Prop. ??; namely, that the pairs of matrices $(\overset{N}{D}_1(x), \overset{N}{D}_2(y))$ and $(\overset{N}{D}_2(y), \overset{N}{D}_1(x))$ have the same *spectral curves*. More specifically, we have the following equalities between their characteristic equations, which actually are identities

$$u_{d_1+1} \det \left[x \mathbf{1}_{d_1+1} - \overset{N}{D}_2(y) \right] = v_{d_2+1} \det \left[y \mathbf{1}_{d_2+1} - \overset{N}{D}_1(x) \right] \quad (1-31)$$

$$u_{d_1+1} \det \left[x \mathbf{1}_{d_1+1} - \overset{N}{D}_2(y) \right] = v_{d_2+1} \det \left[y \mathbf{1}_{d_2+1} - \overset{N}{D}_1(x) \right] . \quad (1-32)$$

(Note that the two integers $d_1 + 1$ and $d_2 + 1$ which determine, in one case, the dimension of the matrix, in the other, its degree as a polynomial in the variable x or y , are interchanged in these equalities, as are the rôles of the variables x and y .)

What is even less obvious, and actually depends on the validity of the Heisenberg commutation relation satisfied by Q and P , is that these curves are not only pairwise equal, but in fact they *all* coincide, since the pairs of matrices $(\overset{N}{D}_1(x), \overset{N}{D}_1(x))$ and $(\overset{N}{D}_2(y), \overset{N}{D}_2(y))$ are conjugate to each other (Theorem ??), so we also have the equalities:

$$\det \left[y \mathbf{1}_{d_2+1} - \overset{N}{D}_1(x) \right] = \det \left[y \mathbf{1}_{d_2+1} - \overset{N}{D}_1(x) \right] \quad (1-33)$$

$$\det \left[x \mathbf{1}_{d_1+1} - \overset{N}{D}_2(y) \right] = \det \left[x \mathbf{1}_{d_1+1} - \overset{N}{D}_2(y) \right] . \quad (1-34)$$

Moreover, the transformations relating them are x and y independent, and are just those defined by the matrices entering in the generalized Christoffel–Darboux relations:

$$\overset{N}{\mathbb{A}} \overset{N}{D}_1(x) = \overset{N}{D}_1(x) \overset{N}{\mathbb{A}}, \quad \overset{N}{\mathbb{B}} \overset{N}{D}_2(y) = \overset{N}{D}_2(y) \overset{N}{\mathbb{B}} . \quad (1-35)$$

These same matrices also relate the system of deformation equations, where they enter as gauge transformations depending on the deformation parameters $\{u_K, v_J\}$.

The key to proving all these results lies in noting (Theorem ??) that, as a consequence of the differential equations and recursion relations satisfied by the ψ_n 's, ϕ_n 's and their Fourier-Laplace transforms, the following quantities are in fact independent of all the variables $N, x, y, \{u_K\}_{K=1\dots d_1+1}$ and $\{v_J\}_{J=1\dots d_2+1}$.

$$\tilde{f}_N(y) := \left(\overset{N-1}{\underline{\Psi}}(y), \overset{N}{\mathbb{B}} \tilde{\Phi}(y) \right), \quad \tilde{g}_N(x) := \left(\overset{N-1}{\tilde{\Phi}}(x), \overset{N}{\mathbb{A}} \tilde{\Psi}(x) \right), \quad (1-36)$$

where $\tilde{\Psi}(x), \tilde{\Phi}(y), \overset{N-1}{\underline{\Psi}}(y), \overset{N-1}{\tilde{\Phi}}(x)$ are *any* solutions of the full system of difference-differential-deformation equations. This allows us to conclude that there exist compatible sequences of fundamental solutions $\overset{N}{\Psi}(x)$ and $\overset{N}{\Phi}(y)$ of the above difference-differential-deformation equations, and the corresponding equations for

the Fourier-Laplace transformed quantities $\overset{N-1}{\underline{\Phi}}, \overset{N-1}{\underline{\Psi}}$ such that

$$\left(\overset{N-1}{\underline{\Phi}}, \overset{N}{\mathbb{A}} \overset{N}{\Psi} \right) \equiv \mathbf{1}, \quad \left(\overset{N-1}{\underline{\Psi}}, \overset{N}{\mathbb{B}} \overset{N}{\Phi} \right) \equiv \mathbf{1}, \quad (1-37)$$

for all values of $\{N, x, y, u_J, v_K\}$.

This fact may be viewed as a form of the bilinear identities implying the existence of τ -functions [?, ?]. The development of this relation, and its connection with the isomonodromic deformation equations, which requires a study of the formal asymptotics of the fundamental solutions near $x = \infty$ and $y = \infty$ will be left to a later work [?].

In the appendix, all the above results are extended to the sequences of multiorthogonal functions that replace the biorthogonal quasi-polynomials in the multi-dimensional case associated to the multi-matrix model discussed above.

2 Biorthogonal Polynomials

2.1 Biorthogonality measure

We first consider sequences of biorthogonal polynomials with respect to the measure arising in the study of the two-matrix model discussed in the introduction. Consideration of the recursion relations obtained by multiplication by or derivation with respect to the independent variable gives rise to representations of the Heisenberg relations (string equation) in terms of pairs of semi-infinite matrices. However, most of the results obtained here may also be shown valid in the fully infinite case.

Let us fix two polynomials, which we refer to as the “potentials”,

$$V_1(x) = \sum_{K=1}^{d_1+1} \frac{u_K}{K} x^K, \quad V_2(y) = \sum_{J=1}^{d_2+1} \frac{v_J}{J} y^J. \quad (2-1)$$

The coupling constants are normalized in a convenient way so that the derivatives are

$$V_1'(x) = \sum_{K=1}^{d_1+1} u_K x^{K-1}. \quad (2-2)$$

We may define two sequences of mutually orthogonal monic polynomials $\pi_n(x), \sigma_n(y)$ of degree n such that

$$\int_{\Gamma_x} dx \int_{\tilde{\Gamma}_y} dy \pi_n(x) \sigma_m(y) e^{-V_1(x) - V_2(y) + xy} = h_n \delta_{mn}, \quad (2-3)$$

$$\pi_n(x) = x^n + \dots, \quad \sigma_n(y) = y^n + \dots. \quad (2-4)$$

In order that the integrals be convergent, one should suitably define the two closed contours of integration $\Gamma_x, \tilde{\Gamma}_y$. If we require that these be the real axis, the degrees of the potentials must be even with the leading coefficient having positive real part. In applications of random matrices to string theory however, the integral is not convergent on the real axis, and the contour should approach ∞ in some appropriate Stokes sector in the complex plane.

It is more convenient to deal with the quasi-polynomials defined by

$$\psi_n(x) = \frac{1}{\sqrt{h_n}} \pi_n(x) e^{-V_1(x)}, \quad \phi_n(y) = \frac{1}{\sqrt{h_n}} \sigma_n(y) e^{-V_2(y)}, \quad (2-5)$$

and their Fourier–Laplace transforms

$$\underline{\psi}_n(y) = \int_{\Gamma_x} dx e^{xy} \psi_n(x), \quad \underline{\phi}_n(x) = \int_{\Gamma_y} dy e^{xy} \phi_n(y). \quad (2-6)$$

The choice of normalization is somewhat arbitrary. We have here chosen the most “symmetric” one, in which the leading coefficients in the various recursion relations are the same. In this notation the orthogonality relations take on a simpler form.

$$\int dx \psi_n(x) \underline{\phi}_m(x) = \int dy \underline{\psi}_n(y) \phi_m(y) = \int \int dx dy \psi_n(x) \phi_m(y) e^{xy} = \delta_{mn} . \quad (2-7)$$

(We suppress for the present the specification of the contour of integration.) We shall think of the spaces spanned by the $\psi_n(x)$ ’s and $\phi_n(y)$ ’s as infinite graded spaces in duality through the pairing in eq. (??). It can easily be seen, using these relations and integration by parts, that multiplication of the ψ_n ’s by x produces a linear combination of ψ_m ’s with $n - d_2 \leq m \leq n + 1$ and multiplication of the ϕ_n ’s by y produces a linear combination of ϕ_j ’s with $n - d_1 \leq j \leq n + 1$. Moreover it is clear (through integration by parts) that multiplication of the ψ_n ’s by x is dual to application of ∂_y to the ϕ_n ’s and vice-versa.

2.2 Recursion relations and generalized Christoffel–Darboux formulae

We denote by Q and P the semi-infinite matrices which implement multiplication and differentiation by x on the space spanned by the ψ_n quasi-polynomials. Introducing the semi-infinite column vectors

$$\underline{\Psi} := [\psi_0, \dots, \psi_n, \dots]^t \quad \text{and} \quad \underline{\Phi} := [\phi_0, \dots, \phi_n, \dots]^t , \quad (2-8)$$

the above remarks imply that

$$\begin{aligned} x \underline{\Psi} &:= Q \underline{\Psi} , & \frac{\partial}{\partial x} \underline{\Psi} &:= P \underline{\Psi} , \\ y \underline{\Phi} &= -P^t \underline{\Phi} , & \frac{\partial}{\partial y} \underline{\Phi} &= -Q^t \underline{\Phi} , \end{aligned} \quad (2-9)$$

where P and Q are semi-infinite matrices of the form

$$Q := \begin{bmatrix} \alpha_0(0) & \gamma(0) & 0 & 0 & \dots \\ \alpha_1(1) & \alpha_0(1) & \gamma(1) & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \alpha_{d_2}(d_2) & \alpha_{d_2-1}(d_2) & \dots & \alpha_0(d_2) & \gamma(d_2) \\ 0 & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \quad (2-10)$$

$$-P := \begin{bmatrix} \beta_0(0) & \beta_1(1) & \dots & \beta_{d_1}(d_1) & \dots \\ \gamma(0) & \beta_0(1) & \beta_1(2) & \ddots & \beta_{d_1}(d_1+1) \\ 0 & \gamma(1) & \beta_0(2) & \ddots & \ddots \\ 0 & 0 & \gamma(2) & \beta_0(3) & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \quad (2-11)$$

satisfying the string equation.

$$[P, Q] = \mathbf{1} . \quad (2-12)$$

The fact that both matrices have a finite band size as indicated in (??), (??) follows from the fact that they are related polynomially to each other through the potentials V_1 and V_2 as follows.

Lemma 2.1 *The two matrices P and Q satisfy the following relations*

$$(P + V_1'(Q))_{\geq 0} = 0, \quad (2-13)$$

$$(-Q^t + V_2'(-P^t))_{\geq 0} = 0, \quad (2-14)$$

where the subscript ≥ 0 means the part above the main diagonal (main diagonal included).

Proof. It is obvious that

$$\begin{aligned} \left[Q \Psi_{\infty} \right]_n &= x \psi_n(x) = x \frac{1}{\sqrt{h_n}} \pi_n(x) e^{-V_1(x)} \\ &= \sqrt{\frac{h_{n+1}}{h_n}} \psi_{n+1}(x) + \text{lower terms.} \end{aligned} \quad (2-15)$$

$$\begin{aligned} \left[(P + V_1'(Q)) \Psi_{\infty} \right]_n &= \left(\frac{\partial}{\partial x} + V_1'(x) \right) \psi_n(x) = \frac{1}{\sqrt{h_n}} \pi_n'(x) e^{-V_1(x)} \\ &= n \sqrt{\frac{h_{n-1}}{h_n}} \psi_{n-1}(x) + \text{lower terms.} \end{aligned} \quad (2-16)$$

Eq. (??) means that Q has only one diagonal above the main diagonal with entries given by

$$\gamma(n) = \sqrt{h_{n+1}/h_n}. \quad (2-17)$$

Eq. (??) then implies that eq. (??) holds and P has d_1 diagonals above the main one. Repeating the argument for the ϕ_n quasi-polynomials similarly shows that $-P^t$ (i.e. multiplication by y) has one diagonal above the main one, eq. (??) holds and $-Q^t$ (i.e. differentiation by y) has d_2 upper diagonals. This proves the lemma and also shows that P and Q are of the finite band sizes indicated in (??), (??). Q.E.D.

Eqs. (??) are just equivalent to the following set of recursion relations

$$\left[Q \Psi_{\infty} \right]_n = x \psi_n(x) = \gamma(n) \psi_{n+1}(x) + \sum_{j=0}^{d_2} \alpha_j(n) \psi_{n-j}(x) \quad (2-18)$$

$$\left[P \Psi_{\infty} \right]_n = \frac{\partial}{\partial x} \psi_n(x) = -\gamma(n-1) \psi_{n-1}(x) - \sum_{j=0}^{d_1} \beta_j(n+j) \psi_{n+j}(x) \quad (2-19)$$

$$\left[-P^t \Phi_{\infty} \right]_n = y \phi_n(y) = \gamma(n) \phi_{n+1}(y) + \sum_{j=0}^{d_1} \beta_j(n) \phi_{n-j}(y) \quad (2-20)$$

$$\left[-Q^t \Phi_{\infty} \right]_n = \frac{\partial}{\partial y} \phi_n(y) = -\gamma(n-1) \phi_{n-1}(y) - \sum_{j=0}^{d_2} \alpha_j(n+j) \phi_{n+j}(y), \quad (2-21)$$

In the following we also define

$$\alpha_{-1}(n) := \gamma(n) =: \beta_{-1}(n), \quad \alpha_j(n) := 0, \quad \forall j \notin [-1, d_2], \quad \beta_j(n) := 0, \quad \forall j \notin [-1, d_1]. \quad (2-22)$$

Defining similarly the semi-infinite row vectors consisting of the Fourier-Laplace transformed functions

$$\underline{\Psi}_{\infty} := [\underline{\psi}_0, \dots, \underline{\psi}_n, \dots] \quad \text{and} \quad \underline{\Phi}_{\infty} := [\underline{\phi}_0, \dots, \underline{\phi}_n, \dots], \quad (2-23)$$

it follows from the dual pairing (??), and integration by parts that:

Lemma 2.2

$$\left[-\frac{\Psi}{\infty} P^t \right]_n = y \underline{\psi}_n(y) = \sum_{j=-1}^{d_1} \beta_j(n+j) \underline{\psi}_{n+j}(y), \quad (2-24)$$

$$\left[\frac{\Psi}{\infty} Q^t \right]_n = \partial_y \underline{\psi}_n(y) = \sum_{l=-1}^{d_2} \alpha_l(n) \underline{\psi}_{n-l}(y), \quad (2-25)$$

$$\left[\frac{\Phi}{\infty} Q \right]_n = x \underline{\phi}_n(x) = \sum_{l=-1}^{d_2} \alpha_l(n+l) \underline{\phi}_{n+l}(x), \quad (2-26)$$

$$\left[-\frac{\Phi}{\infty} P \right]_n = \partial_x \underline{\phi}_n(x) = \sum_{j=-1}^{d_1} \beta_j(n) \underline{\phi}_{n-j}(x). \quad (2-27)$$

Now, we introduce the shift matrices

$$\Lambda := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \ddots \end{bmatrix}, \quad \Lambda^{-1} := \Lambda^t = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \ddots & \cdots \end{bmatrix} \quad (2-28)$$

(The notation Λ^{-1} is a convenient shorthand for the transpose Λ^t , but only signifies that Λ^{-1} is the *right* inverse of Λ . It leads to the abbreviated form $\Lambda^{-j} := (\Lambda^t)^j$.) Introducing the diagonal semi-infinite matrices

$$\alpha_j := \text{diag}(\alpha_j(0), \alpha_j(1), \dots), \quad \beta_j := \text{diag}(\beta_j(0), \beta_j(1), \dots), \quad (2-29)$$

(where we set $\alpha_j(n) := 0$ and $\beta_j(n) := 0$ when $n < j$), eqs. (??), (??) can be concisely written as

$$Q := \sum_{j=-1}^{d_2} \Lambda^{-j} \alpha_j; \quad -P := \sum_{j=-1}^{d_1} \Lambda^j \beta_j. \quad (2-30)$$

The commutation relation in eq. (??) gives in particular the following quadratic relations between the coefficients $\{\alpha_j, \beta_k\}$,

$$\sum_{j=l}^{d_1} \beta_j(n+j) \alpha_{j-l-1}(n+j) = \sum_{j=l}^{d_1} \alpha_{j-l-1}(n) \beta_j(n+l+1), \quad \forall n, \forall l \in [0, d_1] \quad (2-31)$$

$$\sum_{k=l}^{d_2} \beta_{k-l-1}(n+k) \alpha_k(n+k) = \sum_{k=l}^{d_2} \alpha_k(n+l+1) \beta_{k-l-1}(n), \quad \forall n, \forall l \in [0, d_2]. \quad (2-32)$$

Our first objective is to define a set of closed differential and difference systems for vectors consisting of finite sequences of the functions $\{\psi_n\}$, $\{\phi_n\}$ and their Fourier-Laplace transforms, equivalent to the systems (??)-(??), and to study their properties and relations. Consider first the sequence of functions $\{\psi_n(x)\}$. For these we have a multiplicative recursion relation defined by the coefficients $\{\alpha_j(n)\}$ and a differential recursion relation defined by the coefficients $\{\beta_j(n)\}$. We shall show presently that we can define *closed* systems of first order linear ODE's for any consecutive sequence of $d_2 + 1$ functions $(\psi_{N-d_2}, \dots, \psi_N)$ (or $(\underline{\phi}_{N-1}, \dots, \underline{\phi}_{N+d_2-1})$) with coefficients that are polynomials in x . Similar systems can be constructed for any sequence $(\phi_{N-d_1}, \dots, \phi_N)$ (or $(\underline{\psi}_{N-1}, \dots, \underline{\psi}_{N+d_1-1})$). We introduce the following definitions and notations

Definition 2.1 A window of size \mathbf{d}_1 or \mathbf{d}_2 is any subset of d_1 or d_2 consecutive elements of type ψ_n , ϕ_n , $\underline{\phi}_n$ or $\underline{\psi}_n$, with the notations

$$\underline{\Psi}_N := [\psi_{N-d_2}, \dots, \psi_N]^t, \quad N \geq d_2, \quad \underline{\Phi}_N := [\phi_{N-d_1}, \dots, \phi_N]^t, \quad N \geq d_1 \quad (2-33)$$

$$\underline{\Psi}_N := [\psi_N, \dots, \psi_{N+d_1}]^t, \quad N \geq 0, \quad \underline{\Phi}_N := [\phi_N, \dots, \phi_{N+d_2}]^t, \quad N \geq 0 \quad (2-34)$$

$$\underline{\Psi}_N := [\underline{\psi}_{N-d_2}, \dots, \underline{\psi}_N], \quad N \geq d_2, \quad \underline{\Phi}_N := [\underline{\phi}_{N-d_1}, \dots, \underline{\phi}_N], \quad N \geq d_1 \quad (2-35)$$

$$\underline{\Psi}_N := [\underline{\psi}_N, \dots, \underline{\psi}_{N+d_1}], \quad N \geq 0, \quad \underline{\Phi}_N := [\underline{\phi}_N, \dots, \underline{\phi}_{N+d_2}], \quad N \geq 0. \quad (2-36)$$

Notice the difference in the positioning of the windows for the vectors constructed from the ψ_n 's and the ϕ_n 's, and the fact that the barred quantities are defined to be row vectors while the unbarred ones are column vectors.

Definition 2.2 For any N for which these are defined, the pairs of windows $(\underline{\Psi}_N, \underline{\Phi}_N)$ as well as $(\underline{\Phi}_N, \underline{\Psi}_N)$ of dimensions $d_2 + 1$ and $d_1 + 1$, respectively, will be called **dual windows**.

The reason for identifying these particular windows as dual will appear in the sequel.

Let us now consider the kernels

$$K_{11}^N(x, x') := \sum_{n=0}^{N-1} \psi_n(x) \underline{\phi}_n(x'), \quad K_{22}^N(y', y) := \sum_{n=0}^{N-1} \underline{\psi}_n(y') \phi_n(y), \quad (2-37)$$

$$K_{12}^N(x, y) := \sum_{n=0}^{N-1} \psi_n(x) \phi_n(y), \quad K_{21}^N(y', x') := \sum_{n=0}^{N-1} \underline{\psi}_n(y') \underline{\phi}_n(x'), \quad (2-38)$$

that appear in the computation of correlation functions for 2-matrix models [?].

Define the following pair of matrices, which will play an important rôle in what follows:

$$\mathbb{A}_N := \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & -\gamma(N-1) \\ \alpha_{d_2}(N) & \cdots & \alpha_2(N) & \alpha_1(N) & 0 \\ 0 & \alpha_{d_2}(N+1) & \cdots & \alpha_1(N+1) & 0 \\ 0 & 0 & \alpha_{d_2}(N+2) & \cdots & 0 \\ 0 & 0 & 0 & \alpha_{d_2}(N+d_2-1) & 0 \end{array} \right]; \quad (2-39)$$

$$\mathbb{B}_N := \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & -\gamma(N-1) \\ \beta_{d_1}(N) & \cdots & \beta_2(N) & \beta_1(N) & 0 \\ 0 & \beta_{d_1}(N+1) & \cdots & \beta_1(N+1) & 0 \\ 0 & 0 & \beta_{d_1}(N+2) & \cdots & 0 \\ 0 & 0 & 0 & \beta_{d_1}(N+d_1-1) & 0 \end{array} \right] \quad (2-40)$$

For any N , the recursion relations (??), (??), (??), (??) and the differential relations (??), (??), (??), (??) imply that the following generalized Christoffel–Darboux formulae, as well as their “differential” analogs are satisfied.

Proposition 2.1 Generalized Christoffel–Darboux relations:

$$(x - x') K_{11}^N(x, x') = \gamma(N-1) \psi_N \underline{\phi}_{N-1} - \sum_{j=1}^{d_2} \sum_{k=0}^{j-1} \alpha_j(N+k) \underline{\phi}_{N+k} \psi_{N+k-j}$$

$$= - \left(\underline{\Phi}^{N-1}(x'), \overset{N}{\mathbb{A}} \overset{N}{\Psi}(x) \right), \quad (2-41)$$

$$\begin{aligned} (y' - y) \overset{N}{K}_{22}(y', y) &= -\gamma(N-1) \underline{\psi}_{N-1} \phi_N + \sum_{j=1}^{d_1} \sum_{k=0}^{j-1} \beta_j(N+k) \underline{\psi}_{N+k} \phi_{N-j+k} \\ &= \left(\underline{\Psi}^{N-1}(y'), \overset{N}{\mathbb{B}} \overset{N}{\Phi}(y) \right). \end{aligned} \quad (2-42)$$

“Differential” generalized Christoffel–Darboux relations:

$$(\partial_{x'} + \partial_x) \overset{N}{K}_{11}(x', x) = - \left(\underline{\Phi}^{N-1}(x'), (\overset{N}{\mathbb{B}})^t \overset{N-1}{\Psi}(x) \right), \quad (2-43)$$

$$(\partial_{y'} + \partial_y) \overset{N}{K}_{22}(y', y) = - \left(\underline{\Psi}^{N-1}(y'), (\overset{N}{\mathbb{A}})^t \overset{N-1}{\Phi}(y) \right). \quad (2-44)$$

Proof. Use the relations (??)-(??), (??)-(??) and simplify the telescopic sums by cancellation of common terms. Q.E.D.

Although it will not be needed in the remainder of this paper, for the sake of completeness, we also include the following analogous result for the kernels K_{12} and K_{21} , which may be similarly derived. It is related to the above by applying Fourier-Laplace transforms with respect to one of the variables.

Proposition 2.2

$$(x + \partial_y) \overset{N}{K}_{12}(x, y) = - \left(\underline{\Phi}^{N-1}(y), \overset{N}{\mathbb{A}} \overset{N}{\Psi}(x) \right), \quad (2-45)$$

$$(y + \partial_x) \overset{N}{K}_{12}(x, y) = - \left(\underline{\Psi}^{N-1}(y'), \overset{N}{\mathbb{B}} \overset{N}{\Phi}(y) \right), \quad (2-46)$$

$$(y' - \partial_{x'}) \overset{N}{K}_{21}(y', x') = \left(\underline{\Psi}^{N-1}(y'), \overset{N}{\mathbb{B}} \overset{N}{\Phi}(x') \right), \quad (2-47)$$

$$(\partial_{y'} - x') \overset{N}{K}_{21}(y', x') = - \left(\underline{\Phi}^{N-1}(x'), \overset{N}{\mathbb{A}} \underline{\Psi}^t(y') \right). \quad (2-48)$$

2.3 Folding

We now introduce the sequence of companion–like matrices $\mathbf{a}_N(x)$ and $\mathbf{b}_N(y)$ of sizes $d_2 + 1$ and $d_1 + 1$, respectively.

$$\mathbf{a}_N(x) := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\alpha_{d_2}(N)}{\gamma(N)} & \dots & \frac{-\alpha_1(N)}{\gamma(N)} & \frac{(x-\alpha_0(N))}{\gamma(N)} \end{bmatrix}, \quad N \geq d_2, \quad (2-49)$$

$$\mathbf{b}_N(y) := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\beta_{d_1}(N)}{\gamma(N)} & \dots & \frac{-\beta_1(N)}{\gamma(N)} & \frac{(y-\beta_0(N))}{\gamma(N)} \end{bmatrix}, \quad N \geq d_1. \quad (2-50)$$

We then have the following:

Lemma 2.3 *The sequence of matrices $\mathbf{a}_N, \mathbf{b}_N$ implement the shift $N \mapsto N+1$ in the windows of quasi-polynomials in the sense that*

$$\mathbf{a}_N \Psi(x) = \Psi_{N+1}(x), \quad \mathbf{b}_N \Phi(y) = \Phi_{N+1}(y), \quad (2-51)$$

and in general

$$\Psi_{N+j} = \mathbf{a}_{N+j-1} \cdots \mathbf{a}_N \Psi, \quad \Phi_{N+j} = \mathbf{b}_{N+j-1} \cdots \mathbf{b}_N \Phi. \quad (2-52)$$

Proof. This is nothing but a matricial form of the sequence of recursion relations (??), (??) expressing the higher order polynomials as linear combinations of a fixed subset with polynomial coefficients. Q.E.D.

We will refer to this process of expressing any $\psi_n(x)$ by means of linear combinations of elements in a specific window with polynomial coefficients as *folding* onto the specified window.

The determinants of the matrices \mathbf{a}_N and \mathbf{b}_N are easily computed to be

$$\det(\mathbf{a}_N) = (-1)^{d_2+1} \alpha_{d_2}(N) / \gamma(N), \quad \det(\mathbf{b}_N) = (-1)^{d_1+1} \beta_{d_1}(N) / \gamma(N). \quad (2-53)$$

From eqs. (??), (??) we find the relations

$$\alpha_{d_2}(N) = v_{d_2+1} \prod_{j=1}^{d_2} \gamma(N-j); \quad \beta_{d_1}(N) = u_{d_1+1} \prod_{j=1}^{d_1} \gamma(N-j). \quad (2-54)$$

Since the coefficients $\gamma(N)$ are the square roots of the ratios of normalization factors, they cannot vanish for any N , and neither can $\alpha_{d_2}(N)$ or $\beta_{d_1}(N)$, since the deformation parameters u_{d_1+1}, v_{d_2+1} are the leading coefficients of the polynomials $V_1(x), V_2(y)$ and hence also may not vanish. It follows that the matrices \mathbf{a}_N and \mathbf{b}_N are all invertible. We denote their inverses as follows

$$\mathbf{a}_N^N := \left[\mathbf{a}_N \right]^{-1} = \begin{bmatrix} \frac{-\alpha_{d_2-1}(N)}{\alpha_{d_2}(N)} & \cdots & \frac{x-\alpha_0(N)}{\alpha_{d_2}(N)} & \frac{-\gamma(N)}{\alpha_{d_2}(N)} \\ 1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (2-55)$$

$$\mathbf{b}_N^N := \left[\mathbf{b}_N \right]^{-1} = \begin{bmatrix} \frac{-\beta_{d_1-1}(N)}{\beta_{d_1}(N)} & \cdots & \frac{y-\beta_0(N)}{\beta_{d_1}(N)} & \frac{-\gamma(N)}{\beta_{d_1}(N)} \\ 1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (2-56)$$

The shifts $N \rightarrow N-1$ are thus implemented by the inverse matrices $\mathbf{a}_N^N, \mathbf{b}_N^N$, and the folding may take place in either direction with respect to polynomial degrees.

2.4 Folded linear differential systems

We now define the following sequences of finite diagonal matrices

$${}^N\alpha_j := \text{diag}[\alpha_j(N+j-d_1), \alpha_j(N+j-d_1+1), \dots, \alpha_j(N+j)] , \quad j = -1, \dots, d_2 , \quad (2-57)$$

$${}^N\beta_j := \text{diag}[\beta_j(N+j-d_2), \beta_j(N+j-d_2+1), \dots, \beta_j(N+j)] , \quad j = -1, \dots, d_1 . \quad (2-58)$$

Recall that $\alpha_{-1}(n) = \gamma(n) = \beta_{-1}(n)$ by our previous conventions, but the diagonal matrices ${}^N\alpha_{-1}$ and ${}^N\beta_{-1}$ differ in dimensions. When the context leaves no doubt as to the dimension we will write them as

$${}^N\alpha_{-1} = {}^N\gamma := \text{diag}[\gamma(N-d_1-1), \dots, \gamma(N-1)] \quad (2-59)$$

or

$${}^N\beta_{-1} = {}^N\gamma := \text{diag}[\gamma(N-d_2-1), \gamma(N-d_2), \dots, \gamma(N-1)] . \quad (2-60)$$

In either case, we denote the inverse matrix as

$${}^N\gamma := ({}^N\gamma)^{-1} . \quad (2-61)$$

We can now give the closed differential systems referred to previously.

Lemma 2.4 *The windows of quasi-polynomials Ψ, Φ satisfy the following differential systems*

$$\frac{\partial}{\partial x} {}^N\Psi = -{}^N D_1(x) {}^N\Psi , \quad N \geq d_2 + 1 , \quad (2-62)$$

$$\frac{\partial}{\partial y} {}^N\Phi = -{}^N D_2(y) {}^N\Phi , \quad N \geq d_1 + 1 , \quad (2-63)$$

where

$${}^N D_1(x) := {}^N\gamma {}^N\mathbf{a}^{N-1} + \beta_0 + \sum_{j=1}^{d_1} \beta_j {}^N\mathbf{a}_{N+j-1} {}^N\mathbf{a}_{N+j-2} \cdots {}^N\mathbf{a}_N \in gl_{d_2+1}[x] . \quad (2-64)$$

$${}^N D_2(y) := {}^N\gamma {}^N\mathbf{b}^{N-1} + \alpha_0 + \sum_{j=1}^{d_2} \alpha_j {}^N\mathbf{b}_{N+j-1} {}^N\mathbf{b}_{N+j-2} \cdots {}^N\mathbf{b}_N \in gl_{d_1+1}[y] . \quad (2-65)$$

These, taken together for all N are equivalent to the relations (??), (??) when the recursion relations (??), (??) are taken into account.

Proof. Consider the case of the ψ_n 's. The differential relations (??) may be written by stacking them in a window of size $d_2 + 1$, as follows

$$\frac{\partial}{\partial x} {}^N\Psi = -{}^N\gamma {}^N\Psi_{N-1} - \sum_{j=0}^{d_2} \beta_j {}^N\Psi_{N+j} . \quad (2-66)$$

Using the folding relations eq. (??), we immediately obtain (??), (??). The same procedure applied to eq. (??) yields (??), (??). Q.E.D.

We can repeat a similar procedure for the sequences $\{\underline{\psi}_n(y)\}_{n \in \mathbb{N}}$ and $\{\underline{\phi}_n(x)\}_{n \in \mathbb{N}}$. The corresponding windows are represented as row vectors $\underline{\Psi}^N$ and $\underline{\Phi}^N$ since their components are naturally dual to the ϕ_n 's and ψ_n 's respectively. The matrices defining the relevant folding are now

$$\underline{\mathbf{a}}^N := \begin{bmatrix} \frac{x - \alpha_0(N)}{\gamma(N-1)} & 1 & 0 & 0 \\ \frac{-\alpha_1(N+1)}{\gamma(N-1)} & 0 & \ddots & 0 \\ \vdots & 0 & 0 & 1 \\ \frac{-\alpha_{d_2}(N+d_2)}{\gamma(N-1)} & 0 & 0 & 0 \end{bmatrix}, \quad (2-67)$$

$$\underline{\mathbf{b}}^N := \begin{bmatrix} \frac{y - \beta_0(N)}{\gamma(N-1)} & 1 & 0 & 0 \\ \frac{-\beta_1(N+1)}{\gamma(N-1)} & 0 & \ddots & 0 \\ \vdots & 0 & 0 & 1 \\ \frac{-\beta_{d_1}(N+d_1)}{\gamma(N-1)} & 0 & 0 & 0 \end{bmatrix}, \quad (2-68)$$

and we again denote their inverses as

$$\underline{\mathbf{a}}^N := \left[\underline{\mathbf{a}}^N \right]^{-1}, \quad \underline{\mathbf{b}}^N := \left[\underline{\mathbf{b}}^N \right]^{-1}. \quad (2-69)$$

As previously, we now have:

Lemma 2.5 *The sequence of matrices $\{\underline{\mathbf{a}}^N\}$ $\{\underline{\mathbf{b}}^N\}$ implement the shift $N \mapsto N - 1$*

$$\underline{\Psi}^{N-1} = \underline{\Psi}^N \underline{\mathbf{b}}^N; \quad \underline{\Phi}^{N-1} = \underline{\Phi}^N \underline{\mathbf{a}}^N. \quad (2-70)$$

Similarly to the diagonal matrices α_j , β_j , we define the matrices $\underline{\alpha}_j^N$ and $\underline{\beta}_j^N$ as

$$\underline{\alpha}_j^N := \text{diag}(\alpha_j(N), \alpha_j(N+1), \dots, \alpha_j(N+d_1)) \quad , \quad j = -1, \dots, d_2 \quad (2-71)$$

$$\underline{\beta}_j^N := \text{diag}(\beta_j(N), \beta_j(N+1), \dots, \beta_j(N+d_2)) \quad , \quad j = -1, \dots, d_1. \quad (2-72)$$

As before we have the two definitions

$$\underline{\alpha}_{-1}^N := \underline{\gamma}^N := \text{diag}(\gamma(N), \gamma(N+1), \dots, \gamma(N+d_1)) \quad (2-73)$$

$$\underline{\beta}_{-1}^N := \underline{\gamma}^N := \text{diag}(\gamma(N), \gamma(N+1), \dots, \gamma(N+d_2)) \quad , \quad (2-74)$$

which will be used if there is no ambiguity regarding dimensions.

By repeating a procedure similar to what led to the differential systems in Lemma ??, we find:

Lemma 2.6 *The dual windows of Laplace-transformed quasi-polynomials $\underline{\Psi}^{N-1}$, $\underline{\Phi}^{N-1}$ satisfy the following differential systems*

$$\frac{\partial}{\partial y} \underline{\Psi}^{N-1}(y) = \underline{\Psi}^{N-1}(y) \underline{D}_2^N(y) \quad , \quad N \geq d_1 + 1 \quad , \quad (2-75)$$

$$\frac{\partial}{\partial x} \underline{\Phi}^{N-1}(x) = \underline{\Phi}^{N-1}(x) \underline{D}_1^N(x) \quad , \quad N \geq d_2 + 1 \quad , \quad (2-76)$$

where

$$\underline{D}_2(y) := \underline{\mathbf{b}} \frac{N}{N} \underline{\gamma} + \underline{\alpha}_0 + \sum_{j=1}^{d_2} \underline{\mathbf{b}}^{N-1} \underline{\mathbf{b}}^{N-2} \cdots \underline{\mathbf{b}}^{N-j} \underline{\alpha}_j, \quad (2-77)$$

$$\underline{D}_1(x) := \underline{\mathbf{a}} \frac{N}{N} \underline{\gamma} + \underline{\beta}_0 + \sum_{j=1}^{d_1} \underline{\mathbf{a}}^{N-1} \underline{\mathbf{a}}^{N-2} \cdots \underline{\mathbf{a}}^{N-j} \underline{\beta}_j. \quad (2-78)$$

Summarizing, we have thus obtained four differential systems

Size $(d_2 + 1) \times (d_2 + 1)$	Size $(d_1 + 1) \times (d_1 + 1)$
$\frac{\partial}{\partial x} \underline{\Psi}(x) = -\underline{D}_1(x) \underline{\Psi}(x)$	$\frac{\partial}{\partial y} \underline{\Psi}(y) = \underline{\Psi}(y) \underline{D}_2(y)$
$\frac{\partial}{\partial x} \underline{\Phi}(x) = \underline{\Phi}(x) \underline{D}_1(x)$	$\frac{\partial}{\partial y} \underline{\Phi}(y) = -\underline{D}_2(y) \underline{\Phi}(y)$

(2-79)

It should be noted that the two matrices D_1 and \underline{D}_1 (as well as D_2 and \underline{D}_2) have so far only superficial similarities. In particular they *do not* depend on the same subsets of the coefficients $\{\alpha_j(n)\}$ and $\{\beta_j(n)\}$. On the other hand the pairs (D_1, \underline{D}_2) and (D_2, \underline{D}_1) *do* depend on the same $\alpha_j(n)$'s and $\beta_j(n)$'s although they are of different dimensions.

2.5 Deformation equations

The following Lemma gives the effect of an infinitesimal deformation in the coefficients $\{u_K, v_J\}$ expressed as differential equations for the (semi)-infinite vectors Ψ_∞, Φ_∞ of biorthogonal quasi-polynomials (as well as their Fourier–Laplace transforms) and for the matrices P, Q . (Derivations with different conventions can be found in [?, ?, ?].)

Lemma 2.7

$$\partial_{u_K} \Psi_\infty = U^K \Psi_\infty, \quad (2-80)$$

$$\partial_{v_J} \Psi_\infty = -\left(V^J\right)^t \Psi_\infty, \quad (2-81)$$

$$\partial_{u_K} \Phi_\infty = -\left(U^K\right)^t \Phi_\infty, \quad (2-82)$$

$$\partial_{v_J} \Phi_\infty = V^J \Phi_\infty, \quad (2-83)$$

where

$$U^K := -\frac{1}{K} \left\{ [Q^K]_{>0} + \frac{1}{2} [Q^K]_0 \right\}, \quad V^J := -\frac{1}{J} \left\{ [(-P^t)^J]_{>0} + \frac{1}{2} [(-P^t)^J]_0 \right\}. \quad (2-84)$$

Componentwise these read,

$$\partial_{u_K} \psi_n(x) = \sum_{j=0}^K U_j^K(n) \psi_{n+j}(x), \quad (2-85)$$

$$\partial_{v_J} \psi_n(x) = -\sum_{j=0}^J V_j^J(n-j) \psi_{n-j}(x), \quad (2-86)$$

$$\partial_{u_K} \phi_n(y) = -\sum_{j=0}^K U_j^K(n-j) \phi_{n-j}(y), \quad (2-87)$$

$$\partial_{v_j} \phi_n(y) = \sum_{j=0}^J V_j^J(n) \phi_{n+j}(y) , \quad (2-88)$$

where we have used the notation

$$U_j^K(n) := U_{n,j+n}^K , \quad V_j^J(n) := V_{n,j+n}^J . \quad (2-89)$$

(The same relations hold for the Fourier-Laplace transforms with respect to the variables x and y , since the coefficients do not depend on these variables.)

Moreover, we have the following equations for the matrices P, Q :

$$\partial_{u_K} Q = -[Q, U^K], \quad (2-90)$$

$$\partial_{v_j} Q = [Q, V^{J^t}] , \quad (2-91)$$

$$\partial_{u_K} P = -[P, U^K] , \quad (2-92)$$

$$\partial_{v_j} P = [P, V^{J^t}] . \quad (2-93)$$

Proof: Equations (??) and (??) are just definitions, (??) and (??) follow from (??). Eq. (??) is proved in a way similar to Lemma ???. From the definitions (??), one has:

$$\left[U^K \Psi_\infty \right]_n = \partial_{u_K} \psi_n(x) = -\frac{1}{2} \frac{\partial_{u_K} h_n}{h_n} \psi_n(x) + \frac{1}{\sqrt{h_n}} \partial_{u_K} \pi_n(x) e^{-V_1(x)} - \frac{1}{K} x^K \psi_n(x) , \quad (2-94)$$

and

$$\left[-(U^K)^t \Phi_\infty \right]_n = \partial_{u_K} \phi_n(y) = -\frac{1}{2} \frac{\partial_{u_K} h_n}{h_n} \phi_n(y) + \frac{1}{\sqrt{h_n}} \partial_{u_K} \sigma_n(y) e^{-V_2(y)} . \quad (2-95)$$

Since $\partial_{u_K} \pi_n(x)$ has a degree lower than n , one sees that eq. (??) implies that

$$(U^K)_{>0} = -\frac{1}{K} (Q^K)_{>0} , \quad (2-96)$$

and similarly eq. (??) implies that

$$(U^K)_{<0} = 0 . \quad (2-97)$$

They also imply that the diagonal part must be:

$$U_{n,n}^K = -\frac{1}{2} \frac{\partial_{u_K} h_n}{h_n} - \frac{1}{K} (Q^K)_{n,n} = \frac{1}{2} \frac{\partial_{u_K} h_n}{h_n} = -\frac{1}{2K} (Q^K)_{n,n} , \quad (2-98)$$

which proves (??).

The equations (??)–(??) follow from multiplying (??) and (??) by x and (??) and (??) by y and the linear independence of the component functions forming the vectors. The coefficients of the expansion must vanish, and that is precisely the relations (??) to (??). Q.E.D.

We also require the “folded” version of the deformation equations. This leads to eight equations giving the action of ∂_{u_K} and ∂_{v_j} on Ψ_N , $\underline{\Psi}_N$, Φ_N and $\underline{\Phi}_N$. They are introduced in the following lemma, for which we need to define diagonal matrices which play rôles similar to that of the matrices defined in (??), (??) (??), (??) for the differential equations with respect to x or y .

$$\overset{N,d}{U}_{j,K} := \text{diag}(U_j^K(N-d), \dots, U_j^K(N)) , \quad (2-99)$$

$$\overset{N,d}{V}_{j,J} := \text{diag}(V_j^J(N-d-j), \dots, V_j^J(N-j)) . \quad (2-100)$$

With this notation we have:

Lemma 2.8 *The deformation equations can be written in the folded windows (and dual windows) as*

$$\begin{aligned} \frac{\partial}{\partial u_K} \Psi_N &= \mathbf{U}_K^{N,\Psi} \Psi_N, & \frac{\partial}{\partial u_K} \underline{\Psi}_N &= \underline{\Psi}_N \mathbf{U}_K^{N-1, \underline{\Psi}}, \\ \frac{\partial}{\partial v_J} \Psi_N &= -\mathbf{V}_J^{N,\Psi} \Psi_N, & \frac{\partial}{\partial v_J} \underline{\Psi}_N &= -\underline{\Psi}_N \mathbf{V}_J^{N-1, \underline{\Psi}}, \end{aligned} \quad (2-101)$$

$$\begin{aligned} \frac{\partial}{\partial u_K} \Phi_N &= -\mathbf{U}_K^{N,\Phi} \Phi_N, & \frac{\partial}{\partial u_K} \underline{\Phi}_N &= -\underline{\Phi}_N \mathbf{U}_K^{N-1, \underline{\Phi}}, \\ \frac{\partial}{\partial v_J} \Phi_N &= \mathbf{V}_J^{N,\Phi} \Phi_N, & \frac{\partial}{\partial v_J} \underline{\Phi}_N &= \underline{\Phi}_N \mathbf{V}_J^{N-1, \underline{\Phi}}, \end{aligned} \quad (2-102)$$

where

$$\begin{aligned} \mathbf{U}_K^{N,\Psi} &:= \sum_{j=0}^K U_{j,K}^{N,d_2} \mathbf{a}_{N+j-1} \cdots \mathbf{a}_N, & \underline{\mathbf{U}}_K^{N,\Psi} &:= \sum_{j=0}^K \mathbf{b}_N \cdots \mathbf{b}_{N+j-1} U_{j,K}^{N+d_1-1, d_1}, \\ \mathbf{V}_J^{N,\Psi} &:= \sum_{j=0}^J V_{j,J}^{N,d_2} \mathbf{a}^{N-j} \cdots \mathbf{a}^{N-1}, & \underline{\mathbf{V}}_J^{N,\Psi} &:= \sum_{j=0}^J \mathbf{b}^{N-1} \cdots \mathbf{b}^{N-j} V_{j,J}^{N+d_1-1, d_1}, \end{aligned} \quad (2-103)$$

$$\begin{aligned} \mathbf{U}_K^{N,\Phi} &:= \sum_{j=0}^K U_{j,K}^{N-j, d_1} \mathbf{b}^{N-j} \cdots \mathbf{b}^{N-1}, & \underline{\mathbf{U}}_K^{N,\Phi} &:= \sum_{j=0}^K \mathbf{a}^{N-1} \cdots \mathbf{a}^{N-j} U_{j,K}^{N+d_2-1-j, d_2}, \\ \mathbf{V}_J^{N,\Phi} &:= \sum_{j=0}^J V_{j,J}^{N+j, d_1} \mathbf{b}_{N+j-1} \cdots \mathbf{b}_N, & \underline{\mathbf{V}}_J^{N,\Phi} &:= \sum_{j=0}^J \mathbf{a}_N \cdots \mathbf{a}_{N-1+j} V_{j,J}^{N+j+d_2-1, d_2}. \end{aligned} \quad (2-104)$$

Proof. This follows exactly the same lines as the proof of Lemma ??.

Q.E.D.

3 Compatibility of the finite difference–differential–deformation systems

We want to now prove that the recursion relations (??), (??), the linear differential systems (??) and the systems of deformation equations (??), (??) are all *compatible* in the sense that they admit a *basis* of simultaneous solutions (fundamental systems).

Proposition 3.1 The shifts $N \mapsto N + 1$ in eqs. (??) implemented by \mathbf{a}_N and \mathbf{b}_N and the sequence of differential equations (??), (??), respectively, are compatible as vector differential–difference systems.

That is, there exists a sequence of $(d_2 + 1) \times (d_2 + 1)$ fundamental matrix solutions $\left\{ \Psi_N(x) \right\}_{N \geq d_2+1}$ and

$(d_1 + 1) \times (d_1 + 1)$ fundamental matrix solutions $\left\{ \Phi_N(y) \right\}_{N \geq d_1+1}$ simultaneously satisfying the equations

$$\Psi_{N+1}(x) = \mathbf{a}_N(x) \Psi_N(x), \quad (3-1)$$

$$\frac{\partial}{\partial x} \Psi_N(x) = -D_1^N(x) \Psi_N(x), \quad N \geq d_2 + 1, \quad (3-2)$$

and

$$\Phi_{N+1}(y) = \mathbf{b}_N(y) \Phi_N(y), \quad (3-3)$$

$$\frac{\partial}{\partial y} \Phi_N(y) = -D_2^N(y) \Phi_N(y), \quad N \geq d_1 + 1, \quad (3-4)$$

respectively. The same result holds for the barred quantities and the shifts $N \mapsto N - 1$ implemented by $\underline{\mathbf{a}}^N$ and $\underline{\mathbf{b}}^N$. That is, there exist fundamental solutions $\{\underline{\Psi}^{N-1}(y)\}_{N \geq d_2+1}$ of dimension $(d_2 + 1) \times (d_2 + 1)$ and fundamental solutions $\{\underline{\Phi}^{N-1}(x)\}_{N \geq d_2+1}$ of dimension $(d_1 + 1) \times (d_1 + 1)$ simultaneously satisfying the recursion relations and differential systems

$$\underline{\Psi}^{N-1}(y) = \underline{\Psi}^N(y) \underline{\mathbf{b}}^N(y), \quad (3-5)$$

$$\frac{\partial}{\partial y} \underline{\Psi}^{N-1}(y) = \underline{\Psi}^{N-1}(y) \underline{D}_2^N(y), \quad N \geq d_1 + 1, \quad (3-6)$$

and

$$\underline{\Phi}^{N-1}(x) = \underline{\Phi}^N(x) \underline{\mathbf{a}}^N(x), \quad (3-7)$$

$$\frac{\partial}{\partial x} \underline{\Phi}^{N-1}(x) = \underline{\Phi}^{N-1}(x) \underline{D}_1^N(x), \quad N \geq d_2 + 1, \quad (3-8)$$

respectively.

Proof. We prove the compatibility for only one of the four shift-differential systems, the others being completely analogous.

The statement amounts to proving that

$$\partial_x + \underline{D}_1^N(x) = \underline{\mathbf{a}}^N \circ \left(\partial_x + \underline{D}_1^{N+1}(x) \right) \circ \underline{\mathbf{a}}^N = \partial_x + \underline{\mathbf{a}}^N(x) \underline{D}_1^{N+1}(x) \underline{\mathbf{a}}^N(x) + \underline{\mathbf{a}}^N(x) \frac{d}{dx} \underline{\mathbf{a}}^N(x), \quad (3-9)$$

where the dependence of $\underline{\mathbf{a}}^N$ on x has been emphasized.

Let

$$\tilde{\Psi}_N^t(x) := \left[\tilde{\psi}_{N-d_2}(x), \dots, \tilde{\psi}_N(x) \right]^t, \quad N \geq d_2 \quad (3-10)$$

be *any* solution to the equation

$$\left(\partial_x + \underline{D}_1^N(x) \right) \tilde{\Psi}_N^t(x) = 0. \quad (3-11)$$

At this stage the labeling $N - d_2, \dots, N$ has no particular meaning because there are no $\tilde{\psi}_n(x)$'s with $n \notin [N - d_2, N]$. Nevertheless we can *define*

$$\tilde{\Psi}_{N+j}^t(x) := \underline{\mathbf{a}}_{N+j-1} \cdots \underline{\mathbf{a}}_N \tilde{\Psi}_N^t(x) \quad j \geq 1, \quad (3-12)$$

and

$$\tilde{\Psi}_{N-j}^t(x) := \underline{\mathbf{a}}^{N-j} \cdots \underline{\mathbf{a}}^N \tilde{\Psi}_N^t(x), \quad 0 \leq j \leq N - d_2 \quad (3-13)$$

Because of the recursive structure of the matrices $\underline{\mathbf{a}}^N$, the above definition, e.g., for $\tilde{\Psi}_{N+1}^t$ actually defines not $d_2 + 1$ new functions, but only one new function: $\tilde{\psi}_{N+1}$. Therefore, componentwise, we have defined a sequence of new functions $\tilde{\psi}_m(x)$, which satisfy the recursion relation

$$x \tilde{\psi}_m(x) = \sum_{j=-1}^{d_2} \alpha_j(m) \tilde{\psi}_{m-j}(x), \quad m \geq d_2. \quad (3-14)$$

By this definition and by the structure of the matrix

$$D_1 = \gamma \begin{matrix} N \\ \mathbf{a} \end{matrix} + \beta_0 + \sum_{j=1}^{d_1} \beta_j \begin{matrix} N \\ \mathbf{a}_{N+j-1} \mathbf{a}_{N+j-2} \cdots \mathbf{a}_N \end{matrix}, \quad (3-15)$$

the differential system componentwise now reads

$$\partial_x \tilde{\psi}_n(x) = - \sum_{j=-1}^{d_1} \beta_j (n+j) \tilde{\psi}_{n+j}(x), \quad n = N - d_2, \dots, N, \quad (3-16)$$

where the $\tilde{\psi}_n$'s that fall outside the window $N - d_2, \dots, N$ have been defined above in terms of the ones within the window. Therefore we need to prove that the newly defined function

$$\tilde{\psi}_{N+1}(x) := \frac{x - \alpha_0(N)}{\gamma(N)} \tilde{\psi}_N(x) - \sum_{l=1}^{d_2} \frac{\alpha_l(N)}{\gamma(N)} \tilde{\psi}_{N-l}(x) \quad (3-17)$$

satisfies the same sort of differential equation as the preceding ones. (A simple argument by induction then shows that all $\tilde{\psi}_{N+j}$ satisfy the same sort of differential equation for any $j > 1$). This in turn amounts to proving that

$$\partial_x \tilde{\psi}_{N+1}(x) = - \sum_{j=-1}^{d_1} \beta_j (N+1+j) \tilde{\psi}_{N+1+j}(x). \quad (3-18)$$

To do this we compute

$$\begin{aligned} \gamma(N) \partial_x \tilde{\psi}_{N+1}(x) &= \frac{d}{dx} \left(x \tilde{\psi}_N(x) - \sum_{l=0}^{d_2} \alpha_l(N) \tilde{\psi}_{N-l}(x) \right) \\ &= \sum_{l=0}^{d_2} \sum_{j=-1}^{d_1} \alpha_l(N) \beta_j (N-l+j) \tilde{\psi}_{N-l+j}(x) + \tilde{\psi}_N(x) - x \sum_{j=-1}^{d_1} \beta_j (N+j) \tilde{\psi}_{N+j}(x) = \\ &= \sum_{l=0}^{d_2} \sum_{j=-1}^{d_1} \alpha_l(N) \beta_j (N-l+j) \tilde{\psi}_{N-l+j}(x) + \tilde{\psi}_N(x) \\ &\quad - \sum_{l=-1}^{d_2} \sum_{j=-1}^{d_1} \alpha_l(N+j) \beta_j (N+j) \tilde{\psi}_{N+j-l}(x). \end{aligned} \quad (3-19)$$

Rearranging eqs. (??, ??), we have to prove the identity

$$\begin{aligned} -\gamma(N) \sum_{j=-1}^{d_1} \beta_j (N+1+j) \tilde{\psi}_{N+1+j}(x) &= \sum_{l=0}^{d_2} \sum_{j=-1}^{d_1} \alpha_l(N) \beta_j (N-l+j) \tilde{\psi}_{N-l+j}(x) + \tilde{\psi}_N(x) \\ &\quad - \sum_{l=-1}^{d_2} \sum_{j=-1}^{d_1} \alpha_l(N+j) \beta_j (N+j) \tilde{\psi}_{N+j-l}(x), \end{aligned} \quad (3-20)$$

or equivalently

$$\tilde{\psi}_N(x) = - \sum_{l=-1}^{d_2} \sum_{j=-1}^{d_1} \alpha_l(N) \beta_j (N-l+j) \tilde{\psi}_{N-l+j}(x)$$

$$+ \sum_{l=-1}^{d_2} \sum_{j=-1}^{d_1} \alpha_l(N+j) \beta_j(N+j) \tilde{\psi}_{N+j-l}(x) . \quad (3-21)$$

But this last equation is nothing but the Heisenberg commutation relations in eq. (??) and eq. (??). This means that rearranging the coefficients in front of $\psi_{N+r}(x)$ in the RHS of eq. (??), the only nonvanishing coefficient is that of $\tilde{\psi}_N(x)$ and it is exactly 1. A similar argument may be used to prove that the relations (??) hold also for $1 \leq n < N - d_2$

For future convenience we remark that this verification amounts to the fact that the coefficients of the $\tilde{\psi}_{N+r}(x)$'s are the same as for the orthogonal quasi-polynomials $\psi_{N+r}(x)$, since it relies only on the recursion relations, which are the same. Given that the quasi-polynomials $\psi_{N+r}(x)$ are linearly independent, the equality of LHS and RHS follows also for any other sequence. Q.E.D.

Prop. ?? means that we can define $d_2 + 1$ sequences of functions $\left\{ \psi_n^{(q)}(x) \right\}_{n \in \mathbb{N}, q=0 \dots d_2}$ in such a way that in any "window" of size $d_2 + 1$ they constitute a fundamental system of solutions to the differential system (??). One of these sequences is obviously provided by the orthogonal quasi-polynomials. Each of them satisfies both the recursion relations

$$x \psi_n^{(q)}(x) = \sum_{l=-1}^{d_2} \alpha_l(n) \psi_{n-l}^{(q)}(x) \quad (3-22)$$

and the derivative relations

$$\partial_x \psi_n^{(q)}(x) = - \sum_{j=-1}^{d_1} \beta_j(n+j) \psi_{n+j}^{(q)}(x) . \quad (3-23)$$

Remark 3.1 In principle, in order to define these $d_2 + 1$ sequences one should solve the differential system (??) in a given window and then define recursively the rest of the sequence backwards and forwards. To pass from the semi-infinite case to the infinite one, we may define the full sequence $\psi_n^{(q)}$ for $n \in \mathbb{Z}$ just by application of products of the matrices \mathbf{a}_N and their inverses, provided the $\alpha_j(n)$'s are so defined that all the \mathbf{a}_N 's are invertible.

In a completely parallel manner we can define $d_1 + 1$ sequences of functions $\left\{ \phi_n^{(q)}(y) \right\}_{n \in \mathbb{N}, q=0 \dots d_1}$ which provide fundamental systems satisfying

$$\left[\partial_y + \underline{D}_2(y) \right] \underline{\Phi}(y) = 0 . \quad (3-24)$$

Moreover, with minor modifications, we can construct analogous sequences for the dual systems

$$\partial_y \underline{\Psi}^{N-1}(y) = \underline{\Psi}^{N-1}(y) \underline{D}_2^N(y) \quad (3-25)$$

$$\partial_x \underline{\Phi}^{N-1}(x) = \underline{\Phi}^{N-1}(x) \underline{D}_1^N(x) . \quad (3-26)$$

The only difference is that the matrices \mathbf{a}_N and \mathbf{b}_N now implement the shift $N \rightarrow N - 1$. The barred sequences $\left\{ \underline{\phi}_n^{(q)}(y) \right\}_{n \in \mathbb{N}, q=0 \dots d_1}$ and $\left\{ \underline{\psi}_n^{(q)}(y) \right\}_{n \in \mathbb{N}, q=0 \dots d_1}$ will therefore satisfy the recursion relations

$$x \underline{\phi}_n^{(q)}(x) = \sum_{l=-1}^{d_2} \alpha_l(n+l) \underline{\phi}_{n+l}^{(q)}(x) , \quad \partial_x \underline{\phi}_n^{(q)}(x) = \sum_{j=-1}^{d_1} \beta_j(n) \underline{\phi}_{n-j}^{(q)}(x) , \quad (3-27)$$

$$y \underline{\psi}_n^{(q)}(y) = \sum_{j=-1}^{d_1} \beta_j(n+j) \underline{\psi}_{n+j}^{(q)}(y) , \quad \partial_y \underline{\psi}_n^{(q)}(y) = \sum_{l=-1}^{d_2} \alpha_l(n) \underline{\psi}_{n-l}^{(q)}(y) . \quad (3-28)$$

A completely analogous statement holds for the matrices defining the deformation equations in any window.

Proposition 3.2 The shifts $N \mapsto N + 1$ implemented by \mathbf{a}_N in eq.(??) and the sequence of differential equations

$$\partial_{u_K} \underline{\Psi}_N = \underline{\mathbf{U}}_K^{N,\Psi} \underline{\Psi}_N, \quad \partial_{v_J} \underline{\Psi}_N = -\underline{\mathbf{V}}_J^{N,\Psi} \underline{\Psi}_N, \quad N \geq d_2 + 1 \quad (3-29)$$

are compatible, as are the shifts $N \mapsto N - 1$ implemented by \mathbf{b}_N in (??) and the sequence of differential equations

$$\partial_{u_K} \underline{\Psi}^{N-1} = \underline{\Psi}^{N-1} \underline{\mathbf{U}}_K^{N,\Psi}, \quad \partial_{v_J} \underline{\Psi}^{N-1} = -\underline{\Psi}^{N-1} \underline{\mathbf{V}}_J^{N,\Psi}, \quad N \geq 1. \quad (3-30)$$

Similarly the shifts implemented by \mathbf{b}_N and $\underline{\mathbf{a}}_N$ are compatible with the equations

$$\partial_{u_K} \underline{\Phi}_N = -\underline{\mathbf{U}}_K^{N,\Phi} \underline{\Phi}_N, \quad \partial_{v_J} \underline{\Phi}_N = \underline{\mathbf{V}}_J^{N,\Phi} \underline{\Phi}_N, \quad N \geq d_1 + 1 \quad (3-31)$$

$$\partial_{u_K} \underline{\Phi}^{N-1} = -\underline{\Phi}^{N-1} \underline{\mathbf{U}}_K^{N,\Phi}, \quad \partial_{v_J} \underline{\Phi}^{N-1} = \underline{\Phi}^{N-1} \underline{\mathbf{V}}_J^{N,\Phi}, \quad N \geq 1. \quad (3-32)$$

Proof. We will prove compatibility of only one of the eight kinds of systems with the shift; the remaining cases are proven similarly.

As for Prop. ??, we first define a continuous parametric family (depending on x) of solutions to the system

$$\left[\partial_{u_K} - \underline{\mathbf{U}}_K^{N,\Psi} \right] \tilde{\Psi}_N = 0. \quad (3-33)$$

We then define the shifted functions $\tilde{\Psi}_{N+j} := \mathbf{a}_{N+j-1} \cdots \mathbf{a}_N \tilde{\Psi}_N$ so that the equation reads, componentwise

$$\partial_{u_K} \tilde{\psi}_n = \sum_{j=0}^K U_j^K(n) \tilde{\psi}_{n+j}, \quad n = N - d_2, \dots, N. \quad (3-34)$$

Then we have to check that the newly defined $\tilde{\psi}_{N+1}$ also satisfies

$$\partial_{u_K} \tilde{\psi}_{N+1} = \sum_{j=0}^K U_j^K(N+1) \tilde{\psi}_{N+1+j} \quad (3-35)$$

(and by induction the corresponding equations for $\tilde{\psi}_{N+r}$, $r \geq 1$). As before, we use the relations

$$\tilde{\psi}_{N+1} = \frac{x - \alpha_0(N)}{\gamma(N)} \tilde{\psi}_N(x) - \sum_{l=1}^{d_2} \frac{\alpha_l(N)}{\gamma(N)} \tilde{\psi}_{N-l}(x) \quad (3-36)$$

and the differential system satisfied by the $\tilde{\psi}_{N-d_2}, \dots, \tilde{\psi}_N$. To conclude the equality we can reason as in the remark in the proof of Prop. ??. Q.E.D.

The final proposition in this section assures that the deformation equations are compatible with the x, y differential equations in all windows.

Proposition 3.3 The system of equations

$$\left(\partial_x + D_1^N\right) \Psi_N(x) = 0, \quad (3-37)$$

$$\left(\partial_{u_K} - \mathbf{U}_K^{N,\Psi}\right) \Psi_N(x) = 0, \quad (3-38)$$

$$\left(\partial_{v_J} + \mathbf{V}_{(J)}^{N,\Psi}\right) \Psi_N(x) = 0, \quad (3-39)$$

$$\Psi_{N+1}(x) = \mathbf{a}_N(x) \Psi_N(x), \quad N \geq d_2 + 1, \quad (3-40)$$

is compatible, and hence sequences of fundamental systems of solutions $\left\{\Psi_N(x)\right\}_{N>d_2}$ to all equations exist. The same statement holds for the system

$$\left(\partial_y + D_2^N\right) \Phi_N(y) = 0, \quad (3-41)$$

$$\left(\partial_{u_K} + \mathbf{U}_K^{N,\Phi}\right) \Phi_N(y) = 0, \quad (3-42)$$

$$\left(\partial_{v_J} - \mathbf{V}_{(J)}^{N,\Phi}\right) \Phi_N(y) = 0, \quad (3-43)$$

$$\Phi_{N+1}(y) = \mathbf{b}_N(x) \Phi_N(y), \quad N \geq d_1 + 1. \quad (3-44)$$

The corresponding systems for the barred sequences are also compatible, and hence also admit simultaneous sequences of fundamental solutions $\underline{\Psi}^{N-1}(x)$ and $\underline{\Phi}^{N-1}(y)$.

Proof. The proof will only be given for the system (??)-(??) since the others are proved in the same way. The compatibility follows from Props. ?? and ?? together with a proof of compatibility of the equations (??)-(??). Indeed, from the $d_2 + 1$ functions in

$$\tilde{\Psi}_N = [\tilde{\psi}_{N-d_2}(x, u, v), \dots, \tilde{\psi}_N(x, u, v)]^t \quad (3-45)$$

we can consistently define a whole sequence of functions $\tilde{\psi}_n$'s by means of the x -recursion relations in such a way that componentwise the system reads

$$\partial_x \tilde{\psi}_n = - \sum_{j=-1}^{d_1} \beta_j(n+j) \tilde{\psi}_{n+j}, \quad (3-46)$$

$$\partial_{u_K} \tilde{\psi}_n = \sum_{j=0}^K U_j^K(n) \tilde{\psi}_{n+j}, \quad (3-47)$$

$$\partial_{v_J} \tilde{\psi}_n = - \sum_{j=0}^J V_j^J(n-j) \tilde{\psi}_{n-j}. \quad (3-48)$$

Taking cross derivatives and using these expressions one gets, e.g.,

$$\partial_{u_K} \partial_x \tilde{\psi}_n = - \sum_{k=0}^K \sum_{j=-1}^{d_1} U_k^K(n+j) \beta_j(n+j) \tilde{\psi}_{n+j+k} - \sum_{j=-1}^{d_1} \left(\frac{\partial}{\partial u_K} \beta_j(n+j)\right) \tilde{\psi}_{n+j}, \quad (3-49)$$

$$\partial_x \partial_{u_k} \tilde{\psi}_n = - \sum_{j=-1}^{d_1} \sum_{k=0}^K U_k^K(n) \beta_j(n+j+k) \tilde{\psi}_{n+j+k} . \quad (3-50)$$

The expressions for the derivatives of the coefficients β_j may be obtained from the deformation equation (??) for P and then substituted back into eq. (??). However, to prove the equality of the two cross derivatives it is sufficient to collect the coefficients of $\tilde{\psi}_{n+q}$ and note that exactly the same coefficients appear when the functions $\{\tilde{\psi}_n\}$ are replaced by the orthogonal quasi-polynomials $\{\psi_n\}$, for which the equality of the two expressions certainly holds. Since the orthogonal quasi-polynomials are linearly independent functions, the individual coefficients must agree. (This is essentially the same argument as in the remark at the end of proof of Prop. ??). The mutual compatibility of the ∂_{u_K} and ∂_{v_J} deformations is proved in exactly the same way. One just takes the the cross derivatives in equations (??) and (??) and notes that, since these are the same as for the case of the orthogonal quasi-polynomials $\{\psi_n\}$, the corresponding coefficients in the cross differentiated expression must be equal. Q.E.D.

4 Spectral Duality

The aim of this section is to state and prove some remarkable relations between systems related to dual windows (in the sense of Def. ??), which will justify the terminology. One of the main results will be that the four spectral curves given by the characteristic polynomials of $D_1, \underline{D}_1, D_2, \underline{D}_2$ associated to the four systems (eq. (??)) on the two pairs of dual windows $(\underline{\Psi}, \underline{\Phi})$, $(\Phi, \underline{\Psi})$ are actually the same curve.

4.1 Dual spectral curves

First we need a linear algebra lemma.

Lemma 4.1 *Let T be a square matrix having the block form*

$$T = \begin{bmatrix} 0 & F_1 & 0 & 0 & 0 \\ 0 & 0 & F_2 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & F_d \\ G_0 & G_1 & G_2 & \cdots & G_d \end{bmatrix} , \quad (4-1)$$

where the $d+1$ blocks have compatible sizes and the diagonal blocks are square. Then

$$\det[\mathbf{1} - T] = \det[\mathbf{1} - D] , \quad \text{where} \quad (4-2)$$

$$D := G_d + \sum_{k=0}^{d-1} G_k \cdot F_{k+1} \cdots F_d , \quad (4-3)$$

and $\mathbf{1}$ denotes, according to the context, the unit matrix of appropriate size.

Proof. Let

$$N_F := \begin{bmatrix} 0 & F_1 & 0 & 0 & 0 \\ 0 & 0 & F_2 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & F_d \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} . \quad (4-4)$$

We multiply the matrix $\mathbf{1} - T$ from the right by the matrix

$$(\mathbf{1} - N_F)^{-1} = \left[\begin{array}{c|c|c|c|c} \mathbf{1} & F_1 & F_1 F_2 & F_1 F_2 F_3 & F_1 \cdots F_d \\ \hline 0 & \mathbf{1} & F_2 & F_2 F_3 & F_2 \cdots F_d \\ \hline 0 & 0 & \ddots & \ddots & \vdots \\ \hline 0 & 0 & 0 & \mathbf{1} & F_d \\ \hline 0 & 0 & 0 & 0 & \mathbf{1} \end{array} \right] \quad (4-5)$$

Since the matrix $(\mathbf{1} - N_F)^{-1}$ is unimodular, the determinant of $\mathbf{1} - T$ remains unaffected. Then one computes

$$(\mathbf{1} - T) \cdot (\mathbf{1} - N_F)^{-1} = \left[\begin{array}{c|c|c|c|c} \mathbf{1} & 0 & 0 & 0 & 0 \\ \hline 0 & \mathbf{1} & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{1} & 0 & 0 \\ \hline 0 & 0 & 0 & \ddots & 0 \\ \hline \star & \star & \star & \star & \mathbf{1} - D \end{array} \right], \quad (4-6)$$

from which the statement follows by taking the determinant.

Q.E.D.

Proposition 4.1 The spectral curves associated to the characteristic polynomials of $\underline{D}_1, \underline{D}_2, \underline{D}_2, \underline{D}_1$ are pairwise equal. More precisely, we have the formulae

$$u_{d_1+1} \det \left[x \mathbf{1}_{d_1+1} - \underline{D}_2(y) \right] = v_{d_2+1} \det \left[y \mathbf{1}_{d_2+1} - \underline{D}_1(x) \right], \quad (4-7)$$

$$u_{d_1+1} \det \left[x \mathbf{1}_{d_1+1} - \underline{D}_2(y) \right] = v_{d_2+1} \det \left[y \mathbf{1}_{d_2+1} - \underline{D}_1(x) \right], \quad (4-8)$$

which connect the spectral curves of the differential operators of different dimensions operating on the two pairs of dual windows.

Proof. We will only prove one equality since the other is proved similarly. We start with the computation of the characteristic polynomial of $D_1(x)$

$$\begin{aligned} \det \left[y \mathbf{1} - \underline{D}_1(x) \right] &= \det \left[-\begin{array}{c} N \\ \gamma \end{array} \begin{array}{c} N-1 \\ \mathbf{a} \end{array} \right] \det \left[\mathbf{1} - \begin{array}{c} \mathbf{a} \\ N-1 \end{array} \begin{array}{c} \gamma \\ N \end{array} \left(y \mathbf{1} - \beta_0 - \sum_{j=1}^{d_1} \beta_j \begin{array}{c} \mathbf{a} \\ N+j-1 \end{array} \cdots \begin{array}{c} \mathbf{a} \\ N \end{array} \right) \right] \\ &= \frac{\gamma(N-1)^2}{v_{d_2+1}} \det \left[\mathbf{1} - \begin{array}{c} \mathbf{a} \\ N-1 \end{array} \begin{array}{c} \gamma \\ N \end{array} \left(y \mathbf{1} - \beta_0 - \sum_{j=1}^{d_1} \beta_j \begin{array}{c} \mathbf{a} \\ N+j-1 \end{array} \cdots \begin{array}{c} \mathbf{a} \\ N \end{array} \right) \right], \end{aligned} \quad (4-9)$$

where we have used the identity in eq. (??). Now we use Lemma ?? with the identifications

$$G_{d_1} = \begin{array}{c} \mathbf{a} \\ N-1 \end{array} \begin{array}{c} \gamma \\ N \end{array} \left(y \mathbf{1} - \beta_0 \right), \quad G_k = - \begin{array}{c} \mathbf{a} \\ N-1 \end{array} \begin{array}{c} \gamma \\ N \end{array} \beta_{d_1-k-1}, \quad k = 1 \dots d_1, \quad F_k = \begin{array}{c} \mathbf{a} \\ N+d_1-k \end{array}, \quad (4-10)$$

and thus obtain

$$\det \left[\mathbf{1} - \begin{array}{c} \mathbf{a} \\ N-1 \end{array} \begin{array}{c} \gamma \\ N \end{array} \left(y \mathbf{1} - \beta_0 - \sum_{j=1}^{d_1} \beta_j \begin{array}{c} \mathbf{a} \\ N+j-1 \end{array} \cdots \begin{array}{c} \mathbf{a} \\ N \end{array} \right) \right] = \det \left[\mathbf{1}_{(d_1+1)(d_2+1)} - T_{\mathbf{a}\mathbf{b}} \right]. \quad (4-11)$$

The matrix $T_{\mathbf{a}\mathbf{b}}$ is defined by

$$T_{\mathbf{a}\mathbf{b}} := \begin{bmatrix} 0 & \mathbf{a}_{N+d_1-1} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \mathbf{a}_N \\ -\mathbf{a}_{N-1} \beta_{d_1} \gamma_N & -\mathbf{a}_{N-1} \beta_{d_1-1} \gamma_N & \cdots & \mathbf{a}_{N-1} (y\mathbf{1} - \beta_0) \gamma_N \end{bmatrix} \quad (4-12)$$

$$= \begin{bmatrix} \mathbf{a}_{N+d_1-1} & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \mathbf{a}_N & 0 \\ 0 & 0 & 0 & \mathbf{a}_{N-1} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ -\beta_{d_1} \gamma_N & -\beta_{d_1-1} \gamma_N & \cdots & (y\mathbf{1} - \beta_0) \gamma_N \end{bmatrix}, \quad (4-13)$$

We regard $T_{\mathbf{a}\mathbf{b}}$ as an endomorphism of $\mathbb{C}^{d_2+1} \otimes \mathbb{C}^{d_1+1}$. Let P_{12} be the involution interchanging the two factors of the tensor product and let the matrix C implement the reversal of order endomorphism within the $(d_2 + 1) \times (d_2 + 1)$ blocks:

$$C := \text{Blockdiag}(\overbrace{(R, R, \dots, R)}^{d_2+1 \text{ times}}), \quad R := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \in GL_{d_1+1}. \quad (4-14)$$

A direct inspection shows

$$CP_{12}T_{\mathbf{a}\mathbf{b}}^tP_{12}C^{-1} = \begin{bmatrix} 0 & \mathbf{b}_{N-d_2} & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \mathbf{b}_{N-2} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{b}_{N-1} \\ -\mathbf{b}_{N-1} \alpha_{d_2} \gamma_N & -\mathbf{b}_{N-1} \alpha_{d_2-1} \gamma_N & \cdots & -\mathbf{b}_{N-1} \alpha_1 \gamma_N & \mathbf{b}_{N-1} (x\mathbf{1} - \alpha_0) \gamma_N \end{bmatrix} \quad (4-15)$$

where we now have $d_2 + 1$ square blocks of dimension $d_1 + 1$ (i.e. the number of blocks and the dimensions of the blocks have been interchanged). The barred symbols are precisely those defined in eqs. (??, ??), (??, ??).

We now use Lemma ?? again to get

$$\begin{aligned} \frac{v_{d_2+1}}{\gamma(N-1)^2} \det \left[y\mathbf{1} - \overset{N}{D}_1(x) \right] &= \det \left[\mathbf{1}_{(d_1+1)(d_2+1)} - T_{\mathbf{a}\mathbf{b}} \right] \\ &= \det \left[\mathbf{1}_{(d_1+1)(d_2+1)} - CP_{12}T_{\mathbf{a}\mathbf{b}}^tP_{12}C^{-1} \right] \end{aligned}$$

$$\begin{aligned}
&= \det \left[\mathbf{1}_{d_1+1} - \frac{\gamma}{N-1} \underline{\mathbf{b}}^N \left(x\mathbf{1} - \underline{\alpha}_0^{N-1} - \sum_{j=1}^{d_2} \underline{\mathbf{b}}^{N-1} \cdots \underline{\mathbf{b}}^{N-j} \underline{\alpha}_j^{N-1} \right) \right] \\
&= \det \left[-\frac{\gamma}{N-1} \underline{\mathbf{b}}^N \right] \det \left[x\mathbf{1} - \underline{D}_2(y) \right] \\
&= \frac{u_{d_1+1}}{\gamma(N-1)^2} \det \left[x\mathbf{1} - \underline{D}_2(y) \right]. \tag{4-16}
\end{aligned}$$

This concludes the proof.

Q.E.D.

Remark 4.1 Notice that this proof was based purely on an algebraic reinterpretation of the characteristic equations for $D_1(x)$ and $\underline{D}_2(y)$ in which the same set of recursion parameters $\{\alpha_j(n), \beta_j(n)\}$ appear. No assumption was required about any relations between these parameters, and therefore the equalities (??), (??) are really just identities.

4.2 Duality pairings

In what follows we will derive a deeper form of duality; namely, that the linear differential equations satisfied by dual windows are also dual, in the sense of having the same associated spectral curves. This follows from the generalized Christoffel–Darboux formulae satisfied by kernels $K_{ij}, i, j = 1, 2$ when the biorthogonal polynomials and their Fourier-Laplace transforms are replaced by *any* solution to the equations of Prop. ??.

Proposition 4.2 If $\{\tilde{\psi}_n(y)\}_{n \in \mathbb{N}}$ and $\{\tilde{\phi}_n(y)\}_{n \in \mathbb{N}}$ are two arbitrary sequences of functions satisfying both the recursion relations under multiplication by y and the differential relations under application of ∂_y (constructed as in Prop. ??), then

$$(\partial_{y'} + \partial_y) \left(\underline{\tilde{\Psi}}^N(y'), \underline{\mathbb{B}}^N \tilde{\Phi}^N(y) \right) = (y - y') \left(\underline{\tilde{\Psi}}^N(y'), (\underline{\mathbb{A}})^t \tilde{\Phi}^N(y) \right). \tag{4-17}$$

and

$$(\partial_{x'} + \partial_x) \left(\underline{\tilde{\Phi}}^N(x'), \underline{\mathbb{A}}^N \tilde{\Psi}^N(x) \right) = (x - x') \left(\underline{\tilde{\Phi}}^N(x'), (\underline{\mathbb{B}})^t \tilde{\Psi}^N(x) \right). \tag{4-18}$$

Proof. We shall prove the equality (??) only; since (??) is proved identically. The expressions on either side of eq. (??) read (understanding the $\tilde{\psi}_n$'s to depend on y' and the $\tilde{\phi}_n$'s on y):

$$\begin{aligned}
&(\partial_{y'} + \partial_y) \left(\underline{\tilde{\Psi}}^N(y'), \underline{\mathbb{B}}^N \tilde{\Phi}^N(y) \right) \\
&= (\partial_{y'} + \partial_y) \left(-\gamma(N-1) \tilde{\psi}_{N-1} \tilde{\phi}_N + \sum_{j=1}^{d_1} \sum_{k=0}^{j-1} \beta_j(N+k) \tilde{\psi}_{N+k} \tilde{\phi}_{N-j+k} \right) \\
&= \gamma(N-1) \tilde{\psi}_{N-1} \sum_{l=-1}^{d_2} \alpha_l(N+l) \tilde{\phi}_{N+l} - \gamma(N-1) \sum_{l=-1}^{d_2} \alpha_l(N-1) \tilde{\psi}_{N-1-l} \tilde{\phi}_N \\
&\quad + \sum_{j=1}^{d_1} \sum_{k=0}^{j-1} \sum_{l=-1}^{d_2} \alpha_l(N+k) \beta_j(N+k) \tilde{\psi}_{N+k-l} \tilde{\phi}_{N+k-j}
\end{aligned}$$

$$- \sum_{j=1}^{d_1} \sum_{k=0}^{j-1} \sum_{l=-1}^{d_2} \beta_j(N+k) \alpha_l(N+k-j+l) \underline{\psi}_{N+k} \tilde{\phi}_{N+k-j+l} \quad (4-19)$$

and

$$\begin{aligned} & (y' - y) \left(\underline{\Psi}_N(y'), \mathbb{A}^N \Phi^N(y) \right) \\ = & (y' - y) \left(\gamma(N-1) \underline{\psi}_N \tilde{\phi}_{N-1} - \sum_{j=1}^{d_2} \sum_{k=0}^{j-1} \alpha_j(N+k) \underline{\psi}_{N+k-j} \tilde{\phi}_{N+k} \right) \\ = & \gamma(N-1) \tilde{\phi}_{N-1} \sum_{j=-1}^{d_1} \beta_j(N+j) \underline{\psi}_{N+j} - \gamma(N-1) \underline{\psi}_N \sum_{j=-1}^{d_1} \beta_j(N-1) \tilde{\phi}_{N-1-j} \\ & - \sum_{l=1}^{d_2} \sum_{k=0}^{l-1} \sum_{j=-1}^{d_1} \alpha_l(N+k) \beta_j(N+k-l+j) \underline{\psi}_{N+k-l+j} \tilde{\phi}_{N+k} \\ & + \sum_{l=1}^{d_2} \sum_{k=0}^{l-1} \sum_{j=-1}^{d_1} \alpha_l(N+k) \beta_j(N+k) \underline{\psi}_{N+k-l} \tilde{\phi}_{N+k-j} . \end{aligned} \quad (4-20)$$

The claim now is that these two expressions are the same. There are two ways of proving this. The first is a straightforward computation collecting all bilinear terms of the form

$$F_{pq}(y', y) := \underline{\psi}_{N+p}(y') \tilde{\phi}_{N+q}(y) \quad (4-21)$$

in the difference between (??) and (??) and proving that their coefficients vanish. The coefficient of F_{pq} with $p \geq 0$ and $q \geq 0$ vanish identically, as do the coefficient of F_{pq} with $p < -1$ and $q < -1$. The coefficient of F_{pq} with $p \geq 0$ and $q < -1$ vanishes due to relation (??) with $l = p - q - 1$ and $n = N + q$. The coefficient of F_{pq} with $p < -1$ and $q \geq 0$ vanishes due to relation (??) with $l = q - p - 1$ and $n = N + p$. These cancellations are summarized in Table ??.

The second way does not involve any computation; in fact, we can already conclude that the coefficients of all terms $\underline{\psi}_{N+p}(y') \tilde{\phi}_{N+q}(y)$ must agree. We know that for the polynomial solutions $\{\psi_n(x), \phi_n(y)\}$ and their Fourier-Laplace transforms $\{\underline{\psi}_n(y'), \underline{\phi}_n(x')\}$ the two expressions are the same because

$$\begin{aligned} (\partial_{y'} + \partial_y) \left(\underline{\Psi}^{N-1}(y'), \mathbb{B}^N \Phi^N(y) \right) &= (\partial_{y'} + \partial_y) (y' - y) \sum_{n=0}^{N-1} \underline{\psi}_n(y') \phi_n(y) \\ &= (y' - y) (\partial_{y'} + \partial_y) \sum_{n=0}^{N-1} \underline{\psi}_n(y') \phi_n(y) = (y' - y) \left(\underline{\Psi}_N(y'), \mathbb{A}^N \Phi^N(y) \right) , \end{aligned} \quad (4-22)$$

where we have used the generalized Christoffel–Darboux formulae and the identity

$$[(y' - y), (\partial_{y'} + \partial_y)] = 0 . \quad (4-23)$$

Since the $\underline{\psi}_n(y')$'s are linearly independent and so are the $\phi_n(y)$'s, the functions

$$F_{pq}(y', y) := \underline{\psi}_{N+p}(y') \phi_{N+q}(y) \quad (4-24)$$

are also linearly independent. Considering eqs. (??) and (??) as linear equalities for the $F_{pq}(y', y)$'s, one concludes that the coefficient in the two equations must be equal. Q.E.D.

Before stating the next result, we define new pairings by studying the effect of deformations on the kernels. By using the same Christoffel–Darboux trick one easily computes

$$\frac{\partial}{\partial u_K} \sum_{n=0}^{N-1} \psi_n(x) \underline{\phi}_n(x') = \sum_{j=1}^K \sum_{l=1}^j U_j^K(N-l) \psi_{N-l+j} \underline{\phi}_{N-l}, \quad (4-25)$$

$$\frac{\partial}{\partial v_K} \sum_{n=0}^{N-1} \psi_n(x) \underline{\phi}_n(x') = \sum_{j=1}^K \sum_{l=1}^j V_j^K(N-l) \underline{\phi}_{N-l+j} \psi_{N-l}. \quad (4-26)$$

This pairing does not change if we take Fourier–Laplace transforms of Ψ or Φ , so we can easily write the deformations of the two kernels K_{11} and K_{22} .

Proposition 4.3 For any two sequences satisfying both the deformation equations and the x -recursion relations we have

$$\frac{\partial}{\partial u_K} \left(\underline{\Psi}^{N-1}(y'), \mathbb{B} \underline{\Phi}^N(y) \right) = (y' - y) \left(\sum_{j=1}^K \sum_{l=1}^j U_j^K(N-l) \tilde{\psi}_{N-l+j} \tilde{\phi}_{N-l} \right) \quad (4-27)$$

$$\frac{\partial}{\partial v_J} \left(\underline{\Psi}^{N-1}(y'), \mathbb{B} \underline{\Phi}^N(y) \right) = (y' - y) \left(\sum_{j=1}^J \sum_{l=1}^j V_j^J(N-l) \tilde{\phi}_{N-l+j} \tilde{\psi}_{N-l} \right) \quad (4-28)$$

$$\frac{\partial}{\partial u_K} \left(\underline{\Phi}^{N-1}(x'), \mathbb{A} \underline{\Psi}^N(x) \right) = (x' - x) \left(\sum_{j=1}^K \sum_{l=1}^j U_j^K(N-l) \tilde{\psi}_{N-l+j} \tilde{\phi}_{N-l} \right) \quad (4-29)$$

$$\frac{\partial}{\partial v_J} \left(\underline{\Phi}^{N-1}(x'), \mathbb{A} \underline{\Psi}^N(x) \right) = (x' - x) \left(\sum_{j=1}^J \sum_{l=1}^j V_j^J(N-l) \tilde{\phi}_{N-l+j} \tilde{\psi}_{N-l} \right). \quad (4-30)$$

Proof. We prove only one identity, the others being completely similar. If the two sequences consist of the orthogonal quasi-polynomials (and the corresponding Fourier–Laplace transforms), then the equality follows immediately from:

$$\begin{aligned} \frac{\partial}{\partial u_K} \left(\underline{\Psi}^{N-1}(y'), \mathbb{B} \underline{\Phi}^N(y) \right) &= \frac{\partial}{\partial u_K} (y' - y) \sum_{n=0}^{N-1} \psi_n(y') \phi_n(y) \\ &= (y' - y) \left(\sum_{j=1}^K \sum_{l=1}^j U_j^K(N-l) \psi_{N-l+j} \phi_{N-l} \right). \end{aligned} \quad (4-31)$$

Expanding both sides by means of the recursion relations and the deformation equations in order to obtain linear expressions in $F_{p,q}(y', y)$ in the LHS and RHS one concludes that the coefficients must be the same. But the same final expression relies only on the recursion relations, and hence the equality holds for any pair of sequences of functions $\tilde{\psi}_n(y')$ and $\tilde{\phi}_n(y)$ satisfying these same recursion relations (by the same argument as in the proof of Prop. ??). Q.E.D.

Theorem 4.1 If $\{\tilde{\psi}_n(y)\}_{n \in \mathbb{N}}$ and $\{\tilde{\phi}_n(y)\}_{n \in \mathbb{N}}$ (or $\{\tilde{\phi}_n(x)\}_{n \in \mathbb{N}}$ and $\{\tilde{\psi}_n(x)\}_{n \in \mathbb{N}}$) are arbitrary pairs of sequences of functions satisfying the recursion relations (??), (??), (resp. (??), (??)), the differential

relations (??), (??), (resp. (??), (??)) and the deformation equations (??)-(??), then the bilinear expressions

$$\tilde{f}_N(y) := \left(\begin{array}{c} N-1 \\ \tilde{\Psi}(y), \mathbb{B} \tilde{\Phi}_N(y) \end{array} \right), \quad (4-32)$$

$$\tilde{g}_N(x) := \left(\begin{array}{c} N-1 \\ \tilde{\Phi}(x), \mathbb{A} \tilde{\Psi}_N(x) \end{array} \right), \quad (4-33)$$

are independent of y (resp. x) and N , and also are constant in the deformation parameters $\{u_K, v_J\}$.

Proof. Using Prop. ?? and setting $y = y'$ we find at once that

$$\frac{d}{dy} \tilde{f}_N(y) = 0,$$

i.e. $f_N(y) = f_N$ does not depend on y . A similar computation shows that $\tilde{g}_N(x)$ is independent of x . Now we also compute (for, say, $M < N$)

$$\left(\begin{array}{c} N-1 \\ \tilde{\Psi}(y), \mathbb{B} \tilde{\Phi}_N(y') \end{array} \right) - \left(\begin{array}{c} M-1 \\ \tilde{\Psi}(y), \mathbb{B} \tilde{\Phi}_M(y') \end{array} \right) = (y - y') \sum_{n=M}^{N-1} \tilde{\psi}_n(y') \tilde{\phi}_n(y). \quad (4-34)$$

Letting $y = y'$ we obtain $\tilde{f}_N = \tilde{f}_M$, and similarly for \tilde{g}_N . To prove the independence of the deformation parameters u_K and v_J , we use Prop. ?? in a similar way to the above and then again set $y = y'$ (or $x = x'$) to conclude the proof. Q.E.D.

Theorem ?? allows us to conclude that we can choose fundamental systems of solutions to the pairs of dual differential-difference-deformation equations normalized in such a way that the pairing gives the identity matrix.

Corollary 4.1 There exist two pairs of sequences of fundamental matrix solutions to the difference-differential-deformation equations (??)-(??), (??)-(??) $(\Psi_N, \Phi_N), (\Phi_N, \Psi_N)$ such that

$$\left(\begin{array}{c} N-1 \\ \Phi_N, \mathbb{A} \Psi_N \end{array} \right) \equiv \mathbf{1}, \quad \left(\begin{array}{c} N-1 \\ \Psi_N, \mathbb{B} \Phi_N \end{array} \right) \equiv \mathbf{1}. \quad (4-35)$$

We conclude with the following theorem.

Theorem 4.2 The differential-deformation systems

$$\partial_y \Phi_N = -D_2(y) \Phi_N, \quad \partial_{u_K} \Phi_N = -\mathbf{U}_K \Phi_N, \quad \partial_{v_J} \Phi_N = \mathbf{V}_J \Phi_N, \quad (4-36)$$

$$\partial_y \Psi_N = \mathbf{D}_2 \Psi_N, \quad \partial_{u_K} \Psi_N = \mathbf{U}_K \Psi_N, \quad \partial_{v_J} \Psi_N = -\mathbf{V}_J \Psi_N, \quad (4-37)$$

for $K = 1 \dots d_1 + 1$; $J = 1 \dots d_2 + 1$, are put in duality by the matrix \mathbb{B} .

$$\mathbb{B} D_2(y) = \mathbf{D}_2(y) \mathbb{B}, \quad (4-38)$$

$$\partial_{u_K} \mathbb{B} = \mathbb{B} \mathbf{U}_K(y) - \mathbf{U}_K(y) \mathbb{B}, \quad (4-39)$$

$$\partial_{v_J} \mathbb{B} = \mathbf{V}_J(y) \mathbb{B} - \mathbb{B} \mathbf{V}_J(y). \quad (4-40)$$

In particular, since the matrices $\underline{D}_2(y)$ and $\underline{D}_2(y)$ are conjugate to each other, their spectral curves are the same. Similarly, the differential-deformation systems

$$\partial_x \underline{\Psi}_N = -\underline{D}_1(x) \underline{\Psi}_N, \quad \partial_{u_K} \underline{\Psi}_N = \underline{U}_K^{N,\Psi} \underline{\Psi}_N, \quad \partial_{v_J} \underline{\Psi}_N = -\underline{V}_J^{N,\Psi} \underline{\Psi}_N, \quad (4-41)$$

$$\partial_x \underline{\Phi}^{N-1} = \underline{\Phi}^{N-1} \underline{D}_1(x), \quad \partial_{u_K} \underline{\Phi}^{N-1} = -\underline{\Phi}^{N-1} \underline{U}_K^{N,\Phi}, \quad \partial_{v_J} \underline{\Phi}^{N-1} = \underline{\Phi}^{N-1} \underline{V}_J^{N,\Phi}, \quad (4-42)$$

$K = 1 \dots d_1 + 1$; $J = 1 \dots d_2 + 1$, are put in duality by the matrix $\underline{\mathbb{A}}^N$,

$$\underline{\mathbb{A}}^N \underline{D}_1(x) = \underline{D}_1(x) \underline{\mathbb{A}}^N, \quad (4-43)$$

$$\partial_{v_J} \underline{\mathbb{A}}^N = \underline{\mathbb{A}}^N \underline{V}_J^{N,\Psi}(x) - \underline{V}_J^{N,\Phi}(x) \underline{\mathbb{A}}^N, \quad (4-44)$$

$$\partial_{u_K} \underline{\mathbb{A}}^N = \underline{U}_K^{N,\Phi}(x) \underline{\mathbb{A}}^N - \underline{\mathbb{A}}^N \underline{U}_K^{N,\Psi}(x), \quad (4-45)$$

and hence the spectral curves of $\underline{D}_1(x)$ and $\underline{D}_1(x)$ are also the same.

Proof. The three relations follow easily by taking a fundamental system of solutions for the two compatible differential-deformation systems and using Theorem ??.

Q.E.D.

This theorem together with Prop. ?? proves that the four spectral curves

$$\begin{aligned} \det \left[y \mathbf{1} - \underline{D}_2(x) \right] = 0 & \stackrel{\text{Prop. ??}}{\longleftrightarrow} \det \left[x \mathbf{1} - \underline{D}_1(y) \right] = 0 \\ \uparrow \text{Thm. ??} & \qquad \qquad \qquad \uparrow \text{Thm. ??} \\ \det \left[y \mathbf{1} - \underline{D}_2(x) \right] = 0 & \stackrel{\text{Prop. ??}}{\longleftrightarrow} \det \left[x \mathbf{1} - \underline{D}_1(y) \right] = 0 \end{aligned} \quad (4-46)$$

all coincide.

4.3 Concluding remarks

In this work, the main results concern the compatibility of the difference- differential-deformation systems arising from the “folding” procedure (Proposition ??), and the resulting spectral duality Theorems ?? and ??. The constancy of the bilinear pairings between solutions given by Corollary ?? may be viewed as a form of the bilinear relations for Baker functions which imply the Hirota bilinear equations for the associated tau function of eq. (??).

Another consequence of this compatibility is the fact that the (generalized) monodromy of the co-variant derivative operators $\frac{d}{dx} - \underline{D}_1(x)$, and $\frac{d}{dy} - \underline{D}_2(y)$ is independent of both the continuous deformation parameters $\{u_K, v_J\}$ and the integer N ; i.e., we have a dual pair of differential operators families whose coefficients satisfy differential equations in the parameters $\{u_K, v_J\}$ and difference equations in the discrete parameters N that generate isomonodromic deformations. Associated to such isomonodromic deformation equations, there is a sequence of isomonodromic tau functions in the sense of refs [?, ?].

However, since the highest terms of the polynomial matrices $\underline{D}_1(x)$ and $\underline{D}_2(y)$ have a very degenerate spectrum (in fact, they have rank 1), the standard definition of the isomonodromic tau function does not apply. To introduce a suitable definition for this situation, an analysis of the formal asymptotics at

$x = \infty$ (or $y = \infty$) is required. Also, the systems of Proposition ?? represent in a sense, the “vacuum” isomonodromic deformation systems associated with the Fredholm kernels appearing in Proposition ?. When the corresponding integral operator is supported on a union of intervals, the computation of its resolvent is equivalent to a Riemann-Hilbert problem with discontinuities given across these cuts [?]. The resulting “dressed” Baker functions determine isomonodromic families of covariant derivative operators having, in addition to polynomial parts, poles at the endpoints of the intervals, which may be viewed as new deformation parameters. The associated isomonodromic tau functions are given by the Fredholm determinants of the integral operator supported on the union of intervals. [?, ?].

The study of the formal asymptotics associated to the vacuum systems, the corresponding isomonodromic tau functions and the relation of these to the spectral invariants will be developed in a subsequent work ([?]), as will the study of the $N \rightarrow \infty$ asymptotics of the biorthogonal polynomials and associated Fredholm kernels.

A Appendix: Multi-matrix model

The “multi-matrix-model” is a generalization of the 2-matrix-model, which was introduced in the context of string theory and conformal field theory [?, ?], and has been extensively studied [?, ?, ?]. Our notations in the following mainly follow [?]. Calculations of spectral statistics in this model again involves biorthogonal polynomials which obey linear differential systems of finite rank. It will be shown in this appendix that these also satisfy an extended form of the spectral duality relations derived for the 2-matrix case. The results will be summarized, but only a brief sketch of the proofs will be indicated.

Consider $m \geq 2$ random hermitian $N \times N$ matrices M_1, M_2, \dots, M_m , with the measure

$$d\mu = \prod_{k=1}^m e^{-\text{Tr } V_k(M_k)} \prod_{k=1}^{m-1} e^{\text{Tr } M_k M_{k+1}} \prod_{k=1}^m dM_k, \quad (\text{A-1})$$

where dM_k is the standard Lebesgue measure for hermitian matrices, and the potentials V_k , $k = 1 \dots m$, are polynomials of degrees $d_k + 1$, with coefficients

$$V_k(x) = u_{k,0} + \sum_{l=1}^{d_k+1} \frac{u_{k,l}}{l} x^l. \quad (\text{A-2})$$

As in the 2-matrix case, all the correlation functions and statistical properties of the eigenvalues of the m matrices can be expressed in terms of determinants involving m^2 Fredholm integral kernels, which are constructed from an infinite sequence of biorthogonal polynomials and their integral transforms [?]. In this case, the biorthogonal polynomials

$$\pi_n(x) = x^n + \dots, \quad \sigma_n(y) = y^n + \dots \quad (\text{A-3})$$

are defined to be orthogonal in the following sense:

$$\int \int \dots \int dx_1 dx_2 \dots dx_{m-1} dx_m \prod_{k=1}^m e^{-V_k(x_k)} \prod_{k=1}^{m-1} e^{x_k x_{k+1}} \pi_n(x_1) \sigma_l(x_m) = h_n \delta_{nl}, \quad (\text{A-4})$$

where the integral is convergent on the real axis if all the degrees $d_k + 1$ are even, and the leading coefficients are positive. Otherwise we need to consider other integration paths in the complex plane, without boundaries, so that integration by parts may be done. This uniquely determines the polynomials π_n and σ_n for all n .

From π_n , we define the following m sequences of functions $\{\psi_{1,n}\}_{n=0\dots\infty}, \dots, \{\psi_{m,n}\}_{n=0\dots\infty}$

$$\begin{aligned}\psi_{1,n}(x) &:= \frac{1}{\sqrt{h_n}} \pi_n(x) e^{-V_1(x)}, \\ \psi_{2,n}(x) &:= \int dy \psi_{1,n}(y) e^{xy}, \\ \psi_{k+1,n}(x) &:= \int dy \psi_{k,n}(y) e^{xy} e^{-V_k(y)} \quad \text{for } m-1 \geq k \geq 2,\end{aligned}\tag{A-5}$$

and from σ_n , the following m sequences of functions $\{\phi_{1,n}\}_{n=0\dots\infty}, \dots, \{\phi_{m,n}\}_{n=0\dots\infty}$

$$\begin{aligned}\phi_{m,n}(x) &:= \frac{1}{\sqrt{h_n}} \sigma_n(x) e^{-V_m(x)} \\ \phi_{k-1,n}(x) &:= \int dy \phi_{k,n}(y) e^{xy} e^{-V_{k-1}(x)} \quad \text{for } m \geq k \geq 3 \\ \phi_{1,n}(x) &:= \int dy \phi_{2,n}(x) e^{xy},\end{aligned}\tag{A-6}$$

which are dual bases for the respective spaces they span:

$$\int dx \psi_{k,n}(x) \phi_{k,l}(x) = \delta_{nl}.\tag{A-7}$$

A.1 Recursion relations

We define the semi-infinite matrices P_k and Q_k for each $k = 1, \dots, m$, such that [?, ?]

$$x\psi_{k,n}(x) = \sum_l Q_{k(n,l)} \psi_{k,l}(x), \quad \frac{\partial}{\partial x} \psi_{k,n}(x) = \sum_l P_{k(n,l)} \psi_{k,l}(x),\tag{A-8}$$

where it will be shown below that these only involve finite sums. From the pairing (??), and integration by parts, we have

$$x_k \phi_{k,n}(x) = \sum_l Q_{k(n,l)}^t \phi_{k,l}(x), \quad \frac{\partial}{\partial x} \phi_{k,n}(x) = - \sum_l P_{k(n,l)}^t \phi_{k,l}(x).\tag{A-9}$$

Note that these matrices all satisfy Heisenberg relations [?]

$$[P_k, Q_k] = \mathbf{1}.\tag{A-10}$$

Using the definitions of $\psi_{k,n}$ and $\phi_{k,n}$, we find the following relationships between them.

$$P_{k+1} = Q_k \quad k \in [1, m-1], \quad -P_k = Q_{k+1} - V_k'(Q_k) \quad k \in [2, m-1], \quad -P_1 = Q_2.\tag{A-11}$$

In particular, this implies that

$$Q_{k-1} + Q_{k+1} = V_k'(Q_k) \quad \text{for } k \in [2, m-1].\tag{A-12}$$

These relations are enough to ensure that all the matrices Q_k and P_k are of finite band type.

Proposition A.1 The matrix Q_k has r_k bands above the principal diagonal and s_k bands below the principal diagonal, with

$$r_1 = 1 \quad \text{and} \quad r_k = \prod_{l=1}^{k-1} d_l \quad \text{for } k \in [2, m], \quad (\text{A-13})$$

$$s_m = 1 \quad \text{and} \quad s_k = \prod_{l=k+1}^m d_l \quad \text{for } k \in [1, m-1]. \quad (\text{A-14})$$

Proof: Q_1 multiplies the vector of polynomials $\{\pi_n(x)\}_{n=0\ldots\infty}$ by x , and can therefore raise the degree at most by 1; i.e. Q_1 has at most one line above diagonal. From the same argument, since the multiplication by $P_1 + V_1'(Q_1)$ takes the derivative of the vector polynomial $[\pi_n(x)]_{n=0\ldots\infty}$ with respect to x , it must lower the degree, therefore P_1 has at most $d_1 = \deg V_1'$ lines above diagonal and so does $Q_2 = -P_1$. Using ?? recursively, it follows that Q_k has at most r_k lines above diagonal. Repeating the argument with the polynomials $\{\sigma_n\}$, we see that Q_k^t has at most s_k lines above the diagonal. Q.E.D.

Denoting

$$\alpha_{k,l}(n) := Q_{k(n,n+l)}, \quad (\text{A-15})$$

the recursion relations may be written componentwise as

$$x\psi_{k,n}(x) = \sum_{l=-s_k}^{+r_k} \alpha_{k,l}(n)\psi_{k,n+l}(x), \quad x\phi_{k,n}(x) = \sum_{l=-s_k}^{+r_k} \alpha_{k,l}(n-l)\phi_{k,n-l}(x), \quad (\text{A-16})$$

which implies

$$\frac{\partial}{\partial x}\psi_{1,n}(x) = - \sum_{l=-s_2}^{+r_2} \alpha_{2,l}(n)\psi_{1,n+l}(x), \quad \frac{\partial}{\partial x}\psi_{k,n}(x) = \sum_{l=-s_{k-1}}^{+r_{k-1}} \alpha_{k-1,l}(n)\psi_{k,n+l}(x), \quad k \in [2, m], \quad (\text{A-17})$$

and

$$\frac{\partial}{\partial x}\phi_{1,n}(x) = \sum_{l=-s_2}^{+r_2} \alpha_{2,l}(n-l)\phi_{1,n-l}(x), \quad \frac{\partial}{\partial x}\phi_{k,n}(x) = - \sum_{l=-s_{k-1}}^{+r_{k-1}} \alpha_{k-1,l}(n-l)\phi_{k,n-l}(x), \quad k \in [2, m]. \quad (\text{A-18})$$

Note that

$$\alpha_{1,1}(n) = \alpha_{m,-1}(n+1) = \gamma(n) = \sqrt{\frac{h_{n+1}}{h_n}}. \quad (\text{A-19})$$

A.2 Folding

Again, it is possible to “fold” these recursion relations to form finite rank linear differential systems with polynomial coefficients. For each k , define the following windows of size $r_k + s_k$.

$$\Psi_N^k = \begin{pmatrix} \psi_{k,N-s_k} \\ \vdots \\ \psi_{k,N+r_k-1} \end{pmatrix}, \quad \Phi_N^k = (\phi_{k,N-r_k} \quad \cdots \quad \phi_{k,N+s_k-1}). \quad (\text{A-20})$$

Shift operators: For each k , define “ladder” matrices of size $r_k + s_k$,

$$\begin{aligned} a_k^N(x) &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\alpha_{k,-s_k}(N)}{\alpha_{k,r_k}(N)} & \cdots & \frac{x-\alpha_{k,0}(N)}{\alpha_{k,r_k}(N)} & \cdots & -\frac{\alpha_{k,r_k-1}(N)}{\alpha_{k,r_k}(N)} \end{pmatrix} \\ \tilde{a}_k^N(x) &= \begin{pmatrix} -\frac{\alpha_{k,r_k-1}(N-r_k+1)}{\alpha_{k,r_k}(N-r_k)} & 1 & 0 & 0 \\ \frac{x-\alpha_{k,0}(N)}{\alpha_{k,r_k}(N-r_k)} & 0 & \ddots & 0 \\ \vdots & 0 & 0 & 1 \\ -\frac{\alpha_{k,-s_k}(N+s_k)}{\alpha_{k,r_k}(N-r_k)} & 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (\text{A-21})$$

which implement the shifts⁵ in N .

$$a_k^N(x) \Psi_k^N = \Psi_k^{N+1}, \quad \Phi_k^N \tilde{a}_k^N(x) = \Phi_k^{N-1}. \quad (\text{A-22})$$

Differential systems: The differential systems⁶ satisfied by the vectors $\Psi_{kN}(x)$ and $\Phi_{kN}(x)$ are:

$$\frac{\partial}{\partial x} \Psi_k^N(x) = D_k^N(x) \Psi_k^N(x), \quad \frac{\partial}{\partial x} \Phi_k^N(x) = -\Phi_k^N(x) \tilde{D}_k^N(x) \quad (\text{A-23})$$

where $D_k^N(x)$ and $\tilde{D}_k^N(x)$ are matrices of size $r_k + s_k$, and with polynomial coefficients of degree at most d_k in x .

The matrices $D_k^N(x)$ are given by:

$$\begin{aligned} D_1^N(x) &= -\alpha_{2,0}^{1N} - \sum_{l=1}^{r_2} \alpha_{2,l}^{1N} \prod_{j=1}^l a_{1,N+l-j}(x) - \sum_{l=1}^{s_2} \alpha_{2,-l}^{1N} \prod_{j=0}^{l-1} \left(a_{1,N-l+j}(x) \right)^{-1} \\ D_k^N(x) &= \alpha_{k-1,0}^{kN} + \sum_{l=1}^{r_{k-1}} \alpha_{k-1,l}^{kN} \prod_{j=1}^l a_{k,N+l-j}(x) + \sum_{l=1}^{s_{k-1}} \alpha_{k-1,-l}^{kN} \prod_{j=0}^{l-1} \left(a_{k,N-l+j}(x) \right)^{-1} \end{aligned} \quad (\text{A-24})$$

for $m \geq k \geq 2$,

where

$$\alpha_{j,l}^{kN} := \text{diag}(\alpha_{j,l}(N-s_k), \dots, \alpha_{j,l}(N+r_k-1)). \quad (\text{A-25})$$

Similarly:

$$\begin{aligned} \tilde{D}_1^N(x) &= -\tilde{\alpha}_{2,0}^{1N} - \sum_{l=1}^{r_{k-1}} \left(\prod_{j=0}^{l-1} \tilde{a}_{1,N-j}(x) \right) \tilde{\alpha}_{2,l}^{1N} - \sum_{l=1}^{s_{k-1}} \left(\prod_{j=1}^l \left(\tilde{a}_{1,N+j}(x) \right)^{-1} \right) \tilde{\alpha}_{2,-l}^{1N} \\ \tilde{D}_k^N(x) &= \tilde{\alpha}_{k-1,0}^{kN} + \sum_{l=1}^{r_{k-1}} \left(\prod_{j=0}^{l-1} \tilde{a}_{k,N-j}(x) \right) \tilde{\alpha}_{k-1,l}^{kN} + \sum_{l=1}^{s_{k-1}} \left(\prod_{j=1}^l \left(\tilde{a}_{k,N+j}(x) \right)^{-1} \right) \tilde{\alpha}_{k-1,-l}^{kN} \end{aligned}$$

⁵In the $m = 2$ case, we had $\mathbf{a} = a_1$, $\tilde{\mathbf{a}} = \tilde{a}_1$, $\mathbf{b} = (a_2^t)^{-1}$, $\tilde{\mathbf{b}} = (a_2^t)^{-1}$.

⁶Notice that the notations are changed for $m = 2$. D_2 is now what we called $-D_2^t$ and \tilde{D}_2 is what we previously called $-D_2^t$.

$$\text{for } m \geq k \geq 2 \quad (\text{A-26})$$

where

$$\tilde{\alpha}_{j,l}^{kN} := \text{diag}(\alpha_{j,l}(N-l-r_k), \dots, \alpha_{j,l}(N-l+s_k-1)). \quad (\text{A-27})$$

Christoffel–Darboux matrices:

Consider the kernel

$$K_{N,k,k}(x,y) = \sum_{n=0}^{N-1} \phi_{k,n}(y) \psi_{k,n}(x). \quad (\text{A-28})$$

The generalization of the Christoffel–Darboux theorem for this kernel reads

$$\begin{aligned} (x-y)K_{N,k,k}(x,y) &= \sum_{l=1}^{r_k} \sum_{j=1}^l \alpha_{k,l}(N-j) \psi_{k,N-j+l} \phi_{k,N-j} - \sum_{l=1}^{s_k} \sum_{j=1}^l \alpha_{k,-l}(N-j+l) \psi_{k,N-j} \phi_{k,N-j+l} \\ &= \Phi_k(x) \mathbb{A}_k \Psi_k(y). \end{aligned} \quad (\text{A-29})$$

where \mathbb{A}_k is the $(r_k + s_k) \times (r_k + s_k)$ matrix:

$$\mathbb{A}_k = \begin{pmatrix} 0 & \dots & 0 & \alpha_{k,r_k}(N-r_k) & 0 & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \vdots & \ddots & 0 \\ 0 & \dots & 0 & \alpha_{k,1}(N-1) & \dots & \alpha_{k,r_k}(N-1) \\ -\alpha_{k,-s_k}(N) & \dots & -\alpha_{k,-1}(N) & 0 & \dots & 0 \\ 0 & \ddots & \vdots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -\alpha_{k,+s_k-1}(N+s_k-1) & 0 & 0 & 0 \end{pmatrix} \quad (\text{A-30})$$

There is also a “differential Christoffel–Darboux theorem”, for $k > 1$:

$$\begin{aligned} (\partial_x + \partial_y)K_{N,k,k}(x,y) &= \sum_{l=1}^{r_{k-1}} \sum_{j=1}^l \alpha_{k-1,l}(N-j) \psi_{k,N-j+l} \phi_{k,N-j} \\ &\quad - \sum_{l=1}^{s_{k-1}} \sum_{j=1}^l \alpha_{k-1,-l}(N-j+l) \psi_{k,N-j} \phi_{k,N-j+l} \\ &= \hat{\Phi}_k(x) \mathbb{A}_{k-1} \hat{\Psi}_k(y), \end{aligned} \quad (\text{A-31})$$

where

$$\hat{\Psi}_k = (\psi_{k,N-s_{k-1}} \quad \dots \quad \psi_{k,N+r_{k-1}-1})^t, \quad \hat{\Phi}_k = (\phi_{k,N-r_{k-1}} \quad \dots \quad \phi_{k,N+s_{k-1}-1}), \quad (\text{A-32})$$

and for $k = 1$,

$$(\partial_x + \partial_y)K_{N,1,1}(x,y) = - \sum_{l=1}^{r_2} \sum_{j=1}^l \alpha_{2,l}(N-j) \psi_{1,N-j+l} \phi_{1,N-j}$$

$$\begin{aligned}
& + \sum_{l=1}^{s_2} \sum_{j=1}^l \alpha_{2,-l} (N-j+l) \psi_{1,N-j} \phi_{1,N-j+l} \\
& = - \hat{\Phi}_1(x) \mathbb{A}_2 \hat{\Psi}_1(y) , \tag{A-33}
\end{aligned}$$

where

$$\hat{\Psi}_1 = (\psi_{1,N-s_2} \quad \cdots \quad \psi_{1,N+r_2-1})^t , \quad \hat{\Phi}_1 = (\phi_{1,N-r_2} \quad \cdots \quad \phi_{1,N+s_2-1}) . \tag{A-34}$$

A.3 Duality

All the systems $\overset{N}{D}_k(x)$ and $\underline{\overset{N}{D}}_k(x)$ have the same spectral curve. This result follows again in two steps.

- For each k , we have:

$$\overset{N}{D}_k(x) \mathbb{A}_k = \mathbb{A}_k \overset{N}{D}_k(x) \tag{A-35}$$

which implies that

$$\det \left(y \mathbf{1} - \overset{N}{D}_k(x) \right) = \det \left(y \mathbf{1} - \underline{\overset{N}{D}}_k(x) \right) . \tag{A-36}$$

- The relationship between spectral curves for different k is:

$$\det \left(x \mathbf{1} - \overset{N}{D}_{k+1}(y) \right) \propto \det \left(y \mathbf{1} - V'_k(x) \mathbf{1} + \overset{N}{D}_k(x) \right) \quad \text{for } k > 1 , \tag{A-37}$$

and

$$\det \left(x \mathbf{1} - \overset{N}{D}_2(y) \right) \propto \det \left(y \mathbf{1} + \overset{N}{D}_1(x) \right) . \tag{A-38}$$

Proof: We prove ?? using the same method as for $m = 2$. Here is a sketch of the proof for $k > 1$.

Let $\tilde{\Psi}_{kN}(x)$ and $\tilde{\Phi}_{kN}(y)$ be any solutions of the differential systems

$$\frac{\partial}{\partial x} \tilde{\Psi}_{kN}(x) = \overset{N}{D}_k(x) \tilde{\Psi}_{kN}(x) , \quad \frac{\partial}{\partial y} \tilde{\Phi}_{kN}(y) = - \tilde{\Phi}_{kN}(y) \overset{N}{D}_k(y) . \tag{A-39}$$

We construct the functions $\tilde{\psi}_{k,n}(x)$ with $N - s_k - s_{k-1} \leq n \leq N + r_k + r_{k-1} - 1$ and $\tilde{\phi}_{k,n}(y)$ with $N - r_k - r_{k-1} \leq n \leq N + s_k + s_{k-1} - 1$ by recursively applying the shift operators (which are again compatible with the differential systems). This gives

$$(\partial_x + \partial_y) \tilde{\Phi}_{kN}(y) \mathbb{A}_k \tilde{\Psi}_{kN}(x) = (x - y) \tilde{\Phi}_{kN}(y) \mathbb{A}_{k-1} \tilde{\Psi}_{kN}(x) . \tag{A-40}$$

This equality holds term by term, since the coefficients for each monomial of type $\tilde{\psi}_{k,n}(x) \tilde{\phi}_{k,l}(y)$ are the same as when $\tilde{\psi}_{k,n}(x) = \psi_{k,n}(x)$ and $\tilde{\phi}_{k,l}(y) = \phi_{k,l}(y)$, which are linearly independent functions of x and y . By taking $x = y$ one has, for any $\tilde{\Phi}_{kN}(y)$ and $\tilde{\Psi}_{kN}(x)$,

$$\tilde{\Phi}_{kN}(x) \left(\overset{N}{D}_k(x) \mathbb{A}_k - \mathbb{A}_k \overset{N}{D}_k(x) \right) \tilde{\Psi}_{kN}(x) = 0 . \tag{A-41}$$

Since this holds for any basis of solutions, the factor in brackets (...) must vanish. Q.E.D.

Here is a sketch of the proof of (??); it is very similar to the proof of Prop. ??. First, we show how to prove that

$$\det(y\mathbf{1} + D_1(x)) \propto \det(x\mathbf{1} - D_2(y)), \quad (\text{A-42})$$

the other cases being similar.

First, notice that:

$$\begin{aligned} \det(y\mathbf{1} + D_1(x)) &= \frac{\det\left(-\frac{1N}{\alpha_{2,-s_2}}\right)}{\det\left(\begin{array}{ccc} a_1 & \dots & a_1 \\ N-s_2 & & N-1 \end{array}\right)} \\ &\times \det\left(\mathbf{1} - \begin{array}{cccc} a_1 & y & a_1 & \dots & a_1 \\ N-s_2 & \frac{1N}{\alpha_{2,-s_2}} & N-1 & N-s_2+1 & \\ & & \sum_{l=-s_2+1}^{r_2} & & \\ & & a_1 & \frac{1N}{\alpha_{2,-s_2}} & \\ & & & & a_1 \dots a_1 \\ & & & & N+l-1 & N-s_2+1 \end{array}\right). \end{aligned} \quad (\text{A-43})$$

Using lemma ??, the last determinant can be written as the determinant of a block matrix T_1 of size $(r_1 + s_1) \times (r_2 + s_2)$.

$$\det(y\mathbf{1} + D_1(x)) = c_1 \det(\mathbf{1} - T_1), \quad c_1 = \text{const.}, \quad (\text{A-44})$$

where

$$T_1 := \begin{bmatrix} 0 & \begin{array}{c} a_1 \\ N+r_2-1 \end{array} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \begin{array}{c} a_1 \\ N-s_2+1 \end{array} \\ -\begin{array}{c} a_1 \\ N-s_2 \end{array} \frac{\frac{1N}{\alpha_{2,r_2}}}{\frac{1N}{\alpha_{2,-s_2}}} & -\begin{array}{c} a_1 \\ N-s_2 \end{array} \frac{\frac{1N}{\alpha_{2,r_2-1}}}{\frac{1N}{\alpha_{2,-s_2}}} & \dots a_{1N-s_2} \frac{y - \frac{1N}{\alpha_{2,0}}}{\frac{1N}{\alpha_{2,-s_2}}} \dots & -\begin{array}{c} a_1 \\ N-s_2 \end{array} \frac{\frac{1N}{\alpha_{2,-s_2+1}}}{\frac{1N}{\alpha_{2,-s_2}}} \end{bmatrix}. \quad (\text{A-45})$$

On the other hand, by the same argument, we have

$$\det(x\mathbf{1} - D_2(y)) = c_2 \det(\mathbf{1} - T_2), \quad c_2 = \text{const.}, \quad (\text{A-46})$$

where

$$T_2 := \begin{bmatrix} 0 & \begin{array}{c} a_2 \\ N+r_1-1 \end{array} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \begin{array}{c} a_2 \\ N-s_1+1 \end{array} \\ -\begin{array}{c} a_2 \\ N-s_1 \end{array} \frac{\frac{2N}{\alpha_{1,r_1}}}{\frac{2N}{\alpha_{1,-s_1}}} & -\begin{array}{c} a_2 \\ N-s_1 \end{array} \frac{\frac{2N}{\alpha_{1,r_1-1}}}{\frac{2N}{\alpha_{1,-s_1}}} & \dots a_{2N-s_1} \frac{x - \frac{2N}{\alpha_{1,0}}}{\frac{2N}{\alpha_{1,-s_1}}} \dots & -\begin{array}{c} a_2 \\ N-s_1 \end{array} \frac{\frac{2N}{\alpha_{1,-s_1+1}}}{\frac{2N}{\alpha_{1,-s_1}}} \end{bmatrix}. \quad (\text{A-47})$$

It is easy to see that T_1 and T_2 are equal up to permutations of rows and of columns, and therefore they have the same determinant.

The other equalities with $k > 1$,

$$\det\left(x\mathbf{1} - D_{k+1}^N(y)\right) \propto \det\left(y\mathbf{1} - V_k'(x)\text{Id} + D_k^N(x)\right) \quad \text{for } k > 1, \quad (\text{A-48})$$

are proved by the same method and by induction on k . We define the sequence of functions $x_j(x, y)$, $1 \leq j \leq k + 1$, such that

$$x_k = x \quad , \quad x_{k+1} = y \quad , \quad x_{j-1} = V'_j(x_j) - x_{j+1} \quad 2 \leq j \leq k . \quad (\text{A-49})$$

We then prove by induction on j that

$$\det(x_{j-1} - D_j(x_j)) \propto \det(x_j - D_{j+1}(x_{j+1})) \quad 2 \leq j \leq k . \quad (\text{A-50})$$

Each step of the induction is similar to the method described above for $k = 1$. This completes the proof of ???. Q.E.D.

It can also be proven that all these systems are compatible with the shifts and deformations. It follows that if $\Phi_k(x)$ and $\Psi_k(x)$ denote fundamental solution matrices for the systems (??), it is possible to choose their normalizations such that

$$\Phi_k(x) \underset{N}{\mathbb{A}}_k \Psi_k(x) = \mathbf{1} . \quad (\text{A-51})$$

This may again be viewed as a form of the bilinear identities that allow us to deduce bilinear equations for τ -functions.

Acknowledgements. The authors would like to thank J. Hurtubise for helpful discussions relating to this work. The first two authors (MB, BE) would like to thank the CRM for support throughout the period (2000-2001) in which this work was completed.

References

- [1] M.R. Adams, J. Harnad, and J. Hurtubise, “Dual Moment Maps to Loop Algebras”, *Lett. Math. Phys.* **20**, 294–308 (1990).
- [2] M. Adler and P. Van Moerbeke, “String-orthogonal polynomials, string equations and 2-Toda symmetries”, *Comm. Pure and Appl. Math. J.*, **50** 241-290 (1997).
- [3] M. Adler and P. Van Moerbeke, “The Spectrum of Coupled Random Matrices”, *Ann. Math.* **149**, 921–976 (1999).
- [4] J. Baik, P. Deift and K. Johansson “The Longest increasing subsequence in a random permutation and a unitary random matrix model”, *J. Amer. Math. Soc.* **12**, 1119-1178 (1999).
- [5] T. Banks, W. Fischler, S. H. Shenker, L. Susskind, “M theory as a matrix model: A conjecture”, *Phys. Rev.* **D55**, 5112 (1997), hep-th/9610043.
- [6] M. Bertola, B. Eynard and J. Harnad, “Formal asymptotics of dual isomonodromic families and tau functions associated to two-matrix models”, (in preparation, 2001).
- [7] P.M. Bleher and A.R. Its, eds., “Random Matrix Models and Their Applications”, MSRI Research Publications **40**, Cambridge Univ. Press, (Cambridge, 2001).

- [8] D.V. Boulatov and V.A. Kazakov, “The Ising model on a random planar lattice: the structure of the phase transition and the exact critical exponents”, *Phys. Lett. B* **186**, 379 (1987).
- [9] S. Chadha, G. Mahoux, M.L. Mehta, “A method of integration over matrix variables 2.” *J. Phys. A: Math. Gen.* **14**, 579 (1981).
- [10] J.M. Daul, V. Kazakov, I.K. Kostov, “Rational Theories of 2D Gravity from the Two-Matrix Model”, *Nucl. Phys.* **B409**, 311-338 (1993), hep-th/9303093.
- [11] F. David, “Planar diagrams, two-dimensional lattice gravity and surface models”, *Nucl. Phys.* **B 257 [FS14]** 45 (1985).
- [12] P. Di Francesco, P. Ginsparg, J. Zinn-Justin, “2D Gravity and Random Matrices”, *Phys. Rep.* **254**, 1 (1995).
- [13] M. Douglas, “Strings in less than one dimension and the generalized KdV hierarchies”, *Phys. Lett.* **238B**, 176 (1990).
- [14] Michael R. Douglas, “The two-matrix model”, in: *Random surfaces and quantum gravity* (Cargèse, 1990), 77–83, NATO Adv. Sci. Inst. Ser. B Phys., **262**, Plenum, (New York, 1991).
- [15] B. Dubrovin, “Geometry of 2D topological field theory”, in: *Integrable systems and quantum groups*, M. Francaviglia, S. Greco eds., *Springer Lect. Notes in Math.* **1260**, 120-348 (1996).
- [16] F. Dyson, “Statistical theory of energy levels of complex systems, I, II and III” *Journ. Math. Phys.* **3**, 140-156, 157-165, 166-177 (1962).
- [17] F. Dyson, “Correlations between the eigenvalues of a random matrix” *Commun. Math. Phys.* **19**, 235 (1970).
- [18] N. M. Ercolani and K. T.-R. McLaughlin “Asymptotics and integrable structures for biorthogonal polynomials associated to a random two-matrix model”, *Physica D*, **152-153**, 232-268 (2001).
- [19] B. Eynard, “Eigenvalue distribution of large random matrices, from one matrix to several coupled matrices” *Nucl. Phys. B* **506**, 633 (1997), cond-mat/9707005.
- [20] B. Eynard, “Correlation functions of eigenvalues of multi-matrix models, and the limit of a time dependent matrix”, *J. Phys. A: Math. Gen.* **31**, 8081 (1998), cond-mat/9801075.
- [21] B. Eynard “An introduction to random matrices”, lectures given at Saclay, October 2000, notes available at <http://www-spht.cea.fr/articles/t01/014/>.
- [22] B. Eynard, M.L. Mehta, “Matrices coupled in a chain: eigenvalue correlations”, *J. Phys. A: Math. Gen.* **31**, 4449 (1998), cond-mat/9710230.
- [23] M. Gaudin, “Sur la loi limite de l’espacement des valeurs propres d’une matrice aléatoire”, *Nucl. Phys.* **25**, 447-458 (1960).
- [24] T. Guhr, A. Mueller-Groeling, H.A. Weidenmuller, “Random matrix theories in quantum physics: Common concepts”, *Phys. Rep.* **299**, 189 (1998).
- [25] J. Harnad and Alexander R. Its “Integrable Fredholm Operators and Dual Isomonodromic Deformations”, preprint CRM-2477, submitted to *Commun. Math. Phys.* (to appear 2001).

- [26] J. Harnad, “Dual Isomonodromic Deformations and Moment Maps into Loop Algebras”, *Commun. Math. Phys.* **166**, 337-365 (1994).
- [27] J. Harnad, “Dual Isomonodromic Tau Functions and Determinants of Integrable Fredholm Operators”, in: *Random Matrices and Their Applications*, MSRI Research Publications **40**, Cambridge Univ. Press, eds. P.M. Bleher and A.R. Its (Cambridge, 2001).
- [28] J. Harnad, C.A. Tracy and H. Widom, H., “Hamiltonian Structure of Equations Appearing in Random Matrices”, in: *Low Dimensional Topology and Quantum Field Theory*, ed. H. Osborn, pp. 231-245. (Plenum, New York, 1993).
- [29] A. R. Its, A. G. Izergin, V. E. Korepin, and N. A. Slavnov, “Differential Equations for Quantum Correlation Functions,” *Int. J. Mod. Phys.* **B4**, 1003–1037 (1990).
- [30] C. Itzykson and J.B. Zuber, “The planar approximation (II)”, *J. Math. Phys.* **21**, 411 (1980).
- [31] M. Jimbo, T. Miwa and K. Ueno, “Monodromy Preserving Deformation of Linear Ordinary Differential Equations with Rational Coefficients I.”, *Physica* **2D**, 306-352 (1981).
- [32] M. Jimbo and T. Miwa, “Monodromy Preserving Deformation of Linear Ordinary Differential Equations with Rational Coefficients II, III”, *Physica* **2D**, 407-448 (1981); *ibid.*, **4D**, 26–46 (1981).
- [33] K. Johansson, “Shape fluctuations and random matrices”, *Commun. Math. Phys.* **209**, 437-476 (2000).
- [34] N. Katz and P. Sarnak, “Random Matrices, Frobenius Eigenvalues and Monodromy”, *A.M.S. Colloquium Publications*, Vol. **45**, (1999).
- [35] V.A. Kazakov, “Ising model on a dynamical planar random lattice: exact solution”, *Phys Lett.* **A119**, 140-144 (1986).
- [36] I. Kostov, “Gauge Invariant Matrix Model for the A-D-E Closed Strings” *Phys. Lett.* **B297**, 74 (1992), hep-th/9208053.
- [37] M.L. Mehta, *Random Matrices*, 2nd edition, (Academic Press, New York, 1991).
- [38] M.L. Mehta, “A method of integration over matrix variables”, *Commun. Math. Phys.* **79**, 327 (1981).
- [39] M.L. Mehta and M. Gaudin, “On the density of eigenvalues of a random matrix”, *Nucl. Phys.* **18**, 420-427 (1960).
- [40] M.L. Mehta and P. Shukla, “Two coupled matrices: Eigenvalue correlations and spacing functions”, *J. Phys. A: Math. Gen.* **27**, 7793-7803 (1994).
- [41] A.M. Odlyzko, “On the distribution of spacings between the zeros of the zeta function”, *Math. Comp.* **48**, 273-308 (1987).
- [42] M. Praehofer and H. Spohn, “Universal distributions for growth processes in 1 + 1 dimensions and random matrices”, *Phys. Rev. Lett.* **84** (2000) 4882, cond-mat/9912264.
- [43] Z. Rudnick and P. Sarnak, “Zeros of principal L-functions and random matrix theory”, *Duke Math. J.* **81**, 269-322 (1996).
- [44] G. Szegő “Orthogonal Polynomials”, AMS, Providence, Rhode Island, (1939).

- [45] T. Tada, “ (q, p) critical points from the two-matrix models”, *Phys. Lett. B* **259**, 442 (1991). n
- [46] C.A. Tracy and H. Widom, “Introduction to random matrices” in: *Geometric and Quatum Aspects of Integrable Sysytems*, ed. G.F. Helminck, Springer Lecture Notes in Physics **424**, 103-130 (1993).
- [47] C.A. Tracy and H. Widom, “Fredholm determinants, differential equations and matrix models”, *Commun. Math. Phys.* **161**, 289-309 (1994).
- [48] K. Ueno and K. Takasaki, “Toda Lattice Hierarchy”, *Adv. Studies Pure Math.* **4**, 1–95 (1984).
- [49] A. Userkesm, S. P. Norset, “Christoffel-Darboux-Type Formulae and a Recurrence for Biorthogonal Polynomials”, *Constructive Approximation*, **5**, 437-454 (1989).
- [50] J.J.M. Verbaarshot, “Random matrix model approach to chiral symmetry”, *Nucl. Phys. Proc. Suppl.* **53**, 88 (1997).
- [51] E.P. Wigner, “On the statistical distribution of widths and spacings of nuclear resonancne levels”, *Proc. Cambridge Phil. Soc.* **47**, 790-798 (1951).
- [52] P. Zinn-Justin, “Universality of correlation functions of hermitian random matrices in an external field”, *Commun. Math. Phys.* **194**, 631 (1998), cond-mat/9705044.

Coeff. of	Eq. (??)	Eq. (??)
$\tilde{\psi}_{N-1}\tilde{\phi}_{N-1}$	$\gamma^2(N-1)+\sum_{j=1}^{d_1}\alpha_j(N+j-1)\beta_j(N+j-1)$	$\gamma^2(N-1)+\sum_{j=1}^{d_2}\alpha_j(N+j-1)\beta_j(N+j-1)$
$\tilde{\psi}_N\tilde{\phi}_N$	$-\gamma^2(N-1)-\sum_{j=1}^{d_1}\alpha_j(N)\beta_j(N)$	$-\gamma^2(N-1)-\sum_{j=1}^{d_2}\alpha_j(N)\beta_j(N)$
$\tilde{\psi}_{N+p}\tilde{\phi}_{N+q}$ $p \geq 0, q \geq 0$ $p+q \geq 1$	$-\sum_{l=q+1}^{d_2}\alpha_l(N+q)\beta_{p-q+l}(N+p)$	$-\sum_{l=q+1}^{d_2}\alpha_l(N+q)\beta_{p-q+l}(N+p)$
$\tilde{\psi}_{N-p}\tilde{\phi}_{N-q}$ $p \geq 1, q \geq 1$ $p+q \geq 3$	$\sum_{k=0}\alpha_{k+p}(N+k)\beta_{k+q}(N+k)$	$\sum_{k=0}\alpha_{k+p}(N+k)\beta_{k+q}(N+k)$
$\tilde{\psi}_{N-1}\tilde{\phi}_N$	$\gamma(N-1)\alpha_0(N)-\gamma(N-1)\alpha_0(N-1)$	$\sum_{j=1}\alpha_j(N-1+j)\beta_{j-1}(N-1+j)$ $-\sum_{j=1}\alpha_j(N)\beta_{j-1}(N-1)$
$\tilde{\psi}_N\tilde{\phi}_{N-1}$	$\sum_{j=1}\alpha_{j-1}(N+j-1)\beta_j(N+j-1)$ $-\sum_{j=1}\alpha_{j-1}(N-1)\beta_j(N)$	$\gamma(N-1)\beta_0(N)-\gamma(N-1)\beta_0(N-1)$
$\tilde{\psi}_N\tilde{\phi}_{N-q}$ $q \geq 2$	$\sum_{j=0}\alpha_j(N+j)\beta_{j+q}(N+j)$ $-\sum_{j=-1}\beta_{j+q}(N)\alpha_j(N-q)$	$-\gamma(N-1)\beta_{q-1}(N-1)$
$\tilde{\psi}_{N+p}\tilde{\phi}_{N-1}$ $p \geq 1$	$\sum_{j=-1}\alpha_j(N+p+j)\beta_{j+p+1}(N+p+j)$ $-\sum_{j=0}\alpha_j(N-1)\beta_{j+p+1}(N+p)$	$\gamma(N-1)\beta_p(N+p)$
$\tilde{\psi}_{N-1}\tilde{\phi}_{N+q}$ $q \geq 1$	$\gamma(N-1)\alpha_q(N+q)$	$\sum_{j=p}\alpha_j(N+j-1)\beta_{j-q-1}(N+j-1)$ $-\sum_{j=0}\alpha_{j+q+1}(N+q)\beta_j(N-1)$
$\tilde{\psi}_{N-p}\tilde{\phi}_N$ $p \geq 2$	$-\gamma(N-1)\alpha_{p-1}(N-1)$	$\sum_{j=0}\alpha_{j+p}(N+j)\beta_j(N+j)$ $-\sum_{j=-1}\alpha_{j+p}(N)\beta_j(N-p)$
$\tilde{\psi}_{N+p}\tilde{\phi}_{N-q}$ $q \geq 2, p \geq 0$	$\sum_{j=-1}\alpha_j(N+j+p)\beta_{j+p+q}(N+j+p)$ $-\sum_{j=-1}\beta_{j+p+q}(N+p)\alpha_j(N-q)$	0
$\tilde{\psi}_{N-p}\tilde{\phi}_{N+q}$ $q \geq 0, p \geq 2$	0	$\sum_{j=q-1}\alpha_{j+p}(N+j)\beta_{j-q}(N+j)$ $-\sum_{j=-1}\alpha_{j+p+q}(N+q)\beta_j(N-p)$

We have defined $\alpha_j := 0$ if $j \notin [-1, \dots, d_2]$ and $\beta_j \equiv 0$ if $j \notin [-1, \dots, d_1]$.

Table 1: Comparison of coefficients in eq.(??) and eq.(??).